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Bidirectional and Switched Controlled Teleportations

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Dj.Dalal

Chapter 1

General introduction

Quantum information mainly exploits the effects of superposition and measurement in fields such as computing and cryptography. Superposition and measurement, two complementary channels of quantum mechanics, one enriching evolution and the other limiting it; This way of seeing things sets it apart from classical mechanics, which is defined as completely deterministic , in the latter case, the state of a physical system at any given moment is completely determined by the initial data and in addition the measurement makes it possible to determine it in a perfect way. In quantum mechanics, on the other hand, we admit and generalize this principle of superposition to all physical systems, but we confer on measurement a particular status which delimits the effect of this superposition. Quantum mechanics by its strangeness and its mystery has brought physics to a flawless advance that has been confirmed by properly probing matter in its microscopic states in extraordinary ways. Of course, classical mechanics also made it possible to properly probe nature, but came up against the problem of the interaction of light with matter at the microscopic level: to mention only the radiation of the black body and the spectrum of atoms. The foundation of the new mechanics allowed an almost perfect survey of this phenomenon of matter radiation interaction and a new technology was built on it. A proliferating development in computer science and communications has emerged. Moreover, since its advent an opposition to these new ideas has formed, trying to question the basic of this quantum mechanics. Several of its founders such as Einstein, De Brogly and Schrodinger did not accept it as a complete theory and tried to build physical models showing its incompleteness. Two well-known examples are the EPR paradox and schrödinger's cat. which exploit the principle of superposition in its deepest version: the entangled states(inseparable).

These models then assume the existence of hidden variables that quantum mechanics does not take into account. Bell's inequalities, due to J.S.Bell, are the relations obeyed by measurements on entangled states under the assumption of a deterministic theory with local hidden variables (complete according to the EPR argument). The Alain aspect experiment has shown that Bell's inequalities are violated, and it forces us to once again give up causal and local physics at the same time. J.S Bell then argues for a non-local deterministic physics like Bohm's deterministic mechanics. The status of quantum mechanics remains sovereign and the physicists of the new generation have preferred to continue this philosophical debate in a more defiant atmosphere which is that of experiments. We are now attending a new era of nanoscals technology allowing by its tiny smallness, to ensure even more the validity of quantum concepts and to lead us to incredible feats such as quantum teleportation. In addition, scientists have begun to imagine the fallout from this these miniaturized achievements of quantum mechanics and we are seeing the birth of new sciences such as quantum computing and quantum cryptography allowing unprecedented power in comparison with that has been done previously. The objective of this thesis is an initiation to quantum computation and presentation of the essential properties of teleportation.

In the second chapter,we present essential notions to the understanding of the quantum information begin with defining the quantum bit and some quantum gates and the rules that must be respected considering a quantum system ,or 'the postulates of quantum mechanics, .in the next two chapter we take two exemplar of bidirectional teleportation with and without control.

In the chapter number five, we consider a switched controlled teleportation , we examine this teleportation according to the two formalisms, that of kets, then the protocol with noise on density matrix formalism, where we measure the quality of the teleportation by defining the coefficient F of average fidelity. we introduced the fidelity deviation in the next chapter, where we take the example of an arbitrary two-qubit state, then we derive the formula for fidelity deviation for a noisy five-qubit state following the protocol of switched teleportation (in chapter5).Finally, we present a general conclusion.

Chapter 2

A brief introduction to quantum information

2.1 Introduction

In this chapter we will introduce some essential notions to the understanding of the quantum information, and the quantum rules that must be respected for a quantum state.

2.2 Quantum information

2.2.1 Quantum bit

In the classical theory of information,the indivisible unite is called "bit", it takes the values 0 and 1.;a message is a succession of this values. In the quantum information, we define in a similar way the fundamental concept of the quantum bit or briefly the 'qubit' which has two possible states $|0\rangle$ and $|1\rangle$, the difference is that the qubit is in a superposed state.

$\Rightarrow$ The qubit is a physical system in superposition of two states

represented by an unitary vector $|\Psi\rangle$ in a two dimensional space as

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \tag{2.1}$$

where α and β are complex numbers, and $\left\{|0\rangle = \binom{1}{0}, |1\rangle = \binom{0}{1}\right\}$ the base vectors.

When we perform a measurement on it, we will get either $|0\rangle$ with a probability $|\alpha|^2$,or the result $|1\rangle$ with probability $|\beta|^2$,where $|\alpha|^2 + |\beta|^2$ =1 (the condition of normalization)

The qubit could be represented as a point$(0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi)$ in an unite sphere, called Bloch sphere, in polar coordinates :

$$|\Psi\rangle = e^{i\gamma}\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right] \tag{2.2}$$

neglecting the phase factor $e^{i\gamma}$, (2.2) will be:

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle \tag{2.3}$$

2.2.2 Quantum gates:

Unitary operation I the unitary matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv |0\rangle\langle 0| + |1\rangle\langle 1|$$

this operation represented by the following transformations

$$\begin{aligned} I &: |0\rangle \longrightarrow |0\rangle \\ I &: |1\rangle \longrightarrow |1\rangle \end{aligned}$$

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.4}$$

defined by the transformations:

$$\left\{\sigma_x \text{ (not-gate)}: \begin{matrix} |0\rangle \longrightarrow |1\rangle \\ |1\rangle \longrightarrow |0\rangle \end{matrix}\right\} \quad \left\{\sigma_y : \begin{matrix} |0\rangle \longrightarrow -|1\rangle \\ |1\rangle \longrightarrow |0\rangle \end{matrix}\right\} \quad \left\{\sigma_z : \begin{matrix} |0\rangle \longrightarrow |0\rangle \\ |1\rangle \longrightarrow -|1\rangle \end{matrix}\right\} \tag{2.5}$$

Hadamard gate:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \equiv \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \tag{2.6}$$

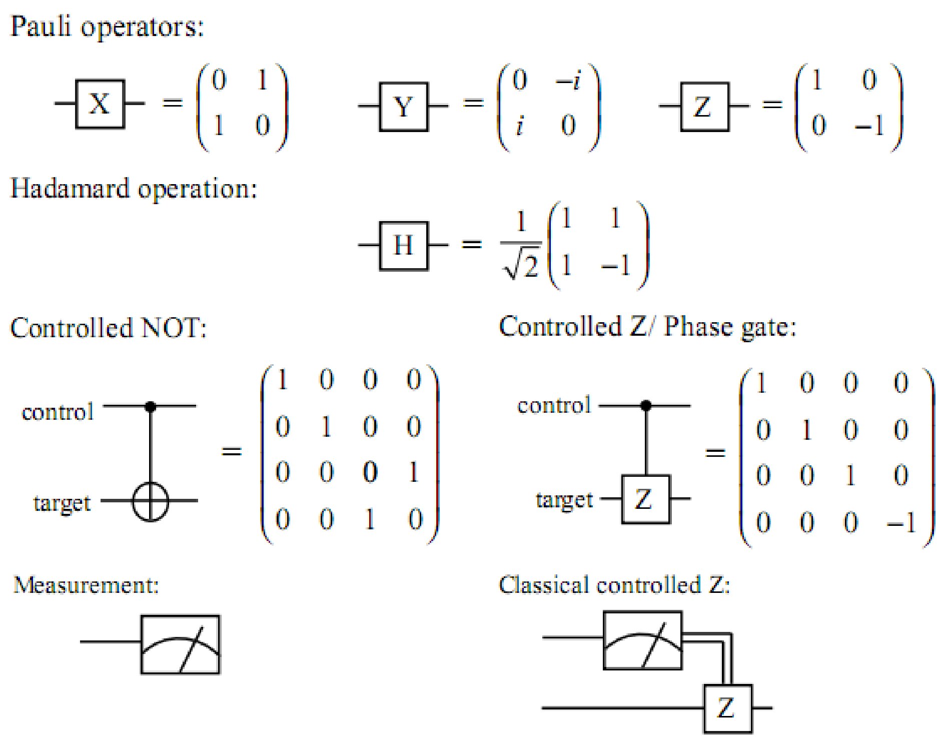
$$\begin{aligned} H &: |0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ H &: |1\rangle \longrightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned} \tag{2.7}$$

Controlled-not gate:

$$U_{cnot} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \equiv |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| \tag{2.8}$$

$$U_{cnot} : \left\{ \begin{matrix} |00\rangle \longrightarrow |00\rangle \\ |01\rangle \longrightarrow |01\rangle \\ |10\rangle \longrightarrow |11\rangle \\ |11\rangle \longrightarrow |10\rangle \end{matrix} \right\} \tag{2.9}$$

Symbols of quantum gates in circuits:



2.3 The postulates of quantum mechanics

The following postulates are for the description of an isolated system, where there is no environment in interaction with the system(quantum postulates for a closed system)

The formalism of quantum mechanics will be modified if we consider the problem of describing a system in interaction with the environment, in this case, we have to introduce the formalism of the density operator, and a general concept of measurement: POVM (Positive Operator Valued Measure).

Postulate 1:

we associate with every quantum system a complex linear space having the structure of a Hilbert space.

the state of the system represented by a vector of norm equal to 1,or a qubit.

$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $\langle\Psi| |\Psi\rangle = |\alpha|^2 + |\beta|^2 = 1$

which is the fundamental state of a system in the quantum information, it can be realized physically in different ways: states of atoms, spin, photons,...

Postulate 2:

To every observable (property) of a quantum system, which can be measured , there corresponds a linear operator $\{A\}$ act on the state space of that system,

$$\begin{aligned} A &: |\Psi\rangle \rightarrow A|\Psi\rangle \\ A(a|\Psi\rangle + b|\Phi\rangle) &= a(A|\Psi\rangle) + b(A|\Phi\rangle) \\ \langle\Phi| |A\Psi\rangle &= \left\langle A^{+}\Phi\right| |\Psi\rangle \end{aligned}$$

It can be developed as:

$$A = \sum_n a_n P_n$$

where a_n is the eigenvalue, and P_n is the projector in the corresponding subspace, it verifies the following relations:

$$\begin{aligned} P_n P_m &= \delta_{nm} P_n \\ P_n^{+} &= P_n \\ \sum_n P_n &= 1 \\ \sum_n P_n^{+} P_n &= 1 \end{aligned}$$

Postulate 3:

A quantum measurement is described by a set $\{M_n\}$ of measurement operators, which act on the state space of the quantum system.

If the system is in the state $|\Psi\rangle$ initially, after measurement the probability of finding the system in the state (n) as a result is:

$$P(n) = \langle\Psi| M_n^+ M_n |\Psi\rangle \tag{2.10}$$

And it will be in the state:

$$\left|\tilde{\Psi}\right\rangle = \frac{M_n |\Psi\rangle}{\sqrt{\langle\Psi| M_n^+ M_n |\Psi\rangle}} \tag{2.11}$$

Postulate 4:

The evolution of a closed quantum system described by an unitary operation

$$|\Psi(t_2)\rangle = U(t_1, t_2) |\Psi(t_2)\rangle \tag{2.12}$$

Where, $U(t_1, t_2)$ is an unitary operator $(U^+U = I)$

More precisely, the evolution in time described by the Schrödinger equation

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle \tag{2.13}$$

Where Ĥ is the Hamiltonian

$$U(t_1, t_2) \equiv \exp\left[-i\frac{\hat{H}}{\hbar}(t_2 - t_1)\right] \tag{2.14}$$

Projective measurement:

The set $\{M_n\}$ of measurement operators admits an interesting particular case which is projective measurements.

A projective measurement described by an observable M in which the set of projectors is $\{P_n\}$,with

$$M = \sum_n a_n P_n = \sum_n n P_n \tag{2.15}$$

The probability of finding a_n is

$$P(a_n) \equiv P(n) = \langle\Psi |P_n| \Psi\rangle = \left\langle\Psi \left|P_n^+ P_n\right| \Psi\right\rangle \tag{2.16}$$

After the measurement, we have:

$$|\Psi\rangle \rightarrow \left|\tilde{\Psi}\right\rangle = \frac{P_n |\Psi\rangle}{\sqrt{P_n}} \tag{2.17}$$

This set of measurement projectors is a special case of the set $\{M_n\}$of operators.

The average value of projective measurement: Let M the observable of measure ,the average value of this observable is:

$$\begin{aligned}\varepsilon(M) &= \sum_n a_n P(n) = \sum_n a_n \langle\Psi| P_n |\Psi\rangle = \langle\Psi| \sum_n a_n P_n |\Psi\rangle \\ &= \langle\Psi| M |\Psi\rangle = \langle M\rangle_\Psi \end{aligned} \tag{2.18}$$

The quadratic deviation is defined by:

$$\Delta(M) = \sqrt{\langle M^2\rangle_\Psi - \langle M\rangle^2{}_\Psi} \tag{2.19}$$

2.4 Density operator:

The quantum mechanics could be formed in the density operator(the density matrix) formalism, which is more suitable and more compatible with almost all the scenarios that we can meet.

Suppose that the state of the system is not completely known, it can be in a state $|\Psi_i\rangle$ with the probability P_i, so we have:$\{P_i, |\Psi_i\rangle\}$ the set of pure states

We define the density operator or the density matrix by the equation:

$$\begin{aligned}\rho &: H \longrightarrow H \\ \rho &: \sum_i P_i |\Psi_i\rangle \langle\Psi_i| \end{aligned} \tag{2.20}$$

$$\rho^2 = |\Psi\rangle \underbrace{\langle\Psi| |\Psi\rangle} \langle\Psi| = |\Psi\rangle \langle\Psi| = \rho \tag{2.21}$$

Let U a quantum operation :

$$\begin{aligned}|\Psi\rangle &\rightarrow \left|\tilde{\Psi}\right\rangle = U|\Psi\rangle \\ \rho &\rightarrow \rho^{(U)} = \sum_i P_i \left|\tilde{\Psi}\right\rangle \left\langle\tilde{\Psi}\right| \\ &= \sum_i P_i U |\Psi\rangle \langle\Psi| U \\ &= U\rho U^+ \\ \rho &\rightarrow \tilde{\rho} = U\rho U^+ \end{aligned} \tag{2.22}$$

The system is initially in the state $|\Psi_i\rangle$,the probability of finding the result (m) is:

$$P(m|i) = \langle\Psi_i| M_m^+ M_m |\Psi_i\rangle$$

or:

$$
\begin{aligned}
P(m|i) &= \sum_{\lambda} \langle\Psi_i| \,|\lambda\rangle \langle\lambda| M_m^{+} M_m |\Psi_i\rangle \\
&= \sum_{\lambda} \langle\lambda| M_m^{+} M_m |\Psi_i\rangle \langle\Psi_i| \,|\lambda\rangle \\
&= Tr(M_m^{+} M_m |\Psi_i\rangle \langle\Psi_i|)
\end{aligned}
$$

If the initial state is not completely known, the probability is:

$$
\begin{aligned}
P(m) &= \sum_{i} P_i P(m|i) \\
&= \sum_{i} P_i Tr(M_m^{+} M_m |\Psi_i\rangle \langle\Psi_i|) \\
&= Tr(M_m^{+} M_m \rho)
\end{aligned}
$$

Just after the measurement, the system will be in the state:

$$
|\Psi_i\rangle^m = \frac{M_m |\Psi_i\rangle}{\sqrt{\langle\Psi_i| M_m^{+} M_m |\Psi_i\rangle}}
$$

In the density matrix formalism we have:

$$
\begin{aligned}
\rho_{(m)} &= \sum_{i} P(m|i) |\Psi_i^m\rangle \langle\Psi_i^m| \\
&= \sum_{i} P(m|i) \frac{M_m |\Psi_i\rangle \langle\Psi_i| M_m^{+}}{\langle\Psi_i| M_m^{+} M_m |\Psi_i\rangle} \\
&= \sum \frac{P(m|i) P_i}{P(m)} \frac{M_m |\Psi_i\rangle \langle\Psi_i| M_m^{+}}{P(m|i)} \\
&= \frac{M_m \rho M_m^{+}}{Tr(M_m \rho M_m^{+})}
\end{aligned}
\tag{2.23}
$$

In the case where the system is in a pure state,or the state of the system is completely known:

$$
\begin{aligned}
\rho &= |\psi\rangle \langle\psi| \\
\rho^2 &= |\psi\rangle \underbrace{\langle\psi| \,|\psi\rangle} \langle\psi| = \rho \\
Tr\rho &= Tr\rho^2 = \sum_{\lambda} \langle\lambda| \,|\psi\rangle \langle\psi| \,|\lambda\rangle \\
&= \langle\psi| \sum_{\lambda} |\lambda\rangle \langle\lambda| \,|\psi\rangle \\
&= \langle\psi| \,|\psi\rangle = 1
\end{aligned}
$$

In the case of mixed state:

$$
\begin{aligned}
\rho &= \sum_i P_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right| \\
Tr(\rho) &= Tr(\sum_i P_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right|) \\
&= \sum_{j=1} \left\langle j \sum_i P_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right| j \right\rangle \\
&= \sum_i P_i \sum_{j=1} \left\langle j \left|\Psi_i\right\rangle \left\langle\Psi_i\right| j \right\rangle \\
&= \sum_i \sum_j P_i \left\langle\Psi_i | j\right\rangle \left\langle j | \Psi_i\right\rangle \\
&= \sum_i P_i \left\langle\Psi_i\right| \sum_j \left|j\right\rangle \left\langle j\right| \left|\Psi_i\right\rangle = \sum_i P_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right| \\
&= \sum_i P_i = 1
\end{aligned}
$$

Reduced density matrix:

In the composed system$\{AB\}$ which described by the density matrix ρ^{AB}, we can describe every part by a density matrix

$$
\begin{aligned}
\rho^A &= Tr_B(\rho^{AB}) \\
\rho^B &= Tr_A(\rho^{AB})
\end{aligned}
$$

where Tr_B is the partial trace on B and Tr_A is the partial trace on A, we define in the subspaces $\mathcal{H}_A$ and $\mathcal{H}_B$ as follows

$$Tr_B(\rho^{AB}) = \sum_{j=1}^{N_B} (I_A \otimes \left\langle\phi_j\right|)\rho^{AB}(I_A \otimes \left|\phi_j\right\rangle) \tag{2.24}$$

and

$$Tr_A(\rho^{AB}) = \sum_{j=1}^{N_A} (\left\langle\psi_j\right| \otimes I_B)\rho^{AB}(\left|\psi_j\right\rangle \otimes I_B) \tag{2.25}$$

where I_A and I_B are the identity operators in $\mathcal{H}_A$ and $\mathcal{H}_B$,$\left|\psi_j\right\rangle (j = 1, 2, ..N_A)$ and $\left|\phi_j\right\rangle (j = 1, 2, ..N_B)$ are orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

2.5 Quantum entanglement:

Quantum entanglement is one of the most surprising aspects in quantum physics, its results are contrary to the common sense, which led Einstein, Podolsky and Rosen to affirm that the quantum mechanics is incomplete, in conclusion some variables (hidden: local-realism theories) are missed in quantum mechanics;it is about "strange states"in which two particles (or more) are profoundly linked in the quantum level that share the same 'existence even at great distance. If a measurement is taken on one of them, the state of the other is also altered(changed) instantly, in order to be compatible with the first one.

This debate between the quantum mechanics and the theories of additional hidden variables (response of Neils Bohr on the article EPR) motivated John Bell to introduce, in 1964, a set of inequalities before being proved by any theory that is local and realistic against this predictions and limits attempted by the property of the reality-locality, the quantum mechanics foresees a violation of its inequalities.

The inequalities are worth changing the debate from epistemology to experimental physics,thus, in the early years of 1980, the experiences conducted by Alain Aspect, Jean Dalibad, Philipe Grangier and Gerard Roger([6][7][8]) had verified, in a clear way, the forecasts of the quantum mechanics.

Despite the success of quantum mechanics; the human spirit or rather say the classic philosophy does not let go and reinvents artefacts(sometimes useless)for trying to restore the properties"local" and "realistic" at a time.

A review of two weak points or "loopholes" has been established and it affects the experimental device. Consequently negative, tests of "ideal experiments" would not allow a definitive and a complete validation of the quantum mechanics against other theories.

The first of this weak points is called "loophole of locality" intervenes as soon as the separation between systems of detections is not sufficient to reject any possibility of information exchange from lower speed to the speed of light during the measurement([9][10]). The second weak point or "loophole of the effectiveness of detections" appears when the effectiveness of the system of detections is not sufficiently high so that the measured events could be not representative to the statistic of the whole([11][12]).

In 1982, Alain Aspect and his collaborators([8]) carried out a first experience of Bell's inequalities test with analyzers which vary quickly compared to the time of flight of entangled photons. This allows to separate the detectors by intervals type of space and by enclosing the locality loophole, this experience had been repeated in 1998 by Antoon Zeilingr and his team([13]) who used a separation of detectors 400m and fast analysers.These experiences allowed to validate the predictions of quantum physics compared to locality loophole, but the effectiveness of employed detectors weren't enough to enclose the second loophole.

2.5.1 Bell's states:

The EPR argument:

In the 1935 article, Albert Einstein, Bolish Podolsky and Nathan Rosen, enunciate:'without disturbing a system, we can predict with certanty,that is to say with probability equal to the unit, the value of a physical quantity, then there exist an element of the physical reality corresponding to this physical quantity.

In the EPR argument that disputes the possibility of a distance correlation between particles by making the hypothesis of a missing interpretation that can resolve this paradox.

Bell's theory:

In 1964, on a paper in the physics journal on "the EPR paradox", John Bell shows that, in the context of the quantum physics a measurement by a measuring device in one region can influence the measurement of another device from a second region even at a great distance.In his argument he uses the spin instead of the position and momentum, to show that the correlation between two spins is subject to an inequality which can be violated by the postulates of quantum physics. The article has highlighted the entanglement, and the entangled states bringing its effect into play are called "Bell's states",which are defined by

$$|B_{xy}\rangle = \frac{1}{\sqrt{2}}(|0y\rangle + (-1)^x |1\bar{y}\rangle) \tag{2.26}$$

with x=0,1 and y=0,1. The four possible states are:

$$|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.27}$$

$$|B_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{2.28}$$

$$|B_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{2.29}$$

$$|B_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{2.30}$$

2.6 Fidelity:

The quality of a transmission canal in teleportation is judged by comparing the initial state and the final state, characterized by the fidelity which is a quantity f takes the values between 0 and 1 ($0 \leq f \leq 1$).

The mean value of fidelity is considered as the standard quantity of teleportation, which is a measure of the expected proximity between an input state and its target state [14], given by

$$f = Tr\left[\sqrt{\hat{\tau}}\hat{\varrho}_\phi\sqrt{\hat{\tau}}\right] \tag{2.31}$$

where $\hat{\tau}$ is the density matrix of the target state, and $\hat{\varrho}_{\phi}$ is the transformed state of the input state$|\phi\rangle$

but it tells nothing about its fluctuations. such fluctuation can be quantified by the deviation of fidelity ovell all input states.

When the initial and the final state are equivalent: $f = 1$, when the canal is maximally entangled.

For quantum teleportation$f > \frac{2}{3}$, f=$\frac{2}{3}$ is the maximally value of fidelity in classical teleportation which can not use the entanglement, its lower value in this case is $\frac{1}{2}$.

Chapter 3

Bidirectional quantum teleportation

Two users transmit simultaneously an unknown single-qubit state to each other[15]

the two users (Alice and Bob), have the single-qubits:

$$|\phi\rangle_A = \alpha_0 |0\rangle + \alpha_1 |1\rangle \tag{3.1}$$

$$|\phi\rangle_B = \beta_0 |0\rangle + \beta_1 |1\rangle \tag{3.2}$$

They share the following quantum channel :

$$|\phi\rangle_{a1b1a2b2} = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)_{a1b1a2b2} \tag{3.3}$$

where the qubits a_1, a_2 belong to Alice and b_1, b_2 belong to Bob,

The general state is

$$\begin{aligned} |\Psi\rangle_{a1b1a2b2AB} &= |\Phi\rangle_{a1b1a2b2} \otimes |\Phi\rangle_A \otimes |\Phi\rangle_B \\ &= \frac{1}{2}[(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)_{a1b1a2b2}\alpha_0\beta_0 |00\rangle_{AB} \\ &+(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)_{a1b1a2b2}\alpha_0\beta_1 |01\rangle_{AB} \\ &+(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)_{a1b1a2b2}\alpha_1\beta_0 |10\rangle_{AB} \\ &+(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)_{a1b1a2b2}\alpha_1\beta_1 |11\rangle_{AB}]. \end{aligned} \tag{3.4}$$

Alice and Bob perform a controlled-not operation with A and B as control qubits, $a_1$and b_2 as targets respectively.

we obtain:

$$\begin{aligned}\left|\Psi'\right\rangle_{a1b1a2b2AB} &= \frac{1}{2}[(|0000\rangle+|0011\rangle+|1100\rangle+|1111\rangle)_{a1b1a2b2}\alpha_0\beta_0\left|00\right\rangle_{AB}\\&+(|0001\rangle+|0010\rangle+|1101\rangle+|1110\rangle)_{a1b1a2b2}\alpha_0\beta_1\left|01\right\rangle_{AB}\\&+(|1000\rangle+|1011\rangle+|0100\rangle+|0111\rangle)_{a1b1a2b2}\alpha_1\beta_0\left|10\right\rangle_{AB}\\&+(|1001\rangle+|1010\rangle+|0101\rangle+|0110\rangle)_{a1b1a2b2}\alpha_1\beta_1\left|11\right\rangle_{AB}]. \end{aligned} \tag{3.5}$$

we apply single-qubit measurement in the Z-basis on qubits $a_1, b_2$and in the X-basis on qubits A and B respectively.

by projecting on all the possible results, where the base vectors of the Z and the X basis are $\{|0\rangle, |1\rangle\}$ and$\left\{|+\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle), |-\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right\}$respectively.

(1)_If the result of Alice is: $|0\rangle_{a_1}|+\rangle_A$ and Bob's result is $|0\rangle_{b_1}|+\rangle_B$:

$$\begin{aligned}\left|\tilde{\Psi}'_1\right\rangle_{a1b1a2b2AB} &= {}_{a1}\langle 0|_A\langle +|_{b2}\langle 0|_B\langle +|\left|\Psi'\right\rangle_{a1b1a2b2AB}=\frac{1}{2}{}_{a_1}\langle 0|_A(\langle 0|+\langle 1|)_{b2}\langle 0|_B(\langle 0|+\langle 1|)\left|\Psi'\right\rangle_{a1b1a2b2AB}\\&= \frac{1}{4}(\alpha_0\beta_0|00\rangle+\alpha_0\beta_1|01\rangle+\alpha_1\beta_0|10\rangle+\alpha_1\beta_1|11\rangle)_{b_1a_2}\end{aligned} \tag{3.6}$$

The collapsed state of the qubits b_1a_2 is:

$$\begin{aligned}|\Omega\rangle_1 &= \frac{\left|\tilde{\Psi}'_1\right\rangle}{\sqrt{\left\langle\tilde{\Psi}'_1\middle|\tilde{\Psi}'_1\right\rangle}}=\frac{\frac{1}{4}(\alpha_0\beta_0|00\rangle+\alpha_0\beta_1|01\rangle+\alpha_1\beta_0|10\rangle+\alpha_1\beta_1|11\rangle)}{\frac{1}{4}\sqrt{|\alpha_0|^2|\beta_0|^2+|\alpha_0|^2|\beta_1|^2+|\alpha_1|^2|\beta_0|^2+|\alpha_1|^2|\beta_1|^2}}\\&= (\alpha_0\beta_0|00\rangle+\alpha_0\beta_1|01\rangle+\alpha_1\beta_0|10\rangle+\alpha_1\beta_1|11\rangle)_{b_1a_2}\end{aligned} \tag{3.7}$$

The remaining possible results of measurement are :

(2)_If the result of Alice is: $|0\rangle_{a_1}|+\rangle_A$ and Bob's result is $|0\rangle_{b_1}|-\rangle_B$:

$$\begin{aligned}\left|\tilde{\Psi}'_2\right\rangle &= {}_{a_1}\langle 0|_A\langle +|_{b2}\langle 0|_B\langle -|\left|\Psi'\right\rangle_{a1b1a2b2AB}\\&= \frac{1}{2}{}_{a_1}\langle 0|_A(\langle 0|+\langle 1|)_{b2}\langle 0|_B(\langle 0|-\langle 1|)\left|\Psi'\right\rangle_{a1b1a2b2AB}\\|\Omega\rangle_2 &= (\alpha_0\beta_0|00\rangle-\alpha_0\beta_1|01\rangle+\alpha_1\beta_0|10\rangle-\alpha_1\beta_1|11\rangle)_{b_1a_2}\end{aligned} \tag{3.8}$$

(3)_If the result of Alice is: $|0\rangle_{a_1}|-\rangle_A$ and Bob's result is $|0\rangle_{b_1}|+\rangle_B$:

$$\begin{aligned}\left|\tilde{\Psi}'_3\right\rangle &= {}_{a_1}\langle 0|_A\langle -|_{b2}\langle 0|_B\langle +|\left|\Psi'\right\rangle_{a1b1a2b2AB}\\&= \frac{1}{2}{}_{a_1}\langle 0|_A(\langle 0|-\langle 1|)_{b2}\langle 0|_B(\langle 0|+\langle 1|)\left|\Psi'\right\rangle_{a1b1a2b2AB}\\|\Omega\rangle_3 &= (\alpha_0\beta_0|00\rangle+\alpha_0\beta_1|01\rangle-\alpha_1\beta_0|10\rangle-\alpha_1\beta_1|11\rangle)_{b_1a_2}\end{aligned} \tag{3.9}$$

(4)_If the result of Alice is: $|0\rangle_{a_1}|-\rangle_A$ and Bob's result is $|0\rangle_{b_1}|-\rangle_B$:

$$\begin{aligned}\left|\tilde{\Psi}'_4\right\rangle &= {}_{a_1}\langle 0|_A\langle -|_{b2}\langle 0|_B\langle -|\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ &= \frac{1}{2}{}_{a_1}\langle 0|_A\left(\langle 0|-\langle 1|\right)_{b2}\langle 0|_B\left(\langle 0|-\langle 1|\right)\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ |\Omega\rangle_4 &= \left(\alpha_0\beta_0|00\rangle-\alpha_0\beta_1|01\rangle-\alpha_1\beta_0|10\rangle+\alpha_1\beta_1|11\rangle\right)_{b_1a_2}\end{aligned} \tag{3.10}$$

(5)_If the result of Alice is: $|0\rangle_{a_1}|+\rangle_A$ and Bob's result is $|1\rangle_{b_1}|+\rangle_B$:

$$\begin{aligned}\left|\tilde{\Psi}'_5\right\rangle &= {}_{a_1}\langle 0|_A\langle +|_{b2}\langle 1|_B\langle +|\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ &= \frac{1}{2}{}_{a_1}\langle 0|_A\left(\langle 0|+\langle 1|\right)_{b2}\langle 1|_B\left(\langle 0|+\langle 1|\right)\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ |\Omega\rangle_5 &= \left(\alpha_0\beta_0|01\rangle+\alpha_0\beta_1|00\rangle+\alpha_1\beta_0|11\rangle+\alpha_1\beta_1|10\rangle\right)_{b_1a_2}\end{aligned} \tag{3.11}$$

(6)_If the result of Alice is: $|0\rangle_{a_1}|+\rangle_A$ and Bob's result is $|1\rangle_{b_1}|-\rangle_B$:

$$\begin{aligned}\left|\tilde{\Psi}'_6\right\rangle &= {}_{a_1}\langle 0|_A\langle +|_{b2}\langle 1|_B\langle -|\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ &= \frac{1}{2}{}_{a_1}\langle 0|_A\left(\langle 0|+\langle 1|\right)_{b2}\langle 1|_B\left(\langle 0|-\langle 1|\right)\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ |\Omega\rangle_6 &= \left(\alpha_0\beta_0|01\rangle-\alpha_0\beta_1|00\rangle+\alpha_1\beta_0|11\rangle-\alpha_1\beta_1|10\rangle\right)_{b_1a_2}\end{aligned} \tag{3.12}$$

(7)_If the result of Alice is: $|0\rangle_{a_1}|-\rangle_A$ and Bob's result is $|1\rangle_{b_1}|+\rangle_B$:

$$\begin{aligned}\left|\tilde{\Psi}'_7\right\rangle &= {}_{a_1}\langle 0|_A\langle -|_{b2}\langle 1|_B\langle +|\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ &= \frac{1}{2}{}_{a_1}\langle 0|_A\left(\langle 0|-\langle 1|\right)_{b2}\langle 1|_B\left(\langle 0|+\langle 1|\right)\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ |\Omega\rangle_7 &= \left(\alpha_0\beta_0|01\rangle+\alpha_0\beta_1|00\rangle-\alpha_1\beta_0|11\rangle-\alpha_1\beta_1|10\rangle\right)_{b_1a_2}\end{aligned} \tag{3.13}$$

(8)_If the result of Alice is: $|0\rangle_{a_1}|-\rangle_A$ and Bob's result is $|1\rangle_{b_1}|-\rangle_B$:

$$\begin{aligned}\left|\tilde{\Psi}'_8\right\rangle &= {}_{a_1}\langle 0|_A\langle -|_{b2}\langle 1|_B\langle -|\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ &= \frac{1}{2}{}_{a_1}\langle 0|_A\left(\langle 0|-\langle 1|\right)_{b2}\langle 1|_B\left(\langle 0|-\langle 1|\right)\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ |\Omega\rangle_8 &= \left(\alpha_0\beta_0|01\rangle-\alpha_0\beta_1|00\rangle-\alpha_1\beta_0|11\rangle+\alpha_1\beta_1|10\rangle\right)_{b_1a_2}\end{aligned} \tag{3.14}$$

(9)_If the result of Alice is: $|1\rangle_{a_1}|+\rangle_A$ and Bob's result is $|0\rangle_{b_1}|+\rangle_B$:

$$\begin{aligned}\left|\tilde{\Psi}'_9\right\rangle &= {}_{a_1}\langle 1|_A\langle +|_{b2}\langle 0|_B\langle +|\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ &= \frac{1}{2}{}_{a_1}\langle 1|_A\left(\langle 0|+\langle 1|\right)_{b2}\langle 0|_B\left(\langle 0|+\langle 1|\right)\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ |\Omega\rangle_9 &= \left(\alpha_0\beta_0|10\rangle+\alpha_0\beta_1|11\rangle+\alpha_1\beta_0|00\rangle+\alpha_1\beta_1|01\rangle\right)_{b_1a_2}\end{aligned} \tag{3.15}$$

(10)_If the result of Alice is: $|1\rangle_{a_1}|+\rangle_A$ and Bob's result is $|0\rangle_{b_1}|-\rangle_B$:

$$\begin{aligned} \left|\tilde{\Psi}'_{10}\right\rangle &= {}_{a_1}\langle 1|_A\langle +|_{b2}\langle 0|_B\langle -|\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ &= \frac{1}{2}{}_{a_1}\langle 1|_A(\langle 0|+\langle 1|)_{b2}\langle 0|_B(\langle 0|-\langle 1|)\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ |\Omega\rangle_{10} &= (\alpha_0\beta_0|10\rangle-\alpha_0\beta_1|11\rangle+\alpha_1\beta_0|00\rangle-\alpha_1\beta_1|01\rangle)_{b_1a_2} \end{aligned} \tag{3.16}$$

(11)_If the result of Alice is: $|1\rangle_{a_1}|-\rangle_A$ and Bob's result is $|0\rangle_{b_1}|+\rangle_B$:

$$\begin{aligned} \left|\tilde{\Psi}'_{11}\right\rangle &= {}_{a_1}\langle 1|_A\langle -|_{b2}\langle 0|_B\langle +|\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ &= \frac{1}{2}{}_{a_1}\langle 1|_A(\langle 0|-\langle 1|)_{b2}\langle 0|_B(\langle 0|+\langle 1|)\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ |\Omega\rangle_{11} &= (\alpha_0\beta_0|10\rangle+\alpha_0\beta_1|11\rangle-\alpha_1\beta_0|00\rangle-\alpha_1\beta_1|01\rangle)_{b_1a_2} \end{aligned} \tag{3.17}$$

(12)_If the result of Alice is: $|1\rangle_{a_1}|-\rangle_A$ and Bob's result is $|0\rangle_{b_1}|-\rangle_B$:

$$\begin{aligned} \left|\tilde{\Psi}'_{12}\right\rangle &= {}_{a_1}\langle 1|_A\langle -|_{b2}\langle 0|_B\langle -|\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ &= \frac{1}{2}{}_{a_1}\langle 1|_A(\langle 0|-\langle 1|)_{b2}\langle 0|_B(\langle 0|-\langle 1|)\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ |\Omega\rangle_{12} &= (\alpha_0\beta_0|10\rangle-\alpha_0\beta_1|11\rangle+\alpha_1\beta_0|00\rangle-\alpha_1\beta_1|01\rangle)_{b_1a_2} \end{aligned} \tag{3.18}$$

(13)_If the result of Alice is: $|1\rangle_{a_1}|+\rangle_A$ and Bob's result is $|1\rangle_{b_1}|+\rangle_B$:

$$\begin{aligned} \left|\tilde{\Psi}'_{13}\right\rangle &= {}_{a_1}\langle 1|_A\langle +|_{b2}\langle 1|_B\langle +|\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ &= \frac{1}{2}{}_{a_1}\langle 1|_A(\langle 0|+\langle 1|)_{b2}\langle 1|_B(\langle 0|+\langle 1|)\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ |\Omega\rangle_{13} &= (\alpha_0\beta_0|11\rangle+\alpha_0\beta_1|10\rangle+\alpha_1\beta_0|01\rangle+\alpha_1\beta_1|00\rangle)_{b_1a_2} \end{aligned} \tag{3.19}$$

(14)_If the result of Alice is: $|1\rangle_{a_1}|+\rangle_A$ and Bob's result is $|1\rangle_{b_1}|-\rangle_B$:

$$\begin{aligned} \left|\tilde{\Psi}'_{14}\right\rangle &= {}_{a_1}\langle 1|_A\langle +|_{b2}\langle 1|_B\langle -|\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ &= \frac{1}{2}{}_{a_1}\langle 0|_A(\langle 0|+\langle 1|)_{b2}\langle 0|_B(\langle 0|-\langle 1|)\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ |\Omega\rangle_{14} &= (\alpha_0\beta_0|11\rangle-\alpha_0\beta_1|10\rangle+\alpha_1\beta_0|01\rangle-\alpha_1\beta_1|00\rangle)_{b_1a_2} \end{aligned} \tag{3.20}$$

(15)_If the result of Alice is: $|1\rangle_{a_1}|-\rangle_A$ and Bob's result is $|1\rangle_{b_1}|+\rangle_B$:

$$\begin{aligned} \left|\tilde{\Psi}'_{15}\right\rangle &= {}_{a_1}\langle 1|_A\langle -|_{b2}\langle 1|_B\langle +|\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ &= \frac{1}{2}{}_{a_1}\langle 1|_A(\langle 0|-\langle 1|)_{b2}\langle 1|_B(\langle 0|+\langle 1|)\left|\Psi'\right\rangle_{a1b1a2b2AB} \\ |\Omega\rangle_{15} &= (\alpha_0\beta_0|11\rangle+\alpha_0\beta_1|10\rangle-\alpha_1\beta_0|01\rangle-\alpha_1\beta_1|00\rangle)_{b_1a_2} \end{aligned} \tag{3.21}$$

(16)_If the result of Alice is: $|1\rangle_{a_1}|-\rangle_A$ and Bob's result is $|1\rangle_{b_1}|-\rangle_B$:

$$\begin{aligned}\left|\tilde{\Psi}'_{16}\right\rangle &= {}_{a_1}\langle 1|_A\langle -|_{b2}\langle 1|_B\langle -|\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ &= \frac{1}{2}{}_{a_1}\langle 1|_A(\langle 0|-\langle 1|)_{b2}\langle 1|_B(\langle 0|-\langle 1|)\left|\Psi'\right\rangle_{a1b1a2b2AB}\\ |\Omega\rangle_{16} &= (\alpha_0\beta_0|11\rangle-\alpha_0\beta_1|10\rangle-\alpha_1\beta_0|01\rangle+\alpha_1\beta_1|00\rangle)_{b_1a_2}\end{aligned} \tag{3.22}$$

the last step is applying suitable unitary operations on measured states to correct the results and obtain the initial state $|\Omega\rangle$

$$|\Omega\rangle_{b_1a_2}=[\alpha_0\beta_0|00\rangle+\alpha_0\beta_1|01\rangle+\alpha_1\beta_0|10\rangle+\alpha_1\beta_1|11\rangle]_{b_1a_2} \tag{3.23}$$

The correction operation of Bob for each result:

$$|\Omega\rangle_1=(\alpha_0|0\rangle+\alpha_1|1\rangle)\otimes(\beta_0|0\rangle+\beta_1|1\rangle)\Longrightarrow U_1=I\otimes I \tag{3.24}$$

$$|\Omega\rangle_2=(\alpha_0|0\rangle+\alpha_1|1\rangle)\otimes(\beta_0|0\rangle-\beta_1|1\rangle)\Longrightarrow U_2=I\otimes Z \tag{3.25}$$

$$|\Omega\rangle_3=(\alpha_0|0\rangle-\alpha_1|1\rangle)\otimes(\beta_0|0\rangle+\beta_1|1\rangle)\Longrightarrow U_3=Z\otimes I \tag{3.26}$$

$$|\Omega\rangle_4=(\alpha_0|0\rangle-\alpha_1|1\rangle)\otimes(\beta_0|0\rangle-\beta_1|1\rangle)\Longrightarrow U_4=Z\otimes Z \tag{3.27}$$

$$|\Omega\rangle_5=(\alpha_0|0\rangle+\alpha_1|1\rangle)\otimes(\beta_0|1\rangle+\beta_1|0\rangle)\Longrightarrow U_5=I\otimes X \tag{3.28}$$

$$|\Omega\rangle_6=(\alpha_0|0\rangle+\alpha_1|1\rangle)\otimes(\beta_0|1\rangle-\beta_1|0\rangle)\Longrightarrow U_6=I\otimes(-iY) \tag{3.29}$$

$$|\Omega\rangle_7=(\alpha_0|0\rangle-\alpha_1|1\rangle)\otimes(\beta_0|1\rangle+\beta_1|0\rangle)\Longrightarrow U_7=Z\otimes X \tag{3.30}$$

$$|\Omega\rangle_8=(\alpha_0|0\rangle-\alpha_1|1\rangle)\otimes(\beta_0|1\rangle-\beta_1|0\rangle)\Longrightarrow U_8=Z\otimes(-iY) \tag{3.31}$$

$$|\Omega\rangle_9=(\alpha_0|1\rangle+\alpha_1|0\rangle)\otimes(\beta_0|0\rangle+\beta_1|1\rangle)\Longrightarrow U_9=X\otimes I \tag{3.32}$$

$$|\Omega\rangle_{10}=(\alpha_0|1\rangle+\alpha_1|0\rangle)\otimes(\beta_0|0\rangle-\beta_1|1\rangle)\Longrightarrow U_{10}=X\otimes Z \tag{3.33}$$

$$|\Omega\rangle_{11} = (\alpha_0 |1\rangle - \alpha_1 |0\rangle) \otimes (\beta_0 |0\rangle + \beta_1 |1\rangle) \Longrightarrow U_{11} = (-iY) \otimes I \tag{3.34}$$

$$|\Omega\rangle_{12} = (\alpha_0 |1\rangle - \alpha_1 |0\rangle) \otimes (\beta_0 |0\rangle - \beta_1 |1\rangle) \Longrightarrow U_{12} = (-iY) \otimes Z \tag{3.35}$$

$$|\Omega\rangle_{13} = (\alpha_0 |1\rangle + \alpha_1 |0\rangle) \otimes (\beta_0 |1\rangle + \beta_1 |0\rangle) \Longrightarrow U_{13} = X \otimes X \tag{3.36}$$

$$|\Omega\rangle_{14} = (\alpha_0 |1\rangle + \alpha_1 |0\rangle) \otimes (\beta_0 |1\rangle - \beta_1 |0\rangle) \Longrightarrow U_{14} = X \otimes (-iY) \tag{3.37}$$

$$|\Omega\rangle_{15} = (\alpha_0 |1\rangle - \alpha_1 |0\rangle) \otimes (\beta_0 |1\rangle + \beta_1 |0\rangle) \Longrightarrow U_{15} = (-iY) \otimes X \tag{3.38}$$

$$|\Omega\rangle_{16} = (\alpha_0 |1\rangle - \alpha_1 |0\rangle) \otimes (\beta_0 |1\rangle - \beta_1 |0\rangle) \Longrightarrow U_{16} = (-iY) \otimes (-iY) \tag{3.39}$$

We conclude the relation between measurement results and the correction operations in the table
:

Measurement result	Correction
$\|0\rangle \|+\rangle$	I
$\|0\rangle \|-\rangle$	Z
$\|1\rangle \|+\rangle$	X
$\|1\rangle \|-\rangle$	$-iY$

Chapter 4

Bidirectional quantum controlled teleportation

The protocol proposed of bidirectional teleportation between two users want to teleport single-qubit to each other under the permission of a controller,Charlie, using five-qubits entangled state as a quantum channel[16].

