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THÈME

On some systems of difference equations, solutions form, stability and periodicity

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ملخّص

تعنى هذه الرسالة بدراسة بعض عائلات جمل المعادلات بالفروقات أين نقوم في كل مرة بعرض الحلول بصيغتها الصريحة.

في الفصل الأول قمنا بدراسة لجمل معادلات من درجات مختلفة حيث قدمنا الحلول بالإستعانة ببعض متتاليات الأعداد الشهيرة، كمتتالية فيبوناشي، بادوفان، تريبوناشي وتريبوناشي المعممة.

الفصل الثّاني كرس لدراسة وحل جملة لثلاث معادلات معرفة بدوال متجانسة، أما في الفصلين الثّالث والرابع فعرضنا دراسة لجملة لثلاث معادلات برتبة عليا وجملة معادلتين من نوع ماكس.

في كل مرة نعرض الصّيغة الصريحة للحلول، كما تطرقنا إلى بعض الحالات الخاصة ودراسة نوعية الحلول والنقاط الصّامدة لها، من تقارب واستقرار وكذا الدورية والتذبذب.

الكلمات المفتاحية :
جملة المعادلات بالفروقات، صيغة الحلول، التوازن، الدورية، التذبذب، الدالة المتجانسة، جمل المعادلات بالفروقات من نوع ماكس، متتالية فيبوناشي، تريبوناشي....

Abstract

This thesis concerns the study of certain classes of systems of nonlinear difference equations where each time we present the solutions on the closed form.

In the first chapter, we study systems of difference equations with different degrees, where we have presented the solutions using well-known number sequences such as, Fibonacci numbers, Padovan, Tribonacci and generalized Tribonacci numbers.

The second chapter is devoted to the study and the resolution of a system of three difference equations defined by homogeneous functions. As for the third and fourth chapter, we presented a study of higher-order of system of three difference equations and a two dimensional Max-type system of difference equations.

Each time, we present the explicit form of the solutions, and a qualitative study of the solutions and their equilibrium points of some particular cases is discussed, including convergence, local and global asymptotic stability, as well as periodicity and oscillatory.

Key Words: System of difference equations, form of solutions, stability, periodicity, oscillation, homogeneous function, Max-type system of difference equations, Fibonacci sequence, Tribonacci ...

Résumé

Cette thèse porte sur l'étude de certaines classes de systèmes d'équations aux différences non-linéaires où à chaque fois nous présentons les solutions sous la forme fermée.

Dans le premier chapitre, nous étudions des systèmes d'équations aux différences de différents degrés, où nous avons présenté les solutions utilisant des suites de nombres bien connues telles que, les nombres de Fibonacci, Padovan, Tribonacci et les nombres généralisés de Tribonacci.

Le deuxième chapitre est consacré à l'étude et à la résolution d'un système de trois équations aux différences définies par des fonctions homogènes. Comme pour le troisième et quatrième chapitre, nous avons présenté une étude d'un système de trois équations aux différences d'ordre supérieur et d'un système de deux équations de type Max.

À chaque fois, nous présentons la forme explicite des solutions, et une étude qualitative des solutions et de leurs points d'équilibres de certains cas particuliers est abordée, y compris la convergence, la stabilité asymptotique locale et globale, ainsi que la périodicité et l'oscillation.

Mots Clés: Système d'équations aux différences, forme des solutions, stabilité, periodicité, oscillation, fonction homogène, système d'équations aux différences type-Max, suite de Fibonacci, Tribonacci....

Contents

General introduction

Difference equations are used to describe real discrete models in various branches of modern sciences such as, for example, biology, economy, control theory. This explains why a big number of papers is devoted to this subject, see for example [2, 3, 7, 8, 44, 46, 53, 54, 55, 56, 65, 66, 67, 92, 96, 95, 104, 105, 115, 124, 127].
In particular non-linear difference equations and their systems is a very hot subject that attract the attention of several researchers. A numerous papers are devoted to this line of research, as examples in the following papers [2, 7, 9, 24, 25, 26, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 53, 63, 91, 93, 95, 96, 104, 106, 107, 109, 110, 111, 116, 117, 118, 119, 124, 127],

we can find some concrete models of such equations and systems, but also to understand the techniques and the methods used in solving and studying the behavior of the solutions of these models.

It is clear that if we want to understand our models, we need to know the behavior of the solutions of the equations of the models, and this fact will be possible if we can solve in closed form these equations.

Generally, it is difficult to determine methods to solve non linear equations and their systems. However, by the help of some change of variables, non linear difference equations or systems are transformed to very simple one, with known form of the solutions. Knowing the closed form of the solutions, provides more information about the behavior of the solutions, like periodicity, oscillation, boundedness, asymptotic behavior,...

One can find in the literature a lot of works on difference equations where explicit formulas of the solutions are given, see for instance [2, 3, 44, 53, 55, 65, 66, 67, 92, 95, 96, 104, 105, 124, 127]. Such type of difference equations and systems is called solvable difference equations.

The first chapter contains three essential sections. In the first section we solve in closed form the system of difference equations

$$x_{n+1}=\frac{ay_nx_{n-1}+bx_{n-1}+c}{y_nx_{n-1}},\ y_{n+1}=\frac{ax_ny_{n-1}+by_{n-1}+c}{x_ny_{n-1}},\ n=0,1,....$$

In particular we represent the solutions of some particular cases of this system in terms of Tribonacci and Padovan numbers and we prove the global stability of the corresponding positive equilibrium points. The results obtained here extend those obtained in some papers (see [7, 44, 92] and [124]).

In the second section we extend the results obtained in the first one, and we show that the system of difference equations

$$x_{n+1}=\frac{ay_{n-2}x_{n-1}y_n+bx_{n-1}y_{n-2}+cy_{n-2}+d}{y_{n-2}x_{n-1}y_n},\ y_{n+1}=\frac{ax_{n-2}y_{n-1}x_n+by_{n-1}x_{n-2}+cx_{n-2}+d}{x_{n-2}y_{n-1}x_n},$$

can be solved in a closed form. We will see that when $a=b=c=d=1$ the solutions are expressed using the famous Tetranacci numbers.

In the third section, we give explicit formulas of the solutions of the two classes of nonlinear systems of difference equations

$$\begin{cases}x_{n+1}=f^{-1}\left(ag(y_n)+bf(x_{n-1})+cg(y_{n-2})+df(x_{n-3})\right),\\ y_{n+1}=g^{-1}\left(af(x_n)+bg(y_{n-1})+cf(x_{n-2})+dg(y_{n-3})\right),\end{cases}$$

and

$$\begin{cases}x_{n+1}=f^{-1}\left(a+\frac{b}{g(y_n)}+\frac{c}{g(y_n)f(x_{n-1})}+\frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})}\right),\\ y_{n+1}=g^{-1}\left(a+\frac{b}{f(x_n)}+\frac{c}{f(x_n)g(y_{n-1})}+\frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})}\right),\end{cases}$$

where $n\in\mathbb{N}_0,\ f,g:D\longrightarrow\mathbb{R}$ are a "$1-1$" continuous functions on D, $D\subseteq\mathbb{R}$, where the results considerably extend some existing results in the literature.

The second chapter is devoted to study the following second order system of difference equations

$$x_{n+1}=f(y_n,y_{n-1}),\ y_{n+1}=g(z_n,z_{n-1}),\ z_{n+1}=h(x_n,x_{n-1})$$

where the functions $f,\ g,\ h:\ (0,+\infty)^2\rightarrow(0,+\infty)$ are continuous and homogeneous. In this study, we establish results on local stability of the unique equilibrium point and to deal with the global attractivity, and so the global stability, some general convergence theorems are provided. Necessary and sufficient conditions on existence of prime period two solutions of our system are given. Also, a result on oscillatory solutions is proved. As applications of the obtained results, concrete models of systems of difference equations defined by homogeneous functions of degree zero are investigated. Our system generalizes some existing works in the literature (eg: [62, 114]) and our results can be applied to study new models of systems of difference equations.

The goal of the third chapter is to derive the solution form and study of the system of nonlinear difference equations

$$x_{n+1} = \frac{x^p_{n-k+1}y_n}{\alpha y^p_{n-k} + \beta y_n},\ y_{n+1} = \frac{y^p_{n-k+1}z_n}{az^p_{n-k} + bz_n},\ z_{n+1} = \frac{z^p_{n-k+1}x_n}{Ax^p_{n-k} + Bx_n},\ n \in \mathbb{N}_0,\ p, k \in \mathbb{N}.$$

Furthermore, the behavior of solutions of the aforementioned system when $p = 1$ is examined. This work generalize the results obtained in [112] and [35].

In the same line of the third chapter, we give in the forth one the closed form solutions of the max-type rational system of non linear difference equations

$$x_{n+1} = \max\left(x_{n-1}, \frac{x_n y_{n-1}}{y_{n-2}}\right),\ y_{n+1} = \max\left(y_{n-1}, \frac{y_n x_{n-1}}{x_{n-2}}\right),$$

and giving the periodicity character of the solutions in a particular cases.

Chapter 1

On some systems of difference equations related to remarkable sequences

1.1 Introduction

We find in the literature many studies that concern the representation of the solutions of some remarkable linear sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan, and Perrin (see, e.g., [4, 28, 50, 59, 64, 71, 72, 123]). Solving in closed form non linear difference equations and systems is a subject that highly attract the attention of researchers (see, e.g.,[25, 26, 24, 42, 44, 63, 92, 104, 105, 124]) and the reference cited therein, where we find very interesting formulas of the solutions. A large range of these formulas are expressed in terms of famous numbers like Fibonacci and Padovan, (see, e.g., [44, 91, 104]). For solving in closed form non linear difference equations and systems generally we use some change of variables that transformed nonlinear equations and systems in linear ones. The paper of Stević [75] has considerably motivated this line of research.

In the second section we solve in closed form the system of difference equations

$$x_{n+1} = \frac{ay_nx_{n-1} + bx_{n-1} + c}{y_nx_{n-1}},\ y_{n+1} = \frac{ax_ny_{n-1} + by_{n-1} + c}{x_ny_{n-1}},\ n = 0, 1, ...,$$

where the initial values x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers and the parameters a, b and c are arbitrary real numbers with $c \neq 0$. In particular we represent the solutions of some particular cases of this system in terms of Tribonacci and Padovan numbers and we prove the global stability of the corresponding positive equilibrium points. The results obtained here extend those obtained in some papers (see [7, 44, 92] and [124]).

