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Thème

Analysis and Application of a Stabilized Collocated Finite Volume Method for Stokes-Darcy Problems

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ملخص

هذه الأطروحة مكرسة لحل مسائل من نوع ستوكس- دارسي في الشكلين المنفصل والمتصل. يتم تطوير مخططات جديدة للأحجام المنتهية وتحليلها نظريًا وذلك بدمج طريقة الأحجام المنتهية مع تقنية الاستقرار بحيث الأولى مبنية على القيم المتمركزة في الخلايا والثانية على قفزات الضغط. وبالتالي، يمكن تحقيق الاستقرار والتقارب الأمثل. بالإضافة إلى ذلك تقدم النتائج العددية المحصل عليها بعد حل بعض مسائل اختبارية مثلى. هذه النتائج تشهد على الأداء الحسابي الملحوظ للمخططات المقترحة.

الكلمات المفتاحية: الأحجام المنتهية، التقطيع الرديف، معادلات ستوكس، معادلات دارسي، الطرق المستقرة.

Abstract

This thesis is devoted to finite volume solution of Stokes-Darcy type problems both in decoupled and coupled forms. Combining finite volume methodology and a stabilization procedure based respectively on collocation and pressure jumps control, novel finite volume schemes are developed and theoretically analyzed. Hence, stability and optimal convergence are achieved. Numerical results are presented for some standard test problems. These attest the remarkable computational performance of proposed schemes.

Keywords: Finite volumes, Collocated discretization, Stokes equations, Darcy equations, Stabilized methods.

Résumé

Cette thèse est consacrée à la résolution par volumes finis des problèmes de type Stokes-Darcy sous formes découplée et couplée. En combinant la méthodologie des volumes finis et une procédure de stabilisation basée respectivement sur la collocation dans les cellules et le contrôle des sauts de pression, des schémas nouveaux de volumes finis sont développés et analysés théoriquement. Ainsi, la stabilité et la convergence optimale sont établies. Les résultats numériques sont présentés pour certains problèmes tests standards. Ceux-ci témoignent de la performance calculatoire remarquable des schémas proposés.

Mots-clés : Volumes finis, Discrétisations groupées, Equations de Stokes, Equations de Darcy, Méthodes stabilisées.

Contents

Introduction

The finite volume method (FVM) is a discretization method which is well suited for the numerical simulation of partial differential equations for conservation and balance laws. It was initially developed as an efficient middle ground between the finite difference and finite element methods. Recently, it has been extensively used in several engineering fields such as fluid mechanics, heat and mass transfer, semi-conductor technologies or petroleum engineering [1, 2, 3]. Some of the important features of FVM are that it may be used on arbitrary geometries, using structured or unstructured meshes, as well as it leads to robust numerical schemes. An additional feature is the local conservativity of the numerical fluxes. This last feature makes FVM quite attractive when modelling problems for which the flux is of importance, such as in the areas previously mentioned.

Starting from the partial differential equation itself, a balance equation is generated by a consistent approximation of the fluxes defined. Different FVM schemes were attemptedly developed by the use of finite element ideas in order to achieve a more rigourous FVM methodology. Among these new approaches, the collocated FVM schemes seem to attract CFD researchers attention for several reasons (see [4, 5, 6, 7]). The colocated schemes are used in commercial codes (Fluent, CFX,...) or industrial codes like those in nuclear safety [8]. Without covering all main reasons why this scheme is so popular, we can cite:

- colocated arrangement of the unknowns
- very cheap assembling step (no numerical integration to perform)
- easy coupling with other systems of equations.

The purpose of the present work is to model numerically two different problems for Stokes and Darcy equations occurring in fluid infiltration phenomena, like the ones encountered in water flowing across soil, oil filtering through sand or rocks, groundwater flows and biological flows in medical research (sugar transportation to a tumor, drug delivery to arteries and passage of oxygen in the brain). Hence, the considered simplified problem is solved for a Stokes flow in one part of the domain and a Darcy flow in the other part. It then becomes crucial to work with a unified FVM that may successfully solve both problems separately. Then, the same FVM is adapted to solve the so-called coupled Stokes-Darcy problem with the same optimal convergence rates in both regimes. Similar studies were undertaken in [9, 10] for finite element methods. However, robust and efficient numerical algorithms for this type of flows are challenging because viscous and porous features require different numerical strategies. In this regard, we can also cite for the finite element methods: [11, 12, 13, 14, 15, 16, 17], the domain decomposition methods: [18, 19, 20, 21, 22], the mortar finite element methods: [23, 24, 25], the discontinuous Galerkin methods: [26, 27, 28], the multigrid methods: [29, 30, 31], and the finite volume methods: [32, 33]. In this thesis, we first focus on building a unified FVM for treating the above cited infiltration problems for Stokes and Darcy equations. Then, the same approach is extended to a coupled Stokes-Darcy problem. Discretization is achieved using a collocated FVM which is applied for both discrete velocity and pressure approximations. However, the collocated FVM usually lead to unstable schemes as pointed by Rhie and Chow [7]. This difficulty is then handled by a stabilization technique. Later, mathematicians were able to prove the stability and convergence for this class of schemes (see, e.g., [34, 35, 36, 37, 38]). In the proposed finite volume scheme, we present an alternative strategy, based on adding of a consistent symmetric stabilization term which penalize the pressure jumps accross the volume boundaries. This term may be seen as a finite volume analogue of the classical stabilization finite element technique introduced in [39] for the Stokes equations.

The thesis is organized as follows. In Chapter 1, we recall some general results of the theory of mixed problems. Without going to the details, we also give a brief presentation of the relevant abstract approximation theory. Chapter 2 is devoted to the description and discussion of considered mathematical models alongside with some appropriate boundary conditions. In particular, weak formulations are derived for the Stokes, Darcy and coupled Stokes-Darcy problems. Then, we give an adaptation of the results obtained in Chapter 1 to show the well-posedness of these formulations.

In Chapter 3, we present a detailed construction of the proposed FVM. First, we display all assumptions needed on the discretization meshs and define appropriate discrete notions (forms, inner products, norms, space). Further, some interpolation results are proved and consistency residuals are established for discrete fluxes.

Chapter 4, which is the heart of the thesis, is aimed to present and analyze two novel stabilized finite volume schemes for the Stokes, Darcy and coupled Stokes-Darcy problems. The proposed schemes are first discussed and then recast under a discrete variational form. Among many important results, the schemes are shown to satisfy stability condition and erreur estimates in classical norms for all considered problems.

In Chapter 5, we aim to illustrate numerically theoretical results developed in Chapter 4. The computational behavior of the proposed schemes is assessed on benchmark test problems. In particular, it is observed that the expected optimal rates of convergence are achieved. Finally, concluding remarks are drawn.

Chapter 1

Mixed problems

1.1 Functional spaces

In order to investigate mathematically the Stokes, Darcy and Stokes-Darcy equations, we introduce the usual methodology for deriving weak formulations in appropriate Sobolev spaces [40, 41]. Hence, we introduce the Lebesgue space $L^2(\Omega)$ which consists of functions that are square-integrable over the domain Ω [42] with respect to the inner product

$$(f, g)_{0,\Omega} = \int_\Omega f(x)g(x)dx \tag{1.1}$$

and the corresponding norm

$$\|f\|_{0,\Omega} = (f, f)_{0,\Omega}^{1/2} \tag{1.2}$$

for all $f, g \in L^2(\Omega)$.

By $L_0^2(\Omega)$ we will denote the subspace of $L^2(\Omega)$ functions with zero mean over Ω, i.e.

$$L_0^2(\Omega) = \left\{ f \in L^2(\Omega) : \int_\Omega f dx = 0 \right\} \tag{1.3}$$

Using multi-indices $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ for the space dimensions $d \in \{2, 3\}$ with $|\alpha| = \alpha_1 + ... + \alpha_d$, and using the multi-index partial derivative $\partial^\alpha = \partial_1^{\alpha_1} ... \partial_d^{\alpha_d}$, where $\partial_i^{\alpha_i} = \partial^{\alpha_i}/\partial x_i^{\alpha_i}$. We define the Sobolev spaces $H^m(\Omega)$

$$H^m(\Omega) = \left\{ f \in L^2(\Omega) : \partial^\alpha f \in L^2(\Omega), \quad 0 \le |\alpha| \le m \right\} \tag{1.4}$$

The natural number $m \in \mathbb{N}$ is called the order of the Sobolev space $H^m(\Omega)$. We equip $H^m(\Omega)$ with the inner product

$$(f, g)_{m,\Omega} = \sum_{|\alpha| \le m} \int_\Omega \partial^\alpha f(x) \partial^\alpha g(x) dx \tag{1.5}$$

and the norm

$$\|f\|_{m,\Omega} = (f,f)_{m,\Omega}^{1/2} \tag{1.6}$$

The elements of $H^m(\Omega)$ are the m times weakly differentiable functions on the domain Ω.

The above spaces turn out to be Hilbert spaces. Next, by $H_0^m(\Omega)$ we denote the Hilbert subspace of $H^m(\Omega)$ which is the closure of $C_c^\infty(\Omega)$ (indefinitely continuously differentiable functions with compact support in Ω) with respect to the norm of $H^m(\Omega)$.

We also define the following Sobolev seminorm on $H^m(\Omega)$ by

$$|f|_{m,\Omega} = \left(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha f(x)|^2 \, dx \right)^{1/2} \qquad \forall f \in H^m(\Omega) \tag{1.7}$$

Clearly, $H^0(\Omega) = L^2(\Omega)$. In particular, the space $H^1(\Omega)$ consists of square-integrable functions with square-integrable first derivatives and has the following important subspace $H_0^1(\Omega)$. In the case of a bounded domain Ω, we have

$$H_0^1(\Omega) = \left\{ f \in H^1(\Omega) : f = 0 \qquad \text{on } \partial\Omega \quad \text{in trace sense} \right\} \tag{1.8}$$

The space $H^1(\Omega)$ has the associated norm

$$\|f\|_{1,\Omega} = \left\{ \|f\|_{0,\Omega}^2 + \sum_{i=1}^d \left\| \frac{\partial f}{\partial x_i} \right\|_{0,\Omega}^2 \right\}^{1/2} \tag{1.9}$$

and the seminorm

$$|f|_{1,\Omega} = \left\{ \sum_{i=1}^d \left\| \frac{\partial f}{\partial x_i} \right\|_{0,\Omega}^2 \right\}^{1/2} \tag{1.10}$$

The latter is actually a norm equivalent to (1.9) in $H_0^1(\Omega)$.

For vector valued functions, we introduce the following spaces of vector functions, which are natural generalizations of the corresponding scalar spaces. For example, the inner product for vector valued fonctions belonging to

$$\mathbf{L}^2(\Omega) = \left[L^2(\Omega) \right]^d \tag{1.11}$$

is given by

$$(\mathbf{f}, \mathbf{g})_{0,\Omega} = \sum_{i=1}^d \int_\Omega f_i \; g_i \; dx \tag{1.12}$$

and the associated norm

$$\|\mathbf{f}\|_{0,\Omega} = \left(\sum_{i=1}^d \|f_i\|_{0,\Omega}^2 \right)^{1/2} \tag{1.13}$$

for any $\mathbf{f}, \mathbf{g} \in \mathbf{L}^2(\Omega)$.

Likewise, $\mathbf{H}^m(\Omega)$ consists of vector valued functions each of whose components belongs to $H^m(\Omega)$, i.e.

$$\mathbf{H}^m(\Omega) = [H^m(\Omega)]^d \tag{1.14}$$

This is equipped with the norm

$$\|\mathbf{f}\|_{m,\Omega} = \left(\sum_{i=1}^{d} \|f_i\|_{m,\Omega}^2\right)^{1/2} \tag{1.15}$$

for any $\mathbf{f} \in \mathbf{H}^m(\Omega)$. Similarly

$$\mathbf{H}_0^1(\Omega) = \left[H_0^1(\Omega)\right]^d \tag{1.16}$$

has the equivalent norm

$$|\mathbf{f}|_{1,\Omega} = \left(\sum_{i=1}^{d} |f_i|_{1,\Omega}^2\right)^{1/2} \tag{1.17}$$

We will also need the following Hilbert space $\mathbf{H}(\mathrm{div};\Omega)$ defined as

$$\mathbf{H}(\mathrm{div};\Omega) = \left\{\mathbf{w} \in \mathbf{L}^2(\Omega) : \quad \mathrm{div}\,\mathbf{w} \in L^2(\Omega)\right\} \tag{1.18}$$

It is equipped with the norm

$$\|\mathbf{w}\|_{\mathrm{div}} = \left(\|\mathbf{w}\|_{0,\Omega}^2 + \|\mathrm{div}\,\mathbf{w}\|_{0,\Omega}^2\right)^{1/2} \tag{1.19}$$

Further, we will denote by $\mathbf{H}_0(\mathrm{div};\Omega)$ the following sub-space of $\mathbf{H}(\mathrm{div};\Omega)$

$$\mathbf{H}_0(\mathrm{div};\Omega) = \{\mathbf{w} \in \mathbf{H}(\mathrm{div};\Omega) : \quad \mathbf{w}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\} \tag{1.20}$$

1.2 Abstract mixed formulation

The analysis presented in this section concerns one classical theorem stated here without proof. Details can be found by the reader in [40, 43]. For simplicity, some parts are omitted.

Let $\mathbf{X}$ and M be two real Hilbert spaces with norms $\|\cdot\|_{\mathbf{X}}$ and $\|\cdot\|_M$ respectively and their topological dual spaces $\mathbf{X}'$ and M'.

Let also given two bilinear forms

$$a(\cdot,\cdot) : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$$
$$b(\cdot,\cdot) : \mathbf{X} \times M \to \mathbb{R}$$

We shall assume that these bilinear functionals are bounded in the sense that there exist positive constants C_a and C_b such that

$$\begin{aligned} |a(\mathbf{u},\mathbf{v})| &\leq C_a \|\mathbf{u}\|_{\mathbf{X}} \|\mathbf{v}\|_{\mathbf{X}} \qquad \forall \mathbf{u},\mathbf{v} \in \mathbf{X} \\ |b(\mathbf{v},q)| &\leq C_b \|\mathbf{v}\|_{\mathbf{X}} \|q\|_M \qquad \forall \mathbf{v} \in \mathbf{X}, \forall q \in M \end{aligned}$$

Now, consider the following variational problem:

Find $(\mathbf{u},p) \in \mathbf{X} \times M$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X} \\ b(\mathbf{u},q) &= 0 \qquad\quad \forall q \in M \end{aligned} \tag{Q}$$

where $\mathbf{f} \in \mathbf{X}'$.

Additionnally, we define two continuous linear operators associated with $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$

$$\begin{aligned} A &: \mathbf{X} \to \mathbf{X}', \langle A\mathbf{u},\mathbf{v}\rangle = a(\mathbf{u},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{X} \\ B &: \mathbf{X} \to M', \langle B\mathbf{v},q\rangle = b(\mathbf{v},q) \quad \forall q \in M \end{aligned}$$

Analogously, let B' denote the dual of the operator B

$$B' : M \to \mathbf{X}', \langle B'q,\mathbf{v}\rangle = \langle B\mathbf{v},q\rangle = b(\mathbf{v},q) \quad \forall \mathbf{v} \in \mathbf{X}$$

Then, problem (Q) can be reformulated as:

Find $(\mathbf{u},p) \in \mathbf{X} \times M$ satisfying

$$\begin{aligned} A\mathbf{u} + B'p = \mathbf{f} &\quad \text{in } \mathbf{X}' \\ B\mathbf{u} = 0 &\quad \text{in } M' \end{aligned}$$

Next, set

$$\mathbf{Z} = KerB = \{\mathbf{v} \in \mathbf{X} : \quad b(\mathbf{v},q) = 0 \quad \forall q \in M\} \tag{1.21}$$

We associate the following problem to (Q):

Find $\mathbf{u} \in \mathbf{Z}$ such that

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{Z} \tag{P}$$

Clearly, if $(\mathbf{u},p) \in \mathbf{X} \times M$ is a solution of (Q), then $\mathbf{u} \in \mathbf{Z}$ is a solution of (P).

The question is now: what are suitable conditions ensuring the converse of the last statement?

Theorem 1 (Existence and uniqueness)**.** *Under the following hypotheses:*
*H1 (**Z**-ellipticity of a). There exists a constant* $\alpha > 0$ *such that*

$$a(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|_{\mathbf{X}}^2 \quad \forall \mathbf{v} \in \mathbf{Z} \tag{1.22}$$

H2 (LBB condition). There exists a constant $\beta > 0$ *such that: given any* $q \in M$

$$\sup_{\mathbf{v}\in\mathbf{X}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{X}}} \geq \beta \|q\|_M \tag{1.23}$$

problem (P) *has a unique solution* $\mathbf{u} \in \mathbf{Z}$ *and there exists a unique* $p \in M$ *such that the pair* $(\mathbf{u}, p)$ *is the unique solution of problem* (Q)*.*

The statment (1.23) is often referred to as the stability condition.
It is also instructive to rewrite Problem (Q) in the following form:

Find $(\mathbf{u}, p) \in \mathbf{X} \times M$ such that

$$B[(\mathbf{u}, p), (\mathbf{v}, q)] = (\mathbf{f}, \mathbf{v}) \qquad \forall (\mathbf{v}, q) \in \mathbf{X} \times M \tag{1.24}$$

where

$$B[(\mathbf{u}, p), (\mathbf{v}, q)] = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) \pm b(\mathbf{u}, q) \tag{1.25}$$

The primary motivation for using this equivalent form of Problem (Q) comes from the celebrated and more general result proved by Babuška in [44].

Theorem 2. *Let* $\mathbf{W}_1$ *and* $\mathbf{W}_2$ *be two Hilbert spaces with norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *respectively. Further, let* $B[\cdot,\cdot]$ *be a bilinear form on* $\mathbf{W}_1 \times \mathbf{W}_2$ *such that*

$$\begin{aligned} |B[\mathbf{s}, \mathbf{r}]| &\leq C_1 \|\mathbf{s}\|_1 \|\mathbf{r}\|_2 && \forall (\mathbf{s}, \mathbf{r}) \in \mathbf{W}_1 \times \mathbf{W}_2 \\ \sup_{\mathbf{s}\in\mathbf{W}_1} \frac{|B[\mathbf{s}, \mathbf{r}]|}{\|\mathbf{s}\|_1} &\geq C_2 \|\mathbf{r}\|_2 && \forall \mathbf{r} \in \mathbf{W}_2 \\ \sup_{\mathbf{r}\in\mathbf{W}_2} \frac{|B[\mathbf{s}, \mathbf{r}]|}{\|\mathbf{r}\|_2} &\geq C_3 \|\mathbf{s}\|_1 && \forall \mathbf{s} \in \mathbf{W}_1 \end{aligned}$$

with $C_2, C_3 > 0, C_1 < \infty$*. Further, let* $\mathbf{f} \in \mathbf{W}_2'$*. Then, there exists exactly one element* $\mathbf{s}_0 \in \mathbf{W}_1$ *such that*

$$B[\mathbf{s}_0, \mathbf{r}] = (\mathbf{f}, \mathbf{v})$$

for all $\mathbf{r} \in \mathbf{W}_2$ *and*

$$\|\mathbf{s}_0\|_1 \leq \frac{\|\mathbf{f}\|_*}{C_3}$$

where

$$\|\mathbf{f}\|_* = \sup_{\mathbf{t}\in\mathbf{W}_2} \frac{|(\mathbf{f}, \mathbf{t})|}{\|\mathbf{t}\|_2}$$

We note that this theorem is very useful in the analysis of finite volume approximations of the continuous Stokes, Darcy and Stokes-Darcy formulations.

1.3 Approximation of mixed problems

This section is devoted to the approximation of abstract mixed problems discussed in the previous section. We keep the same notation here. Our presentation slightly varies from that of Ref. [40, 43], since their results are more general.

Let h now denote a discretization parameter tending to zero and, for each h, let $\mathbf{X}_h \subset \mathbf{X}$ and $M_h \subset M$ be two finite-dimensional spaces. Let us define the closed linear subspace $\mathbf{Z}_h$ of the linear space $\mathbf{X}_h$ analogue to (1.21), defined by

$$\mathbf{Z}_h = \{\mathbf{v}_h \in \mathbf{X}_h : \quad b\left(\mathbf{v}_h, q_h\right) = 0 \quad \forall q_h \in M_h\} \tag{1.26}$$

Since $0 \in \mathbf{Z}_h$, the set $\mathbf{Z}_h$ is nonempty. In general, we do not have necessarily $\mathbf{Z}_h \subset \mathbf{Z}$.

For the same reason, if $b\left(\cdot,\cdot\right)$ featured in (Q) satisfies the stability condition (1.23), it does not automatically satisfy the analogous discrete stability condition with $\mathbf{X}_h$ and M_h. This fact turns out be a source of difficulties in the construction of viable finite space approximations to mixed variational problems. The validity of a discrete stability condition has to be independently verified for each particular choice of spaces $\left(\mathbf{X}_h, M_h\right)$.

Following the Galerkin methodology, we now approximate Problem (Q) by the following:

Find $\mathbf{u}_h \in \mathbf{X}_h$ and $p_h \in M_h$ such that

$$\begin{aligned} a\left(\mathbf{u}_h, \mathbf{v}_h\right) + b\left(\mathbf{v}_h, p_h\right) &= \left(\mathbf{f}, \mathbf{v}_h\right) \quad && \forall \mathbf{v}_h \in \mathbf{X}_h \\ b\left(\mathbf{u}_h, q_h\right) &= 0 && \forall q_h \in M_h \end{aligned} \tag{Q$_h$}$$

with the associated restricted problem:

Find $\mathbf{u}_h \in \mathbf{Z}_h$ such that

$$a\left(\mathbf{u}_h, \mathbf{v}_h\right) = \left(\mathbf{f}, \mathbf{v}_h\right) \quad \forall \mathbf{v}_h \in \mathbf{Z}_h \tag{P$_h$}$$

As $\mathbf{Z}_h \nsubseteq \mathbf{Z}$ in general, (P$_h$) may be viewed as an external approximation of (P).

Here, again, the first component $\mathbf{u}_h$ of any solution $\left(\mathbf{u}_h, p_h\right)$ of (Q$_h$) is also a solution of (P$_h$). The converse is treated in the following theorem which represent the discrete version of Theorem 1.

Theorem 3 (Existence and uniqueness)**.** *Assume the following hypotheses:*

H1 ($\mathbf{Z}_h$-ellipticity of a). There exists a constant $\alpha^* > 0$ such that

$$a\left(\mathbf{v}_h, \mathbf{v}_h\right) \geq \alpha^* \left\|\mathbf{v}_h\right\|_{\mathbf{X}}^2 \quad \forall \mathbf{v}_h \in \mathbf{Z}_h \tag{1.27}$$

H2 (Discrete LBB condition). There exists a constant $\beta^* > 0$, independent of h, such that: given any $q_h \in M_h$

$$\sup_{\mathbf{v}\in\mathbf{X}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{X}}} \geq \beta^* \|q_h\|_M \tag{1.28}$$

Then, problem (P_h) has a unique solution $\mathbf{u}_h \in \mathbf{Z}_h$ and there exists a unique $p_h \in M_h$ such that the pair $(\mathbf{u}_h, q_h)$ is the unique solution of problem (Q_h). Moreover, there exists a constant $C > 0$, independent on h, such that

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{X}} + \|p - p_h\|_M \leq C \left(\inf_{\mathbf{v}_h\in\mathbf{X}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{X}} + \inf_{q_h\in M_h} \|p - q_h\|_M \right) \tag{1.29}$$

Chapter 2

Mathematical models

In this chapter, we apply the above abstract results to solve the incompressible Stokes, Darcy and coupled Stokes-Darcy equations and to establish the existence of a unique weak solution in each case.