The single-qubit of Alice $|\phi\rangle_A$and Bob's $|\phi\rangle_B$:

$$|\phi\rangle_A = \alpha_0 |0\rangle + \alpha_1 |1\rangle \tag{4.1}$$

$$|\phi\rangle_B = \beta_0 |0\rangle + \beta_1 |1\rangle \tag{4.2}$$

Using four different states as quantum channels with five qubits,encoded with two classical bits(1 and 0) .

(I) _ **The channel (1) encoded with 00:**

$$|\Psi_1\rangle_{a_0 b_0 a_1 c b_1} = \frac{1}{2}(|00000\rangle + |00101\rangle + |11000\rangle + |11101\rangle)_{a_0 b_0 a_1 c b_1} \tag{4.3}$$

The general state of the system $|\Phi_1\rangle$

$$\begin{aligned}
|\Phi_1\rangle_{a_0b_0a_1cb_1AB} &= |\Psi_{00}\rangle_{a_0b_0a_1cb_1} \otimes |\phi\rangle_A \otimes |\phi\rangle_B \\
&= \frac{1}{2}(|00000\rangle + |00101\rangle + |11000\rangle + |11101\rangle)_{a_0b_0a_1cb_1} \\
&\quad \otimes(\alpha_0|0\rangle + \alpha_1|1\rangle)_A \otimes (\beta_0|0\rangle + \beta_1|1\rangle)_B \\
&= \frac{1}{2}[\alpha_0\beta_0(|00000\rangle + |00101\rangle + |11000\rangle + |11101\rangle)|00\rangle \\
&\quad +\alpha_0\beta_1(|00000\rangle + |00101\rangle + |11000\rangle + |11101\rangle)|01\rangle \\
&\quad +\alpha_1\beta_0(|00000\rangle + |00101\rangle + |11000\rangle + |11101\rangle)|10\rangle \\
&\quad +\alpha_1\beta_1(|00000\rangle + |00101\rangle + |11000\rangle + |11101\rangle)|11\rangle]. \qquad (4.4)
\end{aligned}$$

Alice and Bob perform a controlled-not operation with A and B as control qubits, $a_0$and b_1 as targets respectively.The general state will be:

$$\begin{aligned}
|\Phi_1'\rangle_{a_0b_0a_1cb_1AB} &= \frac{1}{2}[\alpha_0\beta_0(|00000\rangle + |00101\rangle + |11000\rangle + |11101\rangle)|00\rangle \\
&\quad +\alpha_0\beta_1(|00001\rangle + |00100\rangle + |11001\rangle + |11100\rangle)|01\rangle \\
&\quad +\alpha_1\beta_0(|10000\rangle + |10101\rangle + |01000\rangle + |01101\rangle)|10\rangle \\
&\quad +\alpha_1\beta_1(|10001\rangle + |10100\rangle + |01001\rangle + |01100\rangle)|11\rangle].
\end{aligned}$$

Measurement in the Z-basis : The measurement in the z-basis on the qubits $a_0$and b_1,
(1)_ When Alice measurement result is $|0\rangle$ and Bob's result is $|0\rangle$

$$_{a_0}\langle 0|_{b_1}\langle 0| \, |\Phi_1'\rangle_{a_0b_0a_1cb_1AB} = \frac{1}{2}[\alpha_0\beta_0|00000\rangle + \alpha_0\beta_1|01001\rangle + \alpha_1\beta_0|10010\rangle + \alpha_1\beta_1|11011\rangle]_{b_0a_1cAB} \qquad (4.5)$$

The factor of normalization:

$$\begin{aligned}
\left[\langle\Phi_1'| \, |0\rangle_{a_0}|0\rangle_{b_1} \, {}_{a_0}\langle 0|_{b_1}\langle 0| \, |\Phi_1'\rangle\right]^{\frac{1}{2}} &= \sqrt{\frac{1}{4}|\alpha_0|^2|\beta_0|^2 + |\alpha_0|^2|\beta_1|^2 + |\alpha_1|^2|\beta_0|^2 + |\alpha_1|^2|\beta_1|^2} \qquad (4.6) \\
&= \frac{1}{2} \qquad (4.7)
\end{aligned}$$

$\Longrightarrow$

$$\left|\tilde{\Phi}_1'\right\rangle^{00}_{b_0a_1cAB} = \alpha_0\beta_0|00000\rangle + \alpha_0\beta_1|01001\rangle + \alpha_1\beta_0|10010\rangle + \alpha_1\beta_1|11011\rangle \qquad (4.8)$$

Alice's result $|0\rangle$ and Bob's result $|1\rangle$

$$|\Phi_1'\rangle^{01}_{b_0a_1cAB} = [\alpha_0\beta_0|01000\rangle + \alpha_0\beta_1|00001\rangle + \alpha_1\beta_0|11010\rangle + \alpha_1\beta_1|10011\rangle]_{b_0a_1cAB} \qquad (4.9)$$

(3)_Alice's result $|1\rangle$ and Bob's result (0)

$$\left|\Phi_1'\right\rangle_{b_0a_1cAB}^{10} = [\alpha_0\beta_0\,|10000\rangle + \alpha_0\beta_1\,|11001\rangle + \alpha_1\beta_0\,|00010\rangle + \alpha_1\beta_1\,|01011\rangle]_{b_0a_1cAB} \tag{4.10}$$

(4)_Alice's result $|1\rangle$,and Bob's result $|1\rangle$

$$\left|\Phi_1'\right\rangle_{b_0a_1cAB}^{11} = [\alpha_0\beta_0\,|11000\rangle + \alpha_0\beta_1\,|10001\rangle + \alpha_1\beta_0\,|01010\rangle + \alpha_1\beta_1\,|00011\rangle]_{b_0a_1cAB} \tag{4.11}$$

X unitary operation: In this stage X unitary operation applied by the two users (Alice and Bob) on their qubits a_1 andb_0 respectively.

$\{X\,|0\rangle = |1\rangle\,, X\,|1\rangle = |0\rangle\}$

$$[\alpha_0\beta_0\,|00000\rangle + \alpha_0\beta_1\,|01001\rangle + \alpha_1\beta_0\,|10010\rangle + \alpha_1\beta_1\,|11011\rangle]_{b_0a_1cAB} = |\Theta\rangle$$

Alice's result	Bob's result	OpeartionX on qubits (b_0)(a_1)
0	0	$I \otimes I$
0	1	$I \otimes X$
1	0	$X \otimes I$
1	0	$X \otimes X$

Measurement in the X basis: On the qubits A and B in the corrected state $|\Theta\rangle$,

The case where the results of measurement are

$\cdot\,|A\rangle = |+\rangle$ and $|B\rangle = |+\rangle$:

$$\left|\tilde{\Theta}\right\rangle_1 = \frac{{}_A\langle +|_B\langle +|\,|\Theta\rangle}{\sqrt{({}_A\langle +|_B\langle +|\,|\Theta\rangle)^*({}_A\langle +|_B\langle +|\,|\Theta\rangle)}} \tag{4.12}$$

$$\begin{aligned} {}_A\langle +|_B\langle +|\,|\Theta\rangle &= \frac{1}{2}((\langle 0| + \langle 1|) \otimes (\langle 0| + \langle 1|)\,|\Theta\rangle \\ &= \frac{1}{2}[\alpha_0\beta_0\,|000\rangle + \alpha_0\beta_1\,|010\rangle + \alpha_1\beta_0\,|100\rangle + \alpha_1\beta_1\,|110\rangle] \\ Then, \left|\tilde{\Theta}\right\rangle_1 &= \frac{\frac{1}{2}[\alpha_0\beta_0\,|000\rangle + \alpha_0\beta_1\,|010\rangle + \alpha_1\beta_0\,|100\rangle + \alpha_1\beta_1\,|110\rangle]}{\sqrt{\frac{1}{4}(|\alpha_0|^2|\beta_0|^2 + |\alpha_0|^2|\beta_1|^2 + |\alpha_1|^2|\beta_0|^2 + |\alpha_1|^2|\beta_1|^2)}} \\ &= [\alpha_0\beta_0\,|000\rangle + \alpha_0\beta_1\,|010\rangle + \alpha_1\beta_0\,|100\rangle + \alpha_1\beta_1\,|110\rangle]_{b_0a_1c} \end{aligned} \tag{4.13}$$

$\cdot\,|A\rangle = |+\rangle$ and $|B\rangle = |-\rangle$:

$$\left|\tilde{\Theta}\right\rangle_2 = [\alpha_0\beta_0\,|000\rangle - \alpha_0\beta_1\,|010\rangle + \alpha_1\beta_0\,|100\rangle - \alpha_1\beta_1\,|110\rangle]_{b_0a_1c}$$

$\cdot\,|A\rangle = |-\rangle$ and $|B\rangle = |+\rangle$:

$$\left|\tilde{\Theta}\right\rangle_3 = [\alpha_0\beta_0\,|000\rangle + \alpha_0\beta_1\,|010\rangle - \alpha_1\beta_0\,|100\rangle - \alpha_1\beta_1\,|110\rangle]_{b_0a_1c} \tag{4.14}$$

$\cdot \left|A\right\rangle = \left|-\right\rangle$ and $\left|B\right\rangle = \left|-\right\rangle$:

$$\left|\tilde{\Theta}\right\rangle_4 = [\alpha_0\beta_0 \left|000\right\rangle - \alpha_0\beta_1 \left|010\right\rangle - \alpha_1\beta_0 \left|100\right\rangle + \alpha_1\beta_1 \left|110\right\rangle]_{b_0 a_1 c} \tag{4.15}$$

Z unitary operation:

We apply Z operation to the qubits($b_{0,}a_1$) to otain the qubits ($b_{0,}a_1, c$) in the state:

$$\alpha_0\beta_0 \left|000\right\rangle + \alpha_0\beta_1 \left|010\right\rangle + \alpha_1\beta_0 \left|100\right\rangle + \alpha_1\beta_1 \left|110\right\rangle \tag{4.16}$$

Alice's result	Bob's result	OpeartionZ on qubits (b_0)(a_1)
+	+	$I \otimes I$
+	−	$I \otimes Z$
−	+	$Z \otimes I$
−	−	$Z \otimes Z$

The final state of qubits ($b_{0,}a_1$): On the state(4.16), we measure the qubit of the controller (c),in the x basis , we find two different states of the qubits ($b_{0,}a_1$),

(1)_In the case where the result of measurement of the qubit c is $\left|+\right\rangle$:

$$\begin{aligned} \left|\tilde{\Phi}_1\right\rangle^{(+)} &= \frac{1}{\sqrt{2}}(\left\langle 0\right| + \left\langle 1\right|)(\alpha_0\beta_0 \left|000\right\rangle + \alpha_0\beta_1 \left|010\right\rangle + \alpha_1\beta_0 \left|100\right\rangle + \alpha_1\beta_1 \left|110\right\rangle) \\ &= \alpha_0\beta_0 \left|00\right\rangle + \alpha_0\beta_1 \left|01\right\rangle + \alpha_1\beta_0 \left|10\right\rangle + \alpha_1\beta_1 \left|11\right\rangle \\ &= (\alpha_0 \left|0\right\rangle + \alpha_1 \left|1\right\rangle) \otimes (\beta_0 \left|0\right\rangle + \beta_1 \left|1\right\rangle) \quad (4.17) \\ \Longrightarrow\ & U_1^{(+)} = I \otimes I \quad (4.18) \end{aligned}$$

(2)_If the result of measurement of the qubit c is $\left|-\right\rangle$:

$$\begin{aligned} \left|\tilde{\Phi}_1\right\rangle^{(-)} &= \frac{1}{\sqrt{2}}(\left\langle 0\right| - \left\langle 1\right|)(\alpha_0\beta_0 \left|000\right\rangle + \alpha_0\beta_1 \left|010\right\rangle + \alpha_1\beta_0 \left|100\right\rangle + \alpha_1\beta_1 \left|110\right\rangle) \\ &= \alpha_0\beta_0 \left|00\right\rangle + \alpha_0\beta_1 \left|01\right\rangle + \alpha_1\beta_0 \left|10\right\rangle + \alpha_1\beta_1 \left|11\right\rangle \\ &= (\alpha_0 \left|0\right\rangle + \alpha_1 \left|1\right\rangle) \otimes (\beta_0 \left|0\right\rangle + \beta_1 \left|1\right\rangle) \quad (4.19) \\ \Longrightarrow\ & U_1^{(-)} = I \otimes I \quad (4.20) \end{aligned}$$

For the rest channels the treatment it totally the same,

(II) _ **The channel encoded with 01**

$$|\Psi_2\rangle_{a_0b_0a_1cb_1} = \frac{1}{2}(|00000\rangle + |00111\rangle + |11000\rangle + |11111\rangle)$$

The general state of the system:

$$|\Phi_2\rangle_{a_0b_0a_1cb_1AB} = |\Psi_2\rangle_{a_0b_0a_1cb_1} \otimes |\phi\rangle_A \otimes |\phi\rangle_B$$

Alice and Bob perform controlled-not operation with A and B as control qubits, $a_0$and b_1 as targets respectively.

The general state will be $|\Phi_2'\rangle$

$$\begin{aligned} |\Phi_2'\rangle_{a_0b_0a_1cb_1AB} &= Ucnot_{Aa_1} \otimes Ucnot_{Bb_2} \otimes I_2 \otimes I_2 \otimes I_2 \otimes |\Phi\rangle_{a_0b_0a_1cb_1AB} \\ &= \frac{1}{2}[\alpha_0\beta_0(|00000\rangle + |00111\rangle + |11000\rangle + |11111\rangle)\,|00\rangle \\ &\quad +\alpha_0\beta_1(|00001\rangle + |00110\rangle + |11001\rangle + |11110\rangle)\,|01\rangle \\ &\quad +\alpha_1\beta_0(|10000\rangle + |10111\rangle + |01000\rangle + |01111\rangle)\,|10\rangle \\ &\quad +\alpha_1\beta_1(|10001\rangle + |10110\rangle + |01001\rangle + |01110\rangle)\,|11\rangle]. \end{aligned} \tag{4.21}$$

Measurement in the Z-basis of the qubits a_0b_1 : The four possible outcomes, and the corresponding corrections to transfer the obtained states into the state:

$$\alpha_0\beta_0\,|00000\rangle + \alpha_0\beta_1\,|01101\rangle + \alpha_1\beta_0\,|10010\rangle + \alpha_1\beta_1\,|11111\rangle \tag{4.22}$$

or

$$\alpha_0\beta_0\,|00100\rangle + \alpha_0\beta_1\,|01001\rangle + \alpha_1\beta_0\,|10110\rangle + \alpha_1\beta_1\,|11011\rangle \tag{4.23}$$

.

Alice's result	Bob's result	the state of the qubits (b_0a_1cAB)	OpeartionX
0	0	$\alpha_0\beta_0\,|00000\rangle + \alpha_0\beta_1\,|01101\rangle + \alpha_1\beta_0\,|10010\rangle + \alpha_1\beta_1\,|11111\rangle$	$I \otimes I$
0	1	$\alpha_0\beta_0\,|01100\rangle + \alpha_0\beta_1\,|00001\rangle + \alpha_1\beta_0\,|11110\rangle + \alpha_1\beta_1\,|10011\rangle$	$I \otimes X$
1	0	$\alpha_0\beta_0\,|10000\rangle + \alpha_0\beta_1\,|11101\rangle + \alpha_1\beta_0\,|00010\rangle + \alpha_1\beta_1\,|01111\rangle$	$X \otimes I$
1	0	$\alpha_0\beta_0\,|11100\rangle + \alpha_0\beta_1\,|10001\rangle + \alpha_1\beta_0\,|01110\rangle + \alpha_1\beta_1\,|00011\rangle$	$X \otimes X$

Measurement in the X basis: the states obtained after measuring the qubits A and B, and the corresponding Z operation on (b_0)$and(a_1)$ that allows the unmeasured qubits to be in the state:

$$(\alpha_0\beta_0|000\rangle + \alpha_0\beta_1|011\rangle + \alpha_1\beta_0|100\rangle + \alpha_1\beta_1|111\rangle)_{b_0a_1c} \tag{4.24}$$

Meas of A	Meas of B	states of the qubits b_0a_1c	Operation Z
$+$	$+$	$(\alpha_0\beta_0\|000\rangle + \alpha_0\beta_1\|011\rangle + \alpha_1\beta_0\|100\rangle + \alpha_1\beta_1\|111\rangle)$	$I \otimes I$
$-$	$+$	$(\alpha_0\beta_0\|000\rangle - \alpha_0\beta_1\|011\rangle + \alpha_1\beta_0\|100\rangle - \alpha_1\beta_1\|111\rangle)$	$Z \otimes I$
$+$	$-$	$(\alpha_0\beta_0\|000\rangle + \alpha_0\beta_1\|011\rangle - \alpha_1\beta_0\|100\rangle - \alpha_1\beta_1\|111\rangle)$	$I \otimes Z$
$-$	$-$	$(\alpha_0\beta_0\|000\rangle - \alpha_0\beta_1\|011\rangle - \alpha_1\beta_0\|100\rangle + \alpha_1\beta_1\|111\rangle)$	$Z \otimes Z$

The final state of qubits ($b_{0,}a_1$): The controller measures his qubit c in the X basis, so the finale states and the correction operations of the two users Alice and Bob are:

(1)_

$$\begin{aligned}
\left|\tilde{\Phi}_2\right\rangle^{(+)} &= \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)(\alpha_0\beta_0|000\rangle + \alpha_0\beta_1|011\rangle + \alpha_1\beta_0|100\rangle + \alpha_1\beta_1|111\rangle) \\
&= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle \\
&= (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle) && (4.25) \\
\Longrightarrow\ & U_2^{(+)} = I \otimes I && (4.26)
\end{aligned}$$

(2)_

$$\begin{aligned}
\left|\tilde{\Phi}_2\right\rangle^{(-)} &= \frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)(\alpha_0\beta_0|000\rangle + \alpha_0\beta_1|011\rangle + \alpha_1\beta_0|100\rangle + \alpha_1\beta_1|111\rangle) \\
&= \alpha_0\beta_0|00\rangle - \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle - \alpha_1\beta_1|11\rangle \\
&= (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle - \beta_1|1\rangle) && (4.27) \\
\Longrightarrow\ & U_2^{(+)} = I \otimes Z && (4.28)
\end{aligned}$$

(II) _ **The channel encoded with 10**

$$|\Psi_3\rangle_{a_0b_0a_1cb_1} = \frac{1}{2}(|00000\rangle + |00101\rangle + |11010\rangle + |11111\rangle)$$

The state of the system:

$$|\Phi_3\rangle_{a_0b_0a_1cb_1AB} = |\Psi_3\rangle_{a_0b_0a_1cb_1} \otimes |\phi\rangle_A \otimes |\phi\rangle_B$$

C-not operation:

$$\left|\grave{\Phi}_3\right\rangle_{a_0b_0a_1cb_1AB} = (Ucnot_{Aa_0} \otimes Ucnot_{Bb_1} \otimes I \otimes I \otimes I)\left|\Phi_3\right\rangle_{a_0b_0a_1cb_1AB}$$
$$= \frac{1}{2}[\alpha_0\beta_0(|00000\rangle + |00101\rangle + |11010\rangle + |11111\rangle)|00\rangle$$
$$+\alpha_0\beta_1(|00001\rangle + |00100\rangle + |11011\rangle + |11110\rangle)|01\rangle$$
$$+\alpha_1\beta_0(|10000\rangle + |10101\rangle + |01010\rangle + |01111\rangle)|10\rangle$$
$$+\alpha_1\beta_1(|10001\rangle + |10100\rangle + |01011\rangle + |01110\rangle)|11\rangle].$$

Measurement in the Z basis and X operations: The results of measurements and the corrections that let the remaining qubits be in the state:

$$\alpha_0\beta_0|00000\rangle + \alpha_0\beta_1|01001\rangle + \alpha_1\beta_0|10110\rangle + \alpha_1\beta_1|11111\rangle \quad (4.29)$$

Alice's result	Bob's result	the state of the qubits (b_0a_1cAB)	OpeartionX
0	0	$\alpha_0\beta_0\|00000\rangle + \alpha_0\beta_1\|01001\rangle + \alpha_1\beta_0\|10110\rangle + \alpha_1\beta_1\|11111\rangle$	$I \otimes I$
0	1	$\alpha_0\beta_0\|01000\rangle + \alpha_0\beta_1\|00001\rangle + \alpha_1\beta_0\|11110\rangle + \alpha_1\beta_1\|10111\rangle$	$I \otimes X$
1	0	$\alpha_0\beta_0\|10100\rangle + \alpha_0\beta_1\|11101\rangle + \alpha_1\beta_0\|00010\rangle + \alpha_1\beta_1\|01011\rangle$	$X \otimes I$
1	0	$\alpha_0\beta_0\|11100\rangle + \alpha_0\beta_1\|10101\rangle + \alpha_1\beta_0\|01010\rangle + \alpha_1\beta_1\|00011\rangle$	$X \otimes X$

Measurement in the X basis: on the qubits A and B, and the operations Z on the qubits b_0a_1, to tranfer the the obtained states to the state:

$$(\alpha_0\beta_0|000\rangle + \alpha_0\beta_1|010\rangle + \alpha_1\beta_0|101\rangle + \alpha_1\beta_1|111\rangle) \quad (4.30)$$

Meas of A	Meas of B	states of the qubits b_0a_1c	Operation Z
+	+	$(\alpha_0\beta_0\|000\rangle + \alpha_0\beta_1\|010\rangle + \alpha_1\beta_0\|101\rangle + \alpha_1\beta_1\|111\rangle)$	$I \otimes I$
+	−	$(\alpha_0\beta_0\|000\rangle - \alpha_0\beta_1\|010\rangle + \alpha_1\beta_0\|101\rangle - \alpha_1\beta_1\|111\rangle)$	$Z \otimes I$
−	+	$(\alpha_0\beta_0\|000\rangle + \alpha_0\beta_1\|010\rangle - \alpha_1\beta_0\|101\rangle - \alpha_1\beta_1\|111\rangle)$	$I \otimes Z$
−	−	$(\alpha_0\beta_0\|000\rangle - \alpha_0\beta_1\|010\rangle - \alpha_1\beta_0\|101\rangle + \alpha_1\beta_1\|111\rangle)$	$Z \otimes Z$

on the state(4.30) ,the controller measures his qubit in the X basis,the finale states are:

(1)_

$$\left|\tilde{\Phi}_3\right\rangle^{(+)} = \frac{1}{\sqrt{2}}((\langle 0| + \langle 1|)(\alpha_0\beta_0|000\rangle + \alpha_0\beta_1|010\rangle + \alpha_1\beta_0|101\rangle + \alpha_1\beta_1|111\rangle)$$
$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$
$$= (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle) \quad (4.31)$$
$$\Longrightarrow U_3^{(+)} = I \otimes I \quad (4.32)$$

(2)_

$$\left|\tilde{\Phi}_3\right\rangle^{(-)} = \frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)(\alpha_0\beta_0|000\rangle + \alpha_0\beta_1|010\rangle + \alpha_1\beta_0|101\rangle + \alpha_1\beta_1|111\rangle)$$
$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle - \alpha_1\beta_0|10\rangle - \alpha_1\beta_1|11\rangle$$
$$= (\alpha_0|0\rangle - \alpha_1|1\rangle)\otimes(\beta_0|0\rangle + \beta_1|1\rangle) \quad (4.33)$$
$$\Longrightarrow U_3^{(+)} = Z\otimes I \quad (4.34)$$

The channel encoded with [11]**:**

$$|\Psi_4\rangle_{a_0b_0a_1cb_1} = \frac{1}{2}(|00000\rangle + |00111\rangle + |11010\rangle + |11101\rangle)$$

The state of the system:

$$|\Phi_4\rangle_{a_0b_0a_1cb_1AB} = |\Psi_4\rangle_{a_0b_0a_1cb_1}\otimes|\phi\rangle_A\otimes|\phi\rangle_B$$

After the operation C-NOT it transfered to:

$$\left|\acute{\Phi}_4\right\rangle_{a_0b_0a_1cb_1AB} = \frac{1}{2}[\alpha_0\beta_0(|00000\rangle + |00111\rangle + |11010\rangle + |11101\rangle)|00\rangle$$
$$+\alpha_0\beta_1(|00001\rangle + |00110\rangle + |11011\rangle + |11100\rangle)|01\rangle$$
$$+\alpha_1\beta_0(|10000\rangle + |10111\rangle + |01010\rangle + |01101\rangle)|10\rangle$$
$$+\alpha_1\beta_1(|10001\rangle + |10110\rangle + |01011\rangle + |01100\rangle)|11\rangle]. \quad (4.35)$$

Measurement in the Z basis : after measurement we apply X operation on the unmeasured qubits (b_0a_1), to transfer all the states to :

$$\alpha_0\beta_0|00000\rangle + \alpha_0\beta_1|01101\rangle + \alpha_1\beta_0|10110\rangle + \alpha_1\beta_1|11011\rangle \quad (4.36)$$

Alice's result	Bob's result	the state of the qubits (b_0a_1cAB)	OpeartionX
0	0	$\alpha_0\beta_0\|00000\rangle + \alpha_0\beta_1\|01101\rangle + \alpha_1\beta_0\|10110\rangle + \alpha_1\beta_1\|11011\rangle$	$I\otimes I$
0	1	$\alpha_0\beta_0\|01100\rangle + \alpha_0\beta_1\|00001\rangle + \alpha_1\beta_0\|11010\rangle + \alpha_1\beta_1\|10111\rangle$	$I\otimes X$
1	0	$\alpha_0\beta_0\|10100\rangle + \alpha_0\beta_1\|11001\rangle + \alpha_1\beta_0\|00010\rangle + \alpha_1\beta_1\|01111\rangle$	$X\otimes I$
1	0	$\alpha_0\beta_0\|11000\rangle + \alpha_0\beta_1\|10101\rangle + \alpha_1\beta_0\|01110\rangle + \alpha_1\beta_1\|00011\rangle$	$X\otimes X$

Measurement in the X basis in this stage, the the qubits $b_0 a_1 c$ will be in the state:

$$\alpha_0\beta_0 |000\rangle + \alpha_0\beta_1 |011\rangle + \alpha_1\beta_0 |101\rangle + \alpha_1\beta_1 |110\rangle \tag{4.37}$$

Meas of A	Meas of B	states of the qubits $b_0 a_1 c$	Operation Z
+	+	$(\alpha_0\beta_0 \|000\rangle + \alpha_0\beta_1 \|011\rangle + \alpha_1\beta_0 \|101\rangle + \alpha_1\beta_1 \|110\rangle)$	$I \otimes I$
−	+	$(\alpha_0\beta_0 \|000\rangle - \alpha_0\beta_1 \|011\rangle + \alpha_1\beta_0 \|101\rangle - \alpha_1\beta_1 \|110\rangle)$	$Z \otimes I$
+	−	$(\alpha_0\beta_0 \|000\rangle + \alpha_0\beta_1 \|011\rangle - \alpha_1\beta_0 \|101\rangle - \alpha_1\beta_1 \|110\rangle)$	$I \otimes Z$
−	−	$(\alpha_0\beta_0 \|000\rangle - \alpha_0\beta_1 \|011\rangle - \alpha_1\beta_0 \|101\rangle + \alpha_1\beta_1 \|110\rangle)$	$Z \otimes Z$

The finale states after the controller's measurement:

(1)_

$$\begin{aligned}
\left|\tilde{\Phi}_4\right\rangle^{(+)} &= \frac{1}{\sqrt{2}}((\langle 0| + \langle 1|)(\alpha_0\beta_0 |000\rangle + \alpha_0\beta_1 |011\rangle + \alpha_1\beta_0 |101\rangle + \alpha_1\beta_1 |110\rangle)) \\
&= \alpha_0\beta_0 |00\rangle + \alpha_0\beta_1 |01\rangle + \alpha_1\beta_0 |10\rangle + \alpha_1\beta_1 |11\rangle \\
&= (\alpha_0 |0\rangle + \alpha_1 |1\rangle) \otimes (\beta_0 |0\rangle + \beta_1 |1\rangle) \qquad (4.38) \\
&\Longrightarrow U_4^{(+)} = I \otimes I \qquad (4.39)
\end{aligned}$$

(2)_

$$\begin{aligned}
\left|\tilde{\Phi}_4\right\rangle^{(-)} &= \frac{1}{\sqrt{2}}((\langle 0| - \langle 1|)(\alpha_0\beta_0 |000\rangle + \alpha_0\beta_1 |011\rangle + \alpha_1\beta_0 |101\rangle + \alpha_1\beta_1 |110\rangle)) \\
&= \alpha_0\beta_0 |00\rangle - \alpha_0\beta_1 |01\rangle - \alpha_1\beta_0 |10\rangle + \alpha_1\beta_1 |11\rangle \\
&= (\alpha_0 |0\rangle - \alpha_1 |1\rangle) \otimes (\beta_0 |0\rangle - \beta_1 |1\rangle) \qquad (4.40) \\
&\Longrightarrow U_4^{(+)} = Z \otimes Z \qquad (4.41)
\end{aligned}$$

Chapter 5

Switched quantum teleportation:

In this chapter, we follow a protocol of switched teleportation[17] , where we have two initial qubits and a controller who choose which qubit to teleport to the receiver with a five-qubits entangled state as a quantum channel.

we will use the ket's formalism and the density matrix formalism where a noise impinge on the protocol and we will calculate the fidelity factor F in this case.

5.1 Ket's formalism

Two emitters Alice1 and Alice2, each one have a single-qubit to teleport to a receiver Bob,

their initial states are :

$$|\Phi\rangle_1 = \alpha_1 |0\rangle + \beta_1 |1\rangle \tag{5.1}$$

$$|\Phi\rangle_2 = \alpha_2 |0\rangle + \beta_2 |1\rangle \tag{5.2}$$

The quantum channel linking the users is a five-qubit entangled state $|C\rangle$

$$|C\rangle_{A_1A_2C_3C_4B_5} = \frac{1}{2}\left(|00000\rangle + |01011\rangle + |10101\rangle + |11110\rangle\right) \tag{5.3}$$

where qubits $A_1$and A_2 belong to Alice1 and Alice2, C_3 and C_4 belong to Charlie ,B_5 to Bob.

The global state is:

$$\begin{aligned} |GS\rangle_{12A_1A_2C_3C_4B_5} &= |\Phi\rangle_1 \otimes |\Phi\rangle_2 \otimes |C\rangle_{A_1A_2C_3C_4B_5} \\ &= \frac{1}{2}[\alpha_1\alpha_2 |00\rangle (|00000\rangle + |01011\rangle + |10101\rangle + |11110\rangle) \\ &+\alpha_1\beta_2 |01\rangle (|00000\rangle + |01011\rangle + |10101\rangle + |11110\rangle) \\ &+\beta_1\alpha_2 |10\rangle (|00000\rangle + |01011\rangle + |10101\rangle + |11110\rangle) \\ &+\beta_1\beta_2 |11\rangle (|00000\rangle + |01011\rangle + |10101\rangle + |11110\rangle)]. \end{aligned} \tag{5.4}$$

$\Longrightarrow$

$$\begin{aligned} |GS\rangle_{1A_12A_2C_3C_4B_5} &= \frac{1}{2}[\alpha_1\alpha_2 (|0000000\rangle + |0001011\rangle + |0100101\rangle + |0101110\rangle) \\ &+\alpha_1\beta_2 (|0010000\rangle + |0011011\rangle + |0110101\rangle + |0111110\rangle) \\ &+\beta_1\alpha_2 (|1000000\rangle + |1001011\rangle + |1100101\rangle + |1101110\rangle) \\ &+\beta_1\beta_2 (|1010000\rangle + |1011011\rangle + |1110101\rangle + |1111110\rangle)]. \end{aligned} \tag{5.5}$$

Operations of Alice1 and Alice2: Bell-states measurements on qubits $(1, A_1)$and $(2, A_2)$.

after measurement:

$$\left|\tilde{G}S\right\rangle_{C_3C_4B_5} = \frac{{}_{1A_1}\langle B_{x_1y_1}|_{2A_2}\langle B_{x_2y_2}|\,|GS\rangle_{1A_12A_2C_3C_4B_5}}{\sqrt{\langle GS|\,|B_{x_1y_1}\rangle\,|B_{x_2y_2}\rangle\,\langle B_{x_1y_1}|\,\langle B_{x_2y_2}|\,|GS\rangle}}; (x_1, x_2, y_1, y_2 = 0; 1)$$

The remaining qubits $C_3C_4B_5$ will collapse into one of the following states:

(1)_

$$\begin{aligned} {}_{1A_1}\langle B_{00}|_{2A_2}\langle B_{00}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} &= \frac{1}{2}{}_{1A_12A_2}(\langle 0000| + \langle 0011| + \langle 1100| + \langle 1111|)\,|GS\rangle_{1A_12A_2C_3C_4B_5} \\ &= (\alpha_1\alpha_2 |000\rangle + \alpha_1\beta_2 |011\rangle + \beta_1\alpha_2 |101\rangle + \beta_1\beta_2 |110\rangle) \end{aligned} \tag{5.6}$$

(2)_

$$\begin{aligned} {}_{1A_1}\langle B_{00}|_{2A_2}\langle B_{01}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} &= \frac{1}{2}{}_{1A_12A_2}(\langle 0001| + \langle 0010| + \langle 1101| + \langle 1110|)\,|GS\rangle_{1A_12A_2C_3C_4B_5} \\ &= (\alpha_1\alpha_2 |011\rangle + \alpha_1\beta_2 |000\rangle + \beta_1\alpha_2 |110\rangle + \beta_1\beta_2 |101\rangle) \end{aligned} \tag{5.7}$$

(3)_

$$\begin{aligned} {}_{1A_1}\langle B_{00}|_{2A_2}\langle B_{10}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} &= \frac{1}{2}{}_{1A_12A_2}(\langle 0000| - \langle 0011| + \langle 1100| - \langle 1111|)\,|GS\rangle_{1A_12A_2C_3C_4B_5} \\ &= (\alpha_1\alpha_2 |000\rangle - \alpha_1\beta_2 |011\rangle + \beta_1\alpha_2 |101\rangle - \beta_1\beta_2 |110\rangle) \end{aligned} \tag{5.8}$$

(4)_

$$ {}_{1A_1}\langle B_{00}|_{2A_2}\langle B_{11}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}\left(\langle 0001| - \langle 0010| + \langle 1101| - \langle 1110|\right)|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{4}\left(\alpha_1\alpha_2|011\rangle - \alpha_1\beta_2|000\rangle + \beta_1\alpha_2|110\rangle - \beta_1\beta_2|101\rangle\right)_{C_3C_4B_5} \tag{5.9} $$

(5)_

$$ {}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{00}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}\left(\langle 0100| + \langle 0111| + \langle 1000| + \langle 1011|\right)|GS\rangle_{1A_12A_2C_3C_4B_5} = \left(\alpha_1\alpha_2|101\rangle + \alpha_1\beta_2|110\rangle + \beta_1\alpha_2|000\rangle + \beta_1\beta_2|011\rangle\right)_{C_3C_4B_5} \tag{5.10} $$

(6)_

$$ {}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{01}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}\left(\langle 0101| + \langle 0110| + \langle 1001| + \langle 1010|\right)|GS\rangle_{1A_12A_2C_3C_4B_5} = \left(\alpha_1\alpha_2|110\rangle + \alpha_1\beta_2|101\rangle + \beta_1\alpha_2|011\rangle + \beta_1\beta_2|000\rangle\right)_{C_3C_4B_5} \tag{5.11} $$

(7)_

$$ {}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{10}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}\left(\langle 0100| - \langle 0111| + \langle 1000| - \langle 1011|\right)|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{4}\left(\alpha_1\alpha_2|101\rangle - \alpha_1\beta_2|110\rangle + \beta_1\alpha_2|000\rangle - \beta_1\beta_2|011\rangle\right)_{C_3C_4B_5} \tag{5.12} $$

(8)_

$$ {}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{11}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}\left(\langle 0101| - \langle 0110| + \langle 1001| - \langle 1010|\right)|GS\rangle_{1A_12AC_3C_4B_5} = \left(\alpha_1\alpha_2|110\rangle - \alpha_1\beta_2|101\rangle + \beta_1\alpha_2|011\rangle - \beta_1\beta_2|000\rangle\right)_{C_3C_4B_5} \tag{5.13} $$

(9)_

$$ {}_{1A_1}\langle B_{10}|_{2A_2}\langle B_{00}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}\left(\langle 0000| + \langle 0011| - \langle 1100| - \langle 1111|\right)|GS\rangle_{1A_12A_2C_3C_4B_5} = \left(\alpha_1\alpha_2|000\rangle + \alpha_1\beta_2|011\rangle - \beta_1\alpha_2|101\rangle - \beta_1\beta_2|110\rangle\right)_{C_3C_4B_5} \tag{5.14} $$

(10)_

$$ {}_{1A_1}\langle B_{10}|_{2A_2}\langle B_{01}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}\left(\langle 0001| + \langle 0010| - \langle 1101| - \langle 1110|\right)|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{4}\left(\alpha_1\alpha_2|011\rangle + \alpha_1\beta_2|000\rangle - \beta_1\alpha_2|110\rangle - \beta_1\beta_2|101\rangle\right)_{C_3C_4B_5} \tag{5.15} $$

(11)_

$$ {}_{1A_1}\langle B_{10}|_{2A_2}\langle B_{10}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}\left(\langle 0000| - \langle 0011| - \langle 1100| + \langle 1111|\right)|GS\rangle_{1A_12A_2C_3C_4B_5} = \left(\alpha_1\alpha_2|000\rangle - \alpha_1\beta_2|011\rangle - \beta_1\alpha_2|101\rangle + \beta_1\beta_2|110\rangle\right)_{C_3C_4B_5} \tag{5.16} $$