In the third section we extend the results obtained in the first one, and we show the the system of difference equations

$$x_{n+1}=\frac{ay_{n-2}x_{n-1}y_n+bx_{n-1}y_{n-2}+cy_{n-2}+d}{y_{n-2}x_{n-1}y_n},\ y_{n+1}=\frac{ax_{n-2}y_{n-1}x_n+by_{n-1}x_{n-2}+cx_{n-2}+d}{x_{n-2}y_{n-1}x_n},$$

where $n\in\mathbb{N}_0$, the initial values x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} and y_0 are arbitrary nonzero real numbers and the parameters a, b, c and d are arbitrary real numbers with $d\neq 0$, can be solved in a closed form. We will see that when $a=b=c=d=1$ the solutions are expressed using the famous Tetranacci numbers.

In the forth section, we give explicit formulas of the solutions of the two classes of non-linear systems of difference equations

$$\begin{cases} x_{n+1}=f^{-1}\left(ag(y_n)+bf(x_{n-1})+cg(y_{n-2})+df(x_{n-3})\right), \\ y_{n+1}=g^{-1}\left(af(x_n)+bg(y_{n-1})+cf(x_{n-2})+dg(y_{n-3})\right), \end{cases}$$

and

$$\begin{cases} x_{n+1}=f^{-1}\left(a+\frac{b}{g(y_n)}+\frac{c}{g(y_n)f(x_{n-1})}+\frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})}\right), \\ y_{n+1}=g^{-1}\left(a+\frac{b}{f(x_n)}+\frac{c}{f(x_n)g(y_{n-1})}+\frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})}\right), \end{cases}$$

where $n\in\mathbb{N}_0$, $f,g:D\longrightarrow\mathbb{R}$ are a "$1-1$" continuous functions on D, $D\subseteq\mathbb{R}$, the initial values x_{-i}, y_{-i}, $i=0,1,2,3$ are arbitrary real numbers in D and the parameters a, b, c and d are arbitrary real numbers, where the results considerably extend some existing results in the literature.

Now, recall some known definitions and results about stability, which will be very useful for the sequel, for more details see for example [11, 16, 29, 58].

Let $F:(0,+\infty)^k\rightarrow(0,+\infty)^k$ be a continuous function and consider the system of difference equations

$$Y_{n+1}=F(Y_n),\ n\in\mathbb{N}_0 \tag{1.1}$$

where the initial value $Y_0\in(0,+\infty)^k$. Let $\overline{Y}$ be an equilibrium point of (1.1), that is a solution in $(0,+\infty)^k$ of $\overline{Y}=F(\overline{Y})$.

Definition 1.1. *Let $\overline{Y}$ be an equilibrium point of system (1.1), and let $\|.\|$ any convenient vector norm.*

1. *We say that the equilibrium point $\overline{Y}$ is stable (or locally stable) if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every initial condition Y_0: $\|Y_0 - \overline{Y}\| < \delta$ implies $\|Y_n - \overline{Y}\| < \epsilon$. Otherwise, the equilibrium $\overline{Y}$ is unstable.*

2. *We say that the equilibrium point $\overline{Y}$ is asymptotically stable (or locally asymptotically stable) if it is stable and there exists $\gamma > 0$ such that $\|Y_0 - \overline{Y}\| < \gamma$ implies*

$$\lim_{n\to\infty} Y_n = \overline{Y}.$$

3. *We say that the equilibrium point $\overline{Y}$ is a global attractor if for every Y_0,*

$$\lim_{n\to\infty} Y_n = \overline{Y}.$$

4. *We say that the equilibrium point $\overline{Y}$ is globally (asymptotically) stable if it is stable and a global attractor.*

Assume that F is C^1 on $(0, +\infty)^k$. To system (1.1), we associate a linear system, about the equilibrium point $\overline{Y}$, given by

$$Z_{n+1} = F_J(\overline{Y})Z_n, \quad n \in \mathbb{N}_0,\, Z_n = Y_n - \overline{Y}$$

where F_J is the Jacobian matrix of the function F evaluated at the equilibrium point $\overline{Y}$. To study the stability of the equilibrium point $\overline{Y}$, we need the following theorem.

Theorem 1.1. *Let $\overline{Y}$ be an equilibrium point of system (1.1). Then, the following statements are true:*

(i) If all the eigenvalues of the Jacobian matrix F_J lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{Y}$ is asymptotically stable.

(ii) If at least one eigenvalue of F_J has absolute value greater than one, then the equilibrium $\overline{Y}$ is unstable.

1.2 A second order system

The difference equation

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}}$$

was studied by Azizi in [7]. Noting that the same equation was the subject of a very recent paper by Stevic [92].

In [124] the authors studied the system

$$x_{n+1} = \frac{1 + x_{n-1}}{y_n x_{n-1}},\ y_{n+1} = \frac{1 + y_{n-1}}{x_n y_{n-1}},$$

Motivated by [124], Halim et al. in [44], got the form of the solutions of the following difference equation

$$x_{n+1} = \frac{a + bx_{n-1}}{x_n x_{n-1}},$$

and the system

$$x_{n+1} = \frac{a + bx_{n-1}}{y_n x_{n-1}},\ y_{n+1} = \frac{a + by_{n-1}}{x_n y_{n-1}},$$

Here and motivated by the above mentioned papers we are interested in the following system of difference equations

$$x_{n+1} = \frac{ay_n x_{n-1} + bx_{n-1} + c}{y_n x_{n-1}},\ y_{n+1} = \frac{ax_n y_{n-1} + by_{n-1} + c}{x_n y_{n-1}},\ n = 0, 1, ..., \tag{1.2}$$

where x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero real numbers, a, b and c are arbitrary real numbers with $c \neq 0$. Clearly our system generalized the equations and systems studied in [7, 44, 92] and [124].

1.2.1 Homogeneous third order linear difference equation with constant coefficients.

Consider the homogeneous third order linear difference equation

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2},\ n = 0, 1, ..., \tag{1.3}$$

where the initial values R_0, R_{-1} and R_{-2} and the constant coefficients a, b and c are real numbers with $c \neq 0$. This equation will be of great importance for our study, so we will solve it in closed form. As it is well known, the solution $(R_n)_{n=-2}^{+\infty}$ of equation (1.3) is usually expressed in terms of the roots α, β and γ of the characteristic equation

$$\lambda^3 - a\lambda^2 - b\lambda - c = 0. \tag{1.4}$$

Here we express the solutions of the equation (1.3) using terms of the sequence $(J_n)_{n=0}^{+\infty}$ defined by the recurrent relation

$$J_{n+3} = aJ_{n+2} + bJ_{n+1} + cJ_n, \quad n \in \mathbb{N}, \tag{1.5}$$

and the special initial values

$$J_0 = 0, \quad J_1 = 1 \text{ and } J_2 = a. \tag{1.6}$$

Noting that $(R_n)_{n=-2}^{+\infty}$ and $(J_n)_{n=0}^{+\infty}$ have the same characteristic equation. Also if $a = b = c = 1$, then the equation (1.5) is nothing other then the famous Tribonacci sequence $(T_n)_{n=0}^{+\infty}$.

The closed form of the solutions of $\{J_n\}_{n=0}^{+\infty}$ and many proprieties of them are well known in the literature, for the interest of the readers and for the purpose of our work, we show how we can get the formula of the solutions and we give also a result on the limit

$$\lim_{n\to\infty} \frac{J_{n+1}}{J_n}.$$

For the roots α, β and γ of the characteristic equation (1.4), we have

$$\begin{cases} \alpha+\beta+\gamma = a \\ \alpha\beta+\alpha\gamma+\beta\gamma = -b \\ \alpha\beta\gamma = c. \end{cases} \tag{1.7}$$

We have:

Case 1: If all roots are equal. In this case

$$J_n = \left(c_1 + c_2 n + c_3 n^2\right)\alpha^n.$$

Now using (1.7) and the fact that $J_0 = 0$, $J_1 = 1$ and $J_2 = a$, we obtain

$$J_n = \left(\frac{n}{2\alpha} + \frac{n^2}{2\alpha}\right)\alpha^n. \tag{1.8}$$

Case 2: If two roots are equal, say $\beta = \gamma$. In this case

$$J_n = c_1\alpha^n + (c_2 + c_3 n)\,\beta^n.$$

Using (1.7) and the fact that $J_0 = 0$, $J_1 = 1$ and $J_2 = a$, we obtain

$$J_n = \frac{\alpha}{(\beta-\alpha)^2}\alpha^n + \left(\frac{-\alpha}{(\beta-\alpha)^2} + \frac{n}{\beta-\alpha}\right)\beta^n. \tag{1.9}$$

Case 3: If the roots are all different. In this case

$$J_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 1$ and $J_2 = a$, we obtain

$$J_n = \frac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n. \tag{1.10}$$

In this case we can get two roots of (1.4) complex conjugates say $\gamma = \overline{\beta}$ and the third one real and the formula of J_n will be

$$J_n = \frac{\alpha}{(\overline{\beta}-\alpha)(\beta-\alpha)}\alpha^n + \frac{-\beta}{(\overline{\beta}-\beta)(\beta-\alpha)}\beta^n + \frac{\overline{\beta}}{(\overline{\beta}-\alpha)(\overline{\beta}-\beta)}\overline{\beta}^n. \tag{1.11}$$

