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Cosmology of a Class Models in Teleparallel Gravity

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Contents

Chapter 1

Introduction

It is probably the greatest scientific discovery. This is how Dirac, the pioneer of quantum mechanics and one of the greatest symbols of physics in the last century, described the beauty of relativity. The concept of relativity is not new, as it dates back to the days of Galileo and Newton. One of the most important concepts that fall under this concept is what is known as space-time, which is simply the distance between two points or two events where we enter time as a fourth dimension in it. It was introduced by Hermann Minkowski, Professor of Einstein, and it became known as the 4-D Minkowski space. Since Maxwell discovered his mathematical equations that unify electricity and magnetism in a theory of electromagnetism, problems heaped on the doors of physics, but in1904 Hendrik Lorentz came with his transformations that solve these problems. However biggest impact of these solutions was the share of time and space. Thanks to these transformations, and after mountainous efforts, Poincaré, Lorentz and Einstein reached a theory known as the theory special relativity based on two important principles. As long as relativity prolongs the concepts of time and space, then a new structure must be rebuilt, and this is what Minkowski did. Einstein later realized that this theory is incomplete, and it is only a special case of a more general and deep theory. Einstein was convinced that the special relativity theory was incomplete, and though that it is nothing but a part from more general and deeper equations. Since it applies to a flat space only , it needs to be modified. This happened in 1915 when he developed his general theory of gravity, wich completely changed the concept of gravity. This new theory replaced the older "Newton's theory of Gravitation", and it remains to this day our best descrition of how gravity works. Gravity here is nothing but the free fall of objects on curved surfaces, which are the curved surfaces of Galileo. It was a successful theory due to its ability to explain many of the things that Newton theory didn't, like the anomalous precession of the prehelion of Mercury, the deflection of light by gravitational fields, and the gravitational red-shift of light with an unpecedented precision. In addition, general relativityu theory is considered as the corner stone of cosmology, wich is a new science born in the late of the sixteen. The beginig of this new science started with the discovery that the earth is not at the centre of the universe. Trying to give a sence to cosmology, the science of our whole universe's past, present and futur, physicists need to understand the universe expantion rate. The evidence of this expantion was given in 1998 from studying type Ia Supernoae (more details can be found in [1, 2]). Cosmologists since Hublle had been trying to measure the slowing expantion of our universe as they predicted due the gravity effects. The discovery of cosmic acceleration creat another challenge to find the reason of this acceleration. The simplest explanation that they found was to a cosmological constant to Einstein equations. If we consider the universe as filled with ordinary matter or radiation this should lead us to a slowing of the expantion. This fact lead them to take two posibilties, the first is that there is another form of energy density in the universe they named as Dark enrgy and it take 75% of the universe content, the other one is that the general relativity theory breaks down on cosmological scales and must be modified with a more complete theory of gravity. They knew that a mysterious substance called dark energy is causing the universe to expand (at an ever increasing rate) in all directions. When astronomers point thier telescopes into space to measure the Hubble constant, the number that describes how fast the universe expanding at diffrent distances from us, they come up with disagreement in their findings compared to the theory.

Several issues and shortcomings emerged in the last thirty years leading to the conclusion that Einstein's General Relativity is not the final theory of gravitational interaction. The goal of this master thesis is to combine two majors theories of cosmology, the galileon field model and a modified teleparallel theory of general relativity to test the cosmological evolution of the universe.

The concept of the teleparallelism was first proposed by Einstein to unify gravity and electromagnetism into a unified field theory in 1928 [1]. Unlike general relativity (GR) in which the Levi-Civita connection gives rise to curvature but vanishing torsion, in teleparallelism space-time is endowed with a connection with vanishing curvature, but nonzero torsion. Since the curvature is identically zero, parallel transport of a vector is independent of the path. This is the origin of the name teleparallel, which means "parallel at a distance" (Einstein's quest for unified field theory via teleparallelism is an interesting history and can be found in [2]). It has since been established that GR can in fact be re-cast into teleparallel language [3–6], known as the Teleparallel Equivalent of General Relativity (TEGR). For an interesting formulation of TEGR as higher gauge theory, see [7]. Due to the need to understand the acceleration of the universe, various theories of modified gravity have been introduced, among which is the attempt to generalize TEGR to $f(T)$theory of modified gravity in the same spirit as generalizing general relativity to $f(R)$ gravity.

Given the spatially flat FLRW (Friedmann-Lemaıtre-Robertson-Walker space time) with suitable choice of the scale factor that describes an accelerating universe, allow an understanding of the evolution of structures on the FLRW background. This has been discussed from the observational point of view in [10]. On the theoretical level, perturbation theory is a useful tool for understanding the evolution of structures as it can reveal some properties of the dynamical modes of a gravitational theory. In the case of $f(T)$ theory, Li et. al [11] have shown that there are generically, 5 degrees of freedom in $f(T)$ gravity. Comparing with GR, which has only 2 degrees of freedom, there are 3 extra degrees of freedom, which the authors suggest could correspond to either one massive vector field or one massless vector field together with one scalar field. Due to the high symmetry of FLRW metric, there is no extra degree of freedom at the background level. This corresponds to the fact that the equations to solve for the background have the same number of initial conditions as those of GR. Note that from the Hamiltonian perspective FLRW geometry has no dynamical degrees of freedom since the two degrees of freedom of gravitational waves are not excited in an isotropic universe. Indeed the only dynamics for FLRW universe is its expansion (or contraction). That is, the dynamics is completely determined by the Hubble parameter. This is the case in both GR and f(T) gravity.

Chapter 2

Introduction to general relativity

2.1 General relativity

For a hundred years the general theory of relativity has been a pillar of modern physics. It is a dynamical theory of space-time. The basic idea of it is so elegant and easy to understand. GR theory is simply a set of physical and geometric principles, wich lead us to a set of field equations that determine the gravitational field, and to the geodesic equation that descripe the propagation of light and the motion of particles on the background.

- GR follows from three postulates :
    1. Space time is a 4-dimensional differentiable manifold.
    2. Einstein's principle of Equivalence.
    3. Einstein's equations.

The first postulat means that a 4-dimentional manifold is a topological space that locally looks like euclidian space with all of its usual topology. In other words each point of an n-dimentional manifold has a neighbourhood that is homeomorphic to euclidian space of n-dimensions.

The principle of equivalence that was mentioned in the second postulate, is one of the most important ingredients of GR theory. It was introduced by Eienstein, and it is actually put in two seperate statements: "The Laws of physics in a gravitational frame are equivalent to those in an accelerating frame" and "The Laws of physics in a non accelerating, or free fall, frame are locally those of special relativity". This principle meant that gravity could be reinterperted as a curvature of spacetime. Einstein's principle of equivalence is only half the story. Because he determines how particles must move in spacetime of given curvature, but it doesn't determine how spacetime is itself curved by mass. That was a much more difficult problem. The eventual solution was Einstein's equations.

2.2 Mathematical structure of general relativity

Since GR is a theory based on the assumption of curved spacetime caused by matter and or energy, Euclidean geometry is not sufficient to fully describe the theory, we need a more general kind of geometry allowing non flat euclidean spaces. This section is to define some basic mathematical quantities used in GR theory.

2.2.1 Space time metric

In order to mesure angles between curves and distences between points in non trivial space-times, one needs to define the "metric tensor". It is a rank 2 symmetric tensor defined on a smooth manifold, labelled as $g_{\mu\nu}$ via the associated line element

$$ds^2 = g_{\mu\nu}(x) dx^{\mu} dx^{\nu}. \tag{2.1}$$

The quantity $g_{\mu\nu}$ is called the metric space, and it contains all the information we need to describe the place or the curved surface. The metric is required to be nondegenerate with signature (- + + +). In general it depends on coordinates, and one finds that under a coordinate transformation $x^{\mu} \to y^{\mu}$ the metric transforms as:

$$g_{\alpha\beta}(x) = g_{\mu\nu}(x) \frac{\partial x^{\mu}}{\partial y^{\alpha}} \frac{\partial x^{\nu}}{\partial y^{\beta}}. \tag{2.2}$$

In GR the world line of a particle free from all external non gravitational forces is a particular type of geodesic in curved space-time, in other words , a falling particle always moves along a geodesic given by the equation

$$\frac{dx^{2\mu}}{d\tau^2} + \Gamma^{\mu}_{K\lambda} \frac{dx^k}{d\tau} \frac{dx^{\lambda}}{d\tau}, \tag{2.3}$$

where

$$\Gamma^{\mu}_{k\lambda} = \frac{1}{2} g^{\mu\nu} (\partial_\lambda g_{\nu k} + \partial g_{\lambda k} - \partial_\nu g_{k\lambda}), \tag{2.4}$$

are connection coefficients dependent on coordinates called the Christoffel symbols. By construction, they are symmetric in their lower indices. This connection $\Gamma^{\mu}_{k\lambda}$ is compatible with the metric $g_{\mu\nu}$ if the covariant derivative of the metric tensor is identically zero

$$\nabla_{;k} g_{\mu\nu} = 0, \tag{2.5}$$

To define a covariant derivative, we need to put a connection on a manifold, wich is specified in some coordinate system by a set of coefficients $\Gamma^{\mu}_{k\lambda}$ ($n^3 = 64$ independents components in n=4 dim) wich transform according to the the law

$$\Gamma^{\mu'}_{k'\lambda'} = \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\mu'}}{\partial x^\mu} \Gamma^{\mu}_{k\lambda} - \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{2\mu'}}{\partial x^\lambda}. \tag{2.6}$$

In general the connection symbols does not necessarly depend on the metric. The connection symbols encode all of the information necessary to take into the covariant derivative of a tensor of arbtrary rank. The formula giving the covariant derivative of a general tensor is given by

$$\begin{aligned} \nabla_{;\sigma} T^{\mu_1\mu_2..\mu_K}_{\nu_1\nu_2..\nu_l} = \partial_\sigma T^{\mu_1\mu_2..\mu_K}_{\nu_1\nu_2..\nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2..\mu_K}_{\nu_1\nu_2..\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda..\mu_K}_{\nu_1\nu_2..\nu_l} \\ - \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2..\mu_K}_{\lambda\nu_2..\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2..\mu_K}_{\nu_1\lambda..\nu_l}. \end{aligned} \tag{2.7}$$

In standard flat spacetimes, a vector ϑ^i remains constant along a line if it satisfies

$$\frac{d\vartheta^i}{d\lambda} = 0 \tag{2.8}$$

where λ is the affine parameter used to characterise the curve. Since GR is based on curved space-time, the notion of parallel transporting vectors along a curve needs to be changed. In this context, one needs to introduce the "parallel transportation" that allows to define such kind of transformation. In that we said that a vector is parallel transported along a curve if its covariant derivative vanishes

$$\nabla_{;\sigma} v^\rho = 0. \tag{2.9}$$

An arbitrary tensor of rank $\begin{pmatrix} k \\ l \end{pmatrix}$ is parallel transported along a curve if its covariant derivative vanishes

$$\nabla_\sigma T^{\mu_1\mu_2..\mu_K}_{\nu_1\nu_2..\nu_l} = 0. \tag{2.10}$$

2.3 Einstein's Equations

Before deriving Einstein equations we first introduce the essentials building blocks of these equations.

2.3.1 Riemann curvature tensor

The conection $\Gamma^\mu_{k\lambda}$ measure an apparent gravitational field wich can always be canceled locally by a coordinate transformation. How to mesure a real gravitational field (wich can not be canceled every where)?. The answer is given by considering the geodesic deviation between two test particles wich follow two neighboring geodesics Γ and $\delta\Gamma$. The geodesic is defined by:

$$\frac{Dv^\mu}{D\lambda} = v^\nu D_\nu v^\mu = 0. \tag{2.11}$$

where $v^\mu = \partial_\lambda x^\mu$ is the four velocity and λ an affine parameter. The neighboring geodesic obeys the same equation where the spacetime point x^μ is replaced by $x^\mu + \delta X^\mu$. In The first order, the difference between the equations of the two geodesics is therfore :

$$0 = \delta(v^\gamma D_\gamma v^\mu) = \delta x^\lambda (v^\gamma D_\gamma v^\mu). \tag{2.12}$$

Let us define the curvature tensor, or Riemann tensor $R^{\mu_{\rho\lambda\nu}}$ by:

$$[D_\lambda, D_\nu]A^\mu \equiv R^\mu_{\rho\lambda\nu} A^\rho, \tag{2.13}$$

we see that the Eq (2.11) takes the newtonian form relating the acceleration to the force

$$\frac{D^2\delta x^\mu}{D\tau^2} = R^\mu_{\rho\nu\lambda} U^\rho U^\nu \delta x^\lambda. \tag{2.14}$$

Using the definitions of the covariante derivatives of a vector and of a mixed tensor, we obtain the components of the Riemann tensor:

$$R^{\mu}_{\rho\nu\lambda} = \partial_\nu \Gamma^{\mu}_{\rho\lambda} - \partial_\lambda \Gamma^{\mu}_{\rho\nu} + \Gamma^{\mu}_{\sigma\nu}\Gamma^{\sigma}_{\rho\lambda} - \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\sigma}_{\rho\nu}. \tag{2.15}$$

This tensor satisfies the following Symmetry properties:
By construction, the Riemann tensor is a type tensor (1 ,3) which is anti-symmetric in its last two indices

$$R^{\mu}_{\rho\nu\lambda} = -R^{\mu}_{\rho\lambda\nu}. \tag{2.16}$$

Two other properties are (i) the antisymmetry of the first two lowered indices:

$$R_{\mu\rho\nu\lambda} = R_{\rho\mu\nu\lambda}. \tag{2.17}$$

and (ii) the symmetry by permutation of the first and second pairs of indices

$$R_{\mu\rho\nu\lambda} = R_{\nu\lambda\mu\rho}. \tag{2.18}$$

cyclic symmetry on the last three indices

$$R^{\mu}_{\rho\nu\lambda} + R^{\mu}_{\nu\lambda\rho} + R^{\mu}_{\lambda\rho\nu} = 0. \tag{2.19}$$

All these symmetries reduce the number of independent components of the Riemann tensor to 20 (instead of$4^4 = 256$). Finally, the covariant derivatives of the Riemann tensor verify a very important property, called Bianchi identity.

2.3.2 Einstein's Equations

Einstein's equation are the fondamental equations of GR, they describe how the mass, energy, momentum and pressure are distributed throughout the universe, as well as how they bend space-time. It is this curvature of spacetime that influences how mater, energy are spread as illustrated by John Wheeler citation "Space-time tells matter how to move, Matter tells space-time how to curve". Einstein's field equations can be derived from Einstein's Hilbert Action as we're going to see below.

We introduce the Einstein Hilbert action

$$S_{GR} = \int dx^4 \sqrt{-g} \left(\frac{1}{2\kappa}(R - 2\wedge) + L_m \right), \tag{2.20}$$

where $R = R_{ab}g^{ab}$, and R_{ab} is the Ricci tensor and R is the Ricci scalar and

$$g = \det(g_{\mu\nu}). \tag{2.21}$$

The $\wedge$ term is what called the cosmological constant, we will discuss it later, and L_m is the Lagrangian density of matter.