2.1 Introduction to fluid flows

The Navier-Stokes equations are the fondamental partial differential equations that describe the flow of incompressible fluids. They consist in an equation of motion derived from the conservation of momentum originated from Newton's second law and an incompressibility equation deduced from the conservation of mass in the system. A complete derivation of the Navier-Stokes equations can be found in [45].

Now, let $\Omega_t = \Omega \times [0, \infty[$; $\Omega \subset \mathbb{R}^d$ being an open and bounded domain with Lipschitz boundary $\partial\Omega$ and $d \in \{2, 3\}$.

In compact form the Navier-Stokes equations can be written in the following form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \cdot \mathbf{u} = \mathbf{f} - \nabla p + 2\nu \nabla \cdot \varepsilon(\mathbf{u}) \quad \text{in } \Omega_t \tag{2.1a}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_t \tag{2.1b}$$

where $\mathbf{u}$ denotes the flow velocity, p the fluid pressure, $\mathbf{f}$ an applied body force and $\nu > 0$ the constant kinematic viscosity. Furthermore, $\varepsilon(\mathbf{u})$, called the stress tensor, represents the symmetrical part of the velocity gradient

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) \tag{2.2}$$

It should be emphasized that the constant density ρ has been absorbed into the pressure. Whenever $\mathbf{u}$ and p represent nondimensionalised variables, then ν is the inverse of the Reynold number Re which is usually given by

$$Re = \frac{\rho U L}{\mu} \tag{2.3}$$

where μ is the fluid viscosity, U and L are respectively a representative velocity and a representative lenght associated with the fluid.

In this thesis the fluid motion is assumed to be steady, i.e. $\frac{\partial \mathbf{u}}{\partial t} = 0$.

A particularly important case of equations (2.1a) is obtained by considering the motion of creeping flows (low velocity). The non-linear terms in (2.1a) can then be neglected. The resulting system is commonly known as the Stokes equations. Although these are linear, they deserve special attention because of the incompressibility condition (2.1b).

In addition, assuming the viscous resisting force is linear with respect to the velocity, we obtain what are known as the Darcy equations. The Stokes and Darcy equations are discussed in details below. For a complete derivation, see [46, 47].

2.2 Stokes equations

The Stokes equations can be written as

$$-2\mu\nabla \cdot \varepsilon(\mathbf{u}) + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{2.4a}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{2.4b}$$

Substituting for the stress tensor (2.2) and using (2.1b), the Stokes equations can be simplified to the equivalent form

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{2.5a}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{2.5b}$$

We note that the system (2.4) is said to be non-conservative in contrast to the system (2.5) which is conservative.

In order to obtain a properly defined problem, the equations (2.4) or (2.5) must be supplemented with boundary conditions on $\partial\Omega$. The suitable choice of the latter, necessary and sufficient to define a well-posed problem, is not easy, alghough physics and experience can provide guidelines.

For the purpose of the present work, attention is restricted to the simple type below. Owing to the nature of the differential equations which are elliptic for the velocity, appropriate boundary conditions may consist in imposing the velocity on the whole boundary $\partial\Omega$. Thus, the imposed velocity condition is

$$\mathbf{u} = \mathbf{g} \qquad \text{on } \partial\Omega \tag{2.6}$$

where $\mathbf{g}$ is sufficiently smooth on $\partial\Omega$, and satisfies the compatibility condition

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}\, ds = 0$$

A particular case of this type of boundary condition is known as the non-slip condition. At a fixed wall, for example, the velocity vanishes, i.e.

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega \tag{2.7}$$

Now, equations (2.5) supplemented by (2.6) or (2.7) form what is called a Stokes problem. Except in degenerate cases, this problem cannot be solved exactly. Before going further, let us note that the Stokes problem shows that the pressure p is only defined to an additive constant. So, assume that the pressure p is normalized to belong to $L_0^2(\Omega)$.

The framework presented in Chapter 1 may be applied to establish existence and uniqueness of a solution in a weak sense. To this end, let us transform the Stokes problem (2.5)-(2.6) to a variational form by multiplying (2.5a) and (2.5b) respectively by test functions $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and $q \in L_0^2(\Omega)$ and integrating over Ω. This yields

$$\mu \int_\Omega \nabla\mathbf{u} \cdot \nabla\mathbf{v} - \int_\Omega p \nabla \cdot \mathbf{v} = \int_\Omega \mathbf{f}\ \mathbf{v}$$
$$\int_\Omega q \nabla \cdot \mathbf{u} = 0$$

On the other hand, by an extension result proved in [40], there exists $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ such that

$$\mathbf{u}_0 = \mathbf{g} \quad \text{on } \partial\Omega \qquad \text{and} \qquad \nabla \cdot \mathbf{u}_0 = 0 \quad \text{in } \Omega$$

Consequently, the corresponding weak formulation reads as:

Find $(\mathbf{u} - \mathbf{u}_0, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\mu \left(\nabla\left(\mathbf{u} - \mathbf{u}_0\right), \nabla\mathbf{v}\right)_{0,\Omega} - \left(p, \nabla \cdot \mathbf{v}\right)_{0,\Omega} = \left(\mathbf{f}, \mathbf{v}\right)_{0,\Omega} \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \tag{2.8a}$$

$$\left(q, \nabla \cdot \mathbf{u}\right)_{0,\Omega} = 0 \qquad \forall q \in L_0^2(\Omega) \tag{2.8b}$$

If the non-conservative form (2.4) was considered instead of (2.5), the corresponding weak formulation would have been:

Find $(\mathbf{u}-\mathbf{u}_0, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\mu\left(\varepsilon\left(\mathbf{u}-\mathbf{u}_0\right), \varepsilon\left(\mathbf{v}\right)\right)_{0,\Omega} - (p, \nabla\cdot\mathbf{v})_{0,\Omega} = (\mathbf{f}, \mathbf{v})_{0,\Omega} \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \tag{2.9a}$$

$$(q, \nabla\cdot\mathbf{u})_{0,\Omega} = 0 \qquad \forall q \in L_0^2(\Omega) \tag{2.9b}$$

Theorem 4 (Existence and uniqueness)**.** *The weak Stokes problem* (2.8) *has a unique solution* $(\mathbf{u}-\mathbf{u}_0, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$.

Proof. The proof is an application of Theorem 1 (for details, refer to [40]). □

2.3 Darcy equations

First of all, let us note that Darcy equations can be obtained as a particular case of Stokes equations by assuming that the viscous resisting force is linear with respect to the velocity. If, now, $\mathbf{u}$ denotes the flow velocity, then

$$\mu\mathbf{k}^{-1}\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{2.10a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \tag{2.10b}$$

where, $\mathbf{k}$ is symmetric and uniformly positive definite tensor representing the Darcy permeability.

Another way to write (2.10a) may be done by means of Darcy's law in porous media [48]

$$\mathbf{q} = -\frac{\mathbf{k}}{\mu}\nabla p \tag{2.11}$$

This law is based on a proportionality relationship between $\mathbf{q}$, referred to as the Darcy flux and ∇p through the permeability $\mathbf{k}$, which measures the ability of a porous material for fluid passing, the fluid viscosity μ. The Darcy flux, is not the velocity which the fluid traveling through the pores is experiencing. The flow velocity $\mathbf{u}$ is related to the Darcy flux $\mathbf{q}$ by the porosity φ, which is a measure of the void space in a material. As the material only permits flow through the void space, $\mathbf{q}$ is divided by the porosity to obtain the actual fluid velocity

$$\mathbf{u} = \frac{\mathbf{q}}{\varphi} \tag{2.12}$$

Darcy's law (2.11) in combination with the porous media analogue of the continuity equation (2.10b) gives the Darcy equations (2.10). Moreover, the source term $\mathbf{f}$ has been added to take into account body forces such as gravity.

A complete derivation of Darcy's law can be found in [49].

In contrast to the Stokes framework, Darcy equations are usually supplemented by a Neumann condition on the domain boundary

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \tag{2.13}$$

The obtained problem (2.10)-(2.13) is refered to as a Darcy problem. Owing to the nature of this problem, let us take now functions $\mathbf{v} \in \mathbf{H}_0(\text{div};\Omega)$ and $q \in L_0^2(\Omega)$. Multiplying both equations in (2.10) by respective test fonctions $\mathbf{v}$ and q, and integrating over Ω yield the following variational formulation:

Find $(\mathbf{u},p) \in \mathbf{H}_0(\text{div};\Omega) \times L_0^2(\Omega)$ such that

$$\mu \left(\mathbf{k}^{-1}\mathbf{u}, \mathbf{v}\right)_{0,\Omega} - (p, \nabla \cdot \mathbf{v})_{0,\Omega} = (\mathbf{f}, \mathbf{v})_{0,\Omega} \qquad \forall \mathbf{v} \in \mathbf{H}_0(\text{div};\Omega) \tag{2.14a}$$

$$(q, \nabla \cdot \mathbf{u})_{0,\Omega} = 0 \qquad \forall q \in L_0^2(\Omega) \tag{2.14b}$$

Recall that owing to the positive definiteness of $\mathbf{k}$, there exist two positives constants λ_1, λ_2 such that

$$\lambda_1 \|\mathbf{v}\|_{0,\Omega}^2 \leq \mu \left(\mathbf{k}^{-1}\mathbf{v}, \mathbf{v}\right)_{0,\Omega} \leq \lambda_2 \|\mathbf{v}\|_{0,\Omega}^2 \qquad \forall \mathbf{v} \in \mathbf{H}_0(\text{div};\Omega) \tag{2.15}$$

Likewise, we get the following

Theorem 5. *The weak Darcy problem* (2.14) *has a unique solution* $(\mathbf{u},p) \in \mathbf{H}_0(\text{div};\Omega) \times L_0^2(\Omega)$.

Proof. The proof is an application of Theorem 1. □

2.4 Coupled Stokes-Darcy equations

Now, let us consider an interface problem between a fluid flow, governed by Stokes equations, and the flow in a porous medium, governed by Darcy equations. Let us assume that the bounded domain $\Omega = \Omega_s \cup \Omega_d \subset \mathbb{R}^d, (d = 2 \text{ or } 3)$ is made up by two non overlapping subdomains, consisting of a fluid flow region Ω_s and a porous medium region Ω_d. Boundaries of both subdomains have a non-empty intersection $\Gamma = \partial\Omega_s \cap \partial\Omega_d$. Moreover, let $\mathbf{n}_l$ be the unit vector of outer normal to Ω_l (see Fig. (2.1)). In the whole of Ω, we denote by $\mathbf{u}$ the fluid velocity $\mathbf{u}$ and by p the fluid pressure. Moreover, we use notations $\mathbf{u}_s$ and $\mathbf{u}_d$ to designate $\mathbf{u}$ in Ω_s and Ω_d respectively.

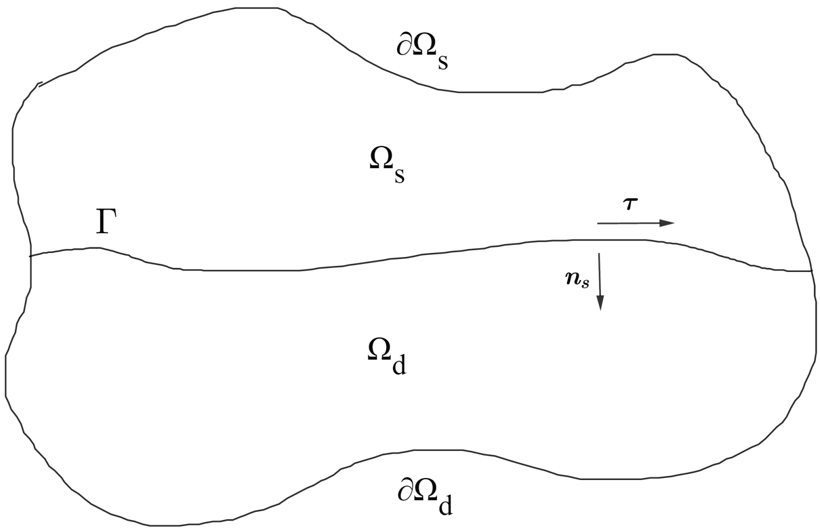


Figure 2.1: Computational domain

One obtains the equations

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega_s \\ \mu\mathbf{k}^{-1}\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega_d \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \end{cases} \tag{2.16}$$

supplemented by the following homogeneous conditions on the external boundary

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega_s \qquad \text{and} \qquad \mathbf{u}\cdot\mathbf{n}_d = 0 \quad \text{on } \partial\Omega_d \tag{2.17}$$

On the interface Γ we consider the following boundary conditions

Continuity of normal velocity

This expresses the mass conservation across the interface Γ. It has the form

$$\mathbf{u}_s\cdot\mathbf{n}_s + \mathbf{u}_d\cdot\mathbf{n}_d = 0 \tag{2.18}$$

Continuity of the fluid normal stress

This is a balance of normal forces across Γ [50]. It is expressed by

$$-\mu\mathbf{n}_s\cdot\nabla\mathbf{u}_s\cdot\mathbf{n}_s + p_s = p_d \tag{2.19}$$

Beaver-Joseph-Saffman condition

Since the fluid flow is viscous, an additional condition on the tangential fluid velocity on Γ must be given. Let τ denote the tangent unit vector on Γ. The simplest assumption is no-slippage along Γ, i.e., $\mathbf{u}_s \cdot \tau = 0$. Nevertheless, this is not in good agreement with experiment. A boundary condition in better agreement with experimental evidence was developed by Beavers and Joseph [51]. Mathematically, this condition can be represented by

$$-\mathbf{n}_s \cdot \nabla \mathbf{u}_s \cdot \tau = \frac{\alpha}{\sqrt{k}}(\mathbf{u}_s - \mathbf{u}_d) \cdot \tau$$

where $k = (\mathbf{k}\tau) \cdot \tau$ and α is a posifive parameter, so-called slip coefficient, determined by experimental evidence.

However, it is still unclear if this leads to a well-posed problem and it has been observed that the term on the left-hand side $\mathbf{u}_d \cdot \tau$ is much smaller than the other terms. Thus, its inclusion in this linear approximation is unclear. The most accepted interface condition was derived by Saffman [52] using a statistical approach and the Brinkman approximation, and also by Jones [53]. Another relevant work was also done by Jäger and Mikelic [54]. The new boundary condition is now known as the Beavers–Joseph–Saffman law and is thus given by

$$-\mathbf{n}_s \cdot \nabla \mathbf{u}_s \cdot \tau = \frac{\alpha}{\sqrt{k}} \mathbf{u}_s \cdot \tau \tag{2.20}$$

The problem defined by (2.16) to (2.20) will be refered to as the coupled Stokes-Darcy problem. Now, let us turn to developing a suitable weak formulation. The purpose of this weak formulation is twofold. Firstly, it is used to show well-posedness of (2.16) to (2.20). Secondly, it is suitable for splitting efficiently the coupled problem into two subproblems.

In order to construct the weak formulation of the coupled Stokes-Darcy problem, define the following new space for the velocity

$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{H}\left(\text{div}; \Omega\right) : \mathbf{v}_s \in \mathbf{H}^1\left(\Omega_s\right), \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_s, \mathbf{v} \cdot \mathbf{n}_d = 0 \text{ on } \partial\Omega_d \right\} \tag{2.21}$$

The first step for obtaining a weak formulation consists in multiplying the first two equations in (2.16) by a test function $\mathbf{v} \in \mathbf{V}$, and the incompressibility equation by a test function $q \in L_0^2(\Omega)$. The next step is to integrate the obtained equations over the entire domain Ω. This results into the equations

$$-\mu \left(\Delta \mathbf{u}, \mathbf{v}\right)_{0,\Omega_s} + \mu \left(\mathbf{k}^{-1}\mathbf{u}, \mathbf{v}\right)_{0,\Omega_d} + \left(\nabla p, \mathbf{v}\right)_{0,\Omega} = \left(\mathbf{f}, \mathbf{v}\right)_{0,\Omega}$$

$$-\left(\nabla \cdot \mathbf{v}, q\right)_{0,\Omega} = 0$$

On one hand, the interface conditions (2.19) and (2.20) yield

$$-\mu\left(\Delta \mathbf{u}, \mathbf{v}\right)_{0,\Omega_s} = \mu\left(\nabla \mathbf{u}, \nabla \mathbf{v}\right)_{0,\Omega_s} + \left(p_d - p_s, \mathbf{v}_s \cdot \mathbf{n}_s\right)_{0,\Gamma} + \frac{\mu\alpha}{\sqrt{k}}\left(\mathbf{u}_s \cdot \tau, \mathbf{v}_s \cdot \tau\right)_{0,\Gamma}$$

On the other hand, we have

$$\left(\nabla p, \mathbf{v}\right)_{0,\Omega} = -\left(p, \nabla \cdot \mathbf{v}\right)_{0,\Omega_s} - \left(p, \nabla \cdot \mathbf{v}\right)_{0,\Omega_d} - \left(p_d - p_s, \mathbf{v} \cdot \mathbf{n}_s\right)_{0,\Gamma}$$

Gathering the above calculations leads to the variational formulation:

Find $(\mathbf{u}, p) \in \mathbf{V} \times L_0^2(\Omega)$ such that

$$\begin{aligned} a\left(\mathbf{u}, \mathbf{v}\right) + b\left(\mathbf{v}, p\right) &= \left(\mathbf{f}, \mathbf{v}\right) \quad && \forall \mathbf{v} \in \mathbf{V} \\ b\left(\mathbf{u}, q\right) &= 0 && \forall q \in L_0^2(\Omega) \end{aligned} \tag{2.22}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \mu\left(\nabla \mathbf{u}, \nabla \mathbf{v}\right)_{0,\Omega_s} + \mu\left(\mathbf{k}^{-1}\mathbf{u}, \mathbf{v}\right)_{0,\Omega_d} + \frac{\mu\alpha}{\sqrt{k}}\left(\mathbf{u}_s \cdot \tau, \mathbf{v}_s \cdot \tau\right)_{0,\Gamma} \tag{2.23}$$

$$b(\mathbf{v}, p) = -\left(p, \nabla \cdot \mathbf{v}\right)_{0,\Omega}$$

Theorem 6 (Existence and uniqueness)**.** *There exists a unique solution* $(\mathbf{u}, p) \in \mathbf{V} \times L_0^2(\Omega)$ *to the weak formulation* (2.22)*.*

Proof. (c.f. [50]). □

Chapter 3

Finite volume approximation

Due to the impossibility of finding exact solutions to the models discussed in Chapter 2, several numerical methods have been developed and extensively analyzed to approximate them. Although the finite difference and finite element methods have been the first techniques applied in computational fluid dynamics, the finite volume methods appeard to be more convenient during the last two decates. With a given discretization mesh, a finite volume scheme is based on the approximation of the flux conservation across boundaries. First, we give the assumptions needed on the lying discretization mesh and we define some special discrete forms, inner products, norms and the discrete space. We also prove interpolation results on the discrete space. Then, we present a thorough study on discrete fluxes and establish consistency residuals which will be applied in the next chapter.

3.1 Discrete aspects

In general, domain discretization can be unconditionnally performed in different ways. However, present work suggests the use of what is called admissible finite volume meshes. These involve some more technical assumptions. This was motivated by works of [55, 56, 57] on diffusion problems.

3.1.1 Discretization and discrete functional spaces

Let Ω be an open bounded polygonal subset of $\mathbb{R}^d, d = 2$ or 3. According to litterature just mentioned, an admissible finite volume mesh of Ω, denoted by $\mathcal{D}_h$, is given by a family of disjoint non-empty convex subdomains (control volumes) K of Ω such that $\overline{\Omega} = \cup_{K \in \mathcal{D}_h} K$. Let us denote by $\partial K = \overline{K} \backslash K$ and $|K|$ respectively the boundary and measure of any $K \in \mathcal{D}_h$.

On the other hand, let us denote by $\mathcal{E}_{int}$, $\mathcal{E}_{ext}$ the finite sets of volume boundaries σ (edges or faces), with measures $|\sigma|$, which are respectively internal to the domain Ω and on $\partial\Omega$.

Further, let $\{\mathbf{x}_K\}_{K\in\mathcal{D}_h}$ be a family of points of $\overline{\Omega}$ such that, for all $K \in \mathcal{D}_h$, $\mathbf{x}_K \in K$ (see Fig. (3.1)). The coordinates of $\mathbf{x}_K$ are denoted by $x_K^{(i)}, i = 1, ..., d$.

Assume K and L to be two neighbouring control volumes of the mesh. A consistent discretization of the flux over the interface of K and L may be performed with a differential quotient involving values of the unknown located on the orthogonal line to the interface between K and L, on either side of this interface. Here, we note that any internal edge, separating two control volumes K and L, is orthogonal to the line segment $[\mathbf{x}_K, \mathbf{x}_L]$ from $\mathbf{x}_K$ to $\mathbf{x}_L$ at the point $\mathbf{x}_\sigma$. We also denote by d_{KL} the distance between $\mathbf{x}_K$ and $\mathbf{x}_L$, and by $d_{K\sigma}$ the distance between $\mathbf{x}_K$ and $\mathbf{x}_\sigma$.

We also denote by h_K the diameter of each control volume and we set

$$h = \sup_{K\in\mathcal{D}_h} h_K$$

which will be refered to as the mesh parameter. Regularity of the mesh is measured by the function regul $(\mathcal{D}_h)$ defined as follows

$$\begin{aligned}\text{regul}\,(\mathcal{D}_h) = \inf\Bigg(&\left\{\frac{d_{K\sigma}}{diam(K)}; K \in \mathcal{D}_h : \sigma \in \partial K\right\}\\ &\cup \left\{\frac{d_{K\sigma}}{d_{KL}}; K, L \in \mathcal{D}_h : \sigma \in \mathcal{E}_{int} \cap \partial K\right\} \cup \left\{\frac{1}{card\,(\partial K)}\right\}\Bigg)\end{aligned}$$

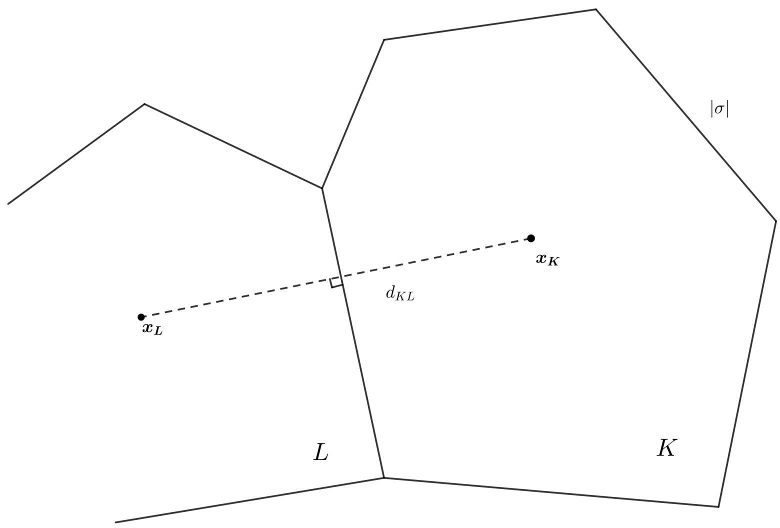


Figure 3.1: Two neighbouring control volumes

Two examples of admissible meshes are now given.