Consider the following linear third order difference equation

$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2},\ n = 0, 1, ..., \tag{1.12}$$

the constant coefficients a, b and c and the initial values S_0, S_{-1} and S_{-2} are real numbers. As for the equation (1.3), we will express the solutions of (1.12) using terms of (1.13). To do this let us consider the difference equation

$$j_{n+3} = -aj_{n+2} + bj_{n+1} - cj_n, \quad n \in \mathbb{N}, \tag{1.13}$$

and the special initial values

$$j_0 = 0, \quad j_1 = 1 \text{ and } j_2 = -a. \tag{1.14}$$

The characteristic equation of (1.12) and (1.13) is

$$\lambda^3 + a\lambda^2 - b\lambda + c = 0. \tag{1.15}$$

Clearly the roots of (1.15) are $-\alpha$, $-\beta$ and $-\gamma$. Now following the same procedure in solving $\{J_n\}$, we get that

$$j_n = (-1)^{n+1}J_n.$$

Lemma 1.2. *Let α, β and γ be the roots of (1.4), assume that α is a real root with* $\max(|\alpha|; |\beta|; |\gamma|) = |\alpha|$. *Then,*

$$\lim_{n\to\infty} \frac{J_{n+1}}{J_n} = \alpha.$$

Proof. If α, β and γ are real and distinct then,

$$\begin{aligned}
\lim_{n\to\infty}\frac{J_{n+1}}{J_n} &= \lim_{n\to\infty}\frac{\frac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}\alpha^{n+1}+\frac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\beta^{n+1}+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}}{\frac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}\alpha^{n}+\frac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\beta^{n}+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n}}\\
&= \lim_{n\to\infty}\frac{\alpha^{n+1}}{\alpha^n}\frac{\frac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}\frac{\alpha^{n+1}}{\alpha^{n+1}}+\frac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\frac{\beta^{n+1}}{\alpha^{n+1}}+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\frac{\gamma^{n+1}}{\alpha^{n+1}}}{\frac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}\frac{\alpha^{n}}{\alpha^{n}}+\frac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\frac{\beta^{n}}{\alpha^{n}}+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\frac{\gamma^{n}}{\alpha^{n}}}\\
&= \lim_{n\to\infty}\alpha\frac{\frac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}+\frac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\left(\frac{\beta}{\alpha}\right)^{n+1}+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\left(\frac{\gamma}{\alpha}\right)^{n+1}}{\frac{\alpha}{(\gamma-\alpha)(\beta-\alpha)}+\frac{-\beta}{(\gamma-\beta)(\beta-\alpha)}\left(\frac{\beta}{\alpha}\right)^{n}+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\left(\frac{\gamma}{\alpha}\right)^{n}}\\
&= \alpha.
\end{aligned}$$

The proof of the other cases of the roots, that is when $\alpha=\beta=\gamma$ or β, γ are complex conjugate, is similar to the first one and will be omitted. □

Remark 1.2.1. If α is a real root and β, γ are complex conjugate with

$$\max(|\alpha|;|\beta|;|\overline{\beta}|)=|\beta|=|\overline{\beta}|,$$

then $\lim\limits_{n\to\infty}\frac{J_{n+1}}{J_n}$ doesn't exist.

In the following result, we solve in closed form the equations (1.3) and (1.12) in terms of the sequence $(J_n)_{n=0}^{+\infty}$. The obtained formula will be very useful to obtain the formula of the solutions of system (1.2).

Lemma 1.3. *We have for all $n\in\mathbb{N}_0$,*

$$R_n = cJ_nR_{-2}+(J_{n+2}-aJ_{n+1})R_{-1}+J_{n+1}R_0, \tag{1.16}$$

$$S_n=(-1)^n\left[cJ_nS_{-2}+(-J_{n+2}+aJ_{n+1})S_{-1}+J_{n+1}S_0\right]. \tag{1.17}$$

Proof. Assume that α, β and γ are the distinct roots of the characteristic equation (1.4), so

$$R_n=c_1'\alpha^n+c_2'\beta^n+c_3'\gamma^n,\ n=-2,-1,0,\ldots.$$

Using the initial values R_0, R_{-1} and R_{-2}, we get

$$\begin{cases}\frac{1}{\alpha^2}c_1'+\frac{1}{\beta^2}c_2'+\frac{1}{\gamma^2}c_3' &= R_{-2}\\ \frac{1}{\alpha}c_1'+\frac{1}{\beta}c_2'+\frac{1}{\gamma}c_3' &= R_{-1}\\ c_1'+c_2'+c_3' &= R_0\end{cases} \tag{1.18}$$

after some calculations we get

$$\begin{aligned} c_1' &= \frac{\alpha^2\beta\gamma}{(\gamma-\alpha)(\beta-\alpha)}R_{-2} - \frac{(\gamma+\beta)\alpha^2}{(\gamma-\alpha)(\beta-\alpha)}R_{-1} + \frac{\alpha^2}{(\gamma-\alpha)(\beta-\alpha)}R_0 \\ c_2' &= -\frac{\alpha\beta^2\gamma}{(\gamma-\beta)(\beta-\alpha)}R_{-2} + \frac{(\alpha+\gamma)\beta^2}{(\gamma-\beta)(\beta-\alpha)}R_{-1} - \frac{\beta^2}{(\gamma-\beta)(\beta-\alpha)}R_0 \\ c_3' &= \frac{\alpha\beta\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}R_{-2} - \frac{(\alpha+\beta)\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}R_{-1} + \frac{\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}R_0 \end{aligned}$$

that is,

$$\begin{aligned} R_n &= \left(\frac{\alpha^2\beta\gamma}{(\gamma-\alpha)(\beta-\alpha)}\alpha^n - \frac{\alpha\beta^2\gamma}{(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{\alpha\beta\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n\right)R_{-2} \\ &+ \left(-\frac{(\gamma+\beta)\alpha^2}{(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{(\alpha+\gamma)\beta^2}{(\gamma-\beta)(\beta-\alpha)}\beta^n - \frac{(\alpha+\beta)\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n\right)R_{-1} \\ &+ \left(\frac{\alpha^2}{(\gamma-\alpha)(\beta-\alpha)}\alpha^n - \frac{\beta^2}{(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n\right)R_0 \end{aligned}$$

$$R_n = cJ_nR_{-2} + (J_{n+2} - aJ_{n+1})R_{-1} + J_{n+1}R_0.$$

The proof of the other cases is similar and will be omitted.

Let $A := -a$ and $B := b$, $C := -c$, then equation (1.12) takes the form of (1.3) and the equation (1.13) takes the form of (1.5). Then analogous to the formula of (1.3) we obtain

$$S_n = Cj_nS_{-2} + (j_{n+2} - Aj_{n+1})S_{-1} + j_{n+1}S_0.$$

Using the fact that $j_n = (-1)^{n+1}J_n$, $A = -a$ and $C := -c$ we get

$$S_n = (-1)^n\left(cJ_nS_{-2} - (J_{n+2} - aJ_{n+1})S_{-1} + J_{n+1}S_0\right).$$

□

1.2.2 Closed form of well defined solutions

In this section, we solve through an analytical approach the system (1.2) with $c \neq 0$ in closed form. By a well defined solution of system (1.2), we mean a solution that satisfies $x_ny_n \neq 0$, $n = -1, 0, \cdots$. Clearly if we choose the initial values and the parameters a, b and c positif, then every solution of (1.2) will be well defined.

The following result give an explicit formula for well defined solutions of the system (1.2).

Theorem 1.4. *Let* $\{x_n, y_n\}_{n\geq -1}$ *be a well defined solution of* (1.2). *Then, for* $n = 0, 1, \ldots,$ *we have*

$$x_{2n+1} = \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_{-1}y_0}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})x_{-1} + J_{2n+1}x_{-1}y_0},$$

$$x_{2n+2} = \frac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})y_{-1} + J_{2n+3}x_0y_{-1}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})y_{-1} + J_{2n+2}x_0y_{-1}},$$

$$y_{2n+1} = \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})y_{-1} + J_{2n+2}x_0y_{-1}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})y_{-1} + J_{2n+1}x_0y_{-1}},$$

$$y_{2n+2} = \frac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})x_{-1} + J_{2n+3}x_{-1}y_0}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_{-1}y_0}$$

where the initial conditions x_{-1}, x_0, y_{-1} *and* $y_0 \in (\mathbb{R} - \{0\}) - F$, *with* F *is the Forbidden set of system* (1.2) *given by*

$$F = \bigcup_{n=0}^{\infty} \{(x_{-1}, x_0, y_{-1}, y_0) \in (\mathbb{R} - \{0\}) : A_n = 0 \, or \, B_n = 0\},$$

where

$$A_n = J_{n+1}y_0x_{-1} + (J_{n+2} - aJ_{n+1})x_{-1} + cJ_n, \;\; B_n = J_{n+1}x_0y_{-1} + (J_{n+2} - aJ_{n+1})y_{-1} + cJ_n.$$

Proof. Putting

$$x_n = \frac{u_n}{v_{n-1}}, \quad y_n = \frac{v_n}{u_{n-1}}, \; n = -1, 0, 1, ..., \tag{1.19}$$

we get the following linear third order system of difference equations

$$u_{n+1} = av_n + bu_{n-1} + cv_{n-2}, \quad v_{n+1} = au_n + bv_{n-1} + cu_{n-2}, \quad n = 0, 1, ..., \tag{1.20}$$

where the initial values $u_{-2}, u_{-1}, u_0, v_{-2}, v_{-1}, v_0$ are nonzero real numbers.
From(1.20) we have for $n = 0, 1, ...,$

$$\begin{cases} u_{n+1} + v_{n+1} = a(v_n + u_n) + b(u_{n-1} + v_{n-1}) + c(v_{n-2} + u_{n-2}), \\ u_{n+1} - v_{n+1} = a(v_n - u_n) + b(u_{n-1} - v_{n-1}) + c(v_{n-2} - u_{n-2}). \end{cases}$$

Putting again

$$R_n = u_n + v_n, \quad S_n = u_n - v_n, \; n = -2, -1, 0, ..., \tag{1.21}$$

we obtain two homogeneous linear difference equations of third order:

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2}, \; n = 0, 1, \cdots,$$

and

$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2},\ n = 0, 1, \cdots. \tag{1.22}$$

Using (1.21), we get for $n = -2, -1, 0, ...,$

$$u_n = \frac{1}{2}(R_n + S_n),\ v_n = \frac{1}{2}(R_n - S_n).$$

From Lemma 1.3 we obtain,

$$\begin{cases} u_{2n-1} = \dfrac{1}{2}\left[cJ_{2n-1}(R_{-2} - S_{-2}) + (J_{2n+1} - aJ_{2n})(R_{-1} + S_{-1}) + J_{2n}(R_0 - S_0)\right], n = 1, 2, \cdots, \\ u_{2n} = \dfrac{1}{2}\left[cJ_{2n}(R_{-2} + S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} - S_{-1}) + J_{2n+1}(R_0 + S_0)\right], n = 0, 1, \cdots, \end{cases} \tag{1.23}$$

$$\begin{cases} v_{2n-1} = \dfrac{1}{2}\left[cJ_{2n-1}(R_{-2} + S_{-2}) + (J_{2n+1} - aJ_{2n})(R_{-1} - S_{-1}) + J_{2n}(R_0 + S_0)\right], n = 1, 2, \cdots, \\ v_{2n} = \dfrac{1}{2}\left[cJ_{2n}(R_{-2} - S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} + S_{-1}) + J_{2n+1}(R_0 - S_0)\right], n = 0, 1, \cdots, \end{cases} \tag{1.24}$$