(12)_

$$ {}_{1A_1}\langle B_{10}|_{2A_2}\langle B_{11}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}(\langle 0001| - \langle 0010| - \langle 1101| + \langle 1110|)\,|GS\rangle_{1A_12A_2C_3C_4B_5} $$
$$ = (\alpha_1\alpha_2|011\rangle - \alpha_1\beta_2|000\rangle - \beta_1\alpha_2|110\rangle + \beta_1\beta_2|101\rangle)_{C_3C_4B_5} \quad (5.17) $$

(13)_

$$ {}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{00}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}(\langle 0100| + \langle 0111| - \langle 1000| - \langle 1011|)\,|GS\rangle_{1A_12A_2C_3C_4B_5} $$
$$ = \frac{1}{4}(\alpha_1\alpha_2|101\rangle + \alpha_1\beta_2|110\rangle - \beta_1\alpha_2|000\rangle - \beta_1\beta_2|011\rangle)_{C_3C_4B_5} \quad (5.18) $$

(14)_

$$ {}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{01}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}(\langle 0101| - \langle 0110| + \langle 1001| - \langle 1010|)\,|GS\rangle_{1A_12AC_3C_4B_5} $$
$$ = (\alpha_1\alpha_2|110\rangle + \alpha_1\beta_2|101\rangle - \beta_1\alpha_2|011\rangle - \beta_1\beta_2|000\rangle)_{C_3C_4B_5} \quad (5.19) $$

(15)_

$$ {}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{10}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}(\langle 0100| - \langle 0111| - \langle 1000| + \langle 1011|)\,|GS\rangle_{1A_12A_2C_3C_4B_5} $$
$$ = (\alpha_1\alpha_2|101\rangle - \alpha_1\beta_2|110\rangle - \beta_1\alpha_2|000\rangle + \beta_1\beta_2|011\rangle)_{C_3C_4B_5} \quad (5.20) $$

(16)_

$$ {}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{11}|\,|GS\rangle_{1A_12A_2C_3C_4B_5} = \frac{1}{2}{}_{1A_12A_2}(\langle 0101| - \langle 0110| - \langle 1001| + 1010)\,|GS\rangle_{1A_12AC_3C_4B_5} $$
$$ = (\alpha_1\alpha_2|110\rangle - \alpha_1\beta_2|101\rangle - \beta_1\alpha_2|011\rangle + \beta_1\beta_2|000\rangle)_{C_3C_4B_5} \quad (5.21) $$

First case:

When Charlie decides to transmit the qubit of Alice1, he performs Hadamard gate on his qubit C_3,

$$ \left|\tilde{G}S\right\rangle_{C_3C_4B_5} \Longrightarrow (H\otimes I\otimes I)\left|\tilde{G}S\right\rangle_{C_3C_4B_5} $$

then he measures his two qubits in the Z-basis.

after Charlie's operations, Bob's qubit B_5will be in the following states:

$H(C_3)$ on (1) $\rightarrow$

$$ |\Omega_1\rangle_{H_1} = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|000\rangle + \alpha_1\alpha_2|100\rangle + \alpha_1\beta_2|011\rangle + \alpha_1\beta_2|111\rangle $$
$$ +\beta_1\alpha_2|001\rangle - \beta_1\alpha_2|101\rangle + \beta_1\beta_2|010\rangle - \beta_1\beta_2|110\rangle]_{C_3C_4B_5}. \quad (5.22) $$

(1)_ when the result of measurement is $|C_3\rangle = |0\rangle$, $|C_4\rangle = |0\rangle$:

$$
\begin{aligned}
|\Omega_1\rangle_{H_1}^{00} &= \frac{{}_{C_3}\langle 0|_{C_4}\langle 0|\,|\Omega_1\rangle_{H_1}}{\sqrt{({}_{C_3}\langle 0|_{C_4}\langle 0|\,|\Omega_1\rangle_{H_1})^*_{C_3}\langle 0|_{C_4}\langle 0|\,|\Omega_1\rangle_{H_1}}} = \frac{\frac{1}{\sqrt{2}}\alpha_1(\alpha_1|0\rangle+\beta_1|1\rangle)_{B_5}}{\frac{1}{\sqrt{2}}|\alpha_1|} \\
&= (\alpha_1|0\rangle+\beta_1|1\rangle)_{B_5} = |\Phi_1\rangle \qquad (5.23)\\
&\Longrightarrow U_1^{00} = I \qquad (5.24)
\end{aligned}
$$

(2)_ $|C_3\rangle = |0\rangle\,,\ \ |C_4\rangle = |1\rangle :$

$$
\begin{aligned}
|\Omega_1\rangle_{H_1}^{01} &= \frac{{}_{C_3}\langle 0|_{C_4}\langle 1|\,|\Omega_1\rangle_{H_1}}{\sqrt{(\langle\Omega_1|_{H_1}\ |0\rangle_{C_3}|1\rangle_{C_4}\,{}_{C_3}\langle 0|_{C_4}\langle 1|\,|\Omega_1\rangle_{H_1}}} \qquad (5.25)\\
&= (\alpha_1|1\rangle+\beta_1|0\rangle)_{B_5} \qquad (5.26)\\
&\Longrightarrow U_1^{01} = X \qquad (5.27)
\end{aligned}
$$

(3)− $|C_3\rangle = |1\rangle\,,\ \ |C_4\rangle = |0\rangle :$

$$
\begin{aligned}
|\Omega_1\rangle_{H_1}^{10} &= \frac{{}_{C_3}\langle 1|_{C_4}\langle 0|\,|\Omega_1\rangle_{H_1}}{\sqrt{(\langle\Omega_1|_{H_1}\ |1\rangle_{C_3}|0\rangle_{C_4}\,{}_{C_3}\langle 1|_{C_4}\langle 0|\,|\Omega_1\rangle_{H_1}}}\\
&= (\alpha_1|0\rangle-\beta_1|1\rangle)_{B_5}\\
&\Longrightarrow U_1^{10} = Z
\end{aligned}
$$

(4)− $|C_3\rangle = |1\rangle\,,\ \ |C_4\rangle = |1\rangle :$

$$
\begin{aligned}
|\Omega_1\rangle_{H_1}^{11} &= \frac{{}_{C_3}\langle 1|_{C_4}\langle 1|\,|\Omega_1\rangle_{H_1}}{\sqrt{(\langle\Omega_1|_{H_1}\ |1\rangle_{C_3}|1\rangle_{C_4}\,{}_{C_3}\langle 1|_{C_4}\langle 1|\,|\Omega_1\rangle_{H_1}}}\\
&= (\alpha_1|1\rangle-\beta_1|0\rangle)_{B_5}\\
&\Longrightarrow U_1^{11} = iY
\end{aligned}
$$

$\underline{H(C_3)\text{ on }(2)\rightarrow}$

$$
\begin{aligned}
|\Omega_2\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|011\rangle+\alpha_1\alpha_2|111\rangle+\alpha_1\beta_2|000\rangle+\alpha_1\beta_2|100\rangle\\
&\quad+\beta_1\alpha_2|010\rangle-\beta_1\alpha_2|110\rangle+\beta_1\beta_2|001\rangle-\beta_1\beta_2|101\rangle]_{C_3C_4B_5}. \qquad (5.28)
\end{aligned}
$$

The final states and Bob's corrections:

.

$\vert C_3\rangle$	$\vert C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1\vert 0\rangle+\beta_1\vert 1\rangle$	I
0	1	$\alpha_1\vert 1\rangle+\beta_1\vert 0\rangle$	X
1	0	$\alpha_1\vert 0\rangle-\beta_1\vert 1\rangle$	Z
1	1	$\alpha_1\vert 1\rangle-\beta_1\vert 0\rangle$	IY

$H(C_3)$ on (3) →

$$|\Omega_3\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|000\rangle + \alpha_1\alpha_2|100\rangle - \alpha_1\beta_2|011\rangle - \alpha_1\beta_2|111\rangle + \beta_1\alpha_2|001\rangle - \beta_1\alpha_2|101\rangle - \beta_1\beta_2|010\rangle + \beta_1\beta_2|110\rangle]_{C_3C_4B_5}. \quad (5.29)$$

The result of Charlie's measurement

.

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1\|0\rangle + \beta_1\|1\rangle$	I
0	1	$-\alpha_1\|1\rangle - \beta_1\|0\rangle$	X
1	0	$\alpha_1\|0\rangle - \beta_1\|0\rangle$	Z
1	1	$-\alpha_1\|1\rangle + \beta_1\|0\rangle$	iY

$$|\Omega_4\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|011\rangle + \alpha_1\alpha_2|111\rangle - \alpha_1\beta_2|000\rangle - \alpha_1\beta_2|100\rangle + \beta_1\alpha_2|010\rangle - \beta_1\alpha_2|110\rangle - \beta_1\beta_2|001\rangle + \beta_1\beta_2|101\rangle]_{C_3C_4B_5}. \quad (5.30)$$

The result of Charlie's measurement

.

Table

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$-\alpha_1\|0\rangle - \beta_1\|1\rangle$	I
0	1	$\alpha_1\|1\rangle + \beta_1\|0\rangle$	X
1	0	$-\alpha_1\|0\rangle + \beta_1\|1\rangle$	Z
1	1	$\alpha_1\|1\rangle - \beta_1\|0\rangle$	iY

$$|\Omega_5\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|001\rangle - \alpha_1\alpha_2|101\rangle + \alpha_1\beta_2|010\rangle - \alpha_1\beta_2|110\rangle + \beta_1\alpha_2|000\rangle + \beta_1\alpha_2|100\rangle + \beta_1\beta_2|011\rangle + \beta_1\beta_2|111\rangle]_{C_3C_4B_5}. \quad (5.31)$$

The result of Charlie's measurement

.

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1\|1\rangle + \beta_1\|0\rangle$	X
0	1	$\alpha_1\|0\rangle + \beta_1\|1\rangle$	I
1	0	$-\alpha_1\|1\rangle + \beta_1\|0\rangle$	iY
1	1	$-\alpha_1\|0\rangle + \beta_1\|1\rangle$	Z

$H(C_3)$ on (6) →

$$|\Omega_6\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|010\rangle - \alpha_1\alpha_2|110\rangle + \alpha_1\beta_2|001\rangle - \alpha_1\beta_2|101\rangle + \beta_1\alpha_2|011\rangle + \beta_1\alpha_2|111\rangle + \beta_1\beta_2|000\rangle + \beta_1\beta_2|100\rangle]_{C_3C_4B_5}. \quad (5.32)$$

The result of Charlie's measurement

.

$\vert C_3\rangle$	$\vert C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1 \vert 1\rangle + \beta_1 \vert 0\rangle$	X
0	1	$\alpha_1 \vert 0\rangle + \beta_1 \vert 1\rangle$	I
1	0	$-\alpha_1 \vert 1\rangle + \beta_1 \vert 0\rangle$	iY
1	1	$-\alpha_1 \vert 0\rangle + \beta_1 \vert 1\rangle$	Z

$\underline{H(C_3) \text{ on } (7) \rightarrow}$

$$|\Omega_7\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |001\rangle - \alpha_1\alpha_2 |101\rangle - \alpha_1\beta_2 |010\rangle + \alpha_1\beta_2 |110\rangle + \beta_1\alpha_2 |000\rangle + \beta_1\alpha_2 |100\rangle - \beta_1\beta_2 |011\rangle - \beta_1\beta_2 |111\rangle]_{C_3C_4B_5}. \quad (5.33)$$

The result of Charlie's measurement

.

$\vert C_3\rangle$	$\vert C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1 \vert 1\rangle + \beta_1 \vert 0\rangle$	X
0	1	$-\alpha_1 \vert 0\rangle - \beta_1 \vert 1\rangle$	I
1	0	$-\alpha_1 \vert 1\rangle + \beta_1 \vert 0\rangle$	iY
1	1	$\alpha_1 \vert 0\rangle - \beta_1 \vert 1\rangle$	Z

$\underline{H(C_3) \text{ on } (8) \rightarrow}$

$$|\Omega_8\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |010\rangle - \alpha_1\alpha_2 |110\rangle - \alpha_1\beta_2 |001\rangle + \alpha_1\beta_2 |101\rangle + \beta_1\alpha_2 |011\rangle + \beta_1\alpha_2 |111\rangle - \beta_1\beta_2 |000\rangle - \beta_1\beta_2 |100\rangle]_{C_3C_4B_5}. \quad (5.34)$$

The result of Charlie's measurement

.

$\vert C_3\rangle$	$\vert C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$-\alpha_1 \vert 1\rangle - \beta_1 \vert 0\rangle$	X
0	1	$\alpha_1 \vert 0\rangle + \beta_1 \vert 1\rangle$	I
1	0	$\alpha_1 \vert 1\rangle - \beta_1 \vert 0\rangle$	iY
1	1	$-\alpha_1 \vert 0\rangle + \beta_1 \vert 1\rangle$	Z

$\underline{H(C_3) \text{ on } (9) \rightarrow}$

$$|\Omega_9\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |000\rangle + \alpha_1\alpha_2 |100\rangle + \alpha_1\beta_2 |011\rangle + \alpha_1\beta_2 |111\rangle - \beta_1\alpha_2 |001\rangle + \beta_1\alpha_2 |101\rangle - \beta_1\beta_2 |010\rangle + \beta_1\beta_2 |110\rangle]_{C_3C_4B_5}. \quad (5.35)$$

The result of Charlie's measurement

.

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1\|0\rangle - \beta_1\|1\rangle$	Z
0	1	$\alpha_1\|1\rangle - \beta_1\|0\rangle$	iY
1	0	$\alpha_1\|0\rangle + \beta_1\|1\rangle$	I
1	1	$\alpha_1\|1\rangle + \beta_1\|0\rangle$	X

$\underline{H(C_3) \text{ on } (10)}\rightarrow$

$$|\Omega_{10}\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|011\rangle + \alpha_1\alpha_2|111\rangle + \alpha_1\beta_2|000\rangle + \alpha_1\beta_2|100\rangle \\ -\beta_1\alpha_2|010\rangle + \beta_1\alpha_2|110\rangle - \beta_1\beta_2|001\rangle + \beta_1\beta_2|101\rangle]_{C_3C_4B_5}. \quad (5.36)$$

The result of Charlie's measurement:

.

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1\|0\rangle - \beta_1\|1\rangle$	Z
0	1	$\alpha_1\|1\rangle - \beta_1\|0\rangle$	iY
1	0	$\alpha_1\|0\rangle + \beta_1\|1\rangle$	I
1	1	$\alpha_1\|1\rangle - \beta_1\|0\rangle$	X

$\underline{H(C_3) \text{ on } (11) \rightarrow}$

$$|\Omega_{11}\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|000\rangle - \alpha_1\beta_2|011\rangle + \beta_1\alpha_2|101\rangle - \beta_1\beta_2|110\rangle \\ +\alpha_1\alpha_2|100\rangle - \alpha_1\beta_2|111\rangle - \beta_1\alpha_2|001\rangle + \beta_1\beta_2|010\rangle]_{C_3C_4B_5}. \quad (5.37)$$

The result of Charlie's measurement

.

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1\|0\rangle - \beta_1\|1\rangle$	Z
0	1	$-\alpha_1\|1\rangle + \beta_1\|0\rangle$	iY
1	0	$\alpha_1\|0\rangle + \beta_1\|1\rangle$	I
1	1	$-\alpha_1\|1\rangle - \beta_1\|0\rangle$	X

$\underline{H(C_3) \text{ on } (12) \rightarrow}$

$$|\Omega_{12}\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|011\rangle - \alpha_1\beta_2|000\rangle - \beta_1\alpha_2|010\rangle + \beta_1\beta_2|001\rangle \\ +\alpha_1\alpha_2|111\rangle - \alpha_1\beta_2|100\rangle + \beta_1\alpha_2|110\rangle - \beta_1\beta_2|101\rangle]_{C_3C_4B_5}. \quad (5.38)$$

The result of Charlie's measurement

.

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$-\alpha_1 \|0\rangle + \beta_1 \|1\rangle$	Z
0	1	$\alpha_1 \|1\rangle - \beta_1 \|0\rangle$	iY
1	0	$-\alpha_1 \|0\rangle - \beta_1 \|1\rangle$	I
1	1	$\alpha_1 \|1\rangle + \beta_1 \|0\rangle$	X

$\underline{H(C_3) \text{ on } (13) \rightarrow}$

$$\begin{aligned} |\Omega_{13}\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |001\rangle + \alpha_1\beta_2 |010\rangle - \beta_1\alpha_2 |000\rangle - \beta_1\beta_2 |011\rangle \\ &\quad -\alpha_1\alpha_2 |101\rangle - \alpha_1\beta_2 |110\rangle - \beta_1\alpha_2 |100\rangle - \beta_1\beta_2 |111\rangle]_{C_3C_4B_5}. \end{aligned} \tag{5.39}$$

The result of Charlie's measurement

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1 \|1\rangle - \beta_1 \|0\rangle$	iY
0	1	$\alpha_1 \|0\rangle - \beta_1 \|1\rangle$	Z
1	0	$-\alpha_1 \|1\rangle - \beta_1 \|0\rangle$	X
1	1	$-\alpha_1 \|0\rangle - \beta_1 \|1\rangle$	I

$\underline{H(C_3) \text{ on } (14) \rightarrow}$

$$\begin{aligned} |\Omega_{14}\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |010\rangle + \alpha_1\beta_2 |001\rangle - \beta_1\alpha_2 |011\rangle - \beta_1\beta_2 |000\rangle \\ &\quad -\alpha_1\alpha_2 |110\rangle - \alpha_1\beta_2 |101\rangle - \beta_1\alpha_2 |111\rangle - \beta_1\beta_2 |100\rangle]_{C_3C_4B_5}. \end{aligned} \tag{5.40}$$

The result of Charlie's measurement

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1 \|1\rangle - \beta_1 \|0\rangle$	iY
0	1	$\alpha_1 \|0\rangle - \beta_1 \|1\rangle$	Z
1	0	$\alpha_1 \|1\rangle + \beta_1 \|0\rangle$	X
1	1	$\alpha_1 \|0\rangle + \beta_1 \|1\rangle$	I

$\underline{H(C_3) \text{ on } (15) \rightarrow}$

$$\begin{aligned} |\Omega_{15}\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |001\rangle - \alpha_1\beta_2 |010\rangle - \beta_1\alpha_2 |000\rangle + \beta_1\beta_2 |011\rangle \\ &\quad -\alpha_1\alpha_2 |101\rangle + \alpha_1\beta_2 |110\rangle - \beta_1\alpha_2 |100\rangle + \beta_1\beta_2 |111\rangle]_{C_3C_4B_5}. \end{aligned} \tag{5.41}$$

The result of Charlie's measurement

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1 \|1\rangle - \beta_1 \|0\rangle$	iY
0	1	$-\alpha_1 \|0\rangle + \beta_1 \|1\rangle$	Z
1	0	$-\alpha_1 \|1\rangle - \beta_1 \|0\rangle$	X
1	1	$\alpha_1 \|0\rangle + \beta_1 \|1\rangle$	I

$\underline{H(C_3) \text{ on } (16) \rightarrow}$

$$\begin{aligned} |\Omega_{16}\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |110\rangle - \alpha_1\beta_2 |101\rangle - \beta_1\alpha_2 |011\rangle + \beta_1\beta_2 |000\rangle \\ &\quad -\alpha_1\alpha_2 |110\rangle - \alpha_1\beta_2 |101\rangle - \beta_1\alpha_2 |011\rangle + \beta_1\beta_2 |000\rangle]_{C_3C_4B_5}. \end{aligned} \tag{5.42}$$

Measurement in the z basis and the coresponding corrections :

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_1 \|1\rangle - \beta_1 \|0\rangle$	iY
0	1	$-\alpha_1 \|0\rangle + \beta_1 \|1\rangle$	Z
1	0	$-\alpha_1 \|1\rangle - \beta_1 \|0\rangle$	X
1	1	$\alpha_1 \|0\rangle + \beta_1 \|1\rangle$	I

we can concludethe relation between the measurements of Charlie and Alice1 and bob's corrections:

Charlie's results	$_{1A_1}\langle B_{00}\|$	$_{1A_1}\langle B_{01}\|$	$_{1A_1}\langle B_{10}\|$	$_{1A_1}\langle B_{11}\|$
0 0	I	X	Z	iY
0 1	X	I	iY	Z
1 0	Z	iY	I	X
1 1	iY	Z	X	I

Second case: when Charlie decides to transmit the qubit of Alice2, he performs Hadamard gate on his qubit C_4,then he mesures his two qubits in the Z-basis, we obtain the final states and the corresponding corrections of Bob:

$\underline{H(C_4) \text{ on } (1) \rightarrow}$

$$\begin{aligned} |\Omega_1\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |000\rangle + \alpha_1\alpha_2 |010\rangle + \alpha_1\beta_2 |001\rangle - \alpha_1\beta_2 |011\rangle \\ &\quad +\beta_1\alpha_2 |101\rangle + \beta_1\alpha_2 |111\rangle + \beta_1\beta_2 |100\rangle - \beta_1\beta_2 |110\rangle]_{C_3C_4B_5}. \end{aligned} \tag{5.43}$$

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2 \|0\rangle + \beta_2 \|1\rangle$	I
0	1	$\alpha_2 \|0\rangle - \beta_2 \|1\rangle$	Z
1	0	$\alpha_2 \|1\rangle + \beta_2 \|0\rangle$	X
1	1	$\alpha_2 \|1\rangle - \beta_2 \|0\rangle$	iY

$\underline{H(C_4) \text{ on } (2) \rightarrow}$

$$|\Omega_2\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|001\rangle - \alpha_1\alpha_2|011\rangle + \alpha_1\beta_2|000\rangle + \alpha_1\beta_2|010\rangle$$
$$+\beta_1\alpha_2|100\rangle - \beta_1\alpha_2|110\rangle + \beta_1\beta_2|101\rangle + \beta_1\beta_2|111\rangle]_{C_3C_4B_5}. \quad (5.44)$$

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2\|1\rangle + \beta_2\|0\rangle$	X
0	1	$-\alpha_2\|1\rangle + \beta_1\|0\rangle$	iY
1	0	$\alpha_2\|0\rangle + \beta_2\|1\rangle$	I
1	1	$-\alpha_2\|0\rangle + \beta_2\|1\rangle$	Z

$\underline{H(C_4) \text{ on } (3) \rightarrow}$

$$|\Omega_3\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|000\rangle + \alpha_1\alpha_2|010\rangle - \alpha_1\beta_2|001\rangle + \alpha_1\beta_2|011\rangle$$
$$+\beta_1\alpha_2|101\rangle + \beta_1\alpha_2|111\rangle - \beta_1\beta_2|100\rangle + \beta_1\beta_2|110\rangle]_{C_3C_4B_5}. \quad (5.45)$$

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2\|0\rangle - \beta_2\|1\rangle$	Z
0	1	$\alpha_2\|0\rangle + \beta_2\|1\rangle$	I
1	0	$\alpha_2\|1\rangle - \beta_2\|0\rangle$	iY
1	1	$\alpha_2\|1\rangle + \beta_2\|0\rangle$	X

$\underline{H(C_4) \text{ on } (4) \rightarrow}$

$$|\Omega_4\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|001\rangle - \alpha_1\alpha_2|011\rangle - \alpha_1\beta_2|000\rangle - \alpha_1\beta_2|010\rangle$$
$$+\beta_1\alpha_2|100\rangle - \beta_1\alpha_2|110\rangle - \beta_1\beta_2|101\rangle - \beta_1\beta_2|111\rangle]_{C_3C_4B_5}. \quad (5.46)$$

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2\|1\rangle - \beta_2\|0\rangle$	iY
0	1	$-\alpha_2\|1\rangle - \beta_2\|0\rangle$	X
1	0	$\alpha_2\|0\rangle - \beta_2\|1\rangle$	Z
1	1	$-\alpha_2\|0\rangle - \beta_2\|1\rangle$	I

$\underline{H(C_4) \text{ on } (5) \rightarrow}$

$$|\Omega_5\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|101\rangle + \alpha_1\alpha_2|111\rangle + \alpha_1\beta_2|100\rangle - \alpha_1\beta_2|110\rangle$$
$$+\beta_1\alpha_2|000\rangle + \beta_1\alpha_2|010\rangle + \beta_1\beta_2|001\rangle - \beta_1\beta_2|011\rangle]_{C_3C_4B_5}. \quad (5.47)$$

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2 \|0\rangle + \beta_2 \|1\rangle$	I
0	1	$\alpha_2 \|0\rangle - \beta_1 \|1\rangle$	Z
1	0	$\alpha_2 \|1\rangle + \beta_2 \|0\rangle$	X
1	1	$\alpha_2 \|1\rangle - \beta_2 \|0\rangle$	iY

$H(C_4)$ on (6) $\rightarrow$

$$|\Omega_6\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|100\rangle - \alpha_1\alpha_2|110\rangle + \alpha_1\beta_2|101\rangle + \alpha_1\beta_2|111\rangle + \beta_1\alpha_2|001\rangle - \beta_1\alpha_2|011\rangle + \beta_1\beta_2|000\rangle + \beta_1\beta_2|010\rangle]_{C_3C_4B_5}. \quad (5.48)$$

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2 \|1\rangle + \beta_2 \|0\rangle$	X
0	1	$-\alpha_2 \|1\rangle + \beta_2 \|0\rangle$	iY
1	0	$\alpha_2 \|0\rangle + \beta_2 \|1\rangle$	I
1	1	$-\alpha_2 \|0\rangle + \beta_2 \|1\rangle$	Z

$H(C_4)$ on (7) $\rightarrow$

$$|\Omega_7\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|101\rangle + \alpha_1\alpha_2|111\rangle - \alpha_1\beta_2|100\rangle + \alpha_1\beta_2|110\rangle + \beta_1\alpha_2|000\rangle + \beta_1\alpha_2|010\rangle - \beta_1\beta_2|001\rangle + \beta_1\beta_2|011\rangle]_{C_3C_4B_5}. \quad (5.49)$$

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2 \|0\rangle - \beta_2 \|1\rangle$	Z
0	1	$\alpha_2 \|0\rangle + \beta_2 \|1\rangle$	I
1	0	$\alpha_2 \|1\rangle - \beta_2 \|0\rangle$	iY
1	1	$\alpha_2 \|1\rangle + \beta_2 \|0\rangle$	X

$H(C_4)$ on (8) $\rightarrow$

$$|\Omega_8\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|100\rangle - \alpha_1\alpha_2|110\rangle - \alpha_1\beta_2|101\rangle - \alpha_1\beta_2|111\rangle + \beta_1\alpha_2|001\rangle - \beta_1\alpha_2|011\rangle - \beta_1\beta_2|000\rangle - \beta_1\beta_2|010\rangle]_{C_3C_4B_5}. \quad (5.50)$$

The result of Charlie's measurement

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2 \|1\rangle - \beta_2 \|0\rangle$	iY
0	1	$-\alpha_2 \|1\rangle - \beta_2 \|0\rangle$	X
1	0	$\alpha_2 \|0\rangle - \beta_2 \|1\rangle$	I
1	1	$-\alpha_2 \|0\rangle - \beta_2 \|1\rangle$	X

$\underline{H(C_4) \text{ on } (9) \rightarrow}$

$$|\Omega_9\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|000\rangle + \alpha_1\alpha_2|010\rangle + \alpha_1\beta_2|001\rangle - \alpha_1\beta_2|011\rangle$$
$$-\beta_1\alpha_2|101\rangle - \beta_1\alpha_2|111\rangle - \beta_1\beta_2|100\rangle + \beta_1\beta_2|110\rangle]_{C_3C_4B_5}. \quad (5.51)$$

The result of Charlie's measurement

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2\|0\rangle + \beta_2\|1\rangle$	I
0	1	$\alpha_2\|0\rangle - \beta_2\|1\rangle$	Z
1	0	$-\alpha_2\|1\rangle - \beta_2\|0\rangle$	X
1	1	$-\alpha_2\|1\rangle + \beta_2\|0\rangle$	iY

$\underline{H(C_4) \text{ on } (10) \rightarrow}$

$$|\Omega_{10}\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|001\rangle - \alpha_1\alpha_2|011\rangle + \alpha_1\beta_2|000\rangle + \alpha_1\beta_2|010\rangle$$
$$-\beta_1\alpha_2|100\rangle + \beta_1\alpha_2|110\rangle - \beta_1\beta_2|101\rangle - \beta_1\beta_2|111\rangle]_{C_3C_4B_5}. \quad (5.52)$$

The result of Charlie's measurement

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2\|1\rangle + \beta_2\|0\rangle$	X
0	1	$-\alpha_2\|1\rangle + \beta_2\|0\rangle$	iY
1	0	$-\alpha_2\|0\rangle - \beta_2\|1\rangle$	I
1	1	$\alpha_2\|0\rangle - \beta_2\|1\rangle$	Z

$\underline{H(C_4) \text{ on } (11) \rightarrow}$

$$|\Omega_{11}\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|000\rangle - \alpha_1\beta_2|001\rangle - \beta_1\alpha_2|101\rangle - \beta_1\beta_2|110\rangle$$
$$+\alpha_1\alpha_2|010\rangle + \alpha_1\beta_2|011\rangle - \beta_1\alpha_2|111\rangle + \beta_1\beta_2|100\rangle]_{C_3C_4B_5}. \quad (5.53)$$

The result of Charlie's measurement

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2\|0\rangle - \beta_2\|1\rangle$	Z
0	1	$\alpha_2\|0\rangle + \beta_2\|1\rangle$	I
1	0	$-\alpha_2\|1\rangle + \beta_2\|0\rangle$	iY
1	1	$-\alpha_2\|1\rangle - \beta_2\|0\rangle$	X

$\underline{H(C_4) \text{ on } (12) \rightarrow}$

$$|\Omega_{12}\rangle = \frac{1}{\sqrt{2}}[\alpha_1\alpha_2|001\rangle - \alpha_1\beta_2|000\rangle - \beta_1\alpha_2|100\rangle + \beta_1\beta_2|101\rangle$$
$$-\alpha_1\alpha_2|011\rangle - \alpha_1\beta_2|010\rangle + \beta_1\alpha_2|110\rangle + \beta_1\beta_2|111\rangle]_{C_3C_4B_5}. \quad (5.54)$$

$\Longrightarrow$

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2 \|1\rangle - \beta_2 \|0\rangle$	iY
0	1	$-\alpha_2 \|1\rangle - \beta_2 \|0\rangle$	X
1	0	$-\alpha_2 \|0\rangle + \beta_2 \|1\rangle$	Z
1	1	$\alpha_2 \|0\rangle + \beta_2 \|1\rangle$	I

$\underline{H(C_4) \text{ on } (13) \rightarrow}$

$$\begin{aligned} |\Omega_{13}\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |101\rangle + \alpha_1\beta_2 |100\rangle - \beta_1\alpha_2 |000\rangle - \beta_1\beta_2 |001\rangle \\ &+\alpha_1\alpha_2 |111\rangle - \alpha_1\beta_2 |110\rangle - \beta_1\alpha_2 |010\rangle + \beta_1\beta_2 |011\rangle]_{C_3C_4B_5}. \end{aligned} \tag{5.55}$$

The result of Charlie's measurement

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$-\alpha_2 \|0\rangle - \beta_2 \|1\rangle$	I
0	1	$-\alpha_2 \|0\rangle + \beta_2 \|1\rangle$	Z
1	0	$\alpha_2 \|1\rangle + \beta_2 \|0\rangle$	X
1	1	$\alpha_2 \|1\rangle - \beta_2 \|0\rangle$	iY

$\underline{H(C_4) \text{ on } (14) \rightarrow}$

$$\begin{aligned} |\Omega_{14}\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |100\rangle + \alpha_1\beta_2 |101\rangle - \beta_1\alpha_2 |001\rangle - \beta_1\beta_2 |000\rangle \\ &-\alpha_1\alpha_2 |110\rangle + \alpha_1\beta_2 |111\rangle + \beta_1\alpha_2 |011\rangle - \beta_1\beta_2 |010\rangle]_{C_3C_4B_5}. \end{aligned} \tag{5.56}$$

The result of Charlie's measurement

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$\alpha_2 \|1\rangle + \beta_2 \|0\rangle$	X
0	1	$\alpha_2 \|1\rangle - \beta_2 \|0\rangle$	iY
1	0	$\alpha_2 \|0\rangle + \beta_2 \|1\rangle$	I
1	1	$\alpha_2 \|0\rangle - \beta_2 \|1\rangle$	Z

$\underline{H(C_4) \text{ on } (15) \rightarrow}$

$$\begin{aligned} |\Omega_{15}\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |101\rangle - \alpha_1\beta_2 |100\rangle - \beta_1\alpha_2 |000\rangle + \beta_1\beta_2 |001\rangle \\ &+\alpha_1\alpha_2 |111\rangle + \alpha_1\beta_2 |110\rangle - \beta_1\alpha_2 |010\rangle - \beta_1\beta_2 |011\rangle]_{C_3C_4B_5}. \end{aligned} \tag{5.57}$$

$\Longrightarrow$

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$-\alpha_2 \|0\rangle + \beta_2 \|1\rangle$	Z
0	1	$-\alpha_2 \|0\rangle - \beta_1 \|1\rangle$	I
1	0	$\alpha_2 \|1\rangle + \beta_2 \|0\rangle$	X
1	1	$\alpha_2 \|1\rangle - \beta_2 \|0\rangle$	iY

$\underline{H(C_4) \text{ on } (16) \rightarrow}$

$$\begin{aligned} |\Omega_{16}\rangle &= \frac{1}{\sqrt{2}}[\alpha_1\alpha_2 |100\rangle - \alpha_1\beta_2 |101\rangle + \beta_1\alpha_2 |011\rangle + \beta_1\beta_2 |000\rangle \\ &-\alpha_1\alpha_2 |110\rangle - \alpha_1\beta_2 |111\rangle - \beta_1\alpha_2 |001\rangle + \beta_1\beta_2 |010\rangle]_{C_3C_4B_5}. \end{aligned} \tag{5.58}$$

$\Longrightarrow$

$\|C_3\rangle$	$\|C_4\rangle$	the state of Bob's qubit B_5	The correction operation
0	0	$-\alpha_2 \|1\rangle + \beta_2 \|0\rangle$	iY
0	1	$\alpha_2 \|1\rangle + \beta_1 \|0\rangle$	X
1	0	$\alpha_2 \|0\rangle - \beta_2 \|1\rangle$	Z
1	1	$-\alpha_2 \|0\rangle - \beta_2 \|1\rangle$	I

.

the relation between the results of Alice2 &nd Charlies measurement and the correction opeartions:

Charlie's results	${}_{2A_2}\langle B_{00}\|$	${}_{2A_2}\langle B_{01}\|$	${}_{2A_2}\langle B_{10}\|$	${}_{2A_2}\langle B_{11}\|$
0 0	I	X	Z	iY
0 1	Z	iY	I	X
1 0	X	I	iY	Z
1 1	iY	Z	X	I

Using the density matrix formalism, in the case of pure state, we will get similar results to which we obtain using the ket's formalism and will get the unit fidelity F=1, so we are interested in the case of noisy quantum channel.

5.2 The Protocol with noisy quantum channel

The channel transformed as:

$$\bar{\rho}_c = \left[(1-\lambda)\,\rho_C + \frac{\lambda}{32} I_{32}\right]$$

where $(0 \le \lambda \le 1)$,and ρ_C is the density operator of the channel $|C\rangle$:

$$\begin{aligned} \rho_C &= |C\rangle\langle C| = \frac{1}{4}\left[|00000\rangle + |01011\rangle + |10101\rangle + |11110\rangle\right] \otimes \left[\langle 00000| + \langle 01011| + \langle 10101| + \langle 11110|\right] \\ &= \frac{1}{4}[|00000\rangle\langle 00000| + |00000\rangle\langle 01011| + |00000\rangle\langle 10101| + |00000\rangle\langle 11110| + |01011\rangle\langle 00000| \\ &+ |01011\rangle\langle 01011| + |01011\rangle\langle 10101| + |01011\rangle\langle 11110| + |10101\rangle\langle 00000| + |10101\rangle\langle 01011| \\ &+ |10101\rangle\langle 10101| + |10101\rangle\langle 11110| + |11110\rangle\langle 00000| + |11110\rangle\langle 01011| + |11110\rangle\langle 10101| \\ &+ |11110\rangle\langle 11110|] \end{aligned} \tag{5.59}$$

the density operators of the teleported states are:

$$\begin{aligned}\rho_1 &= |\Phi_1\rangle\langle\Phi_1| = (\alpha_1|0\rangle + \beta_1|1\rangle)\otimes(\alpha_1^*\langle 0| + \beta_1^*\langle 1|) \\ &= \Big[|\alpha_1|^2|0\rangle\langle 0| + \alpha_1\beta_1^*|0\rangle\langle 1| + \beta_1\alpha_1^*|1\rangle\langle 0| + |\beta_1|^2|1\rangle\langle 1|\Big] \end{aligned} \tag{5.60}$$

$$\begin{aligned}\rho_2 &= |\Phi_2\rangle\langle\Phi_2| = (\alpha_2|0\rangle + \beta_2|1\rangle)\otimes(\alpha_2^*\langle 0| + \beta_2^*\langle 1|) \\ &= \Big[|\alpha_2|^2|0\rangle\langle 0| + \alpha_2\beta_2^*|0\rangle\langle 1| + \beta_2\alpha_2^*|1\rangle\langle 0| + |\beta_2|^2|1\rangle\langle 1|\Big] \end{aligned} \tag{5.61}$$

$$\begin{aligned}\rho_{12} = \; & \rho_1\otimes\rho_2 = |\alpha_1|^2|\alpha_2|^2|00\rangle\langle 00| + |\alpha_1|^2\alpha_2\beta_2^*|00\rangle\langle 01| + |\alpha_1|^2\beta_2\alpha_2^*|01\rangle\langle 00| \\ & + |\alpha_1|^2|\beta_2|^2|01\rangle\langle 01| + \alpha_1\beta_1^*|\alpha_2|^2|00\rangle\langle 10| + \alpha_1\beta_1^*\alpha_2\beta_2^*|00\rangle\langle 11| + \alpha_1\beta_1^*\beta_2\alpha_2^*|01\rangle\langle 10| \\ & + \alpha_1\beta_1^*|\beta_2|^2|01\rangle\langle 11| + \beta_1\alpha_1^*|\alpha_2|^2|10\rangle\langle 00| + \beta_1\alpha_1^*\alpha_2\beta_2^*|10\rangle\langle 01| + |\beta_1|^2|\alpha_2|^2|10\rangle\langle 10| \\ & + |\beta_1|^2\alpha_2\beta_2^*|10\rangle\langle 11| + \beta_1\alpha_1^*\beta_2\alpha_2^*|11\rangle\langle 00| + \beta_1\alpha_1^*|\beta_2|^2|11\rangle\langle 01| + |\beta_1|^2\beta_2\alpha_2^*|11\rangle\langle 10| \\ & + |\beta_1|^2|\beta_2|^2|11\rangle\langle 11| \end{aligned} \tag{5.62}$$

The density operator of the general state is

$$\begin{aligned}\rho &= \rho_1\otimes\rho_2\otimes\bar{\rho}_c \\ &= [|\alpha_1|^2|\alpha_2|^2|00\rangle\langle 00| + |\alpha_1|^2\alpha_2\beta_2^*|00\rangle\langle 01| + |\alpha_1|^2\beta_2\alpha_2^*|01\rangle\langle 00| + |\alpha_1|^2|\beta_2|^2|01\rangle\langle 01| \\ & +\alpha_1\beta_1^*|\alpha_2|^2|00\rangle\langle 10| + \alpha_1\beta_1^*\alpha_2\beta_2^*|00\rangle\langle 11| + \alpha_1\beta_1^*\beta_2\alpha_2^*|01\rangle\langle 10| + \alpha_1\beta_1^*|\beta_2|^2|01\rangle\langle 11| \\ & +\beta_1\alpha_1^*|\alpha_2|^2|10\rangle\langle 00| + \beta_1\alpha_1^*\alpha_2\beta_2^*|10\rangle\langle 01| + |\beta_1|^2|\alpha_2|^2|10\rangle\langle 10| + |\beta_1|^2\alpha_2\beta_2^*|10\rangle\langle 11| \\ & +\beta_1\alpha_1^*\beta_2\alpha_2^*|11\rangle\langle 00| + \beta_1\alpha_1^*|\beta_2|^2|11\rangle\langle 01| + |\beta_1|^2\beta_2\alpha_2^*|11\rangle\langle 10| + |\beta_1|^2|\beta_2|^2|11\rangle\langle 11|] \\ & \otimes[\frac{1}{4}(1-\lambda)(|00000\rangle\langle 00000| + |00000\rangle\langle 01011| + |00000\rangle\langle 10101| \\ & + |00000\rangle\langle 11110| + |01011\rangle\langle 00000| + |01011\rangle\langle 01011| + |01011\rangle\langle 10101| + |01011\rangle\langle 11110| \\ & + |10101\rangle\langle 00000| + |10101\rangle\langle 01011| + |10101\rangle\langle 10101| + |10101\rangle\langle 11110| + |11110\rangle\langle 00000| \\ & + |11110\rangle\langle 01011| + |11110\rangle\langle 10101| + |11110\rangle\langle 11110|) + \frac{\lambda}{32}I_{32}] \end{aligned} \tag{5.63}$$

Bell states measurement:

Alice1 and Alice2 measured their qubits in the Bell basis $\{B_{xy}\}$:

$$\rho_m = \frac{M^+ \rho M}{tr(M^+ \rho M)} \tag{5.64}$$

where

$$M = |B_{xy}\rangle_{1A_1} |B_{\acute{x}\acute{y}}\rangle_{2A_2 1A_1} \langle B_{xy}|_{2A_2} \langle B_{\acute{x}\acute{y}}| \otimes I_2 \otimes I_2 \otimes I_2$$

1)_

$$M_1 = M_1^+ = |B_{00}\rangle_{1A_1} |B_{00}\rangle_{2A_2 1A_1} \langle B_{00}|_{2A_2} \langle B_{00}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned} M_1^+ \rho M_1 &= |B_{00}\rangle_{1A_1} |B_{00}\rangle_{2A_2 1A_1} \langle B_{00}|_{2A_2} \langle B_{00}| \rho |B_{00}\rangle_{1A_1} |B_{00}\rangle_{2A_2 1A_1} \langle B_{00}|_{2A_2} \langle B_{00}| \\ &= \frac{1}{4}[(1-\lambda)\frac{1}{4}(|\alpha_1|^2 |\alpha_2|^2 |000\rangle \langle 000| + |\alpha_1|^2 \alpha_2 \beta_2^* |000\rangle \langle 011| + |\alpha_1|^2 \beta_2 \alpha_2^* |011\rangle \langle 000| \\ &+ |\alpha_1|^2 |\beta_2|^2 |011\rangle \langle 011| + \alpha_1 \beta_1^* |\alpha_2|^2 |000\rangle \langle 101| + \alpha_1 \beta_1^* \alpha_2 \beta_2^* |000\rangle \langle 110| + \alpha_1 \beta_1^* \beta_2 \alpha_2^* |011\rangle \langle 101| \\ &+ \alpha_1 \beta_1^* |\beta_2|^2 |011\rangle \langle 110| + \beta_1 \alpha_1^* |\alpha_2|^2 |101\rangle \langle 000| + \beta_1 \alpha_1^* \alpha_2 \beta_2^* |101\rangle \langle 011| + |\beta_1|^2 |\alpha_2|^2 |101\rangle \langle 101| \\ &+ |\beta_1|^2 \alpha_2 \beta_2^* |101\rangle \langle 110| + \beta_1 \alpha_1^* \beta_2 \alpha_2^* |110\rangle \langle 000| + \beta_1 \alpha_1^* |\beta_2|^2 |110\rangle \langle 011| + |\beta_1|^2 \beta_2 \alpha_2^* |110\rangle \langle 101| \\ &+ |\beta_1|^2 |\beta_2|^2 |110\rangle \langle 110|) + \frac{\lambda}{32} I_8] \end{aligned} \tag{5.65}$$

$$tr(M_1^+ \rho M_1) = \frac{1}{4}[\frac{1}{4}(1-\lambda)(|\alpha_1|^2 |\alpha_2|^2 + |\alpha_1|^2 |\beta_2|^2 + |\beta_1|^2 |\alpha_2|^2 + |\beta_1|^2 |\beta_2|^2) + \frac{\lambda}{32} tr(I_8)] = \frac{1}{16} \tag{5.66}$$

$\Longrightarrow$

$$\begin{aligned} \rho_{m1} &= (1-\lambda)[|\alpha_1|^2 |\alpha_2|^2 |000\rangle \langle 000| + |\alpha_1|^2 \alpha_2 \beta_2^* |000\rangle \langle 011| + \alpha_1 \beta_1^* |\alpha_2|^2 |000\rangle \langle 101| \\ &+ \alpha_1 \beta_1^* \alpha_2 \beta_2^* |000\rangle \langle 110| + |\alpha_1|^2 \beta_2 \alpha_2^* |011\rangle \langle 000| + |\alpha_1|^2 |\beta_2|^2 |011\rangle \langle 011| + \alpha_1 \beta_1^* \beta_2 \alpha_2^* |011\rangle \langle 101| \\ &+ \alpha_1 \beta_1^* |\beta_2|^2 |011\rangle \langle 110| + \beta_1 \alpha_1^* |\alpha_2|^2 |101\rangle \langle 000| + \beta_1 \alpha_1^* \alpha_2 \beta_2^* |101\rangle \langle 011| + |\beta_1|^2 |\alpha_2|^2 |101\rangle \langle 101| \\ &+ |\beta_1|^2 \alpha_2 \beta_2^* |101\rangle \langle 110| + \beta_1 \alpha_1^* \beta_2 \alpha_2^* |110\rangle \langle 000| + \beta_1 \alpha_1^* |\beta_2|^2 |110\rangle \langle 011| + |\beta_1|^2 \beta_2 \alpha_2^* |110\rangle \langle 101| \\ &+ |\beta_1|^2 |\beta_2|^2 |110\rangle \langle 110|] + \frac{\lambda}{8} I_8. \end{aligned} \tag{5.67}$$

(2)_

$$M = M^+ = |B_{00}\rangle_{1A_1} |B_{01}\rangle_{2A_2 1A_1} \langle B_{00}|_{2A_2} \langle B_{01}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}
\rho_{m2} &= \frac{|B_{00}\rangle_{1A_1}|B_{01}\rangle_{2A_21A_1}\langle B_{00}|_{2A_2}\langle B_{01}|\rho|B_{00}\rangle_{1A_1}|B_{01}\rangle_{2A_21A_1}\langle B_{00}|_{2A_2}\langle B_{01}|}{tr(|B_{00}\rangle_{1A_1}|B_{01}\rangle_{2A_21A_1}\langle B_{00}|_{2A_2}\langle B_{01}|\rho|B_{00}\rangle_{1A_1}|B_{01}\rangle_{2A_21A_1}\langle B_{00}|_{2A_2}\langle B_{01}|)}\\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|011\rangle\langle 011| + |\alpha_1|^2\alpha_2\beta_2^*|011\rangle\langle 000| + \alpha_1\beta_1^*|\alpha_2|^2|011\rangle\langle 110|\\
&+\alpha_1\beta_1^*\alpha_2\beta_2^*|011\rangle\langle 101| + |\alpha_1|^2|\beta_2|^2|000\rangle\langle 000| + |\alpha_1|^2\beta_2\alpha_2^*|000\rangle\langle 011| + \alpha_1\beta_1^*\beta_2\alpha_2^*|000\rangle\langle 110|\\
&+\alpha_1\beta_1^*|\beta_2|^2|000\rangle\langle 101| + \beta_1\alpha_1^*|\alpha_2|^2|110\rangle\langle 011| + \beta_1\alpha_1^*\alpha_2\beta_2^*|110\rangle\langle 000| + |\beta_1|^2|\alpha_2|^2|110\rangle\langle 110|\\
&+|\beta_1|^2\alpha_2\beta_2^*|110\rangle\langle 101| + \beta_1\alpha_1^*\beta_2\alpha_2^*|101\rangle\langle 011| + \beta_1\alpha_1^*|\beta_2|^2|101\rangle\langle 000| + |\beta_1|^2\beta_2\alpha_2^*|101\rangle\langle 110|\\
&+|\beta_1|^2|\beta_2|^2|101\rangle\langle 101|] + \frac{\lambda}{8}I_8. \qquad (5.68)
\end{aligned}$$

(3)_

$$M = M^+ = |B_{00}\rangle_{1A_1}|B_{10}\rangle_{2A_21A_1}\langle B_{00}|_{2A_2}\langle B_{10}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}
\rho_{m3} &= \frac{{}_{1A_1}\langle B_{00}|_{2A_2}\langle B_{10}|\rho|B_{00}\rangle_{1A_1}|B_{10}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{00}|_{2A_2}\langle B_{10}|\rho|B_{00}\rangle_{1A_1}|B_{10}\rangle_{2A_2})}\\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|000\rangle\langle 000| - |\alpha_1|^2\alpha_2\beta_2^*|000\rangle\langle 011| + \alpha_1\beta_1^*|\alpha_2|^2|000\rangle\langle 101|\\
&-\alpha_1\beta_1^*\alpha_2\beta_2^*|000\rangle\langle 110| - |\alpha_1|^2\beta_2\alpha_2^*|011\rangle\langle 000| + |\alpha_1|^2|\beta_2|^2|011\rangle\langle 011| - \alpha_1\beta_1^*\beta_2\alpha_2^*|011\rangle\langle 101|\\
&+\alpha_1\beta_1^*|\beta_2|^2|011\rangle\langle 110| + \beta_1\alpha_1^*|\alpha_2|^2|101\rangle\langle 000| - \beta_1\alpha_1^*\alpha_2\beta_2^*|101\rangle\langle 011| + |\beta_1|^2|\alpha_2|^2|101\rangle\langle 101|\\
&-|\beta_1|^2\alpha_2\beta_2^*|101\rangle\langle 110| - \beta_1\alpha_1^*\beta_2\alpha_2^*|110\rangle\langle 000| + \beta_1\alpha_1^*|\beta_2|^2|110\rangle\langle 011| - |\beta_1|^2\beta_2\alpha_2^*|110\rangle\langle 101|\\
&+|\beta_1|^2|\beta_2|^2|110\rangle\langle 110|] + \frac{\lambda}{8}I_8. \qquad (5.69)
\end{aligned}$$

(4)_

$$M = M^+ = |B_{00}\rangle_{1A_1}|B_{11}\rangle_{2A_21A_1}\langle B_{00}|_{2A_2}\langle B_{11}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}
\rho_{m4} &= \frac{{}_{1A_1}\langle B_{00}|_{2A_2}\langle B_{11}|\rho|B_{00}\rangle_{1A_1}|B_{11}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{00}|_{2A_2}\langle B_{11}|\rho|B_{00}\rangle_{1A_1}|B_{11}\rangle_{2A_2})}\\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|011\rangle\langle 011| - |\alpha_1|^2\alpha_2\beta_2^*|011\rangle\langle 000| + \alpha_1\beta_1^*|\alpha_2|^2|011\rangle\langle 110|\\
&-\alpha_1\beta_1^*\alpha_2\beta_2^*|011\rangle\langle 101| + |\alpha_1|^2|\beta_2|^2|000\rangle\langle 000| - |\alpha_1|^2\beta_2\alpha_2^*|000\rangle\langle 011| - \alpha_1\beta_1^*\beta_2\alpha_2^*|000\rangle\langle 110|\\
&+\alpha_1\beta_1^*|\beta_2|^2|000\rangle\langle 101| + \beta_1\alpha_1^*|\alpha_2|^2|110\rangle\langle 011| - \beta_1\alpha_1^*\alpha_2\beta_2^*|110\rangle\langle 000| + |\beta_1|^2|\alpha_2|^2|110\rangle\langle 110|\\
&-|\beta_1|^2\alpha_2\beta_2^*|110\rangle\langle 101| - \beta_1\alpha_1^*\beta_2\alpha_2^*|101\rangle\langle 011| + \beta_1\alpha_1^*|\beta_2|^2|101\rangle\langle 000| - |\beta_1|^2\beta_2\alpha_2^*|101\rangle\langle 110|\\
&+|\beta_1|^2|\beta_2|^2|101\rangle\langle 101|] + \frac{\lambda}{8}I_8. \qquad (5.70)
\end{aligned}$$

(5)_

$$M = M^+ = |B_{01}\rangle_{1A_1}|B_{00}\rangle_{2A_21A_1}\langle B_{01}|_{2A_2}\langle B_{00}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}
\rho_{m5} &= \frac{{}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{00}|\,\rho\,|B_{01}\rangle_{1A_1}|B_{00}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{00}|\,\rho\,|B_{01}\rangle_{1A_1}|B_{00}\rangle_{2A_2})}\\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|101\rangle\langle 101| + |\alpha_1|^2\alpha_2\beta_2^*|101\rangle\langle 110| + \alpha_1\beta_1^*|\alpha_2|^2|101\rangle\langle 000|\\
&+\alpha_1\beta_1^*\alpha_2\beta_2^*|101\rangle\langle 011| + |\alpha_1|^2|\beta_2|^2|110\rangle\langle 110| + |\alpha_1|^2\beta_2\alpha_2^*|110\rangle\langle 101| + \alpha_1\beta_1^*\beta_2\alpha_2^*|110\rangle\langle 000|\\
&+\alpha_1\beta_1^*|\beta_2|^2|110\rangle\langle 011| + \beta_1\alpha_1^*|\alpha_2|^2|000\rangle\langle 101| + \beta_1\alpha_1^*\alpha_2\beta_2^*|000\rangle\langle 110| + |\beta_1|^2|\alpha_2|^2|000\rangle\langle 000|\\
&+|\beta_1|^2\alpha_2\beta_2^*|000\rangle\langle 011| + \beta_1\alpha_1^*\beta_2\alpha_2^*|011\rangle\langle 101| + \beta_1\alpha_1^*|\beta_2|^2|011\rangle\langle 110| + |\beta_1|^2\beta_2\alpha_2^*|011\rangle\langle 000|\\
&+|\beta_1|^2|\beta_2|^2|011\rangle\langle 011|] + \frac{\lambda}{8}I_8.
\end{aligned}$$

(6)_

$$M = M^+ = |B_{01}\rangle_{1A_1}|B_{01}\rangle_{2A_2 1A_1}\langle B_{01}|_{2A_2}\langle B_{01}|\otimes I_2\otimes I_2\otimes I_2$$

$$\begin{aligned}
\rho_{m6} &= \frac{{}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{01}|\,\rho\,|B_{01}\rangle_{1A_1}|B_{01}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{01}|\,\rho\,|B_{01}\rangle_{1A_1}|B_{01}\rangle_{2A_2})}\\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|110\rangle\langle 110| + |\alpha_1|^2\alpha_2\beta_2^*|110\rangle\langle 101| + \alpha_1\beta_1^*|\alpha_2|^2|110\rangle\langle 011|\\
&+\alpha_1\beta_1^*\alpha_2\beta_2^*|110\rangle\langle 000| + |\alpha_1|^2\beta_2\alpha_2^*|101\rangle\langle 110| + |\alpha_1|^2|\beta_2|^2|101\rangle\langle 101| + \alpha_1\beta_1^*\beta_2\alpha_2^*|101\rangle\langle 000|\\
&+\alpha_1\beta_1^*|\beta_2|^2|101\rangle\langle 000| + \beta_1\alpha_1^*|\alpha_2|^2|011\rangle\langle 110| + \beta_1\alpha_1^*\alpha_2\beta_2^*|011\rangle\langle 101| + |\beta_1|^2|\alpha_2|^2|011\rangle\langle 011|\\
&+|\beta_1|^2\alpha_2\beta_2^*|011\rangle\langle 000| + \beta_1\alpha_1^*\beta_2\alpha_2^*|000\rangle\langle 110| + \beta_1\alpha_1^*|\beta_2|^2|000\rangle\langle 101| + |\beta_1|^2\beta_2\alpha_2^*|000\rangle\langle 011|\\
&+|\beta_1|^2|\beta_2|^2|000\rangle\langle 000|] + \frac{\lambda}{8}I_8.
\end{aligned}$$

(7)_

$$M = M^+ = |B_{01}\rangle_{1A_1}|B_{10}\rangle_{2A_2 1A_1}\langle B_{01}|_{2A_2}\langle B_{10}|\otimes I_2\otimes I_2\otimes I_2$$

$$\begin{aligned}
\rho_{m7} &= \frac{{}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{10}|\,\rho\,|B_{01}\rangle_{1A_1}|B_{10}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{10}|\,\rho\,|B_{01}\rangle_{1A_1}|B_{10}\rangle_{2A_2})}\\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|101\rangle\langle 101| - |\alpha_1|^2\alpha_2\beta_2^*|101\rangle\langle 110| + \alpha_1\beta_1^*|\alpha_2|^2|101\rangle\langle 000|\\
&-\alpha_1\beta_1^*\alpha_2\beta_2^*|101\rangle\langle 011| + |\alpha_1|^2|\beta_2|^2|110\rangle\langle 110| - |\alpha_1|^2\beta_2\alpha_2^*|110\rangle\langle 101| - \alpha_1\beta_1^*\beta_2\alpha_2^*|110\rangle\langle 000|\\
&+\alpha_1\beta_1^*|\beta_2|^2|110\rangle\langle 011| + \beta_1\alpha_1^*|\alpha_2|^2|000\rangle\langle 101| - \beta_1\alpha_1^*\alpha_2\beta_2^*|000\rangle\langle 110| + |\beta_1|^2|\alpha_2|^2|000\rangle\langle 000|\\
&-|\beta_1|^2\alpha_2\beta_2^*|000\rangle\langle 011| - \beta_1\alpha_1^*\beta_2\alpha_2^*|011\rangle\langle 101| + \beta_1\alpha_1^*|\beta_2|^2|011\rangle\langle 110| - |\beta_1|^2\beta_2\alpha_2^*|011\rangle\langle 000|\\
&+|\beta_1|^2|\beta_2|^2|011\rangle\langle 011|] + \frac{\lambda}{8}I_8.
\end{aligned}$$

(8)_

$$M = M^+ = |B_{01}\rangle_{1A_1}|B_{11}\rangle_{2A_2 1A_1}\langle B_{01}|_{2A_2}\langle B_{11}|\otimes I_2\otimes I_2\otimes I_2$$

$$\begin{aligned}\rho_{m8} &= \frac{{}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{11}|\rho|B_{01}\rangle_{1A_1}|B_{11}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{11}|\rho|B_{01}\rangle_{1A_1}|B_{11}\rangle_{2A_2})}\\ &= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|110\rangle\langle 110| - |\alpha_1|^2\alpha_2\beta_2^*|110\rangle\langle 101| + \alpha_1\beta_1^*|\alpha_2|^2|110\rangle\langle 011|\\ &\quad -\alpha_1\beta_1^*\alpha_2\beta_2^*|110\rangle\langle 000| + |\alpha_1|^2|\beta_2|^2|101\rangle\langle 101| - |\alpha_1|^2\beta_2\alpha_2^*|101\rangle\langle 110| - \alpha_1\beta_1^*\beta_2\alpha_2^*|101\rangle\langle 011|\\ &\quad +\alpha_1\beta_1^*|\beta_2|^2|101\rangle\langle 000| + \beta_1\alpha_1^*|\alpha_2|^2|011\rangle\langle 110| - \beta_1\alpha_1^*\alpha_2\beta_2^*|011\rangle\langle 101| + |\beta_1|^2|\alpha_2|^2|011\rangle\langle 011|\\ &\quad -|\beta_1|^2\alpha_2\beta_2^*|011\rangle\langle 000| - \beta_1\alpha_1^*\beta_2\alpha_2^*|000\rangle\langle 110| + \beta_1\alpha_1^*|\beta_2|^2|000\rangle\langle 101| - |\beta_1|^2\beta_2\alpha_2^*|000\rangle\langle 011|\\ &\quad +|\beta_1|^2|\beta_2|^2|000\rangle\langle 000|] + \frac{\lambda}{8}I_8. \qquad (5.71)\end{aligned}$$

(9)_

$$M = M^+ = |B_{10}\rangle_{1A_1}|B_{00}\rangle_{2A_2 1A_1}\langle B_{10}|_{2A_2}\langle B_{00}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}\rho_{m9} &= \frac{{}_{1A_1}\langle B_{10}|_{2A_2}\langle B_{00}|\rho|B_{10}\rangle_{1A_1}|B_{00}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{10}|_{2A_2}\langle B_{00}|\rho|B_{10}\rangle_{1A_1}|B_{00}\rangle_{2A_2})}\\ &\quad (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|000\rangle\langle 000| + |\alpha_1|^2\alpha_2\beta_2^*|000\rangle\langle 011| - \alpha_1\beta_1^*|\alpha_2|^2|000\rangle\langle 101|\\ &\quad -\alpha_1\beta_1^*\alpha_2\beta_2^*|000\rangle\langle 110| + |\alpha_1|^2\beta_2\alpha_2^*|011\rangle\langle 000| + |\alpha_1|^2|\beta_2|^2|011\rangle\langle 011| - \alpha_1\beta_1^*\beta_2\alpha_2^*|011\rangle\langle 101|\\ &\quad -\alpha_1\beta_1^*|\beta_2|^2|011\rangle\langle 110| - \beta_1\alpha_1^*|\alpha_2|^2|101\rangle\langle 000| - \beta_1\alpha_1^*\alpha_2\beta_2^*|101\rangle\langle 011| + |\beta_1|^2|\alpha_2|^2|101\rangle\langle 101|\\ &\quad +|\beta_1|^2\alpha_2\beta_2^*|101\rangle\langle 110| - \beta_1\alpha_1^*\beta_2\alpha_2^*|110\rangle\langle 000| - \beta_1\alpha_1^*|\beta_2|^2|110\rangle\langle 011| + |\beta_1|^2\beta_2\alpha_2^*|110\rangle\langle 101|\\ &\quad +|\beta_1|^2|\beta_2|^2|110\rangle\langle 110|] + \frac{\lambda}{8}I_8.\end{aligned}$$

(10)_

$$M = M^+ = |B_{10}\rangle_{1A_1}|B_{01}\rangle_{2A_2 1A_1}\langle B_{10}|_{2A_2}\langle B_{01}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}\rho_{m10} &= \frac{{}_{1A_1}\langle B_{10}|_{2A_2}\langle B_{01}|\rho|B_{10}\rangle_{1A_1}|B_{01}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{10}|_{2A_2}\langle B_{01}|\rho|B_{10}\rangle_{1A_1}|B_{01}\rangle_{2A_2})}\\ &= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|011\rangle\langle 011| + |\alpha_1|^2\alpha_2\beta_2^*|011\rangle\langle 000| - \alpha_1\beta_1^*|\alpha_2|^2|011\rangle\langle 110|\\ &\quad -\alpha_1\beta_1^*\alpha_2\beta_2^*|011\rangle\langle 101| + |\alpha_1|^2|\beta_2|^2|000\rangle\langle 000| + |\alpha_1|^2\beta_2\alpha_2^*|000\rangle\langle 011| - \alpha_1\beta_1^*\beta_2\alpha_2^*|000\rangle\langle 110|\\ &\quad -\alpha_1\beta_1^*|\beta_2|^2|000\rangle\langle 101| - \beta_1\alpha_1^*|\alpha_2|^2|110\rangle\langle 011| - \beta_1\alpha_1^*\alpha_2\beta_2^*|110\rangle\langle 000| + |\beta_1|^2|\alpha_2|^2|110\rangle\langle 110|\\ &\quad +|\beta_1|^2\alpha_2\beta_2^*|110\rangle\langle 101| - \beta_1\alpha_1^*\beta_2\alpha_2^*|101\rangle\langle 011| - \beta_1\alpha_1^*|\beta_2|^2|101\rangle\langle 000| + |\beta_1|^2\beta_2\alpha_2^*|101\rangle\langle 110|\\ &\quad +|\beta_1|^2|\beta_2|^2|101\rangle\langle 101|] + \frac{\lambda}{8}I_8. \qquad (5.72)\end{aligned}$$

(11)_

$$M = M^+ = |B_{10}\rangle_{1A_1}|B_{10}\rangle_{2A_2 1A_1}\langle B_{10}|_{2A_2}\langle B_{10}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}
\rho_{m11} &= \frac{{}_{1A_1}\langle B_{10}|_{2A_2}\langle B_{10}|\,\rho\,|B_{10}\rangle_{1A_1}|B_{10}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{10}|_{2A_2}\langle B_{10}|\,\rho\,|B_{10}\rangle_{1A_1}|B_{10}\rangle_{2A_2})} \\
& (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|000\rangle\langle 000| - |\alpha_1|^2\alpha_2\beta_2^*|000\rangle\langle 011| - \alpha_1\beta_1^*|\alpha_2|^2|000\rangle\langle 101| \\
& +\alpha_1\beta_1^*\alpha_2\beta_2^*|000\rangle\langle 110| - |\alpha_1|^2\beta_2\alpha_2^*|011\rangle\langle 000| + |\alpha_1|^2|\beta_2|^2|011\rangle\langle 011| + \alpha_1\beta_1^*\beta_2\alpha_2^*|011\rangle\langle 101| \\
& -\alpha_1\beta_1^*|\beta_2|^2|011\rangle\langle 110| - \beta_1\alpha_1^*|\alpha_2|^2|101\rangle\langle 000| + \beta_1\alpha_1^*\alpha_2\beta_2^*|101\rangle\langle 011| + |\beta_1|^2|\alpha_2|^2|101\rangle\langle 101| \\
& -|\beta_1|^2\alpha_2\beta_2^*|101\rangle\langle 110| + \beta_1\alpha_1^*\beta_2\alpha_2^*|110\rangle\langle 000| - \beta_1\alpha_1^*|\beta_2|^2|110\rangle\langle 011| - |\beta_1|^2\beta_2\alpha_2^*|110\rangle\langle 101| \\
& +|\beta_1|^2|\beta_2|^2|110\rangle\langle 110|] + \frac{\lambda}{8}I_8.
\end{aligned}$$

(12)_

$$M = M^+ = |B_{10}\rangle_{1A_1}|B_{11}\rangle_{2A_2 1A_1}\langle B_{10}|_{2A_2}\langle B_{11}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}
\rho_{m12} &= \frac{{}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{11}|\,\rho\,|B_{01}\rangle_{1A_1}|B_{11}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{01}|_{2A_2}\langle B_{11}|\,\rho\,|B_{01}\rangle_{1A_1}|B_{11}\rangle_{2A_2})} \\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|011\rangle\langle 011| - |\alpha_1|^2\alpha_2\beta_2^*|011\rangle\langle 000| - \alpha_1\beta_1^*|\alpha_2|^2|011\rangle\langle 110| \\
& +\alpha_1\beta_1^*\alpha_2\beta_2^*|011\rangle\langle 101| + |\alpha_1|^2|\beta_2|^2|000\rangle\langle 000| - |\alpha_1|^2\beta_2\alpha_2^*|000\rangle\langle 011| + \alpha_1\beta_1^*\beta_2\alpha_2^*|000\rangle\langle 110| \\
& -\alpha_1\beta_1^*|\beta_2|^2|000\rangle\langle 101| - \beta_1\alpha_1^*|\alpha_2|^2|110\rangle\langle 011| + \beta_1\alpha_1^*\alpha_2\beta_2^*|110\rangle\langle 000| + |\beta_1|^2|\alpha_2|^2|110\rangle\langle 110| \\
& -|\beta_1|^2\alpha_2\beta_2^*|110\rangle\langle 101| + \beta_1\alpha_1^*\beta_2\alpha_2^*|101\rangle\langle 011| - \beta_1\alpha_1^*|\beta_2|^2|101\rangle\langle 000| - |\beta_1|^2\beta_2\alpha_2^*|101\rangle\langle 110| \\
& +|\beta_1|^2|\beta_2|^2|101\rangle\langle 101|] + \frac{\lambda}{8}I_8.
\end{aligned} \tag{5.73}$$

(13)_

$$M = M^+ = |B_{11}\rangle_{1A_1}|B_{00}\rangle_{2A_2 1A_1}\langle B_{11}|_{2A_2}\langle B_{00}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}
\rho_{m13} &= \frac{{}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{00}|\,\rho\,|B_{11}\rangle_{1A_1}|B_{00}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{00}|\,\rho\,|B_{11}\rangle_{1A_1}|B_{00}\rangle_{2A_2})} \\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|101\rangle\langle 101| + |\alpha_1|^2\alpha_2\beta_2^*|101\rangle\langle 110| - \alpha_1\beta_1^*|\alpha_2|^2|101\rangle\langle 000| \\
& -\alpha_1\beta_1^*\alpha_2\beta_2^*|101\rangle\langle 011| + |\alpha_1|^2|\beta_2|^2|110\rangle\langle 110| + |\alpha_1|^2\beta_2\alpha_2^*|110\rangle\langle 101| - \alpha_1\beta_1^*\beta_2\alpha_2^*|110\rangle\langle 000| \\
& -\alpha_1\beta_1^*|\beta_2|^2|110\rangle\langle 011| - \beta_1\alpha_1^*|\alpha_2|^2|000\rangle\langle 101| - \beta_1\alpha_1^*\alpha_2\beta_2^*|000\rangle\langle 110| + |\beta_1|^2|\alpha_2|^2|000\rangle\langle 000| \\
& +|\beta_1|^2\alpha_2\beta_2^*|000\rangle\langle 011| - \beta_1\alpha_1^*\beta_2\alpha_2^*|011\rangle\langle 101| - \beta_1\alpha_1^*|\beta_2|^2|011\rangle\langle 110| + |\beta_1|^2\beta_2\alpha_2^*|011\rangle\langle 000| \\
& +|\beta_1|^2|\beta_2|^2|011\rangle\langle 011|] + \frac{\lambda}{8}I_8.
\end{aligned}$$

(14)_

$$M = M^+ = |B_{11}\rangle_{1A_1}|B_{01}\rangle_{2A_2 1A_1}\langle B_{11}|_{2A_2}\langle B_{01}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}
\rho_{m14} &= \frac{{}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{01}|\,\rho\,|B_{11}\rangle_{1A_1}|B_{01}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{01}|\,\rho\,|B_{11}\rangle_{1A_1}|B_{01}\rangle_{2A_2})}\\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|110\rangle\langle 110| + |\alpha_1|^2\alpha_2\beta_2^*|110\rangle\langle 101| - \alpha_1\beta_1^*|\alpha_2|^2|110\rangle\langle 011|\\
&\quad -\alpha_1\beta_1^*\alpha_2\beta_2^*|110\rangle\langle 000| + |\alpha_1|^2|\beta_2|^2|101\rangle\langle 101| + |\alpha_1|^2\beta_2\alpha_2^*|101\rangle\langle 110| - \alpha_1\beta_1^*\beta_2\alpha_2^*|101\rangle\langle 011|\\
&\quad -\alpha_1\beta_1^*|\beta_2|^2|101\rangle\langle 000| - \beta_1\alpha_1^*|\alpha_2|^2|011\rangle\langle 110| - \beta_1\alpha_1^*\alpha_2\beta_2^*|011\rangle\langle 101| + |\beta_1|^2|\alpha_2|^2|011\rangle\langle 011|\\
&\quad +|\beta_1|^2\alpha_2\beta_2^*|011\rangle\langle 000| - \beta_1\alpha_1^*\beta_2\alpha_2^*|000\rangle\langle 110| - \beta_1\alpha_1^*|\beta_2|^2|000\rangle\langle 101| + |\beta_1|^2\beta_2\alpha_2^*|000\rangle\langle 011|\\
&\quad +|\beta_1|^2|\beta_2|^2|000\rangle\langle 000|] + \frac{\lambda}{8}I_8.
\end{aligned}$$

(15)_

$$M = M^+ = |B_{11}\rangle_{1A_1}|B_{10}\rangle_{2A_2 1A_1}\langle B_{11}|_{2A_2}\langle B_{10}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}
\rho_{m15} &= \frac{{}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{10}|\,\rho\,|B_{11}\rangle_{1A_1}|B_{10}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{10}|\,\rho\,|B_{11}\rangle_{1A_1}|B_{10}\rangle_{2A_2})}\\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|101\rangle\langle 101| - |\alpha_1|^2\alpha_2\beta_2^*|101\rangle\langle 110| + \alpha_1\beta_1^*|\alpha_2|^2|101\rangle\langle 000|\\
&\quad -\alpha_1\beta_1^*\alpha_2\beta_2^*|101\rangle\langle 011| + |\alpha_1|^2|\beta_2|^2|110\rangle\langle 110| - |\alpha_1|^2\beta_2\alpha_2^*|110\rangle\langle 101| - \alpha_1\beta_1^*\beta_2\alpha_2^*|110\rangle\langle 000|\\
&\quad +\alpha_1\beta_1^*|\beta_2|^2|110\rangle\langle 011| + \beta_1\alpha_1^*|\alpha_2|^2|000\rangle\langle 101| - \beta_1\alpha_1^*\alpha_2\beta_2^*|000\rangle\langle 110| + |\beta_1|^2|\alpha_2|^2|000\rangle\langle 000|\\
&\quad -|\beta_1|^2\alpha_2\beta_2^*|000\rangle\langle 011| - \beta_1\alpha_1^*\beta_2\alpha_2^*|011\rangle\langle 101| + \beta_1\alpha_1^*|\beta_2|^2|011\rangle\langle 110| - |\beta_1|^2\beta_2\alpha_2^*|011\rangle\langle 000|\\
&\quad +|\beta_1|^2|\beta_2|^2|011\rangle\langle 011|] + \frac{\lambda}{8}I_8. \qquad (5.74)
\end{aligned}$$

(16)_

$$M = M^+ = |B_{11}\rangle_{1A_1}|B_{11}\rangle_{2A_2 1A_1}\langle B_{11}|_{2A_2}\langle B_{11}| \otimes I_2 \otimes I_2 \otimes I_2$$

$$\begin{aligned}
\rho_{m16} &= \frac{{}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{11}|\,\rho\,|B_{11}\rangle_{1A_1}|B_{11}\rangle_{2A_2}}{tr({}_{1A_1}\langle B_{11}|_{2A_2}\langle B_{11}|\,\rho\,|B_{11}\rangle_{1A_1}|B_{11}\rangle_{2A_2})}\\
&= (1-\lambda)[|\alpha_1|^2|\alpha_2|^2|110\rangle\langle 110| - |\alpha_1|^2\alpha_2\beta_2^*|110\rangle\langle 101| - \alpha_1\beta_1^*|\alpha_2|^2|110\rangle\langle 011|\\
&\quad +\alpha_1\beta_1^*\alpha_2\beta_2^*|110\rangle\langle 000| + |\alpha_1|^2|\beta_2|^2|101\rangle\langle 101| - |\alpha_1|^2\beta_2\alpha_2^*|101\rangle\langle 110| - \alpha_1\beta_1^*\beta_2\alpha_2^*|101\rangle\langle 011|\\
&\quad +\alpha_1\beta_1^*|\beta_2|^2|101\rangle\langle 000| + \beta_1\alpha_1^*|\alpha_2|^2|011\rangle\langle 110| - \beta_1\alpha_1^*\alpha_2\beta_2^*|011\rangle\langle 101| + |\beta_1|^2|\alpha_2|^2|011\rangle\langle 011|\\
&\quad -|\beta_1|^2\alpha_2\beta_2^*|011\rangle\langle 000| - \beta_1\alpha_1^*\beta_2\alpha_2^*|000\rangle\langle 110| + \beta_1\alpha_1^*|\beta_2|^2|000\rangle\langle 101| - |\beta_1|^2\beta_2\alpha_2^*|000\rangle\langle 011|\\
&\quad +|\beta_1|^2|\beta_2|^2|000\rangle\langle 000|] + \frac{\lambda}{8}I_8. \qquad (5.75)
\end{aligned}$$

The controllers's operations:

When the teleported state is Alice1's state: An Hadamard operation applied on the first qubit of the controller (C_3) in the matrices ρ_m;

then we perform measurement in the Z basis on qubits (C3,C4).

$$\rho_m \rightarrow \rho_m^{(H_1)} = H\left(C_3\right)\rho_{m1} H^{+}\left(C_3\right) \equiv (H \otimes I_2 \otimes I_2)\rho_m (H \otimes I_2 \otimes I_2) \tag{5.76}$$

$\underline{\boldsymbol{\rho}_{m1}}$:

$$\begin{aligned}
\rho_{1(H_1)} &= H\left(C_3\right)\rho_{m1} H^{+}\left(C_3\right) \\
&= \frac{(1-\lambda)}{2}[\left|\alpha_1\right|^2 \left|\alpha_2\right|^2 (|000\rangle + |100\rangle)(\langle 000| + \langle 100|) + \left|\alpha_1\right|^2 \alpha_2\beta_2^*(|000\rangle + |100\rangle)(\langle 011| + \langle 111|) \\
&\quad +\alpha_1\beta_1^* \left|\alpha_2\right|^2 (|000\rangle + |100\rangle)(\langle 001| - \langle 101|) + \alpha_1\beta_1^*\alpha_2\beta_2^*(|000\rangle + |100\rangle)(\langle 010| - \langle 110|) \\
&\quad + \left|\alpha_1\right|^2 \beta_2\alpha_2^*(|011\rangle + |111\rangle)(\langle 000| + \langle 100|) + \left|\alpha_1\right|^2 \left|\beta_2\right|^2 |011\rangle + |111\rangle)(\langle 011| + \langle 111|) \\
&\quad +\alpha_1\beta_1^*\beta_2\alpha_2^* |011\rangle + |111\rangle)(\langle 001| - \langle 101|) + \alpha_1\beta_1^* \left|\beta_2\right|^2 |011\rangle + |111\rangle)(\langle 010| - \langle 110|) \\
&\quad +\beta_1\alpha_1^* \left|\alpha_2\right|^2 (|001\rangle - |101\rangle)(\langle 000| + \langle 100|) + \beta_1\alpha_1^*\alpha_2\beta_2^*(|001\rangle - |101\rangle)(\langle 011| + \langle 111|) \\
&\quad + \left|\beta_1\right|^2 \left|\alpha_2\right|^2 (|001\rangle - |101\rangle)(\langle 001| - \langle 101|) + \left|\beta_1\right|^2 \alpha_2\beta_2^*(|001\rangle - |101\rangle)(\langle 010| - \langle 110|) \\
&\quad +\beta_1\alpha_1^*\beta_2\alpha_2^*(|010\rangle - |110\rangle)(\langle 000| + \langle 100|) + \beta_1\alpha_1^* \left|\beta_2\right|^2 (|010\rangle - |110\rangle)(\langle 011| + \langle 111|) \\
&\quad + \left|\beta_1\right|^2 \beta_2\alpha_2^*(|010\rangle - |110\rangle)(\langle 001| - \langle 101|) + \left|\beta_1\right|^2 \left|\beta_2\right|^2 (|010\rangle - |110\rangle)(\langle 010| - \langle 110|) \\
&\quad +\frac{\lambda}{32} I_8
\end{aligned} \tag{5.77}$$

Measurement in the Z basis:

$$\begin{aligned}
\tilde{\rho} &= \frac{(M^{ab})\rho_{1(H_1)}(M^{ab})^{+}}{tr((M^{ab})\rho_{1(H_1)}(M^{ab})^{+})}; \quad a = 0,1/b = 0,1 \\
M^{ab} &= (|a\rangle\langle a| \otimes |b\rangle\langle b| \otimes I)
\end{aligned}$$

Depending on the four possibilities of measurement,the results that Bob will receive and his correction operations (U) are:

(1)_ the case where the result of measurement is $|0\rangle_{C_3}\ |0\rangle_{C_4}$ or $(a = 0, b = 0)$:

$$\rho_{1(H_1)}^{00} = \frac{(M^{00})\rho_{1(H_1)}(M^{00})^{+}}{tr((M^{00})\rho_{1(H_1)}(M^{00})^{+})} \tag{5.78}$$

$$\begin{aligned}
(M^{00})\rho_{1(H_1)}(M^{00})^{+} &= (|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes I)\rho_{1(H_1)}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes I) \\
&= \frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^2 |0\rangle\langle 0| + \alpha_1\beta_1^* |0\rangle\langle 1| + \beta_1\alpha_1^* |1\rangle\langle 0| \\
&\quad + \left|\beta_1\right|^2 |1\rangle\langle 1|) + \frac{\lambda}{8} I
\end{aligned} \tag{5.79}$$

$$tr((M^{00})\rho_{1(H_1)}(M^{00})^+) = \frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4} \tag{5.80}$$

$\rightarrow$

$$\begin{aligned}
\rho^{00}_{1(H_1)} &= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right| + \alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| + \beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right| + \left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|) + \frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\alpha_2\right|^2\left|\Phi_1\right\rangle\left\langle\Phi_1\right| + \frac{\lambda}{8}I\right] \\
&\Longrightarrow U^{00}_{1(H_1)} = I
\end{aligned}$$

The fidelity:

$$\begin{aligned}
F &= Tr\left[\left|\Phi_1\right\rangle\left\langle\Phi_1\right|\cdot(U^{00}_{1(H_1)})\rho^{00}_{1(H_1)}(U^{00}_{1(H_1)})^+\right] \qquad (5.81)\\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}Tr\{(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right| + \alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| + \beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right| + \left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|)[(\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right| \\
&\quad + \alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| + \beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right| + \left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|) + \frac{\lambda}{8}I]\} \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}Tr\{\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^4\left|0\right\rangle\left\langle 0\right| + \left|\alpha_1\right|^2\alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| + \left|\alpha_1\right|^2\left|\beta_1\right|^2\left|0\right\rangle\left\langle 0\right| + \left|\beta_1\right|^2\alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| \\
&\quad + \left|\alpha_1\right|^2\beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right| + \left|\alpha_1\right|^2\left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right| + \left|\beta_1\right|^4\left|1\right\rangle\left\langle 1\right|) + \frac{\lambda}{8}(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right| + \alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| \\
&\quad + \beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right| + \left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|)\} \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^4 + \left|\alpha_1\right|^2\left|\beta_1\right|^2 + \left|\alpha_1\right|^2\left|\beta_1\right|^2 + \left|\beta_1\right|^4) + \frac{\lambda}{8}(\left|\alpha_1\right|^2 + \left|\beta_1\right|^2)\right] \\
&\Longrightarrow F = \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{8}\right] \qquad (5.82)
\end{aligned}$$