Example 7 (Triangular meshes)**.** *If Ω is an open bounded polygonal subset of $\mathbb{R}^2$, then $\mathcal{D}_h$ can be a family of disjoint open triangular subsets of Ω having an edge as a common boundary. Assume that all angles of triangles are acute. This last condition is sufficient for the orthogonal bisectors to intersect inside each triangle, thus naturally defining the points $\mathbf{x}_K \in K$ (see Fig. (3.2)).*

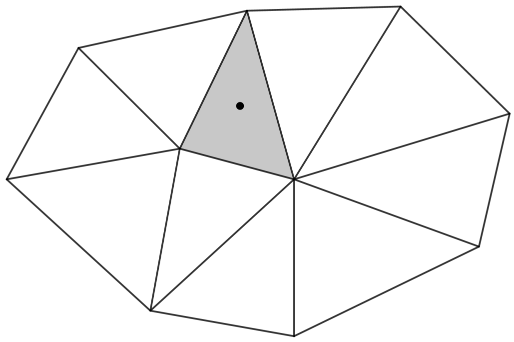


Figure 3.2: Triangular control volume (in Grey)

Example 8 (Voronoï meshes)**.** *If Ω be an open bounded polygonal subset of $\mathbb{R}^d$, then an admissible finite volume mesh can be constructed using the so-called "Voronoi" technique. Let $\{P_i; i = 1, ..., n\}$ be a finite family of points in Ω. The Voronoï mesh control volumes K_i are defined by*

$$K_i = \{y \in \Omega, \|P_i - y\| < \|P_j - y\|, j \neq i\} \qquad i, j = 1, ..., n$$

where $\|\cdot\|$ denotes the Euclidean norm (see Fig. (3.3)).

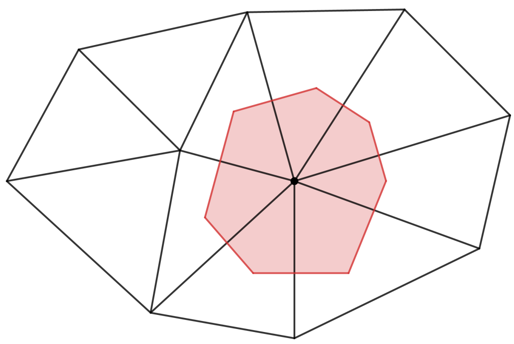


Figure 3.3: 2D Voronoï control volume (in Red) associated with a triangulation

Now, given an admissible mesh $\mathcal{D}_h$ let us introduce the discrete space $X_h \subset L^2(\Omega)$ of piecewise constant functions associated to $\mathcal{D}_h$ and some equivalent discrete forms of the H_0^1 norm on X_h. Likewise, we also make use of the discrete space

$$X_h^0 = X_h \cap L_0^2(\Omega)$$

For all $v_h \in X_h$, we denote by $v_{h,K}$ the value (constant) of v_h in any $K \in \mathcal{D}_h$.

For $v_h, w_h \in X_h$, we set

$$[v_h, w_h]_h = \sum_{\sigma\in\mathcal{E}_{int}} \frac{|\sigma|}{d_{KL}} (v_{h,L} - v_{h,K})(w_{h,L} - w_{h,K}) + \sum_{\sigma\in\mathcal{E}_{ext}} \frac{|\sigma|}{d_{K\sigma}} v_{h,K} w_{h,K} \tag{3.1}$$

This is clearly an inner product on X_h. The associated norm is

$$\|v_h\|_h = [v_h, v_h]_h^{1/2}$$

We also define the following bilinear form

$$\langle v_h, w_h\rangle_h = \sum_{\sigma\in\mathcal{E}_{int}} \frac{|\sigma|}{d_{KL}} (v_{h,L} - v_{h,K})(w_{h,L} - w_{h,K}) \tag{3.2}$$

which generates the semi-norm

$$|v_h|_h = \langle v_h, v_h\rangle_h^{1/2}$$

Of course, these definitions extend naturally to vector valued functions as follows. For $\mathbf{v}_h = \left(v_h^{(i)}\right)_{i=1,\dots,d} \in \mathbf{X}_h = {X_h}^d$ and $\mathbf{w}_h = \left(w_h^{(i)}\right)_{i=1,\dots,d} \in \mathbf{X}_h$, we set

$$[\mathbf{v}_h, \mathbf{w}_h]_h = \sum_{i=1}^{d} \left[v_h^{(i)}, w_h^{(i)}\right]_h$$

and

$$\|\mathbf{v}_h\|_h = \left(\sum_{i=1}^{d} \left[v_h^{(i)}, v_h^{(i)}\right]_h\right)^{1/2}$$

Analogously to the well-known Poincaré inequality the continuous case, we get

Lemma 9. *([41, 58]) The following discrete Poincaré inequalities hold*

$$\begin{aligned} \|\mathbf{v}_h\|_{0,\Omega} &\le C\|\mathbf{v}_h\|_h \qquad \forall \mathbf{v}_h \in \mathbf{X}_h \\ \|\mathbf{v}_h\|_{0,\Omega} &\le C|\mathbf{v}_h|_h \qquad \forall \mathbf{v}_h \in \mathbf{X}_h^0 \end{aligned} \tag{3.3}$$

where C depends only on Ω.

The following result deals with behavior on volume boundary. A proof can be found in [59].

Lemma 10 (Trace inequalities)**.** *Let $K \in \mathcal{D}_h$. If $u \in H^1(K)$, then*

$$\|u\|_{0,\partial K} \leq \left(C\frac{1}{h}\right)^{1/2} \left(\|u\|_{0,K} + h_K \|u\|_{1,K}\right) \tag{3.4}$$

Similarly, if $u \in H^2(K)$, we have

$$\|u\|_{0,\partial K} \leq \left(C\frac{1}{h}\right)^{1/2} \left(\|u\|_{1,K} + h_K \|u\|_{2,K}\right) \tag{3.5}$$

3.1.2 Interpolation in X_h

In order to establish some relevant approximation results for X_h, we adapt well-known estimates about approximation properties in Sobolev spaces. (see, e.g., [60, 61]). To this end, let u be a function in $L^2(\Omega)$. For each $K \in \mathcal{D}_h$, we first define w_K as the convex hull of $\left\{K, L; \overline{L} \cap \overline{K} \in \partial K\right\}$. If $\mathbb{P}_1(\omega)$ is the space of all polynomials defined on $\omega \subset \Omega$ in d variables of degree at most 1 and $\phi_K \in \mathbb{P}_1(K)$ satisfies

$$\int_{w_K} (u - \phi_K)\, \psi = 0 \qquad \forall \psi \in \mathbb{P}_1(w_K)$$

then we can define the linear operator $\pi(u) \in X_h$ by $\pi(u)|_K = \phi_K(\mathbf{x}_K)$ for all $K \in \mathcal{D}_h$.

This definition is easily extendable to vector-valued functions with the same notations.

We get the following result whose proof is based on Jackson's type inequalities as suggested in [61].

Lemma 11. *Let $u \in L^2(\Omega)$ and ϕ_K be defined as above.*

If $u \in H^1(w_K)$, then

$$\|u - \phi_K\|_{0,w_K} \leq Ch\, |u|_{1,\Omega} \tag{3.6}$$

$$|u - \phi_K|_{1,w_K} \leq C|u|_{1,\Omega} \tag{3.7}$$

If $u \in H^2(w_K)$, then

$$\|u - \phi_K\|_{0,w_K} \leq Ch^2 |u|_{2,\Omega} \tag{3.8}$$

$$|u - \phi_K|_{1,w_K} \leq Ch\, |u|_{2,\Omega} \tag{3.9}$$

We note that the constants in Lemma 11 depend only on those in Jackson's type inequalities.

Boundedness of π is now addressed.

Proposition 12. *Let $u \in H^1_0(\Omega)$. Then, the following estimate holds*

$$\|\pi(u)\|_h \leq C|u|_{1,\Omega} \tag{3.10}$$

If, in addition, $u \in H^1(\Omega)$, then

$$|\pi(u)|_h \leq C|u|_{1,\Omega} \tag{3.11}$$

Proof. Let $u \in H_0^1(\Omega)$. First, by the definition of $\pi(u)|_K = \phi_K(\mathbf{x}_K)$ and $\pi(u)|_L = \phi_L(\mathbf{x}_L)$, we have

$$\begin{aligned}
\|\pi(u)\|_h^2 &= \sum_{\sigma\in\mathcal{E}_{int}\cap\partial K} \frac{|\sigma|}{d_{KL}}(\phi_L(\mathbf{x}_L) - \phi_K(\mathbf{x}_K))^2 + \sum_{\substack{\sigma\in\mathcal{E}_{ext}\\(\sigma\in\varepsilon_K)}} \frac{|\sigma|}{d_{K\sigma}}\phi_K(\mathbf{x}_K)^2 \\
&\leq 2\sum_{\sigma\in\mathcal{E}_{int}\cap\partial K} \frac{|\sigma|}{d_{KL}}(\phi_K(\mathbf{x}_L) - \phi_K(\mathbf{x}_K))^2 + 2\sum_{\substack{\sigma\in\mathcal{E}_{int}\\(\sigma=K|L)}} \frac{|\sigma|}{d_{KL}}(\phi_L(\mathbf{x}_L) - \phi_K(\mathbf{x}_L))^2 \\
&+ 2\sum_{\sigma\in\mathcal{E}_{ext}\cap\partial K} \frac{|\sigma|}{d_{K\sigma}}(\phi_K(\mathbf{x}_K) - \phi_K(\mathbf{x}_\sigma))^2 + 2\sum_{\sigma\in\mathcal{E}_{ext}\cap\partial K} \frac{|\sigma|}{d_{K\sigma}}\phi_K(\mathbf{x}_\sigma)^2 \\
&= T_1 + T_2 + T_3 + T_4
\end{aligned}$$

We now bound each term $T_i, i = 1, ..., 4$. Hence, we have

$$T_1 = 2\sum_{\sigma\in\mathcal{E}_{int}\cap\partial K} |\sigma|\, d_{KL}(\nabla\phi_K \cdot n_{KL})^2$$

The quantity $|\sigma|\, d_{KL}$ can be seen as the measure of a domain included in $K \cup L$, and is, therefore, lower than the measure of w_K, so using Lemma 11, we get

$$T_1 \leq C_1|u|_{1,\Omega}^2 \tag{3.12}$$

Now, using the fact that $L^\infty(K)$ and $L^2(K)$ norms are equivalent on $\mathbb{P}_1(K)$ and Lemma 11, the term T_2 can be estimated as follows

$$T_2 \leq C_2|u|_{1,\Omega}^2 \tag{3.13}$$

Using the same arguments as for the bound of T_1 (replace d_{KL} by $d_{K\sigma}$), a similar result can be obtained for the third term

$$T_3 \leq C_3|u|_{1,\Omega}^2 \tag{3.14}$$

Finally, using the fact that u vanishes on $\partial\Omega$, the trace inequality (3.5) and Lemma 11 yields

$$T_4 \leq C_4|u|_{1,\Omega}^2 \tag{3.15}$$

Gathering (3.12) to (3.15) yields the result.

The proof of the second inequality of the proposition can be easily derived from the proof of the first one. □

Proposition 13. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Then, the following estimate holds*

$$\|\pi(u)\|_h \leq Ch|u|_{2,\Omega} \tag{3.16}$$

Proof. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$. We will follow the methodology used above in the proof of Proposition 12 we have

$$\|\pi(u)\|_h^2 \leq T_1 + T_2 + T_3 + T_4$$

where $T_i, i = 1, ..., 4$ defined as previous. Using Lemma 11, we get

$$T_1 \leq C_1 h^2 |u|_{2,\Omega}^2 \tag{3.17}$$

The term T_2 can be estimated as follows

$$T_2 \leq C_2 h^2 |u|_{2,\Omega}^2$$

The term T_3

$$T_3 \leq C_3 h^2 |u|_{2,\Omega}^2 \tag{3.18}$$

Finally,

$$T_4 \leq C_4 h^2 |u|_{2,\Omega}^2 \tag{3.19}$$

Gathering (3.17) to (3.19) yields the result. □

The following result gives more insight into the way $\pi(u)$ approximates the function u itself.

Proposition 14. *Let $u \in H^1(\Omega)$. Then the following estimate holds*

$$\|\pi(u) - u\|_{0,\Omega} \leq Ch|u|_{1,\Omega} \tag{3.20}$$

Let us now suppose that $u \in H^2(\Omega) \cap H_0^1(\Omega)$. As a consequence, u is continuous and we can define $u_h \in X_h$ by $u_{h,K} = u(\mathbf{x}_K), K \in \mathcal{D}_h$. Then we have

$$\|\pi(u) - u\|_h \leq Ch|u|_{2,\Omega} \tag{3.21}$$

The next lemma is a classical consequence of a lemma due to Nečas [62].

Lemma 15. *For every $q \in L_0^2(\Omega)$ there is a $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ such that*

$$\nabla \cdot \mathbf{u} = q \qquad \textit{and} \qquad |\mathbf{u}|_{1,\Omega} \leq C\|q\|_{0,\Omega} \tag{3.22}$$

Furthermore, boundedness of π (Proposition 12) gives

$$\|\pi(\mathbf{u})\|_h \leq C\|q\|_{0,\Omega} \tag{3.23}$$

3.2 Discrete fluxes and consistency residuals

Finite volume schemes are classically presented as discrete balance equations with a suitable approximation of the fluxes, see [1, 2, 55, 63]. However, in this section some residual estimates are derived.

3.2.1 Discrete operators

The fluxes need to be approximated as a function of the discrete unknowns $\mathbf{u}_{h,K}$ associated with each control volume $K \in \mathcal{D}_h$. A straightforward choice is to approximate the fluxes on the edges $\sigma \in \partial K$ for all $K \in \mathcal{D}_h$.

Let us begin by defining the discrete divergence operator $(\nabla_h \cdot) : \mathbf{X}_h \to X_h$ by

$$(\nabla_h \cdot \mathbf{u}_h)_K := \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_{int} \cap \partial K} |\sigma| \frac{\mathbf{u}_{h,L} + \mathbf{u}_{h,K}}{2} \cdot \mathbf{n}_\sigma \tag{3.24}$$

The adjoint of this discrete divergence with respect to the discrete L^2 inner product defines a discrete gradient ∇_h. Thus, for any $p_h \in X_h$, we define its discrete gradient $\nabla_h \, p_h \in \mathbf{X}_h$ by

$$(\nabla_h \, p_h)_K := \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_{int} \cap \partial K} |\sigma| \frac{p_{h,L} - p_{h,K}}{2} \mathbf{n}_\sigma \tag{3.25}$$

It remains to give a finite volume discretization of the viscous stress tensor $\varepsilon(\mathbf{u})$. To this end, let us note that the divergence of the stress tensor can be written as

$$\nabla \cdot \varepsilon(\mathbf{u}) := \frac{1}{2} (\Delta \mathbf{u} + \nabla (\nabla \cdot \mathbf{u})) \tag{3.26}$$

Further, for any given function $\mathbf{u}_h \in \mathbf{X}_h$, let $(\Delta_h \mathbf{u}_h) \in X_h$ be the function defined by

$$(\Delta_h \mathbf{u}_h)_K := \frac{1}{|K|} \left(\sum_{\sigma \in \mathcal{E}_{int} \cap \partial K} |\sigma| \frac{\mathbf{u}_{h,L} - \mathbf{u}_{h,K}}{d_{LK}} - \sum_{\sigma \in \mathcal{E}_{ext} \cap \partial K} |\sigma| \frac{\mathbf{u}_{h,K}}{d_{K\sigma}} \right) \tag{3.27}$$

which is called the discrete Laplace operator.

For discretizing the second summation in (3.26), we propose the following discrete operator $(\nabla_h (\nabla_h \cdot \mathbf{u}_h))_K$

$$(\nabla_h (\nabla_h \cdot \mathbf{u}_h))_K := \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_{int} \cap \partial K} |\sigma| \frac{(\nabla_h \cdot \mathbf{u}_h)_L - (\nabla_h \cdot \mathbf{u}_h)_K}{2} \mathbf{n}_{\sigma,K} \tag{3.28}$$

As already mentioned in the introduction, it is very important to emphasize that this version has not been analyzed and applied elsewhere.

3.2.2 Consistency residuals

Here, we successively establish estimates for the consistency residuals associated to the diffusive term, the pressure gradient term and, the velocity divergence term. These estimates are based on consistency results for local quantities.

Lemma 16. *Let $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$. Denote by R_Δ the following quantity*

$$\text{if } \sigma \in \mathcal{E}_{int} \cap \partial K, \qquad R_\Delta(\mathbf{u}) = \frac{|\sigma|}{d_{K\sigma}} \left(\pi(\mathbf{u})|_L - \pi(\mathbf{u})|_K \right) - \int_\sigma \nabla \mathbf{u} \cdot \mathbf{n}_\sigma$$

$$\text{if } \sigma \in \mathcal{E}_{ext} \cap \partial K, \qquad R_\Delta(\mathbf{u}) = -\frac{|\sigma|}{d_{K\sigma}} \pi(\mathbf{u})|_K - \int_\sigma \nabla \mathbf{u} \cdot \mathbf{n}_\sigma$$

Then,

$$|R_\Delta(\mathbf{u})| \leq Ch\, |\mathbf{u}|_{2,\Omega} \tag{3.29}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$.

Proof. Let us begin with the case of $\sigma \in \mathcal{E}_{int} \cap \partial K$. By the definition of $\pi(\mathbf{u})$ the quantity R_Δ reads as

$$\begin{aligned} R_\Delta(\mathbf{u}) &= \left(\frac{|\sigma|}{d_\sigma} \left(\phi_K(\mathbf{x}_L) - \phi_K(\mathbf{x}_K) \right) - \int_\sigma \nabla \mathbf{u} \cdot \mathbf{n}_\sigma \right) + \frac{|\sigma|}{d_\sigma} \left(\phi_L(\mathbf{x}_L) - \phi_K(\mathbf{x}_L) \right) \\ &= T_1 + T_2 \end{aligned}$$

Since ϕ_K is a linear polynomial, the first term of the right hand side of the preceding relation can be expressed as

$$T_1 = |\sigma| \nabla \phi_K \cdot \frac{\overrightarrow{\mathbf{x}_L \mathbf{x}_K}}{d_\sigma} - \int_\sigma \nabla \mathbf{u} \cdot \mathbf{n}_\sigma = - \int_\sigma \nabla (\mathbf{u} - \phi_K) \cdot \mathbf{n}_\sigma$$

Applying the Cauchy-Schwarz inequality followed by (3.5) and Lemma 11, we obtain

$$|T_1| \leq C_1 h\, |\mathbf{u}|_{2,\Omega}$$

On the other hand, using the fact that the vector space $\mathbb{P}_1(K)$ is finite-dimensional, on which the $L^\infty(K)$ and $L^2(K)$ norms are equivalent, the fact that L is included in both w_L and w_K and Lemma 11, we get

$$|T_2| \leq C_2 h\, |\mathbf{u}|_{2,\Omega}$$

The proof is then easily completed by collecting the above bounds of T_1 and T_2.

On $\sigma \in \mathcal{E}_{ext} \cap \partial K$, we have

$$\begin{aligned} R_\Delta(\mathbf{u}) &= \frac{|\sigma|}{d_{K\sigma}} \left(-\phi_K(\mathbf{x}_K) \right) - \int_\sigma \nabla \mathbf{u} \cdot \mathbf{n}_\sigma \\ &= \left(\frac{|\sigma|}{d_{K\sigma}} \left(\phi_K(\mathbf{x}_\sigma) - \phi_K(\mathbf{x}_K) \right) - \frac{1}{|\sigma|} \int_\sigma \nabla \mathbf{u} \cdot \mathbf{n}_\sigma \right) - \frac{|\sigma|}{d_{K\sigma}} \left(\phi_K(\mathbf{x}_\sigma) \right) \\ &= T_1 + T_2 \end{aligned}$$

T_1 can be estimated similarly to the first part. As the function $\mathbf{u}$ vanishes on $\partial\Omega$, we also have

$$T_2 = -\frac{1}{d_{K\sigma}} \int_\sigma \phi_K = -\frac{1}{d_{K\sigma}} \int_\sigma (\phi_K - \mathbf{u})$$

Now, combining the Cauchy-Schwarz inequality, the trace inequality (3.5) and Lemma 11 yields

$$|T_2| \leq C_3 h \, |\mathbf{u}|_{2,\Omega}$$

Once again, (3.29) follows from the bounds for T_1 and T_2. □

Lemma 17. *Let $p \in H^1(\Omega)$. Denote by R_∇ the following quantity*

$$\text{if } \sigma \in \mathcal{E}_{int} \cap \partial K, \quad R_\nabla(p) = \frac{1}{2}\left(\pi(p)|_L + \pi(p)|_K\right)\mathbf{n}_\sigma - \frac{1}{|\sigma|}\int_\sigma p\mathbf{n}_\sigma$$

$$\text{if } \sigma \in \mathcal{E}_{ext} \cap \partial K, \quad R_\nabla(p) = \pi(p)|_K \,\mathbf{n}_\sigma - \frac{1}{|\sigma|}\int_\sigma p\mathbf{n}_\sigma$$

Then,

$$|R_\nabla(p)| \leq Ch \, |p|_{1,\Omega} \tag{3.30}$$

Proof. The quantity R_∇ can be decomposed as follows

$$\begin{aligned} R_\nabla(p) &= \frac{|\sigma|}{2}\left(\phi_K(\mathbf{x}_K) + \phi_L(\mathbf{x}_L)\right) n_\sigma - \int_\sigma p n_\sigma \\ &= |\sigma|\left(\phi_K\left(\frac{\mathbf{x}_K + \mathbf{x}_L}{2}\right) - \phi_K(\mathbf{x}_\sigma)\right) n_\sigma + \int_\sigma (\phi_K - p)\, n_\sigma + |\sigma|\left(\phi_L(\mathbf{x}_L) - \phi_K(\mathbf{x}_L)\right) n_\sigma \\ &= T_1 + T_2 + T_3 \end{aligned}$$

Next, let us bound successively the three terms T_1, T_2 and T_3

First, we have

$$T_1 = |\sigma|\left(\nabla\phi_K \cdot \overrightarrow{\mathbf{x}_\sigma \mathbf{x}_G}\right)\mathbf{n}_\sigma$$

where

$$\mathbf{x}_G = \frac{\mathbf{x}_K + \mathbf{x}_L}{2}$$

As the distance between $\mathbf{x}_\sigma$ and $\mathbf{x}_G$ is smaller than h, by Lemma 11, we get

$$|T_1| \leq C_1 h \, |p|_{1,\Omega}$$

The term T_2 is estimated by applying successively the Cauchy-Schwarz inequality, (3.4) and Lemma 11

$$|T_2| \leq C_2 h \, |p|_{1,\Omega}$$

Now, using the fact that $L^\infty(K)$ and $L^2(K)$ norms are equivalent on $\mathbb{P}_1(K)$ and Lemma 11, the term T_3, is bounded by

$$|T_3| \leq C_3 h \, |p|_{1,\Omega}$$

Collecting the bounds of T_1, T_2 and T_3 completes the proof. □

Lemma 18. *Let* $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$*. Denote by* R_{div} *the following quantity*

$$\text{for } \sigma \in \mathcal{E}_{int} \cap \partial K, \quad R_{\text{div}}(\mathbf{u}) = \left(\frac{|\sigma|}{2} \left(\pi(\mathbf{u})|_L + \pi(\mathbf{u})|_K \right) - \int_\sigma \mathbf{u} \right) \cdot \mathbf{n}_\sigma$$

Then,

$$|R_{\text{div}}(\mathbf{u})| \leq ch\, |\mathbf{u}|_{1,\Omega} \tag{3.31}$$

Furthermor, if $\mathbf{u} \in \mathbf{H}^2(\Omega)$*, then*

$$|R_{\text{div}}(\mathbf{u})| \leq ch^2\, |\mathbf{u}|_{2,\Omega} \tag{3.32}$$

Proof. By the definition of π. We have

$$\begin{aligned} R_{\text{div}}(\mathbf{u}) &= \left(\frac{|\sigma|}{2} \left(\phi_K(\mathbf{x}_K) + \phi_L(\mathbf{x}_L) \right) - \int_\sigma \mathbf{u} \right) \cdot \mathbf{n}_\sigma \\ &= \left(\frac{|\sigma|}{2} \left(\phi_K(\mathbf{x}_K) + \phi_K(\mathbf{x}_L) \right) \cdot \mathbf{n}_\sigma - \int_\sigma \mathbf{u} \cdot \mathbf{n}_\sigma \right) + \frac{|\sigma|}{2} \left(\phi_L(\mathbf{x}_L) - \phi_K(\mathbf{x}_L) \right) \cdot \mathbf{n}_\sigma \\ &= T_1 + T_2 \end{aligned}$$

We have

$$T_1 = |\sigma|\, \phi_K\left(\frac{\mathbf{x}_K + \mathbf{x}_L}{2} \right) \cdot \mathbf{n}_\sigma - \int_\sigma \mathbf{u} \cdot \mathbf{n}_\sigma = \int_\sigma (\mathbf{u} - \phi_K) \cdot \mathbf{n}_\sigma$$

Applying the Cauchy-Schwarz inequality, (3.5) and Lemma 11 gives the results for $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ or for $\mathbf{u} \in \mathbf{H}^2(\Omega)$.