Substituting (1.23) and (1.24) in (1.19), we get for $n = 0, 1, ...,$

$$\begin{cases} x_{2n+1} = \dfrac{cJ_{2n+1}(R_{-2} - S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} + S_{-1}) + J_{2n+2}(R_0 - S_0)}{cJ_{2n}(R_{-2} - S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} + S_{-1}) + J_{2n+1}(R_0 - S_0)}, \\ x_{2n+2} = \dfrac{cJ_{2n+2}(R_{-2} + S_{-2}) + (J_{2n+4} - aJ_{2n+3})(R_{-1} - S_{-1}) + J_{2n+3}(R_0 + S_0)}{cJ_{2n+1}(R_{-2} + S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} - S_{-1}) + J_{2n+2}(R_0 + S_0)}, \end{cases} \tag{1.25}$$

$$\begin{cases} y_{2n+1} = \dfrac{cJ_{2n+1}(R_{-2} + S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} - S_{-1}) + J_{2n+2}(R_0 + S_0)}{cJ_{2n}(R_{-2} + S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} - S_{-1}) + J_{2n+1}(R_0 + S_0)}, \\ y_{2n+2} = \dfrac{cJ_{2n+2}(R_{-2} - S_{-2}) + (J_{2n+4} - aJ_{2n+3})(R_{-1} + S_{-1}) + J_{2n+3}(R_0 - S_0)}{cJ_{2n+1}(R_{-2} - S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} + S_{-1}) + J_{2n+2}(R_0 - S_0)}. \end{cases} \tag{1.26}$$

Then,

$$\begin{cases} x_{2n+1} = \dfrac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\dfrac{R_{-1} + S_{-1}}{R_{-2} - S_{-2}} + J_{2n+2}\dfrac{R_0 - S_0}{R_{-2} - S_{-2}}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})\dfrac{R_{-1} + S_{-1}}{R_{-2} - S_{-2}} + J_{2n+1}\dfrac{R_0 - S_0}{R_{-2} - S_{-2}}}, \\ x_{2n+2} = \dfrac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})\dfrac{R_{-1} - S_{-1}}{R_{-2} + S_{-2}} + J_{2n+3}\dfrac{R_0 + S_0}{R_{-2} + S_{-2}}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\dfrac{R_{-1} - S_{-1}}{R_{-2} + S_{-2}} + J_{2n+2}\dfrac{R_0 + S_0}{R_{-2} + S_{-2}}}, \end{cases} \tag{1.27}$$

$$\begin{cases} y_{2n+1} = \dfrac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\dfrac{R_{-1} - S_{-1}}{R_{-2} + S_{-2}} + J_{2n+2}\dfrac{R_0 + S_0}{R_{-2} + S_{-2}}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})\dfrac{R_{-1} - S_{-1}}{R_{-2} + S_{-2}} + J_{2n+1}\dfrac{R_0 + S_0}{R_{-2} + S_{-2}}}, \\ y_{2n+2} = \dfrac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})\dfrac{R_{-1} + S_{-1}}{R_{-2} - S_{-2}} + J_{2n+3}\dfrac{R_0 - S_0}{R_{-2} - S_{-2}}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})\dfrac{R_{-1} + S_{-1}}{R_{-2} - S_{-2}} + J_{2n+2}\dfrac{R_0 - S_0}{R_{-2} - S_{-2}}}. \end{cases} \tag{1.28}$$

We have

$$x_{-1}=\frac{u_{-1}}{v_{-2}}=\frac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}},\ x_0=\frac{u_0}{v_{-1}}=\frac{R_0+S_0}{R_{-1}-S_{-1}}, \tag{1.29}$$

$$y_{-1}=\frac{v_{-1}}{u_{-2}}=\frac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}},\ y_0=\frac{v_0}{u_{-1}}=\frac{R_0-S_0}{R_{-1}+S_{-1}} \tag{1.30}$$

From (1.29), (1.30) it follows that,

$$\begin{cases}\dfrac{R_0-S_0}{R_{-2}-S_{-2}}=\dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}}\times\dfrac{R_0-S_0}{R_{-1}+S_{-1}}=x_{-1}y_0\\ \dfrac{R_0+S_0}{R_{-2}+S_{-2}}=\dfrac{R_0+S_0}{R_{-1}-S_{-1}}\times\dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}}=x_0y_{-1}\end{cases} \tag{1.31}$$

Using (1.27), (1.28), (1.29), (1.30) and (1.31), we obtain the closed form of the solutions of (1.2), that is for $n=0,1,...,$ we have

$$\begin{cases}x_{2n+1}=\dfrac{cJ_{2n+1}+(J_{2n+3}-aJ_{2n+2})x_{-1}+J_{2n+2}x_{-1}y_0}{cJ_{2n}+(J_{2n+2}-aJ_{2n+1})x_{-1}+J_{2n+1}x_{-1}y_0},\\ x_{2n+2}=\dfrac{cJ_{2n+2}+(J_{2n+4}-aJ_{2n+3})y_{-1}+J_{2n+3}x_0y_{-1}}{cJ_{2n+1}+(J_{2n+3}-aJ_{2n+2})y_{-1}+J_{2n+2}x_0y_{-1}},\end{cases}$$

$$\begin{cases}y_{2n+1}=\dfrac{cJ_{2n+1}+(J_{2n+3}-aJ_{2n+2})y_{-1}+J_{2n+2}x_0y_{-1}}{cJ_{2n}+(J_{2n+2}-aJ_{2n+1})y_{-1}+J_{2n+1}x_0y_{-1}},\\ y_{2n+2}=\dfrac{cJ_{2n+2}+(J_{2n+4}-aJ_{2n+3})x_{-1}+J_{2n+3}x_{-1}y_0}{cJ_{2n+1}+(J_{2n+3}-aJ_{2n+2})x_{-1}+J_{2n+2}x_{-1}y_0}.\end{cases}$$

□

Remark 1.2.2. Writing system (1.2) in the form

$$\begin{cases}x_{n+1}= & f\left(x_n,x_{n-1},y_n,y_{n-1}\right)=\frac{ay_nx_{n-1}+bx_{n-1}+c}{y_nx_{n-1}},\\ y_{n+1}= & g\left(x_n,x_{n-1},y_n,y_{n-1}\right)=\frac{ax_ny_{n-1}+by_{n-1}+c}{x_ny_{n-1}}.\end{cases}$$

So it follows that points (α,α), (β,β) and (γ,γ) are solutions of the of system

$$\begin{cases}\bar{x}=\dfrac{a\bar{y}\bar{x}+b\bar{x}+c}{\bar{y}\bar{x}},\\ \bar{y}=\dfrac{a\bar{x}\bar{y}+b\bar{y}+c}{\bar{x}\bar{y}}\end{cases}$$

where α, β and γ are the roots of (1.4).

Theorem 1.5. *Under the same conditions in Lemma 1.2, for every well defined solution of system (1.2), we have*

$$\lim_{n\to+\infty}x_{2n+1}=\lim_{n\to+\infty}x_{2n+2}=\lim_{n\to+\infty}y_{2n+1}=\lim_{n\to+\infty}y_{2n+2}=\alpha.$$

Proof. We have

$$
\begin{aligned}
\lim_{n\to\infty} x_{2n+1} &= \lim_{n\to\infty} \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}y_0x_{-1}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})x_{-1} + J_{2n+1}y_0x_{-1}} \\
&= \lim_{n\to\infty} \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}y_0x_{-1}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})x_{-1} + J_{2n+1}y_0x_{-1}} \\
&= \lim_{n\to\infty} \frac{c\frac{J_{2n+1}}{J_{2n}} + \left(\frac{J_{2n+3}}{J_{2n+2}} \times \frac{J_{2n+2}}{J_{2n+1}} \times \frac{J_{2n+1}}{J_{2n}} - a\frac{J_{2n+2}}{J_{2n+1}} \times \frac{J_{2n+1}}{J_{2n}}\right)x_{-1} + \frac{J_{2n+2}}{J_{2n+1}} \times \frac{J_{2n+1}}{J_{2n}}y_0x_{-1}}{c\frac{J_{2n}}{J_{2n}} + \left(\frac{J_{2n+2}}{J_{2n+1}} \times \frac{J_{2n+1}}{J_{2n}} - a\frac{J_{2n+1}}{J_{2n}}\right)x_{-1} + \frac{J_{2n+1}}{J_{2n}}y_0x_{-1}} \\
&= \frac{c\alpha + (\alpha^3 - a\alpha^2)x_{-1} + \alpha^2 y_0x_{-1}}{c + (\alpha^2 - a\alpha)x_{-1} + \alpha y_0x_{-1}} \\
&= \alpha\ .
\end{aligned}
$$

In the same way we show that

$$\lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} y_{2n+2} = \alpha.$$

□

1.2.3 Particular cases

Here we are interested in some particular cases of system (1.2). Some of these particular cases have been the subject of some recent papers.

1.2.3.1 The solutions of the equation $x_{n+1} = \frac{ax_nx_{n-1}+bx_{n-1}+c}{x_nx_{n-1}}$

If we choose $y_{-1} = x_{-1}$ and $y_0 = x_0$, then system (1.2) is reduced to the equation

$$x_{n+1} = \frac{ax_nx_{n-1} + bx_{n-1} + c}{x_nx_{n-1}}, \quad n \in \mathbb{N}_0. \tag{1.32}$$

The following results are respectively direct consequences of Theorem 1.4 and Theorem 1.5.

Corollary 1.6. *Let $\{x_n\}_{n\geq -1}$ be a well defined solution of the equation (1.32). Then for $n = 0, 1, \ldots,$ we have*

$$x_{2n+1} = \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_{-1}x_0}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})x_{-1} + J_{2n+1}x_{-1}x_0},$$

$$x_{2n+2} = \frac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})x_{-1} + J_{2n+3}x_0x_{-1}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_0x_{-1}}.$$

Corollary 1.7. *Under the same conditions in Lemma 1.2, for every well defined solution of equation* (1.32)*, we have*

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} x_{2n+2} = \alpha.$$

The equation (1.32) was been studied by Azizi in [7] and Stevic in [92].