(2)_ if $(a = 0, b = 1)$

$$\rho^{01}_{1(H_1)} = \frac{(M^{01})\rho_{1(H_1)}(M^{01})^+}{tr((M^{01})\rho_{1(H_1)}(M^{01})^+)} \tag{5.83}$$

$$\begin{aligned}
(M^{01})\rho_{1(H_1)}(M^{01})^+ &= (\left|0\right\rangle\left\langle 0\right| \otimes \left|1\right\rangle\left\langle 1\right| \otimes I)\rho_{1(H_1)}(\left|0\right\rangle\left\langle 0\right| \otimes \left|1\right\rangle\left\langle 1\right| \otimes I) \\
&= \frac{1-\lambda}{2}\left|\beta_2\right|^2(\left|\alpha_1\right|^2\left|1\right\rangle\left\langle 1\right| + \alpha_1\beta_1^*\left|1\right\rangle\left\langle 0\right| + \beta_1\alpha_1^*\left|0\right\rangle\left\langle 1\right| + \left|\beta_1\right|^2\left|0\right\rangle\left\langle 0\right|) + \frac{\lambda}{8}I
\end{aligned}$$

$$tr(M^{01})\rho_{1(H_1)}(M^{01})^+ = \frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{4}$$

$$\begin{aligned}
\rho^{01}_{1(H_1)} &= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\beta_2\right|^2(\left|\alpha_1\right|^2\left|1\right\rangle\left\langle 1\right|+\alpha_1\beta_1^*\left|1\right\rangle\left\langle 0\right|+\beta_1\alpha_1^*\left|0\right\rangle\left\langle 1\right|+\left|\beta_1\right|^2\left|0\right\rangle\left\langle 0\right|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}X\left[\frac{1-\lambda}{2}\left|\beta_2\right|^2\left|\Phi_1\right\rangle\left\langle\Phi_1\right|+\frac{\lambda}{8}I\right]X^+ \\
&\Longrightarrow U^{01}_{1(H_1)}=X
\end{aligned}$$

Fidelity:

$$\begin{aligned}
F &= Tr[\left|\Phi_1\right\rangle\left\langle\Phi_1\right|\cdot\left(U^{01}_{1(H_1)}\right)\rho^{01}_{1(H_1)}\left(U^{01}_{1(H_1)}\right)^+]=Tr[\left|\Phi_1\right\rangle\left\langle\Phi_1\right|\cdot(X)\,\rho^{01}_{1(H_1)}(X)^+] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}Tr\{(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right|+\alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right|+\beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right|+\left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|)(\left|0\right\rangle\left\langle 1\right|+\left|1\right\rangle\left\langle 0\right|) \\
&\quad (\frac{1-\lambda}{2}\left|\beta_2\right|^2(\left|\alpha_1\right|^2\left|1\right\rangle\left\langle 1\right|+\alpha_1\beta_1^*\left|1\right\rangle\left\langle 0\right|+\beta_1\alpha_1^*\left|0\right\rangle\left\langle 1\right|+\left|\beta_1\right|^2\left|0\right\rangle\left\langle 0\right|)+\frac{\lambda}{8}I)(\left|0\right\rangle\left\langle 1\right|+\left|1\right\rangle\left\langle 0\right|)\} \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}Tr\{\frac{1-\lambda}{2}\left|\beta_2\right|^2(\left|\alpha_1\right|^4+\left|\alpha_1\right|^2\left|\beta_1\right|^2+\left|\alpha_1\right|^2\left|\beta_1\right|^2+\left|\beta_1\right|^4)+\frac{\lambda}{8}(\left|\alpha_1\right|^2+\left|\beta_1\right|^2)\} \\
F &= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{8}\right]
\end{aligned}$$

(3)_$(a=1,b=0)$

$$\rho^{10}_{1(H_1)}=\frac{(M^{10})\rho_{1(H_1)}(M^{10})^+}{tr((M^{10})\rho_{1(H_1)}(M^{10})^+)} \tag{5.84}$$

$$\begin{aligned}
(M^{10})\rho_{1(H_1)}(M^{10})^+ &= (\left\langle 1\right|\left\langle 0\right|\otimes I)\rho_{1(H_1)}(\left|1\right\rangle\left|0\right\rangle\otimes I) \\
&= \frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right|-\alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right|-\beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right|+\left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|)+\frac{\lambda}{8}I.
\end{aligned}$$

$$tr((M^{10})\rho_{1(H_1)}(M^{10})^+)=\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{4}$$

$$\begin{aligned}
\rho^{10}_{1(H_1)} &= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right|-\alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right|-\beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right| \\
&\quad +\left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|)+\frac{\lambda}{8}I]=\frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{4}}(Z)\left[\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\Phi_1\right\rangle\left\langle\Phi_1\right|+\frac{\lambda}{8}I\right](Z^+) \\
&\Longrightarrow U^{10}_{1(H_1)}=Z
\end{aligned}$$

$$\begin{aligned}
F &= Tr[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{10}_{1(H_1)}\right)\rho^{10}_{1(H_1)}\left(U^{10}_{1(H_1)}\right)^{+}] = Tr[|\Phi_1\rangle\langle\Phi_1|\cdot(Z)\,\rho^{10}_{1(H_1)}(Z)^{+}] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}Tr\{(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1|+\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)(|0\rangle\langle 0|-|1\rangle\langle 1|) \\
&\quad (\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle 0|-\alpha_1\beta_1^*|0\rangle\langle 1|-\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I)(|0\rangle\langle 0|-|1\rangle\langle 1|)\} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}Tr\{\frac{1-\lambda}{2}(|\alpha_2|^2(|\alpha_1|^4+|\alpha_1|^2|\beta_1|^2+|\alpha_1|^2|\beta_1|^2+|\beta_1|^4)+\frac{\lambda}{8}(|\alpha_1|^2+|\beta_1|^2)\} \\
F &= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]
\end{aligned}$$

(4)_ if $(a=1,b=1)$:

$$\rho^{11}_{1(H_1)} = \frac{(M^{11})\rho_{1(H_1)}(M^{11})^{+}}{tr((M^{11})\rho_{1(H_1)}(M^{11})^{+})} \tag{5.85}$$

$$\begin{aligned}
(M^{11})\rho_{1(H_1)}(M^{11})^{+} &= (\langle 1|\langle 1|\otimes I)\rho_{1(H_1)}(|1\rangle|1\rangle\otimes I) \\
&= \frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0|-\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I
\end{aligned}$$

$$tr((M^{10})\rho_{1(H_1)}(M^{10})^{+}) = \frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}$$

$$\begin{aligned}
\rho^{11}_{1(H_1)} &= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0|-\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I\right] \\
&\Longrightarrow U^{11}_{1(H_1)} = iY
\end{aligned}$$

Fidelity:

$$\begin{aligned}
F &= Tr[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{11}_{1(H_1)}\right)\rho^{11}_{1(H_1)}\left(U^{11}_{1(H_1)}\right)^{+}] = Tr[|\Phi_1\rangle\langle\Phi_1|\cdot(iY)\,\rho^{11}_{1(H_1)}(iY)^{+}] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}Tr\{(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1|+\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)(|0\rangle\langle 1|-|1\rangle\langle 0|) \\
&\quad (\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0|+\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I)(|0\rangle\langle 1|-|1\rangle\langle 0|)\} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}Tr\{\frac{1-\lambda}{2}(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0|-\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}(|\alpha_1|^2+|\beta_1|^2)\} \\
F &= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]
\end{aligned}$$

$\boldsymbol{\rho}_{m2}$:

$$\begin{aligned}
\rho_{2(H_1)} &= H(C_3)\,\rho_{m2} H^{+}(C_3) \\
&= \frac{(1-\lambda)}{2}[|\alpha_1|^2 |\alpha_2|^2 (|011\rangle + |111\rangle)(\langle 011| + \langle 111|) + |\alpha_1|^2 \alpha_2\beta_2^* |(011+111\rangle)(\langle 000| + \langle 100|) \\
&\quad +\alpha_1\beta_1^* |\alpha_2|^2 (|011\rangle + |111\rangle)(\langle 010| - \langle 110|) + \alpha_1\beta_1^*\alpha_2\beta_2^*(|011\rangle + |111\rangle)(\langle 001| - \langle 101|) \\
&\quad + |\alpha_1|^2 |\beta_2|^2 (|000\rangle + |100\rangle)(\langle 000| + \langle 100|) + |\alpha_1|^2 \beta_2\alpha_2^*(|000\rangle + |100\rangle)(\langle 011| + \langle 111|) \\
&\quad +\alpha_1\beta_1^*\beta_2\alpha_2^*(|000\rangle + |100\rangle)(\langle 010| - \langle 110|) + \alpha_1\beta_1^* |\beta_2|^2 (|000\rangle + |100\rangle)(\langle 001| - \langle 101|) \\
&\quad +\beta_1\alpha_1^* |\alpha_2|^2 (|010\rangle - |110\rangle)(\langle 011| + \langle 111|) + \beta_1\alpha_1^*\alpha_2\beta_2^*(|010\rangle - |110\rangle)(\langle 000| + \langle 100|) \\
&\quad + |\beta_1|^2 |\alpha_2|^2 (|010\rangle - |110\rangle)(\langle 010| - \langle 110|) + |\beta_1|^2 \alpha_2\beta_2^*(|010\rangle - |110\rangle)(\langle 001| - \langle 101|) \\
&\quad +\beta_1\alpha_1^*\beta_2\alpha_2^*(|001\rangle - |101\rangle)(\langle 011| + \langle 111|) + \beta_1\alpha_1^* |\beta_2|^2 (|001\rangle - |101\rangle)(\langle 000| + \langle 100|) \\
&\quad + |\beta_1|^2 \beta_2\alpha_2^*(|001\rangle - |101\rangle)(\langle 010| - \langle 110|) + |\beta_1|^2 |\beta_2|^2 (|001\rangle - |101\rangle)(\langle 001| - \langle 101|)] \\
&\quad +\frac{\lambda}{8} I_8. \qquad (5.86)
\end{aligned}$$

Measurement of the qubits C_3 and C_4 in the z-basis:

(1)_$(a = 0, b = 0)$

$$\rho^{00}_{2(H_1)} = \frac{(M^{00})\rho_{2(H_1)}(M^{00})^{+}}{tr\left[(M^{00})\rho_{2(H_1)}(M^{00})^{+}\right]} \qquad (5.87)$$

$$\begin{aligned}
(M^{00})\rho_{2(H_1)}(M^{00})^{+} &= (\langle 0|\langle 0| \otimes I)\rho_{2(H_1)}(|0\rangle|0\rangle \otimes I) \\
&= \frac{1-\lambda}{2} |\beta_2|^2 (|\alpha_1|^2 |0\rangle\langle 0| + \alpha_1\beta_1^* |0\rangle\langle 1| + \beta_1\alpha_1^* |1\rangle\langle 0| + |\beta_1|^2 |1\rangle\langle 1|) + \frac{\lambda}{8} \qquad (5.88)
\end{aligned}$$

$$tr((M^{00})\rho_{2(H_1)}(M^{00})^{+}) = \frac{1-\lambda}{2} |\beta_2|^2 + \frac{\lambda}{4} \qquad (5.89)$$

$$\begin{aligned}
\rho^{00}_{2(H_1)} &= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2} |\beta_2|^2 (|\alpha_1|^2 |0\rangle\langle 0| + \alpha_1\beta_1^* |0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^* |1\rangle\langle 0| + |\beta_1|^2 |1\rangle\langle 1|) + \frac{\lambda}{8} I] \qquad (5.90) \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2} |\beta_2|^2 |\Phi_1\rangle\langle\Phi_1| + \frac{\lambda}{8} I\right] \qquad (5.91) \\
&\Longrightarrow U^{00}_{2(H_1)} = I \qquad (5.92)
\end{aligned}$$

Fidelity:

$$\begin{aligned}
F &= Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U^{00}_{2(H_1)})\rho^{00}_{2(H_1)}(U^{00}_{2(H_1)})^{+}\right] \qquad (5.93)\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}Tr\{(|\alpha_1|^2|0\rangle\langle0|+\alpha_1\beta_1^*|0\rangle\langle1|+\beta_1\alpha_1^*|1\rangle\langle0|+|\beta_1|^2|1\rangle\langle1|)[(\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle0|\\
&\quad+\alpha_1\beta_1^*|0\rangle\langle1|+\beta_1\alpha_1^*|1\rangle\langle0|+|\beta_1|^2|1\rangle\langle1|)+\frac{\lambda}{8}I]\}\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}Tr\{\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^4|0\rangle\langle0|+|\alpha_1|^2\alpha_1\beta_1^*|0\rangle\langle1|+|\alpha_1|^2|\beta_1|^2|0\rangle\langle0|+|\beta_1|^2\alpha_1\beta_1^*|0\rangle\langle1|\\
&\quad+|\alpha_1|^2\beta_1\alpha_1^*|1\rangle\langle0|+|\alpha_1|^2|\beta_1|^2|1\rangle\langle1|+|\beta_1|^4|1\rangle\langle1|)+\frac{\lambda}{8}(|\alpha_1|^2|0\rangle\langle0|+\alpha_1\beta_1^*|0\rangle\langle1|\\
&\quad+\beta_1\alpha_1^*|1\rangle\langle0|+|\beta_1|^2|1\rangle\langle1|)\}\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^4+|\alpha_1|^2|\beta_1|^2+|\alpha_1|^2|\beta_1|^2+|\beta_1|^4)+\frac{\lambda}{8}(|\alpha_1|^2+|\beta_1|^2)\right]\\
\Longrightarrow\ & F=\frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right] \qquad (5.94)
\end{aligned}$$

(2)_$(a=0,b=1)$

$$\begin{aligned}
\rho^{01}_{2(H_1)} &= \frac{(M^{01})\rho_{2(H_1)}(M^{01})^{+}}{tr((M^{01})\rho_{2(H_1)}(M^{01})^{+})}\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle1|+\alpha_1\beta_1^*|1\rangle\langle0|\\
&\quad+\beta_1\alpha_1^*|0\rangle\langle1|+|\beta_1|^2|0\rangle\langle0|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[X[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I]X^{+}\right]\\
&\Longrightarrow\ U^{01}_{2(H_1)}=X
\end{aligned}$$

Fidelity:

$$\begin{aligned}
F &= Tr[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{01}_{2(H_1)}\right)\rho^{01}_{2(H_1)}\left(U^{01}_{2(H_1)}\right)^{+}]=Tr[|\Phi_1\rangle\langle\Phi_1|\cdot(X)\,\rho^{01}_{2(H_1)}(X)^{+}]\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}Tr\{(|\alpha_1|^2|0\rangle\langle0|+\alpha_1\beta_1^*|0\rangle\langle1|+\beta_1\alpha_1^*|1\rangle\langle0|+|\beta_1|^2|1\rangle\langle1|)(|0\rangle\langle1|+|1\rangle\langle0|)\\
&\quad(\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle1|+\alpha_1\beta_1^*|1\rangle\langle0|+\beta_1\alpha_1^*|0\rangle\langle1|+|\beta_1|^2|0\rangle\langle0|)+\frac{\lambda}{8}I)(|0\rangle\langle1|+|1\rangle\langle0|)\}\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}Tr\{\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^4+|\alpha_1|^2|\beta_1|^2+|\alpha_1|^2|\beta_1|^2+|\beta_1|^4)+\frac{\lambda}{8}(|\alpha_1|^2+|\beta_1|^2)\}\\
F &= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]
\end{aligned}$$

(3)_$(a = 1, b = 0)$

$$\begin{aligned}
\rho^{10}_{2(H_1)} &= \frac{(M^{10})\rho_{2(H_1)}(M^{10})^{+}}{tr((M^{10})\rho_{2(H_1)}(M^{10})^{+})} \\
&= \frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{4}[\frac{1-\lambda}{2}\left|\beta_2\right|^2(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right| - \alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| \\
&\quad -\beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right| + \left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|) + \frac{\lambda}{8}I] \qquad (5.95)\\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{4}}\left[Z(\frac{1-\lambda}{2}\left|\beta_2\right|^2\left|\Phi_1\right\rangle\left\langle\Phi_1\right| + \frac{\lambda}{8}I)Z^{+}\right] \\
&\Longrightarrow U_2^{10} = Z \qquad (5.96)
\end{aligned}$$

Fidelity:

$$\begin{aligned}
F &= Tr[\left|\Phi_1\right\rangle\left\langle\Phi_1\right| \cdot \left(U^{10}_{2(H_1)}\right)\rho^{10}_{2(H_1)}\left(U^{10}_{2(H_1)}\right)^{+}] = Tr[\left|\Phi_1\right\rangle\left\langle\Phi_1\right| \cdot (Z)\,\rho^{10}_{2(H_1)}(Z)^{+}] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{4}} Tr\{(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right| + \alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| + \beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right| + \left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|)(\left|0\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|) \\
&\quad (\frac{1-\lambda}{2}\left|\beta_2\right|^2(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right| - \alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| - \beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right| + \left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|) + \frac{\lambda}{8}I)(\left|0\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|)\} \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{4}} Tr\{\frac{1-\lambda}{2}(\left|\beta_2\right|^2(\left|\alpha_1\right|^4 + \left|\alpha_1\right|^2\left|\beta_1\right|^2 + \left|\alpha_1\right|^2\left|\beta_1\right|^2 + \left|\beta_1\right|^4) + \frac{\lambda}{8}(\left|\alpha_1\right|^2 + \left|\beta_1\right|^2)\} \\
F &= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{8}\right]
\end{aligned}$$

(4)_$(a = 1, b = 1)$

$$\begin{aligned}
\rho^{11}_{2(H_1)} &= \frac{(M^{11})\rho_{2(H_1)}(M^{11})^{+}}{tr((M^{11})\rho_{2(H_1)}(M^{11})^{+})} \\
&= (\frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}})[\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^2\left|1\right\rangle\left\langle 1\right| - \alpha_1\beta_1^*\left|1\right\rangle\left\langle 0\right| \\
&\quad -\beta_1\alpha_1^*\left|0\right\rangle\left\langle 1\right| + \left|\beta_1\right|^2\left|0\right\rangle\left\langle 0\right|) + \frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}\left[(iY)(\frac{1-\lambda}{2}\left|\alpha_2\right|^2\left|\Phi_1\right\rangle\left\langle\Phi_1\right| + \frac{\lambda}{8}I)(iY)^{+}\right] \\
&\Longrightarrow U^{11}_{2(H_1)} = iY
\end{aligned}$$

$$
\begin{aligned}
F &= Tr[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{11}_{2(H_1)}\right)\rho^{11}_{2(H_1)}\left(U^{11}_{2(H_1)}\right)^{+}] = Tr[|\Phi_1\rangle\langle\Phi_1|\cdot(iY)\,\rho^{11}_{2(H_1)}(iY)^{+}] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}} Tr\{(|\alpha_1|^2|0\rangle\langle0|+\alpha_1\beta_1^*|0\rangle\langle1|+\beta_1\alpha_1^*|1\rangle\langle0|+|\beta_1|^2|1\rangle\langle1|)(|0\rangle\langle1|-|1\rangle\langle0|) \\
&\quad (\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle1|+\alpha_1\beta_1^*|1\rangle\langle0|+\beta_1\alpha_1^*|0\rangle\langle1|+|\beta_1|^2|0\rangle\langle0|)+\frac{\lambda}{8}I)(|0\rangle\langle1|-|1\rangle\langle0|)\} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}} Tr\{\frac{1-\lambda}{2}(|\alpha_1|^2|1\rangle\langle1|-\alpha_1\beta_1^*|1\rangle\langle0|-\beta_1\alpha_1^*|0\rangle\langle1|+|\beta_1|^2|0\rangle\langle0|)+\frac{\lambda}{8}(|\alpha_1|^2+|\beta_1|^2)\} \\
F &= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]
\end{aligned}
$$

$\underline{\boldsymbol{\rho}_{m3}}$:

$$
\begin{aligned}
\rho_{3(H_1)} &= H(C_3)\,\rho_{m3}H^{+}(C_3) \\
&= \frac{(1-\lambda)}{2}[|\alpha_1|^2|\alpha_2|^2(|000\rangle+|100\rangle)(\langle000|+\langle100|)-|\alpha_1|^2\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle011|+\langle111|) \\
&\quad +\alpha_1\beta_1^*|\alpha_2|^2(|000\rangle+|100\rangle)(\langle001|-\langle101|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle010|-\langle110|) \\
&\quad -|\alpha_1|^2\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle000|+\langle100|)+|\alpha_1|^2|\beta_2|^2(|011\rangle+|111\rangle)(\langle011|+\langle111|) \\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle001|-\langle101|)+\alpha_1\beta_1^*|\beta_2|^2(|011\rangle+|111\rangle)(\langle010|-\langle110|) \\
&\quad +\beta_1\alpha_1^*|\alpha_2|^2(|001\rangle-|101\rangle)(\langle000|+\langle100|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle011|+\langle111|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|001\rangle-|101\rangle)(\langle001|-\langle101|)-|\beta_1|^2\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle010|-\langle110|) \\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle000|+\langle100|)+\beta_1\alpha_1^*|\beta_2|^2(|010\rangle-|110\rangle)(\langle011|+\langle111|) \\
&\quad -|\beta_1|^2\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle001|-\langle101|)+|\beta_1|^2|\beta_2|^2(|010\rangle-|110\rangle)(\langle010|-\langle110|)] \\
&\quad +\frac{\lambda}{8}I_8. \qquad (5.97)
\end{aligned}
$$

(1)_When Charlie's result of measurement is $|0\rangle_{C_3}|0\rangle_{C_4}$:

$$
\begin{aligned}
\rho^{00}_{3(H_1)} &= \frac{(M^{00})\rho_{3(H_1)}(M^{00})^{+}}{tr((M^{00})\rho_{3(H_1)}(M^{00})^{+})} \qquad (5.98) \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle0|+\alpha_1\beta_1^*|0\rangle\langle1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle0|+|\beta_1|^2|1\rangle\langle1|)+\frac{\lambda}{8}I] \qquad (5.99) \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[(\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)] \\
&\Longrightarrow U^{00}_{3}=I \qquad (5.100)
\end{aligned}
$$

$$
\begin{aligned}
F &= Tr[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{00}_{3(H_1)}\right)\rho^{00}_{3(H_1)}\left(U^{00}_{3(H_1)}\right)^+] = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot\rho^{00}_{3(H_1)}\right] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right] \qquad (5.101)
\end{aligned}
$$

(2)_if the result is $|0\rangle_{C_3}|1\rangle_{C_4}$:

$$
\begin{aligned}
\rho^{01}_{3(H_1)} &= \frac{(M^{01})\rho_{3(H_1)}(M^{01})^+}{tr((M^{01})\rho_{3(H_1)}(M^{01})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle1|+\alpha_1\beta_1^*|1\rangle\langle0| \\
&\quad +\beta_1\alpha_1^*|0\rangle\langle1|+|\beta_1|^2|0\rangle\langle0|)+\frac{\lambda}{8}I] \qquad (5.102) \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[X(\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)X^+\right] \\
&\Longrightarrow U^{01}_3 = X \qquad (5.103)
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
F &= Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{01}_{3(H_1)}\right)\rho^{01}_{3(H_1)}\left(U^{01}_{3(H_1)}\right)^+\right] = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(X)\rho^{01}_{3(H_1)}(X)^+\right] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right] \qquad (5.104)
\end{aligned}
$$

(3)_if the result is $|1\rangle_{C_3}|0\rangle_{C_4}$:

$$
\begin{aligned}
\rho^{10}_{3(H_1)} &= \frac{(M^{10})\rho_{3(H_1)}(M^{10})^+}{tr((M^{10})\rho_{3(H_1)}(M^{10})^+)} \qquad (5.105) \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle0|-\alpha_1\beta_1^*|0\rangle\langle1| \\
&\quad -\beta_1\alpha_1^*|1\rangle\langle0|+|\beta_1|^2|1\rangle\langle1|)+\frac{\lambda}{8}I] \qquad (5.106) \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[Z(\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)Z^+\right] \qquad (5.107) \\
&\Longrightarrow U^{10}_3 = Z \qquad (5.108)
\end{aligned}
$$

$$
\begin{aligned}
F &= Tr[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{10}_{3(H_1)}\right)\rho^{10}_{3(H_1)}\left(U^{10}_{3(H_1)}\right)^+] = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(Z)\rho^{10}_{3(H_1)}(Z)^+\right] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right] \qquad (5.109)
\end{aligned}
$$

(4)_if the result is $|1\rangle_{C_3}|1\rangle_{C_4}$:

$$\rho^{11}_{3(H_1)} = \frac{(M^{11})\rho_{3(H_1)}(M^{11})^+}{tr((M^{11})\rho_{3(H_1)}(M^{11})^+)} \tag{5.110}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0| -\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.111}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[(iY)(\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)(iY^+)\right] \tag{5.112}$$

$$\Longrightarrow U^{11}_3 = iY \tag{5.113}$$

$$F = Tr[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{11}_{3(H_1)}\right)\rho^{11}_{3(H_1)}\left(U^{11}_{3(H_1)}\right)^+] = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(iY)\rho^{11}_{3(H_1)}(iY)^+\right]$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right] \tag{5.114}$$

$\underline{\boldsymbol{\rho}_{m4}}$:

$$\rho_{4(H_1)} = H(C_3)\rho_{m4}H^+(C_3)$$

$$\begin{aligned} =& \frac{(1-\lambda)}{2}[|\alpha_1|^2|\alpha_2|^2(|011\rangle+|111\rangle)(\langle 011|+\langle 111|)-|\alpha_1|^2\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 000|+\langle 100|) \\ &+\alpha_1\beta_1^*|\alpha_2|^2(|011\rangle+|011\rangle)(\langle 110|-\langle 110|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 001|-\langle 101|) \\ &+|\alpha_1|^2|\beta_2|^2(|000\rangle+|100\rangle)(\langle 000|+\langle 100|)-|\alpha_1|^2\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 011|+\langle 111|) \\ &-\alpha_1\beta_1^*\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 010|-\langle 110|)+\alpha_1\beta_1^*|\beta_2|^2(|000\rangle+|100\rangle)(\langle 001|-\langle 101|) \\ &+\beta_1\alpha_1^*|\alpha_2|^2(|010\rangle-|110\rangle)(\langle 011|+\langle 111|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 000|+\langle 100|) \\ &+|\beta_1|^2|\alpha_2|^2(|010\rangle-|110\rangle)(\langle 010|-\langle 110|)-|\beta_1|^2\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 001|-\langle 101|) \\ &-\beta_1\alpha_1^*\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 011|+\langle 111|)+\beta_1\alpha_1^*|\beta_2|^2(|001\rangle-|101\rangle)(\langle 000|+\langle 100|) \\ &-|\beta_1|^2\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 010|-\langle 110|)+|\beta_1|^2|\beta_2|^2(|001\rangle-|101\rangle)(\langle 001|-\langle 101|)] \\ &+\frac{\lambda}{8}I_8. \end{aligned} \tag{5.115}$$

(1)_if the result of measurement is $|0\rangle_{C_3}|0\rangle_{C_4}$:

$$\begin{aligned}
\rho^{00}_{4(H_1)} &= \frac{(M^{00})\rho_{4(H_1)}(M^{00})^+}{tr((M^{00})\rho_{4(H_1)}(M^{00})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I] \\
&\Longrightarrow U_4^{00}=I
\end{aligned}$$

$$\begin{aligned}
F &= Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{00}_{4(H_1)}\right)\rho^{00}_{4(H_1)}\left(U^{00}_{4(H_1)}\right)^+\right] = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(I)\rho^{00}_{4(H_1)}(I)\right] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right] \qquad (5.116)
\end{aligned}$$

(2)_for the result $|0\rangle_{C_3}|1\rangle_{C_4}$ or $(a=0,b=1)$:

$$\begin{aligned}
\rho^{01}_{4(H_1)} &= \frac{(M^{01})\rho_{4(H_1)}(M^{01})^+}{tr((M^{01})\rho_{4(H_1)}(M^{01})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[X(\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)X^+] \\
&\Longrightarrow U_4^{01}=X
\end{aligned}$$

$$\begin{aligned}
F &= Tr[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{01}_{4(H_1)}\right)\rho^{01}_{4(H_1)}\left(U^{01}_{4(H_1)}\right)^+] = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(X)\rho^{01}_{4(H_1)}(X)^+\right] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right] \qquad (5.117)
\end{aligned}$$

(3)_ $(a=1,b=0)$

$$\begin{aligned}
\rho^{10}_{4(H_1)} &= \frac{(M^{10})\rho_{4(H_1)}(M^{10})^{+}}{tr((M^{10})\rho_{4(H_1)}(M^{10})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\beta_2\right|^2(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right|-\alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| \\
&\quad -\beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right|+\left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|)+\frac{\lambda}{8}I] && (5.118)\\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}\left|\beta_2\right|^2\left|\Phi_1\right\rangle\left\langle\Phi_1\right|+\frac{\lambda}{8}I)Z^{+}] && (5.119)\\
&\Longrightarrow U_4^{10}=Z && (5.120)
\end{aligned}$$

$$\begin{aligned}
F &= Tr[\left|\Phi_1\right\rangle\left\langle\Phi_1\right|\cdot\left(U^{10}_{4(H_1)}\right)\rho^{10}_{4(H_1)}\left(U^{10}_{4(H_1)}\right)^{+}] = Tr\left[\left|\Phi_1\right\rangle\left\langle\Phi_1\right|\cdot(Z)\rho^{10}_{4(H_1)}(Z)^{+}\right] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{8}\right] && (5.121)
\end{aligned}$$

(4)_ $(a=1,b=1)$:

$$\begin{aligned}
\rho^{11}_{4(H_1)} &= \frac{(M^{11})\rho_{4(H_1)}(M^{11})^{+}}{tr((M^{11})\rho_{4(H_1)}(M^{11})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^2\left|1\right\rangle\left\langle 1\right|-\alpha_1\beta_1^*\left|1\right\rangle\left\langle 0\right| \\
&\quad -\beta_1\alpha_1^*\left|0\right\rangle\left\langle 1\right|+\left|\beta_1\right|^2\left|0\right\rangle\left\langle 0\right|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{4}}\left[(iY)(\frac{1-\lambda}{2}\left|\alpha_2\right|^2\left|\Phi_1\right\rangle\left\langle\Phi_1\right|+\frac{\lambda}{8}I)(iY)^{+}\right] \\
&\Longrightarrow U_4^{11}=iY
\end{aligned}$$

$$\begin{aligned}
F &= Tr\left[\left|\Phi_1\right\rangle\left\langle\Phi_1\right|\cdot\left(U^{11}_{4(H_1)}\right)\rho^{11}_{4(H_1)}\left(U^{11}_{4(H_1)}\right)^{+}\right] = Tr\left[\left|\Phi_1\right\rangle\left\langle\Phi_1\right|\cdot(iY)\rho^{11}_{4(H_1)}(iY)^{+}\right] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{8}\right] && (5.122)
\end{aligned}$$

$\underline{\boldsymbol{\rho}_{m_5}:}$

$$\begin{aligned}
\rho_{5(H_1)} &= H(C_3)\,\rho_{m5} H^{+}(C_3) \\
&= \frac{(1-\lambda)}{2}[|\alpha_1|^2|\alpha_2|^2(|001\rangle - |101\rangle)(\langle 001| - \langle 101|) + |\alpha_1|^2\alpha_2\beta_2^*(|001\rangle - |101\rangle)(\langle 010| - \langle 110|) \\
&\quad +\alpha_1\beta_1^*|\alpha_2|^2(|001\rangle - |101\rangle)(\langle 000| + \langle 100|) + \alpha_1\beta_1^*\alpha_2\beta_2^*(|001\rangle - |101\rangle)(\langle 011| + \langle 111|) \\
&\quad +|\alpha_1|^2|\beta_2|^2(|010\rangle - |110\rangle)(\langle 010| - \langle 110|) + |\alpha_1|^2\beta_2\alpha_2^*(|010\rangle - |110\rangle)(\langle 001| - \langle 101|) \\
&\quad +\alpha_1\beta_1^*\beta_2\alpha_2^*(|010\rangle - |110\rangle)(\langle 000| + \langle 100|) + \alpha_1\beta_1^*|\beta_2|^2(|010\rangle - |110\rangle)(\langle 011| + \langle 111|) \\
&\quad +\beta_1\alpha_1^*|\alpha_2|^2(|000\rangle + |100\rangle)(\langle 001| - \langle 101|) + \beta_1\alpha_1^*\alpha_2\beta_2^*(|000\rangle + |100\rangle)(\langle 010| - \langle 110|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|000\rangle + |100\rangle)(\langle 000| + \langle 100|) + |\beta_1|^2\alpha_2\beta_2^*(|000\rangle + |100\rangle)(\langle 011| + \langle 111|) \\
&\quad +\beta_1\alpha_1^*\beta_2\alpha_2^*(|011\rangle + |111\rangle)(\langle 001| - \langle 101|) + \beta_1\alpha_1^*|\beta_2|^2(|011\rangle + |111\rangle)(\langle 010| - \langle 110|) \\
&\quad +|\beta_1|^2\beta_2\alpha_2^*(|011\rangle + |111\rangle)(\langle 000| + \langle 100|) + |\beta_1|^2|\beta_2|^2(|011\rangle + |111\rangle)(\langle 011| + \langle 111|)] \\
&\quad +\frac{\lambda}{8}I_8. \qquad (5.123)
\end{aligned}$$

(1)_When Charlie's result of measurement is $|0\rangle_{C_3}|0\rangle_{C_4}$:

$$\begin{aligned}
\rho_{5(H_1)}^{00} &= \frac{(M^{00})\rho_{5(H_1)}(M^{00})^{+}}{tr((M^{00})\rho_{5(H_1)}(M^{00})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1| + \alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1| + |\beta_1|^2|0\rangle\langle 0|) + \frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2 + \frac{\lambda}{4}}[(X)(\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1| + \frac{\lambda}{8}I)(X)^{+}] \\
&\Longrightarrow U_5^{00} = X
\end{aligned}$$

$$\begin{aligned}
F &= Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U_{5(H_1)}^{00}\right)\rho_{5(H_1)}^{00}\left(U_{5(H_1)}^{00}\right)^{+}\right] = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(X)\rho_{5(H_1)}^{00}(X)^{+}\right] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2 + \frac{\lambda}{8}\right] \qquad (5.124)
\end{aligned}$$

(2)_When Charlie's result of measurement is $|0\rangle_{C_3}|1\rangle_{C_4}$:

$$\begin{aligned}
\rho^{01}_{5(H_1)} &= \frac{(M^{01})\rho_{5(H_1)}(M^{01})^+}{tr((M^{01})\rho_{5(H_1)}(M^{01})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[(\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)\right] \\
&\Longrightarrow U^{01}_5 = I
\end{aligned}$$

$$\begin{aligned}
F &= Tr[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{01}_{5(H_1)}\right)\rho^{01}_{5(H_1)}\left(U^{01}_{5(H_1)}\right)^+] = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot\rho^{01}_{5(H_1)}\right] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]
\end{aligned} \tag{5.125}$$

(3)_When Charlie's result of measurement is $|1\rangle_{C_3}|0\rangle_{C_4}$:

$$\begin{aligned}
\rho^{10}_{5(H_1)} &= \frac{(M^{10})\rho_{5(H_1)}(M^{10})^+}{tr((M^{10})\rho_{5(H_1)}(M^{10})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad -\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[(iY)(\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)(iY)^+\right] \\
&\Longrightarrow U^{10}_5 = iY
\end{aligned}$$

$$\begin{aligned}
F &= Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{10}_{5(H_1)}\right)\rho^{10}_{5(H_1)}\left(U^{10}_{5(H_1)}\right)^+\right] = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(iY)\rho^{10}_{5(H_1)}(iY)^+\right] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]
\end{aligned} \tag{5.126}$$

(4)_When Charlie's result of measurement is $|1\rangle_{C_3}|1\rangle_{C_4}$:

$$\begin{aligned}
\rho^{11}_{5(H_1)} &= \frac{(M^{11})\rho_{5(H_1)}(M^{11})^{+}}{tr((M^{11})\rho_{5(H_1)}(M^{11})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0|-\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad -\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \qquad (5.127)\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[(Z)(\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)(Z)^{+}\right] \qquad (5.128)\\
&\Longrightarrow U^{11}_{5}=Z \qquad (5.129)
\end{aligned}$$

$$\begin{aligned}
F &= Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot\left(U^{11}_{5(H_1)}\right)\rho^{11}_{5(H_1)}\left(U^{11}_{5(H_1)}\right)^{+}\right]=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(Z)\rho^{01}_{5(H_1)}(Z)^{+}\right]\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right] \qquad (5.130)
\end{aligned}$$

$\underline{\boldsymbol{\rho}_{m6}}$:

$$\begin{aligned}
\rho_{6(H_1)} &= H(C_3)\rho_{m6}H^{+}(C_3)\\
&= \frac{(1-\lambda)}{2}[|\alpha_1|^2|\alpha_2|^2(|010\rangle-|110\rangle)(\langle 010|-\langle 110|)+|\alpha_1|^2\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 001|-\langle 101|)\\
&\quad +\alpha_1\beta_1^*|\alpha_2|^2(|010\rangle-|110\rangle)(\langle 011|+\langle 111|)+\alpha_1\beta_1^*\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 000|+\langle 100|)\\
&\quad +|\alpha_1|^2\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 010|-\langle 110|)+|\alpha_1|^2|\beta_2|^2(|001\rangle-|101\rangle)(\langle 001|-\langle 101|)\\
&\quad +\alpha_1\beta_1^*\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 011|+\langle 111|)+\alpha_1\beta_1^*|\beta_2|^2(|001\rangle-|101\rangle)(\langle 000|+\langle 100|)\\
&\quad +\beta_1\alpha_1^*|\alpha_2|^2(|011\rangle+|111\rangle)(\langle 010|-\langle 110|)+\beta_1\alpha_1^*\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 001|-\langle 101|)\\
&\quad +|\beta_1|^2|\alpha_2|^2(|011\rangle+|111\rangle)(\langle 011|+\langle 111|)+|\beta_1|^2\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 000|+\langle 100|)\\
&\quad +\beta_1\alpha_1^*\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 010|-\langle 110|)+\beta_1\alpha_1^*|\beta_2|^2(|000\rangle+|100\rangle)(\langle 001|-\langle 101|)\\
&\quad +|\beta_1|^2\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 011|+\langle 111|)+|\beta_1|^2|\beta_2|^2(|000\rangle+|100\rangle)(\langle 000|+\langle 100|)]\\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_if Charlie's result of measurement is $|0\rangle_{C_3}|0\rangle_{C_4}$:

$$\begin{aligned}
\rho^{00}_{6(H_1)} &= \frac{(M^{00})\rho_{6(H_1)}(M^{00})^{+}}{tr((M^{00})\rho_{6(H_1)}(M^{00})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[(X)[\frac{1-\lambda}{2}|\beta_2|^2(|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](X^{+})\right] \\
&\Longrightarrow U^{00}_{6(H_1)} = X
\end{aligned}$$

So the fidelity:

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(X)\rho^{00}_{6(H_1)}(X^{+})\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

(2)_if Charlie's result of measurement is $|0\rangle_{C_3}|1\rangle_{C_4}$:

$$\begin{aligned}
\rho^{01}_{6(H_1)} &= \frac{(M^{01})\rho_{6(H_1)}(M^{01})^{+}}{tr((M^{01})\rho_{6(H_1)}(M^{01})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[(X)[\frac{1-\lambda}{2}|\alpha_2|^2(|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](X^{+})\right] \\
&\Longrightarrow U^{01}_{6(H_1)} = I
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot\rho^{01}_{6(H_1)}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right] \tag{5.131}$$

(3)_if Charlie's result of measurement is $|1\rangle_{C_3}|0\rangle_{C_4}$:

$$\begin{aligned}
\rho^{10}_{6(H_1)} &= \frac{(M^{10})\rho_{6(H_1)}(M^{10})^{+}}{tr((M^{10})\rho_{6(H_1)}(M^{10})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad -\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[(iY)[\frac{1-\lambda}{2}|\beta_2|^2(|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](iY^{+})\right] \\
&\Longrightarrow U^{10}_{6(H_1)} = iY
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(iY)\rho^{10}_{6(H_1)}(iY)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

(4)_if Charlie's result of measurement is $|1\rangle_{C_3}|1\rangle_{C_4}$:

$$\begin{aligned}
\rho^{11}_{6(H_1)} &= \frac{(M^{11})\rho_{6(H_1)}(M^{11})^{+}}{tr((M^{11})\rho_{6(H_1)}(M^{11})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle 0|-\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad -\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \qquad (5.132)\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[(Z)(\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)(Z^{+})\right] \qquad (5.133)\\
&\Longrightarrow U^{11}_{6(H_1)}=Z \qquad (5.134)
\end{aligned}$$

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(Z)\rho^{11}_{6(H_1)}(Z^{+})\right]=\frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