On the other hand, we get

$$T_2 = \frac{|\sigma|}{2} \left(\phi_L(\mathbf{x}_L) - \phi_K(\mathbf{x}_L) \right) \cdot \mathbf{n}_\sigma$$

Using the equivalence of the $L^\infty(K)$ and $L^2(K)$ norms on $\mathbb{P}_1(K)$ and since K is included in both w_K and w_L, we obtain the results. □

Chapter 4

New stabilized FVM for Stokes-Darcy problems

This chapter, which is the principal contribution of the present work, is devoted to the introduction and study of the two new schemes for the numerical solution of the Stokes and Darcy equations by means of stabilized finite volume methods. These are based on collocated approximation of the velocity and pressure unknowns. First, the equations are separately considered and a unified stable and convergent technique is shown to work in both situations. We prove that the obtained stabilized FV formulations satisfy a discrete stability condition and error estimates in classical L^2 norms. Then, this is extended to solve the coupled Stokes-Darcy problem with the same convergence rates. Main results of this chapter have already been the objects of two papers [64, 65].

4.1 Unified stabilized FVM for the Stokes and Darcy equations

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be an open bounded domain with a polygonal or polyhedral boundary $\partial\Omega$. For discretizing, let assume $\mathcal{D}_h$ to be a family of volumes as previously defined with regularity $\text{regul}\left(\mathcal{D}_h\right) > \theta > 0$.

In order to formulate the FVM scheme of interest, let us recall the compact weak formulation of the Stokes and Darcy equations discussed in Chapter 2 :

Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\alpha \left(\mathbf{u}, \mathbf{v}\right)_{0,\Omega} + \mu \left(\varepsilon\left(\mathbf{u}\right), \varepsilon\left(\mathbf{v}\right)\right)_{0,\Omega} - \left(p, \nabla \cdot \mathbf{v}\right)_{0,\Omega} = \left(\mathbf{f}, \mathbf{v}\right)_{0,\Omega} \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \qquad (4.1a)$$

$$\left(q, \nabla \cdot \mathbf{u}\right)_{0,\Omega} = 0 \qquad \forall q \in L_0^2(\Omega) \qquad (4.1b)$$

where the case $\alpha = 0$ corresponds to the Stokes setting and the case $\mu = 0$ is the Darcy one. Here, for simplicity of presentation, we consider a non-slip boundary condition.

Let us set $\mathbf{W}_h = \mathbf{X}_h \times X_h^0$. The product space $\mathbf{W}_h$ will be equipped with the following norm

$$\|(\mathbf{u}_h, p_h)\|_{\mathbf{W}_h}^2 = \|\mathbf{u}_h\|_l^2 + \|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2 + \|p_h\|_{0,\Omega}^2$$

with

$$l = \begin{cases} h & \text{for Stokes} \\ 0, d & \text{for Darcy} \end{cases}$$

Following the Petrov-Galerkin methodology, a discrete formulation of (4.1) reads:

Find $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla_h \cdot \mathbf{v}_h)_{0,\Omega} = (\mathbf{f}, \mathbf{v}_h)_{0,\Omega} \qquad \forall \mathbf{v}_h \in \mathbf{X}_h \tag{4.2a}$$

$$(q_h, \nabla_h \cdot \mathbf{u}_h)_{0,\Omega} + J(p_h, q_h) = 0 \qquad \forall q \in X_h \tag{4.2b}$$

where

$$a(\mathbf{u}_h, \mathbf{v}_h) = \begin{cases} \mu[\mathbf{u}_h, \mathbf{v}_h]_h + \mu(\nabla_h \cdot \mathbf{u}_h, \nabla_h \cdot \mathbf{v}_h)_{0,\Omega} & \text{for Stokes} \\ \alpha(\mathbf{u}_h, \mathbf{v}_h)_{0,\Omega} & \text{for Darcy} \end{cases}$$

and

$$J(p_h, q_h) = \delta \sum_K \int_{\partial K \setminus \partial\Omega} h_{\partial K} [p_h][q_h] ds$$

is a stabilization term, with $\delta > 0$. Here, $[\cdot]$ denotes the jump through interior edges, whereas the scalar product $[\cdot,\cdot]_h$ and the discrete divergence $\nabla_h \cdot$ are as in (3.1) and (3.24) respectively.

Now, the natural corresponding stabilized FV scheme for the solution of (4.2) consists in finding $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ such that for each control volume $K \in \mathcal{D}_h$

$$\int_K (A_h(\mathbf{u}_h))_K + \int_K (\nabla_h\, p_h)_K = \int_K \mathbf{f} \tag{4.3a}$$

$$\int_K (\nabla_h \cdot \mathbf{u}_h)_K + \delta \sum_{\sigma \in \mathcal{E}_{int} \cap \partial K} \int_\sigma h_{\partial K} [p_h] = 0 \tag{4.3b}$$

where

$$(A_h(\mathbf{u}_h))_K = \begin{cases} -\mu(\Delta_h \mathbf{u}_h)_K - \mu(\nabla_h(\nabla_h \cdot \mathbf{u}_h))_K & \text{for Stokes} \\ \alpha \int_K \mathbf{u}_{h,K} & \text{for Darcy} \end{cases}$$

and the operators ∇_h, Δ_h and $\nabla_h(\nabla_h \cdot)$ are as in (3.25), (3.27) and (3.28) respectively. In addition, for insuring discrete pressure uniqueness, the system (4.3) is supplemented by the relation

$$\sum_K \int_K p_{h,K} = 0 \tag{4.4}$$

We also need to introduce the generalized bilinear form B defined on $\mathbf{W}_h$ by

$$B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{v}_h, q_h\right)\right] = a\left(\mathbf{u}_h, \mathbf{v}_h\right) - \left(p_h, \nabla_h \cdot \mathbf{v}_h\right)_{0,\Omega} + \left(q_h, \nabla_h \cdot \mathbf{u}_h\right)_{0,\Omega} + J\left(p_h, q_h\right) \quad (4.5)$$

The latter definition yields the following equivalent form (4.2):

Find $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ such that

$$B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{v}_h, q_h\right)\right] = \left(\mathbf{f}, \mathbf{v}_h\right)_{0,\Omega} \qquad \forall\left(\mathbf{v}_h, q_h\right) \in \mathbf{W}_h \quad (4.6)$$

4.1.1 Study of the scheme

Discrete solution regularity

Let us first state a boundedness result of the discrete divergence in the L^2 norm for any $\mathbf{u}_h \in \mathbf{X}_h$.

Lemma 19 (Boundedness of $\nabla_h \cdot \mathbf{u}_h$)**.** *There exists C such that for all $\mathbf{u}_h \in \mathbf{X}_h$ we have*

$$\|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega} \leq C\|\mathbf{u}_h\|_h \quad (4.7)$$

Proof. For any $\mathbf{u}_h \in \mathbf{X}_h$ we get

$$\begin{aligned}
\|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2 &= \sum_K \frac{1}{|K|}\left(\sum_{\sigma\in\mathcal{E}_{int}\cap\partial K} |\sigma| \frac{\mathbf{u}_{h,L} + \mathbf{u}_{h,K}}{2} \cdot \mathbf{n}_\sigma - \sum_{\sigma\in\partial K} |\sigma|\, \mathbf{u}_{h,K} \cdot \mathbf{n}_\sigma\right)^2 \\
&\leq 4\sum_K \frac{1}{|K|}\left(\sum_{\sigma\in\mathcal{E}_{int}\cap\partial K}\left(|\sigma| \frac{\mathbf{u}_{h,L} - \mathbf{u}_{h,K}}{2}\right)^2 + \sum_{\sigma\in\mathcal{E}_{ext}\cap\partial K}\left(|\sigma|\, \mathbf{u}_{h,K} \cdot \mathbf{n}_\sigma\right)^2\right) \\
&= \sum_K \sum_{\sigma\in\mathcal{E}_{int}\cap\partial K} \frac{|\sigma|^2}{|K|}\left(\mathbf{u}_{h,L} - \mathbf{u}_{h,K}\right)^2 + 4\sum_K \sum_{\sigma\in\mathcal{E}_{ext}\cap\partial K} \frac{|\sigma|^2}{|K|}\left(\mathbf{u}_{h,K}\right)^2
\end{aligned}$$

Further, the usual regularity yields

$$\|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2 \leq C^2\left(\sum_{\sigma\in\mathcal{E}_{int}} \frac{|\sigma|}{d_{KL}}(\mathbf{u}_{h,L} - \mathbf{u}_{h,K})^2 + \sum_{\sigma\in\mathcal{E}_{ext}} \frac{|\sigma|}{d_{K\sigma}}(\mathbf{u}_{h,K})^2\right)$$

as required. □

Proposition 20 (Discrete estimate)**.** *Let $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ be a solution to (4.6). Then, there exists C, which depends only on d, Ω, the following inequalities hold*

$$\|\mathbf{u}_h\|_l \leq C\|\mathbf{f}\|_{0,\Omega} \quad (4.8)$$

$$J\left(p_h, p_h\right) \leq C\|\mathbf{f}\|_{0,\Omega}^2 \quad (4.9)$$

Proof. Setting $\mathbf{v} = \mathbf{u}_h$ and $q_h = p_h$ in (4.6), we get

$$a(\mathbf{u}_h, \mathbf{u}_h) + J(p_h, p_h) = (\mathbf{f}, \mathbf{u}_h)_{0,\Omega}$$

Hence, for Stokes we have

$$\mu\|\mathbf{u}_h\|_h^2 + \mu\|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2 + J(p_h, p_h) = (\mathbf{f}, \mathbf{u}_h)_{0,\Omega}$$

Using Young's inequality followed by the Poincaré inequality (3.3), we get

$$\mu\|\mathbf{u}_h\|_h^2 + J(p_h, p_h) \leq \frac{\text{diam}(\Omega)^2}{2\varepsilon}\|\mathbf{f}\|_{0,\Omega}^2 + \frac{\varepsilon}{2}\|\mathbf{u}_h\|_h^2$$

which leads to

$$\left(\mu - \frac{\varepsilon}{2}\right)\|\mathbf{u}_h\|_h^2 + J(p_h, p_h) \leq \frac{\text{diam}(\Omega)^2}{2\varepsilon}\|\mathbf{f}\|_{0,\Omega}^2$$

if $\varepsilon < 2\mu$.

For Darcy, the choice of $\varepsilon < 2\alpha$ gives

$$\alpha\|\mathbf{u}_h\|_{0,\Omega}^2 + J(p_h, p_h) \leq \frac{1}{2\varepsilon}\|\mathbf{f}\|_{0,\Omega}^2 + \frac{\varepsilon}{2}\|\mathbf{u}_h\|_{0,\Omega}^2$$

Consequently, taking $\varepsilon < 2\min\{\mu, \alpha\}$ yields the required bounds. □

Proposition 21 (L^2 pressure estimate). *Let $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ be a solution to (4.6). Then, there exists C, depending only on d, Ω, μ and θ,such that the following inequality holds*

$$\|p_h\|_{0,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega} \tag{4.10}$$

Proof. Let $p_h \in X_h^0$ be given. According to Lemma 15, there exists $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ with

$$\nabla \cdot \mathbf{w} = p_h \qquad \text{and} \qquad |\mathbf{w}|_{1,\Omega} \leq C\,\|p_h\|_{0,\Omega} \tag{4.11}$$

Next, we set

$$w_{h,K}^{(i)} = \frac{1}{|K|}\int_K w^{(i)}(\mathbf{x})\,dx, \qquad \forall K \in \mathcal{D}_h, \quad i = 1, \ldots, d \tag{4.12}$$

and

$$w_{h,\sigma}^{(i)} = \frac{1}{|\sigma|}\int_\sigma w^{(i)}(\mathbf{x})\,d\gamma(x), \quad \forall \sigma \in \mathcal{E}_{int}, \quad i = 1, \ldots, d \tag{4.13}$$

where $\mathbf{w} = \left(w^{(i)}\right)_{i=1,\ldots,d}$.

So, we have

$$(p_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} = \sum_K p_{h,K} \sum_{\sigma\in\mathcal{E}_{int}\cap\partial K} \int_\sigma \mathbf{w}_{h,\sigma}\cdot\mathbf{n}_\sigma + \sum_K p_{h,K} \sum_{\sigma\in\mathcal{E}_{int}\cap\partial K} \int_\sigma \left(\frac{\mathbf{w}_{h,L} + \mathbf{w}_{h,K}}{2} - \mathbf{w}_{h,\sigma}\right)\cdot\mathbf{n}_\sigma$$

and by using (4.13) we get

$$(p_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} = \|p_h\|_{0,\Omega}^2 + \sum_{\sigma\in\mathcal{E}_{int}} \int_\sigma (p_{h,K} - p_{h,L}) \left(\frac{\mathbf{w}_{h,L} + \mathbf{w}_{h,K}}{2} - \mathbf{w}_{h,\sigma}\right) \cdot \mathbf{n}_\sigma$$

Applying the results given in [55], there exists $C > 0$ such that

$$\forall K \in \mathcal{D}_h, \quad \forall \sigma \in \partial K, \quad |\mathbf{w}_{h,K} - \mathbf{w}_{h,\sigma}|^2 \leq Ch^{2-d} \int_K |\nabla \mathbf{w}(\mathbf{x})|^2 d\mathbf{x} \tag{4.14}$$

Now, combining the Cauchy-Schwarz inequality, the obvious inequality

$$\left(\frac{\mathbf{w}_{h,L} + \mathbf{w}_{h,K}}{2} - \mathbf{w}_{h,\sigma}\right)^2 \leq \frac{1}{2}\left((\mathbf{w}_{h,K} - \mathbf{w}_{h,\sigma})^2 + \left((\mathbf{w}_{h,L} - \mathbf{w}_{h,\sigma})^2\right)\right)$$

and (4.14) yields

$$\left|\sum_{\sigma\in\mathcal{E}_{int}} \int_\sigma (p_{h,K} - p_{h,L}) \left(\frac{\mathbf{w}_{h,L} + \mathbf{w}_{h,K}}{2} - \mathbf{w}_{h,\sigma}\right) \cdot \mathbf{n}_\sigma\right| \leq CJ(p_h, p_h)^{\frac{1}{2}} \|\mathbf{w}\|_{1,\Omega}$$

On the other hand, applying (3.23) gives

$$(p_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} \geq \|p_h\|_{0,\Omega}^2 - CJ(p_h, p_h)^{\frac{1}{2}} \|p_h\|_{0,\Omega}$$

Next, by setting $\mathbf{v}_h = \mathbf{w}_h$ in (4.2a), we get

$$(p_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} = a(\mathbf{u}_h, \mathbf{w}_h) - (\mathbf{f}, \mathbf{w}_h)_{0,\Omega}$$

so that

$$\|p_h\|_{0,\Omega}^2 - CJ(p_h, p_h)^{\frac{1}{2}} \|p_h\|_{0,\Omega} \leq a(\mathbf{u}_h, \mathbf{w}_h) - (\mathbf{f}, \mathbf{w}_h)_{0,\Omega}$$

Now, it remains to bound each term of the right-hand side of this relation. To this end, using (3.23) we have

$$(\mathbf{f}, \mathbf{w}_h)_{0,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega} \|p_h\|_{0,\Omega}$$

For Stokes we have

$$a(\mathbf{u}_h, \mathbf{w}_h) \leq \mu \|\mathbf{u}_h\|_h \|\mathbf{w}_h\|_h + \mu \|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega} \|\nabla_h \cdot \mathbf{w}_h\|_{0,\Omega}$$

Applying inequalities (4.7),(4.8) and (3.23), we get

$$a(\mathbf{u}_h, \mathbf{w}_h) \leq C\|\mathbf{f}\|_{0,\Omega} \|p_h\|_{0,\Omega}$$

For Darcy, applying Cauchy-Schwarz inequality,(3.3),(4.8) and (3.23), we get

$$a(\mathbf{u}_h, \mathbf{w}_h) \leq C\|\mathbf{f}\|_{0,\Omega} \|p_h\|_{0,\Omega}$$

which yields

$$\|p_h\|_{0,\Omega}^2 - CJ(p_h, p_h)^{\frac{1}{2}} \|p_h\|_{0,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega} \|p_h\|_{0,\Omega}$$

so that

$$\|p_h\|_{0,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega} + CJ(p_h, p_h)^{\frac{1}{2}}$$

Finally, applying inequality (4.9) gives the claimed estimate. □

Scheme stability

The crucial point is to show that the stabilizing term $J(p_h, p_h)$ enhances sufficiently the degrees of freedom in the pressure field so that a stability condition is satisfied. In the analysis, we will use the following composite norm

$$||| \, (\mathbf{u}_h, p_h) \, |||^2 = \|(\mathbf{u}_h, p_h)\|^2_{\mathbf{W}_h} + J\left(p_h, p_h\right) \qquad \forall \left(\mathbf{u}_h, p_h\right) \in \mathbf{W}_h$$

Note that the triple norm contains of course the L^2-norm of $\nabla_h \cdot \mathbf{u}_h$. This term seems superfluous for Stokes since we already control the discrete norm of the velocities, but of vital importance for Darcy. In fact, the control of the divergence is what allows us to prove optimal error estimates in the energy norm for sufficiently regular solutions. The main result of this section is the following theorem, assuring the well-posedness of the proposed approximation.

Theorem 22. *The finite volume formulation* (4.6) *satisfies the following stability condition*

$$\gamma \, ||| \, (\mathbf{u}_h, p_h) \, ||| \leq \sup_{(\mathbf{v}_h, q_h) \in \mathbf{W}_h} \frac{B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{v}_h, q_h\right)\right]}{||| \, (\mathbf{v}_h, q_h) \, |||} \qquad \forall \left(\mathbf{u}_h, p_h\right) \in \mathbf{W}_h \tag{4.15}$$

Proof. Let $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$.

1) Control of $\|\mathbf{u}_h\|_l^2$:

Taking $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, p_h)$ in (4.5) we get

$$B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{u}_h, p_h\right)\right] = a\left(\mathbf{u}_h, \mathbf{u}_h\right) + J\left(p_h, p_h\right)$$

Hence, we have for Stokes

$$a\left(\mathbf{u}_h, \mathbf{u}_h\right) \geq \mu \|\mathbf{u}_h\|_h^2$$

and for Darcy

$$a\left(\mathbf{u}_h, \mathbf{u}_h\right) = \|\mathbf{u}_h\|_{0,\Omega}^2$$

Thus,

$$B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{u}_h, p_h\right)\right] \geq C_a \|\mathbf{u}_h\|_l^2 + J\left(p_h, p_h\right) \tag{4.16}$$

where

$$C_a = \begin{cases} \mu & \text{for Stokes} \\ 1 & \text{for Darcy} \end{cases}$$

2) Control of $\|p_h\|_{0,\Omega}^2$:

We will follow the methodology used above in the proof of Proposition 21 by applying Lemma 15. Let $\mathbf{w}_h \in \mathbf{X}_h$ defined as in (4.12) and (4.13) for the given p_h.

Taking $(\mathbf{v}_h, q_h) = (\mathbf{w}_h, 0)$ in (4.5) we obtain

$$B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{w}_h, 0\right)\right] = a\left(\mathbf{u}_h, \mathbf{w}_h\right) - \left(p_h, \nabla_h \cdot \mathbf{w}_h\right)_{0,\Omega} \tag{4.17}$$

To proceed, we have to bound each term of the right hand side of this relation. So,
a) for Stokes:

$$\left|a\left(\mathbf{u}_h, \mathbf{w}_h\right)\right| \leq \mu \|\mathbf{u}_h\|_h \|\mathbf{w}_h\|_h + \mu \|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega} \|\nabla_h \cdot \mathbf{w}_h\|_{0,\Omega}$$

Applying inequality (4.7) and (3.23) yields

$$\left|a\left(\mathbf{u}_h, \mathbf{w}_h\right)\right| \leq \frac{2\mu^2}{\varepsilon} \|\mathbf{u}_h\|_h^2 + \varepsilon C \|p_h\|_{0,\Omega}^2$$

b) for Darcy:
Applying inequality (3.3) and (3.23) yields

$$\left|a\left(\mathbf{u}_h, \mathbf{w}_h\right)\right| \leq \frac{1}{\varepsilon} \|\mathbf{u}_h\|_{0,\Omega}^2 + \varepsilon C \|p_h\|_{0,\Omega}^2$$

In both cases, we get

$$a\left(\mathbf{u}_h, \mathbf{w}_h\right) \geq -\frac{C_b}{\varepsilon} \|\mathbf{u}_h\|_l^2 - \varepsilon C \|p_h\|_{0,\Omega}^2$$

where

$$C_b = \begin{cases} 2\mu^2 & \text{for Stokes} \\ 1 & \text{for Darcy} \end{cases}$$

Furthermore, it follows

$$\left(p_h, \nabla_h \cdot \mathbf{w}_h\right)_{0,\Omega} \leq -\|p_h\|_{0,\Omega}^2 + \frac{1}{\varepsilon} J\left(p_h, p_h\right) + \varepsilon C \|p_h\|_{0,\Omega}^2$$

so that

$$-\left(p_h, \nabla_h \cdot \mathbf{w}_h\right)_{0,\Omega} \geq \left(1 - \varepsilon C\right) \|p_h\|_{0,\Omega}^2 - \frac{1}{\varepsilon} J\left(p_h, p_h\right)$$

Gathering all above results, we finally get

$$B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{w}_h, 0\right)\right] \geq -\frac{C_b}{\varepsilon} \|\mathbf{u}_h\|_l^2 \left(1 - \varepsilon C\right) \|p_h\|_{0,\Omega}^2 - \frac{1}{\varepsilon} J\left(p_h, p_h\right) \tag{4.18}$$

3) Control of $\|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2$:

Taking $(\mathbf{v}_h, q_h) = (\mathbf{0}, \nabla_h \cdot \mathbf{u}_h)$ in (4.5) gives

$$B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{0}, \nabla_h \cdot \mathbf{u}_h\right)\right] = \|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2 + J\left(p_h, \nabla_h \cdot \mathbf{u}_h\right)$$

By the definition of discrete divergence operator, $\nabla_h \cdot \mathbf{u}_h \in X_h$ is constant on each volume and hence we have

$$C \|\nabla_h \cdot \mathbf{u}_h\|_{0,K}^2 \geq h \left\|\left(\nabla_h \cdot \mathbf{u}_h\right)_K\right\|_{0,\partial K}^2$$

Using the latter together with Cauchy-Schwarz and Young inequalities allows the following bound for the stabilization term

$$|J(p_h, \nabla_h \cdot \mathbf{u}_h)| \leq \frac{1}{\varepsilon} J(p_h, p_h) + \varepsilon C \|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2$$

So,

$$B[(\mathbf{u}_h, p_h), (\mathbf{0}, \nabla_h \cdot \mathbf{u}_h)] \geq (1 - \varepsilon C) \|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2 - \frac{1}{\varepsilon} J(p_h, p_h) \tag{4.19}$$

Finally, we conclude by taking $(\mathbf{v}_h, q_h) = (\beta \mathbf{u}_h + \mathbf{w}_h, \beta p_h + \nabla_h \cdot \mathbf{u}_h)$ with

$$\beta \geq \left(\frac{C_b}{C_a} + 2\right) \frac{1}{\varepsilon} + \left(\frac{1}{C_a} + 1\right)(1 - \varepsilon C)$$

By combining the latter with (4.16),(4.18) and (4.19), it follows

$$B[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)] \geq \min\left\{\beta C_a - \frac{C_b}{\varepsilon}, 1 - \varepsilon C, \beta - \frac{2}{\varepsilon}\right\} |||\, (\mathbf{u}_h, p_h) \,|||^2$$

Therefore,

$$B[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)] \geq (1 - \varepsilon C) \,|||\, (\mathbf{u}_h, p_h) \,|||^2$$

The required result follows by taking ε sufficiently small, e.g. $\varepsilon < \frac{1}{C}$, and noting that there exists C such that

$$|||\, (\mathbf{u}_h, p_h) \,||| \geq C \,|||\, (\mathbf{v}_h, q_h) \,|||$$

□

4.1.2 Error estimates

Here, we establish error estimates for the discrete FV solution in the usual norms. Let us start with the Stokes case. The fundamendal result of error estimates is a consequence based on the following intermediate proposition.