1.2.3.2 The solutions of the system $x_{n+1} = \frac{y_n x_{n-1}+x_{n-1}+1}{y_n x_{n-1}}$, $y_{n+1} = \frac{x_n y_{n-1}+y_{n-1}+1}{x_n y_{n-1}}$

Consider the system

$$x_{n+1} = \frac{y_n x_{n-1} + x_{n-1} + 1}{y_n x_{n-1}},\ y_{n+1} = \frac{x_n y_{n-1} + y_{n-1} + 1}{x_n y_{n-1}}\quad n \in \mathbb{N}_0. \tag{1.33}$$

Clearly system (1.33) is a particular case of system (1.2) with $a = b = c = 1$. In this case the sequence $\{J_n\}$ is the famous classical sequence of Tribonacci numbers $\{T_n\}$, that is

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \quad n \in \mathbb{N}, \quad \text{where } T_0 = 0,\ T_1 = 1 \text{ and } T_2 = 1,$$

and we have

$$T_n = \frac{\alpha^{n+1}}{(\beta-\alpha)(\gamma-\alpha)} - \frac{\beta^{n+1}}{(\beta-\alpha)(\gamma-\beta)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \qquad n = 0, 1, ...,$$

with

$$\alpha = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3},\ \beta = \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\gamma = \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3},\ \omega = \frac{-1+i\sqrt{3}}{2}.$$

Numerically we have $\alpha = 1.839286755$ and the two complex conjugate are

$$-0.4196433777 + 0.6062907300i,\ -0.4196433777 - 0.6062907300i$$

with $i^2 = -1$.

The following results follows respectively from Theorem 1.4 and Theorem 1.5.

Corollary 1.8. *Let* $\{x_n, y_n\}_{n\geq -1}$ *be a well defined solution of* (1.33)*. Then, for* $n = 0, 1, 2, 3, \ldots,$ *we have*

$$x_{2n+1} = \frac{cT_{2n+1} + (T_{2n+3} - aT_{2n+2})x_{-1} + T_{2n+2}x_{-1}y_0}{cT_{2n} + (T_{2n+2} - aT_{2n+1})x_{-1} + T_{2n+1}x_{-1}y_0},$$

$$x_{2n+2} = \frac{cT_{2n+2} + (T_{2n+4} - aT_{2n+3})y_{-1} + T_{2n+3}x_0y_{-1}}{cT_{2n+1} + (T_{2n+3} - aT_{2n+2})y_{-1} + T_{2n+2}x_0y_{-1}},$$

$$y_{2n+1} = \frac{cT_{2n+1} + (T_{2n+3} - aT_{2n+2})y_{-1} + T_{2n+2}x_0y_{-1}}{cT_{2n} + (T_{2n+2} - aT_{2n+1})y_{-1} + T_{2n+1}x_0y_{-1}},$$

$$y_{2n+2} = \frac{cT_{2n+2} + (T_{2n+4} - aT_{2n+3})x_{-1} + T_{2n+3}x_{-1}y_0}{cT_{2n+1} + (T_{2n+3} - aT_{2n+2})x_{-1} + T_{2n+2}x_{-1}y_0}$$

Corollary 1.9. *For every well defined solution of system* (1.2)*, we have*

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} x_{2n+2} = \lim_{n\to+\infty} y_{2n+1} = \lim_{n\to+\infty} y_{2n+2} = \alpha.$$

For the equation

$$x_{n+1} = \frac{x_nx_{n-1} + x_{n-1} + 1}{x_nx_{n-1}}, \quad n \in \mathbb{N}_0. \tag{1.34}$$

we have the following results.

Corollary 1.10. *Let* $\{x_n\}_{n\geq -1}$ *be a well defined solution of the equation* (1.34)*. Then for* $n = 0, 1, \ldots,$ *we have*

$$x_{2n+1} = \frac{T_{2n+1} + (T_{2n+3} - T_{2n+2})x_{-1} + T_{2n+2}x_{-1}x_0}{T_{2n} + (T_{2n+2} - T_{2n+1})x_{-1} + T_{2n+1}x_{-1}x_0},$$

$$x_{2n+2} = \frac{T_{2n+2} + (T_{2n+4} - T_{2n+3})x_{-1} + T_{2n+3}x_0x_{-1}}{T_{2n+1} + (T_{2n+3} - T_{2n+2})x_{-1} + T_{2n+2}x_0x_{-1}}.$$

Corollary 1.11. *Under the same conditions in Lemma 1.2, for every well defined solution of the equation* (1.34)*, we have*

$$\lim_{n\to+\infty} x_{2n+1} = \lim_{n\to+\infty} x_{2n+2} = \alpha.$$

Let $I = (0, +\infty)$, $J = (0, +\infty)$ and choosing x_{-1}, x_0, y_{-1} and $y_0 \in (0, +\infty)$. Then clearly the system

$$\overline{x} = f(\overline{x}, \overline{y}) = \frac{\overline{xy} + \overline{x} + 1}{\overline{xy}}, \ \overline{y} = g(\overline{x}, \overline{y}) = \frac{\overline{xy} + \overline{y} + 1}{\overline{xy}}$$

has a unique solution $(\alpha, \alpha) \in I \times J$, that is (α, α) is the unique equilibrium point (fixed point) of our system

$$x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}) = \frac{y_nx_{n-1} + x_{n-1} + 1}{y_nx_{n-1}},$$

$$y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}) = \frac{x_n y_{n-1} + y_{n-1} + 1}{x_n y_{n-1}}.$$

Clearly the functions

$$f : I^2 \times J^2 \longrightarrow I \qquad \text{and} \qquad g : I^2 \times J^2 \longrightarrow I$$

defined by

$$f(u_0; u_1; v_0; v_1) = \frac{v_0 u_1 + u_1 + 1}{v_0 u_1} \qquad \text{and} \qquad g(u_0; u_1; v_0; v_1) = \frac{u_0 v_1 + v_1 + 1}{u_0 v_1}$$

are continuously differentiable.

In the following result we prove that the unique equilibrium point (α, α) of (1.33) is locally asymptotically stable.

Theorem 1.12. *The equilibrium point* (α, α) *is locally asymptotically stable.*

Proof. The Jacobian matrix associated to the system (1.33) around the equilibrium point (α, α), is given by

$$A = \begin{pmatrix} 0 & -\frac{1}{\alpha^3} & -\frac{\alpha+1}{\alpha^3} & 0 & \\ 1 & 0 & 0 & 0 & 0 \\ -\frac{\alpha+1}{\alpha^3} & 0 & 0 & -\frac{1}{\alpha^3} & \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the characteristic polynomial of A is

$$P(\lambda) = \lambda^4 + \frac{(2\alpha^3 - \alpha^2 - 2\alpha - 1)}{\alpha^6}\lambda^2 + \frac{1}{\alpha^6}$$

and the roots of $P(\lambda)$ are

$$\lambda_1 = \frac{1}{2} \times \frac{1 + \alpha + \sqrt{-4\alpha^3 + \alpha^2 + 2\alpha + 1}}{\alpha^3}, \ \lambda_2 = -\frac{1}{2} \times \frac{1 + \alpha + \sqrt{-4\alpha^3 + \alpha^2 + 2\alpha + 1}}{\alpha^3},$$

$$\lambda_3 = \frac{1}{2} \times \frac{-1 - \alpha + \sqrt{-4\alpha^3 + \alpha^2 + 2\alpha + 1}}{\alpha^3}, \ \lambda_4 = -\frac{1}{2} \times \frac{-1 - \alpha + \sqrt{-4\alpha^3 + \alpha^2 + 2\alpha + 1}}{\alpha^3}.$$

We have $|\lambda_i| < 1, \ i = 1, 2, 3, 4$, so the equilibrium point (α, α) is locally asymptotically stable. □

The following result is a direct consequence of Theorem 1.12 and Corollary 1.9.

Theorem 1.13. *The equilibrium point* (α, α) *is globally asymptotically stable .*

Let $I = (0, +\infty)$ and choosing x_{-1}, $x_0 \in (0, +\infty)$. Writing the equation (1.34) as

$$x_{n+1} = h(x_n, x_{n-1}) = \frac{x_n x_{n-1} + x_{n-1} + 1}{x_n x_{n-1}} \tag{1.35}$$

where

$$h : I^2 \longrightarrow I$$

is defined by

$$h(u_0; u_1) = \frac{u_0 u_1 + u_1 + 1}{u_0 u_1}.$$

The function h is continuously differentiable. The equation $\overline{x} = h(\overline{x}, \overline{x})$ has the unique solution $\overline{x} = \alpha$ in $(0, +\infty)$. The linear equation associated to the equation (1.35) about the equilibrium point $\overline{x} = \alpha$ is given by

$$y_{n+1} = \frac{\partial h}{\partial u_0}(\alpha, \alpha)\, y_n + \frac{\partial h}{\partial u_1}(\alpha, \alpha)\, y_{n-1},$$

the last equation has as characteristic polynomial

$$Q(\lambda) = \lambda^2 - \frac{\partial h}{\partial u_0}(\alpha, \alpha)\, \lambda - \frac{\partial h}{\partial u_1}(\alpha, \alpha)\, .$$

In the following result we show that the unique equilibrium point $\overline{x} = \alpha$ is globally stable.