According to the principle of stationary action variation:

$$\delta S_{GR} = \delta \int dx^4 \sqrt{-g} \left(\frac{1}{2\kappa}(R - 2\wedge) + L_m \right) = 0, \tag{2.22}$$

we obtain

$$\int [\frac{1}{2k} \frac{\delta(R\sqrt{-g})}{\delta g^{\mu\nu}} - \frac{\wedge}{k} \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} + \frac{\delta(L_m\sqrt{-g})}{\delta g^{\mu\nu}}] \delta g^{\mu\nu} dx^4 = 0, \tag{2.23}$$

or

$$\int \left(\frac{\delta R}{\delta g^{\mu\nu}} - \frac{R}{2(-g)} \frac{\delta g}{\delta g^{\mu\nu}} + \frac{\wedge}{-g} \frac{\delta g}{\delta g^{\mu\nu}} + 2\kappa [\frac{\delta L_m}{\delta g^{\mu\nu}} - \frac{L_m}{2(-g)} \frac{\delta g}{\delta g^{\mu\nu}} \right) \sqrt{-g} \delta g^{\mu\nu} dx^4 = 0., \tag{2.24}$$

For a non degenerate matrix A we use a formula for differentiation of determinants :

$$\frac{\delta \det[A]}{\delta A_{ij}} = \det[A][A^{-1}], \tag{2.25}$$

which allows us to derive

$$g^{\mu\nu} \delta g_{\mu\nu} = -g_{\mu\nu} \delta g^{\mu\nu}, \tag{2.26}$$

$$\delta g_{ab} = -g_{\mu b} \delta g^{\mu\nu}) g, \tag{2.27}$$

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu}, \tag{2.28}$$

Substituting in the formula (2.24) we get :

$$\int \left(\frac{\delta R}{\delta g^{\mu\nu}} - \frac{R}{2} g_{\mu\nu} + \wedge g_{\mu\nu} - \kappa \left(-2 \frac{\delta L_m}{\delta g^{\mu\nu}} + L_m g_{\mu\nu} \right) \right) \delta g^{\mu\nu} dx^4 = 0, \tag{2.29}$$

We identify the terms involving the non gravitational Lagrangian as the matter source tensor

$$-2 \frac{\delta L_m}{\delta g^{\mu\nu}} + L_m g_{\mu\nu} = T_{\mu\nu}, \tag{2.30}$$

$$\int \left(\frac{\delta R}{\delta g^{\mu\nu}} - \frac{1}{2} R g_{\mu\nu} + \wedge g^{\mu\nu} - \kappa T_{\mu\nu} \right) \delta g^{\mu\nu} dx^4 = 0, \tag{2.31}$$

It remains to calculate $\frac{\delta R}{\delta g^{\mu\nu}}$. We know that :

$$R = R_{ab} g^{ab}, \tag{2.32}$$

then we

$$\int \left(R_{\mu\nu} + \frac{g^{ab} \delta R_{ab}}{\delta g^{\mu\nu}} - \frac{1}{2} R g_{\mu\nu} + \wedge g_{\mu\nu} - \kappa T_{\mu\nu} \right) \delta g^{\mu\nu} dx^4 = 0, \tag{2.33}$$

The Palantini identity can then be used to write $g^{ab}\delta g_{ab}$ as a total derivative thus causing the whole term to vanish by stokes theorem . The Palatini identity is given by :

$$\delta R_{ab} = \nabla_a \delta \Gamma^{\rho}_{b\rho} - \nabla_\rho \Gamma^{\rho}_{ab}. \tag{2.34}$$

Then

$$\int \left(R_{\mu\nu} + \frac{\nabla_\sigma [g^{\sigma b} \delta \Gamma^{\rho}_{b\rho} - g^{ab} \delta \Gamma^{\sigma}_{ab}]}{\delta g^{\mu\nu}} - \frac{1}{2} R g_{\mu\nu} + \wedge g_{\mu\nu} - \kappa T_{\mu\nu} \right) \sqrt{-g} \delta g^{\mu\nu} dx^4 = 0, \tag{2.35}$$

By stokes theorem the total derivative doesn't contribute to the integral and we have

$$\int \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \wedge g_{\mu\nu} - \kappa \right) T_{\mu\nu} \sqrt{-g} \delta g^{\mu\nu} dx^4 = 0, \tag{2.36}$$

Since $\delta g^{\mu\nu}$are arbitrary we finally obtain the Einstein's field Equation are:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \wedge g_{\mu\nu} = \frac{8\Pi G}{C^4} T_{\mu\nu}. \tag{2.37}$$

where the constant κ is

$$\kappa = \frac{8\Pi G}{C^4} T_{\mu\nu}. \tag{2.38}$$

With G is Newton's gravitational constant, and $T_{\mu\nu}$ the stess energy tensor. Hence Einstein's tensor[] defined as:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{2.39}$$

with $G_{\mu\nu}$ is determined by the curvature of space and time at a particular point in space and time, and is equated with the energy and momentum at that point.
The solutions to these equations are the components of the metric tensor$g_{\mu\nu}$, which specifies the spacetime geometry. The inertial trajectories of particles can then be found using the geodesic equation.

Chapter 3

Introduction to cosmology

Cosmology is one of the most important application of the theory of general relativity. Understanding the evolution of the universe is what cosmology aim to reach, by studiying the large scale proprties of the universe, aiming to explain the origin and evolution of the entire contents of the univers, the underlying physical prosses, and thereby to obtain a deeper understanding of the laws of physics assumed to hold throughout the univers. The roots of this science go back to ancient times, where it was studied on philosophical and religious foundations, and it remained so, until Einstein published his general theory of relativity that changed everything. The mathematical structure of this science is similar to what expected. It is relatively easy at least if compared to other topics in GR as black holes for exemble. This relative ease is due to two reasons, the first is that gravity governs the entire universe in a wide range, and this means that we don't need to take into account those local influences coming from other forces in the universe, the second one is that in a sufficiently large scale the universe becomes completely homogeneous and isotropic. Homogeneity is the property of being identical everywhere in space, while isotropy is the property of looking the same in every direction. This two properties is what called "The cosmological Principle " which the cornerstone principle of cosmology. The cosmological principle also says that those properties must be valid from the begining of the universe, and will remain so for ever.

3.1 Metric and Friedmann Equations

Previously we've got the theory of space and time (Einstein's theory of GR), and the idea was that matter controle's the metric, wich tell's you how far apart two little elements are close toghether in space actually are. And the metric may not be the normal one, it can be curved wich causes things to move in different way, but what is now the issue in cosmology ?.

3.1.1 The FRW Metric

If we stick to general relativity, then the dynamics of the universe is described by Einstein's field equations, these are non linear partial differential equations. However, they can be simplified under some symmetry conditions. The Friedmann-Robertson-Wlaker (FRW) metric allows us to do this with powerful assumptions like isotropy and homogeneity. It was A. Friedmann who went throughout and demonstrated for the first time that A. Einstein's general theory of relativity admits nonstatic solution. He described an expanding, contracting, or collapsing universe, using his metic given by the following general form:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{1}{1-Kr^2}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right), \tag{3.1}$$

where the curvature parameter K tells us wich metric to use depending on the nature of the curvature. For an isotropic, homogeneous universe we must exist in one of these possible universes:

$$\begin{cases} K=+1 & \text{for} \quad \text{spherical universe} \\ K=0 & \text{for} \quad \text{flat universe} \\ K=-1 & \text{for} \quad \text{hyperbolic universe} \end{cases} \tag{3.2}$$

If it is flat then the universre will expand for ever and ever with an decreasing rate, if it is spherical it is a closed and will eventually colapse back in a big crunch, while if it's hyperbolic it is open and will expand for ever with an increasing rate. The function $a(t)$ is the scale factor of the expanding or contracting universe. the vector (t, r, θ, ϕ) are comoving coordinates, r is a dimensionless radius, and t is the cosmic time. The redshift z undergone by radiation from a comoving object as it travels to us today is related to the scale factor at wich it was emitted

$$\frac{a}{a_0} = \frac{1}{(1+z)} \tag{3.3}$$

or $z \equiv (a_0/a) - 1$. Evaluating the Einstein equations of motion for this metric gives the Friedmann equations. This begins by calculating the non zero christofel symbols.

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \tag{3.4}$$

$$\Gamma^{0}_{11} = -\frac{1}{2} g^{00} \partial_0 g_{11} = \frac{a\dot{a}}{1 - Kr^2}, \tag{3.5}$$

$$\Gamma^{0}_{22} = -\frac{1}{2} g^{00} \partial_0 g_{22} = a\dot{a}r^2, \tag{3.6}$$

$$\Gamma^{0}_{33} = -\frac{1}{2} g^{00} \partial_0 g_{33} = a\dot{a}r^2 \sin(\theta)^2, \tag{3.7}$$

$$\Gamma^{1}_{10} = \Gamma^{1}_{01} = \Gamma^{2}_{20} = \Gamma^{2}_{02} = \Gamma^{3}_{30} = \Gamma^{3}_{03} = \frac{1}{2} g^{11} \partial_0 g_{11} = \frac{\dot{a}}{a}, \tag{3.8}$$

$$\Gamma^{1}_{11} = \frac{1}{2} g^{11} \partial_1 g_{11} = \frac{Kr}{1 - Kr^2}, \tag{3.9}$$

$$\Gamma^{1}_{22} = -\frac{1}{2} g^{11} \partial_1 g_{22} = -r(1 - Kr^2), \tag{3.10}$$

$$\Gamma^{1}_{33} = -\frac{1}{2} g^{11} \partial_1 g_{33} = -r(1 - Kr^2)\sin^2(\theta), \tag{3.11}$$

$$\Gamma^{2}_{12} = \Gamma^{2}_{21} = \Gamma^{3}_{13} = \Gamma^{3}_{31} = \frac{1}{r}, \tag{3.12}$$

$$\Gamma^{2}_{33} = -\frac{1}{2} g^{22} \partial_2 g_{33} = -\sin(\theta)\cos(\theta), \tag{3.13}$$

$$\Gamma^{3}_{23} = \Gamma^{3}_{32} = \cot(\theta). \tag{3.14}$$

Next, the computation of the non zero components of the Ricci curvature tensor gives

$$R_{00} = -\partial_0\Gamma^1_{01} - \partial_0\Gamma^2_{02} - \partial_0\Gamma^3_{03} - \Gamma^1_{01}\Gamma^1_{10} - \Gamma^2_{02}\Gamma^2_{20} - \Gamma^3_{03}\Gamma^3_{03}, \tag{3.15}$$

$$= -3\partial_t\frac{\dot{a}}{a} - 3(\frac{\dot{a}}{a})^2 = -3\frac{\ddot{a}}{a}, \tag{3.16}$$

$$R_{11} = \partial_0\Gamma^0_{11} - \partial_1\Gamma^1_{12} - \partial_1\Gamma^3_{13} + \Gamma^0_{11}\Gamma^2_{02} + \Gamma^0_{11}\Gamma^3_{03} - \Gamma^1_{10}\Gamma^0_{11} + \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^3_{13} - \Gamma^2_{21}\Gamma^2_{12} - \Gamma^3_{31}\Gamma^3_{13}, \tag{3.17}$$

$$= \frac{a\ddot{a} + 2\dot{a}^2 + 2K}{1 - Kr^2}, \tag{3.18}$$

$$R_{22} = \partial_0\Gamma^0_{22} + \partial_1\Gamma^1_{12} - \partial_2\Gamma^3_{23} + \Gamma^0_{22}\Gamma^1_{01} + \Gamma^0_{22}\Gamma^3_{03} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^1_{22}\Gamma^3_{31} - \Gamma^2_{20}\Gamma^0_{22} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{32}\Gamma^3_{23} \tag{3.19}$$

$$= r^2[a\ddot{a} + 2\dot{a}^2 + 2K], \tag{3.20}$$

$$R_{33} = \frac{a\ddot{a} + 2\dot{a}^2 + 2K}{1 - Kr^2}. \tag{3.21}$$

$$R_{22} = \partial_0\Gamma^0_{33} + \partial_1\Gamma^1_{33} + \partial_2\Gamma^2_{33} + \Gamma^0_{33}\Gamma^1_{01} + \Gamma^0_{33}\Gamma^2_{02} + \Gamma^1_{33}\Gamma^1_{11} + \Gamma^1_{33}\Gamma^2_{21} - \Gamma^3_{30}\Gamma^0_{33} - \Gamma^3_{31}\Gamma^1_{33} - \Gamma^3_{32}\Gamma^2_{33}, \tag{3.22}$$

$$= r^2[a\ddot{a} + 2\dot{a}^2 + 2K]\sin\theta^2. \tag{3.23}$$

The scalar curvature is then given by :

$$R = R_{\mu\nu}g^{\mu\nu} = -6[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}], \tag{3.24}$$

where $H(t)$ is the hubble parameter which gives the rate of expansion of the universe

$$H = \frac{\dot{a}}{a}, \tag{3.25}$$

and the overdot denotes a derivative with respect to the proper time t and

$$\dot{H} = \frac{\ddot{a}}{a} - H^2, \tag{3.26}$$

The first friedman equation comes from evaluating the $(0,0)$ component of the Einstein equations

$$G_{00} = R_{00} - \frac{1}{2}Rg_{00} = 3\left[(\frac{\dot{a}}{a})^2 + \frac{K}{a^2}\right]. \tag{3.27}$$

Now consider the component (i, j), we find the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P). \tag{3.28}$$

3.1.2 The Stress energy Tensor

The energy momentum sources may be modeled as a perfect fluid, specified by an energy density ρ and isotropic pressure P in its rest frame. The energy-momentum tensor of such fluid is given by:

$$T^{\mu\nu} = (P+\rho)U^{\mu}U^{\nu} - Pg^{\mu\nu}, \tag{3.29}$$

where:

$$U^{\mu} = \frac{dx^{\mu}}{ds} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.30}$$

is the fluid 4-velocity of an observer comoving with the expansion, and verify$U^{\mu}U_{\mu} = -1$. Explicitly the energy momentum tensor reads as :

$$T^{\mu}_{\nu} = \text{diag}(-\rho, P, P, P), \tag{3.31}$$

The trace is

$$T = T^{\mu}_{\mu} = -\rho + 3P,$$

Once we have the form of the stress-energy tensor, the generalization of the energy conservation law is given by

$$\nabla_{\nu} T_{\mu\nu} = 0, \tag{3.32}$$

$$\partial_{\nu} T^{\mu\nu} + \Gamma^{\mu}_{\rho\nu} + \Gamma^{\nu}_{\rho\nu} T^{\rho\mu} = 0. \tag{3.33}$$

Settingh $\nu = 0$, yields the evolution equation for $\rho(t)$, which is the welle known continuity equation

$$\dot{\rho}(t) + 3H(P(t) + \rho(t)) = 0. \tag{3.34}$$

Know taking the Einstein field equation (2.37), wich can be written in the form:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + g^{\mu\nu}\wedge, \tag{3.35}$$

$$G_{00} - g^{00}\wedge = 8\pi G T_{00}, \tag{3.36}$$

$$3\left[(\frac{\dot{a}}{a})^2 + \frac{K}{a^2}\right] = 8\pi G\rho + \wedge, \tag{3.37}$$

The first equation of Friedmann then:

$$3H^2 = \frac{8\pi G\rho_i}{c^2} - \frac{K}{a^2} + \wedge., \tag{3.38}$$

The second equation of Friedmann follows from the second form of Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \wedge g_{\mu\nu} + \frac{8\Pi G}{C^4}T_{\mu\nu}, \tag{3.39}$$

Taking the trace of both members we obtain :

$$-R = 8\pi G\rho + 4\wedge, \tag{3.40}$$

and then

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}) - g_{\mu\nu}\wedge. \tag{3.41}$$

Taking the $(0,0)$ component

$$R_{00} = 8\pi G(T_{00} - \frac{1}{2}T g_{00}) - g_{OO}\wedge, \tag{3.42}$$

we get:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\wedge}{3}. \tag{3.43}$$

These equations admit a static solution with positive spatial curvature and all the parameters ρ, p, and $\wedge$ nonnegative. This called the "Eeinstein static universe". Einstein was intrested in finding a static ($\dot{a} = 0$) solutions, both due to his hope that general relativity would embody Mach's principle that matter determines inertia, and simply to account for the astronomical data as they were z understood at the time ,The discovery by hubble that the universe is expanding eliminated the empirical need for a static world model. Friedmann equations introduced above are considered as the basic equations of relativistic cosmology. Their solutions describe how the universe expanding as function of time, and therefore how the distance between any two objects can be calculated.

In order to solve equation Eqs. (3.38-3.43) we need to define the behaviour of the mass and energy density. To do so we use the equation of state of a perfect fluid wich relates the pressure with its energy density as follows:

$$P_i = \omega_i \rho, \tag{3.44}$$

where ω is the constant known as the equation of parameter state. Then we can write

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}, \tag{3.45}$$

and by integration we get :

$$\rho \propto a^{-3(1+\omega)}, \tag{3.46}$$

This equation determines the change in density linked to the variation of the scale factor we have three spicial cases:

3.1.2.1 Non-Relativistic Matter ($\omega = 0$)

In case the universe dominated by matter

$$P = 0 \tag{3.47}$$

Substituting p in fluid equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \frac{p}{C^2}) = 0. \tag{3.48}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \quad \Rightarrow \quad \frac{1}{a^3}\frac{\partial}{\partial t}(\rho a^3) = 0 \quad \Rightarrow \quad \rho \propto a^{-3}. \tag{3.49}$$

3.1.2.2 Relativistic Matter ($\omega = \frac{1}{3}$)

Known as the Radiation dominated ear where

$$P = \frac{\rho c^2}{3}. \tag{3.50}$$

Hence

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0 \quad \Rightarrow \quad \frac{1}{a^4}\frac{\partial}{\partial t}(\rho a^4) = 0 \quad \Rightarrow \quad \rho \propto a^{-4}. \tag{3.51}$$

3.1.2.3 Dark Energ ($\omega = -1$)

$$P = -\rho c^2. \tag{3.52}$$

From the Einstien equations with cosmological constant $\wedge$

$$G_{\mu\nu} - 8\pi G T_{\mu\nu}. \tag{3.53}$$

In the vacuum

$$T_{\mu\nu} = -g_\mu \nu \frac{\wedge}{8\pi G}. \tag{3.54}$$

The EOS of the universe

$$\rho = -P = \frac{\wedge}{8\pi G}. \tag{3.55}$$

3.1.3 Cosmological parameters

The most important parameters describing the expanding universe are the following:

3.1.3.1 The Hubble constant

The second principle or second clue to cosmology .It comes from the study of spectrum ,when looking at distant galaxies. It was found that the dark lines of the spectra has been shifted when compared to the emision spectra, this shift in wavelength was realiseed that is due to to the doppele effect .

$$z = \frac{\triangle\lambda}{\lambda\ _0}, \tag{3.56}$$

With z the redshift and $\Delta\lambda$ the shift in wavelength whish is the obsrved wave length minus the current wavelength spectral measured in the lab on earth.

$$\lambda_{obs} = \lambda_0,$$

The galaxies are moving away from as with a velocity V .wich could be found from

$$v = cz,$$

If we plot the red shift against distance we'll see that the recession velocity is proportional to distance galaxies. This relation is state in Hubble's law. A very simple equation

$$\vartheta = H_0 D, \tag{3.57}$$

This what E.Hubble discovered in 1929, this equation define the Hubble constasnt, it is the Hubble parameter measured today, we denoted it's value by H_0, current estimates are in the range of $H_0 = 65 - 75 Km/sMpc$.it is often written as:

$$\triangle \vec{x} = \vec{x_b} - \vec{x_a}, \tag{3.58}$$

Velocity of galaxy b seen from galaxy a is:

$$\triangle \vec{\vartheta} = \vec{\vartheta_b} - \vec{\vartheta_a} = H_0 \vec{x_b} - H_0 \vec{x_a} = H_0 \triangle \vec{x}, \tag{3.59}$$

So no matter where you are, everything appears to move away from us in just the same.