$\boldsymbol{\rho}_{m7}$:

$$\begin{aligned}
\rho_{7(H_1)} &= H(C_3)\rho_{m7}H^{+}(C_3)\\
&= \frac{(1-\lambda)}{2}[|\alpha_1|^2|\alpha_2|^2(|001\rangle-|101\rangle)(\langle 001|-\langle 101|)-|\alpha_1|^2\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle 010|-\langle 110|)\\
&+\alpha_1\beta_1^*|\alpha_2|^2(|001\rangle-|101\rangle)(\langle 000|+\langle 100|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle 011|+\langle 111|)\\
&+|\alpha_1|^2|\beta_2|^2(|010\rangle-|110\rangle)(\langle 010|-\langle 110|)-|\alpha_1|^2\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle 001|-\langle 101|)\\
&-\alpha_1\beta_1^*\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle 000|+\langle 100|)+\alpha_1\beta_1^*|\beta_2|^2(|010\rangle-|110\rangle)(\langle 011|+\langle 111|)\\
&+\beta_1\alpha_1^*|\alpha_2|^2(|000\rangle+|100\rangle)(\langle 001|-\langle 101|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle 010|-\langle 110|)\\
&+|\beta_1|^2|\alpha_2|^2(|000\rangle+|100\rangle)(\langle 000|+\langle 100|)-|\beta_1|^2\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle 011|+\langle 111|)\\
&-\beta_1\alpha_1^*\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle 001|-\langle 101|)+\beta_1\alpha_1^*|\beta_2|^2(|011\rangle+|111\rangle)(\langle 010|-\langle 110|)\\
&-|\beta_1|^2\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle 000|+\langle 100|)+|\beta_1|^2|\beta_2|^2(|011\rangle+|111\rangle)(\langle 011|+\langle 111|)]\\
&+\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_if the result of measurement is $|0\rangle_{C_3}|0\rangle_{C_4}$ or $(a=0,b=0)$:

$$\begin{aligned}
\rho^{00}_{7(H_1)} &= \frac{(M^{00})\rho_{7(H_1)}(M^{00})^{+}}{tr((M^{00})\rho_{7(H_1)}(M^{00})^{+})}\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0|\\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I]\\
&\Longrightarrow U^{00}_{7(H_1)}=X
\end{aligned}$$

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(Z)\rho^{11}_{6(H_1)}(Z^{+})\right]=\frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

(2)_$(a = 0, b = 1)$

$$\begin{aligned}
\rho^{01}_{7(H_1)} &= \frac{(M^{01})\rho_{7(H_1)}(M^{01})^+}{tr((M^{01})\rho_{7(H_1)}(M^{01})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0| + \alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0| + |\beta_1|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1| + \frac{\lambda}{8}I] \\
&\Longrightarrow U^{01}_{7(H_1)} = I
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot\rho^{01}_{7(H_1)}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{8}\right]$$

(3)_ $(a = 1, b = 0)$

$$\begin{aligned}
\rho^{10}_{7(H_1)} &= \frac{(M^{10})\rho_{7(H_1)}(M^{10})^+}{tr((M^{10})\rho_{7(H_1)}(M^{10})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1| - \alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad -\beta_1\alpha_1^*|0\rangle\langle 1| + |\beta_1|^2|0\rangle\langle 0|) + \frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2 + \frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1| + \frac{\lambda}{8}I)(iY)^+] \\
&\Longrightarrow U^{10}_{7(H_1)} = iY
\end{aligned}$$

(4)_$(a = 1, b = 1)$

$$\begin{aligned}
\rho^{11}_{7(H_1)} &= \frac{(M^{11})\rho_{7(H_1)}(M^{11})^+}{tr((M^{11})\rho_{7(H_1)}(M^{11})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0| - \alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad -\beta_1\alpha_1^*|1\rangle\langle 0| + |\beta_1|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \qquad (5.135) \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}[(Z)(\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1| + \frac{\lambda}{8}I)(Z)^+] \\
&\Longrightarrow U^{11}_{7(H_1)} = Z \qquad (5.136)
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(Z)\rho^{11}_{7(H_1)}(Z)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{8}\right] \qquad (5.137)$$

$\boldsymbol{\rho}_{m8:}$

$$
\begin{aligned}
\rho_{8(H_1)} &= H\left(C_3\right)\rho_{m8}H^{+}\left(C_3\right) \\
&= \frac{(1-\lambda)}{2}(\left|\alpha_1\right|^2\left|\alpha_2\right|^2(|010\rangle-|110\rangle)(\langle 010|-\langle 110|)-\left|\alpha_1\right|^2\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 001|-\langle 101|) \\
&\quad +\alpha_1\beta_1^*\left|\alpha_2\right|^2(|010\rangle-|110\rangle)(\langle 011|+\langle 111|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 000|+\langle 100|) \\
&\quad +\left|\alpha_1\right|^2\left|\beta_2\right|^2(|001\rangle-|101\rangle)(\langle 001|-\langle 101|)-\left|\alpha_1\right|^2\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 010|-\langle 110|) \\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 011|+\langle 111|)+\alpha_1\beta_1^*\left|\beta_2\right|^2(|001\rangle-|101\rangle)(\langle 000|+\langle 100|) \\
&\quad +\beta_1\alpha_1^*\left|\alpha_2\right|^2(|011\rangle+|111\rangle)(\langle 010|-\langle 110|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 001|-\langle 101|) \\
&\quad +\left|\beta_1\right|^2\left|\alpha_2\right|^2(|011\rangle+|111\rangle)(\langle 011|+\langle 111|)-\left|\beta_1\right|^2\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 000|+\langle 100|) \\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 010|-\langle 110|)+\beta_1\alpha_1^*\left|\beta_2\right|^2(|000\rangle+|100\rangle)(\langle 001|-\langle 101|) \\
&\quad -\left|\beta_1\right|^2\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 011|+\langle 111|)+\left|\beta_1\right|^2\left|\beta_2\right|^2(|000\rangle+|100\rangle)(\langle 000|+\langle 100|)) \\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}
$$

(1)_if the result of measurement is $|0\rangle_{C_3}|0\rangle_{C_4}$:

$$
\begin{aligned}
\rho_{8(H_1)}^{00} &= \frac{(M^{00})\rho_{8(H_1)}(M^{00})^{+}}{tr((M^{00})\rho_{8(H_1)}(M^{00})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\beta_2\right|^2(\left|\alpha_1\right|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+\left|\beta_1\right|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}(X)[\frac{1-\lambda}{2}\left|\beta_2\right|^2(|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](X^{+}) \\
&\Longrightarrow U_{8(H_1)}^{00}=X
\end{aligned}
$$

So the fidelity:

$$
F=Tr\left\{|\Phi_1\rangle\langle\Phi_1|\cdot(X)\rho_{8(H_1)}^{00}(X^{+})\right\}=\frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{8}\right]
$$

(2)_if the result of measurement is $|0\rangle_{C_3}|1\rangle_{C_4}$:

$$
\begin{aligned}
\rho_{8(H_1)}^{01} &= \frac{(M^{01})\rho_{8(H_1)}(M^{01})^{+}}{tr((M^{01})\rho_{8(H_1)}(M^{01})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+\left|\beta_1\right|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_2\right|^2(|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I] \\
&\Longrightarrow U_{8(H_1)}^{01}=I
\end{aligned}
$$

$$F = Tr\left[\left|\Phi_1\right\rangle\left\langle\Phi_1\right| \cdot \rho^{01}_{8(H_1)}\right] = \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{8}\right]$$

(3)_if the result of measurement is $\left|1\right\rangle_{C_3}\left|0\right\rangle_{C_4}$:

$$\begin{aligned}
\rho^{10}_{8(H_1)} &= \frac{(M^{10})\rho_{8(H_1)}(M^{10})^+}{tr((M^{10})\rho_{8(H_1)}(M^{10})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\beta_2\right|^2(\left|\alpha_1\right|^2\left|1\right\rangle\left\langle 1\right| - \alpha_1\beta_1^*\left|1\right\rangle\left\langle 0\right| \\
&\quad -\beta_1\alpha_1^*\left|0\right\rangle\left\langle 1\right| + \left|\beta_1\right|^2\left|0\right\rangle\left\langle 0\right|) + \frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{4}}(iY)[\frac{1-\lambda}{2}\left|\beta_2\right|^2\left|\Phi_1\right\rangle\left\langle\Phi_1\right| + \frac{\lambda}{8}I](iY^+)
\end{aligned}$$

$$F = Tr\left[\left|\Phi_1\right\rangle\left\langle\Phi_1\right| \cdot (iY)\rho^{10}_{8(H_1)}(iY)^+\right] = \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_2\right|^2 + \frac{\lambda}{8}\right]$$

(4)_if the result of measurement is $\left|1\right\rangle_{C_3}\left|1\right\rangle_{C_4}$:

$$\begin{aligned}
\rho^{11}_{8(H_1)} &= \frac{(M^{11})\rho_{8(H_1)}(M^{11})^+}{tr((M^{11})\rho_{8(H_1)}(M^{11})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^2\left|0\right\rangle\left\langle 0\right| - \alpha_1\beta_1^*\left|0\right\rangle\left\langle 1\right| \\
&\quad -\beta_1\alpha_1^*\left|1\right\rangle\left\langle 0\right| + \left|\beta_1\right|^2\left|1\right\rangle\left\langle 1\right|) + \frac{\lambda}{8}I] \qquad (5.138) \\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}(Z)[\frac{1-\lambda}{2}\left|\alpha_2\right|^2\left|\Phi_1\right\rangle\left\langle\Phi_1\right| + \frac{\lambda}{8}I](Z^+) \qquad (5.139)
\end{aligned}$$

$$F = Tr\left[\left|\Phi_1\right\rangle\left\langle\Phi_1\right| \cdot (Z)\rho^{11}_{8(H_1)}(Z^+)\right] = \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\alpha_2\right|^2 + \frac{\lambda}{8}\right] \qquad (5.140)$$

$\underline{\boldsymbol{\rho}_{m9}}$:

$$\begin{aligned}
\rho_{m9(H_1)} &= H(C_3)\,\rho_{m9}H^+(C_3)\\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|000\rangle+|100\rangle)(\langle 000|+\langle 100|)+|\alpha_1|^2\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle 011|+\langle 111|)\\
&\quad -\alpha_1\beta_1^*|\alpha_2|^2(|000\rangle+|100\rangle)(\langle 001|-\langle 101|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle 010|-\langle 110|)\\
&\quad +|\alpha_1|^2\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle 000|+\langle 100|)+|\alpha_1|^2|\beta_2|^2(|011\rangle+|111\rangle)(\langle 011|+\langle 111|)\\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle 001|-\langle 101|)-\alpha_1\beta_1^*|\beta_2|^2(|011\rangle+|111\rangle)(\langle 010|-\langle 110|)\\
&\quad -\beta_1\alpha_1^*|\alpha_2|^2(|001\rangle-|101\rangle)(\langle 000|+\langle 100|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle 011|+\langle 111|)\\
&\quad +|\beta_1|^2|\alpha_2|^2(|001\rangle-|101\rangle)(\langle 001|-\langle 101|)+|\beta_1|^2\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle 010|-\langle 110|)\\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle 000|+\langle 100|)-\beta_1\alpha_1^*|\beta_2|^2(|010\rangle-|110\rangle)(\langle 011|+\langle 111|)\\
&\quad +|\beta_1|^2\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle 001|-\langle 101|)+|\beta_1|^2|\beta_2|^2(|010\rangle-|110\rangle)(\langle 010|-\langle 110|))\\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_ for the results $|0\rangle_{C_3}|0\rangle_{C_4}$:

$$\begin{aligned}
\rho^{00}_{9(H_1)} &= \frac{(M^{00})\rho_{9(H_1)}(M^{00})^+}{tr((M^{00})\rho_{9(H_1)}(M^{00})^+)}\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle 0|-\alpha_1\beta_1^*|0\rangle\langle 1|\\
&\quad -\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}(Z)[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](Z^+)\\
&\Longrightarrow U_9^{00}=Z
\end{aligned}$$

The fidelity:

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_9^{00})\rho^{00}_{9(H_1)}(U_9^{00})^+\right]=\frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right] \tag{5.141}$$

(2)_

$$\begin{aligned}
\rho^{01}_{9(H_1)} &= \frac{(M^{01})\rho_{9(H_1)}(M^{01})^+}{tr((M^{01})\rho_{9(H_1)}(M^{01})^+)}\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0|\\
&\quad -\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}(iY)[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](iY^+)\\
&\Longrightarrow U_9^{01}=iY
\end{aligned}$$

$\Longrightarrow$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1| \cdot (U_9^{01})\rho_{9(H_1)}^{01}(U_9^{01})^+\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{8}\right]$$

(3)_

$$\begin{aligned}
\rho_{9(H_1)}^{10} &= \frac{(M^{10})\rho_{9(H_1)}(M^{10})^+}{tr((M^{10})\rho_{9(H_1)}(M^{10})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle 0| + \alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad + \beta_1\alpha_1^*|1\rangle\langle 0| + |\beta_1|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1| + \frac{\lambda}{8}I] \\
&\Longrightarrow U_9^{10} = I
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1| \cdot (U_9^{10})\rho_{9(H_1)}^{10}(U_9^{10})^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2 + \frac{\lambda}{8}\right]$$

(4)_

$$\begin{aligned}
\rho_{9(H_1)}^{11} &= \frac{(M^{11})\rho_{9(H_1)}(M^{11})^+}{tr((M^{11})\rho_{9(H_1)}(M^{11})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1| + \alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad + \beta_1\alpha_1^*|0\rangle\langle 1| + |\beta_1|^2|0\rangle\langle 0|) + \frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}(X)[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1| + \frac{\lambda}{8}I](X^+) \\
&\Longrightarrow U_{8(H_1)}^{00} = X
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1| \cdot (U_9^{11})\rho_{9(H_1)}^{11}(U_9^{11})^+\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2 + \frac{\lambda}{8}\right]$$

$\boldsymbol{\rho}_{m10}$:

$$\begin{aligned}
\rho_{10(H_1)} &= H\left(C_3\right)\rho_{m10}H^{+}\left(C_3\right)\\
&= \frac{(1-\lambda)}{2}(\left|\alpha_1\right|^2\left|\alpha_2\right|^2(|011\rangle+|111\rangle)(\langle 011|+\langle 111|)+\left|\alpha_1\right|^2\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 000|+\langle 100|)\\
&\quad -\alpha_1\beta_1^*\left|\alpha_2\right|^2(|011\rangle+|111\rangle)\langle 110|-\alpha_1\beta_1^*\alpha_2\beta_2^*(|011\rangle+|111\rangle)\langle 101|\\
&\quad +\left|\alpha_1\right|^2\left|\beta_2\right|^2(|000\rangle+|100\rangle)(\langle 000|+\langle 100|)+\left|\alpha_1\right|^2\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 011|+\langle 111|)\\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|000\rangle+|100\rangle)\langle 110|-\alpha_1\beta_1^*\left|\beta_2\right|^2(|000\rangle+|100\rangle)\langle 101|\\
&\quad -\beta_1\alpha_1^*\left|\alpha_2\right|^2|110\rangle(\langle 011|+\langle 111|)-\beta_1\alpha_1^*\alpha_2\beta_2^*|110\rangle(\langle 000|+\langle 100|)\\
&\quad +\left|\beta_1\right|^2\left|\alpha_2\right|^2(|010\rangle-|110\rangle)(\langle 010|-\langle 110|)+\left|\beta_1\right|^2\alpha_2\beta_2^*(|010\rangle-|110\rangle)\langle 101|\\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*|101\rangle(\langle 011|+\langle 111|)-\beta_1\alpha_1^*\left|\beta_2\right|^2(|010\rangle-|110\rangle)(\langle 000|+\langle 100|)\\
&\quad +\left|\beta_1\right|^2\beta_2\alpha_2^*|101\rangle\langle 110|+\left|\beta_1\right|^2\left|\beta_2\right|^2(|001\rangle-|101\rangle)(\langle 001|-\langle 101|))\\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\begin{aligned}
\rho^{00}_{10(H_1)} &= \frac{(M^{00})\rho_{10(H_1)}(M^{00})^{+}}{tr((M^{00})\rho_{10(H_1)}(M^{00})^{+})}\\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\beta_2\right|^2(\left|\alpha_1\right|^2|0\rangle\langle 0|-\alpha_1\beta_1^*|0\rangle\langle 1|\\
&\quad -\beta_1\alpha_1^*|1\rangle\langle 0|+\left|\beta_1\right|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}(Z)[\frac{1-\lambda}{2}\left|\beta_2\right|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](Z^{+})\\
&\Longrightarrow\quad U^{00}_{10}=Z
\end{aligned}$$

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U^{00}_{10})\rho^{00}_{10(H_1)}(U^{00}_{10})^{+}\right]=\frac{1}{\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_2\right|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\begin{aligned}
\rho^{01}_{10(H_1)} &= \frac{(M^{01})\rho_{9(H_1)}(M^{01})^{+}}{tr((M^{01})\rho_{10(H_1)}(M^{01})^{+})}\\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_2\right|^2(\left|\alpha_1\right|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0|\\
&\quad -\beta_1\alpha_1^*|0\rangle\langle 1|+\left|\beta_1\right|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_2\right|^2+\frac{\lambda}{4}}(iY)[\frac{1-\lambda}{2}\left|\alpha_2\right|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](iY^{+})\\
&\Longrightarrow\quad U^{01}_{10(H_1)}=iY
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{10}^{01})\rho_{10(H_1)}^{01}(U_{10}^{01})^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\begin{aligned}
\rho_{10(H_1)}^{10} &= \frac{(M^{10})\rho_{10(H_1)}(M^{10})^+}{tr((M^{10})\rho_{10(H_1)}(M^{10})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I] \\
&\Longrightarrow U_{12(H_1)}^{11} = I
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{10}^{10})\rho_{10(H_1)}^{10}(U_{10}^{10})^+\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\begin{aligned}
\rho_{10(H_1)}^{11} &= \frac{(M^{11})\rho_{10(H_1)}(M^{11})^+}{tr((M^{11})\rho_{10(H_1)}(M^{11})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}(X)[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](X^+) \\
&\Longrightarrow U_{10(H_1)}^{11} = X
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{10}^{11})\rho_{10(H_1)}^{11}(U_{10}^{11})^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m11}}$

$$\begin{aligned}
\rho_{11(H_1)} &= H(C_3)\,\rho_{m11}H^+(C_3)\\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|000\rangle+|100\rangle)(\langle000|+\langle100|)-|\alpha_1|^2\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle011|+\langle111|)\\
&\quad -\alpha_1\beta_1^*|\alpha_2|^2(|000\rangle+|100\rangle)(\langle001|-\langle101|)+\alpha_1\beta_1^*\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle010|-\langle110|)\\
&\quad -|\alpha_1|^2\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle000|+\langle100|)+|\alpha_1|^2|\beta_2|^2(|011\rangle+|111\rangle)(\langle011|+\langle111|)\\
&\quad +\alpha_1\beta_1^*\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle001|-\langle101|)-\alpha_1\beta_1^*|\beta_2|^2(|011\rangle+|111\rangle)(\langle010|-\langle110|)\\
&\quad -\beta_1\alpha_1^*|\alpha_2|^2(|001\rangle-|101\rangle)(\langle000|+\langle100|)+\beta_1\alpha_1^*\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle011|+\langle111|)\\
&\quad +|\beta_1|^2|\alpha_2|^2(|001\rangle-|101\rangle)(\langle001|-\langle101|)-|\beta_1|^2\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle010|-\langle110|)\\
&\quad +\beta_1\alpha_1^*\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle000|+\langle100|)-\beta_1\alpha_1^*|\beta_2|^2(|010\rangle-|110\rangle)(\langle011|+\langle111|)\\
&\quad -|\beta_1|^2\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle001|-\langle101|)+|\beta_1|^2|\beta_2|^2(|010\rangle-|110\rangle)(\langle010|-\langle110|))\\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\begin{aligned}
\rho^{00}_{11(H_1)} &= \frac{(M^{00})\rho_{11(H_1)}(M^{00})^+}{tr((M^{00})\rho_{11(H_1)}(M^{00})^+)}\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle0|-\alpha_1\beta_1^*|0\rangle\langle1|\\
&\quad -\beta_1\alpha_1^*|1\rangle\langle0|+|\beta_1|^2|1\rangle\langle1|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}(Z)[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](Z^+)\\
&\Longrightarrow U^{00}_{11}=Z
\end{aligned}$$

The fidelity:

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U^{00}_{11})\rho^{00}_{11(H_1)}(U^{00}_{11})^+\right]=\frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right] \tag{5.142}$$

(2)_

$$\begin{aligned}
\rho^{01}_{11(H_1)} &= \frac{(M^{01})\rho_{11(H_1)}(M^{01})^+}{tr((M^{01})\rho_{11(H_1)}(M^{01})^+)}\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle1|-\alpha_1\beta_1^*|1\rangle\langle0|\\
&\quad -\beta_1\alpha_1^*|0\rangle\langle1|+|\beta_1|^2|0\rangle\langle0|)+\frac{\lambda}{8}I] \qquad (5.143)\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)(iY)^+]\\
U^{01}_{11} &= iY \qquad (5.144)
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{11}^{01})\rho_{11(H_1)}^{01}(U_{11}^{01})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right] \tag{5.145}$$

(3)_

$$\begin{aligned}
\rho_{11(H_1)}^{10} &= \frac{(M^{10})\rho_{11(H_1)}(M^{10})^{+}}{tr((M^{10})\rho_{11(H_1)}(M^{10})^{+})}\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1|\\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \qquad (5.146)\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I]\\
&\Longrightarrow U_{11}^{10} = I \qquad (5.147)
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{11}^{10})\rho_{11(H_1)}^{10}(U_{11}^{10})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right] \tag{5.148}$$

(4)_

$$\begin{aligned}
\rho_{9(H_1)}^{11} &= \frac{(M^{11})\rho_{9(H_1)}(M^{11})^{+}}{tr((M^{11})\rho_{9(H_1)}(M^{11})^{+})}\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0|\\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[(X)(\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I)(X^{+})]\\
&\Longrightarrow U_{8(H_1)}^{00} = X
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{11}^{11})\rho_{11(H_1)}^{11}(U_{11}^{11})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

$\boldsymbol{\rho}_{m12}$

$$\begin{aligned}
\rho_{12(H_1)} &= H(C_3)\,\rho_{m12}H^{+}(C_3)\\
&= (1-\lambda)(|\alpha_1|^2|\alpha_2|^2(|011\rangle+|111\rangle)(\langle 011|+\langle 111|)-|\alpha_1|^2\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 000|+\langle 100|)\\
&\quad -\alpha_1\beta_1^*|\alpha_2|^2(|011\rangle+|111\rangle)(\langle 010|-\langle 110|)+\alpha_1\beta_1^*\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 001|-\langle 101|)\\
&\quad +|\alpha_1|^2|\beta_2|^2(|000\rangle+|100\rangle)(\langle 000|+\langle 100|)-|\alpha_1|^2\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 011|+\langle 111|)\\
&\quad +\alpha_1\beta_1^*\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 010|-\langle 110|)-\alpha_1\beta_1^*|\beta_2|^2(|000\rangle+|100\rangle)(\langle 001|-\langle 101|)\\
&\quad -\beta_1\alpha_1^*|\alpha_2|^2(|010\rangle-|110\rangle)(\langle 011|+\langle 111|)+\beta_1\alpha_1^*\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 000|+\langle 100|)\\
&\quad +|\beta_1|^2|\alpha_2|^2(|010\rangle-|110\rangle)(\langle 010|-\langle 110|)-|\beta_1|^2\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 001|-\langle 101|)\\
&\quad +\beta_1\alpha_1^*\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 011|+\langle 111|)-\beta_1\alpha_1^*|\beta_2|^2(|001\rangle-|101\rangle)(\langle 000|+\langle 100|)\\
&\quad -|\beta_1|^2\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 010|-\langle 110|)+|\beta_1|^2|\beta_2|^2(|001\rangle-|101\rangle)(\langle 001|-\langle 101|))\\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\begin{aligned}
\rho^{00}_{12(H_1)} &= \frac{(M^{00})\rho_{12(H_1)}(M^{00})^{+}}{tr((M^{00})\rho_{12(H_1)}(M^{00})^{+})}\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0|-\alpha_1\beta_1^*|0\rangle\langle 1|\\
&\quad -\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}(Z)[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](Z^{+})\\
&\Longrightarrow U^{00}_{12}=Z
\end{aligned}$$

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U^{00}_{12})\rho^{00}_{12(H_1)}(U^{00}_{12})^{+}\right]=\frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\begin{aligned}
\rho^{01}_{12(H_1)} &= \frac{(M^{01})\rho_{12(H_1)}(M^{01})^{+}}{tr((M^{01})\rho_{12(H_1)}(M^{01})^{+})}\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0|\\
&\quad -\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}(iY)[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](iY^{+})\\
&\Longrightarrow U^{01}_{12(H_1)}=iY
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{12}^{01})\rho_{12(H_1)}^{01}(U_{12}^{01})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\begin{aligned}
\rho_{12(H_1)}^{10} &= \frac{(M^{10})\rho_{12(H_1)}(M^{10})^{+}}{tr((M^{10})\rho_{12(H_1)}(M^{10})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I] \\
&\Longrightarrow U_{12(H_1)}^{11} = I
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{12}^{10})\rho_{12(H_1)}^{10}(U_{12}^{10})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\begin{aligned}
\rho_{12(H_1)}^{11} &= \frac{(M^{11})\rho_{10(H_1)}(M^{11})^{+}}{tr((M^{11})\rho_{10(H_1)}(M^{11})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}(X)[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](X^{+}) \\
&\Longrightarrow U_{12(H_1)}^{11} = X
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{12}^{11})\rho_{12(H_1)}^{11}(U_{12}^{11})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m13}}$:

$$\begin{aligned}
\rho_{13(H_1)} &= H(C_3)\,\rho_{m13}H^{+}(C_3) \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|001\rangle-|101\rangle)(\langle 001|-\langle 101|)+|\alpha_1|^2\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle 010|-\langle 110|) \\
&\quad -\alpha_1\beta_1^*|\alpha_2|^2(|001\rangle-|101\rangle)(\langle 000|+\langle 100|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle 011|+\langle 111|) \\
&\quad +|\alpha_1|^2|\beta_2|^2(|010\rangle-|110\rangle)(\langle 010|-\langle 110|)+|\alpha_1|^2\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle 001|-\langle 101|) \\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle 000|+\langle 100|)-\alpha_1\beta_1^*|\beta_2|^2(|010\rangle-|110\rangle)(\langle 011|+\langle 111|) \\
&\quad -\beta_1\alpha_1^*|\alpha_2|^2(|000\rangle+|100\rangle)(\langle 001|-\langle 101|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle 010|-\langle 110|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|000\rangle+|100\rangle)(\langle 000|+\langle 100|)+|\beta_1|^2\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle 011|+\langle 111|) \\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle 001|-\langle 101|)-\beta_1\alpha_1^*|\beta_2|^2(|011\rangle+|111\rangle)(\langle 010|-\langle 110|) \\
&\quad +|\beta_1|^2\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle 000|+\langle 100|)+|\beta_1|^2|\beta_2|^2(|011\rangle+|111\rangle)(\langle 011|+\langle 111|)) \\
&\quad +\frac{\lambda}{8}I_8. \qquad (5.149)
\end{aligned}$$

(1)_

$$\begin{aligned}
\rho_{13(H_1)}^{00} &= \frac{(M^{00})\rho_{13(H_1)}(M^{00})^{+}}{tr((M^{00})\rho_{13(H_1)}(M^{00})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad -\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}(iY)[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](iY)^{+} \\
\Longrightarrow &\quad U_{13}^{00}=iY
\end{aligned}$$

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{13}^{00})\rho_{13(H_1)}^{00}(U_{13}^{00})^{+}\right]=\frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\begin{aligned}
\rho_{13(H_1)}^{01} &= \frac{(M^{01})\rho_{13(H_1)}(M^{01})^{+}}{tr((M^{01})\rho_{13(H_1)}(M^{01})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0|-\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad -\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}(Z)[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](Z)^{+} \\
\Longrightarrow &\quad U_{13}^{01}=iY
\end{aligned}$$

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{13}^{00})\rho_{13(H_1)}^{00}(U_{13}^{00})^+\right]=\frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\begin{aligned}\rho_{13(H_1)}^{10} &= \frac{(M^{10})\rho_{13(H_1)}(M^{10})^+}{tr((M^{10})\rho_{13(H_1)}(M^{10})^+)}\\ &= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0|\\ &\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I]\\ &= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}(X)[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](X)^+\\ \Longrightarrow &\quad U_{13}^{10}=X\end{aligned}$$

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{13}^{10})\rho_{13(H_1)}^{10}(U_{13}^{10})^+\right]=\frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\begin{aligned}\rho_{13(H_1)}^{11} &= \frac{(M^{11})\rho_{13(H_1)}(M^{11})^+}{tr((M^{11})\rho_{13(H_1)}(M^{11})^+)}\\ &= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1|\\ &\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I]\\ &= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I]\\ \Longrightarrow &\quad U_{13}^{11}=I\end{aligned}$$

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{13}^{11})\rho_{13(H_1)}^{11}(U_{13}^{11})^+\right]=\frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m14}}$:

$$
\begin{aligned}
\rho_{14(H_1)} &= H(C_3)\,\rho_{m14}H^+(C_3) \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|010\rangle-|110\rangle)(\langle 010|-\langle 110|)+|\alpha_1|^2\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 001|-\langle 101|) \\
&\quad -\alpha_1\beta_1^*|\alpha_2|^2(|010\rangle-|110\rangle)(\langle 011|+\langle 111|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 000|+\langle 100|) \\
&\quad +|\alpha_1|^2|\beta_2|^2(|001\rangle-|101\rangle)(\langle 001|-\langle 101|)+|\alpha_1|^2\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 010|-\langle 110|) \\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 011|+\langle 111|)-\alpha_1\beta_1^*|\beta_2|^2(|001\rangle-|101\rangle)(\langle 000|+\langle 100|) \\
&\quad -\beta_1\alpha_1^*|\alpha_2|^2(|011\rangle+|111\rangle)(\langle 010|-\langle 110|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 001|-\langle 101|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|011\rangle+|111\rangle)(\langle 011|+\langle 111|)+|\beta_1|^2\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 000|+\langle 100|) \\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 010|-\langle 110|)-\beta_1\alpha_1^*|\beta_2|^2(|000\rangle+|100\rangle)(\langle 001|-\langle 101|) \\
&\quad +|\beta_1|^2\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 011|+\langle 111|)+|\beta_1|^2|\beta_2|^2(|000\rangle+|100\rangle)(\langle 000|+\langle 100|)) \\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}
$$

(1)_

$$
\begin{aligned}
\rho^{00}_{14(H_1)} &= \frac{(M^{00})\rho_{14(H_1)}(M^{00})^+}{tr((M^{00})\rho_{14(H_1)}(M^{00})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad -\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[(iY)[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](iY)^+\right] \\
&\Longrightarrow\quad U^{00}_{14}=iY
\end{aligned}
$$

$$
F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U^{00}_{14})\rho^{00}_{14(H_1)}(U^{00}_{14})^+\right]=\frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]
$$

(2)_

$$
\begin{aligned}
\rho^{01}_{14(H_1)} &= \frac{(M^{01})\rho_{14(H_1)}(M^{01})^+}{tr((M^{01})\rho_{14(H_1)}(M^{01})^+)} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle 0|-\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad -\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I] \\
&\Longrightarrow\quad U^{01}_{14}=Z
\end{aligned}
$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{14}^{01})\rho_{14(H_1)}^{01}(U_{14}^{01})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\begin{aligned}
\rho_{14(H_1)}^{10} &= \frac{(M^{10})\rho_{14(H_1)}(M^{10})^{+}}{tr((M^{10})\rho_{14(H_1)}(M^{10})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[(X)[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](X)^{+}\right] \\
&\Longrightarrow U_{14}^{10} = X
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{14}^{10})\rho_{14(H_1)}^{10}(U_{14}^{10})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\begin{aligned}
\rho_{14(H_1)}^{11} &= \frac{(M^{11})\rho_{14(H_1)}(M^{11})^{+}}{tr((M^{11})\rho_{14(H_1)}(M^{11})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I] \\
&\Longrightarrow U_{14}^{11} = I
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{14}^{11})\rho_{14(H_1)}^{11}(U_{14}^{11})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m15}}$:

$$\begin{aligned}
\rho_{15(H_1)} &= H(C_3)\,\rho_{m15}H^+(C_3)\\
&= (1-\lambda)(|\alpha_1|^2|\alpha_2|^2(|001\rangle-|101\rangle)(\langle001|-\langle101|)-|\alpha_1|^2\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle010|-\langle110|)\\
&\quad+\alpha_1\beta_1^*|\alpha_2|^2(|001\rangle-|101\rangle)(\langle000|+\langle100|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|001\rangle-|101\rangle)(\langle011|+\langle111|)\\
&\quad+|\alpha_1|^2|\beta_2|^2(|010\rangle-|110\rangle)(\langle010|-\langle110|)-|\alpha_1|^2\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle001|-\langle101|)\\
&\quad-\alpha_1\beta_1^*\beta_2\alpha_2^*(|010\rangle-|110\rangle)(\langle000|+\langle100|)+\alpha_1\beta_1^*|\beta_2|^2(|010\rangle-|110\rangle)(\langle011|+\langle111|)\\
&\quad+\beta_1\alpha_1^*|\alpha_2|^2(|000\rangle+|100\rangle)(\langle001|-\langle101|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle010|-\langle110|)\\
&\quad+|\beta_1|^2|\alpha_2|^2(|000\rangle+|100\rangle)(\langle000|+\langle100|)-|\beta_1|^2\alpha_2\beta_2^*(|000\rangle+|100\rangle)(\langle011|+\langle111|)\\
&\quad-\beta_1\alpha_1^*\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle001|-\langle101|)+\beta_1\alpha_1^*|\beta_2|^2(|011\rangle+|111\rangle)(\langle010|-\langle110|)\\
&\quad-|\beta_1|^2\beta_2\alpha_2^*(|011\rangle+|111\rangle)(\langle000|+\langle100|)+|\beta_1|^2|\beta_2|^2(|011\rangle+|111\rangle)(\langle011|+\langle111|))\\
&\quad+\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\begin{aligned}
\rho^{00}_{15(H_1)} &= \frac{(M^{00})\rho_{1((H_1)}(M^{00})^+}{tr((M^{00})\rho_{15(H_1)}(M^{00})^+)}\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle1|-\alpha_1\beta_1^*|1\rangle\langle0|\\
&\quad-\beta_1\alpha_1^*|0\rangle\langle1|+|\beta_1|^2|0\rangle\langle0|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[(iY)[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](iY)^+\right]\\
&\Longrightarrow U^{00}_{15}=iY
\end{aligned}$$

$$F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U^{00}_{15})\rho^{00}_{15(H_1)}(U^{00}_{13})^+\right]=\frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\begin{aligned}
\rho^{01}_{15(H_1)} &= \frac{(M^{01})\rho_{15(H_1)}(M^{01})^+}{tr((M^{01})\rho_{15(H_1)}(M^{01})^+)}\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle0|-\alpha_1\beta_1^*|0\rangle\langle1|\\
&\quad-\beta_1\alpha_1^*|1\rangle\langle0|+|\beta_1|^2|1\rangle\langle1|)+\frac{\lambda}{8}I]\\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[(Z)[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](Z)^+\right]\\
&\Longrightarrow U^{01}_{15}=iY
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{15}^{00})\rho_{15(H_1)}^{00}(U_{15}^{00})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\begin{aligned}
\rho_{15(H_1)}^{10} &= \frac{(M^{10})\rho_{13(H_1)}(M^{10})^{+}}{tr((M^{10})\rho_{13(H_1)}(M^{10})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[(X)[\frac{1-\lambda}{2}|\alpha_2|^2(|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](X)^{+}\right] \\
\Longrightarrow &\quad U_{15}^{10} = X
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{15}^{10})\rho_{15(H_1)}^{10}(U_{15}^{10})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\begin{aligned}
\rho_{15(H_1)}^{11} &= \frac{(M^{11})\rho_{15(H_1)}(M^{11})^{+}}{tr((M^{11})\rho_{15(H_1)}(M^{11})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I] \\
\Longrightarrow &\quad U_{15}^{11} = I
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{15}^{11})\rho_{15(H_1)}^{11}(U_{15}^{11})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m16}}$:

$$
\begin{aligned}
\rho_{16(H_1)} &= H(C_3)\,\rho_{m16}H^{+}(C_3) \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|010\rangle-|110\rangle)(\langle 010|-\langle 110|)-|\alpha_1|^2\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 001|-\langle 101|) \\
&\quad -\alpha_1\beta_1^*|\alpha_2|^2(|010\rangle-|110\rangle)(\langle 011|+\langle 111|)+\alpha_1\beta_1^*\alpha_2\beta_2^*(|010\rangle-|110\rangle)(\langle 000|+\langle 100|) \\
&\quad +|\alpha_1|^2|\beta_2|^2(|001\rangle-|101\rangle)(\langle 001|-\langle 101|)-|\alpha_1|^2\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 010|-\langle 110|) \\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|001\rangle-|101\rangle)(\langle 011|+\langle 111|)+\alpha_1\beta_1^*|\beta_2|^2(|001\rangle-|101\rangle)(\langle 000|+\langle 100|) \\
&\quad +\beta_1\alpha_1^*|\alpha_2|^2(|011\rangle+|111\rangle)(\langle 010|-\langle 110|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 001|-\langle 101|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|011\rangle+|111\rangle)(\langle 011|+\langle 111|)-|\beta_1|^2\alpha_2\beta_2^*(|011\rangle+|111\rangle)(\langle 000|+\langle 100|) \\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 010|-\langle 110|)+\beta_1\alpha_1^*|\beta_2|^2(|000\rangle+|100\rangle)(\langle 001|-\langle 101|) \\
&\quad -|\beta_1|^2\beta_2\alpha_2^*(|000\rangle+|100\rangle)(\langle 011|+\langle 111|)+|\beta_1|^2|\beta_2|^2(|000\rangle+|100\rangle)(\langle 000|+\langle 100|)) \\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}
$$

(1)_

$$
\begin{aligned}
\rho^{00}_{16(H_1)} &= \frac{(M^{00})\rho_{16(H_1)}(M^{00})^{+}}{tr((M^{00})\rho_{16(H_1)}(M^{00})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|-\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad -\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[(iY)[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](iY)^{+}\right] \\
&\Longrightarrow U^{00}_{16}=iY
\end{aligned}
$$

$$
F=Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U^{00}_{16})\rho^{00}_{16(H_1)}(U^{00}_{16})^{+}\right]=\frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]
$$

(2)_

$$
\begin{aligned}
\rho^{01}_{16(H_1)} &= \frac{(M^{01})\rho_{16(H_1)}(M^{01})^{+}}{tr((M^{01})\rho_{16(H_1)}(M^{01})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle 0|-\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad -\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I] \\
&\Longrightarrow U^{01}_{16}=Z
\end{aligned}
$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{16}^{01})\rho_{16(H_1)}^{01}(U_{16}^{01})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\begin{aligned}
\rho_{16(H_1)}^{10} &= \frac{(M^{10})\rho_{16(H_1)}(M^{10})^{+}}{tr((M^{10})\rho_{16(H_1)}(M^{10})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|1\rangle\langle 1|+\alpha_1\beta_1^*|1\rangle\langle 0| \\
&\quad +\beta_1\alpha_1^*|0\rangle\langle 1|+|\beta_1|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[(X)[\frac{1-\lambda}{2}|\beta_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I](X)^{+}\right] \\
&\Longrightarrow U_{16}^{10} = X
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{16}^{10})\rho_{16(H_1)}^{10}(U_{16}^{10})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\begin{aligned}
\rho_{16(H_1)}^{11} &= \frac{(M^{11})\rho_{16(H_1)}(M^{11})^{+}}{tr((M^{11})\rho_{16(H_1)}(M^{11})^{+})} \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2(|\alpha_1|^2|0\rangle\langle 0|+\alpha_1\beta_1^*|0\rangle\langle 1| \\
&\quad +\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \\
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_2|^2|\Phi_1\rangle\langle\Phi_1|+\frac{\lambda}{8}I] \\
&\Longrightarrow U_{16}^{11} = I
\end{aligned}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(U_{16}^{11})\rho_{16(H_1)}^{11}(U_{16}^{11})^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}\right]$$

When the controller choose to teleport the qubit of Alice2: he applies Hadamard operation on his second qubit C_4,then performs measurement in the Zbasis on his two qubits.

$$\rho_m \rightarrow \rho_{m(H_2)} = H(C_4)\,\rho_m H^{+}(C_4) \equiv (I_2\otimes H\otimes I_2)\rho_m(I_2\otimes H\otimes I_2) \tag{5.150}$$

The final states and the corrections of Bob for each measurement possibility are:

$\underline{\boldsymbol{\rho}_{m1}}$:

$$
\begin{aligned}
\rho_{1(H_2)} &= H(C_4)\rho_{m1}H^{+}(C_4) \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)+|\alpha_1|^2\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|) \\
&\quad +\alpha_1\beta_1^*|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|)+\alpha_1\beta_1^*\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|) \\
&\quad +|\alpha_1|^2\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|)+|\alpha_1|^2|\beta_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|) \\
&\quad +\alpha_1\beta_1^*\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|)+\alpha_1\beta_1^*|\beta_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|) \\
&\quad +\beta_1\alpha_1^*|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|)+\beta_1\alpha_1^*\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)+|\beta_1|^2\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|) \\
&\quad +\beta_1\alpha_1^*\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|)+\beta_1\alpha_1^*|\beta_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|) \\
&\quad +|\beta_1|^2\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|)+|\beta_1|^2|\beta_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)) \\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}
$$

(1)_ If the result of Charlie's measurement is $|0\rangle_{C_3}|0\rangle_{C_4}$:

$$
\rho_{1(H_2)}^{00} = \frac{(M^{00})\rho_{1(H_2)}(M^{00})^{+}}{tr((M^{00})\rho_{1(H_2)}(M^{00})^{+})} \tag{5.151}
$$

$$
= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1| +\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.152}
$$

$$
= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I] \tag{5.153}
$$

$$
F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho_{1(H_2)}^{00}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]
$$