Proposition 23. *Let* $(\mathbf{u}, p) \in (\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)) \times H^1(\Omega)$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ *be the respective solutions of* (2.4) *and* (4.6) *for Stokes. Then, for any* $\varepsilon < \frac{1}{2}\min\{\mu, 1\}$ *there exists a constant* C*, depending only on* d, μ, Ω *and* θ*, such that*

$$\|\mathbf{u}_h - \pi(\mathbf{u})\|_h \leq Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.20}$$

$$J(p_h, p_h)^{1/2} \leq Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.21}$$

Proof. First, let $(\hat{\mathbf{u}}_h, \hat{p}_h) \in \mathbf{X}_h \times X_h^0$ be defined by $\hat{\mathbf{u}}_h = \pi(\mathbf{u})$ and $\hat{p}_h = \pi(p)$. Integrating (2.4a) on $K \in \mathcal{D}_h$ and using the fact that $\nabla \cdot \mathbf{u} = 0$ gives

$$-\mu \sum_{\sigma \in \partial K} \int_\sigma \nabla \mathbf{u} \cdot \mathbf{n}_\sigma + \sum_{\sigma \in \partial K} \int_\sigma p \mathbf{n}_\sigma = \int_K \mathbf{f} \tag{4.22}$$

i.e.

$$-\mu \int_K \Delta_h \hat{\mathbf{u}}_h + \int_K \nabla_h \hat{p}_h = \int_K \mathbf{f} + \int_K R_K(\mathbf{u}, p)$$

where

$$R_K(\mathbf{u}, p) = -\mu \sum_{\sigma \in \partial K} R_\Delta(\mathbf{u}) + \sum_{\sigma \in \partial K} R_\nabla(p)$$

Set $\mathbf{e}_h = \hat{\mathbf{u}}_h - \mathbf{u}_h$ and $\epsilon_h = \hat{p}_h - p_h$. Then, subtracting (4.3a) for Stokes equation from the above equation, we get

$$-\mu \int_K \Delta_h \mathbf{e}_h + \mu \left(\nabla_h \left(\nabla_h \cdot \mathbf{u}_h\right)\right)_K + \int_K \nabla_h (\epsilon_h) = \int_K R_K(\mathbf{u}, p)$$

For any $\mathbf{v}_h \in \mathbf{X}_h$, we get

$$\mu[\mathbf{e}_h, \mathbf{v}_h]_h - \mu \left(\nabla_h \cdot \mathbf{u}_h, \nabla_h \cdot \mathbf{v}_h\right)_{0,\Omega} - \left(\epsilon_h, \nabla_h \cdot \mathbf{v}_h\right)_{0,\Omega} = \left(R(\mathbf{u}, p), \mathbf{v}_h\right)_{0,\Omega} \tag{4.23}$$

Setting $\mathbf{v}_h = \mathbf{e}_h$ in this last relation yields

$$\mu\|\mathbf{e}_h\|_h^2 + \mu\|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2 - \left(\epsilon_h, \nabla_h \cdot \mathbf{e}_h\right)_{0,\Omega} = \left(R(\mathbf{u}, p), \mathbf{e}_h\right)_{0,\Omega} + \mu\left(\nabla_h \cdot \mathbf{u}_h, \nabla_h \cdot \hat{\mathbf{u}}_h\right)_{0,\Omega} \tag{4.24}$$

Now, let us integrate (2.4b) on $K \in \mathcal{D}_h$. This gives

$$\sum_{\sigma \in \partial K} \int_\sigma \mathbf{u} \cdot \mathbf{n}_\sigma = 0$$

Since $\mathbf{u}$ vanishes on the boundary of Ω, we obtain

$$\int_K \nabla_h \cdot \hat{\mathbf{u}}_h = \sum_{\sigma \in \mathcal{E}_{int} \cap \partial K} R_{\text{div}}(\mathbf{u}) \qquad \forall K \in \mathcal{D}_h$$

Then, subtracting (4.3b) for Stokes equation from the above equation gives

$$\int_K \nabla_h \cdot \mathbf{e}_h = \sum_{\sigma \in \mathcal{E}_{int} \cap \partial K} R_{\text{div}}(\mathbf{u}) + 2\delta \sum_{\sigma \in \mathcal{E}_{int} \cap \partial K} \int_\sigma h\,[p_h]$$

This yields

$$\left(q_h, \nabla_h \cdot \mathbf{e}_h\right)_{0,\Omega} = \sum_{\sigma \in \mathcal{E}_{int}} R_{\text{div}}(\mathbf{u}) \left(q_{h,K} - q_{h,L}\right) + J(p_h, q_h)$$

and setting $q_h = \epsilon_h$ in this relation gives

$$\left(\epsilon_h, \nabla_h \cdot \mathbf{e}_h\right)_{0,\Omega} = \sum_{\sigma \in \mathcal{E}_{int}} R_{\text{div}}(\mathbf{u}) \left(\epsilon_{h,K} - \epsilon_{h,L}\right) + J(p_h, \epsilon_h) \tag{4.25}$$

Gathering (4.24) and (4.25), we get

$$\begin{aligned} \mu\|\mathbf{e}_h\|_h^2 + \mu\|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2 + J(p_h, p_h) &= \left(R(\mathbf{u}, p), \mathbf{e}_h\right)_{0,\Omega} + \mu\left(\nabla_h \cdot \mathbf{u}_h, \nabla_h \cdot \hat{\mathbf{u}}_h\right)_{0,\Omega} \\ &+ \sum_{\sigma \in \mathcal{E}_{int}} R_{\text{div}}(\mathbf{u}) \left(\epsilon_{h,K} - \epsilon_{h,L}\right) + J(p_h, \widehat{p}_h) \end{aligned} \tag{4.26}$$

Next, let us study the terms in the right-hand side of (4.26). The first term is

$$(R(\mathbf{u},p),\mathbf{e}_h)_{0,\Omega} = -\mu\sum_K\sum_{\sigma\in\partial K} R_\Delta(\mathbf{u})\cdot\mathbf{e}_{h,K} + \sum_K\sum_{\sigma\in\partial K} R_\nabla(p)\cdot\mathbf{e}_{h,K} \tag{4.27}$$

Using the Cauchy-Schwarz inequality and the consistencly result (3.29), it follows

$$\left|-\mu\sum_K\sum_{\sigma\in\partial K} R_\Delta(\mathbf{u})\cdot\mathbf{e}_{h,K}\right| \le Ch\|\mathbf{e}_h\|_h\|\mathbf{u}\|_{2,\Omega}$$

Likewise, by using the Cauchy-Schwarz inequality and the consistencly result (3.30), we get

$$\left|\sum_K\sum_{\sigma\in\partial K} R_\nabla(p)\cdot\mathbf{e}_{h,K}\right| \le Ch\|\mathbf{e}_h\|_h\|p\|_{1,\Omega}$$

Getting back to (4.27), we have

$$(R(\mathbf{u},p),\mathbf{e}_h)_{0,\Omega} \le C\frac{h^2}{\varepsilon}\|\mathbf{u}\|^2_{2,\Omega} + 2\varepsilon\|\mathbf{e}_h\|^2_h + C_5\frac{h^2}{\varepsilon}\|p\|^2_{1,\Omega} \tag{4.28}$$

for all $\varepsilon > 0$.

Using the Cauchy-Schwarz inequality,(3.16) and (4.7) on the second term in (4.26), we get

$$\mu(\nabla_h\cdot\mathbf{u}_h,\nabla_h\cdot\hat{\mathbf{u}}_h)_{0,\Omega} \le C\frac{h^2}{\varepsilon}\|\mathbf{u}\|^2_{2,\Omega} + \varepsilon\|\nabla_h\cdot\mathbf{u}_h\|^2_{0,\Omega} \tag{4.29}$$

Let us decompose the third term in (4.26) to get

$$\sum_{\sigma\in\mathcal{E}_{int}} R_{\text{div}}(\mathbf{u})(\epsilon_{h,K}-\epsilon_{h,L}) = \sum_{\sigma\in\mathcal{E}_{int}} R_{\text{div}}(\mathbf{u})(\widehat{p}_{h,K}-\widehat{p}_{h,L}) + \sum_{\sigma\in\mathcal{E}_{int}} R_{\text{div}}(\mathbf{u})(p_{h,K}-p_{h,L})$$

We have

$$\sum_{\sigma\in\mathcal{E}_{int}} R_{\text{div}}(\mathbf{u})(\widehat{p}_{h,K}-\widehat{p}_{h,L}) \le \left(\sum_{\sigma\in\mathcal{E}_{int}}\frac{1}{d_{KL}}(\widehat{p}_{h,K}-\widehat{p}_{h,L})^2\right)^{\frac{1}{2}}\left(\sum_{\sigma\in\mathcal{E}_{int}} d_{KL}\left(R_{\text{div}}(\mathbf{u})\right)^2\right)^{\frac{1}{2}}$$

Using the consistencly result (3.31), we get

$$\sum_{\sigma\in\mathcal{E}_{int}} R_{\text{div}}(\mathbf{u})(\widehat{p}_{h,K}-\widehat{p}_{h,L}) \le Ch^2\left(\frac{1}{\varepsilon}\|\mathbf{u}\|^2_{2,\Omega} + \varepsilon\|p\|^2_{1,\Omega}\right) \tag{4.30}$$

so that

$$\sum_{\sigma\in\mathcal{E}_{int}} R_{\text{div}}(\mathbf{u})(p_{h,K}-p_{h,L}) \le C\frac{h^2}{\varepsilon}\|\mathbf{u}\|^2_{2,\Omega} + \varepsilon J(p_h,p_h) \tag{4.31}$$

Finally, the Cauchy-Schwarz inequality implies

$$J(p_h,\hat{p}_h) \le \varepsilon J(p_h,p_h) + C\frac{h^2}{\varepsilon}\|p\|^2_{1,\Omega} \tag{4.32}$$

Gathering (4.28) to (4.32) yields the control error inequality

$$(\mu-\varepsilon)\|\mathbf{e}_h\|^2_h + (\mu-\varepsilon)\|\nabla_h\cdot\mathbf{u}_h\|^2_{0,\Omega} + (1-2\varepsilon)J(p_h,p_h) \le Ch^2\left(\|\mathbf{u}\|^2_{2,\Omega} + \|p\|^2_{1,\Omega}\right)$$

By choosing ε sufficiently small, e.g. $\varepsilon < \frac{1}{2}\min\{\mu,1\}$, it is clear that the latter implies (4.20) and (4.21). □

Theorem 24. *In addition to assumptions of Proposition 23, let $\mathbf{u}_{\mathcal{D}} \in \mathbf{X}_h$ defined by: $\mathbf{u}_{\mathcal{D}K} = \mathbf{u}\left(\mathbf{x}_K\right), \quad K \in \mathcal{D}_h$. Then*

$$\|\mathbf{u}_h - \mathbf{u}_{\mathcal{D}}\|_h \leq Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.33}$$

and

$$\|\mathbf{u}_h - \mathbf{u}\|_{0,\Omega} \leq Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.34}$$

$$\|p_h - p\|_{0,\Omega} \leq Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.35}$$

Proof. First of all, note that (4.33) is straightforwardly deduced from the definition of $\mathbf{u}_{\mathcal{D}}$ and (4.20) by applying the triangular inequality and a classical interpolation result . Likewise, (4.34) follows by similar arguments from the triangular inequality, interpolation results and the discrete Poincaré inequality (3.3). It remains to establish (4.35).

Here again, we will use Lemma 15. Like in Proposition 21, there is a $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ satisfying

$$\nabla \cdot \mathbf{w} = \epsilon_h \quad \text{and} \quad \|\mathbf{w}\|_{1,\Omega} \leq C_{15}\|\epsilon_h\|_{0,\Omega}$$

We again define $\mathbf{w}_h = \pi\left(\mathbf{w}\right)$ satisfying (4.12) and (4.13). We have

$$\|\epsilon_h\|_{0,\Omega}^2 \leq (\epsilon_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} + J(\epsilon_h, \epsilon_h)^{\frac{1}{2}} C\|\epsilon_h\|_{0,\Omega}$$

which yields

$$\|\epsilon_h\|_{0,\Omega}^2 \leq (\epsilon_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} + C\frac{h^2}{\varepsilon}\|p\|_{1,\Omega}^2 + \frac{1}{\varepsilon}J\left(p_h, p_h\right) + \varepsilon C\|\epsilon_h\|_{0,\Omega}^2$$

We now use $\mathbf{w}_h$ as a test function in (4.23). We get

$$(\epsilon_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} = \mu[\mathbf{e}_h, \mathbf{w}_h]_h - \mu(\nabla_h \cdot \mathbf{u}_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} - \left(R\left(\mathbf{u}, p\right), \mathbf{w}_h\right)_{0,\Omega}$$

Gathering the latter with the above relations yields

$$\begin{aligned}
\|\epsilon\|_{0,\Omega}^2 \leq\ & C\|\mathbf{e}_h\|_h\|\epsilon_h\|_{0,\Omega} + C\|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_h\|\epsilon_h\|_{0,\Omega} + C\|\widehat{\mathbf{u}}_h\|_h\|\epsilon_h\|_{0,\Omega} \\
& + Ch\|\epsilon_h\|_h\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) + C\frac{h^2}{\varepsilon}\|p\|_{1,\Omega}^2 \\
& + \frac{1}{\varepsilon}J\left(p_h, p_h\right) + \varepsilon C_{12}\|\epsilon_h\|_{0,\Omega}^2
\end{aligned}$$

Finally, applying successively the Young inequality, (3.16), (4.20), (4.21) and the continuity of the interpolation operator leads to the desired result. □

The corresponding result on error estimates for the Darcy part is now established.

Theorem 25. *Let* $(\mathbf{u}, p) \in \left(\mathbf{H}^1(\Omega) \cap \mathbf{H}_0^1(\Omega)\right) \times H^1(\Omega)$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ *be the Darcy solutions of* (2.10) *and* (4.6) *repectively. Then, there exists* C*, which depends only on* d, Ω, δ *and* θ*, such that*

$$\|\mathbf{u}_h - \pi(\mathbf{u})\|_{0,\Omega} \leq Ch\left(\|\mathbf{u}\|_{1,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.36}$$

$$\|\mathbf{u}_h - \mathbf{u}\|_{0,\Omega} \leq Ch\left(\|\mathbf{u}\|_{1,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.37}$$

$$\|p_h - p\|_{0,\Omega}^2 \leq Ch\left(\|\mathbf{u}\|_{1,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.38}$$

Proof. 1) Control equation for errors.

First, let $(\widehat{\mathbf{u}}_h, \hat{p}_h) \in \mathbf{X}_h \times X_h^0$ be defined by $\widehat{\mathbf{u}}_h = \pi(\mathbf{u})$ and $\hat{p}_h = \pi(p)$. Integrating (2.10a) on $K \in \mathcal{D}_h$ gives

$$\int_K \mathbf{u} + \sum_{\sigma \in \partial K} \int_\sigma p\mathbf{n}_\sigma = \int_K \mathbf{f} \tag{4.39}$$

We introduce for all $K \in \mathcal{D}_h$ the following consistency residuals

$$R_{0,K}(\mathbf{u}) = \widehat{\mathbf{u}}_h - \frac{1}{|K|}\int_K \mathbf{u}$$

From (4.39) we get

$$\int_K \widehat{\mathbf{u}}_{h,K} + \int_K \nabla_h \, \widehat{p}_h = \int_K \mathbf{f} + \int_K R_K(\mathbf{u}, p)$$

with

$$R_K(\mathbf{u}, p) = R_{0,K}(\mathbf{u}) + \frac{1}{|K|}\sum_{\sigma \in \partial K} \int_\sigma R_\nabla(p)\,\mathbf{n}_\sigma$$

Set $\mathbf{e}_h = \widehat{\mathbf{u}}_h - \mathbf{u}_h$ and $\epsilon_h = \widehat{p}_h - p_h$. Subtracting equation (4.3a) for Darcy equation from the above equation yields

$$\int_K \mathbf{e}_{h,K} + \int_K \nabla_h \, \epsilon_h = \int_K R_K(\mathbf{u}, p)$$

For all $\mathbf{v}_h \in \mathbf{X}_h$, we get

$$\int_\Omega \mathbf{e}_h \cdot \mathbf{v}_h - \int_\Omega \epsilon_h (\nabla_h \cdot \mathbf{v}_h) = \int_\Omega R(\mathbf{u}, p) \cdot \mathbf{v}_h$$

and setting $\mathbf{v}_h = \mathbf{e}_h$ in this relation, it follows

$$\|\mathbf{e}_h\|_{0,\Omega}^2 - \int_\Omega \epsilon_h (\nabla_h \cdot \mathbf{e}_h) = \int_\Omega R(\mathbf{u}, p) \cdot \mathbf{e}_h$$

Using (4.25), we get

$$\|\mathbf{e}_h\|_{0,\Omega}^2 + J(p_h, p_h) = \int_\Omega R(\mathbf{u}, p) \cdot \mathbf{e}_h + \sum_{\sigma \in \mathcal{E}_{int}} |\sigma|\, R_{\text{div}}(\epsilon_{h,K} - \epsilon_{h,L}) + J(p_h, \widehat{p}_h)$$

2) Proof of bounds.

Let us study the terms at the right-hand side of the above equation. The first term can be written as

$$\begin{aligned}\int_\Omega R(\mathbf{u},p)\cdot\mathbf{e}_h &= \sum_K |K|\, R_{0,K}(\mathbf{u})\cdot\mathbf{e}_{h,K} + \sum_K\sum_{\sigma\in\partial K}|\sigma|\, R_\nabla(p)\,\mathbf{n}_\sigma\cdot\mathbf{e}_{h,K}\\ &\leq \left(\sum_K |K|\,(R_{0,K}(\mathbf{u}))^2\right)^{\frac{1}{2}}\|\mathbf{e}_h\|_{0,\Omega} + Ch\|\mathbf{e}_h\|_{0,\Omega}\|p\|_{1,\Omega}\end{aligned}$$

from the result shown in (4.28). We have

$$\int_\Omega R(\mathbf{u},p)\cdot\mathbf{e}_h \leq C_3\frac{h^2}{\varepsilon}\|\mathbf{u}\|_{1,\Omega}^2 + 2\varepsilon\|\mathbf{e}_h\|_{0,\Omega}^2 + C\frac{h^2}{\varepsilon}\|p\|_{1,\Omega}^2 \tag{4.40}$$

Finally, using like (4.30),(4.31) and (4.32) gives (4.36). Using the same technique as in Proposition 23, we deduce (4.37) and (4.38). This completes the proof of the theorem. □

4.2 A FVM for a coupled Stokes-Darcy problem

Now, let us turn to the coupled Stokes-Darcy problem (2.16) to (2.20). Here, recall that the domain Ω is split into two parts Ω_s and Ω_d for the Stokes-Darcy system as presented in [64].

For discretizing Ω let assume $\mathcal{D}_h = \mathcal{D}_{h,s}\cup\mathcal{D}_{h,d}$ where $\mathcal{D}_{h,s}$ and $\mathcal{D}_{h,d}$ are two famillies of regular volumes for the partitioning of Ω_s and Ω_d as previously defined. Further, denote by $\mathcal{E}_\Gamma$ the finite set of volume boundaries σ (edges or faces) on the interface Γ.

Let us define the global space of velocities $\mathbf{V}_h$ analogue to (2.21), defined by

$$\mathbf{V}_h = \mathbf{X}_h \cap \mathbf{V}$$

equipped with the norm

$$\|\mathbf{u}\|_{\mathbf{V}_h} = \left(\|\mathbf{u}\|_h^2 + \|\mathbf{u}\|_{0,\Omega_d}^2\right)^{\frac{1}{2}}$$

We will also denote by $\mathbf{W}_h$ the product space $\mathbf{V}_h \times X_h^0$.

The proposed discrete approximation of (2.16) is :

Find $(\mathbf{u}_h, p_h)\in\mathbf{W}_h$ such that

$$a(\mathbf{u}_h,\mathbf{v}_h) + b(\mathbf{v}_h,p_h) = (\mathbf{f},\mathbf{v}_h)_{0,\Omega} \qquad \forall\mathbf{v}_h\in\mathbf{V}_h \tag{4.41a}$$

$$b(\mathbf{u}_h,q_h) - J(p_h,q_h) = 0 \qquad \forall q_h\in X_h \tag{4.41b}$$

where

$$\begin{aligned} a(\mathbf{u}_h,\mathbf{v}_h) &= \mu[\mathbf{u}_h,\mathbf{v}_h]_h + \mu\left(\mathbf{k}^{-1}\mathbf{u}_h,\mathbf{v}_h\right)_{0,\Omega_d} + \frac{\mu\alpha}{\sqrt{k}}(\mathbf{u}_{h,s}\cdot\tau,\mathbf{v}_{h,s}\cdot\tau)_{0,\Gamma}\\ b(\mathbf{v}_h,p_h) &= -(p_h,\nabla_h\cdot\mathbf{v}_h)_{0,\Omega}\\ J(p_h,q_h) &= J_s(p_h,q_h) + J_d(p_h,q_h)\end{aligned}$$

with

$$J_l\left(p_h, q_h\right) = \delta_l \sum_K \int_{\partial K \backslash \Gamma} h_{\partial K}\left[p_h\right]\left[q_h\right] ds \qquad l = s, d$$

and $\delta_l > 0$ is a stabilization parameter.

The stabilized coupled formulation (4.41) can be written in the global form:

Find $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ such that

$$B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{v}_h, q_h\right)\right] = \left(\mathbf{f}, \mathbf{v}_h\right)_{0,\Omega} \tag{4.42}$$

where

$$B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{v}_h, q_h\right)\right] = a\left(\mathbf{u}_h, \mathbf{v}_h\right) + b\left(\mathbf{v}_h, p_h\right) + b\left(\mathbf{u}_h, q_h\right) - J\left(p_h, q_h\right) \tag{4.43}$$

We equip $\mathbf{W}_h$ with the following norm

$$|||(\mathbf{u}_h, p_h)|||^2 = \|\mathbf{u}_h\|^2_{\mathbf{V}_h} + \|\nabla_h \cdot \mathbf{u}_h\|^2_{0,\Omega} + \|p_h\|^2_{0,\Omega} + J\left(p_h, p_h\right) \tag{4.44}$$

4.2.1 Study of the scheme

Discrete solution regularity

This section is concerned with the existence and uniqueness of the finite volume solution. First, let us establish some estimates which express continuity (control) of discrete velocity and pressure with respect to the external source $\mathbf{f}$.