3.1.3.2 The Matter density parameter

The mass density ρ of the Universe and the value of the cosmological constant are dynamical properties of the Universe, acting the time evolution of the metric. They can be made into dimensionless density parameters.
Using the friedmann equation for $\wedge = 0$ and in the presence of curvature

$$H^2 - \frac{8}{3}\pi G\rho = -\frac{Kc^2}{a}, \tag{3.60}$$

The universe is flat if $K = 0$ or if it has a critical density given by

$$\rho_{crit} = \frac{3H^2}{8\pi G}, \tag{3.61}$$

We define the dimensionless density parameter Ω_i by

$$\Omega_i \equiv \frac{\rho_i}{\rho_c} = \frac{8\pi G}{3H^2}\rho_i, \tag{3.62}$$

The current physical value of the critical density is

$$\rho_{0,crit} = 0,921 \times 10^{-29} h_{70}^2 g.cm^{-3}$$

In the case of a universe dominated by a mixture of matter, radiation and dark energy, the total density is given by:

$$\rho = \rho_M + \rho_R + \rho_{DE} = \frac{3H^2}{8\pi G}(\Omega_M + \Omega_R + \Omega_{DE}), \tag{3.63}$$

Using the dimensionless parameter densities :

$$\rho_M = \frac{3H^2\Omega_M}{8\pi G}, \qquad \rho_R = \frac{3H^2\Omega_R}{8\pi G}, \qquad \rho_{DE} = \frac{3H^2\Omega_{DE}}{8\pi G}, \tag{3.64}$$

we obtain:

$$\Omega_M + \Omega_R + \Omega_{DE} = 1. \tag{3.65}$$

and

$$\rho = \frac{3H_0^2}{8\pi G}\left[\Omega_{DE_0}\frac{\rho_{DE}(a)}{\rho_{DE_O}} + \Omega_{M_0}\left(\frac{a_0}{a}\right)^3 + \Omega_{R_0}\left(\frac{a_0}{a}\right)^4\right], \tag{3.66}$$

If we consider the dark matter represent the the vacuum energy

$$\rho = \frac{3H_0^2}{8\pi G}\left[\Omega_{\wedge_0} + \Omega_{M_0}(\frac{a_0}{a})^3 + \Omega_{R_0}(\frac{a_0}{a})^4\right], \tag{3.67}$$

where$\Omega_{\wedge_0}$ the vacuum density parameter,using 1.72 and the first equation of friedman we can write the Hubble parameter in terms of Ω

$$H = H_0\sqrt{\Omega_{DE_0}\frac{\rho_{DE(a)}}{\rho_{DE_O}} + \Omega_{M_0}(\frac{a_0}{a})^3 + \Omega_{R_0}(\frac{a_0}{a})^4}, \tag{3.68}$$

With

$$dt = \frac{dt}{H_0 a\sqrt{\Omega_{DE_0}\frac{\rho_{DE(a)}}{\rho_{DE_O}} + \Omega_{M_0}(\frac{a_0}{a})^3 + \Omega_{R_0}(\frac{a_0}{a})^4}}. \tag{3.69}$$

3.1.3.3 The decceleration parameter

The deecceleration parameter is a measure of the acceleration or decceleration of the universe expansion and is defined by

$$q = -\frac{a\ddot{a}}{\dot{a}^2}, \tag{3.70}$$

Using the second Friedmann equation we write

$$q = \frac{4\pi G}{3H^2}(3P+\rho), \tag{3.71}$$

and using the Friedman second equation we obtain :

$$\begin{aligned} q &= \frac{4\pi G\rho_M}{3H^2} + \frac{8\pi G\rho_R}{3H^2} + \frac{4\pi G\rho_{DE}}{3H^2}(1+3\omega_{DE}), \\ &= \frac{1}{2}[\Omega_M + 2\Omega_R + \Omega_{DE}(1+3\omega_{DE})], \end{aligned} \tag{3.72}$$

for $\omega_{DE} = -1$ we get

$$q = \frac{1}{2}(\Omega_M + 2\Omega_R - 2\Omega_\wedge).$$

3.2 Fundamentals of Standard cosmology

The “Concordance cosmological Model”, or $\wedge$ CDM model is a parametrization of the Big Bang Cosmological model in wich the universe contains a cosmological constant, denoted by Lamda, associated with radiation and cold dark matter. It is frequently refered as the standard model of Big Bang, because it is the simplest model that provides a complete description of the cosmological evolution of the universe from the Big Bang to the present day.

As we have seen previously from the cosmological principle that man does not occupy a privileged place in the universe. Thus the universe must be spatially homogeneous and isotropic. This leads to the metric shape of Friedmann, Roberstson and Walker. Einstein introduced in 1917 a cosmological constant in his equations because he thought that he would thus find a closed static universe which would be in accordance with Mach's principle, inertia is meaningless in an empty universe. However, the discovery of the solutions of an expanding universe without cosmological constant by Friedmann in 1922 and in the observation of the remoteness of galaxies by Hubble in 1929 ended by completing the existence of a cosmological constant. The model of a static universe had just collapsed and the constant cosmological was therefore no longer useful. She was "put away in the closet" for a while, before being reborn again.

However, during all these years it was not totally forgotten. In fact, to remedy the problem of primordial singularity in Friedmann's solution, Lemaitre had developed in 1927 a model of a universe of static origin in which the scale factor is constant. It was only in 1968 that Yakov Borisovich Zelídovich [24] considered the importance of the cosmological constant by making the connection with the energy of the vacuum. In fact, its calculation is a loop of the vacuum resulting from renormalization gives an energy tensor of the vacuum energy vacuum which has the same form as that of the cosmological constant.

3.2.1 Red shift and cosmological expansion

The frequency of the shift of the light can provide us with valuable information about the scale factor a. How do we relate the cosmological redshift to the expansion of the Universe? We start with the FRW metric. In this case the null geodesic for photons has $ds = 0$

$$c^2 dt^2 = a(t)^2 dr^2,$$

Hence

$$c\int_{t_e+\lambda_e/c}^{t_0+\lambda_0/c} \frac{dt}{a(t)} = \int_0^r dr = r, \tag{3.73}$$

Tis means that the intarval between emission and observation are the same then:

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+\lambda_e/c}^{t_0+\lambda_0/c} \frac{dt}{a(t)}, \tag{3.74}$$

Substracting $\int_{t_e+\lambda_e/c}^{t_0} dt/a(t)$ from both sides we get

$$\int_{t_e}^{t_e+\lambda_e/c} \frac{dt}{a(t)} = \int_{t_0}^{t_0+\lambda_0/c} \frac{dt}{a(t)}, \tag{3.75}$$

at small time dt and that $a \approx cont$ we write

$$\frac{1}{a(t_e)} \int_{t_e}^{t_e+\lambda_e/c} dt = \frac{1}{a(t_0)} \int_{t_0}^{t_0+\lambda_0/c} dt, \tag{3.76}$$

giving

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)}, \tag{3.77}$$

The red shift is defined by

$$z = \frac{(\lambda_0 - \lambda_e)}{\lambda_e}, \tag{3.78}$$

and then

$$1 + z = \frac{a(t_0)}{a(t_e)}. \tag{3.79}$$

3.2.2 Time and distance in cosmology

How do astronomers can study things that are so far away in space?. The expansion of the universe means that the distance between two objects is contineously changing. In order to solve this, astronomers defined different kinds of distances .

Starting from :

$$ds^2 = -c^2 dt^2 - a(t)^2 \left[dr^2 + S_K(r) d\Omega^2\right], \tag{3.80}$$

where

$$S_k(r) = \begin{cases} R\sin(r/R) & \text{for} \quad k = +1 \\ r & \text{for} \quad k = 0 \\ R\sinh(r/R) & \text{for} \quad k = -1 \end{cases} \tag{3.81}$$

Along a spatial geodesic

$$ds = a(t)dr, \tag{3.82}$$

The propre distance at a fixed point is given by

$$d_p(t) = a(t)\int_0^r dr = a(t)r, \tag{3.83}$$

$$d_p(t) = a(t)\int_0^r dr = a(t)R_0 \arcsin\left(\frac{r}{R_O}\right), \quad for \quad (K=1) \tag{3.84}$$

$$= a(t)r, \qquad for \qquad (K=0) \tag{3.85}$$

$$= a(t)R_0 \text{arcsinh}\left(\frac{r}{R_O}\right), \quad for \quad (K=-1) \tag{3.86}$$

Then

$$\dot{d}_p = \dot{a}r = \frac{\dot{a}}{a}d_p, \qquad \vartheta_p(t_0) = H_0 d_p(t_0), \tag{3.87}$$

The inverse of the Hubble constant is the Hubble time t_H

$$t_{H_0} = \frac{1}{H_0}, \tag{3.88}$$

and the speed of light c times the Hubble time is the Hubble distance D_{H_0}

$$D_{H_0} = \frac{c}{H_0}, \tag{3.89}$$

The comoving distance to a red shift z is given by

$$D_c = D_{H_0}\int_0^z \frac{dz}{E(z)}, \tag{3.90}$$

Where :

$$E(Z) = \sqrt{\Omega_M(1+z)^3 + \Omega_K(1+z)^2 + \Lambda}. \tag{3.91}$$

3.2.2.1 Luminosity distance

The luminosity distance D_L is defined by the relationship between bolometric integrated over all frequencies. The flux and bolometric luminosity L are related by

$$F = \frac{L}{4\pi D^2} \tag{3.92}$$

In general the flux=Luminosity/Area, where the area is

$$A_p(t_0) = 4\pi S_k(r)^2$$

And L for luminosity

$$L = \frac{E}{t} = \frac{Energy}{time}$$

where

$$E_0 = \frac{E_e}{1+z} \quad dt_0 = \frac{dt_e}{1+z} \tag{3.93}$$

then

$$F = \frac{L}{4\pi S_k(r)(1+z)^2} \tag{3.94}$$

a factor $(1+z)$ is due to the energy loss of photons ,and the other one is due to the time dialation of the photon rate.
A luminosity distance is defined as:

$$d_L = S_k(r)(1+z) \tag{3.95}$$

the distance luminusity in terms of the curvature and Hubble parameter becomes:

$$d_L = \frac{D_{H_0}(1+z)}{\sqrt{\Omega_{k_0}}} \sinh\left(\sqrt{\Omega_{K_0}} \int \frac{dz}{E(z)}\right) \quad for \quad (K > 0) \tag{3.96}$$

$$d_L = D_{H_0}(1+z) \int \frac{dz}{E(z)} \quad for \quad (K = 0) \tag{3.97}$$

$$d_L = \frac{D_{H_0}(1+z)}{\sqrt{-\Omega_{k_0}}} \sin\left(\sqrt{-\Omega_{K_0}} \int \frac{dz}{E(z)}\right) \quad for \quad (K<0) \tag{3.98}$$

In a numerical calculations it is much faster to calculate the distance luminosity from the following differential equations

$$\frac{d}{dz}[d_L] - \frac{d_L}{1+z} = \frac{D_{H_0}\sqrt{\frac{\Omega_{K_0} d_L^2}{D_{H_0}} + (z+1)^2}}{E(z)} \tag{3.99}$$

$$\frac{d}{dz}[d_L] - \frac{d_L}{1+z} = \frac{D_{H_0}(z+1)}{E(z)} \tag{3.100}$$

$$\frac{d}{dz}[d_L] - \frac{d_L}{1+z} = \frac{D_{H_0}\sqrt{(z+1)^2 - \frac{\Omega_{k_0} d_L^2}{D_{H_0^2}}}}{E(z)} \tag{3.101}$$

The intial value of the luminosity distance set top the present day is zero. In observational data on Supernovae Ia, we use the modulus distance defined as the diffrence between the apparent magnitude m, and the absolute magnitude M

$$\mu(z, H_0) = m - M = 5\log_{10}\left(\frac{d_L}{10pc}\right) \tag{3.102}$$

3.2.2.2 Angular distances

An other important distance in cosmology is the angular distance used in CMB anisotropies observations. If we consider a light travel's from $(r; \theta, \phi)$ to the origin we have the distance

$$ds = a(t_e) S_k(r) \delta\theta \tag{3.103}$$

or

$$ds = la(t_e) = \frac{1}{1+z} \tag{3.104}$$

so

$$l = S_k(r) \frac{\delta\theta}{1+z} \tag{3.105}$$

and the angular distance is given by

$$d_A = \frac{l}{\delta}\theta = \frac{S_k(r)}{1+z} = \frac{d_L}{(1+z)^2} \tag{3.106}$$

3.2.3 Age of the universe

Several different methods have been used to calculate the age of universe, (different physics, different measurement), all of them agree that the lowwer limit to the age of the universe is in the interval $\sim 12-13$ Gyrs:

$$t_0 - t_i = t_{H_0} \int \frac{dz}{E(z)(1+z)} \tag{3.107}$$

$$t_0 = \frac{t_{H_0}}{3\sqrt{1-\Omega_{m_0}}} \ln\left(\frac{1+\sqrt{1-\Omega_{m_0}}}{1+\sqrt{1-\Omega_{m_0}}}\right) \tag{3.108}$$

3.2.4 Problems of the Lamda CDM Model

The SBB cosmological model has been very successful in explaining, among other things, the Hubble expansion of the universe, the existence of the CBR and the abundances of the light elements which were formed during primordial nucleosynthesis. Despite its great successes, this model had a number of long-standing shortcomings which we will now summarize:

3.2.4.1 Horizon problem

The CBR, which we receive now, was emitted at the time of 'decoupling' of matter and radiation when the cosmic temperature was $T_d \approx 3.000K$. The decoupling time (at temperature T_d), t_d, can be calculated from

$$\frac{T_0}{T_d} = \frac{2,73K}{3,000K} = \frac{a(t_d)}{a(t_0)} = \left(\frac{t_d}{t_0}\right)^{2/3} \tag{3.109}$$

It turns out that $t_d \approx 200.000h^{-1}$ years.
The universe displays a pronounced degree of large-scale homogeneity. CMB is the observational evidence. Measurements show that it has a thermal blackbody spectrum with a temperature highly homogeneous.

The horizon (or homogeneity) problem arises because the cosmic microwave background radiation (CMBR) has the same temperature whichever direction we look. Now the CMBR has its origin at the furthest point from which light has time to travel since the beginning of the universe. Currently, this is at around $46,5$ billion light-year's from us. So two points of the CMBR separated by 180% from our point of view will be 93 billion light-years apart and no form of communication or causal effects will have had time to travel between the two since the beginning of the universe. However, the temperature of both points is, on average, identical, suggesting that these two points must have been in contact at some time in the past. The current best solution to this problem is inflation paradigm, where different parts of the universe were close enough together for their temperatures to even out before inflation drove them apart at a rate far in excess of the speed of light.

3.2.4.2 Flatness Problem

Consider the Friedmann equation in the form:

$$\Omega - 1 = \frac{k}{aH^2} \tag{3.110}$$

The comoving Hubble radius $(aH)^{-1}$ grows with time , and thus $\Omega = 1$ is an unstable points. Indeed

$$\frac{|\Omega - 1|_{pl}}{|\Omega - 1|_0} \sim (\frac{a_{pl}}{a_0})^2 \sim (\frac{T_0}{T_{pl}})^2 \sim O(10^{-64}) \tag{3.111}$$

where $T_{pl}$is Planck temperature and $T_0 = 2.725$ K is the today temperature of the CMBR. Where in the first step we took a radiation dominated universe and in the last step we set $|\Omega - 1|_0 = O(1)$, to have a flat universe at present. The value of Ω at earlier times need to be extremely fine tuned.

When the strong energy condition of GR, $(1+3\omega) > 1$, is satisfied, $\omega = 1$ is an unstable fixed point:

$$\dot{\omega} = -\frac{2K}{aH^3}(H\dot{a} + \dot{H}a) \tag{3.112}$$

using

$$\dot{H} = \frac{\ddot{a}}{a} - H^2 = -\frac{1}{2}(\rho + P) + \frac{K}{a^2} \tag{3.113}$$

we obatin

$$\dot{\omega} = -\frac{2K}{aH^3}\frac{a}{6}(\rho + 3P) = H(\Omega - 1)\Omega(1 + 3\omega)$$

In the last step we used $\Omega = \frac{\rho}{\rho_{crit}}$ with $\rho_{crit} = 3H$

$$\frac{d|\Omega - 1|}{d\ln a} = \Omega|\Omega - 1|(1 + 3\omega). \quad (3.114)$$

It follows that in an expanding universe $|\Omega - 1|$ grows if $(1 + 3\omega) > 0$ or $\omega > -\frac{1}{3}$. The strong condition is satisfied for normal matter and radiation.

3.3 The Cosmological Constant

The physical nature of the cosmological constant remains a persistent enigma immanent to Einstein's Theory of General Relativity. It is generally thought to represent the gravitational contribution of vacuum fluctuations, when it was introduced with aim of describing the accelerating expansion of the universe, then using observations one can constrain its value.