(2)_ If the result of Charlie's measurement is $|0\rangle_{C_3}|1\rangle_{C_4}$:

$$
\rho_{1(H_2)}^{01} = \frac{(M^{01})\rho_{1(H_2)}(M^{01})^{+}}{tr((M^{01})\rho_{1(H_2)}(M^{01})^{+})} \tag{5.154}
$$

$$
= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0|-\alpha_2\beta_2^*|0\rangle\langle 1| -\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.155}
$$

$$
= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)Z^{+}] \tag{5.156}
$$

$$
F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(Z)\rho_{1(H_2)}^{01}(Z)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}(\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8})
$$

(3)_ If the result of Charlie's measurement is $|1\rangle_{C_3}|0\rangle_{C_4}$:

$$\rho^{10}_{1(H_2)} = \frac{(M^{10})\rho_{1(H_2)}(M^{10})^+}{tr((M^{10})\rho_{1(H_2)}(M^{10})^+)} \tag{5.157}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0|$$
$$+\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.158}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[X(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)X^+\right] \tag{5.159}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{10}_{1(H_2)}(X)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(4)_ If the result of Charlie's measurement is $|1\rangle_{C_3}|1\rangle_{C_4}$:

$$\rho^{11}_{1(H_2)} = \frac{(M^{11})\rho_{1(H_2)}(M^{11})^+}{tr((M^{11})\rho_{1(H_2)}(M^{11})^+)} \tag{5.160}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1|-\alpha_2\beta_2^*|1\rangle\langle 0|$$
$$-\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.161}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(iY^+)] \tag{5.162}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{11}_{1(H_2)}(iY)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}(\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8})$$

$\underline{\boldsymbol{\rho}_{m2}}$:

$$\rho_{2(H_2)} = H(C_4)\rho_{m2}H^+(C_4)$$
$$= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)+|\alpha_1|^2\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|)$$
$$+\alpha_1\beta_1^*|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|)+\alpha_1\beta_1^*\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|)$$
$$+|\alpha_1|^2|\beta_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)+|\alpha_1|^2\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|)$$
$$+\alpha_1\beta_1^*\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|)+\alpha_1\beta_1^*|\beta_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|)$$
$$+\beta_1\alpha_1^*|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|)+\beta_1\alpha_1^*\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|)$$
$$+|\beta_1|^2|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)+|\beta_1|^2\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|)$$
$$+\beta_1\alpha_1^*\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|)+\beta_1\alpha_1^*|\beta_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|)$$
$$+|\beta_1|^2\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|)+|\beta_1|^2|\beta_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|))$$
$$+\frac{\lambda}{8}I_8.$$

(1)_ If the result of Charlie's measurement is $|0\rangle_{C_3}|0\rangle_{C_4}$:

$$\rho^{00}_{2(H_2)} = \frac{(M^{00})\rho_{2(H_2)}(M^{00})^+}{tr((M^{00})\rho_{2(H_2)}(M^{00})^+)} \tag{5.163}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0| +\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.164}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(X)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(X)] \tag{5.165}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{00}_{2(H_2)}(X)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}(\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8})$$

(2)_ If the result of Charlie's measurement is $|0\rangle_{C_3}|1\rangle_{C_4}$:

$$\rho^{01}_{2(H_2)} = \frac{(M^{01})\rho_{2(H_2)}(M^{01})^+}{tr((M^{01})\rho_{2(H_2)}(M^{01})^+)} \tag{5.166}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|-\alpha_2\beta_2^*|1\rangle\langle 0| -\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.167}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(iY)^+] \tag{5.168}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|(iY)\rho^{01}_{2(H_2)}(iY)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(3)_ If the result of Charlie's measurement is $|1\rangle_{C_3}|0\rangle_{C_4}$:

$$\rho^{10}_{2(H_2)} = \frac{(M^{10})\rho_{2(H_2)}(M^{10})^+}{tr((M^{10})\rho_{2(H_2)}(M^{10})^+)} \tag{5.169}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1| +\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.170}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)] \tag{5.171}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{10}_{2(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(4)_ If the result of Charlie's measurement is $|1\rangle_{C_3}|1\rangle_{C_4}$:

$$\rho^{11}_{2(H_2)} = \frac{(M^{11})\rho_{2(H_2)}(M^{11})^+}{tr((M^{11})\rho_{2(H_2)}(M^{11})^+)} \tag{5.172}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0| - \alpha_2\beta_2^*|0\rangle\langle 1|$$

$$-\beta_2\alpha_2^*|1\rangle\langle 0| + |\beta_2|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \tag{5.173}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(Z)(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)(Z)] \tag{5.174}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot Z\rho^{11}_{2(H_2)}Z\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m3}}$:

$$\rho_{3(H_2)} = H(C_4)\rho_{m3}H^+(C_4)$$

$$= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|) - |\alpha_1|^2\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|)$$
$$+\alpha_1\beta_1^*|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|) - \alpha_1\beta_1^*\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|)$$
$$-|\alpha_1|^2\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|) + |\alpha_1|^2|\beta_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)$$
$$-\alpha_1\beta_1^*\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|) + \alpha_1\beta_1^*|\beta_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|)$$
$$+\beta_1\alpha_1^*|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|) - \beta_1\alpha_1^*\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|)$$
$$+|\beta_1|^2|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|) - |\beta_1|^2\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|)$$
$$-\beta_1\alpha_1^*\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|) + \beta_1\alpha_1^*|\beta_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|)$$
$$-|\beta_1|^2\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|) + |\beta_1|^2|\beta_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|))$$
$$+\frac{\lambda}{8}I_8.$$

(1)_ If the result of Charlie's measurement is $|0\rangle_{C_3}|0\rangle_{C_4}$:

$$\rho^{00}_{3(H_2)} = \frac{(M^{00})\rho_{3(H_2)}(M^{00})^+}{tr((M^{00})\rho_{3(H_2)}(M^{00})^+)} \tag{5.175}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0| - \alpha_2\beta_2^*|0\rangle\langle 1|$$

$$-\beta_2\alpha_2^*|1\rangle\langle 0| + |\beta_2|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \tag{5.176}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(Z)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)(Z)^+] \tag{5.177}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(Z)\rho^{00}_{3(H_2)}(Z)\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(2)_ If the result of Charlie's measurement is $|0\rangle_{C_3}|1\rangle_{C_4}$:

$$\rho^{01}_{3(H_2)} = \frac{(M^{01})\rho_{3(H_2)}(M^{01})^{+}}{tr((M^{01})\rho_{3(H_2)}(M^{01})^{+})} \quad (5.178)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1|$$
$$+\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \quad (5.179)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I] \quad (5.180)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{01}_{3(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(3)_ If the result of Charlie's measurement is $|1\rangle_{C_3}|0\rangle_{C_4}$:

$$\rho^{10}_{3(H_2)} = \frac{(M^{10})\rho_{3(H_2)}(M^{10})^{+}}{tr((M^{10})\rho_{3(H_2)}(M^{10})^{+})} \quad (5.181)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1|-\alpha_2\beta_2^*|1\rangle\langle 0|$$
$$-\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \quad (5.182)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(iY)] \quad (5.183)$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot(iY)\rho^{10}_{3(H_2)}(iY)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(4)_ If the result of Charlie's measurement is $|1\rangle_{C_3}|1\rangle_{C_4}$:

$$\rho^{11}_{3(H_2)} = \frac{(M^{11})\rho_{3(H_2)}(M^{11})^{+}}{tr((M^{11})\rho_{3(H_2)}(M^{11})^{+})} \quad (5.184)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0|$$
$$+\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \quad (5.185)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(X)(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(X)] \quad (5.186)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{11}_{3(H_2)}(X)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m_4}}$:

$$\begin{aligned}
\rho_{4(H_2)} &= H(C_4)\rho_{m_4}H^+(C_4) \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)-|\alpha_1|^2\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|) \\
&\quad +\alpha_1\beta_1^*|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|) \\
&\quad +|\alpha_1|^2|\beta_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)-|\alpha_1|^2\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|) \\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|)+\alpha_1\beta_1^*|\beta_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|) \\
&\quad +\beta_1\alpha_1^*|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)-|\beta_1|^2\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|) \\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|)+\beta_1\alpha_1^*|\beta_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|) \\
&\quad -|\beta_1|^2\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|)+|\beta_1|^2|\beta_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)) \\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_ If the result of Charlie's measurement is $|0\rangle_{C_3}|0\rangle_{C_4}$:

$$\rho^{00}_{4(H_2)} = \frac{(M^{00})\rho_{4(H_2)}(M^{00})^+}{tr((M^{00})\rho_{4(H_2)}(M^{00})^+)} \tag{5.187}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|-\alpha_2\beta_2^*|1\rangle\langle 0| -\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.188}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(iY)] \tag{5.189}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{00}_{4(H_2)}(iY)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(2)_ If the result of Charlie's measurement is $|0\rangle_{C_3}|1\rangle_{C_4}$:

$$\rho^{01}_{4(H_2)} = \frac{(M^{01})\rho_{4(H_2)}(M^{01})^+}{tr((M^{01})\rho_{4(H_2)}(M^{01})^+)} \tag{5.190}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0| +\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.191}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(X)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(X)^+] \tag{5.192}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|(X)\rho^{01}_{4(H_2)}(X)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(3)_ If the result of Charlie's measurement is $|1\rangle_{C_3}|0\rangle_{C_4}$:

$$\rho^{10}_{4(H_2)} = \frac{(M^{10})\rho_{4(H_2)}(M^{10})^+}{tr((M^{10})\rho_{4(H_2)}(M^{10})^+)} \quad (5.193)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0|-\alpha_2\beta_2^*|0\rangle\langle 1|$$

$$-\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \quad (5.194)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(Z)(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(Z)] \quad (5.195)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(Z)\rho^{10}_{4(H_2)}(Z)\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(4)_ If the result of Charlie's measurement is $|1\rangle_{C_3}|1\rangle_{C_4}$:

$$\rho^{11}_{4(H_2)} = \frac{(M^{11})\rho_{4(H_2)}(M^{11})^+}{tr((M^{11})\rho_{4(H_2)}(M^{11})^+)} \quad (5.196)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1|$$

$$+\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \quad (5.197)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)] \quad (5.198)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{11}_{4(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

$\boldsymbol{\rho}_{m5}$:

$$\begin{aligned}
\rho_{5(H_2)} &= H(C_4)\rho_{m5}H^+(C_4) \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)+|\alpha_1|^2\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|) \\
&+\alpha_1\beta_1^*|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|)+\alpha_1\beta_1^*\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|) \\
&+|\alpha_1|^2|\beta_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)+|\alpha_1|^2\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|) \\
&+\alpha_1\beta_1^*\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|)+\alpha_1\beta_1^*|\beta_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|) \\
&+\beta_1\alpha_1^*|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|)+\beta_1\alpha_1^*\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|) \\
&+|\beta_1|^2|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)+|\beta_1|^2\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|) \\
&+\beta_1\alpha_1^*\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|)+\beta_1\alpha_1^*|\beta_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|) \\
&+|\beta_1|^2\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|)+|\beta_1|^2|\beta_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)) \\
&+\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\rho^{00}_{5(H_2)} = \frac{(M^{00})\rho_{5(H_2)}(M^{00})^{+}}{tr((M^{00})\rho_{5(H_2)}(M^{00})^{+})} \tag{5.199}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1|$$

$$+\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.200}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I] \tag{5.201}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{00}_{5(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{5(H_2)} = \frac{(M^{01})\rho_{5(H_2)}(M^{01})^{+}}{tr((M^{01})\rho_{5(H_2)}(M^{01})^{+})} \tag{5.202}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0|-\alpha_2\beta_2^*|0\rangle\langle 1|$$

$$-\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.203}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)Z^{+}] \tag{5.204}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(Z)\rho^{01}_{5(H_2)}(Z)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{5(H_2)} = \frac{(M^{10})\rho_{5(H_2)}(M^{10})^{+}}{tr((M^{10})\rho_{5(H_2)}(M^{10})^{+})} \tag{5.205}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0|$$

$$+\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.206}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[X(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)X^{+}] \tag{5.207}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{10}_{5(H_2)}(X)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{5(H_2)} = \frac{(M^{11})\rho_{5(H_2)}(M^{11})^{+}}{tr((M^{11})\rho_{5(H_2)}(M^{11})^{+})} \tag{5.208}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1| - \alpha_2\beta_2^*|1\rangle\langle 0|$$

$$-\beta_2\alpha_2^*|0\rangle\langle 1| + |\beta_2|^2|0\rangle\langle 0|) + \frac{\lambda}{8}I] \tag{5.209}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)(iY^{+})] \tag{5.210}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{11}_{5(H_2)}(iY)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

$\boldsymbol{\rho}_{m6}$:

$$\begin{aligned}
\rho_{6(H_2)} &= H(C_4)\rho_{m6}H^{+}(C_4)\\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|) + |\alpha_1|^2\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|)\\
&+\alpha_1\beta_1^*|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|) + \alpha_1\beta_1^*\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|)\\
&+|\alpha_1|^2\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|) + |\alpha_1|^2|\beta_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)\\
&+\alpha_1\beta_1^*\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|) + \alpha_1\beta_1^*|\beta_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|)\\
&+\beta_1\alpha_1^*|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|) + \beta_1\alpha_1^*\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|)\\
&+|\beta_1|^2|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|) + |\beta_1|^2\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|)\\
&+\beta_1\alpha_1^*\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|) + \beta_1\alpha_1^*|\beta_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|)\\
&+|\beta_1|^2\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|) + |\beta_1|^2|\beta_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|))\\
&+\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\rho^{00}_{6(H_2)} = \frac{(M^{00})\rho_{6(H_2)}(M^{00})^{+}}{tr((M^{00})\rho_{6(H_2)}(M^{00})^{+})} \tag{5.211}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1| + \alpha_2\beta_2^*|1\rangle\langle 0|$$

$$+\beta_2\alpha_2^*|0\rangle\langle 1| + |\beta_2|^2|0\rangle\langle 0|) + \frac{\lambda}{8}I] \tag{5.212}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[X(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)X] \tag{5.213}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot X\rho^{00}_{6(H_2)}X\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{6(H_2))} = \frac{(M^{01})\rho_{6(H_2)}(M^{01})^{+}}{tr((M^{01})\rho_{6(H_2)}(M^{01})^{+})} \tag{5.214}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\beta_1\right|^2(\left|\alpha_2\right|^2\left|1\right\rangle\left\langle 1\right|-\alpha_2\beta_2^*\left|1\right\rangle\left\langle 0\right|$$

$$-\beta_2\alpha_2^*\left|0\right\rangle\left\langle 1\right|+\left|\beta_2\right|^2\left|0\right\rangle\left\langle 0\right|)+\frac{\lambda}{8}I] \tag{5.215}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{4}}[iY(\frac{1-\lambda}{2}\left|\beta_1\right|^2\left|\Phi_2\right\rangle\left\langle\Phi_2\right|+\frac{\lambda}{8}I)(iY)^{+}] \tag{5.216}$$

$$F = Tr\left[\left|\Phi_2\right\rangle\left\langle\Phi_2\right|\cdot(iY)\rho^{01}_{6(H_2)}(iY)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{6(H_2)} = \frac{(M^{10})\rho_{6(H_2)}(M^{10})^{+}}{tr((M^{10})\rho_{6(H_2)}(M^{10})^{+})} \tag{5.217}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_1\right|^2(\left|\alpha_2\right|^2\left|0\right\rangle\left\langle 0\right|+\alpha_2\beta_2^*\left|0\right\rangle\left\langle 1\right|$$

$$+\beta_2\alpha_2^*\left|1\right\rangle\left\langle 0\right|+\left|\beta_2\right|^2\left|1\right\rangle\left\langle 1\right|)+\frac{\lambda}{8}I] \tag{5.218}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_1\right|^2\left|\Phi_2\right\rangle\left\langle\Phi_2\right|+\frac{\lambda}{8}I] \tag{5.219}$$

$$F = Tr\left[\left|\Phi_2\right\rangle\left\langle\Phi_2\right|\cdot\rho^{10}_{6(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{6(H_2)} = \frac{(M^{11})\rho_{6(H_2)}(M^{11})^{+}}{tr((M^{11})\rho_{6(H_2)}(M^{11})^{+})} \tag{5.220}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_1\right|^2(\left|\alpha_2\right|^2\left|0\right\rangle\left\langle 0\right|-\alpha_2\beta_2^*\left|0\right\rangle\left\langle 1\right|$$

$$-\beta_2\alpha_2^*\left|1\right\rangle\left\langle 0\right|+\left|\beta_2\right|^2\left|1\right\rangle\left\langle 1\right|)+\frac{\lambda}{8}I] \tag{5.221}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{4}}[(Z)(\frac{1-\lambda}{2}\left|\alpha_1\right|^2\left|\Phi_2\right\rangle\left\langle\Phi_2\right|+\frac{\lambda}{8}I)(Z^{+})] \tag{5.222}$$

$$F = Tr\left[\left|\Phi_2\right\rangle\left\langle\Phi_2\right|\cdot(Z)\rho^{11}_{6(H_2)}(Z)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m7}:}$

$$\begin{aligned}
\rho_{7(H_2)} &= (H_2)\rho_{m7}(H_2)^+ \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|101\rangle+|111\rangle)(\langle101|+\langle111|)-|\alpha_1|^2\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle100|-\langle110|) \\
&\quad +\alpha_1\beta_1^*|\alpha_2|^2(|101\rangle+|111\rangle)(\langle000|+\langle010|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle001|-\langle011|) \\
&\quad +|\alpha_1|^2|\beta_2|^2(|100\rangle-|110\rangle)(\langle100|-\langle110|)-|\alpha_1|^2\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle101|+\langle111|) \\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle000|+\langle010|)+\alpha_1\beta_1^*|\beta_2|^2(|100\rangle-|110\rangle)(\langle001|-\langle011|) \\
&\quad +\beta_1\alpha_1^*|\alpha_2|^2(|000\rangle+|010\rangle)(\langle101|+\langle111|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle100|-\langle110|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|000\rangle+|010\rangle)(\langle000|+\langle010|)-|\beta_1|^2\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle001|-\langle011|) \\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle101|+\langle111|)+\beta_1\alpha_1^*|\beta_2|^2(|001\rangle-|011\rangle)(\langle100|-\langle110|) \\
&\quad -|\beta_1|^2\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle000|+\langle010|)+|\beta_1|^2|\beta_2|^2(|001\rangle-|011\rangle)(\langle001|-\langle011|)) \\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\rho^{00}_{7(H_2)} = \frac{(M^{00})\rho_{7(H_2)}(M^{00})^+}{tr((M^{00})\rho_{7(H_2)}(M^{00})^+)} \tag{5.223}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle0|-\alpha_2\beta_2^*|0\rangle\langle1| -\beta_2\alpha_2^*|1\rangle\langle0|+|\beta_2|^2|1\rangle\langle1|)+\frac{\lambda}{8}I] \tag{5.224}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)Z] \tag{5.225}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot Z\rho^{00}_{7(H_2)}Z\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{7(H_2)} = \frac{(M^{01})\rho_{7(H_2)}(M^{01})^+}{tr((M^{01})\rho_{7(H_2)}(M^{01})^+)} \tag{5.226}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle0|+\alpha_2\beta_2^*|0\rangle\langle1| +\beta_2\alpha_2^*|1\rangle\langle0|+|\beta_2|^2|1\rangle\langle1|)+\frac{\lambda}{8}I] \tag{5.227}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I] \tag{5.228}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{01}_{7(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{7(H_2)} = \frac{(M^{10})\rho_{7(H_2)}(M^{10})^+}{tr((M^{10})\rho_{7(H_2)}(M^{10})^+)} \quad (5.229)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|-\alpha_2\beta_2^*|1\rangle\langle 0|$$

$$-\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \quad (5.230)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(iY)] \quad (5.231)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{10}_{7(H_2)}(iY)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{7(H_2)} = \frac{(M^{11})\rho_{7(H_2)}(M^{11})^+}{tr((M^{11})\rho_{7(H_2)}(M^{11})^+)} \quad (5.232)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0|$$

$$+\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \quad (5.233)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(X)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(X)] \quad (5.234)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{11}_{7(H_2)}(X)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

$\boldsymbol{\rho}_{m8}$:

$$\begin{aligned}
\rho_{8(H_2)} &= (H_2)\rho_{m8}(H_2)^+ \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)-|\alpha_1|^2\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|) \\
&+\alpha_1\beta_1^*|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|) \\
&+|\alpha_1|^2|\beta_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)-|\alpha_1|^2\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|) \\
&-\alpha_1\beta_1^*\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|)+\alpha_1\beta_1^*|\beta_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|) \\
&+\beta_1\alpha_1^*|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|) \\
&+|\beta_1|^2|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)-|\beta_1|^2\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|) \\
&-\beta_1\alpha_1^*\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|)+\beta_1\alpha_1^*|\beta_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|) \\
&-|\beta_1|^2\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|)+|\beta_1|^2|\beta_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)) \\
&+\frac{\lambda}{8}I_8. \quad (5.235)
\end{aligned}$$

(1)_

$$\rho^{00}_{8(H_2)} = \frac{(M^{00})\rho_{8(H_2)}(M^{00})^{+}}{tr((M^{00})\rho_{8(H_2)}(M^{00})^{+})} \tag{5.236}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1| - \alpha_2\beta_2^*|1\rangle\langle 0|$$

$$-\beta_2\alpha_2^*|0\rangle\langle 1| + |\beta_2|^2|0\rangle\langle 0|) + \frac{\lambda}{8}I] \tag{5.237}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)(iY)^{+}] \tag{5.238}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{00}_{8(H_2)}(iY)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{8(H_2)} = \frac{(M^{01})\rho_{8(H_2)}(M^{01})^{+}}{tr((M^{01})\rho_{8(H_2)}(M^{01})^{+})} \tag{5.239}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1| + \alpha_2\beta_2^*|1\rangle\langle 0|$$

$$+\beta_2\alpha_2^*|0\rangle\langle 1| + |\beta_2|^2|0\rangle\langle 0|) + \frac{\lambda}{8}I] \tag{5.240}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[X(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)X^{+}] \tag{5.241}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{01}_{8(H_2)}(X)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{8(H_2)} = \frac{(M^{10})\rho_{8(H_2)}(M^{10})^{+}}{tr((M^{10})\rho_{8(H_2)}(M^{10})^{+})} \tag{5.242}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0| - \alpha_2\beta_2^*|0\rangle\langle 1|$$

$$-\beta_2\alpha_2^*|1\rangle\langle 0| + |\beta_2|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \tag{5.243}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)Z^{+}] \tag{5.244}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(Z)\rho^{10}_{8(H_2)}(Z)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{8(H_2)} = \frac{(M^{11})\rho_{8(H_2)}(M^{11})^{+}}{tr((M^{11})\rho_{8(H_2)}(M^{11})^{+})} \quad (5.245)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1|$$
$$+\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \quad (5.246)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)] \quad (5.247)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{11}_{8(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho_{m9}}}$:

$$\rho_{9(H_2)} = (H_2)\rho_{m9}(H_2)^{+}$$
$$= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)+|\alpha_1|^2\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|)$$
$$-\alpha_1\beta_1^*|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|)$$
$$+|\alpha_1|^2\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|)+|\alpha_1|^2|\beta_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)$$
$$-\alpha_1\beta_1^*\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|)-\alpha_1\beta_1^*|\beta_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|)$$
$$-\beta_1\alpha_1^*|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|)$$
$$+|\beta_1|^2|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)+|\beta_1|^2\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|)$$
$$-\beta_1\alpha_1^*\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|)-\beta_1\alpha_1^*|\beta_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|)$$
$$+|\beta_1|^2\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|)+|\beta_1|^2|\beta_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|))$$
$$+\frac{\lambda}{8}I_8.$$

(1)_

$$\rho^{00}_{9(H_2)} = \frac{(M^{00})\rho_{9(H_2)}(M^{00})^{+}}{tr((M^{00})\rho_{9(H_2)}(M^{00})^{+})} \quad (5.248)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1|$$
$$+\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \quad (5.249)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)] \quad (5.250)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{00}_{9(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{9(H_2))} = \frac{(M^{01})\rho_{9(H_2)}(M^{01})^{+}}{tr((M^{01})\rho_{9(H_2)}(M^{01})^{+})} \tag{5.251}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0| - \alpha_2\beta_2^*|0\rangle\langle 1|$$
$$-\beta_2\alpha_2^*|1\rangle\langle 0| + |\beta_2|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \tag{5.252}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)Z^{+}] \tag{5.253}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(Z)\rho^{01}_{9(H_2)}(Z)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{1(H_2)} = \frac{(M^{10})\rho_{1(H_2)}(M^{10})^{+}}{tr((M^{10})\rho_{1(H_2)}(M^{10})^{+})} \tag{5.254}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1| + \alpha_2\beta_2^*|1\rangle\langle 0|$$
$$+\beta_2\alpha_2^*|0\rangle\langle 1| + |\beta_2|^2|0\rangle\langle 0|) + \frac{\lambda}{8}I] \tag{5.255}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[X(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)X] \tag{5.256}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{10}_{1(H_2)}(X)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{1(H_2)} = \frac{(M^{11})\rho_{1(H_2)}(M^{11})^{+}}{tr((M^{11})\rho_{1(H_2)}(M^{11})^{+})} \tag{5.257}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1| - \alpha_2\beta_2^*|1\rangle\langle 0|$$
$$-\beta_2\alpha_2^*|0\rangle\langle 1| + |\beta_2|^2|0\rangle\langle 0|) + \frac{\lambda}{8}I] \tag{5.258}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)(iY^{+})] \tag{5.259}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{11}_{1(H_2)}(iY)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m10}}$:

$$\begin{aligned}
\rho_{10(H_2)} &= (H_2)\rho_{m10}(H_2)^+ \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)+|\alpha_1|^2\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|) \\
&\quad -\alpha_1\beta_1^*|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|) \\
&\quad +|\alpha_1|^2|\beta_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)+|\alpha_1|^2\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|) \\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|)-\alpha_1\beta_1^*|\beta_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|) \\
&\quad -\beta_1\alpha_1^*|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)+|\beta_1|^2\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|) \\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|)-\beta_1\alpha_1^*|\beta_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|) \\
&\quad +|\beta_1|^2\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|)+|\beta_1|^2|\beta_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)) \\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\rho^{00}_{10(H_2)} = \frac{(M^{00})\rho_{10(H_2)}(M^{00})^+}{tr((M^{00})\rho_{10(H_2)}(M^{00})^+)} \tag{5.260}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0|$$

$$+\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.261}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[X(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)X] \tag{5.262}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot X\rho^{00}_{10(H_2)}X\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{10(H_2)} = \frac{(M^{01})\rho_{10(H_2)}(M^{01})^+}{tr((M^{01})\rho_{1(H_2)}(M^{01})^+)} \tag{5.263}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|-\alpha_2\beta_2^*|1\rangle\langle 0|$$

$$-\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.264}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(iY)^+] \tag{5.265}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{01}_{10(H_2)}(iY)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{10(H_2)} = \frac{(M^{10})\rho_{10(H_2)}(M^{10})^+}{tr((M^{10})\rho_{10(H_2)}(M^{10})^+)} \quad (5.266)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1|$$

$$+\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \quad (5.267)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I] \quad (5.268)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{10}_{10(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{10(H_2)} = \frac{(M^{11})\rho_{10(H_2)}(M^{11})^+}{tr((M^{11})\rho_{10(H_2)}(M^{11})^+)} \quad (5.269)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_2|^2(|\alpha_1|^2|0\rangle\langle 0|-\alpha_1\beta_1^*|0\rangle\langle 1|$$

$$-\beta_1\alpha_1^*|1\rangle\langle 0|+|\beta_1|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \quad (5.270)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_2|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\beta_2|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)Z^+] \quad (5.271)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot Z\rho^{11}_{10(H_2)}Z\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m11}}$:

$$\begin{aligned}
\rho_{11(H_2)} &= (H_2)\rho_{m11}(H_2)^+ \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)-|\alpha_1|^2\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|) \\
&-\alpha_1\beta_1^*|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|)+\alpha_1\beta_1^*\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|) \\
&-|\alpha_1|^2\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|)+|\alpha_1|^2|\beta_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|) \\
&+\alpha_1\beta_1^*\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|)-\alpha_1\beta_1^*|\beta_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|) \\
&-\beta_1\alpha_1^*|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|)+\beta_1\alpha_1^*\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|) \\
&+|\beta_1|^2|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)-|\beta_1|^2\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|) \\
&+\beta_1\alpha_1^*\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|)-\beta_1\alpha_1^*|\beta_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|) \\
&-|\beta_1|^2\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|)+|\beta_1|^2|\beta_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)) \\
&+\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\rho^{00}_{11(H_2)} = \frac{(M^{00})\rho_{11(H_2)}(M^{00})^{+}}{tr((M^{00})\rho_{11(H_2)}(M^{00})^{+})} \tag{5.272}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0|-\alpha_2\beta_2^*|0\rangle\langle 1|$$
$$-\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.273}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)Z] \tag{5.274}$$

$$F = Tr\left[|\Phi_1\rangle\langle\Phi_1|\cdot Z\rho^{00}_{1(H_2)}Z\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{11(H_2)} = \frac{(M^{01})\rho_{11(H_2)}(M^{01})^{+}}{tr((M^{01})\rho_{11(H_2)}(M^{01})^{+})} \tag{5.275}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1|$$
$$+\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.276}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I] \tag{5.277}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{01}_{11(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{11(H_2)} = \frac{(M^{10})\rho_{11(H_2)}(M^{10})^{+}}{tr((M^{10})\rho_{11(H_2)}(M^{10})^{+})} \tag{5.278}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1|-\alpha_2\beta_2^*|1\rangle\langle 0|$$
$$-\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.279}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(iY)] \tag{5.280}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{10}_{11(H_2)}(iY)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{11(H_2)} = \frac{(M^{11})\rho_{11(H_2)}(M^{11})^+}{tr((M^{11})\rho_{11(H_2)}(M^{11})^+)} \tag{5.281}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0| + \beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.282}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(X)(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(X^+)] \tag{5.283}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{11}_{11(H_2)}(X)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m12}}$:

$$\begin{aligned}
\rho_{12(H_2)} &= (H_2)\rho_{m12}(H_2)^+ \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)-|\alpha_1|^2\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|) \\
&-\alpha_1\beta_1^*|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|)+\alpha_1\beta_1^*\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|) \\
&+|\alpha_1|^2|\beta_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)-|\alpha_1|^2\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|) \\
&+\alpha_1\beta_1^*\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|)-\alpha_1\beta_1^*|\beta_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|) \\
&-\beta_1\alpha_1^*|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|)+\beta_1\alpha_1^*\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|) \\
&+|\beta_1|^2|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)-|\beta_1|^2\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|) \\
&+\beta_1\alpha_1^*\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|)-\beta_1\alpha_1^*|\beta_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|) \\
&-|\beta_1|^2\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|)+|\beta_1|^2|\beta_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)) \\
&+\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\rho^{00}_{12(H_2)} = \frac{(M^{00})\rho_{12(H_2)}(M^{00})^+}{tr((M^{00})\rho_{12(H_2)}(M^{00})^+)} \tag{5.284}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0| + \beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.285}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(iY)] \tag{5.286}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{00}_{12(H_2)}(iY)\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{12(H_2)} = \frac{(M^{01})\rho_{12(H_2)}(M^{01})^+}{tr((M^{01})\rho_{12(H_2)}(M^{01})^+)} \tag{5.287}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0|$$
$$+\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \tag{5.288}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[X(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)X^+] \tag{5.289}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{01}_{12(H_2)}(X)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{12(H_2)} = \frac{(M^{10})\rho_{12(H_2)}(M^{10})^+}{tr((M^{10})\rho_{12(H_2)}(M^{10})^+)} \tag{5.290}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0|-\alpha_2\beta_2^*|0\rangle\langle 1|$$
$$-\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.291}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)Z^+] \tag{5.292}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(Z)\rho^{10}_{12(H_2)}(Z)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{12(H_2)} = \frac{(M^{11})\rho_{12(H_2)}(M^{11})^+}{tr((M^{11})\rho_{12(H_2)}(M^{11})^+)} \tag{5.293}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1|$$
$$+\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.294}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I] \tag{5.295}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{11}_{12(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

ρ_{m13}:

$$\begin{aligned}
\rho_{13(H_2)} &= (H_2)\rho_{m13}(H_2)^+ \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)+|\alpha_1|^2\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|) \\
&\quad -\alpha_1\beta_1^*|\alpha_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|) \\
&\quad +|\alpha_1|^2|\beta_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)+|\alpha_1|^2\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|) \\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|)-\alpha_1\beta_1^*|\beta_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|) \\
&\quad -\beta_1\alpha_1^*|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)+|\beta_1|^2\alpha_2\beta_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|) \\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|)-\beta_1\alpha_1^*|\beta_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|) \\
&\quad +|\beta_1|^2\beta_2\alpha_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|)+|\beta_1|^2|\beta_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)) \\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\rho^{00}_{13(H_2)} = \frac{(M^{00})\rho_{13(H_2)}(M^{00})^+}{tr((M^{00})\rho_{13(H_2)}(M^{00})^+)} \tag{5.296}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0|+\alpha_2\beta_2^*|0\rangle\langle 1| +\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.297}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I] \tag{5.298}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{00}_{13(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{13(H_2)} = \frac{(M^{01})\rho_{13(H_2)}(M^{01})^+}{tr((M^{01})\rho_{13(H_2)}(M^{01})^+)} \tag{5.299}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0|-\alpha_2\beta_2^*|0\rangle\langle 1| -\beta_2\alpha_2^*|1\rangle\langle 0|+|\beta_2|^2|1\rangle\langle 1|)+\frac{\lambda}{8}I] \tag{5.300}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)Z] \tag{5.301}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(Z)\rho^{01}_{13(H_2)}(Z)\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{13(H_2)} = \frac{(M^{10})\rho_{13(H_2)}(M^{10})^+}{tr((M^{10})\rho_{13(H_2)}(M^{10})^+)} \quad (5.302)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0|$$

$$+\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \quad (5.303)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[X(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)X^+] \quad (5.304)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{10}_{13(H_2)}(X)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{13(H_2)} = \frac{(M^{11})\rho_{13(H_2)}(M^{11})^+}{tr((M^{11})\rho_{13(H_2)}(M^{11})^+)} \quad (5.305)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|-\alpha_2\beta_2^*|1\rangle\langle 0|$$

$$-\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I] \quad (5.306)$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(iY)] \quad (5.307)$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{11}_{13(H_2)}(iY)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

$\boldsymbol{\rho}_{m14}$:

$$\begin{aligned}
\rho_{14(H_2)} &= (H_2)\rho_{m14}(H_2)^+ \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)+|\alpha_1|^2\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|) \\
&-\alpha_1\beta_1^*|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|)-\alpha_1\beta_1^*\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|) \\
&+|\alpha_1|^2|\beta_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)+|\alpha_1|^2\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|) \\
&-\alpha_1\beta_1^*\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|)-\alpha_1\beta_1^*|\beta_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|) \\
&-\beta_1\alpha_1^*|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|) \\
&+|\beta_1|^2|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)+|\beta_1|^2\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|) \\
&-\beta_1\alpha_1^*\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|)-\beta_1\alpha_1^*|\beta_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|) \\
&+|\beta_1|^2\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|)+|\beta_1|^2|\beta_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)) \\
&+\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\rho^{00}_{14(H_2)} = \frac{(M^{00})\rho_{14(H_2)}(M^{00})^{+}}{tr((M^{00})\rho_{14(H_2)}(M^{00})^{+})} \tag{5.308}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\beta_1\right|^2(\left|\alpha_2\right|^2\left|1\right\rangle\left\langle 1\right| + \alpha_2\beta_2^*\left|1\right\rangle\left\langle 0\right|$$

$$+\beta_2\alpha_2^*\left|0\right\rangle\left\langle 1\right| + \left|\beta_2\right|^2\left|0\right\rangle\left\langle 0\right|) + \frac{\lambda}{8}I] \tag{5.309}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{4}}[X(\frac{1-\lambda}{2}\left|\beta_1\right|^2\left|\Phi_2\right\rangle\left\langle\Phi_2\right| + \frac{\lambda}{8}I)X] \tag{5.310}$$

$$F = Tr\left[\left|\Phi_2\right\rangle\left\langle\Phi_2\right|\cdot X\rho^{00}_{14(H_2)}X^{+}\right] = \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{1(H_2)} = \frac{(M^{01})\rho_{14(H_2)}(M^{01})^{+}}{tr((M^{01})\rho_{14(H_2)}(M^{01})^{+})} \tag{5.311}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\beta_1\right|^2(\left|\alpha_2\right|^2\left|1\right\rangle\left\langle 1\right| - \alpha_2\beta_2^*\left|1\right\rangle\left\langle 0\right|$$

$$-\beta_2\alpha_2^*\left|0\right\rangle\left\langle 1\right| + \left|\beta_2\right|^2\left|0\right\rangle\left\langle 0\right|) + \frac{\lambda}{8}I] \tag{5.312}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}\left|\beta_1\right|^2\left|\Phi_2\right\rangle\left\langle\Phi_2\right| + \frac{\lambda}{8}I)(iY)^{+}] \tag{5.313}$$

$$F = Tr\left[\left|\Phi_2\right\rangle\left\langle\Phi_2\right|\cdot (iY)\rho^{01}_{14(H_2)}(iY)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_1\right|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{14(H_2)} = \frac{(M^{10})\rho_{14(H_2)}(M^{10})^{+}}{tr((M^{10})\rho_{14(H_2)}(M^{10})^{+})} \tag{5.314}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_1\right|^2(\left|\alpha_2\right|^2\left|0\right\rangle\left\langle 0\right| + \alpha_2\beta_2^*\left|0\right\rangle\left\langle 1\right|$$

$$+\beta_2\alpha_2^*\left|1\right\rangle\left\langle 0\right| + \left|\beta_2\right|^2\left|1\right\rangle\left\langle 1\right|) + \frac{\lambda}{8}I] \tag{5.315}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_1\right|^2\left|\Phi_2\right\rangle\left\langle\Phi_2\right| + \frac{\lambda}{8}I] \tag{5.316}$$

$$F = Tr\left[\left|\Phi_2\right\rangle\left\langle\Phi_2\right|\cdot\rho^{10}_{14(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\alpha_1\right|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{14(H_2)} = \frac{(M^{11})\rho_{14(H_2)}(M^{11})^+}{tr((M^{11})\rho_{14(H_2)}(M^{11})^+)} \tag{5.317}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0| - \alpha_2\beta_2^*|0\rangle\langle 1|$$

$$-\beta_2\alpha_2^*|1\rangle\langle 0| + |\beta_2|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \tag{5.318}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2 + \frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)Z] \tag{5.319}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(Z)\rho^{11}_{14(H_2)}(Z)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2 + \frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m15}}$:

$$\rho_{15(H_2)} = (H_2)\rho_{m15}(H_2)^+$$

$$= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|101\rangle + |111\rangle)(\langle 101| + \langle 111|) - |\alpha_1|^2\alpha_2\beta_2^*(|101\rangle + |111\rangle)(\langle 100| - \langle 110|)$$

$$+\alpha_1\beta_1^*|\alpha_2|^2(|101\rangle + |111\rangle)(\langle 000| + \langle 010|) - \alpha_1\beta_1^*\alpha_2\beta_2^*(|101\rangle + |111\rangle)(\langle 001| - \langle 011|)$$

$$+|\alpha_1|^2|\beta_2|^2(|100\rangle - |110\rangle)(\langle 100| - \langle 110|) - |\alpha_1|^2\beta_2\alpha_2^*(|100\rangle - |110\rangle)(\langle 101| + \langle 111|)$$

$$-\alpha_1\beta_1^*\beta_2\alpha_2^*(|100\rangle - |110\rangle)(\langle 000| + \langle 010|) + \alpha_1\beta_1^*|\beta_2|^2(|100\rangle - |110\rangle)(\langle 001| - \langle 011|)$$

$$+\beta_1\alpha_1^*|\alpha_2|^2(|000\rangle + |010\rangle)(\langle 101| + \langle 111|) - \beta_1\alpha_1^*\alpha_2\beta_2^*(|000\rangle + |010\rangle)(\langle 100| - \langle 110|)$$

$$+|\beta_1|^2|\alpha_2|^2(|000\rangle + |010\rangle)(\langle 000| + \langle 010|) - |\beta_1|^2\alpha_2\beta_2^*(|000\rangle + |010\rangle)(\langle 001| - \langle 011|)$$

$$-\beta_1\alpha_1^*\beta_2\alpha_2^*(|001\rangle - |011\rangle)(\langle 101| + \langle 111|) + \beta_1\alpha_1^*|\beta_2|^2(|001\rangle - |011\rangle)(\langle 100| - \langle 110|)$$

$$-|\beta_1|^2\beta_2\alpha_2^*(|001\rangle - |011\rangle)(\langle 000| + \langle 010|) + |\beta_1|^2|\beta_2|^2(|001\rangle - |011\rangle)(\langle 001| - \langle 011|))$$

$$+\frac{\lambda}{8}I_8.$$

(1)_

$$\rho^{00}_{15(H_2)} = \frac{(M^{00})\rho_{15(H_2)}(M^{00})^+}{tr((M^{00})\rho_{15(H_2)}(M^{00})^+)} \tag{5.320}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0| - \alpha_2\beta_2^*|0\rangle\langle 1|$$

$$-\beta_2\alpha_2^*|1\rangle\langle 0| + |\beta_2|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \tag{5.321}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2 + \frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)Z] \tag{5.322}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot Z\rho^{00}_{15(H_2)}Z\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2 + \frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{15(H_2)} = \frac{(M^{01})\rho_{15(H_2)}(M^{01})^{+}}{tr((M^{01})\rho_{15(H_2)}(M^{01})^{+})} \tag{5.323}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\beta_1\right|^2(\left|\alpha_2\right|^2 \left|0\right\rangle\left\langle 0\right| + \alpha_2\beta_2^* \left|0\right\rangle\left\langle 1\right|$$
$$+\beta_2\alpha_2^* \left|1\right\rangle\left\langle 0\right| + \left|\beta_2\right|^2 \left|1\right\rangle\left\langle 1\right|) + \frac{\lambda}{8}I] \tag{5.324}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\beta_1\right|^2 \left|\Phi_2\right\rangle\left\langle\Phi_2\right| + \frac{\lambda}{8}I] \tag{5.325}$$

$$F = Tr\left[\left|\Phi_2\right\rangle\left\langle\Phi_2\right| \cdot \rho^{01}_{1(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_1\right|^2 + \frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{15(H_2)} = \frac{(M^{10})\rho_{15(H_2)}(M^{10})^{+}}{tr((M^{10})\rho_{15(H_2)}(M^{10})^{+})} \tag{5.326}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_1\right|^2(\left|\alpha_2\right|^2 \left|1\right\rangle\left\langle 1\right| - \alpha_2\beta_2^* \left|1\right\rangle\left\langle 0\right|$$
$$-\beta_2\alpha_2^* \left|0\right\rangle\left\langle 1\right| + \left|\beta_2\right|^2 \left|0\right\rangle\left\langle 0\right|) + \frac{\lambda}{8}I] \tag{5.327}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2 + \frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}\left|\alpha_1\right|^2 \left|\Phi_2\right\rangle\left\langle\Phi_2\right| + \frac{\lambda}{8}I)(iY)^{+}] \tag{5.328}$$

$$F = Tr\left[\left|\Phi_2\right\rangle\left\langle\Phi_2\right| \cdot (iY)\rho^{10}_{15(H_2)}(iY)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\alpha_1\right|^2 + \frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{15(H_2)} = \frac{(M^{11})\rho_{15(H_2)}(M^{11})^{+}}{tr((M^{11})\rho_{15(H_2)}(M^{11})^{+})} \tag{5.329}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2 + \frac{\lambda}{4}}[\frac{1-\lambda}{2}\left|\alpha_1\right|^2(\left|\alpha_2\right|^2 \left|1\right\rangle\left\langle 1\right| + \alpha_2\beta_2^* \left|1\right\rangle\left\langle 0\right|$$
$$+\beta_2\alpha_2^* \left|0\right\rangle\left\langle 1\right| + \left|\beta_2\right|^2 \left|0\right\rangle\left\langle 0\right|) + \frac{\lambda}{8}I] \tag{5.330}$$

$$= \frac{1}{\frac{1-\lambda}{2}\left|\alpha_1\right|^2 + \frac{\lambda}{4}}[X(\frac{1-\lambda}{2}\left|\alpha_1\right|^2 \left|\Phi_2\right\rangle\left\langle\Phi_2\right| + \frac{\lambda}{8}I)X] \tag{5.331}$$

$$F = Tr\left[\left|\Phi_2\right\rangle\left\langle\Phi_2\right| \cdot (X)\rho^{11}_{15(H_2)}(X)^{+}\right] = \frac{1}{\frac{1-\lambda}{2}\left|\beta_1\right|^2 + \frac{\lambda}{4}}\left[\frac{1-\lambda}{2}\left|\beta_1\right|^2 + \frac{\lambda}{8}\right]$$