Theorem 1.14. *The equilibrium point $\overline{x} = \alpha$ is globally stable.*

Proof. The linear equation associated to (1.34) about the equilibrium point $\overline{x} = \alpha$ is

$$y_{n+1} = -\frac{\alpha + 1}{\alpha^3} y_n - \frac{1}{\alpha^3} y_{n-1}$$

and the characteristic polynomial is

$$Q(\lambda) = \lambda^2 + \left(\frac{\alpha + 1}{\alpha^3}\right) \lambda + \left(\frac{1}{\alpha^3}\right).$$

We have

$$\left|\frac{\alpha + 1}{\alpha^3}\lambda + \frac{1}{\alpha^3}\right| \leq \left|\frac{\alpha + 1}{\alpha^3}\right| + \left|\frac{1}{\alpha^3}\right| < 1 = \left|\lambda^2\right|, \forall \lambda \in \mathbb{C} : |\lambda| = 1.$$

So, by Rouché's theorem the roots of the characteristic polynomial $Q(\lambda)$ lie in the open unit disk. Then the equilibrium point $\overline{x} = \alpha$ is locally asymptotically stable. Now, from this and Corollary 1.11 the result holds. □

1.2.3.3 The system $x_{n+1} = \frac{bx_{n-1}+c}{y_n x_{n-1}}$, $y_{n+1} = \frac{by_{n-1}+c}{x_n y_{n-1}}$

When $a = 0$, the system (1.2) takes the form

$$x_{n+1} = \frac{bx_{n-1} + c}{y_n x_{n-1}}, \; y_{n+1} = \frac{by_{n-1} + c}{x_n y_{n-1}} \quad n \in \mathbb{N}_0. \tag{1.36}$$

From Theorem 1.4, we get the following result.

Corollary 1.15. *Let* $\{x_n, y_n\}_{n\geq -1}$ *be a well defined solution of* (1.36). *Then, for* $n = 0, 1, \ldots,$ *we have*

$$x_{2n+1} = \frac{cP_{2n+1} + P_{2n+3}x_{-1} + P_{2n+2}x_{-1}y_0}{cP_{2n} + P_{2n+2}x_{-1} + P_{2n+1}x_{-1}y_0},$$

$$x_{2n+2} = \frac{cP_{2n+2} + P_{2n+4}y_{-1} + P_{2n+3}x_0 y_{-1}}{cP_{2n+1} + P_{2n+3}y_{-1} + P_{2n+2}x_0 y_{-1}},$$

$$y_{2n+1} = \frac{cP_{2n+1} + P_{2n+3}y_{-1} + P_{2n+2}x_0 y_{-1}}{cP_{2n} + P_{2n+2}y_{-1} + P_{2n+1}x_0 y_{-1}},$$

$$y_{2n+2} = \frac{cP_{2n+2} + P_{2n+4}x_{-1} + P_{2n+3}x_{-1}y_0}{cP_{2n+1} + P_{2n+3}x_{-1} + P_{2n+2}x_{-1}y_0}.$$

Here we have write $\{P_n\}_n$ instead of $\{J_n\}_n$, as in this case $\{J_n\}_n$ takes the form of a generalized (Padovan) sequence, that is

$$P_{n+3} = bP_{n+1} + cP_n, \quad n \in \mathbb{N},$$

with special values $P_0 = 0$, $P_1 = 1$ and $P_2 = 0$. The system (1.36) was been investigated by Halim et al. in [44] and by Yazlik et al. in [124] with $b = 1$ and $c = \pm 1$. The one dimensional version of system (1.36), that is the equation

$$x_{n+1} = \frac{bx_{n-1} + c}{x_n x_{n-1}}, \quad n \in \mathbb{N}_0. \tag{1.37}$$

was been also investigated by Halim et al. in [44]. Form Corollary 1.15, we get that the well defined solutions of equation (1.37) are given for $n = 0, 1, ...,$ by

$$x_{2n+1} = \frac{cP_{2n+1} + P_{2n+3}x_{-1} + P_{2n+2}x_{-1}x_0}{cP_{2n} + P_{2n+2}x_{-1} + P_{2n+1}x_{-1}x_0},$$

$$x_{2n+2} = \frac{cP_{2n+2} + P_{2n+4}x_{-1} + P_{2n+3}x_0 x_{-1}}{cP_{2n+1} + P_{2n+3}x_{-1} + P_{2n+2}x_0 x_{-1}}.$$

In [124] and [44] we can find additional results on the stability of some equilibrium points.

Remark 1.2.3. If $c=0$, The system (1.2) become

$$x_{n+1}=\frac{ay_n+b}{y_n},\ y_{n+1}=\frac{ax_n+b}{x_n},\ n\in\mathbb{N}_0. \tag{1.38}$$

We note that if also $b=0$, then the solutions of the system (1.38) are given by

$$\{(x_0,y_0),(a,a),(a,a),...,\}.$$

The system (1.38) is a particular case of the more general system

$$x_{n+1}=\frac{ay_n+b}{cy_n+d},\ y_{n+1}=\frac{\alpha x_n+\beta}{\gamma x_n+\lambda},\ n\in\mathbb{N}_0 \tag{1.39}$$

which was been completely solved by Stevic in [91]. So, we refer to this paper for the readers interested in the form of the solutions of the system (1.39) and its particular case system (1.38). As it was proved in [91], the solutions are expressed using the terms of a corresponding generalized Fibonacci sequence. Noting that the papers [63, 104] and [105] deals also with particular cases of the system (1.39) or its one dimensional version.

1.3 A third order system

Recently in [2] and as generalization of the equations and systems studied in [7, 44, 91, 124], we have solved in a closed form the system of difference equations

$$\begin{cases} x_{n+1}=\dfrac{ay_nx_{n-1}y_n+bx_{n-1}+c}{x_{n-1}y_n}, \\ y_{n+1}=\dfrac{ax_ny_{n-1}x_n+by_{n-1}+c}{y_{n-1}x_n}, \end{cases} \tag{1.40}$$

Here and motivated by the above mentioned papers we show that we are able to expressed in closed form the well defined solutions of following system of difference equations

$$\begin{cases} x_{n+1}=\dfrac{ay_{n-2}x_{n-1}y_n+bx_{n-1}y_{n-2}+cy_{n-2}+d}{y_{n-2}x_{n-1}y_n}, \\ y_{n+1}=\dfrac{ax_{n-2}y_{n-1}x_n+by_{n-1}x_{n-2}+cx_{n-2}+d}{x_{n-2}y_{n-1}x_n}, \end{cases} \tag{1.41}$$

where $n\in\mathbb{N}_0$, the initial values x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} and y_0 are arbitrary nonzero real numbers and the parameters a, b, c and d are arbitrary real numbers with $d\neq 0$.

Clearly if $d=0$, then system (1.41) is nothing other than system (1.40). For the readers interested in the solutions of this system, we refer to [2], where the system (1.40) was been completely solved.

Noting also that the system (1.41) can be seen as a generalization of the equation

$$x_{n+1} = \frac{ax_{n-2}x_{n-1}x_n + bx_{n-1}x_{n-2} + cx_{n-2} + d}{x_{n-2}x_{n-1}x_n}, \; n \in \mathbb{N}_0. \tag{1.42}$$

In fact the solutions of (1.42) can be obtained from the solutions of (1.41) by choosing $y_{-i} = x_{-i}$, $i = 0, 1, 2$. The equation (1.42) was been the subject of substantial part of the paper of Azizi [7], which also motivated our present study. The same equation was studied on field of complex Numbers by Stevic in [95].

We will see that the explicit formulas of the well defined solutions of system (1.41) are expressed using the terms of the sequence $(J_n)_{n=0}^{+\infty}$ which are the solutions of the fourth order linear homogeneous difference equation defined by the relation

$$J_{n+4} = aJ_{n+3} + bJ_{n+2} + cJ_{n+1} + dJ_n, \quad n \in \mathbb{N}_0, \tag{1.43}$$

and the special initial values

$$J_0 = 0, \quad J_1 = 0, \quad J_2 = 1 \text{ and } J_3 = a. \tag{1.44}$$

Now we solve in closed form the equation (1.43). This equation (with the same or different initial values and parameters) was the subject of some papers in the literature, see for example [115, 46, 95].

The characteristic equation associated to the equation (1.43) is

$$\lambda^4 - a\lambda^3 - b\lambda^2 - c\lambda - d = 0 \tag{1.45}$$

and let α, β, γ and δ its four roots, then

$$\begin{cases} \alpha + \beta + \gamma + \delta = a \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -b \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = c \\ \alpha\beta\gamma\delta = -d \end{cases} \tag{1.46}$$

We have:

Case 1: If all roots are real and equal. In this case

$$J_n = \left(c_1 + c_2 n + c_3 n^2 + c_4 n^3\right) \alpha^n.$$

Now using (1.46) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, we obtain

$$J_n = \left(\frac{-n + n^3}{6\alpha^2}\right)\alpha^n \tag{1.47}$$

Case 2: If three roots are real and equal, say $\beta = \gamma = \delta$. In this case

$$J_n = c_1\alpha^n + \left(c_2 + c_3 n + c_4 n^2\right)\beta^n.$$

Now using (1.46) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, we obtain

$$J_n = \frac{-\alpha}{(\beta-\alpha)^3}\alpha^n + \left(\frac{\alpha}{(\beta-\alpha)^3} - \frac{n(\alpha+\beta)}{2\beta(\beta-\alpha)^2} + \frac{n^2}{2\beta(\beta-\alpha)}\right)\beta^n, \tag{1.48}$$

Case 3: If two real roots are equal, say $\gamma = \delta$. In this case

$$J_n = c_1\alpha^n + c_2\beta^n + (c_3 + c_4 n)\gamma^n.$$

Now using (1.46) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, we obtain

$$J_n = \frac{-\alpha}{(\gamma-\alpha)^2(\beta-\alpha)}\alpha^n + \frac{\beta}{(\gamma-\beta)^2(\beta-\alpha)}\beta^n + \left(\frac{\alpha\beta-\gamma^2}{(\gamma-\alpha)^2(\gamma-\beta)^2} + \frac{n}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^n, \tag{1.49}$$

Case 4: If double two real roots are equal, say $\alpha = \beta \neq \gamma = \delta$. In this case

$$J_n = (c_1 + c_2 n)\alpha^n + (c_3 + c_4 n)\gamma^n.$$

Now using (1.46) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, we obtain

$$J_n = \left(\frac{\gamma+\alpha}{(\gamma-\alpha)^3} + \frac{n}{(\gamma-\alpha)^2}\right)\alpha^n + \left(-\frac{\gamma+\alpha}{(\gamma-\alpha)^3} + \frac{n}{(\gamma-\alpha)^2}\right)\gamma^n, \tag{1.50}$$

Case 5: If the roots are all real and different. In this case

$$J_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n + c_4\delta^n.$$

Again, using (1.46) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, we obtain

$$\begin{aligned} J_n &= \frac{-\alpha}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{\beta}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{-\gamma}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\ &\quad + \frac{\delta}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n \end{aligned} \tag{1.51}$$