Despite the good precision with the $\wedge$ CDM predicts the current cosmological observations, theoretically there are some issues, one of the most important ones is what some scientists call it "worse prediction of physics"and Einstein's greatest blunder the Cosmological Constant, and when todays theories give an estimated value that is about 120 orders of magnitude larger than the measured value, in quantum field theory.

Chapter 4

Dark energy in scalar-tensor theories

4.1 Horndeski theories

Over the past few decades the theory of Einstein “General Relativity” has shown a remarkable success wether on a theoretical or experimental levels, Despite these, there have been numerous attempts to modify or extend GR, sppured by varios motivations, one of them is to test general relativity quantitatively by constructing a parametrized space of theories around GR, which could be constrained by observations, another motivation is finding a solution to the dark sector of the universe, Dark Energy and Dark Matter. Since the observations allow the variation of the DE equation of state many models been proposed to explain the present accelerated of the universe.

In 1974 Horndeski came up with a theory of extended gravity. He formulated the most general scalar-tensor theory in 4-dimensions that yields second order field equations that respects Ostrogradski’s theorem. His theory remained unnoticed until its recent discovery as generalized covariant galileons, in it is proven that the two theories are actually equivalent in four dimensions. This is not trivial since the construction of the two theories is totally different. Horndeski started by trying to find the unique set of scalar-tensor theories that give up to second order equations in curved four-dimensional spacetime.

Among the scalar field class we find Galileon theory. First proposed by Nicolis et al. It represents a long distance modification of GR on Minkowski space through the addition of the Galileon scalar field ϕ. The field’s designation is born from the fact that the Lagrangian that results from its introduction remains invariant under a generalization of the Galilean invariance given by $\partial_\mu \phi \to \partial_\mu \phi + b_\mu, \phi \to \phi + c$. Galileons are scalar Lagrangians that give purely second order equations of motion. Extending (galileons in flat space) to curved spacetime with the metic $g_{\mu\nu}$ is called covariatization, the resulting theories is known as the covariant galileon.

Any alternative theory has to fulfill a number of requirements in order to be satisfactory. A very strong limitation to the space of possible theories is given by Ostrogradski's theorem for a non-degenerate theory whose Lagrangian contains second or higher derivatives with respect to time, their associated Hamiltonian is unboundedfrom below, making the system unstable and lacking a well-defined vacuum state. Degenerate theories are those for which Ostrogradski's construction does not apply, as it is the case for any theory described by second order equations of motion.

The most general 4-dimensinal scalar-tensor theories that keeping the field equations of motion at second order are described by the linear combinations of the following Lagrangians:

$$S = \sum_{i=2}^{5} \int d^4x \sqrt{-g} \mathscr{L}_i, = \sum_{i=2}^{5} S_i, \tag{4.1}$$

$$S_i = \int d^4x \sqrt{-g} \mathscr{L}_i. \tag{4.2}$$

$$\mathscr{L}_2 = K(\phi, X), \tag{4.3}$$

$$\mathscr{L}_3 = -G_3(\phi, X)\Box\phi, \tag{4.4}$$

$$\mathscr{L}_4 = G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - \left(\nabla_\mu\nabla_\nu\phi\right)^2\right], \tag{4.5}$$

$$\mathscr{L}_5 = G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3(\Box\phi)\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right], \tag{4.6}$$

where g is the determinant of the metric $g_{\mu\nu}$ and:

$$\phi_\mu \equiv \nabla_\mu\phi, \qquad \phi_{\mu\nu} \equiv \nabla_\mu\nabla_\nu\phi, \qquad \Box = \nabla_\mu\nabla^\mu, \tag{4.7}$$

where G_i are a set of four indpendent arbitrary functions of the scalar field its Kinetic energy X, and The derivatives of these functions are

$$X = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi, \quad G_\phi = \frac{\partial G}{\partial \phi}, \quad G_X = \frac{\partial G}{\partial X}, \tag{4.8}$$

The above Lagrangian was first discovered by Horndeski in a different form (), is equivalent to that derived byHorndeski(). The non-minimal couplings to gravity in L_4 and L_5 are necessary to eliminate higher derivatives that would otherwise appear in the field equations.

Many alternative models have been proposedto approach the origine of Dark Matter, we can clasify them into two classes (i) modified matter models and (ii) modified gravity modeles, the first model is introduced to derive the late-time cosmic acceleration. The models that represets this class are:

- Quintessence model:

  The dynamics of quintessence scalar field is governed by an ordinary scalar field, which slowly rolls down the potential. Slow-roll is the condition in which kinetic energy of the system is less than the potential energy, yielding the negative pressure. Its EoS parameter describes accelerated expansion of the universe in the interval $-1 < w_q < \frac{1}{3}$ it takes the choice :

$$K(\phi, X) = X - V(\phi), \qquad G_3(\phi, X) = G_5(\phi, X) = 0, \qquad G_4(\phi, X) = \frac{1}{2}, \tag{4.9}$$

- K-essence model:

  Gives the accelerated expansion of the universe with the help of kinetic energy X and its modified forms

$$G_2(\phi, X) = K(\phi, X). \tag{4.10}$$

  The second class is presented by the following models:

- Brans-Dick Theory:

  The simplest class, in scalar-tensor theories. It is thought to embody the Mach's principle. Mathematically the BD theory is expressed by the following action principle:

$$K(\phi, x) = \frac{\omega_{BD}}{\phi} X - v(\phi), \qquad G_3 = G_5 = 0, \qquad G_4 = \frac{\phi}{2}. \tag{4.11}$$

$$S=\frac{1}{2}\int dx^4\sqrt{|g|}[\phi R-\frac{\omega_{BD}}{\phi}(\partial\phi)^2-2V]. \tag{4.12}$$

- Covariant Galileon: The covariant Galileon without the field potential corresponds to :

$$K=-c_2X, \qquad G_3=c_3, \qquad G_4=\frac{1}{2}-C_4X^2, \qquad G_5=3c_5X^2, \tag{4.13}$$

  Where $C_{2,3,4,5}$ are dimensionless constants For the covariant Galileon there exists a stable de Sitter solution where X= constant.

Kinetic Coupling to Einstein's tensor is anothor particular and very interesting case within the class of the Horndeski theories. It corresponds to the following choice

$$K=X-V, \qquad G_3=0, \qquad G_4=\frac{1}{2}, \qquad G_5=-\frac{\alpha}{2}\phi \tag{4.14}$$

That leads to the action :

$$S=\frac{1}{2}\int dx^4\sqrt{|g|}[R+2(X-V)+\alpha G_{\mu\nu}\partial^\mu\phi\partial^\nu\phi]. \tag{4.15}$$

- Kinetic braiding theory:
  It is described by the Lagranian bellow:

$$K_{KBD}=K(\phi,X)+G_3(\phi,X)\Box\phi+G_4(\phi)R. \tag{4.16}$$

$$S_m=\int dx^4[\sqrt{-g}\rho_m(n)+J^\nu\partial_\nu\ell]. \tag{4.17}$$

ρ_m the energy density, wich depends of the density number of the fluid n, ℓ is a scalar field, and J^ν is a vector such that:

$$U_\alpha=\frac{J^\alpha}{n\sqrt{-g}}, \tag{4.18}$$

using the normalisation $U_\alpha U^\alpha=-1$ we obtain:

$$n=\sqrt{\frac{J^\alpha J^\beta g_{\alpha\beta}}{g}}, \tag{4.19}$$

The Variation of the matter Lagrangian with respect to J^α leads to:

$$\partial_\nu\ell=\frac{\rho_{m,n}J_\nu}{n\sqrt{-g}}. \tag{4.20}$$

4.1.1 Background equations of motion

We are intrested in the late time cosmology in wich the field ϕ responsible for DE. Taking into account the barotopic perfect fluid with the equation of state :

$$w = \frac{P_m}{\rho_m}. \tag{4.21}$$

In the following we focus on non relativistic matter $(w = 0)$ minimally coupled to the field ϕ, the total action is:

$$S = S_m + S_H. \tag{4.22}$$

Let us derive the equations of motion describing the background evolution form, the easiest way is to substitute $\phi = \phi(t)$ and the flat FLRW metric defined as :

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \delta_{ij} dx^i dx^j. \tag{4.23}$$

N(t) is the lapse wich is introduced to derive the Friedmann equation.then it takes the value equals to 1 after the variation of the action S. In this review the scalar field ϕ depends only of t(the cosmic time), where the first component of J^ν is:

$$J^0 = n_0 a^3. \tag{4.24}$$

The schutz-sorkin action in this case reduces to:

$$S_m = -\int dx^4 a^3 (N\rho_m + n_0 \dot{\ell}). \tag{4.25}$$

Variying S_m with respect to ℓ gives

$$N_0 \equiv n_0 a^3 = cte. \tag{4.26}$$

4.1.1.1 Variation with respect to N(t)

This gives the constraint equation corresponding to the Friedmann equation :

$$S_H = -\int d^4x\delta\left(a^3N\mathscr{L}_\rangle\right), \tag{4.27}$$

$$= -\int a^3d^4x\left[\delta NK(\phi,X) + N\delta K(\phi,X)\right], \tag{4.28}$$

$$= -\int a^3d^4x\left[K(\phi,X) + \frac{\partial K}{\partial}X\frac{\partial X}{\partial N}\right]\delta N, \tag{4.29}$$

$$\varepsilon = \sum_{i=2}^{5}\varepsilon_i = \rho_m. \tag{4.30}$$

with

$$E_2 = 2XK_X - K, \tag{4.31}$$

$$E_3 = 6X\dot{\phi}HG_{3X} - 2XG_{3\phi}, \tag{4.32}$$

$$E_4 = -6H^2G_4 + 24H^2X(G_{4X} + XG_{4XXX}) - 12HX\dot{\phi}G_{4\phi X} - 6H\dot{\phi}G_{4\phi}, \tag{4.33}$$

$$E_5 = 2H^3X\dot{\phi}(5G_{5X} + 2XG_{5XX}) - 6H^2X(3G_{5\phi} - 2XG_{5\phi X}), \tag{4.34}$$

The above quantities contain derivatives of the metric and the scalar field up to first order. Variation with respect to a(t), yields the evolution equation

$$P = \sum_{i=2}^{5}P_i = 0, \tag{4.35}$$

$$P_2 = K, \tag{4.36}$$

$$P_3 = -2X(G_{3\phi} + \ddot{\phi}G_{3X}), \tag{4.37}$$

$$\begin{aligned} P_4 = {} & 2(3H^2 + 2\dot{H})G_4 - 12H^2XG_{4X} - 4H\dot{X}G_{4X} - 8\dot{H}XG_{4X} - 8HX\dot{X}G_{4XX} \\ & + 2(\ddot{\phi} + 2H\phi)G_{4\phi} + 4XG_{4\phi\phi} + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4\phi X}, \end{aligned} \tag{4.38}$$

$$\begin{aligned} P_5 = {} & -2X(2H^3\dot{\phi} + 2H\dot{H}\phi + 3H^2\ddot{\phi})G_{5X} - 4H^2X^2\ddot{\phi}G_{5XX} + 4HX(\dot{X} - HX)G_{5\phi X} \\ & + 2[2(\dot{H}X + H\dot{X}) + 3H^2X]G_{5\phi} + 4HX\dot{\phi}G_{5\phi\phi}, \end{aligned} \tag{4.39}$$

The background quantities E_i and P_i are defined in an analogous way in which the energy density and the isotropic pressure of a usual scalar field are defined. In our case the, the distinction between the gravitational and scalar-field portions of the Lagrangian is ambiguous, and hence, in that sense, the gravitational contribution is included in the above expressions. We can write it as :

$$2Q_T\dot{H} - D_{6\ddot{\phi}} + D_7\dot{\phi} = -P_m. \tag{4.40}$$

know the Variation with respect to $\phi(t)$, gives the scalar-field equation of motion it follows that:

$$\frac{1}{a^3}\frac{d}{dt}(a^3J) = P_\phi. \tag{4.41}$$

Where

$$J = \dot{\phi}k_x + 6HXG_{3X} - 2\dot{\phi}G_{3\phi} + 6H^2\dot{\phi}(G_{4X} + 2XG_{4XX}) - 12HXG_{4\phi X} + 2H^3X(3G_{5X} + 2XG_{5XX}) - 6H^2\dot{\phi}(G_{5\phi} + XG_{5\phi X}). \tag{4.42}$$

and

$$P_\phi = K\phi - 2X(G_{3\phi\phi} + \ddot{\phi}G_{3\phi X}) + 6(2H^2 + \dot{H})G_{4\phi} + 6H(\dot{X} + 2HX)G_{4\phi X} - 6H^2XG_{5\phi\phi} + 2H^3X\dot{\phi}G_{5\phi X}. \tag{4.43}$$

Non-relativistic matter obeys the continuity equation :

$$\dot{\rho_m} + 3H(P_m + \rho_m) = 0. \tag{4.44}$$

Dark energy and

$$\rho_{DE} = 2XK_X - 6X\dot{\phi}HG_{3X} - 2XG_{3\phi} - 6H^2(G_4 - \frac{1}{2}) + 24H^2X(G_{4X} + XG_{4XX}) - 12HX\dot{\phi}G_{4\phi X} - 6H\dot{\phi}G_{4\phi} + 2H^3X\dot{\phi}(5G_{5X} + 2XG_{5XX}) - 6H^2X(3G_{5\phi} + 2XG_{5\phi X}). \tag{4.45}$$

$$
\begin{aligned}
P_{DE} = {} & K - 2X(G_{3\phi} + \ddot{\phi} G_{3X}) + 2(3H^2 + 2\dot{H})(G_4 - \frac{1}{2}) - 12H^2 X G_{4X} - 4H\dot{X} G_{4X} - 8\dot{H} X G_{4X} \\
& - 8HX\dot{X} G_{4XX} + 2(\ddot{\phi} + 2H\dot{\phi}) G_{4\phi} + 4X G_{4\phi\phi} + 4X(\ddot{\phi} + 2H\dot{\phi}) G_{4\phi X} - 2X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} \\
& + 3H^2\ddot{\phi}) G_{5X} - 4H^2 X^2 \ddot{\phi} G_{5XX} + 4HX(\dot{X} - HX) G_{5\phi X} + 2(2(\dot{H}X + H\dot{X}) + 3H^2 X) G_{5\phi} \\
& + 4HX\dot{\phi} G_{5\phi\phi}.
\end{aligned} \tag{4.46}
$$

EOS parameter

$$
\omega_{DE} = -1 - \frac{2(2G_4 - 2\dot{\phi}2G_{4X} + \dot{\phi}2G_{5\phi} - H\phi 3G_{5X})\dot{H} - D_6\ddot{\phi} + D_7\dot{\phi}}{\rho_{DE}}. \tag{4.47}
$$

Chapter 5

Dark energy in modified teleparallel scalar-tensor theories

5.1 Teleparallel gravity

The great success of general relativity in describing experimental data does cearly show that we are on the right track, and our geometry theory of gravity works exteremly well, all prediction of GR including gravitational waves, have been experimentally verified, nonetheless, when applied to the entire universe, we are faced with conceptual and observational challenges that are simply summarised as dark enrgy and dark matter problem.

Even though we have a perfectly good model also there ($\wedge$ CDM), it is very unplesant to realise that we don't have reasonable idea as the nature of some 95% of the budget of the universe, this with all the development in other field of physics has motivated scientist to extend or modificat general relativity in a attempt to solve the missing peaces of this universe
Teleparallel gravity or Teleparallel equivalent of General Relativity (TEGR), this approach appeared when Einstein wanted to unify electromagnetism with gravity in 1920. TEGR is completly equivalent to GR at the level of the field equations,and instead of curvature, it uses torsion to describe the gravitational interaction, the basic ingredient in the structure of this theory is the tetrad field e, wich defined on a tangent space at each point of the general manifold, This field has 16 degrees of freedom, while the metric has only 10, the idea then was to exploit the additional degrees of freedom to acommondate the electromagnetic field,

In this frame work, gravity is no longer the effect of geometry of space-time, but rather a force, its equation of motion are identical to those in GR, and their actions only differing by the total derivative term, so why TEGR when it is equivalent to GR. In spite of this equivalence, it shows many distinguished features that make of it a theory worthy studing. First of all, it is a gauge theory of translations, meaning that it can be more easily unified with the other three fundamental forces of the standard mode. As a gauge theory it could even survive in the absence of the equivalence principle. Moreover, in the framework of TEGR one can separate gravitational from inertial effects and because of that, one can define a gravitational energy momentum density.