Proposition 26. *Let $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ be a solution to (4.42). Then, there exists C such that the following inequalities hold*

$$\|\mathbf{u}_h\|_{\mathbf{V}_h} \leq C\|\mathbf{f}\|_{0,\Omega} \tag{4.45}$$

$$J\left(p_h, p_h\right) \leq C\|\mathbf{f}\|^2_{0,\Omega} \tag{4.46}$$

Proof. Setting $\mathbf{v}_h = \mathbf{u}_h$ and $q_h = -p_h$ in (4.42), we get

$$a\left(\mathbf{u}_h, \mathbf{u}_h\right) + J\left(p_h, p_h\right) = \left(\mathbf{f}, \mathbf{v}_h\right)_{0,\Omega}$$

Using (2.15), the Young inequality and Poincaré inequalities (3.3), it follows that

$$\mu\|\mathbf{u}_h\|^2_h + \lambda_1\|\mathbf{u}_h\|^2_{0,\Omega_d} + J\left(p_h, p_h\right) \leq \frac{\operatorname{diam}\left(\Omega\right)^2}{2\varepsilon}\|\mathbf{f}\|^2_{0,\Omega} + \frac{\varepsilon}{2}\|\mathbf{u}_h\|^2_h + \frac{\varepsilon}{2\operatorname{diam}\left(\Omega\right)^2}\|\mathbf{u}_h\|^2_{0,\Omega_d}$$

It is now clear that the choice $\varepsilon < 2\min\left\{\mu, \lambda_1 \operatorname{diam}\left(\Omega\right)^2\right\}$ yields the result. □

Proposition 27. *Let $(\mathbf{u}_h, p_h) \in \mathbf{W}_h$ be a solution to (4.42). Then, there exists C such that the following inequality holds*

$$\|p_h\|_{0,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega} \tag{4.47}$$

Proof. Let $(\mathbf{w}_h, p_h) \in \mathbf{W}_h$, where $\mathbf{w}_h = \pi(\mathbf{w}), \mathbf{w} \in \mathbf{H}_0^1(\Omega)$. Following the same steps as previously, we get

$$(p_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} \geq \|p_h\|_{0,\Omega}^2 - CJ(p_h, p_h)^{1/2} \|p_h\|_{0,\Omega} \tag{4.48}$$

Next, setting $\mathbf{v}_h = \mathbf{w}_h$ in (4.41a) gives

$$(p_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} = a(\mathbf{u}_h, \mathbf{w}_h) - (\mathbf{f}, \mathbf{w}_h)_{0,\Omega}$$

so that

$$\|p_h\|_{0,\Omega}^2 - CJ(p_h, p_h)^{1/2} \|p\|_{0,\Omega} \leq a(\mathbf{u}_h, \mathbf{w}_h) - (\mathbf{f}, \mathbf{w}_h)_{0,\Omega}$$

Now, it remains to bound each term of the right-hand side of this relation. Combining the Cauchy-Schwarz inequality, (3.3) and (3.23) yields

$$(\mathbf{f}, \mathbf{w}_h)_{0,\Omega} \leq \|\mathbf{f}\|_{0,\Omega} \|p_h\|_{0,\Omega} \tag{4.49}$$

Using again the Cauchy-Schwarz inequality and (2.15) leads to

$$a(\mathbf{u}, \mathbf{w}_h) \leq C\|\mathbf{u}_h\|_{\mathbf{V}_h} \|\mathbf{w}_h\|_{\mathbf{V}_h} + C\|\mathbf{u}_{h,s}\|_{0,\Gamma} \|\mathbf{w}_{h,s}\|_{0,\Gamma}$$

Since $\mathbf{u}_h \in \mathbf{V}_h$ is constant on each volume, we have

$$C\|\mathbf{u}_h\|_{0,K}^2 \geq h \|\mathbf{u}_{h,K}\|_{\partial K}^2 \tag{4.50}$$

For $\sigma \in \mathcal{E}_\Gamma$ we can also see that

$$\sum_{\sigma \in \mathcal{E}_\Gamma} \left\| h_\sigma^{1/2} \cdot \mathbf{u}_{h,l} \right\|_{0,\sigma}^2 \leq C\|\mathbf{u}_{h,l}\|_{0,\Omega_l^\Gamma}^2 \leq C\|\mathbf{u}_h\|_{0,\Omega_l}^2 \tag{4.51}$$

where Ω_l^Γ denotes the union of the volumes in Ω_l neighboring the boundary Γ.

Further, using (3.3) and (4.51) we get

$$Ch^{1/2}\|\mathbf{u}_{h,s}\|_{0,\Gamma} \leq \|\mathbf{u}_h\|_{\mathbf{V}_h}$$

To conclude, the use of the trace inequality (3.4) combined with interpolation result leads to

$$h^{-1/2}\|\mathbf{w}_{h,s}\|_{0,\Gamma} \leq \|\mathbf{w}\|_{1,\Omega}$$

So,

$$a(\mathbf{u}_h, \mathbf{w}_h) \leq C\|\mathbf{u}_h\|_{\mathbf{V}_h} \|\mathbf{w}_h\|_{\mathbf{V}_h} + \|\mathbf{u}_h\|_{\mathbf{V}_h} \|\mathbf{w}_h\|_{1,\Omega} \tag{4.52}$$

and by (3.3), (4.45) and (3.23) we get

$$a(\mathbf{u}_h, \mathbf{w}_h) \leq C\|\mathbf{f}\|_{0,\Omega} \|p_h\|_{0,\Omega} \tag{4.53}$$

Gathering (4.49) and (4.53) gives

$$\|p_h\|_{0,\Omega}^2 - CJ\left(p_h, p_h\right)^{1/2}\|p_h\|_{0,\Omega} \le C\|\mathbf{f}\|_{0,\Omega}\|p_h\|_{0,\Omega}$$

and, equivalently,

$$\|p_h\|_{0,\Omega} \le C\|\mathbf{f}\|_{0,\Omega} + CJ\left(p_h, p_h\right)^{1/2}$$

Finally, applying inequality (4.46) gives the claimed result. □

Scheme stability

Theorem 28. *The following stability inequality holds*

$$\gamma\,|||(\mathbf{u}_h,p_h)||| \le \sup_{(\mathbf{v}_h,q_h)\in\mathbf{W}_h} \frac{B\left[\left(\mathbf{u}_h,p_h\right),\left(\mathbf{v}_h,q_h\right)\right]}{|||(\mathbf{v}_h,q_h)|||} \qquad \forall\left(\mathbf{u}_h,p_h\right)\in\mathbf{W}_h \tag{4.54}$$

Proof. 1) Control of $\|\mathbf{u}_h\|_{\mathbf{V}_h}^2$:

Taking $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, -p_h)$ in (4.43) we obtain

$$B\left[\left(\mathbf{u}_h, p_h\right), \left(\mathbf{u}_h, -p_h\right)\right] = a\left(\mathbf{u}_h, \mathbf{u}_h\right) + J\left(p_h, p_h\right)$$

Hence, using (2.15) we have

$$a\left(\mathbf{u}_h, \mathbf{u}_h\right) \ge \min\left\{\mu, \lambda_1\right\}\|\mathbf{u}_h\|_{\mathbf{V}_h}^2$$

Thus,

$$B\left[\left(\mathbf{u}_h, p\right), \left(\mathbf{u}_h, -p_h\right)\right] \ge \min\left\{\mu, \lambda_1\right\}\|\mathbf{u}_h\|_{\mathbf{V}_h}^2 + J\left(p_h, p_h\right) \tag{4.55}$$

2) Control of $\|p_h\|_{0,\Omega}^2$:

Setting $(\mathbf{v}_h, q_h) = (\mathbf{w}_h, 0)$ in (4.43) we obtain

$$B\left[\left(\mathbf{u}_h,p_h\right), \left(\mathbf{w}_h, 0\right)\right] = a\left(\mathbf{u}_h, \mathbf{w}_h\right) + b\left(p_h, \mathbf{w}_h\right)$$

To proceed, we need to bound each term of the right-hand side of tis relation. Thus, from (4.52), we may write

$$a\left(\mathbf{u}_h, \mathbf{w}_h\right) \ge -\frac{\max\left\{\mu^2, \lambda_2^2\right\}}{\varepsilon}\|\mathbf{u}_h\|_{\mathbf{V}_h}^2 - \varepsilon C\|p_h\|_{0,\Omega}^2 \tag{4.56}$$

For $b\left(p_h, \mathbf{w}_h\right)$ we use the same technique as in Proposition (21). It follows

$$b\left(p_h, \mathbf{w}_h\right) \ge \left(1 - \varepsilon C\right)\|p_h\|_{0,\Omega}^2 - \frac{1}{\varepsilon}J\left(p_h, p_h\right) \tag{4.57}$$

Collecting all estimated terms, we obtain

$$B\left[\left(\mathbf{u}_h,p_h\right), \left(\mathbf{w}_h, 0\right)\right] \ge -\frac{\max\left\{\mu^2, \lambda_2^2\right\}}{\varepsilon}\|\mathbf{u}_h\|_{\mathbf{V}_h}^2 + \left(1 - \varepsilon C\right)\|p_h\|_{0,\Omega}^2 - \frac{1}{\varepsilon}J\left(p_h, p_h\right) \tag{4.58}$$

3) Control of $\|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2$:

Taking $(\mathbf{v}_h,q_h) = (\mathbf{0}, -\nabla_h \cdot \mathbf{u}_h)$ in (4.43) gives

$$B\left[(\mathbf{u}_h,p_h), (\mathbf{0}, -\nabla_h \cdot \mathbf{u}_h)\right] = \|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2 + J\left(p_h, \nabla_h \cdot \mathbf{u}_h\right)$$

We apply (4.50) together with the Cauchy-Schwarz and Young inequalities, then

$$|J\left(p_h, \nabla_h \cdot \mathbf{u}_h\right)| \leq \frac{1}{\varepsilon} J\left(p_h, p_h\right) + \varepsilon C \|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2$$

which leads to

$$B\left[(\mathbf{u}_h, p_h), (\mathbf{0}, -\nabla_h \cdot \mathbf{u}_h)\right] \geq (1 - \varepsilon C) \|\nabla_h \cdot \mathbf{u}_h\|_{0,\Omega}^2 - \frac{1}{\varepsilon} J\left(p_h, p_h\right) \tag{4.59}$$

Finally, by taking $(\mathbf{v}_h, q_h) = (\beta \mathbf{u}_h + \mathbf{w}_h, \beta p_h - \nabla_h \cdot \mathbf{u}_h)$ with β sufficiently large and ε conveniently chosen, we get

$$B\left[(\mathbf{u}_h, p_h), (\mathbf{w}_h, q_h)\right] \geq \min\left\{\beta \min\left\{\mu, \lambda_1\right\} - \frac{\max\left\{\mu^2, \lambda_2^2\right\}}{\varepsilon}, 1 - \varepsilon C, \beta - \frac{2}{\varepsilon}\right\} |||(\mathbf{u}_h,p_h)|||^2$$

Therefore,

$$B\left[(\mathbf{u}_h, p_h), (\beta \mathbf{u}_h + \mathbf{v}_h, \beta p_h - \nabla_h \cdot \mathbf{u}_h)\right] \geq (1 - \varepsilon C) |||(\mathbf{u}_h,p_h)|||^2$$

The required result follows by noting that there exists C such that

$$|||(\mathbf{u}_h,p_h)||| \geq C\, |||(\mathbf{v}_h,q_h)|||$$

□

4.2.2 Error estimates

As a consequence of the above stability results, we obtain the error estimates relevant to convergence.

Proposition 29. *Let* $(\mathbf{u},p)$ *and* $(\mathbf{u}_h,p_h)$ *be the solutions of problems* (2.16) *and* (4.42) *respectively. Assume that* $(\mathbf{u}, p) \in \mathbf{H}^2(\Omega) \times H^1(\Omega)$*. Then, there exists* C*, depending only on* $d, \Omega, \mu, \lambda_1$ *and* θ*, such that*

$$\|\pi(\mathbf{u}) - \mathbf{u}_h\|_{\mathbf{V}_h} \leq Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.60}$$

$$J(p_h, p_h)^{\frac{1}{2}} \leq Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.61}$$

Proof. First, let $(\hat{\mathbf{u}}_h, \hat{p}_h) \in \mathbf{X}_h \times X_h^0$ be defined by $\hat{\mathbf{u}}_h = \pi(\mathbf{u})$ and $\hat{p}_h = \pi(p)$.

Multiplying the two first equations in (2.16) by $\mathbf{v} \in \mathbf{V}_h$, integrating by parts on $K \in \mathcal{D}_{h,s}$ and $L \in \mathcal{D}_{h,d}$ respectively, adding the two equations and using the interface condition yields

$$\begin{aligned}&\mu[\widehat{\mathbf{u}}_h, \mathbf{v}_h]_h + \mu\left(\mathbf{k}^{-1}\widehat{\mathbf{u}}_h, \mathbf{v}_h\right)_{0,\Omega_d} - (\widehat{p}_h, \nabla_h \cdot \mathbf{v}_h)_{0,\Omega} + \frac{\mu\alpha}{\sqrt{k}}(\widehat{\mathbf{u}}_{h,s}\cdot\tau, \mathbf{v}_{h,s}\cdot\tau)_{0,\Gamma}\\ &= (\mathbf{f}, \mathbf{v}_h)_{0,\Omega} + R_1(\mathbf{v}_h) + \frac{\mu\alpha}{\sqrt{k}}((\widehat{\mathbf{u}}_{h,s} - \mathbf{u}_s)\cdot\tau, \mathbf{v}_{h,s}\cdot\tau)_{0,\Gamma}\end{aligned}$$

where $R_1(\mathbf{v}_h)$ is the consistency residual defined by

$$\begin{aligned}R_1(\mathbf{v}_h) = &\mu[\widehat{\mathbf{u}}_h, \mathbf{v}_h]_h + \mu\sum_{K\in\mathcal{D}_{h,s}}\sum_{\sigma\in\partial K}\int_{\sigma\backslash\Gamma}\nabla\mathbf{u}\cdot\mathbf{v}_{h,K}\mathbf{n}_\sigma + \mu\left(\mathbf{k}^{-1}\widehat{\mathbf{u}}_h, \mathbf{v}_h\right)_{0,\Omega_d} - \sum_{L\in\mathcal{D}_{h,d}}\int_L \mathbf{k}^{-1}\mathbf{u}_d\cdot\mathbf{v}_{h,L}\\ &- (\widehat{p}_h, \nabla_h\cdot\mathbf{v}_h)_{0,\Omega} - \sum_{K\in\mathcal{D}_h}\sum_{\sigma\in\partial K}\int_{\sigma\backslash\Gamma} p\mathbf{v}_{h,K}\cdot\mathbf{n}_\sigma\end{aligned}$$

Set $\mathbf{e}_h = \widehat{\mathbf{u}}_h - \mathbf{u}_h$ and $\epsilon_h = \hat{p}_h - p_h$. Subtracting (4.41a) from the above equation, we then get

$$\mu[\mathbf{e}_h, \mathbf{v}_h]_h + \mu\left(\mathbf{k}^{-1}\mathbf{e}_h, \mathbf{v}_h\right)_{0,\Omega_d} - (\epsilon_h, \nabla_h\cdot\mathbf{v}_h)_{0,\Omega} + \frac{\mu\alpha}{\sqrt{k}}(\mathbf{e}_{h,s}\cdot\tau, \mathbf{v}_{h,s}\cdot\tau)_{0,\Gamma} \tag{4.62}$$

$$= R_1(\mathbf{v}_h) + \frac{\mu\alpha}{\sqrt{k}}((\widehat{\mathbf{u}}_{h,s} - \mathbf{u}_s)\cdot\tau, \mathbf{v}_{h,s}\cdot\tau)_{0,\Gamma} \tag{4.63}$$

By setting $\mathbf{v}_h = \mathbf{e}_h$ in the last relation, we have

$$\begin{aligned}&\mu\|\mathbf{e}_h\|_h^2 + \mu\left\|\mathbf{k}^{-1/2}\mathbf{e}_h\right\|_{0,\Omega_d}^2 - (\epsilon, \nabla_h\cdot\mathbf{e}_h)_{0,\Omega} + \frac{\mu\alpha}{\sqrt{k}}\|\mathbf{e}_{h,s}\cdot\tau\|_{0,\Gamma}^2\\ =&R_1(\mathbf{e}_h) + \frac{\mu\alpha}{\sqrt{k}}((\widehat{\mathbf{u}}_{h,s} - \mathbf{u}_s)\cdot\tau, \mathbf{e}_{h,s}\cdot\tau)_{0,\Gamma}\end{aligned} \tag{4.64}$$

Now, multiplying the last equation in (2.16) by $q_h \in X_h^0$, integrating by parts on $K \in \mathcal{D}_{h,s}$ and $L \in \mathcal{D}_{h,d}$, and adding the two deduced equations we get

$$(q_h, \nabla_h\cdot\widehat{\mathbf{u}})_{0,\Omega_s} + (q_h, \nabla_h\cdot\widehat{\mathbf{u}})_{0,\Omega_d} = R_2(q_h) - \int_\Gamma \mathbf{u}_s q_{h,K}\cdot\mathbf{n}_s ds - \int_\Gamma \mathbf{u}_d q_{h,L}\cdot\mathbf{n}_d ds$$

where $R_2(q_h)$ is the consistency residual

$$R_2(q_h) = (q_h, \nabla_h\cdot\widehat{\mathbf{u}}_h)_{0,\Omega_s} - \sum_{K\in\mathcal{M}_s}\sum_{\sigma\in\partial K}\int_{\sigma\backslash\Gamma}\mathbf{u}q_{h,K}\cdot\mathbf{n}_\sigma + (q_h, \nabla_h\cdot\widehat{\mathbf{u}}_h)_{0,\Omega_d} - \sum_{L\in\mathcal{M}_d}\sum_{\sigma\in\partial K}\int_{\sigma\backslash\Gamma}\mathbf{u}q_{h,L}\cdot\mathbf{n}_\sigma$$

Next, adding (4.41b) to the above equation gives

$$(q_h, \nabla_h\cdot\mathbf{e}_h)_{0,\Omega} = R_2(q_h) - \int_\Gamma \mathbf{u}_s q_{h,K}\cdot\mathbf{n}_s ds - \int_\Gamma \mathbf{u}_d q_{h,L}\cdot\mathbf{n}_d ds + J(p_h, q_h)$$

and setting $q_h = \epsilon_h$ in this relation yields

$$(\epsilon_h, \nabla_h\cdot\mathbf{e}_h)_{0,\Omega} = R_2(\epsilon_h) - \int_\Gamma \mathbf{u}_s\epsilon_{h,s}\cdot\mathbf{n}_s ds - \int_\Gamma \mathbf{u}_d\epsilon_{h,d}\cdot\mathbf{n}_d ds + J(p_h, \epsilon_h)$$

The latter may be written as well in the equivalent form

$$(\epsilon_h, \nabla_h \cdot \mathbf{e}_h)_{0,\Omega} = R_2(\epsilon_h) - \int_\Gamma \mathbf{u}_s \epsilon_{h,s} \cdot \mathbf{n}_s ds - \int_\Gamma \mathbf{u}_d \epsilon_{h,d} \cdot \mathbf{n}_d ds + J(p_h, \epsilon_h) \tag{4.65}$$

Gathering (4.64) and (4.65) we get

$$\mu \|\mathbf{e}_h\|_h^2 + \mu \left\| \mathbf{k}^{-1/2} \mathbf{e}_h \right\|_{0,\Omega_d}^2 + \frac{\mu\alpha}{\sqrt{k}} \|\mathbf{e}_{h,s} \cdot \tau\|_{0,\Gamma}^2 + J(p_h, p_h)$$

$$= R_1(\mathbf{e}_h) + R_2(\epsilon_h) + \frac{\mu\alpha}{\sqrt{k}} \left((\widehat{\mathbf{u}}_{h,s} - \mathbf{u}_s) \cdot \tau, \mathbf{e}_{h,s} \cdot \tau \right)_{0,\Gamma} \tag{4.66}$$

$$- \int_\Gamma \mathbf{u}_s \epsilon_{h,s} \cdot \mathbf{n}_s ds - \int_\Gamma \mathbf{u}_d \epsilon_{h,d} \cdot \mathbf{n}_d ds + J(p_h, \hat{p}_h) \tag{4.67}$$

Next, let us study the terms in the right-hand side of the above equation.

I) Boundedness of $R_1(\mathbf{e}_h)$ and $R_2(\epsilon_h)$:

Thanks to consistency results we get

$$R_1(\mathbf{e}_h) \leq Ch \left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \right) \|\mathbf{e}_h\|_{V_h} \tag{4.68}$$

$$R_2(\epsilon_h) \leq Ch^2 \left(\frac{1}{\varepsilon} \|\mathbf{u}\|_{2,\Omega}^2 + \varepsilon \|p\|_{1,\Omega}^2 \right) + \varepsilon J(p_h, p_h) \tag{4.69}$$

II) Using (3.5) and interpolation results gives

$$\frac{\mu\alpha}{\sqrt{k}} \left((\widehat{\mathbf{u}}_{h,s} - \mathbf{u}_s) \cdot \tau, \mathbf{e}_{h,s} \cdot \tau \right)_{0,\Gamma} \leq Ch^3 \|\mathbf{u}\|_{2,\Omega}^2 + \frac{\mu\alpha}{\sqrt{k}} \|\mathbf{e}_{h,s} \cdot \tau\|_{0,\Gamma}^2 \tag{4.70}$$

III) Using the Cauchy-Schwarz inequality followed by (3.16) we get

$$J(p_h, \hat{p}_h) \leq J(p_h, p_h)^{\frac{1}{2}} \left(C \sum_{\sigma \in \mathcal{E}_{int}} |\sigma| \, h (\hat{p}_{h,K} - \hat{p}_{h,L})^2 \right)^{\frac{1}{2}}$$

$$\leq \varepsilon J(p_h, p_h) + C \frac{h^2}{\varepsilon} \|p\|_{1,\Omega}^2 \tag{4.71}$$

IV) Using the Cauchy Schwarz inequality followed by the Young inequality we get

$$\left| - \int_\Gamma \mathbf{u}_s \epsilon_{h,s} \cdot \mathbf{n}_s ds - \int_\Gamma \mathbf{u}_d \epsilon_{h,d} \cdot \mathbf{n}_d ds \right| \leq \varepsilon h \|\epsilon_{h,s}\|_{0,\Gamma}^2 + \frac{h^{-1}}{\varepsilon} \|\mathbf{u}_s\|_{0,\Gamma}^2 + \varepsilon h \|\epsilon_{h,d}\|_{0,\Gamma}^2 + \frac{h^{-1}}{\varepsilon} \|\mathbf{u}_d\|_{0,\Gamma}^2$$

By the trace inequality (3.5) we have

$$h^{-1} \|\mathbf{u}_s\|_{0,\Gamma}^2 \leq Ch^2 \|\mathbf{u}_s\|_{2,\Omega_s}^2$$

$$\frac{h^{-1}}{\varepsilon} \|\mathbf{u}_d\|_{0,\Gamma}^2 \leq Ch^2 \|\mathbf{u}_d\|_{2,\Omega_d}^2$$

which, in vertue of (4.51), leads to

$$\left| - \int_\Gamma \mathbf{u}_s \epsilon_{h,s} \cdot \mathbf{n}_s ds - \int_\Gamma \mathbf{u}_d \epsilon_{h,d} \cdot \mathbf{n}_d ds \right| \leq \varepsilon C \|\epsilon_h\|_{0,\Omega}^2 + \frac{C}{\varepsilon} h^2 \|\mathbf{u}\|_{2,\Omega}^2 \tag{4.72}$$

To conclude we need to control $\|\epsilon_h\|_{0,\Omega}^2$.