Case 6: If two real roots are equal, say $\alpha = \beta$ and two roots are complex conjugate ones, say $\delta = \overline{\gamma}$. In this case

$$J_n = (c_1 + c_2 n)\alpha^n + c_3\gamma^n + c_4\overline{\gamma}^n.$$

Again, using (1.46) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, we obtain

$$J_n = \left(\frac{\overline{\gamma}\gamma - \alpha^2}{(\overline{\gamma}-\alpha)^2(\gamma-\alpha)^2} + \frac{n}{(\overline{\gamma}-\alpha)(\gamma-\alpha)}\right)\alpha^n + \frac{-\gamma}{(\overline{\gamma}-\gamma)(\gamma-\alpha)^2}\gamma^n + \frac{\overline{\gamma}}{(\overline{\gamma}-\gamma)(\overline{\gamma}-\alpha)^2}\overline{\gamma}^n \tag{1.52}$$

Case 7: If two real roots α, β are different and two roots are complex conjugate ones, say $\delta = \overline{\gamma}$. In this case

$$J_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n + c_4\overline{\gamma}^n.$$

Again, using (1.46) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, we obtain

$$J_n = \frac{-\alpha}{(\overline{\gamma}-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{\beta}{(\overline{\gamma}-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{-\gamma}{(\overline{\gamma}-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n + \frac{\overline{\gamma}}{(\overline{\gamma}-\gamma)(\overline{\gamma}-\beta)(\overline{\gamma}-\alpha)}\overline{\gamma}^n \tag{1.53}$$

Case 8: If two complex roots are equal, say $\alpha = \gamma$ and $\beta = \delta = \overline{\alpha}$. In this case

$$J_n = (c_1 + c_2 n)\alpha^n + (c_3 + c_4 n)\overline{\alpha}^n.$$

Again, using (1.46) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, we obtain

$$J_n = \left(\frac{\overline{\alpha}+\alpha}{(\overline{\alpha}-\alpha)^3} + \frac{n}{(\overline{\alpha}-\alpha)^2}\right)\alpha^n + \left(\frac{-\overline{\alpha}-\alpha}{(\overline{\alpha}-\alpha)^3} + \frac{n}{(\overline{\alpha}-\alpha)^2}\right)\overline{\alpha}^n \tag{1.54}$$

Case 9: If the roots are all complex and different, say $\beta = \overline{\alpha}$ and $\delta = \overline{\gamma}$. In this case

$$J_n = c_1\alpha^n + c_2\overline{\alpha}^n + c_3\gamma^n + c_4\overline{\gamma}^n.$$

Again, using (1.46) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, we obtain

$$J_n = \frac{-\alpha}{(\overline{\gamma}-\alpha)(\gamma-\alpha)(\overline{\alpha}-\alpha)}\alpha^n + \frac{\overline{\alpha}}{(\overline{\gamma}-\overline{\alpha})(\gamma-\overline{\alpha})(\overline{\alpha}-\alpha)}\overline{\alpha}^n + \frac{-\gamma}{(\overline{\gamma}-\gamma)(\gamma-\overline{\alpha})(\gamma-\alpha)}\gamma^n + \frac{\overline{\gamma}}{(\overline{\gamma}-\gamma)(\overline{\gamma}-\overline{\alpha})(\overline{\gamma}-\alpha)}\overline{\gamma}^n \tag{1.55}$$

1.3.1 Form of the solutions - (Main result)

Here, we give a closed form for the well defined solutions of the system (1.41) with $d \neq 0$. To this end we will use the same change of variables as in [2] to transform the system (1.41) to a linear one and than following the same procedure as in [2] to obtain the closed form of the solutions. To get the solutions of the corresponding linear system we need to solve some fourth order linear difference equations. In particular, we derive from the main result (Main Theorem), for which we leave the proof to the next section, the solutions of some particular systems and equations where their solutions are related to the famous Tetranacci numbers. We recall that by a well defined solutions of system (1.41), we mean a solution that satisfies $x_n y_n \neq 0$, $n \geq -2$. The set of well defined solutions is not empty, in fact it suffices to choose the initial values and the parameters a, b, c and d positive, to see that every solution of (1.41) will be well defined.

1.3.1.1 Closed form of well defined solutions of the system (1.41)

The following result give an explicit formula for well defined solutions of the system (1.41).

Theorem 1.16. *(Main Theorem.) Let $\{x_n, y_n\}_{n\geq -1}$ be a well defined solution of (1.41). Then, for $n \in \mathbb{N}_0$, we have*

$$x_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, y_{-2} + (J_{2n+4} - aJ_{2n+3})\, x_{-1}y_{-2} + J_{2n+3}y_0x_{-1}y_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\, y_{-2} + (J_{2n+3} - aJ_{2n+2})\, x_{-1}y_{-2} + J_{2n+2}y_0x_{-1}y_{-2}},$$

$$x_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\, x_{-2} + (J_{2n+5} - aJ_{2n+4})\, y_{-1}x_{-2} + J_{2n+4}x_0y_{-1}x_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, x_{-2} + (J_{2n+4} - aJ_{2n+3})\, y_{-1}x_{-2} + J_{2n+3}x_0y_{-1}x_{-2}},$$

$$y_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, x_{-2} + (J_{2n+4} - aJ_{2n+3})\, y_{-1}x_{-2} + J_{2n+3}x_0y_{-1}x_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\, x_{-2} + (J_{2n+3} - aJ_{2n+2})\, y_{-1}x_{-2} + J_{2n+2}x_0y_{-1}x_{-2}},$$

$$y_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\, y_{-2} + (J_{2n+5} - aJ_{2n+4})\, x_{-1}y_{-2} + J_{2n+4}y_0x_{-1}y_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, y_{-2} + (J_{2n+4} - aJ_{2n+3})\, x_{-1}y_{-2} + J_{2n+3}y_0x_{-1}y_{-2}}.$$

where the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}$ and $y_0 \in (\mathbb{R} - \{0\}) - F$, with F is the Forbidden set of system (1.41) given by

$$F = \bigcup_{n=0}^{\infty} \{(x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0) \in (\mathbb{R} - \{0\}) : A_n = 0 \; or \, B_n = 0\},$$

where

$$A_n = dJ_{n+1} + (cJ_{n+1} + dJ_n)\, y_{-2} + (J_{n+3} - aJ_{n+2})\, x_{-1}y_{-2} + J_{n+2}y_0x_{-1}y_{-2},$$

$$B_n = dJ_{n+1} + (cJ_{n+1} + dJ_n)\, x_{-2} + (J_{n+3} - aJ_{n+2})\, y_{-1}x_{-2} + J_{n+2}x_0y_{-1}x_{-2}.$$

1.3.1.2 Particular cases

Now, we focus our study on some particular cases of system (1.41).

1.3.1.2.1 The solutions of the equation $x_{n+1} = \frac{ax_{n-2}x_{n-1}x_n + bx_{n-1}x_{n-2} + cx_{n-2} + d}{x_{n-2}x_{n-1}x_n}$ If we choose $y_{-2} = x_{-2}$, $y_{-1} = x_{-1}$ and $y_0 = x_0$, then system (1.41) is reduced to the equation

$$x_{n+1} = \frac{ax_{n-2}x_{n-1}x_n + bx_{n-1}x_{n-2} + cx_{n-2} + d}{x_{n-2}x_{n-1}x_n}, \quad n \in \mathbb{N}_0. \tag{1.56}$$

So, it follows from the Main Theorem that

Corollary 1.17. *Let* $\{x_n\}_{n\geq -1}$ *be a well defined solution of the equation* (1.56). *Then for* $n \in \mathbb{N}_0$, *we have*

$$x_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, x_{-2} + (J_{2n+4} - aJ_{2n+3})\, x_{-1}x_{-2} + J_{2n+3}x_0x_{-1}x_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\, x_{-2} + (J_{2n+3} - aJ_{2n+2})\, x_{-1}x_{-2} + J_{2n+2}x_0x_{-1}x_{-2}},$$

$$x_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\, x_{-2} + (J_{2n+5} - aJ_{2n+4})\, x_{-1}x_{-2} + J_{2n+4}x_0x_{-1}x_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, x_{-2} + (J_{2n+4} - aJ_{2n+3})\, x_{-1}x_{-2} + J_{2n+3}x_0x_{-1}x_{-2}},$$

Noting that this equation was studied in Azizi in [7] and Stevic in [95].

1.3.1.2.2 The solutions of the system (1.41) **with** $a = b = c = d = 1$ Consider the system

$$\begin{cases} x_{n+1} = \dfrac{y_{n-2}x_{n-1}y_n + x_{n-1}y_{n-2} + y_{n-2} + 1}{y_{n-2}x_{n-1}y_n}, \\ y_{n+1} = \dfrac{x_{n-2}y_{n-1}x_n + y_{n-1}x_{n-2} + x_{n-2} + 1}{x_{n-2}y_{n-1}x_n}, \quad n \in \mathbb{N}_0, \end{cases} \tag{1.57}$$

which is a is particular case of the system (1.41) with $a = b = c = d = 1$. In this case the sequence $\{J_n\}$ is nothing other than the sequence of Tetranacci numbers $\{T_n\}$, that is

$$T_{n+4} = T_{n+3} + T_{n+2} + T_{n+1} + T_n,\ n \in \mathbb{N}_0, \quad \text{where } T_0 = T_1 = 0,\ T_2 = 1 \text{ and } T_3 = 1,$$

and we have

$$T_n = \frac{-\alpha}{(\overline{\gamma} - \alpha)(\gamma - \alpha)(\beta - \alpha)}\alpha^n + \frac{\beta}{(\overline{\gamma} - \beta)(\gamma - \beta)(\beta - \alpha)}\beta^n + \frac{-\gamma}{(\overline{\gamma} - \gamma)(\gamma - \beta)(\gamma - \alpha)}\gamma^n + \frac{\overline{\gamma}}{(\overline{\gamma} - \gamma)(\overline{\gamma} - \beta)(\overline{\gamma} - \alpha)}\overline{\gamma}^n, \qquad n \in \mathbb{N}_0,$$

with

$$\alpha = \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},\ \beta = \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},$$

$$\gamma = \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},\ \delta = \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},$$

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{\frac{1}{3}} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{\frac{1}{3}}}.$$

Numerically we have $\alpha = 1.927561975$, $\beta = -0.774804113$ and the two complex conjugate are $\gamma = -0.076378931 + 0.814703647i$, $\delta = \bar{\gamma}$ with $i^2 = -1$.