In this section we introduce the fundamental objects in teleparallel gravity, Here the Latin indices are coordinate on the tangent space, wheras Greek indices correspond to spacetime coordinates ,and both takes values a, $\mu = 0,1,2,3$, with the signature (+, -, -, -) of Minkowski metric η_{ab}

5.1.1 Mathematical stucture

Teleparallelism is a framework to describe gravity ,where the dynamical variables are the vierbein or tetrad field.the tetrad field $e_a(x)$ is a set of four orthonormal vectors at each point of the manifold M,that contains a basis of the tangent space of the space T_pM.Any vector at a spacetime point has components in the coordinate and non-coordinate orthonormal basis .

$$V = V^{;\mu} e_{;\mu} = V^{;a} e_{;a}. \tag{5.1}$$

so its components are related by the vierbein field transformation

$$V_{;\mu} = e^{;\mu}_{;a} V^{;a}, \qquad V_{;a} = e^{;a}_{mu} V^{;\mu}, \tag{5.2}$$

$$V^{;\mu}_{;\nu} = e^{;\mu}_{;a} V^{;a}_{;\nu} = e^{;b}_{;\nu} V^{;\mu}_{;b} = e^{;\mu}_{;a} e^{;b}_{;\nu} V^{;a}_{;b}. \tag{5.3}$$

from this we can see that vierbein field allow us to switch bteen Greek and Latin bases. The dual coframe $e_a(x)$ is the basis of the co-tangent space $T_{p*}M$ they can be decomposed in a coordinate basis as :

$$e^{;a} = e^{;a}_{;\mu} dx^{\mu}, \qquad \text{and} \qquad e_{;a} = e^{;\mu}_{;a} \partial_{\mu}, \tag{5.4}$$

where

$$e^{;a}_{;\mu} e^{;\mu}_{;b} = \delta^{;a}_{;b}, \qquad e^{;a}_{;\mu} e^{;\nu}_{;a} = \delta^{;\nu}_{;\mu}, \tag{5.5}$$

this condition of orthoganility is the link btween the metric and tetrad :

$$g_{\mu\nu} = e^{;a}_{;\mu} e^{;b}_{;\nu} \eta_{;ab}. \tag{5.6}$$

Consequently, there is an infinite set of tetrads that satisfy these conditions. Since the relation metric-tetrad is invariant under local Lorents transformations of the tetrad TEGR theory is based upon weitzenbock connection, witch is curvaturless and metric compatible.

Now we can, if we want, consider every tensor with Latin indices instead of spacetime ones with the relation between the two being understood as

$$T^{a_1..a_m}_{b_1..b_m} \equiv e^{a_1}_{\alpha_1} ... e^{a_n}_{\alpha_n} T^{\alpha_1...a_n}_{\beta_1...b_n} e^{\beta_1}_{b_1} ... e^{\beta_m}_{b_m}. \tag{5.7}$$

One can also say that we have a copy of the tangent space at each point with the canonical metric η_{ab} in it, and the tetrads realise an isomorphism between the two (pseudo) normed linear spaces.

Moreover, we can now have two types of the connection coefficients, $\Gamma^{\alpha}_{\mu\nu}$ for the usual tensors (The affine connection coefficient) and $\omega^{a}_{\mu b}$(The spin connection coefficient) for those with tangent space indices.

The covariant derivative of a vector V in the coordinate basis is:

$$\begin{aligned} \nabla V &= (\nabla_{;\mu} V^{;\nu}) dx^{;\mu} \otimes \partial_{;\nu}, \\ &= (\partial_{;\mu} V^{;a} + \Gamma^{\nu}_{\mu\rho} V^{;\rho}) dx^{;\mu} \otimes \partial_{;\nu}. \end{aligned} \tag{5.8}$$

In the mixed bases we have

$$\begin{aligned}
\nabla V &= (\nabla_{;\mu} V^{;a}) dx^{;\mu} \otimes \partial_{;\nu}, \\
&= (\partial_{;\mu} V^{;a} + \omega^{;a}_{;\mu b} V^{;b}) dx^{;\mu} \otimes \partial_{;\nu}, \\
&= (\partial_{;\mu} (e^{;a}_{;\nu} V^{;\nu}) + \omega^{;a}_{;\mu b} e_{;\rho} e^{;b}_{;\rho} V^{;\sigma}) dx^{;\mu} \otimes (e^{;\sigma}_{;a} \partial_{;\sigma}), \\
&= e^{;\sigma}_{;a} (e^{;a}_{;\nu} \partial_{;\mu} V^{;\nu} + V^{;\nu} \partial^{;\mu} e^{;a}_{;\nu} + \omega^{;a}_{;\mu b} e^{;b}_{;\rho} V^{;\sigma} dx^{;\mu} \otimes \partial_{;\rho}, \\
&= (\partial_{;\mu} V^{;\rho} + e^{;\rho}_{;a} \partial_{;\mu} e^{;a}_{;\nu} V^{;\nu} + e^{;\rho}_{;a} e^{;b}_{;\sigma} \omega^{;a}_{;\mu b} V^{;\sigma}) dx^{;\mu} \otimes \partial_{;\rho}, \\
&= (\partial_{;\mu} V^{;\nu} + e^{;\nu}_{;a} \partial_{;\mu} e^{;a}_{;\sigma} V^{;\sigma} + e^{;\nu}_{;a} e^{;b}_{;\sigma} \omega^{;a}_{;\mu b} V^{;\sigma}) dx^{;\mu} \otimes \partial_{;\nu} (\partial_{;\mu} V^{;\nu} + (e^{;\nu}_{;a} \partial_{;\mu} e^{;a}_{;\sigma} + e^{;\nu}_{;a} e^{;b}_{;\sigma} \omega^{a}_{\mu b} V^{;\sigma})).
\end{aligned} \tag{5.9}$$

From (5.8) and (5.9) we find the affine conection in terms of the spin connection

$$\Gamma^{\nu}_{\mu\sigma} = e^{\nu}_{a} \partial_{\mu} e^{a}_{\sigma} + e^{\nu}_{a} e^{b}_{\sigma} \omega^{a}_{\mu b}. \tag{5.10}$$

Then

$$\omega^{a}_{\mu b} = e^{a}_{\nu} e^{\sigma}_{b} \Gamma^{\nu}_{\mu\sigma} - e^{\sigma}_{b} \partial_{\mu} e^{a}_{\sigma}. \tag{5.11}$$

In order to freely change the nature of the indices by the tetrads, we wish this procedure to commute with taking a covariant derivative. Obviously, this goal would be achieved by the following requirement

$$\partial_{\mu} e^{a}_{\mu} + \omega^{a}_{\mu b} - \Gamma^{\sigma}_{\mu b} e^{a}_{\sigma} = 0. \tag{5.12}$$

Which can be referred to as vanishing of the "full covariant derivative" of the tetrad. With this understanding in mind, we can conveniently use tensors with indices of both types, and the covariant derivatives would be unambiguously defined for a tensor even if we are allowed to transform from one type to another. The recipe is that we use Γ-terms for Greek indices, and ω-terms for Latin indices:

$$\nabla_{;\mu} T^{;a\alpha} = \partial_{;\mu} T^{;a\alpha} + \Gamma^{\alpha}_{\mu\beta} T^{;a\beta} + \omega^{a}_{\mu b} T^{;b\alpha}. \tag{5.13}$$

Condition is solved straightforwardly to obtain:

$$\Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{a} (\partial_{\mu} e^{a}_{\nu} + \omega^{a}_{\mu b} e^{b}_{\nu}) \equiv e^{\alpha}_{a} D_{\mu} e^{a}_{\nu}. \tag{5.14}$$

with D_μ being the Lorentz-covariant (with respect to the Latin index only) derivative. Relation can also be reversed to obtain the spin connection

$$\omega^a_{\mu b} = e^a_\alpha \Gamma^\alpha_{\mu\nu} e^\nu_b - e^\nu_b \partial_\mu e^a_\nu. \tag{5.15}$$

which corresponds to a given affine connection on the manifold. In particular, one can find the spin connection which corresponds to the Levi-Civita connection of a given metric $g^{\mu\nu}$. Basically, if (4) is valid, then both $\Gamma^\alpha_{\mu\nu}$ and ω represent one and the same connection in different disguises. This conclusion is further substantiated by comparing the curvartures for both connections

$$R^a_{b\mu\nu}(\omega) = \partial_{;\mu}\omega^{;a}_{;\nu b}\partial_{;\nu}\omega^{;a}_{;\mu b} + \omega^{;a}_{;\mu c}\omega^c_{\nu b} - \omega^a_{\nu c}\omega^c_{\mu b}. \tag{5.16}$$

and

$$R^a_{\beta\mu\nu}(\Gamma) = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\rho}\Gamma^\rho_{\nu\beta} - \Gamma^\alpha_{\nu\rho}\Gamma^\rho_{\mu\beta}. \tag{5.17}$$

In other words, the two Riemann tensors are related by mere change of the types of indices. Therefore, those are one and the same tensor under our conventions which are common for all the tensors we use.

Note also that the non-metricity in this formalism (with the vanishing of the "full covariant derivative" of the tetrad) is automatically equal to zero because :

$$\begin{aligned} \nabla_\alpha g_{\mu\nu}, &= \eta_{ab}(\partial_\alpha(e^a_\mu e^b_\nu) - \Gamma^\beta_{\alpha\mu} e^a_\beta e^b_\nu - \Gamma^\beta_{\alpha\nu}, \\ &= -e^b_\mu e^c_\nu(\eta_{ab}\omega^a_{\alpha c} + \eta_{ac}\omega^a_{\alpha b}) = 0. \end{aligned} \tag{5.18}$$

A basic idea of teleparallel gravity is to give an equivalent description of general relativity in terms of torsion tensor.

since $\nabla_\alpha g_{\mu\nu} = 0$, one can follow the standard textbook derivation of the Levi-Civita connection and prove that:

$$\tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu}(g) - K^\alpha_{\mu\nu}. \tag{5.19}$$

Where

$$K^{\alpha\mu\nu} = \frac{1}{2}(T^{\alpha\mu\nu} + T^{\nu\alpha\mu} + T^{\mu\alpha\nu}) = -K^{\alpha\nu\mu}. \tag{5.20}$$

is known under the name of contortion, It is obviously antisymmetric with respect to two indices. And $\Gamma^\alpha_{\mu\nu}(g)$ are the usual Christoffel symbols of the symmetric connection, while $\tilde{\Gamma}^\alpha_{\mu\nu}$ is the so called Weitzenbock connection

$$\tilde{\Gamma}^{\alpha}_{\mu\nu} = e^{a}_{\rho}\partial_{\nu}e^{\rho}_{\mu} \tag{5.21}$$

which is a connection without curvature and presenting only torsion, as a direct consequence of this definition, the Weitzenb ock covariant derivative of the tetrad field vanishes identically:

$$\nabla_{\mu}e^{a}_{\nu} \equiv \partial_{\mu}e^{a}_{\nu} - \Gamma^{\rho}_{\nu\mu}e^{a}_{\rho} = 0 \tag{5.22}$$

This is the so called absolute parallelism condition.
Substituting connection () into the the definition of curvature, we get:

$$R^{\alpha}_{\beta\mu\nu} = R^{\alpha}_{\beta\mu\nu} + \nabla_{\mu}K^{\alpha}_{\nu\beta} - \nabla_{\nu}K^{\alpha}_{\mu\beta} + K^{\alpha}_{\mu\beta} + K^{\alpha}_{\mu\beta} + K^{\alpha}_{\mu\rho}K^{\rho}_{\nu\beta} - K^{\rho}_{\nu\rho}K^{\rho}_{\mu\beta}. \tag{5.23}$$

for the Riemann tensor with ∇_{μ} being the covariant derivative associated to $\Gamma^{\alpha}_{\mu\nu}(g)$. Making the necessary contractions we obtain the scalar curvature

$$R = R + 2\nabla_{\mu}T_{\mu} + \mathbf{T}. \tag{5.24}$$

where the torsion vector is :

$$T_{\mu} \equiv T^{\lambda}_{\lambda\mu} = -T^{\lambda}_{\mu\lambda}. \tag{5.25}$$

and the torsion scalar can be written in several equivalent ways:

$$\begin{aligned} T &= \frac{1}{2}K_{\alpha\beta\mu}T^{\beta\alpha\mu}, \\ &= \frac{1}{2}T_{\alpha\beta\mu}S^{\alpha\beta\mu}, \\ &= \frac{1}{4}T_{\alpha\beta\mu}T^{\alpha\beta\mu} + \frac{1}{2}T_{\alpha\beta\mu}T^{\beta\alpha\mu} - T^{\mu}T_{\mu}. \end{aligned} \tag{5.26}$$

The torsion tensor can be defined as :

$$T^{a}_{\mu\nu} = \partial_{\mu}e^{a}_{\nu} - \partial_{\nu}e^{a}_{\mu}. \tag{5.27}$$

In terms of space time we have :

$$T^{\sigma}_{\mu\nu} = e^{\sigma}_{a}T^{a}_{\mu\nu}. \tag{5.28}$$

$$e_a^\sigma(\partial_\mu e_\nu^a - \partial_\nu e_\mu a + \omega_{\mu b}^a e_\nu^b - \omega_{\nu b}^a e_\mu^b) = e_a^\sigma \partial_\mu e_\nu^a + e_a^\sigma \omega_{\mu b}^a e_\nu^b - e_a^\sigma \partial_\nu e_\mu a - e_a^\sigma \omega_{\nu b}^a e_\mu^b. \quad (5.29)$$

we know that

$$\Gamma_{\mu\nu}^\sigma = e_a^\sigma \partial_\mu e_\nu^a + e_a^\sigma \omega_{\mu b}^a e_\nu^b. \quad (5.30)$$

The torsion tensor $T_{\mu\nu}^\sigma$ it the anti-symmetric part of the affine connection coefficients , that is:

$$T_{\mu\nu}^\sigma = \frac{1}{2}(\Gamma_{\mu\nu}^\sigma - \Gamma_{\nu\mu}^\sigma) \equiv T_{[\mu\nu]}^\sigma. \quad (5.31)$$

In GR, it is postulated tha $T_{\mu\nu}^\alpha = 0$. It is a general convention to call U_4 a 4-dimensional space-time manifold endowed with metric and torsion the superpotential :

$$S^{\alpha\mu\nu} = K^{\mu\alpha\nu} + g^{\alpha\mu}T^\nu - g^{\alpha\nu}T^\mu \quad (5.32)$$

Which satisfies the antisymmetry condition $S_{\mu\nu}^\alpha = -S_{\mu\nu}^\alpha$ In teleparallel gravity one exploits the Weitzenbock connection given by $\omega_{\mu\nu}^a = 0$, which is obviously curvature-free, $R_{\beta\mu\nu}^\alpha(\tilde{\Gamma}) = 0$ Note that this definition blatantly breaks local Lorentz invariance because the connection is not a tensor, and it is not a covariant condition that it vanishes.
The TEGR Lagrangian is :

$$\mathscr{L}_T = -\frac{1}{2K}\int d^4xeT. \quad (5.33)$$

Variying this action with respect to tetrads and scalar field where $L(g_{\mu\nu}, \phi, \partial\phi)$, to get to equations of motion then:

We define some notations

$$\delta|e| = ee_a^\mu \delta e_\mu^a, \quad (5.34)$$

$$\delta e_a^\mu = -e_b^\mu e_a^\nu \delta e_\nu^b, \quad (5.35)$$

$$\delta g_{\mu\nu} = \eta_{ab}(e_\mu^a \delta e_\nu^b + e_\nu^a \delta e_\mu^b), \quad (5.36)$$

$$\delta g^{\mu\nu} = -(g^{\mu\nu}e_b^\nu + g^{\nu\mu}e_a^\mu)\delta e_\alpha^a, \quad (5.37)$$

The variation of the action in the teleparallel gravity :

$$\delta S = \frac{1}{2k}\int d^4x\delta(eT), \tag{5.38}$$

$$= \frac{1}{2k}\int d^4x(\delta eT + e\delta T), \tag{5.39}$$

Using (5.26):

$$\delta T = \frac{1}{4}\delta(T^{\alpha\mu\nu}T_{\alpha\mu\nu}) + \frac{1}{2}\delta(T^{\alpha\mu\nu}T_{\alpha\nu\mu}) - \delta(T^{\mu}T_{\mu}). \tag{5.40}$$

Hence

$$T_{\mu} = T^{\alpha}_{\mu\alpha} = e^{\alpha}_{a}(\partial_{\mu}e^{a}_{\lambda} - \partial_{\lambda}e^{a}_{\mu}), \tag{5.41}$$

$$T^{\alpha\mu\nu} = g_{\alpha\gamma}e^{\gamma}_{a}(\partial_{\mu}e^{a}_{\nu} - \partial_{\nu}e^{a}_{\mu}), \tag{5.42}$$

$$T^{\alpha\mu\nu} = e^{\alpha}_{a}(\partial^{\mu}(g^{\nu\gamma}e^{a}_{\gamma}) - \partial^{\nu}(g^{\mu\gamma}e^{a}_{\gamma})), \tag{5.43}$$

$$\delta T^{\alpha}_{\mu\nu} = \delta(e^{\alpha}_{a}\partial_{\mu}e^{a}_{\nu}) - \delta(e^{\alpha}_{a}\partial_{\nu}e^{a}_{\mu}),$$
$$= \delta e^{\alpha}_{a}\partial_{\mu}e^{a}_{\nu} + e^{\alpha}_{a}\partial_{\nu}(\delta e^{a}_{\mu}) - \delta e^{\alpha}_{a}\partial_{\nu}e^{a}_{\mu} - e^{\alpha}_{a}\partial_{\nu}(\delta e^{a}_{\mu}.$$