$\underline{\boldsymbol{\rho}_{m16}}$:

$$\begin{aligned}
\rho_{_{16(H_2)}} &= (H_2)\rho_{m16}(H_2)^+ \\
&= \frac{(1-\lambda)}{2}(|\alpha_1|^2|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 100|-\langle 110|)-|\alpha_1|^2\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 101|+\langle 111|) \\
&\quad -\alpha_1\beta_1^*|\alpha_2|^2(|100\rangle-|110\rangle)(\langle 001|-\langle 011|)+\alpha_1\beta_1^*\alpha_2\beta_2^*(|100\rangle-|110\rangle)(\langle 000|+\langle 010|) \\
&\quad +|\alpha_1|^2|\beta_2|^2(|101\rangle+|111\rangle)(\langle 101|+\langle 111|)-|\alpha_1|^2\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 100|-\langle 110|) \\
&\quad -\alpha_1\beta_1^*\beta_2\alpha_2^*(|101\rangle+|111\rangle)(\langle 001|-\langle 011|)+\alpha_1\beta_1^*|\beta_2|^2(|101\rangle+|111\rangle)(\langle 000|+\langle 010|) \\
&\quad +\beta_1\alpha_1^*|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 100|-\langle 110|)-\beta_1\alpha_1^*\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 101|+\langle 111|) \\
&\quad +|\beta_1|^2|\alpha_2|^2(|001\rangle-|011\rangle)(\langle 001|-\langle 011|)-|\beta_1|^2\alpha_2\beta_2^*(|001\rangle-|011\rangle)(\langle 000|+\langle 010|) \\
&\quad -\beta_1\alpha_1^*\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 100|-\langle 110|)+\beta_1\alpha_1^*|\beta_2|^2(|000\rangle+|010\rangle)(\langle 101|+\langle 111|) \\
&\quad -|\beta_1|^2\beta_2\alpha_2^*(|000\rangle+|010\rangle)(\langle 001|-\langle 011|)+|\beta_1|^2|\beta_2|^2(|000\rangle+|010\rangle)(\langle 000|+\langle 010|)) \\
&\quad +\frac{\lambda}{8}I_8.
\end{aligned}$$

(1)_

$$\rho^{00}_{_{16(H_2)}} = \frac{(M^{00})\rho_{_{16(H_2)}}(M^{00})^+}{tr((M^{00})\rho_{_{16(H_2)}}(M^{00})^+)} \tag{5.332}$$

$$\begin{aligned}
&= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|1\rangle\langle 1|-\alpha_2\beta_2^*|1\rangle\langle 0| \\
&\quad -\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I]
\end{aligned} \tag{5.333}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[(iY)(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)(iY)] \tag{5.334}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(iY)\rho^{00}_{_{16(H_2)}}(iY)\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(2)_

$$\rho^{01}_{_{16(H_2))}} = \frac{(M^{01})\rho_{_{16(H_2)}}(M^{01})^+}{tr((M^{01})\rho_{_{16(H_2)}}(M^{01})^+)} \tag{5.335}$$

$$\begin{aligned}
&= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|1\rangle\langle 1|+\alpha_2\beta_2^*|1\rangle\langle 0| \\
&\quad +\beta_2\alpha_2^*|0\rangle\langle 1|+|\beta_2|^2|0\rangle\langle 0|)+\frac{\lambda}{8}I]
\end{aligned} \tag{5.336}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[X(\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2|+\frac{\lambda}{8}I)X^+] \tag{5.337}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(X)\rho^{01}_{_{1(H_2)}}(X)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

(3)_

$$\rho^{10}_{1(H_2)} = \frac{(M^{10})\rho_{1(H_2)}(M^{10})^+}{tr((M^{10})\rho_{1(H_2)}(M^{10})^+)} \tag{5.338}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\beta_1|^2(|\alpha_2|^2|0\rangle\langle 0| - \alpha_2\beta_2^*|0\rangle\langle 1|$$

$$-\beta_2\alpha_2^*|1\rangle\langle 0| + |\beta_2|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \tag{5.339}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}[Z(\frac{1-\lambda}{2}|\beta_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I)Z^+] \tag{5.340}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot(Z)\rho^{10}_{1(H_2)}(Z)^+\right] = \frac{1}{\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\beta_1|^2+\frac{\lambda}{8}\right]$$

(4)_

$$\rho^{11}_{16(H_2)} = \frac{(M^{11})\rho_{16(H_2)}(M^{11})^+}{tr((M^{11})\rho_{16(H_2)}(M^{11})^+)} \tag{5.341}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2(|\alpha_2|^2|0\rangle\langle 0| + \alpha_2\beta_2^*|0\rangle\langle 1|$$

$$+\beta_2\alpha_2^*|1\rangle\langle 0| + |\beta_2|^2|1\rangle\langle 1|) + \frac{\lambda}{8}I] \tag{5.342}$$

$$= \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}[\frac{1-\lambda}{2}|\alpha_1|^2|\Phi_2\rangle\langle\Phi_2| + \frac{\lambda}{8}I] \tag{5.343}$$

$$F = Tr\left[|\Phi_2\rangle\langle\Phi_2|\cdot\rho^{11}_{16(H_2)}\right] = \frac{1}{\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{4}}\left[\frac{1-\lambda}{2}|\alpha_1|^2+\frac{\lambda}{8}\right]$$

Discussion:

lets take the expression of the fidelity(5.82)

$$F = \frac{1}{\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}}(\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8})$$

$$\Longrightarrow \left(\frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{4}\right)F = \frac{1-\lambda}{2}|\alpha_2|^2+\frac{\lambda}{8}$$

$$\Longrightarrow |\alpha_2|^2 = \frac{\lambda}{4(1-\lambda)}\left(\frac{1-2F}{F-1}\right)$$

from the condition of normalization we have:

$$0 \leq |\alpha_2|^2 \leq 1 \implies 0 \leq \frac{\lambda}{4(1-\lambda)}\left(\frac{1-2F}{F-1}\right) \leq 1$$

since 0≤ F≤ 1,:

$$0 \leq \frac{\lambda}{4(1-\lambda)} \leq \left(\frac{F-1}{1-2F}\right)$$

the fidelity is maximal F=1 when $\lambda = 0$,(when we don't expect any noise)
The channel is useful for quantum teleportation if: F> $\frac{2}{3}$
or

$$\left(\frac{F-1}{1-2F}\right) < 1$$

so

$$\begin{aligned} 0 \quad &\leq \quad \frac{\lambda}{4(1-\lambda)} \leq \left(\frac{F-1}{1-2F}\right) \prec 1 \\ &\rightarrow \quad 0 \leq \frac{\lambda}{4(1-\lambda)} < 1 \\ &\rightarrow \quad 0 \leq \lambda < 4(1-\lambda) \\ &\Longrightarrow \quad 0 \leq \lambda < \frac{4}{5} \end{aligned}$$

For a classical teleportation: $\frac{1}{2}$ <F≤ $\frac{2}{3}$:

$$\frac{4}{5} \leq \lambda < 1$$

the minimum value of fidelity for a classical teleportation is F=$\frac{1}{2}$,when $\lambda = 1$.

Chapter 6

Fidelity deviation:

6.1 introduction:

For a given protocol of teleportation, the average fidelity represents how well the teleportation, but it says nothing about the fluctuation of the fidelity over all the input states, or which states shows lower fidelity than the mean value, particularly in the case where a noise impinge on the protocols.

We introduce the fidelity deviation D to investigate such property for a two-qubit quantum channel as an example, then we will develop the calculation to derive D for the protocol in the previous chapter trough a five-qubit channel.

6.2 Fidelity deviation for a two-qubit state:

Deriving the fidelity deviation defined as([18])

$$\delta = \sqrt{\langle f_\rho^2 \rangle - \langle f_\rho \rangle^2} \tag{6.1}$$

where

$$\langle f_\rho \rangle = \int f_{\Psi,\rho} d\Psi \quad (\text{ and } \langle f_\rho^2 \rangle = \int f_{\Psi,\rho} d\Psi \ )$$

for a canonical two-qubit density matrix (quantum channel)

$$\rho = \frac{1}{4}(I \otimes I + \mathbf{r} \cdot \sigma \otimes I + I \otimes \mathbf{s} \cdot \sigma + \sum_{i,j} t_{ij} \sigma_i \otimes \sigma_j) \tag{6.2}$$

Where $\mathbf{r}$ and $\mathbf{s}$ are unit vectors in R^3,and the coefficients t_{ij} form a real matrix(3×3), we initially follow the standard protocol of teleportation in([19])

The teleported state represented by the density matrix

$$\rho_\Psi = |\Psi\rangle \langle\Psi| \tag{6.3}$$

the initial state is:

$$\rho_i = \rho_\Psi \otimes \rho \tag{6.4}$$

First, the measurement in the Bell basis$\{|\Psi_k\rangle\, ; k = 0, 1, 2, 3\}$, then after the receiver get the result, he performs the correction operations U_k to obtain the state of the sender:

$$\tilde{\rho}_i = \frac{M_k(\rho_\Psi \otimes \rho)M_k^+}{Tr(M_k(\rho_\Psi \otimes \rho)M_k^+)} \tag{6.5}$$

where

$$M_k = M_k^+ = |\Psi_k\rangle \langle\Psi_k| \otimes I \tag{6.6}$$

the correction:

$$\rho_f = U_K \tilde{\rho}_i U_K^+ \tag{6.7}$$

where

$$\begin{aligned} U_k &= I \otimes I \otimes \sigma_k && (6.8)\\ \sigma_k &\equiv I, \sigma_i && (6.9) \end{aligned}$$

The final state,

$$\begin{aligned} \rho_f &= \frac{1}{P_k}(I \otimes I \otimes \sigma_k)(|\Psi_k\rangle \langle\Psi_k| \otimes I)(\rho_\Psi \otimes \rho)(|\Psi_k\rangle \langle\Psi_k| \otimes I)(I \otimes I \otimes \sigma_k) && (6.10)\\ &= \frac{1}{P_k}(|\Psi_k\rangle \langle\Psi_k| \otimes \sigma_k)(\rho_\Psi \otimes \rho)(|\Psi_k\rangle \langle\Psi_k| \otimes \sigma_k) && (6.11)\\ (P_k &= Tr(M_k(\rho_\Psi \otimes \rho)M_k^+)) \end{aligned}$$

Now, we take the partial trace ς_k over Alice's qubit:

$$\begin{aligned} \varsigma_k &= Tr_{Alice}\rho_f\\ &= \sum_{\grave{k}=0}^{3}(\langle\Psi_{\grave{k}}| \otimes I)\rho_f(|\Psi_{\grave{k}}\rangle \otimes I)\\ &= \sum_{\grave{k}=0}^{3}(\langle\Psi_{\grave{k}}| \otimes I)\frac{1}{P_k}(|\Psi_k\rangle \langle\Psi_k| \otimes \sigma_k)(\rho_\Psi \otimes \rho)(|\Psi_k\rangle \langle\Psi_k| \otimes \sigma_k)(|\Psi_{\grave{k}}\rangle \otimes I)\\ &= \frac{1}{P_k}\sum_{\grave{k}=0}^{3}(\underbrace{\langle\Psi_{\grave{k}}|\,|\Psi_k\rangle} \langle\Psi_k| \otimes \sigma_k)(\rho_\Psi \otimes \rho)(|\Psi_k\rangle \underbrace{\langle\Psi_k|\,|\Psi_{\grave{k}}\rangle} \otimes \sigma_k)\\ &= \frac{1}{P_k}\sum_{\grave{k}=0}^{3}(\delta_{k\grave{k}} \langle\Psi_k| \otimes \sigma_k)(\rho_\Psi \otimes \rho)(|\Psi_k\rangle \delta_{k\grave{k}} \otimes \sigma_k)\\ &= \frac{1}{P_k}(\langle\Psi_k| \otimes \sigma_k)(\rho_\Psi \otimes \rho)(|\Psi_k\rangle \otimes \sigma_k) && (6.12) \end{aligned}$$

In the Block sphere representation:

$$\rho_\Psi = |\Psi\rangle\langle\Psi| = \frac{1}{2}(I + \mathbf{a}\cdot\boldsymbol{\sigma}) \tag{6.13}$$

where $\mathbf{a}$ is an unit vector in R^3

$\rightarrow$

$$\begin{aligned}
|\Psi\rangle\langle\Psi| \otimes \rho &= \frac{1}{2}(I + \mathbf{a}\cdot\boldsymbol{\sigma}) \otimes \frac{1}{4}(I\otimes I + \mathbf{r}\cdot\boldsymbol{\sigma}\otimes I + I\otimes\mathbf{s}\cdot\boldsymbol{\sigma} + \sum_{i,j} t_{ij}\sigma_i\otimes\sigma_j) \\
&= \frac{1}{8}(I\otimes I\otimes I + \mathbf{a}\cdot\boldsymbol{\sigma}\otimes I\otimes I + I\otimes\mathbf{r}\cdot\boldsymbol{\sigma}\otimes I + \mathbf{a}\cdot\boldsymbol{\sigma}\otimes\mathbf{r}\cdot\boldsymbol{\sigma}\otimes I \\
&\quad + I\otimes I\otimes\mathbf{s}\cdot\boldsymbol{\sigma} + \mathbf{a}\cdot\boldsymbol{\sigma}\otimes I\otimes\mathbf{s}\cdot\boldsymbol{\sigma} + I\otimes\sum_{i,j} t_{ij}\sigma_i\otimes\sigma_j \\
&\quad + \mathbf{a}\cdot\boldsymbol{\sigma}\otimes\sum_{i,j} t_{ij}\sigma_i\otimes\sigma_j)
\end{aligned} \tag{6.14}$$

substituting (6.14) in (6.12)$\Longrightarrow$

$$\begin{aligned}
\varsigma_k &= \frac{1}{8P_k}[\langle\Psi_k|I\otimes I|\Psi_k\rangle\otimes\sigma_k I\sigma_k + \langle\Psi_k|\mathbf{a}\cdot\boldsymbol{\sigma}\otimes I|\Psi_k\rangle\otimes\sigma_k I\sigma_k \\
&\quad + \langle\Psi_k|I\otimes\mathbf{r}\cdot\boldsymbol{\sigma}|\Psi_k\rangle\otimes\sigma_k I\sigma_k + \langle\Psi_k|\mathbf{a}\cdot\boldsymbol{\sigma}\otimes\mathbf{r}\cdot\boldsymbol{\sigma}|\Psi_k\rangle\otimes\sigma_k I\sigma_k \\
&\quad + \langle\Psi_k|I\otimes I|\Psi_k\rangle\,\sigma_k(\mathbf{s}\cdot\boldsymbol{\sigma})\sigma_k + \langle\Psi_k|\mathbf{a}\cdot\boldsymbol{\sigma}\otimes I|\Psi_k\rangle\,\sigma_k(s\cdot\sigma)\sigma_k \\
&\quad + \left\langle\Psi_k\left|I\otimes\sum_{i,j} t_{ij}\sigma_i\right|\Psi_k\right\rangle\otimes\sigma_k(\sigma_j)\sigma_k + \left\langle\Psi_k\left|a\cdot\sigma\otimes\sum_{i,j} t_{ij}\sigma_i\right|\Psi_k\right\rangle\otimes\sigma_k(\sigma_j)\sigma_k] \\
&= \frac{1}{8P_k}[I + (\mathbf{a}, T_k\mathbf{r})\cdot I + (O_k^+ s)\sigma + (\mathbf{a}, T_k\mathbf{1}T)(O_k^+\cdot 1)\boldsymbol{\sigma}] \\
&= \frac{1}{8P_k}\left\{[1 + (\mathbf{a}, T_k\mathbf{r})]I + O_k^+(\mathbf{s} + T^+T_k\mathbf{a})\boldsymbol{\sigma}\right\}
\end{aligned} \tag{6.15}$$

Where we used([20])for two vectors $\mathbf{n}$ and $\mathbf{m}$ in R^3**:**

$$\langle\Psi_k|\mathbf{n}\cdot\boldsymbol{\sigma}\otimes\mathbf{m}\cdot\boldsymbol{\sigma}|\Psi_k\rangle = (\mathbf{n}, T_k\mathbf{m})$$

the matrices T_k correspond to the projectors $|\Psi_k\rangle\langle\Psi_k|$ given by :

$$T_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, T_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$\sigma_k(\mathbf{n}.\boldsymbol{\sigma})\sigma_k = (O_k^+\mathbf{n})\boldsymbol{\sigma}$$

we have $Tr(\sigma_i)$=0 , so:

$$\begin{aligned}
\langle\Psi_k|\mathbf{a}\cdot\boldsymbol{\sigma}\otimes I|\Psi_k\rangle &= \sum_i\langle\Psi_k|a_i\cdot\sigma_i\otimes I|\Psi_k\rangle \\
&= \langle\Psi_k|a_x\cdot\sigma_x\otimes I|\Psi_k\rangle + \langle\Psi_k|a_y\cdot\sigma_y\otimes I|\Psi_k\rangle + \langle\Psi_k|a_z\cdot\sigma_z\otimes I|\Psi_k\rangle \\
&= a_xTr(\sigma_x\otimes I) + a_yTr(\sigma_y\otimes I) + a_zTr(\sigma_z\otimes I) \\
&= 0
\end{aligned} \tag{6.16}$$

For k=0,1,2,3:

$$O_k^+ = -T_k \tag{6.17}$$

we substitute in (6.15) :

$$\begin{aligned}
\varsigma_k &= \frac{1}{8P_k}\left\{[1+(\mathbf{a},T_k\mathbf{r})]I+(-T_k)(\mathbf{s}+T^+T_k\mathbf{a})\boldsymbol{\sigma}\right\}\\
&= \frac{1}{8P_k}\left\{[1+(\mathbf{a},T_k\mathbf{r})]I+(-T_k\mathbf{s}-T_kT^+T_k\mathbf{a})\boldsymbol{\sigma}\right\}\\
P_k\varsigma_k &= \frac{1}{8P_k}\left\{[1+(\mathbf{a},T_k\mathbf{r})]I-(T_k\mathbf{s}+T^+\mathbf{a})\boldsymbol{\sigma}\right\}
\end{aligned} \tag{6.18}$$

The fidelity:

$$\begin{aligned}
f &= Tr\left[\left(\sum_{k=0}^{3}P_k\varsigma_k\right)(|\Psi\rangle\langle\Psi|)\right]\\
&= \sum_{k=0}^{3}Tr\left[\frac{1}{8}\left\{[1+(\mathbf{a},T_k\mathbf{r})]I-(T_k\mathbf{s}+T^+\mathbf{a})\boldsymbol{\sigma}\right\}\cdot\frac{1}{2}(I+\mathbf{a}\cdot\boldsymbol{\sigma})\right]\\
&= \frac{1}{16}\sum_{k=0}^{3}Tr\left[I+(\mathbf{a},T_k\mathbf{r})I+\mathbf{a}\cdot\boldsymbol{\sigma}+(\mathbf{a},T_k\mathbf{r})\mathbf{a}\cdot\boldsymbol{\sigma}-(T_k\mathbf{s})\boldsymbol{\sigma}-(T_k\mathbf{s})\boldsymbol{\sigma}(\mathbf{a}\cdot\boldsymbol{\sigma})-(T^+\mathbf{a})\boldsymbol{\sigma}-(T^+\mathbf{a})\boldsymbol{\sigma}(\mathbf{a}\cdot\boldsymbol{\sigma})\right]\\
&= \frac{1}{16}\sum_{k=0}^{3}(2+2(\mathbf{a},T_k\mathbf{r})-(T_k\mathbf{s})\mathbf{a}-2(T^+\mathbf{a})\mathbf{a})\\
&= \frac{1}{8}\sum_{k=0}^{3}\left[1+\mathbf{a}^TT_k(\mathbf{r}-\mathbf{s})-\mathbf{a}^T(T\mathbf{a})\right]
\end{aligned}$$

The average fidelity

(we follow exactly the method used in([18]))

$$\langle f_\rho\rangle = \int f_{\Psi,\rho}d\Psi$$

We use Shur's orthogonality lemma on R^d, [**?**] for every matrix X:

$$\int_G dyO_gXO_g = \frac{1}{d}Tr(X)I_d \quad \text{where} \int_G dg = 1 \tag{6.19}$$

O$_g$ is an irreducible orthogonal representation of any element in the group G.

here the group G is the rotation group O(3) and the vectors **a**$\in R$,

we choose to present the vectors **a** by :**a**=R$_a$**z** , where R$_a$ is the matrix of rotation , and **z** is the unit vector.

Then

$$\begin{aligned}
\langle f_\rho \rangle &= \int f_{\mathbf{a},\rho} d\mathbf{a} = \int \frac{1}{8}\sum_{k=0}^{3}\left[1+\mathbf{a}^T T_k(\mathbf{r}-\mathbf{s})-\mathbf{a}^T(T\mathbf{a})\right] d\mathbf{a} \\
&= \frac{1}{8}\sum_{k=0}^{3}\left[\int d\mathbf{a}+\int \mathbf{a}^T T_k(\mathbf{r}-\mathbf{s})d\mathbf{a}-\int \mathbf{a}^T(T\mathbf{a})d\mathbf{a}\right] \\
&= \frac{1}{8}\sum_{k=0}^{3}\left[1-\frac{1}{3}Tr(T)\right] \\
&= \frac{1}{2}\left[1-\frac{1}{3}Tr(T)\right]
\end{aligned} \tag{6.20}$$

Expression for $\left\langle f_\rho^2 \right\rangle$ **:**

$$f_\rho = \frac{1}{8}\sum_{k=0}^{3}\left[1+\mathbf{a}^T x_k - \mathbf{a}^T(T\mathbf{a})\right] = \frac{1}{2}\left(1-\mathbf{a}^T(T\mathbf{a})\right)+\frac{1}{8}\sum_{k=0}^{3}\mathbf{a}^T x_k$$

$\mathrm{x}_k = T_k(\mathbf{r}-\mathbf{s})$,

$$f_\rho^2 = \left(\frac{1}{2}(1-\mathbf{a}^T T\mathbf{a})+\frac{1}{8}\sum_{k=0}^{3}\mathbf{a}^T x_k\right)^2 = \frac{1}{4}(1-\mathbf{a}^T T\mathbf{a})^2+\frac{1}{64}\sum_{\grave{k},k=0}^{3}\mathbf{a}^T x_k \mathbf{a}^T x_{\grave{k}}+\frac{1}{8}(1-\mathbf{a}^T T\mathbf{a})\sum_{k=0}^{3}\mathbf{a}^T x_k$$

$\Longrightarrow$

$$\begin{aligned}
\left\langle f_\rho^2 \right\rangle &= \int f_{\rho,\mathbf{a}}^2 d\mathbf{a} = \int \left(\frac{1}{4}(1-\mathbf{a}^T T\mathbf{a})^2+\frac{1}{64}\sum_{\grave{k},k=0}^{3}\mathbf{a}^T x_k \mathbf{a}^T x_{\grave{k}}+\frac{1}{8}(1-\mathbf{a}^T T\mathbf{a})\sum_{k=0}^{3}\mathbf{a}^T x_k\right) d\mathbf{a} \\
&= \int \left(\frac{1}{4}(1-\mathbf{a}^T T\mathbf{a})^2\right) d\mathbf{a} = \frac{1}{4}\int \left[1+\left(\mathbf{a}^T T\mathbf{a}\right)\left(\mathbf{a}^T T\mathbf{a}\right)-2\mathbf{a}^T T\mathbf{a}\right]
\end{aligned} \tag{6.21}$$

we can write [?]

$$\left(\mathbf{a}^T T\mathbf{a}\right)\left(\mathbf{a}^T T\mathbf{a}\right) = \left(\mathbf{a}^T\otimes\mathbf{a}^T\right)(T\otimes T)(\mathbf{a}\otimes\mathbf{a})$$

so

$$\left\langle f_\rho^2 \right\rangle = \int \left(\frac{1}{4}(1-\mathbf{a}^T T\mathbf{a})^2\right) d\mathbf{a} = \frac{1}{4}\int \left[1+\left(\mathbf{a}^T\otimes\mathbf{a}^T\right)(T\otimes T)(\mathbf{a}\otimes\mathbf{a})-2\mathbf{a}^T T\mathbf{a}\right] d\mathbf{a} \tag{6.22}$$

using the generalization of Schur's lemma for a matrix X on $\mathrm{R}^d\otimes R^d$:

$$\int_G dg(O_g\otimes O_g)X(O_g^T\otimes O_g^T) = AI+BD+CP$$

Where I is the identity, D=$\left(\sum_{i=0}^{d-1}\mathbf{x}_i\otimes\mathbf{x}_i\right)\left(\sum_{j=0}^{d-1}\mathbf{x}_j\otimes\mathbf{x}_j\right)$, ($\{\mathbf{x}_i\}$: is an orthonormal basis of R^d), and P=$\sum_{i,j=0}^{d-1}\left(\mathbf{x}_i\otimes\mathbf{x}_i\right)\left(\mathbf{x}_i\otimes\mathbf{x}_i\right)^T$

The coefficients A,B ,C are given by:

$$A = \frac{(d+1)Tr(X) - Tr(XD) - Tr(XP)}{d(d-1)(d-2)}$$

$$B = \frac{-Tr(X) + (d+1)Tr(XD) - Tr(XP)}{d(d-1)(d-2)}$$

$$C = \frac{-Tr(X) - Tr(XD) + (d+1)Tr(XP)}{d(d-1)(d-2)}$$

so we can evaluate the integral(6.22) :

$$\begin{aligned}\int \left[1 - 2\mathbf{a}^T T\mathbf{a}\right] d\mathbf{a} &= 1 - \frac{2}{3}Tr(T) \ , \left(\int d\mathbf{a} = 1\right) \\ \int \left(\mathbf{a}^T \otimes \mathbf{a}^T\right)(T \otimes T)(\mathbf{a} \otimes \mathbf{a})\, d\mathbf{a} &= \frac{1}{15}\left[Tr(T \otimes T) + Tr(T \otimes TD) + (Tr(T \otimes TP)\right] \\ &= \frac{1}{15}(TrT)^2 + Tr(TT^+) + TrT^2)\end{aligned}$$

$\Longrightarrow$

$$\left\langle f_\rho^2 \right\rangle = \frac{1}{4}\left[1 - \frac{2}{3}Tr(T) + \frac{1}{15}(TrT)^2 + Tr(TT^+) + TrT^2)\right] \tag{6.23}$$

Fidelity deviation:

$$\begin{aligned}\delta_\varrho &= \sqrt{\langle f_\rho^2 \rangle - \langle f_\rho \rangle^2} = \sqrt{\frac{1}{4}\left[1 - \frac{2}{3}Tr(T) + \frac{1}{15}(TrT)^2 + Tr(TT^+) + TrT^2)\right] - \left[\frac{1}{2}\left(1 - \frac{1}{3}Tr(T)\right)\right]^2} \\ &= \sqrt{\frac{1}{4}\left[1 - \frac{2}{3}Tr(T) + \frac{1}{15}(TrT)^2 + Tr(TT^+) + TrT^2)\right] - \left[\frac{1}{4}\left(1 - \frac{2}{3}Tr(T) + \frac{1}{9}(TrT)^2\right)\right]}\end{aligned}$$

$$\Longrightarrow \quad \delta_\varrho = \frac{1}{\sqrt{30}}\sqrt{Tr(T)^2 - \frac{1}{3}(TrT)^2} \tag{6.24}$$

the corresponding T is diagonal, with eigenvalues,λ_i $|t_{ii}|$, i = 1, 2, 3, where $\lambda_i \in$ {-1,+1}. In particular, λ_i = -1 for $t_{ii} = 0$, i = 1, 2, 3,

if det T≤ 0; λ_i, λ_j = -1, λ_k = +1 for any choice of i = j = k $\in$ {1, 2, 3} satisfying$|t_{ii}| \geq |t_{jj}| \geq |t_{kk}|$, if det T > 0.

6.3 Fidelity deviation for a five-qubit state:

Since the protocol is a transmission of a qubit state, we consider the protocol in the previous chapter, of shwitched controlled teleportation, with noisy channel(second section) we will follow the same method of deriving the fidelity deviation for a two-qubit state.

First, the protocol to find the expression of the average fidelity f.

The five-qubit density matrix ρ :

$$\begin{aligned}\rho_C &= (1-\lambda)\left|C\right\rangle\left\langle C\right| + \frac{\lambda}{32}I_{32}\\ &= \frac{1}{4}(1-\lambda)(\left|00000\right\rangle\left\langle 00000\right| + \left|00000\right\rangle\left\langle 01011\right| + \left|00000\right\rangle\left\langle 10101\right| + \left|00000\right\rangle\left\langle 11110\right| + \left|01011\right\rangle\left\langle 00000\right|\\ &+ \left|01011\right\rangle\left\langle 01011\right| + \left|01011\right\rangle\left\langle 10101\right| + \left|01011\right\rangle\left\langle 11110\right| + \left|10101\right\rangle\left\langle 00000\right| + \left|10101\right\rangle\left\langle 01011\right|\\ &+ \left|10101\right\rangle\left\langle 10101\right| + \left|10101\right\rangle\left\langle 11110\right| + \left|11110\right\rangle\left\langle 00000\right| + \left|11110\right\rangle\left\langle 01011\right| + \left|11110\right\rangle\left\langle 10101\right|\\ &+ \left|11110\right\rangle\left\langle 11110\right|) + \frac{\lambda}{32}I_{32} \qquad (6.25)\end{aligned}$$

The two teleported states of Alice1 and Alice2 in the Block representation:

$$\begin{aligned}\left|\Phi_1\right\rangle\left\langle\Phi_1\right| &= \frac{1}{2}(I + \mathbf{a}_1\boldsymbol{\sigma})\\ \left|\Phi_2\right\rangle\left\langle\Phi_2\right| &= \frac{1}{2}(I + \mathbf{a}_2\boldsymbol{\sigma})\end{aligned}$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are unit vectors in R^3.

the general state of the system:

$$\rho = \left|\Phi_1\right\rangle\left\langle\Phi_1\right| \otimes \left|\Phi_2\right\rangle\left\langle\Phi_2\right| \otimes \rho_C \qquad (6.26)$$

The measurements of Alice(1 and 2):

After the measurement :

$$\rho \rightarrow \rho_1 = \frac{M_1\rho M_1^+}{Tr\left(M_1\rho M_1^+\right)}, \quad M_1 = M_1^+ = \left|\Psi_k\right\rangle\left\langle\Psi_k\right| \otimes \left|\Psi_m\right\rangle\left\langle\Psi_m\right| \otimes I \otimes I \otimes I \qquad (6.27)$$

($\Psi_k \equiv B_{xy}$,k=0,1,2,3 correspond to Alice1 , m=0,1,2,3 correspond to alice2)

define $\mathrm{p}_1 = Tr\left(M_1\rho M_1^+\right)$,

$$\rho_1 = \frac{1}{p_1}(\left|\Psi_k\right\rangle\left\langle\Psi_k\right| \otimes \left|\Psi_m\right\rangle\left\langle\Psi_m\right| \otimes I \otimes I \otimes I)\boldsymbol{\rho}(\left|\Psi_k\right\rangle\left\langle\Psi_k\right| \otimes \left|\Psi_m\right\rangle\left\langle\Psi_m\right| \otimes I \otimes I \otimes I)$$

we take the partial trace over Alice1 and Alice2:

$$
\begin{aligned}
Tr_A(\rho_1) &= \sum_{\substack{\acute{m}=0\\ \acute{k}=0}}^{3} \left(\langle\Psi_{\acute{k}}| \otimes \langle\Psi_{\acute{m}}| \otimes I \otimes I \otimes I\right) \rho_1(|\Psi_{\acute{k}}\rangle \otimes |\Psi_{\acute{m}}\rangle \otimes I \otimes I \otimes I) \\
&= \frac{1}{p_1}(\langle\Psi_k| \otimes \langle\Psi_m| \otimes I \otimes I \otimes I)\boldsymbol{\rho}(|\Psi_k\rangle \otimes |\Psi_m\rangle \otimes I \otimes I \otimes I) \\
&= \frac{1}{4p_1}[\frac{1-\lambda}{4}(|000\rangle\langle 000| + |011\rangle\langle 011| + |101\rangle\langle 101| + |110\rangle\langle 110|) \\
&\quad +(-1)^{x_m}(a_1)_x(|000\rangle\langle 011| + |011\rangle\langle 000| + |101\rangle\langle 110| + |110\rangle\langle 101|) \\
&\quad +(-1)^{x_k}(a_2)_x(|000\rangle\langle 101| + |011\rangle\langle 110| + |101\rangle\langle 000| + |110\rangle\langle 011|) \\
&\quad +(-1)^{x_k+x_m}(a_1)_x(a_2)_x(|000\rangle\langle 110| + |011\rangle\langle 101| + |101\rangle\langle 011| + |110\rangle\langle 000|) \\
&\quad +\frac{\lambda}{8}I_8] \qquad (6.28)
\end{aligned}
$$

where :

$$
\begin{aligned}
\langle\Psi_k| I \otimes |0\rangle\langle 0| |\Psi_k\rangle &= \frac{1}{2}((\langle 0y| + (-1)^x\langle 1\bar{y}|)I \otimes |0\rangle\langle 0|(|0y\rangle + (-1)^x|1\bar{y}\rangle) = 1 \\
\langle\Psi_k| I \otimes |1\rangle\langle 1| |\Psi_k\rangle &= 1 \\
\langle\Psi_k| I \otimes |0\rangle\langle 1| |\Psi_k\rangle &= \langle\Psi_k| I \otimes |1\rangle\langle 0| |\Psi_k\rangle = 0 \\
\langle\Psi_k| \mathbf{a}\boldsymbol{\sigma} \otimes |0\rangle\langle 0| |\Psi_k\rangle &= \langle\Psi_k| \mathbf{a}\boldsymbol{\sigma} \otimes |1\rangle\langle 1| |\Psi_k\rangle = 0 \\
\langle\Psi_k| \mathbf{a}\boldsymbol{\sigma} \otimes |0\rangle\langle 1| |\Psi_k\rangle &= \langle\Psi_k| \mathbf{a}\boldsymbol{\sigma} \otimes |1\rangle\langle 0| |\Psi_k\rangle = (-1)^x a_x
\end{aligned}
$$

The controller's operations:

in the case where the teleported state is Alice1's state,

The operation H on the qubit C_3 the density matrix defined by $Tr_A(\rho_1)$:

$$\rho_2 = (H \otimes I \otimes I)Tr_A(\rho_1)(H \otimes I \otimes I)$$

Measurement of Charlie in the Z basis

After measurement

$$\rho_2 \implies \tilde{\rho}_2 = \frac{M_2\rho_2 M}{Tr\,(M_2\rho_2 M)} \quad \text{with } M_2 = (|ab\rangle\langle ab| \otimes I); a = 0,1; b = 0,1$$

Bob performs the corresponding unitary operation σ_{ab}:

$$\tilde{\rho}_B = \frac{1}{p_{ab}}(|ab\rangle\langle ab| \otimes \sigma_{ab})\rho_2(|ab\rangle\langle ab| \otimes \sigma_{ab})$$

where $p_{ab} = Tr\,(M_2\rho_2 M)$

now we take the partial trace over Charlie's qubits:

$$\begin{aligned}
\varsigma_{ab} &= Tr_C(\rho_2) = \sum_{\substack{\grave{a}=0\\ \grave{b}=0}}^{1} \left(\langle \grave{a}| \left\langle \grave{b} \right| \otimes I \right) \tilde{\rho}_B \left(|\grave{a}\rangle \left| \grave{b} \right\rangle \otimes I \right) \\
&= \frac{1}{p_{ab}} (\langle ab| \otimes \sigma_{ab}) \rho_2 (|ab\rangle \otimes \sigma_{ab})
\end{aligned}$$

the fidelity defined by:

$$\begin{aligned}
f_{\psi,\rho_c} &= Tr\left[\left(\sum_{a,b} p_{ab}\varsigma_{ab} \right) |\Phi_1\rangle \langle \Phi_1| \right] \\
&= Tr\left[\left(\sum_{a,b} (\langle ab| \otimes \sigma_{ab}) \rho_2 (|ab\rangle \otimes \sigma_{ab}) \right) \left(\frac{1}{2}(I + \mathbf{a}_1 \boldsymbol{\sigma}) \right) \right] \qquad (6.29)
\end{aligned}$$

we need to find an expression for $\left\langle f_{\rho_c} \right\rangle$ and for $\left\langle f_{\rho_c}^2 \right\rangle$ to obtain:

$$\delta_{\rho_c} = \sqrt{\left\langle f_{\rho_c}^2 \right\rangle - \left\langle f_{\rho_c} \right\rangle^2} \qquad (6.30)$$

Chapter 7

General conclusion

Among the applications of quantum physics in the communication and information theory, the quantum teleportation, which is one of the branches of quantum information theory that allows for the transmition of quantum information to a distatnce location, its essence lies in the non-local correlations or the quantum entanglement.

In the first chapters of this thesis, we have considered the quantum computing in different protocols of bidirectional teleportation with and without control. These laters contain calculations of protocols, already recently published in international journals, which consist in mastering the calculations technique of the protocols.

Then in the next chapter, we have considered a new scheme of switched controlled teleportation, which we have its implementation in the platform of Dr Kh.Khelfaoui, and it is the original protocol proposed by the platform (automatic research). The originality of the this chapter is an analytic and exact verification of the proposed protocol which confirms and supports the reliability of the calculations made by the plateform. Furthermore, a complementary computation cantainning a noise in the entanglement channel is presented, and the fidelity corresponding to the protocol is given.

In the last chapter, we started an original work with the aim of deriving the fidelity and fidelity deviation recently proposed in published literature, for the noisy channel of the protocol of Dr Khelfaoui, but due to the complexity and the requirement of more calculations, the work will be completed incessantly and hopefully it will lead us to meaninful results.

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