Let $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ be given such that:

$$\nabla \cdot \mathbf{w}(\boldsymbol{x}) = \epsilon_h(x) \quad \text{and} \quad \|\mathbf{w}\|_{1,\Omega} \leq C\|\epsilon_h\|_{0,\Omega} \tag{4.73}$$

Like in proposition 21, we have

$$\|\epsilon_h\|_{0,\Omega}^2 \leq (\epsilon_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} + J(\epsilon_h, \epsilon_h)^{\frac{1}{2}} C\|\epsilon_h\|_{0,\Omega} \tag{4.74}$$

We now use $\mathbf{w}_h$ as a test function in (4.63) to get

$$(\epsilon_h, \nabla_h \cdot \mathbf{w}_h)_{0,\Omega} = a(\mathbf{e}_h, \mathbf{w}_h) - R_1(\mathbf{w}_h) \tag{4.75}$$

Taking into account (4.74) and using (4.52) , (4.68) and (4.73) give

$$\|\epsilon_h\|_{0,\Omega}^2 \leq C\|\mathbf{e}_h\|_{\mathbf{V}_h} \|\epsilon_h\|_{0,\Omega} + Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right)\|\epsilon_h\|_{0,\Omega} + CJ(\epsilon_h, \epsilon_h)^{\frac{1}{2}}\|\epsilon\|_{0,\Omega}$$

So,

$$\|\epsilon_h\|_{0,\Omega} \leq C\|\mathbf{e}_h\|_{\mathbf{V}_h} + Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) + CJ(\epsilon_h, \epsilon_h)^{\frac{1}{2}} \tag{4.76}$$

Using the Cauchy-Schwarz inequality and (3.16) we get

$$\|\epsilon_h\|_{0,\Omega}^2 \leq C\|\mathbf{e}_h\|_{\mathbf{V}_h}^2 + Ch^2\left(\|\mathbf{u}\|_{2,\Omega}^2 + \|p\|_{1,\Omega}^2\right) + Ch^2\|p\|_{1,\Omega}^2 + \varepsilon CJ(p_h, p_h)$$

Now, substituting this result in (4.72) gives

$$\begin{aligned}\left|-\int_\Gamma \mathbf{u}_s\epsilon_{h,s}\cdot\mathbf{n}_s ds - \int_\Gamma \mathbf{u}_d\epsilon_{h,d}\cdot\mathbf{n}_d ds\right| &\leq \varepsilon C\|\mathbf{e}_h\|_{\mathbf{V}_h}^2 + \varepsilon Ch^2\left(\|\mathbf{u}\|_{2,\Omega}^2 + \|p\|_{1,\Omega}^2\right)\\ &\quad + \varepsilon CJ(p_h,p_h) + \frac{C}{\varepsilon}h^2\|\mathbf{u}\|_{2,\Omega}^2\end{aligned} \tag{4.77}$$

Gathering all above results, we may rewrite (4.67) as follows:

$$(\mu - C\varepsilon)\|\mathbf{e}_h\|_h^2 + (\lambda_1 - C\varepsilon)\|\mathbf{e}_h\|_{0,\Omega_d}^2 + (1 - C\varepsilon)J(p_h,p_h) \leq Ch^2\left(\|\mathbf{u}\|_{2,\Omega}^2 + \|p\|_{1,\Omega}^2\right)$$

It is now clear that choosing ε sufficiently small, e.g. $\varepsilon < \min\left\{\frac{\mu}{C}, \frac{\lambda_1}{C}, \frac{1}{C}\right\}$, the latter implies (4.60) and (4.61). □

Theorem 30. *In addition to assumptions of Proposition 29, let $\mathbf{u}_\mathcal{D} \in \mathbf{V}_h$ defined by: $\mathbf{u}_{\mathcal{D}K} = \mathbf{u}(\mathbf{x}_K), \quad K \in \mathcal{D}_h$. Then*

$$\|\mathbf{u}_\mathcal{D} - \mathbf{u}_h\|_{\mathbf{V}_h} \leq Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.78}$$

$$\|\mathbf{u}_\mathcal{D} - \mathbf{u}_h\|_{0,\Omega} \leq Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.79}$$

$$\|p - p_h\|_{0,\Omega} \leq Ch\left(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\right) \tag{4.80}$$

Proof. First of all, note that (4.78) is straightforwardly deduced from the definition of $\mathbf{u}_{\mathcal{D}}$ and (4.60) by applying the triangular inequality and interpolation results. Likewise, (4.79) follows by similar arguments from the triangular inequality, interpolation results and the discrete Poincaré inequality (3.3).

Finally, using (4.76), (4.60), (4.61) and interpolation results, we get

$$\|p - p_h\|_{0,\Omega}^2 \leq 2\|p - \pi\,(p)\|_{0,\Omega}^2 + 2\|\pi\,(p) - p_h\|_{0,\Omega}^2 \leq Ch^2\left(\|\mathbf{u}\|_{2,\Omega}^2 + \|p\|_{1,\Omega}^2\right)$$

□

Chapter 5

Numerical tests

This chapter presents several representative results of numerical experiments undertaken for the Stokes, Darcy and coupled Stokes-Darcy models previousely studied using Matlab R2017. The main objective is to confirm the validity of the error estimates established in the theoretical convergence study by means of benchmark test problems with given analytical solutions.

For the convergence analysis, the velocity error is measured by

$$e_{h,K}^{(i)} = u_K^{(i)}(\mathbf{x}_K) - u_{h,K}^{(i)}$$

and the pressure error by

$$\epsilon_{h,K} = p_K(\mathbf{x}_K) - p_{h,K}$$

The following discrete error norms are used for the investigation of convergence rates

1) velocity norm in H_0^1

$$\|\mathbf{e}_h\|_h = \sqrt{\sum_{i=1}^{2} \left[e_h^{(i)}, e_h^{(i)}\right]_h}$$

2) velocity norm in L^2

$$\|\mathbf{e}_h\|_{0,\Omega} = \sqrt{\sum_{i=1}^{2} \int_\Omega \left(e_h^{(i)}\right)^2 d\Omega}$$

3) pressure norm in L^2

$$\|\epsilon_h\|_{0,\Omega} = \sqrt{\int_\Omega \epsilon_h^2 d\Omega}$$

5.1 Decoupled problem

In this section two numerical exemples are analyzed for a study of discrete solution accuracy and convergence rates for both Stokes and Darcy problems. The computational domain is $\Omega =]0,1[\times]0,1[$ and the problem (2.4), (2.10) is to be discretized and solved using uniform partitionings of Ω into $n \times n$ equal squares ($n = 10, ..., 100$). Moreover, in order to restore a unique pressure field a zero mean pressure is imposed on Ω. In all cases, the source term $\mathbf{f}$ is chosen such that equations (2.4), (2.10) hold.

Problem I

The first numerical exemple concerns both Stokes and Darcy problems with the same exact velocity and pressure fields:

$$\begin{aligned} \mathbf{u}_1(x,y) &= 2000(x-x^2)^2\left(y-y^2\right)(1-2y) \\ \mathbf{u}_2(x,y) &= -2000(y-y^2)^2\left(x-x^2\right)(1-2x) \\ p(x,y) &= 100(x^2+y^2-\frac{2}{3}). \end{aligned}$$

We set $\delta = 0.5$. In Fig. (5.1) and Fig. (5.2), the approximate velocity vectors and pressure elevations are shown on the 80×80 partitioning for Stokes with $\mu = 0.1$ and for Darcy with $\alpha = 100$ repectively. The displayed graphs are in excellent agreement with the exact solution plots. The computed convergence rates are presented in Fig. (5.3). Better results than theoretical rates predicted by the above study have been obtained. For Stokes flow, the convergence rates are near to 3/2 for the both the velocity in $\|\cdot\|_h$ norm and pressure in L^2 norm, while the computed rate is close to 2 for the velocity in L^2 norm. For the Darcy problem, the convergence rate is near to 3/2 for the pressure in the L^2 norm, whereas it seems close to 1 for the velocity in the same norm. In fact, This unexpected fact deserves further investigation for explanation.

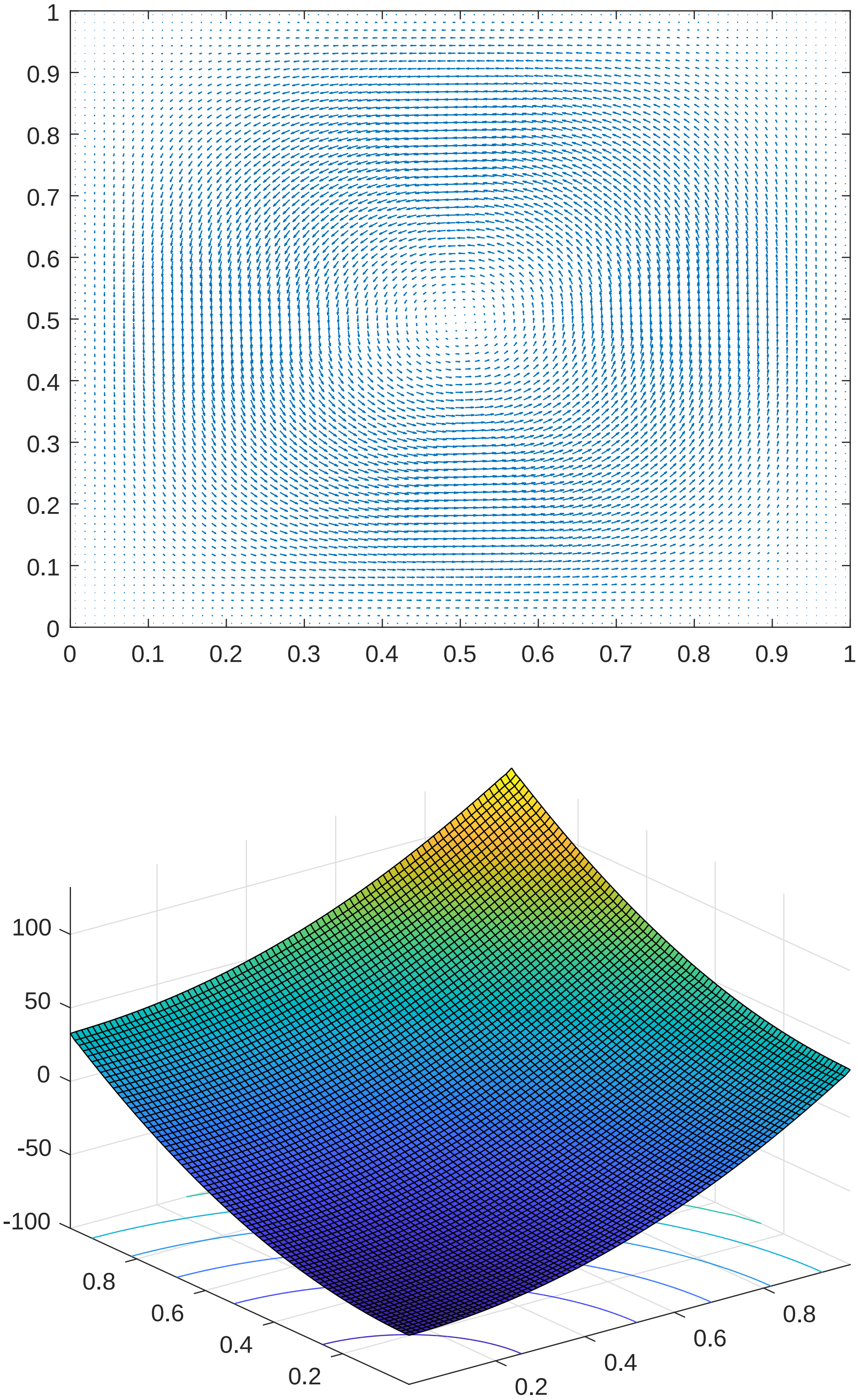


Figure 5.1: Approximate velocity vectors and pressure elevation for Stokes with $\mu = 0.1$.

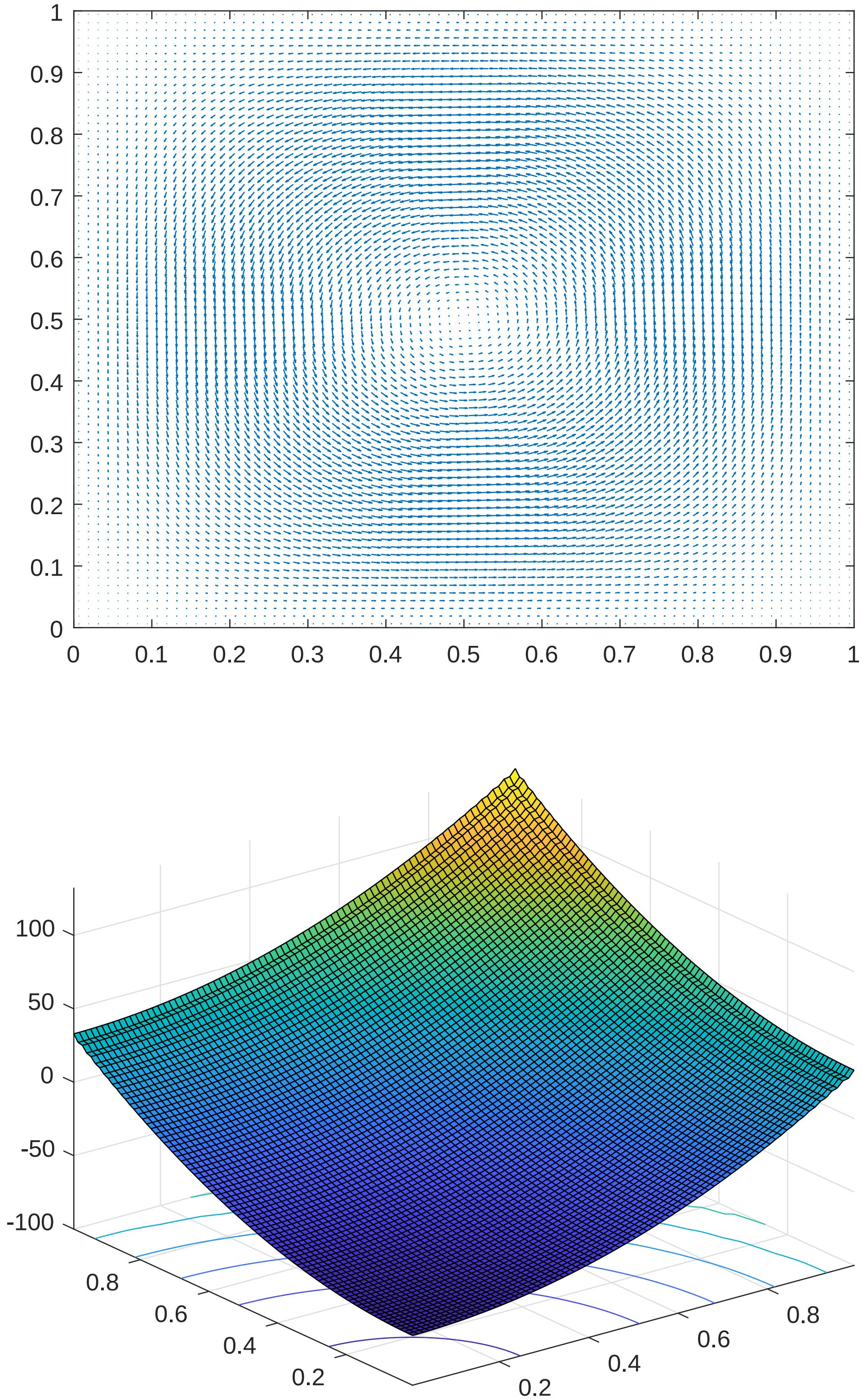


Figure 5.2: Approximate velocity vectors and pressure elevation for Darcy with $\alpha = 100$.

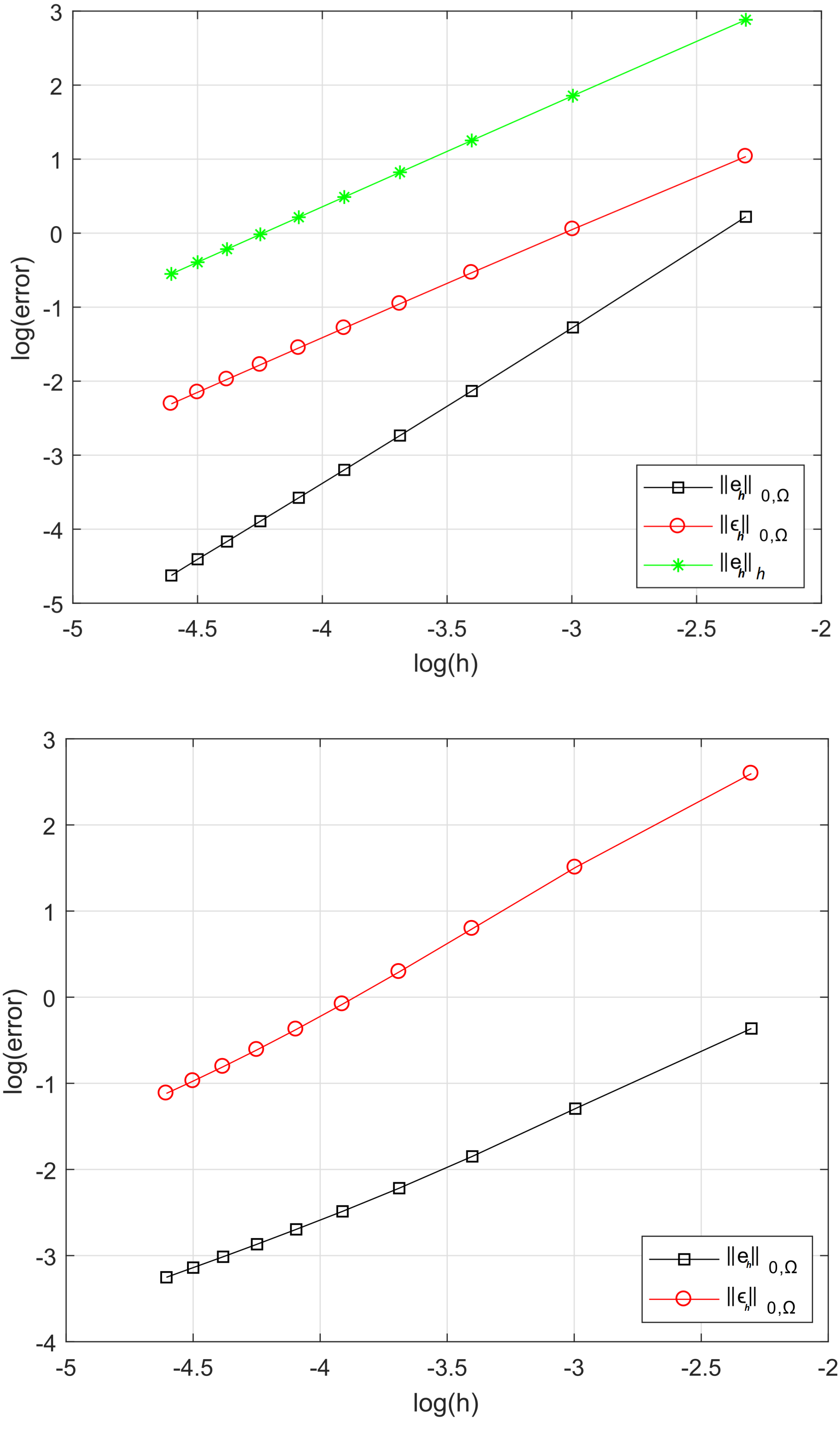


Figure 5.3: Convergence history for Stokes (top) with $\mu = 0.1$ and for Darcy (bottom) with $\alpha = 100$.

Problem II

The second numerical test is devoted to analyzing the same features when the exact velocity and pressure are given for the Stokes flow by:

$$\begin{aligned}
\mathbf{u}_1(x,y) &= \pi \sin(2\pi y)\sin^2(\pi x)\\
\mathbf{u}_2(x,y) &= -\pi \sin(2\pi x)\sin^2(\pi y)\\
p(x,y) &= \sin(2\pi x)\sin(2\pi y)
\end{aligned}$$

and for the Darcy flow by:

$$\begin{aligned}
\mathbf{u}_1(x,y) &= 1/2\sin(2\pi y)\sin^2(\pi x)\\
\mathbf{u}_2(x,y) &= -1/2\sin(2\pi x)\sin^2(\pi y)\\
p(x,y) &= 2x - 4y^3
\end{aligned}$$

Now, we set $\delta = 10$. The approximate velocity vectors and pressure elevations on the 80×80 partitioning for Stokes with $\mu = 1$ and for Darcy with $\alpha = 10$ are displayed in Fig. (5.4) and Fig. (5.5) respectively. The behavior is again remarkable. According to the convergence error history shown in Fig. (5.6), predicted optimal rates are widely achieved for both Stokes and Darcy cases. Here also, better results than the predicted rates are attained.

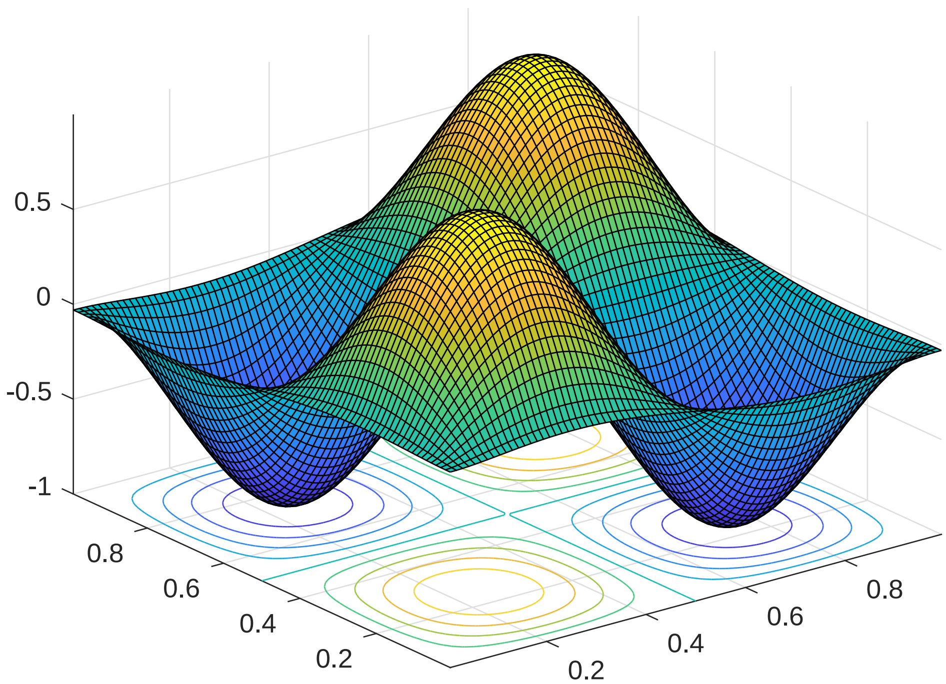


Figure 5.4: Approximate velocity vectors and pressure elevation for Stokes with $\mu = 1$.