Using the notation above and substituting we get :

$$\delta T^{\alpha}_{\mu\nu} \;\; = -e^{\alpha}_{b}(e^{\sigma}_{a}(\partial_{\mu}e^{a}_{\nu} - \partial_{\nu}(\delta e^{a}_{\mu}))\delta e^{b}_{\sigma} + -e^{\alpha}_{a}(\partial_{\mu}\delta e^{a}_{\nu} - \partial_{\nu}(\delta e^{a}_{\mu}), \tag{5.44}$$

$$= -e^{\alpha}_{b}T^{\sigma}_{\mu\nu}\delta e^{b}_{\sigma} + -e^{\alpha}_{a}(\partial_{\mu}\delta e^{a}_{\nu} - \partial_{\nu}(\delta e^{a}_{\mu}). \tag{5.45}$$

$$\delta(T^{\mu}T_{\mu}) = -2(T^{\gamma}T^{\beta}_{\alpha\mu} + T^{\beta}T_{\mu})e^{\mu}_{a}\delta e^{a}_{\beta} + 2(T^{\beta}e^{\mu}_{a} - T^{\mu}e^{\beta}_{a})\partial_{\beta}\delta e^{a}_{\beta}. \tag{5.46}$$

$$\delta(T_{\alpha\mu\nu}T^{\mu\alpha\nu}) = 2(T^{\beta\mu\alpha} - T^{\alpha\mu\beta})T_{\mu\alpha\nu}e^{\nu}_{a}\delta e^{a}_{\beta} + (T^{\alpha\beta}_{\mu} - T^{\beta\alpha}_{\mu})e^{\mu}_{a}\partial_{\alpha}\delta e^{a}_{\beta}. \tag{5.47}$$

$$\delta(T_{\alpha\mu\nu}T^{\alpha\mu\nu}) = -4T^{\alpha\mu\nu}T_{\alpha\mu\beta}e^{\beta}_{a}\delta e^{a}_{\nu} + 4T^{\mu\nu}_{\alpha}e^{\alpha_a}\partial_{\mu}\delta e^{a}_{\nu}. \tag{5.48}$$

it is easy to find the field equations after adding the matter Lagrangian:

$$\frac{1}{e}\partial_{\mu}(ee^{\rho}_{a}S^{\mu\nu}_{\rho}) - e^{\lambda}_{a}S^{\nu\mu}_{\rho}T^{\rho}_{\mu\lambda} + \frac{1}{4}e^{\rho}_{a}T = \frac{k^2}{2}e^{\rho}_{a}T^{\nu}_{\rho}. \tag{5.49}$$

with

$$T^{\nu}_{\rho} = \frac{1}{e}\frac{\delta\mathscr{L}_m}{\delta e^{a}_{\lambda}}. \tag{5.50}$$

and

$$S_{\rho}^{\mu\nu} = \frac{1}{4}\left(T^{\nu\mu}{}_{\rho} - T^{\mu\nu}{}_{\rho} + T_{\rho}^{\mu\nu}\right) + \frac{1}{2}\left(\delta_{\rho}^{\mu} T^{\alpha\nu}{}_{\alpha} - \delta_{\rho}^{\nu} T^{\alpha\mu}{}_{\alpha}\right). \tag{5.51}$$

The Teleparallel Lagrangian suffers from a defecct, because it is unable to govern the dynamics of the entier vierbein. In fact the equivalence between this action and Einstein -Hilbert action tells us that , it govern only the metric, as a result TE action is invariant under local Lorentz transformation.

5.1.2 Extension of TEGR theory

We saw earlier that TEGR can be alternatively formulated as gravitational theory in terms of the torsion .Taking the TEGR as a base we can build up a modified gravity by generalisation of the torsion T to a fonction f(T), inspired by the extension of the Ricci scalar to $f(R)$. As it seen most of works in modifing gravity starts from extension of the Einsteins-Hilbert action using the usual curvature .The reason for $f(T)$ being the best condidat is that the equation of motion are of second ordre unlike $f(R)$, however in $f(T)$ gravity one can obtain more degrees of freedoms compared with $f(R)$ theories.

The $f(T)$ gravity theory generalises T in the lagrangian density to an arbitrary function of T:

$$\mathscr{L} \Rightarrow \mathscr{L}_{f(t)} = \frac{e}{16\pi G} f(t). \tag{5.52}$$

The derivation of field equations is very similar to that described above for teleparallel gavity.

$$\delta S = \int d^4x \frac{e}{16\pi G} f(t) = 0. \tag{5.53}$$

$$\delta S = \frac{1}{16\pi G}\int d^4x \delta(e\delta f(t)), \tag{5.54}$$

$$= \frac{1}{16\pi G}\int d^4x \delta e f(T) + e\delta f(T). \tag{5.55}$$

$$\delta f(T) = \frac{\partial f(T)}{\partial T}\frac{\partial T}{\partial e_{\mu}^{a}}. \tag{5.56}$$

Folowing the same steps , the equations of motion read now :

$$\frac{1}{4}e_a^{\nu}f(T)+e^{-1}\partial_\mu(eS_a^{\mu\nu})f_T(T)+e_a^{\lambda}T^{\rho}_{\mu\lambda}S_\rho^{\mu\nu}f_T(T)+S_a^{\mu\nu}\partial_\mu(T)f_{TT}(T). \tag{5.57}$$

5.1.3 Gravitational interaction in teleparallel gravity

We mentioned earlier that tleparallel gravity can be interpreted as a gauge theory for the translation group The reason for translations can be understood from the gauge paradigm, of which Noether's theorem is a fundamental piece. Recall that the source of the gravitational field is energy and momentum. According to Noether's theorem, the energy–momentum current is covariantly conserved provided the source Lagrangian is invariant under spacetime translations. If gravitation is to present a gauge formulation with energy–momentum as the source, then itmust be a gauge theory for the translation group.

In teleparallel gravity the gauge transformation is defined as alocal tangent space coordinate

$$x^a \longrightarrow x^a+\varepsilon^a(x^4). \tag{5.58}$$

With $\varepsilon^a(x^4)$ the infinitesimal transformation parameter.

The Lagrangian in teleparallel gravity is :

$$L = L_G + L_m \tag{5.59}$$

$$L_G = \frac{e}{16\pi G}\left[\frac{1}{4}T_{\alpha\mu\nu}T^{\alpha\mu\nu}+\frac{1}{2}T_{\alpha\mu\nu}T^{\mu\alpha\nu}-T^{\mu}T_\mu\right] \tag{5.60}$$

From the varaiation of the action with respect to the tetrad e^a_μ, we obtain from the gauge Lagrangian L_G the teleparallel version of the gravitaional field equation using the Euler-Lagrange :

$$\frac{\partial L_G}{\partial e^a_\rho}-\partial_\sigma\frac{\partial L_G}{\partial(\partial_\sigma e^a_\rho)}=0. \tag{5.61}$$

$$\partial_\sigma(eS_a^{\rho\sigma})-eJ_a^{\rho}=eT_a^{\rho(m)}. \tag{5.62}$$

where $S_a^{\rho\sigma}$the superpotential

$$eS_a^{\rho\sigma}=\frac{\partial L_G}{\partial(\partial_\sigma e^a_\rho)}. \tag{5.63}$$

With the term

$$eJ_a^\rho = -\frac{\partial L_G}{(\partial e_\rho^a)}. \tag{5.64}$$

In our case represents the the Noether energy-momentum density of gravitational current the determination of (5.61) we start by deriving L_G

$$\begin{aligned}\frac{\partial}{\partial(\partial_\sigma e_\rho^a)}(T^{\alpha\mu\nu}T_{\alpha\mu\nu}) &= 2T_b^{\mu\nu}\frac{\partial T_{\mu\nu}^b}{\partial(\partial_\sigma e_\rho^a)}, \\ &= 2T_b^{\mu\nu}\delta_a^b(\delta_\mu^\sigma\delta_\nu^\rho - \delta_\nu^\sigma\delta_\mu^\rho), \\ &= T_a^{\sigma\rho}.\end{aligned} \tag{5.65}$$

the second term :

$$\begin{aligned}\frac{\partial}{\partial(\partial_\sigma e_\rho^a)}(T^{\alpha\mu\nu}T_{\mu\alpha\nu}) &= 2T_c^{\nu\mu}\frac{\partial T_{\mu\nu}^c}{\partial(\partial_\sigma e_\rho^a)} + T_{\mu\nu}^c\frac{\partial T_c^{\nu\mu}}{\partial(\partial_\sigma e_\rho^a)}, \\ &= T_a^{\rho\sigma} - T_a^{\sigma\rho}.\end{aligned} \tag{5.66}$$

$$\begin{aligned}\frac{\partial}{\partial(\partial_\sigma e_\rho^a)}(T^\mu T_\mu) &= 2T_\nu^{\nu\mu}e_b^\gamma\frac{\partial T_{\mu\gamma}^b}{\partial(\partial_\sigma e_\rho^a)}, \\ &= 2T_\nu^{\nu\mu}e_b^\gamma\delta_a^b(\delta_\mu^\sigma\delta_\gamma^\rho - \delta_\gamma^\sigma\delta_\mu^\rho)2T_\nu^{\nu\sigma}e_\rho^a - T_\nu^{\nu\rho}e_\sigma^a.\end{aligned} \tag{5.67}$$

The adition of these results gives us :

$$S_a^{\rho\sigma} = \frac{1}{2}(T_a^{\sigma\rho} + T_a^{\rho\sigma} - T_a^{\rho\sigma}) - T_\nu^{\sigma\nu}e_a^\rho + T_\nu^{\nu\rho}e_a^\sigma. \tag{5.68}$$

Now we move to J_a^ρ, and do the same steps but the derivative changes :

$$\begin{aligned}J_a^\rho &= -\frac{1}{e}\frac{\partial L_G}{\partial e_\rho^a}, \\ &= -\frac{1}{16\pi G}\left[\frac{1}{4}(T_b^{\mu\nu}\frac{\partial T_{\mu\nu}^b}{\partial e_\rho^a} + T_{\mu\nu}^b\frac{\partial T_b^{\mu\nu}}{\partial e_\rho^a} + T_{\mu\nu}^b\frac{\partial T_{\nu\mu}^b}{\partial e_\rho^a}) + \frac{1}{2}\frac{\partial T_{\mu\nu}^b}{\partial e_\rho^a} - T_{\gamma\mu}^\gamma\frac{\partial T_\nu^{\nu\mu}}{\partial e_\rho^a} - \frac{\partial_{\gamma\mu}^\gamma}{\partial e_\rho^a}T_\nu^{\nu\mu}\right] + e_a^\rho L_G.\end{aligned} \tag{5.69}$$

We use the following notation

$$\frac{\partial e}{\partial e_a^\rho} = e e_a^\rho, \qquad \frac{\partial e_b^\mu}{\partial e_a^\rho} = -e_a^\mu e_b^\rho, \tag{5.70}$$

$$\frac{\partial g^{\mu\nu}}{\partial e_a^\rho} = -g^{\rho\nu} e_{_b}^{\mu} - g^{\rho\nu} e_b^\nu, \tag{5.71}$$

simplifiying

$$\frac{\partial T_{\mu\nu}^b}{\partial e_\rho^a} = \omega_{a\mu}^b \delta_\mu^\rho - \omega_{a\nu}^c \delta_\nu^\rho, \tag{5.72}$$

$$\begin{aligned}
\frac{\partial T_b^{\mu\nu}}{\partial e_\rho^a} &= \eta_{bc} \frac{\partial}{\partial e_a^\rho} (g^{\mu\alpha} g^{\nu\beta} T_{\alpha\beta}^b), \\
&= (\frac{\partial g^{\mu\alpha}}{\partial e_a^\rho} (g^{\nu\beta} + g^{\mu\alpha} \frac{\partial g^{\nu\beta}}{\partial e_a^\rho}) T_{b\alpha\beta} + \eta_{bc} g^{\mu\alpha} g^{\nu\beta} \frac{\partial T_{\alpha\beta}^b}{\partial e_a^\rho}, \\
&= (-g^{\nu\beta} (g^{\alpha\rho} e_a^\mu - g^{\mu\rho} e_a^\alpha) - g^{\mu\alpha} (g^{\nu\rho} e_a^\beta - g^{\beta\rho} e_a^\nu)) T_{b\alpha\beta} + \eta_{bc} g^{\mu\alpha} g^{\nu\beta} (\omega_{a\alpha}^b \delta_\beta^\rho - \omega_{a\beta}^b \delta_\alpha^\rho), \\
&\quad - e_a^\mu T_b^{\rho\mu} - g^{\mu\rho} T^{ba\nu} - g^{\nu\rho} T_{ba}^\mu - e_a^\nu T_b^{\mu\rho} + \eta_{bc} (g^{\mu\alpha} g^{\nu\rho} \omega_{a\alpha}^c - g^{\mu\rho} g^{\nu\beta} \omega_{a\beta}^c).
\end{aligned} \tag{5.73}$$

$$\begin{aligned}
\frac{\partial T_b^{\nu\mu}}{\partial e_\rho^a} &= \eta_{bc} \frac{\partial}{\partial e_\rho^a} (e_\lambda^\rho T^{\nu\mu\lambda}), \\
&= \eta_{ba} T^{\nu\mu\rho} T^{\nu\mu\rho} + \eta_{bc} e_\lambda^b \frac{\partial T^{\nu\mu\lambda}}{\partial e_\rho^a}.
\end{aligned} \tag{5.74}$$

$$\begin{aligned}
\frac{\partial T^{\nu\mu\lambda}}{\partial e_\rho^a} &= \frac{\partial}{\partial e_\rho^a} (\eta^{cb} e_c^\nu T_c^{\mu\lambda}), \\
&= -\eta^{cb} e_a^\nu - e_b^\rho T_C^{\mu\lambda} + \eta^{cb} e_b^\nu \frac{\partial T_b^{\mu\lambda}}{\partial e_\rho^a}, \\
&= -e_a^\nu T^{\rho\mu\lambda} - \eta^{cb} e_b^\nu (e_a^\mu T_b^\mu T_b^{\rho\lambda} + g^{\mu\rho} T_{ba}^\lambda + g^{\lambda\rho} T_{ba}^\mu + e_a^\lambda T_b^{\mu\rho}), \\
&\quad + \eta^{cb} e_b^\nu \eta_{bd} g^{\mu\alpha} g^{\lambda\rho} \omega_{a\alpha}^d - \eta^{cb} e_b^\nu \eta_{bd} g^{\mu\rho} g^{\lambda\beta} \omega_{\alpha\alpha}^d, \\
&= -e_a^\nu T^{\rho\mu\lambda} - e_a^\mu T^{\nu\rho\lambda} - g^{\mu\lambda} T_a^{\nu\lambda} g^{\alpha\beta} \omega_{\alpha\beta}^c, \\
\frac{\partial T^{\nu\mu}{}_b}{\partial e_\rho^a} &= -e_a{}^\nu T^{\rho\mu}{}_b - e_a{}^\mu T^{\nu\rho}{}_b - e_b{}^\rho T^{\nu\mu}{}_a - g^{\mu\rho} T^\nu{}_{ab} + e_c{}^\nu (g^{\mu\lambda} g^{\sigma\rho} - g^{\mu\rho} g^{\sigma\lambda}) \omega^c{}_{a\lambda}.
\end{aligned} \tag{5.75}$$

The fifth term:

$$\begin{aligned}\frac{\partial T^{\nu\mu}{}_{\nu}}{\partial e^{a}_{\rho}} &= \frac{\partial}{\partial e^{a}_{\rho}}(g_{\nu\lambda}T^{\nu\mu\lambda}) = \frac{\partial g_{\nu\lambda}}{\partial e^{a}_{\rho}}T^{\nu\mu\lambda} + g_{\nu\lambda}\frac{\partial T^{\nu\mu}{}_{\nu}}{\partial e^{a}_{\rho}},\\ &= -T^{\rho\mu}{}_{a} - e_{a}{}^{\mu}T^{\nu\rho}{}_{\nu} - g^{\mu\rho}T^{\nu}{}_{a\nu} + e_{b}{}^{\rho}g^{\mu\nu}\omega^{b}{}_{a\nu} - e^{\nu}_{b}g^{\mu\rho}\omega^{b}_{a\nu}.\end{aligned} \tag{5.76}$$

$$\begin{aligned}\frac{\partial T^{\lambda}_{\lambda\nu}}{\partial e^{a}_{\rho}} &= \frac{\partial}{\partial e^{a}_{\rho}}(e^{\lambda}_{b}T^{b}_{\mu\lambda}) = -e^{\lambda}_{a}e^{\rho}_{b}T^{b}_{\mu\lambda} + e^{\lambda}_{b}\frac{\partial T^{b}_{\mu\lambda}}{\partial e^{a}_{\rho}},\\ &= -T^{\rho}_{\mu a} + e^{\rho}_{b}\omega^{b}_{a\mu} - e^{\lambda}_{b}\omega^{b}_{a\lambda}.\end{aligned} \tag{5.77}$$

$$J^{\rho}_{a} = \frac{1}{8\pi G}\left[e^{\lambda}_{a}S^{\nu\rho}_{b}T^{\nu\lambda}_{b} + \omega^{b}_{a\sigma}S^{\rho\sigma}_{b}\right] - \frac{e^{\rho}_{a}}{e}\mathscr{L}. \tag{5.78}$$

Due to the anti-symmetry of $S^{\rho\sigma}_{b}$ in the last two indices, (eJ^{ρ}_{a}) and,$T^{\nu\lambda}_{b}$is conserved as a consequence of the field equation:

$$\partial_{\rho}(eJ^{\rho}_{a} + eT^{\rho(m)}_{a}) = 0. \tag{5.79}$$

We can write (5.71)in the following form

$$G_{\mu\nu} = T^{m}_{\nu\mu}. \tag{5.80}$$

This result tells us that there is an equivalence between the gravitationl field in the presence of matter source in GR and TEGR. This means that both of the curvature and torsion are related to the same degrees of freedom of gravitation field. Let us know take the geodesic equation of general relativity .

$$\frac{du^{a}}{ds} + \tilde{\Gamma}^{a}_{\mu\nu}u_{\mu}u^{\nu} = 0. \tag{5.81}$$

Using the relation between the Weitzenbock and christoffel connection

$$\Gamma^{a}_{\mu\nu} = \tilde{\Gamma}^{a}_{\mu\nu} + K^{a}_{\mu\nu}. \tag{5.82}$$

$$\frac{du^{a}}{ds} + \Gamma^{a}_{\mu\nu}u^{\mu}u^{\nu} = K^{a}_{\mu\nu}u^{\mu}u^{\nu}. \tag{5.83}$$

5.2 The ghost condensate galileon in modified teleparallel gravity

We consider the cosmology of a galileon field with self cubic interaction and a ghost condensate term in the framework of modified teleparallel gravity. The action describing the model is given by :

$$S=\int dx^4 e(f(T)+L_\phi+L_m+L_r). \tag{5.84}$$

Where L_m and L_r are the Lagrangian of matter and radiation, and L_ϕ the Lagrangian of the ghost condensate galileon

$$L_\phi=c_1X+c_2X^2+c_3X\Box\phi. \tag{5.85}$$

where c_i are constants. The kinetic term of the scalar field ϕ is defined $X=-1/2g^{\mu\nu}\nabla_{;\mu}\phi\nabla_{;\nu}\phi$, $\Box=g_{\mu\nu}\nabla^{;\mu}\nabla^{;\nu}$, is the d'alembertian where and e is the determinant of the tetrad field e^a_μ, T is the scalar of torsion. It is obvious that the action above invariant under the transformation $\phi\rightarrow\phi+c$.

Variying this action with respect to the tetrad fileds and the scalar field ($L(g_{\mu\nu},\phi,\partial\phi)$), we get the equations of motion

$$\delta S=\int dx^4\left[\frac{\delta(ef(T))}{\delta e^a_\mu}\delta e^a_\mu+\frac{\delta(eL_\phi)}{\delta e^a_\mu}+\frac{\delta(eL_m)}{\delta e^a_\mu}+\frac{\delta(eL_r)}{\delta e^a_\mu}+\frac{\delta(eL_\phi)}{\delta\phi}\delta\phi)+\frac{\delta(eL_\phi)}{\delta(\partial\phi)}\delta(\partial\phi)\right]. \tag{5.86}$$

First we start with the variation of $f(T)$. Let us recall the following results :

$$\delta|e|=ee^\mu_a\delta e^a_\mu, \tag{5.87}$$

$$\delta e^\mu_a=-e^\mu_b e^\nu_a\delta e^b_\nu, \tag{5.88}$$

$$\delta g_{\mu\nu}=\eta_{ab}(e^a_\mu\delta e^b_\nu+e^a_\nu\delta e^b_\mu), \tag{5.89}$$

$$\delta g^{\mu\nu}=-(g^{\mu\nu}e^\nu_b+g^{\nu\mu}e^\mu_a)\delta e^a_\alpha, \tag{5.90}$$

The total variation are given by:

$$\delta S=\delta S^1+\delta S^2+\delta S^3+\delta S^4. \tag{5.91}$$

5.2.0.1 Variation of S^1

$$\delta S^1 = \frac{\delta(ef(T))}{\delta e^a_\mu} = ef_T \frac{\delta T}{\delta e^a_\mu} + f(T)\frac{\delta e}{\delta e^a_\mu}. \tag{5.92}$$

$$\frac{\delta e}{\delta e^a_\mu} = ee^\beta_a \frac{\delta e^a_\beta}{\delta e^a_\mu} = ee^\beta_a \delta^\beta_\mu. \tag{5.93}$$

Where f_T is the derivative with respect to the torsion scalar.

5.2.0.2 Variation of S^2

Variation with respect to the scalar field

$$\delta S^{(2)} = \int d^4x\delta\left(\sqrt{-g}c_i L^i_\phi + L_m\right) = 0. \tag{5.94}$$

The variation of the first term of the action is given by:

$$\delta S^{(2)} = \int d^4x\sqrt{-g}\left(c_1\delta X + c_2\delta X^2 + c_3\delta(X\Box\phi) + L_m\right). \tag{5.95}$$

$$\delta S^{(2)}_1 = -\frac{1}{2}\int d^4x\sqrt{-g}c_1\delta((\nabla^{;\mu}\phi)(\nabla_{;\nu}\phi)), \tag{5.96}$$

$$= -\frac{1}{2}\int d^4x\sqrt{-g}c_1[\delta(\nabla^{;\mu}\phi)\nabla_{;\nu}\phi + \nabla^{;\mu}\phi\delta(\nabla_{;\nu}\phi)], \tag{5.97}$$

$$= -\int d^4x\sqrt{-g}c_1\left[(\nabla^{;\mu}\phi\delta(\nabla_{;\mu}\phi))\right], \tag{5.98}$$

$$= -\int d^4x\sqrt{-g}c_1\left[(\nabla^{;\mu}\phi\nabla_{;\nu}(\delta\phi)\right] \tag{5.99}$$

$$= -\int d^4x\sqrt{-g}c_1\left[\nabla_{;\mu}(\nabla^{;\mu}\phi(\delta\phi)) - \Box\phi(\delta\phi)\right]. \tag{5.100}$$

Using the relation :

$$\nabla_{;\mu}W^{;\mu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}W^{;\mu}). \tag{5.101}$$

and the first term is a surface term which vanishes at infinity we obtain:

$$\delta S^{(2)}_1 = \int d^4x\sqrt{-g}c_1\Box\phi(\delta\phi). \tag{5.102}$$

The variation of the second term:

$$\delta S_2^{(2)} = \frac{1}{(4)}\int d^4x\sqrt{-g}c_2\left(\nabla_{;\mu}\phi\nabla^{;\mu}\phi\right), \tag{5.103}$$

$$= \frac{1}{2}\int d^4x\sqrt{-g}c_2\delta\left(\nabla_{;\mu}\phi\nabla^{;\mu}\phi\right)\left(\nabla_{;\mu}\phi\nabla^{;\mu}\phi\right), \tag{5.104}$$

$$= \frac{1}{(4)}\int d^4x\sqrt{-g}c_2[\nabla_{;\mu}(\nabla^{;\mu}\phi(\delta\phi)) - \Box\phi(\delta\phi)](\nabla\phi)^2, \tag{5.105}$$

$$= -\frac{1}{2}\int d^4x\sqrt{-g}c_2\left[(\Box\phi)(\nabla\phi)^2\right]. \tag{5.106}$$

Variation of the third term using the same method before:

$$\delta S_3^{(2)} = -\frac{1}{2}\delta\int d^4x\sqrt{-g}c_3[\{(\nabla^{;\mu}\phi\nabla_{;\nu}\phi)\}\Box\phi], \tag{5.107}$$

$$= -\frac{1}{2}\delta\int d^4x\sqrt{-g}c_3\left\{2\nabla^{;\mu}\phi\nabla_{;\mu}(\delta\phi)\Box\phi + (\nabla^{;\mu}\phi\nabla_{;\nu}\phi\delta(\nabla^{;\mu}\nabla_{;\nu}g^{\mu\nu}))\right\}, \tag{5.108}$$

$$= -\int d^4x\sqrt{-g}c_3\left\{(-(\Box\pi)^2 - \nabla_{;\mu}(\Box\phi) + \nabla_{;\mu}(\nabla^{;\nu\mu}\nabla_{;\nu}))(\delta\phi) - \frac{1}{2}\nabla_{;\mu}(\nabla^{;\mu}((\nabla\phi)^2)\delta\phi)\right\}, \tag{5.109}$$

$$= -\int d^4x\sqrt{-g}c_3\left[(\Box\pi)^2 + (\nabla_{;\nu}\nabla_{;\mu}\phi)(\nabla_{;\nu}\nabla_{;\mu} - \nabla_{;\mu}(\Box\phi) + \Box(\nabla_{;\nu})\right](\delta\phi). \tag{5.110}$$

where we have used :

$$\nabla_{;\mu}(\Box\phi) + \Box(\nabla_{;\nu}\phi) = R^{\mu\nu}\phi_{;\mu}\phi_{;\nu}. \tag{5.111}$$

In what follows we move on to the calculation of the different contributions $T^{(i)}_{\alpha\beta}(i = 1,2,3)$ of the energy momentum of the galileon. Variation with respect to the metric of the first term of the action:

$$\delta S_1^{(2)} = -\frac{1}{2}\int d^4x\left[\delta(\sqrt{-g}\phi^{;\alpha}\phi_{;\alpha})\right], \tag{5.112}$$

$$= -\frac{1}{2}\int d^4x\left[\sqrt{-g}\delta(g^{\alpha\beta}\phi_{;\beta}\phi_{;\alpha})\right], \tag{5.113}$$

$$= -\frac{1}{2}\int d^4x\left[\delta\sqrt{-g}(g^{\alpha\beta}\phi_{;\beta}\phi_{;\alpha}) + \sqrt{-g}\delta(g^{\alpha\beta}\phi_{;\beta}\phi_{;\alpha})\right], \tag{5.114}$$

$$= -\frac{1}{2}\int d^4x\sqrt{-g}\left\{-\frac{1}{2}g_{\alpha\beta}(\phi^{;\alpha}\phi_{;\beta|} + (\phi_{;\alpha}\phi_{;\beta})\delta\right\}g^{\alpha\beta}. \tag{5.115}$$

with

$$\frac{\delta\sqrt{-g}}{\sqrt{-g}} = \frac{1}{2}\frac{\delta\,(-g)}{(-g)} = -\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}. \tag{5.116}$$

$$T_{\alpha\beta} = \phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}(\nabla\phi)^2. \tag{5.117}$$

Variation of the second term:

$$\delta S_2^{(2)} = \int d^4x c_2 \delta\left[\sqrt{-g}\left(-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi\right)^2\right], \tag{5.118}$$

$$= \int d^4x\sqrt{-g}c_2\left[-\frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}\left(-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi\right)^2 + \delta\left(\nabla_\alpha\phi\nabla_\beta\phi g^{\alpha\beta}\right)\left(-\frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi\right)\right], \tag{5.119}$$

$$= -\frac{1}{2}\int d^4x\sqrt{-g}c_2\left[g_{\alpha\beta}\left(-\frac{1}{2}\phi_\alpha\phi^\alpha\right)^2 + \left(\phi_{;\alpha}\phi_{;\beta}\right)(\nabla\phi)^2\right]. \tag{5.120}$$

Then

$$T_{;\alpha\beta} = g_{\alpha\beta}\left(-\frac{1}{2}\phi_{;\alpha}\phi^{;\alpha}\right)^2 + \left(\phi_{;\alpha}\phi_{;\beta}\right)(\nabla\phi)^2. \tag{5.121}$$

The third term:

$$\delta S_2^{(3)} = -\frac{1}{2}\int d^4x\delta\left[\sqrt{-g}(\nabla\phi)^2(\Box\phi)\right], \tag{5.122}$$

$$= -\frac{1}{2}\int d^4x\left[\delta(\sqrt{-g})(\nabla\phi)^2(\Box\phi) + \sqrt{-g}\delta(\nabla_{;\alpha}\phi\nabla_{;\beta}\phi g^{\alpha\beta})(\Box\phi) + \sqrt{-g}(\nabla\phi)^2\delta(\Box\phi)\right], \tag{5.123}$$

$$= -\frac{1}{2}\int d^4x\sqrt{-g}c_3\left[g_{\alpha\beta}(\nabla\phi)^2(\Box\phi) + (\nabla_{;\alpha}\phi\nabla_{;\beta}\phi)(\Box\phi) + (\nabla\phi)^2(\nabla_\alpha\nabla_{;\beta}\phi)\right]\delta g^{\alpha\beta}. \tag{5.124}$$

To calculate the last term in (5.122), we use the the following equation:

$$\delta(\nabla_{;\beta}\phi) = -\delta\Gamma^\mu_{\alpha\beta}(\phi_\mu). \tag{5.125}$$

With

$$\delta\Gamma^\mu_{\alpha\beta} = -\frac{1}{2}g^{\mu\rho}\left[\nabla_{;\beta}(\delta g_{\alpha\rho}) + \nabla_{;\alpha}(\delta g_{\beta\rho}) - \nabla_{;\rho}(\delta g_{\alpha\beta})\right]. \tag{5.126}$$

Then

$$\delta S_4^{(3)} = -\frac{1}{2}\int d^4x\sqrt{-g}(\nabla\phi)^2\delta(\nabla_{;\beta}\phi)g^{\alpha\beta}, \tag{5.127}$$

$$= -\frac{1}{4}\int d^4x\sqrt{-g}\left[(\nabla\phi)^2 g^{\mu\rho}\left[\nabla_{;\beta}(\delta g_{\alpha\rho}) + \nabla_{;\alpha}(\delta g_{\beta\rho}) - \nabla_{;\rho}(\delta g_{\alpha\beta})\right](\phi_\mu)g^{\alpha\beta}\right], \tag{5.128}$$

$$= -\frac{1}{4}\int d^4x\sqrt{-g}\left[(\nabla\phi)^2\phi^{;\rho}(\nabla^{;\alpha}(\delta g_{\alpha\rho}) + \nabla^{;\beta}(\delta g_{\beta\rho}) - \nabla_{;\rho}(\delta g_{\alpha\beta})g^{\alpha\beta})\right]. \tag{5.129}$$

using the Ostrograsdeski theorem ,and after the variation we have :

$$\delta S_4^{(3)} = -\frac{1}{4}\int d^4x\sqrt{-g}\left[-\nabla^{;\alpha}(\nabla\phi)^2\phi^{;\rho}(\delta g_{\alpha\rho}) - (\nabla\phi)^2\phi^{;\rho\alpha}(\delta g_{\alpha\rho}) - \nabla^{;\beta}(\nabla\phi)^2\phi^{;\rho}(\delta g_{\beta\rho})\right], \tag{5.130}$$

$$= -\frac{1}{4}\int d^4x\sqrt{-g}\left[4\phi_{;\mu\alpha}\phi^{;\mu}\phi_{;\beta} + 2(\nabla\phi)^2\phi_{;\alpha\beta} - 2\phi_{;\mu\rho}\phi^{;\mu}\phi^{;\nu\rho}g_{\alpha\beta} - (\nabla\phi)^2(\Box\phi)^2 g_{\alpha\beta}\right]\delta g^{\alpha\beta}. \tag{5.131}$$

$$T_{\alpha\beta} = \frac{1}{4}\left(4\phi_{;\mu\alpha}\phi^{;\mu}\phi_{;\beta} + 2(\nabla\phi)^2\phi_{;\alpha\beta} - 2\phi_{;\mu\rho}\phi^{;\mu}\phi^{;\nu\rho}g_{\alpha\beta} - (\nabla\phi)^2(\Box\phi)^2 g_{\alpha\beta}\right). \tag{5.132}$$

The first therm in the action is nothing but the equation of motion of the tetrad field, while the second one is the equation of motion of the scalar field, we can write the equation of motion as follows:

$$T_a^{\mu(T)}e_\nu^a = T_\mu^{\nu(\phi)} + T_\mu^{\nu(m)} + T_\mu^{\nu(r)}. \tag{5.133}$$

Where

$$T_a^{\mu(T)}e_\nu^a = \frac{1}{4}fe_a^\mu + f_T(\nabla_{;\nu}(e_a^\rho S_\rho^{\nu\mu}) - e_a^\lambda T_{\nu\lambda}^\rho S_\rho^{\nu\mu}) + e_a^\rho\partial_\nu f_T S_\rho^{\mu\nu}. \tag{5.134}$$

$$T_{\mu\nu}^\phi = (c_1X + c_2X^2)g^{\mu\nu} + (c_1 + 2c_{2X})\partial_{;\mu}\phi\partial_{;\nu}\phi + c_3(g_{\mu\nu}\nabla_{;\alpha}\phi\nabla_{;\beta}\phi\partial_{;\alpha}\partial_{;\beta}\phi - \Gamma_{\beta\alpha}^\lambda\partial_\lambda\phi.$$
$$+\Box\nabla_{;\mu}\phi\nabla_{;\nu}\phi - \nabla^{;\alpha}\phi(\nabla^{;\mu}\phi\partial_\alpha\partial_\nu - \Gamma_{\nu\alpha}^\lambda\partial_\lambda\phi)(\partial_\alpha\partial_\mu - \Gamma_{\mu\alpha}^\lambda\partial_\lambda\phi)). \tag{5.135}$$

The energy momentum tensor here consered for each fluid

$$\nabla_{;\mu}T_i^{\mu\nu} = 0. \tag{5.136}$$

leading to

$$T_i^{\mu\nu} = (P_i + \rho_i)U^\mu U^\nu + P_i g^{\mu\nu}. \tag{5.137}$$

where $i = m, r$ represents the matter and radiation in the universe..