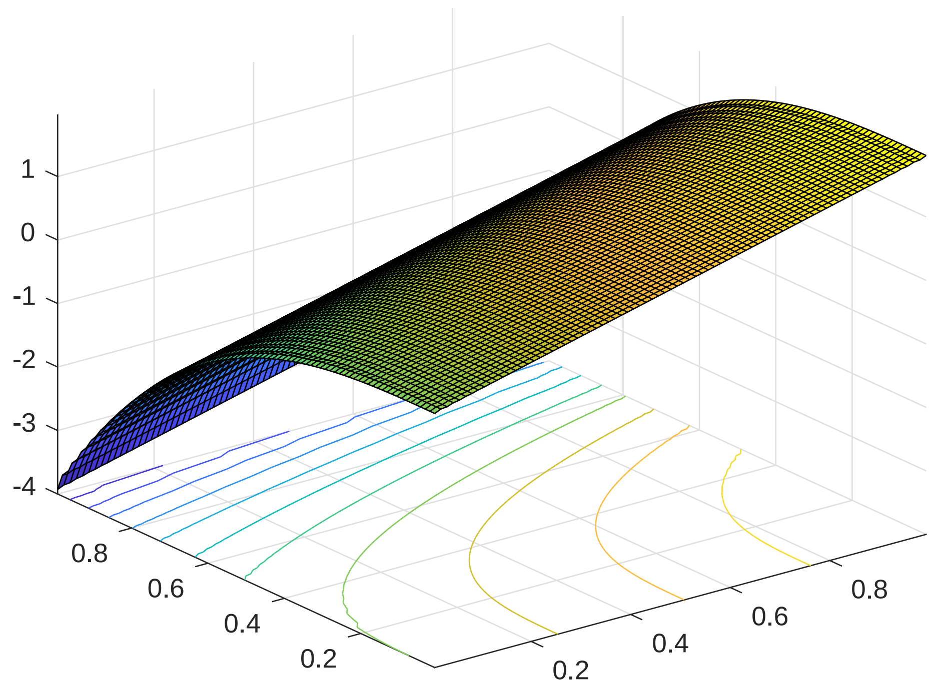
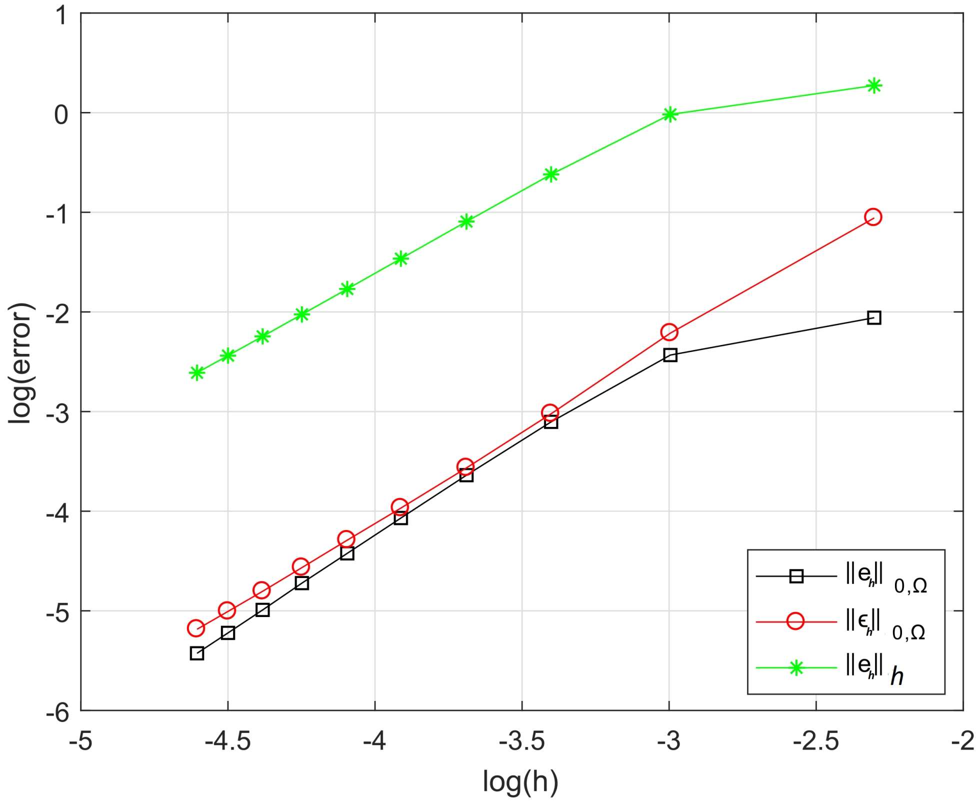


Figure 5.5: Approximate velocity vectors and pressure elevation for Darcy with $\alpha = 10$.



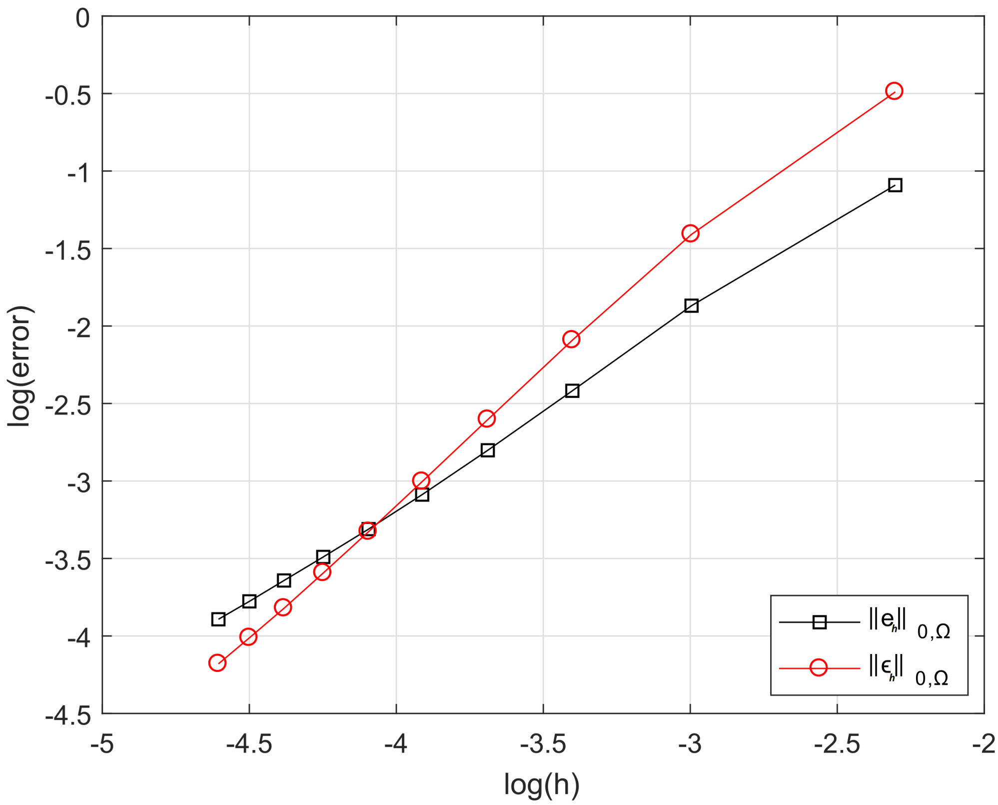


Figure 5.6: Convergence history for Stokes (top) with $\mu = 1$ and for Darcy (bottom) with $\alpha = 10$.

5.2 Coupled problem

Many computational tests were performed for several values of physical and stabilization parameters. In this section, we only report some representative results of numerical experiments for the coupled Stokes-Darcy model of an incompressible fluid. For simplicity, the computational domain is taken to be $\Omega = \Omega_s \cup \Omega_d$, where $\Omega_s = (0,1)\times(1,2)$, $\Omega_d = (0,1)\times(0,1)$ with the interface $\Gamma = (0,1)\times\{1\}$. As the focus of this section is on the properties of convergence and error estimates, the domain is computed using a uniform grid by first dividing it into $n\times n$ equal squares $(n = 10, 20, ..., 70)$.

Here, it is important to emphasize that difficulty resides in finding solutions that satisfy all interface conditions (2.18), (2.19) and (2.20). To this end, we use the same trick of generalizing the equations to include a nonhomogeneous term. Thus, we replace equations (2.19) and (2.20) by

$$-\mu \mathbf{n}_s \cdot \nabla \mathbf{u}_s \cdot \mathbf{n} + p_s = p_d + g_1$$
$$-\mathbf{n}_s \cdot \nabla \mathbf{u}_s \cdot \tau = \frac{\alpha}{\sqrt{k}} \mathbf{u}_s \cdot \tau + g_2$$

where g_1 and g_2 are given functions on Γ according to the analytical solutions. The modified variational formulation has added two terms $-(g_1, \mathbf{v}_s \cdot \mathbf{n}_s)_{0,\Gamma} + \mu (g_2, \mathbf{v}_s \cdot \tau)_{0,\Gamma}$ in the right-hand side of the first equation of (2.22).

We consider two exemples. The first one consists in taking the same pressur solution for both Stokes and Darcy sub-problems with different values of the stabilization parameter, whereas we take different solutions in the second one with the same stabilization parameter.

Problem I

Here, we select the terms $\mathbf{f}$ in the Stokes and Darcy equations, g_1 and g_2 in the interface conditions according to the analytical solutions:

$$\mathbf{u}_s = \left(\left(x-x^2\right)^2\left(3y^2-10y+8\right), -2\left(y^3-5y^2+8y-4\right)\left(x-x^2\right)(1-2x)\right),$$
$$\mathbf{u}_d = \left(\left(x-x^2\right)^2\left(3y^2-2y\right), -2\left(y^3-y^2\right)\left(x-x^2\right)(1-2x)\right),$$
$$p = 10\,(x+y-3/2).$$

by taking $\mu = 0.1$, $k = 0.0001$ and $\alpha = 1$. For the stabilization parameter, we choose $\delta_1 = 5$ for the Stokes regime and $\delta_2 = 0.01$ for the Darcy one. The approximate velocity vectors and pressure elevations for the finest mesh $n = 70$ are displayed in Fig. (5.7). We notice that the plots look very similar to the exact solution plots.

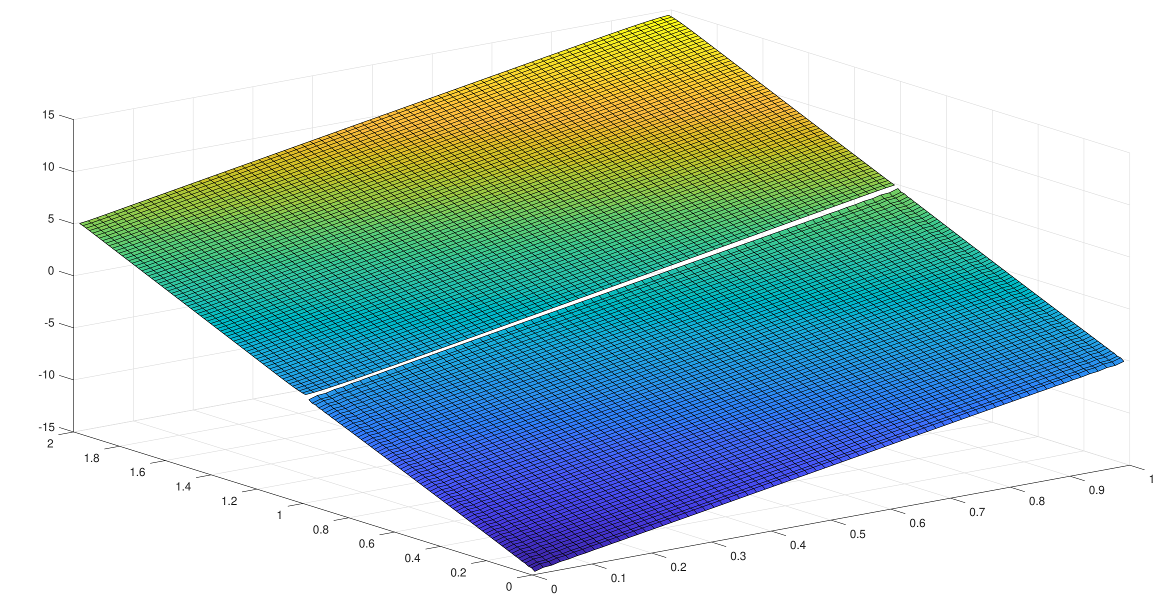


Figure 5.7: Approximate velocity vectors and pressure elevation.

The computed convergence rates are presented in Fig (5.8). It is observed that even better results than above predicted theoretical rates have been obtained in all considered cases.

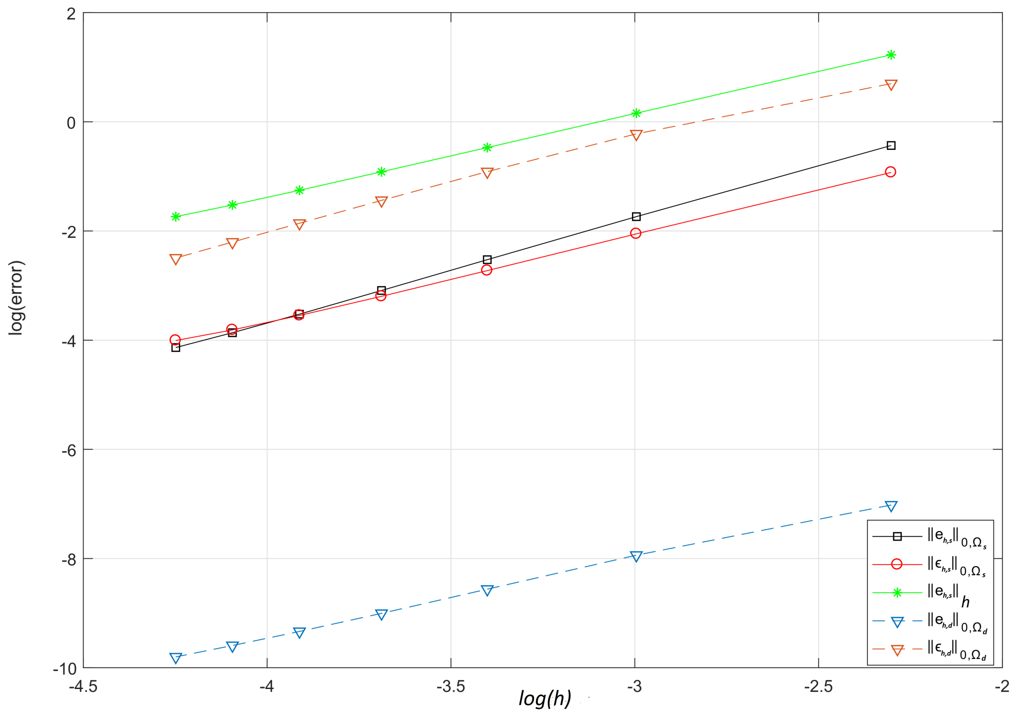


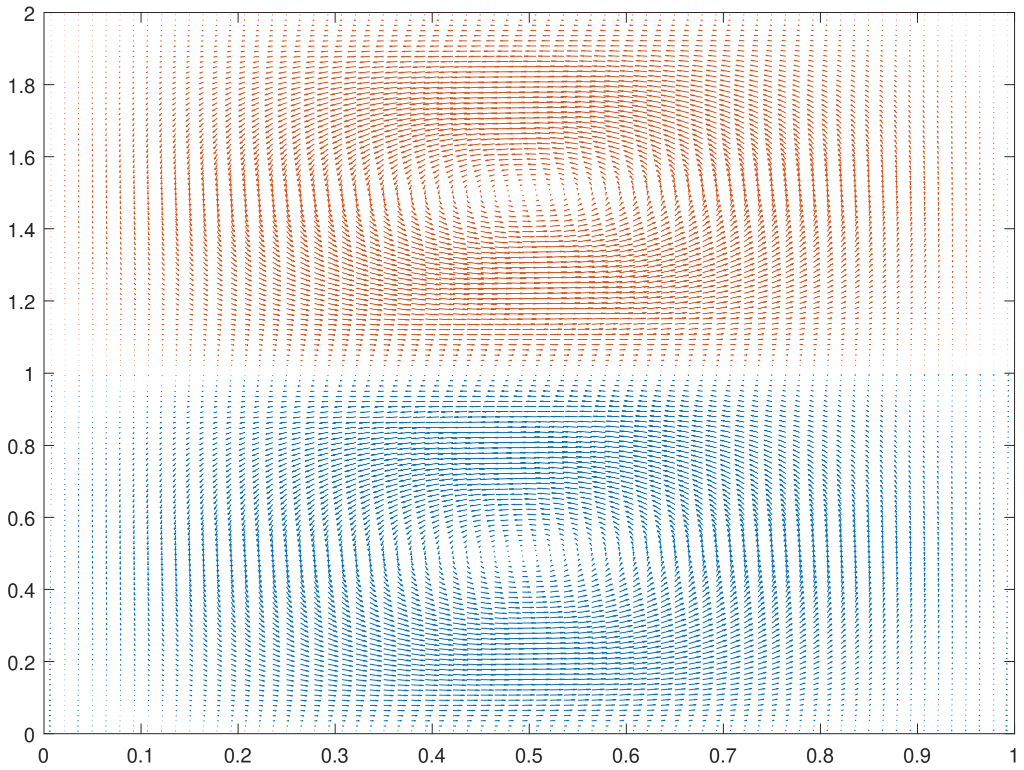
Figure 5.8: Convergence history of the velocity and of the pressure.

Problem II

In this case, we select the terms $\mathbf{f}$ in the Stokes and Darcy equations, g_1 and g_2 in the interface conditions to impose the exact solutions:

$$\mathbf{u}_s = \left(\pi \sin(2\pi y)\sin^2(\pi x), -\pi \sin(2\pi x)\sin^2(\pi y)\right),$$
$$p_s = 2x + \frac{29}{11}y^3 - 73/11,$$
$$\mathbf{u}_d = \left(\sin^2(\pi x)\left(y - y^2\right)(1-2y), -\pi\left(y-y^2\right)^2 \sin(\pi x)\cos(\pi x)\right),$$
$$p_d = 5xy - 3x - 4.$$

by taking $\mu = 0.01, k = 0.01$ and $\alpha = 1$. For the stabilization parameter, we choose the value $\delta = 7$ for both Stokes and Darcy regimes. The approximate velocity vectors and pressure elevations for the finest mesh $n = 70$ are displayed in Fig. (5.9) . Similar plots to the exact solutions plots are anew obtained.



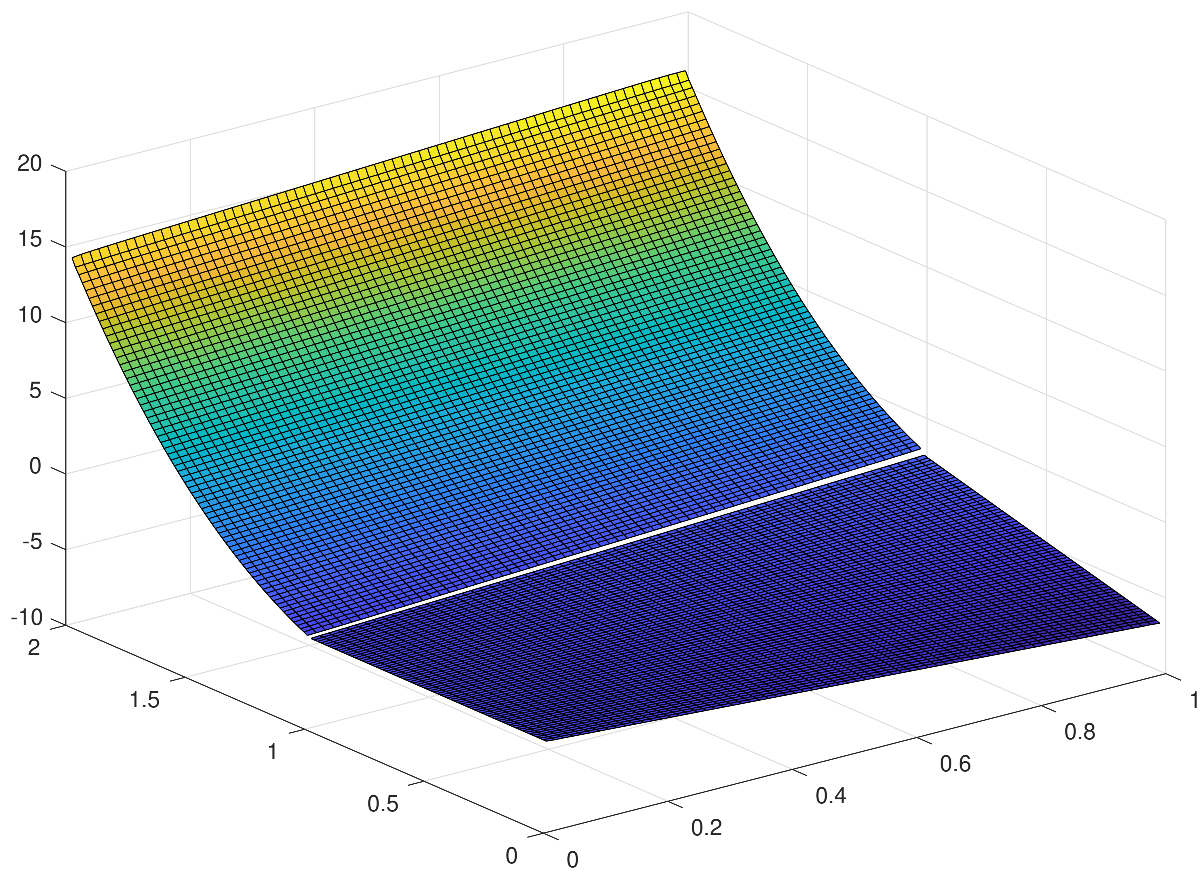


Figure 5.9: Approximate velocity vectors and pressure elevation.

The computed convergence rates are presented in Fig (5.10). It is observed that better results than above predicted theoretical rates have been obtained.

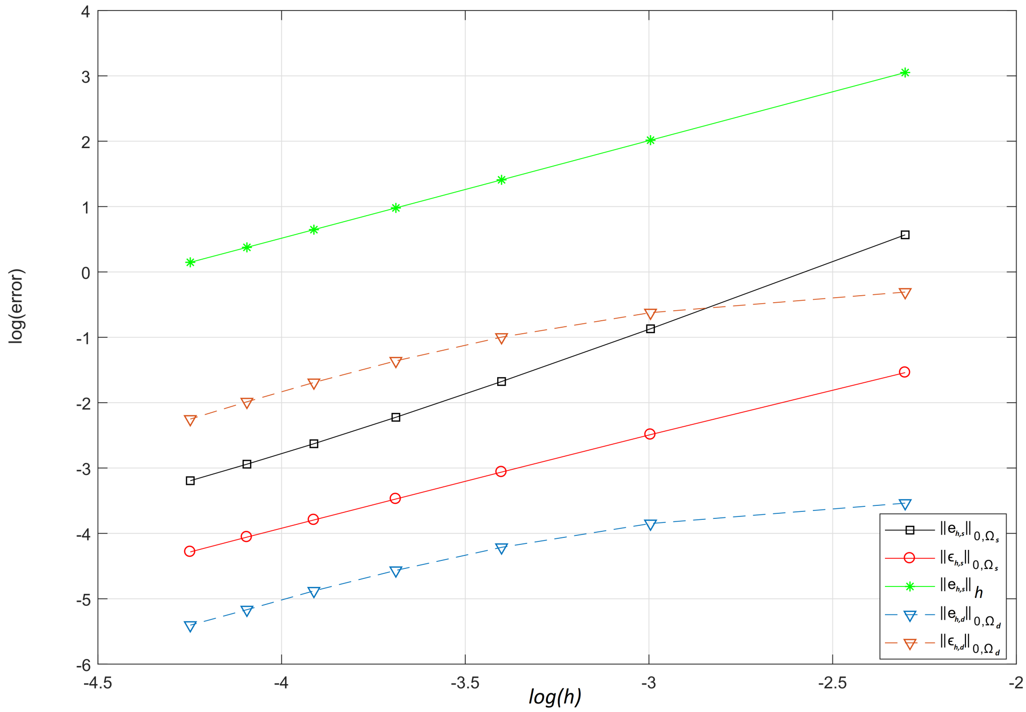


Figure 5.10: Convergence history of the velocity and of the pressure.

Conclusion

The collocated finite volume method stabilized by means of pressure jumps proves to be an efficient and accurate technique for solving Stokes-Darcy problems in both decoupled and coupled forms. Numerical schemes based on the proposed framework are simple to implement and ensure optimal convergence for standard test problems.

As future work, it would be nice to model more realistic problems. The extension of the present technique to the full Navier-Stokes equations combined with the Darcy equations would also be an interesting subject to investigate. Finally, the transient case remains the main target.

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