We know that the energy momentum tensor is symmetric and invariant under local Lorentz transformation, then the anti-symmetric part in the tensor (5.134) must vanish. Thus it leads us to the constraint

$$(S_{\lambda}^{\sigma\nu} g^{\lambda\mu} - S_{\lambda}^{\sigma\nu} g^{\lambda\nu})\partial_{\sigma} f_T = 0. \tag{5.138}$$

Let us know take a flat, homogeneous and isotropic FLRW space-time metric, where

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j. \tag{5.139}$$

then the metric can take the form using the tetrad field:

$$e_{\mu}^{a} = \mathrm{diag}(1, a(t), a(t), a(t)). \tag{5.140}$$

The background equations of motion are given by replacing (5.138) in (5.131), and imposing that the scalar field is an homogeneous function of the cosmic time. The scalar field torsion in this bockground is $T = 6H^2$.

$$-\frac{1}{2}f(T) - \rho_m(t) + Tf(T) - \frac{1}{2}c_1\dot{\phi}(t)^2 + 3c_3H(t)\dot{\phi}^3 - \frac{3}{4}c_2\dot{\phi}^4, \tag{5.141}$$

$$-\frac{1}{2}f(T) + P_m(t) + 6H^2(t)f_T(T) + 2f_T(T)\dot{H}(t) + \frac{1}{2}c_1\dot{\phi}^2 + \frac{1}{4}c_2\dot{\phi}^4 + 24H^2(t)\dot{H}(t)f_{TT}(T) + c_3\dot{\phi}^2\ddot{\phi}(t), \tag{5.142}$$

$$-3c_2H(t)\dot{\phi}^3 + c_1(-3H(t)\dot{\phi}(t) - \ddot{\phi}) - 3c_2\dot{\phi}^2\ddot{\phi} + c_3(9H(t)^2\dot{\phi}^2 + 3\dot{H}(t)\dot{\phi}^2 + 6H(t)\dot{\phi}\ddot{\phi}(t)). \tag{5.143}$$

Where the scalar of Torsion is $T = 6H^2$, $H = a/\dot{a}$ is the Hubble constant and it's derivative $\dot{H}$ with respect to t. We can write the equations (5.140) and (5.141) in a standard way

$$3H^2 = \rho_{DE} + \rho_m + \rho_r. \tag{5.144}$$

$$3H^2 + 2\dot{H}^2 = -(P_{DE} + P_m + P_r). \tag{5.145}$$

Let us consider now the particular choice

$$f(T) = T + kT^m. \tag{5.146}$$

Then

$$f_T(T) = (1 + km(T)^{m-1}) \quad f_{TT}(T) = [(m-1)km(T)^{m-2}]. \tag{5.147}$$

Now the density ρ_{DE} and P_{DE} are given by:

$$\rho_{DE} = 2^{m-1}K(3T)^m - mk(6T)^m + \frac{1}{2}c_1\dot{\phi}^2 - 3c_3H\dot{\phi}^3 + \frac{3}{4}c_2\dot{\phi}^4. \tag{5.148}$$

$$P_{DE} = (2m-1)k(T6)^{m-1}(3H^2 + 2m\dot{H}) + \frac{1}{4}\dot{\phi}^2(2c_1 + c_2\dot{\phi}^2 + 4c_3\ddot{\phi}). \tag{5.149}$$

The EOS parameter is defined

$$\omega_{DE} = \frac{P_{DE}}{\rho_{DE}}. \tag{5.150}$$

5.3 Dynamical system analysis

Let us now write the Friedman equations:

$$3H^2 - 2^{-1+m}3^mKH^{2m} + mk(6H^2)^m - \frac{1}{2}c_1\dot{\phi}(t)^2 + 3c_3H(t)\dot{\phi}^3 - \frac{3}{4}c_2\dot{\phi}^4 - \rho_m. \tag{5.151}$$

$$3H^2 + 2\dot{H} + (2m-1)k(6H^2)^{-1+m}(3H^2 - 2m\dot{H}) + \frac{1}{4}\dot{\phi}^2(2c_1 + c_2\dot{\phi}^2 + 4c_3\ddot{\phi}) + P_m. \tag{5.152}$$

To fix some of the constants of the model we consider de Sitter epoch where $H = H_{ds} = const$ and where $P_m = 0$,$\rho_m = 0$, $\rho_r = 0$ we obtain from Friedman equations :

$$c_1 = -\frac{12H_{ds}^2 + 2^{1+m}3^m(-1+2m)kH_{ds}^{2m} + 3c_3H_{ds}\dot{\phi}_{ds}^2}{\dot{\phi}_{ds}^2}, \tag{5.153}$$

$$c_2 = \frac{2(6H_{ds}^2 + (2m-1)k(6H_{ds}^2)^m) + 3c_3H_{ds}\dot{\phi}_{ds}^3}{\dot{\phi}_{ds}^4}, \tag{5.154}$$

This let us with two free parameters k and c_3.

To study the dynamics of FRLW equations in the framework of dynamical system theory, we define the following dimensionless quantities:

$$x_1 = \frac{c_1\dot{\phi}^2}{6H^2}, \qquad x_2 = \frac{c_2\dot{\phi}^4}{4H^2}, \qquad x_3 = -\frac{c_3\dot{\phi}^3}{H}, \qquad x_4 = \frac{(1-2m)k}{6^{1-m}H^{2-2m}}, \tag{5.155}$$

$$\Omega_r = \frac{\rho_r}{3H^2}, \qquad \Omega_m = \frac{\rho_m}{3H^2},$$

Then it follows that :

$$x_1 + x_2 + x_3 + x_4 = \Omega_{DE}. \tag{5.156}$$

and

$$1 = \Omega_{DE} + \Omega_m + \Omega_r. \tag{5.157}$$

The quantities $x_{1,2,3,4}$ and Ω_r obey the differential equations:

$$x_1' = 2(-\varepsilon_H + \varepsilon_\phi)x_1, \tag{5.158}$$

$$x_2' = -2(\varepsilon_H - 2\varepsilon_\phi)x_2, \tag{5.159}$$

$$x_3' = -(\varepsilon_H - 3\varepsilon_\phi)x_3, \tag{5.160}$$

$$x_4' = 2(-1+m)\varepsilon_H x_4, \tag{5.161}$$

$$\Omega_r' = -2(2+\varepsilon_H)\Omega_r, \tag{5.162}$$

where a prime represents a derivative with respect to $N = \ln a$, and

$$\frac{1}{3}(3 + 3x_1 + x_2 - \varepsilon_\phi x_3 - 3x_4 + \varepsilon_H(2 - 2mx_4) + \Omega_r) = 0, \tag{5.163}$$

$$-2(3+\varepsilon_\phi)x_1 - 4(1+\varepsilon_\phi) + x_2 - (3 + \varepsilon_H + 2\varepsilon_\phi)x_3 = 0, \tag{5.164}$$

Solving in terms of ε_H and ε_ϕ we obtain:

$$\varepsilon_H = \frac{-(6x_1 + 6x_1^2 + 12x_2 + 14x_1x_2 + 4x_2^2 + 6x_3 + 12x_1x_3}{4x_1 + 8x_2 + 4x_3 + x_3^2 - 4mx_1x_4 - 8mx_2x_4 - 4mx_3x_4} \tag{5.165}$$

$$\frac{+6x_2x_3) + 3x_3^2 - 6x_1x_4 - 12x_2x_4 - 6x_3x_4 + 2x_1\Omega_r + 4x_2\Omega_r + 4x_2\Omega_r + 2x_3\Omega_r}{4x_1 + 8x_2 + 4x_3 + x_3^2 - 4mx_1x_4 - 8mx_2x_4 - 4mx_3x_4}. \tag{5.166}$$

$$\varepsilon_\phi = \frac{-12x_1 - 8x_2 - 3x_3 + 3x_1x_3 + x_2x_3 + 12mx_1x_4 + 8mx_2x_4 - 3x_3x_4 + 6mx_3x_4 + x_3\Omega_r}{-4x_1 - 8x_2 - 4x_3 - x_3^2 + 4mx_1x_4 + 8mx_2x_4 + 4mx_3x_3}. \tag{5.167}$$

The equation of state of Dark energy is then given by:

$$\omega_{DE} = \frac{-4(3+2m\varepsilon_H)x_4 + 12x_1 - 4x_3\varepsilon_\phi + 4x_2}{12x_4 + 12x_1 - 12x_3 + 12x_2}. \tag{5.168}$$

We note that we used the constraint $\Omega_{DE} + \Omega_r + \Omega_m = 1$.

We again obatin a relation between the dynamical variables in de Sitter era where:

$$\varepsilon_{\phi ds} = 0, \quad \varepsilon_H = 0, \quad \Omega_r = 0, \tag{5.169}$$

and then $x_{1|ds}$ and $x_{2|ds}$ are given by:

$$x_{1|ds} = \frac{1}{2}(-4 + x_3 + 4x_4)_{|ds}. \tag{5.170}$$

$$x_2|ds = -\frac{3}{2}(-2 + x_3 + 2x_4)_{|ds}. \tag{5.171}$$

5.3.1 Numerical Solution

In the following we proceeds to a numerical integration of the dynamical system equations. We aim to show the evolution of the variables x_i , Ω_i, w_{DE} and H with the cosmic time. The initial conditions are taken at $N_i = \ln a_i = -20$, are $x_{1i} = -0.150524\,10^{-15}, x_{3i} = 2.00539\,10^{-4}, x_{4i} = 10.6157, x_{2i} = 1 - x_{3i} - x_{4i} - \Omega_{ri} - \Omega_{ri}\frac{0.28621}{8.3\,10^{-5}}e^{-20}, \Omega_{ri} = 0.999997$.

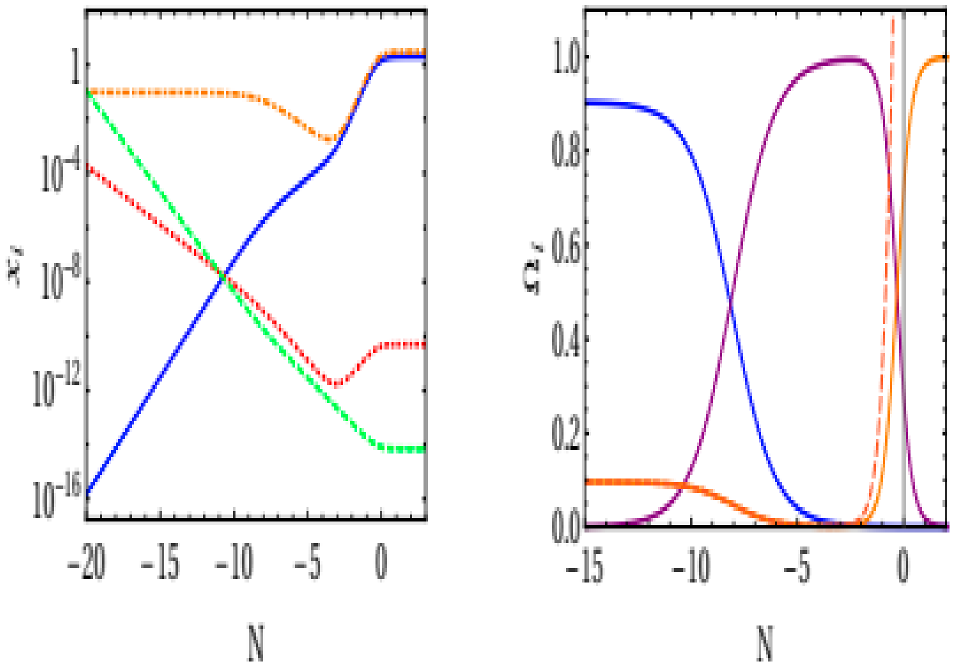

Figure 5.1: Left: variation of the dynamical variables vs N. Right: variation of the dimensionless density energies with N.

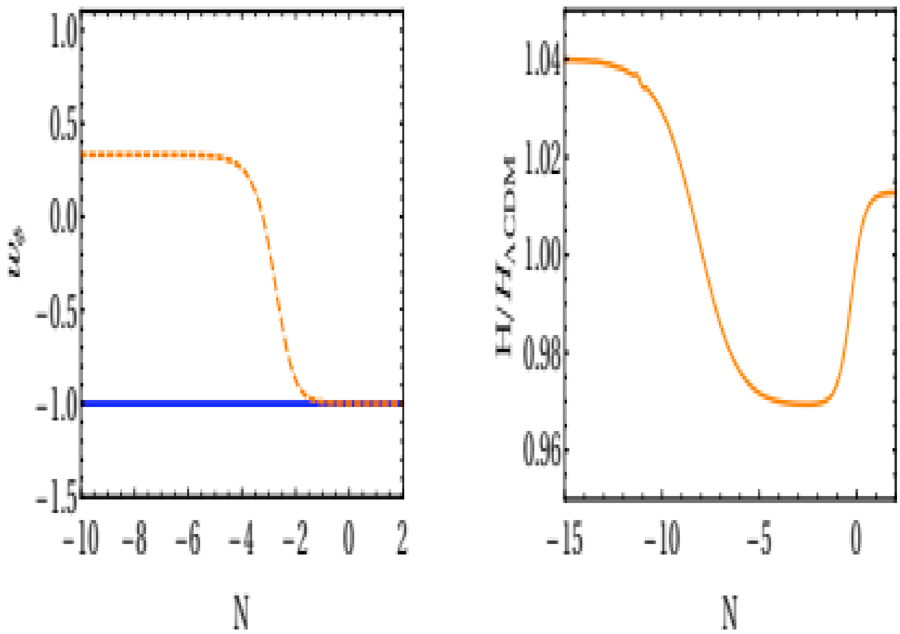

Figure 5.2: Evolution of the EoS dark energy w_{DE} and $H/H_{\Lambda CDM}$.

For the dark energy equation of state parameter it is easy to observe that $w_{DE} \to -1$ when $x_3, x_4 \to 0$. This behavior starts in the matter domnated era and continue in the dark energy energy era, as the cosmological constant do. Thus the ghost galileon field perfectly mimics the constant cosmological in the matter era. This result is of great importance since we have a dynamical dark energy as can be seen from figure.5.1 but with an EOS $w_{DE} = -1$in the matter dominated era. We also plot in figure.5.2 the ration $H/H_{\Lambda CDM}$,and we see a great agrement with the ΛCDMmodel. The maximal error is less then 5%.

Chapter 6

Conclusion

The great theoretical and experimental research activity currently carried out in cosmology and particle physics is entirely devoted to understanding the true nature of dark energy , the cause of the accelerated universe . Although the cosmological constant$\wedge$is the simplest candidate for dark energy, it is still a challenging problem to explain the very small value of $\wedge$ consistent with today's dark energy scale.this was one of the bigest problem's in the lamda CDM model,besides the problem's that we discocted earlear

Acording to the first chapiter we saw that in the theory of Eeinstein there is no gravitational force interaction in general relativity, but gravitational field wich can be expressed by the torsion-less Levi–Civita metric–connection, whose curvature determines the intensity of the gravitational field. On the other hand, in the teleparallel description of gravitation,the presence of a gravitational field is expressed by the flat Weitzenbock connection, whose torsion is now the entity responsible for determining the intensity of the gravitational field. The gravitational interaction, therefore, can be described either, in terms of curvature, or in terms of torsion. Whether gravitation requires a curved or a torsioned spacetime, therefore, is a matter of convention.

In general space time can in principle present two different properties curvature and Torsion but the torsion is absent in GR theory . makes the theory incomplit in some physicits vision and need to be extanded .This extension in GR was by adding exra degrees of freedom like the scalar field ϕ like the Horndesky theories in the third chapitre, wich turned out to be a generalisation of the Galileon fields to curved space .

In the last chapitre we saw that general theory of relativity can be described in terms of tetrad field and of the torsion tensor ,this tetrad field satisfies a field equations that are strictly equivalent to Eeinstein's equation .In this geometrical description .the tetrad field yield several definition that can't not be astablished in the ordinary metric .An important point of the teleparallel equivalent of general relativity is that it allows for the definition of an energy-momentum gauge current J_a^ρ for the gravitational field which is covariant under a spacetime general coordinate transformation, This means essentially that $J_a^\rho$is a true spacetime tensor, but not a tangent–spacetensor. Then, by rewriting the gauge field equation in a purely spacetime form,it becomes Einstein's equation, and the gauge currentJ_a^ρ reduces to the canonical energy-momentum pseudotensor of the gravitational field. Teleparallel gravity,therefore, seems to provide a more appropriate environment to deal with the energy problem since in the ordinary context of general relativity, the energy-momentum density for the gravitational field will always be represented by a pseudotensor.

Many models proposed in an attempt to solve the puzle of Dark energy,chapitre four contain some of these models, as well it contains the scalar tensor theories chapitre to$f(T)$ gravity is the simplest modification of TEGR,by replacing the so-called torsion scalar T with $f(T)$. the equations of motion of f(T) gravity are second-order instead of fourth-order. Secondly, the local Lorentz invariance is violated in f(T) gravity .Therefore, extra degrees of freedom will appear. Till now, it is not clear how many extra degrees of freedom there are in f(T) gravity..

In the last chapitre we did a combination between the modified TEGR and the Galeleon field model introduced by Nicolas et al. this model motivated by the feeding of a class of a scalar mode linked to the brane at 4 dim which appears at the DGP decoupling limit at 5dimensions,the auteurs of one thus constructs the model of the galileon, which describes the dynamics of a scalar field not minimally coupled to the metric and which leaves the equations of motion invariant under the Galilean transformation$\phi \Rightarrow \phi + b_\mu x^\mu + c$. In this paper we the cosmological evolution of an isotropic and homogineous universe in Telleparallel gravity theory with a cubic Galileon field ,we confirmed the existence of an accelerated expansion of the universe with a behavior of the dark energy equation of state in the dark matter identical to the one of the cosmological constant.

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