The one dimensional version of the system (1.57), is the equation

$$x_{n+1} = \frac{x_{n-2}x_{n-1}x_n + x_{n-1}x_{n-2} + x_{n-2} + 1}{x_{n-2}x_{n-1}x_n},\quad n \in \mathbb{N}_0. \tag{1.58}$$

The following results follows respectively from the Main Theorem.

Corollary 1.18. *Let $\{x_n, y_n\}_{n\geq -1}$ be a well defined solution of (1.57). Then, for $n \in \mathbb{N}_0$, we have*

$$x_{2n+1} = \frac{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,y_{-2} + (T_{2n+4} - T_{2n+3})\,x_{-1}y_{-2} + T_{2n+3}y_0x_{-1}y_{-2}}{T_{2n+1} + (T_{2n+1} + T_{2n})\,y_{-2} + (T_{2n+3} - T_{2n+2})\,x_{-1}y_{-2} + T_{2n+2}y_0x_{-1}y_{-2}},$$

$$x_{2n+2} = \frac{T_{2n+3} + (T_{2n+3} + T_{2n+2})\,x_{-2} + (T_{2n+5} - T_{2n+4})\,y_{-1}x_{-2} + T_{2n+4}x_0y_{-1}x_{-2}}{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,x_{-2} + (T_{2n+4} - T_{2n+3})\,y_{-1}x_{-2} + T_{2n+3}x_0y_{-1}x_{-2}},$$

$$y_{2n+1} = \frac{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,x_{-2} + (T_{2n+4} - T_{2n+3})\,y_{-1}x_{-2} + T_{2n+3}x_0y_{-1}x_{-2}}{T_{2n+1} + (T_{2n+1} + T_{2n})\,x_{-2} + (T_{2n+3} - T_{2n+2})\,y_{-1}x_{-2} + T_{2n+2}x_0y_{-1}x_{-2}},$$

$$y_{2n+2} = \frac{T_{2n+3} + (T_{2n+3} + T_{2n+2})\,y_{-2} + (T_{2n+5} - T_{2n+4})\,x_{-1}y_{-2} + T_{2n+4}y_0x_{-1}y_{-2}}{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,y_{-2} + (T_{2n+4} - T_{2n+3})\,x_{-1}y_{-2} + T_{2n+3}y_0x_{-1}y_{-2}}.$$

Corollary 1.19. *Let $\{x_n\}_{n\geq -1}$ be a well defined solution of the equation (1.58). Then for $n \in \mathbb{N}_0$, we have*

$$x_{2n+1} = \frac{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,x_{-2} + (T_{2n+4} - T_{2n+3})\,x_{-1}x_{-2} + T_{2n+3}x_0x_{-1}x_{-2}}{T_{2n+1} + (T_{2n+1} + T_{2n})\,x_{-2} + (T_{2n+3} - T_{2n+2})\,x_{-1}x_{-2} + T_{2n+2}x_0x_{-1}x_{-2}},$$

$$x_{2n+2} = \frac{T_{2n+3} + (T_{2n+3} + T_{2n+2})\,x_{-2} + (T_{2n+5} - T_{2n+4})\,x_{-1}x_{-2} + T_{2n+4}x_0x_{-1}x_{-2}}{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,x_{-2} + (T_{2n+4} - T_{2n+3})\,x_{-1}x_{-2} + T_{2n+3}x_0x_{-1}x_{-2}},$$

Remark 1.3.1. When $a = d = 0$, the system (1.41) takes the form

$$x_{n+1} = \frac{bx_{n-1} + c}{y_nx_{n-1}},\ y_{n+1} = \frac{by_{n-1} + c}{x_ny_{n-1}}\quad n \in \mathbb{N}_0. \tag{1.59}$$

As it is noted in [2], the solutions are expressed using Padovan numbers. This system and same particular cases of it has been the subject of the papers [44, 124].

If $d = c = 0$, The system (1.41) become

$$x_{n+1} = \frac{ay_n + b}{y_n},\ y_{n+1} = \frac{ax_n + b}{x_n},\ n \in \mathbb{N}_0. \tag{1.60}$$

Again it is noted in [2] that:

- The system (1.60) is a particular case of the more general system

$$x_{n+1} = \frac{ay_n + b}{cy_n + d},\ y_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \lambda},\ n \in \mathbb{N}_0 \tag{1.61}$$

which was been completely solved by Stevic in [91] and the solutions are expressed using a generalized Fibonacci sequence.

- Also, particular cases of system (1.61) has been studied in [63, 40, 105, 104].

- If also $b = 0$, then the solutions of the system (1.60) are given by

$$\{(x_0, y_0), (a, a), (a, a), ..., \}.$$

1.3.2 Proof of the Main result

In order to solve the system (1.41), we need firstly to solve the following two homogeneous forth order linear difference equations

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2} + dR_{n-3},\ n \in \mathbb{N}_0, \tag{1.62}$$

$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2} + dS_{n-3},\ n \in \mathbb{N}_0, \tag{1.63}$$

where the initial values $R_0, R_{-1}, R_{-2}, R_{-3}, S_0, S_{-1}, S_{-2}$ and S_{-3} and the constant coefficients a, b, c and d are real numbers with $d \neq 0$. In fact we will express the terms of the sequences $(R_n)_{n=-3}^{+\infty}$ and $(S_n)_{n=-3}^{+\infty}$ using the sequence $(J_n)_{n=0}^{+\infty}$.

The difference equation (1.62) has the same characteristic equation as $(J_n)_{n=0}^{+\infty}$, that is the equation (1.45).

To solve the difference equation (1.63) using terms of (1.43), we need the following fourth order linear difference equation defined by

$$j_{n+4} = -aj_{n+3} + bj_{n+2} - cj_{n+1} + dj_n, \quad n \in \mathbb{N}_0, \tag{1.64}$$

and the special initial values

$$j_0 = 0, \quad j_1 = 0, \quad j_2 = 1 \text{ and } j_3 = -a \tag{1.65}$$

The characteristic equation of (1.63) and (1.64) is

$$\lambda^4 + a\lambda^3 - b\lambda^2 + c\lambda - d = 0. \tag{1.66}$$

Clearly the roots of (1.66) are $-\alpha$, $-\beta$, $-\gamma$ and $-\delta$. Now following the same procedure in solving $\{J_n\}$, it is not hard to see that

$$j_n = (-1)^n J_n.$$

Now, we are able to prove the following result.

Lemma 1.20. *We have for all $n \in \mathbb{N}_0$,*

$$R_n = dJ_{n+1}R_{-3} + (cJ_{n+1} + dJ_n)\, R_{-2} + (J_{n+3} - aJ_{n+2})\, R_{-1} + J_{n+2}R_0 \tag{1.67}$$

$$S_n = (-1)^{n+1}\left[dJ_{n+1}S_{-3} - (cJ_{n+1} + dJ_n)\, S_{-2} + (J_{n+3} - aJ_{n+2})\, S_{-1} - J_{n+2}S_0\right]. \tag{1.68}$$

Proof. Assume that α, β, γ and δ are the distinct roots of the characteristic equation (1.45), so

$$R_n = c'_1\alpha^n + c'_2\beta^n + c'_3\gamma^n + c'_4\delta^n,\ n \geq -3.$$

Using the initial values R_0, R_{-1}, R_{-2} and R_{-3}, we get

$$\begin{cases} \frac{1}{\alpha^3}c'_1 + \frac{1}{\beta^3}c'_2 + \frac{1}{\gamma^3}c'_3 + \frac{1}{\delta^3}c'_4 & = R_{-3} \\ \frac{1}{\alpha^2}c'_1 + \frac{1}{\beta^2}c'_2 + \frac{1}{\gamma^2}c'_3 + \frac{1}{\delta^2}c'_4 & = R_{-2} \\ \frac{1}{\alpha}c'_1 + \frac{1}{\beta}c'_2 + \frac{1}{\gamma}c'_3 + \frac{1}{\delta}c'_4 & = R_{-1} \\ c'_1 + c'_2 + c'_3 + c'_4 & = R_0 \end{cases} \tag{1.69}$$

after some calculations using the Cramer method we get

$$\begin{aligned}
c_1' &= \frac{\beta\gamma\delta\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_{-3} - \frac{(\gamma\beta+\gamma\delta+\beta\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_{-2} \\
&\quad + \frac{(\beta+\gamma+\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_{-1} - \frac{\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_0 \\
c_2' &= -\frac{\alpha\gamma\delta\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_{-3} + \frac{(\gamma\alpha+\gamma\delta+\alpha\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_{-2} \\
&\quad - \frac{(\alpha+\gamma+\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_{-1} + \frac{\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_0 \\
c_3' &= \frac{\alpha\beta\delta\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_{-3} - \frac{(\alpha\beta+\alpha\delta+\beta\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_{-2} \\
&\quad + \frac{(\alpha+\beta+\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_{-1} - \frac{\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_0 \\
c_4' &= -\frac{\alpha\beta\gamma\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_{-3} + \frac{(\gamma\alpha+\gamma\beta+\alpha\beta)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_{-2} \\
&\quad - \frac{(\alpha+\beta+\gamma)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_{-1} + \frac{\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_0
\end{aligned}$$

that is,

$$\begin{aligned}
R_n &= \left(\frac{\beta\gamma\delta\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n - \frac{\alpha\gamma\delta\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{\alpha\beta\delta\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \right.\\
&\quad \left. - \frac{\alpha\beta\gamma\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n\right) R_{-3} \\
&\quad + \left(-\frac{(\gamma\beta+\gamma\delta+\beta\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{(\gamma\alpha+\gamma\delta+\alpha\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n - \frac{(\alpha\beta+\alpha\delta+\beta\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \right.\\
&\quad \left. + \frac{(\gamma\alpha+\gamma\beta+\alpha\beta)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n\right) R_{-2} \\
&\quad + \left(\frac{(\beta+\gamma+\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n - \frac{(\alpha+\gamma+\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{(\alpha+\beta+\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \right.\\
&\quad \left. - \frac{(\alpha+\beta+\gamma)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n\right) R_{-1} \\
&\quad + \left(-\frac{\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n - \frac{\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \right.\\
&\quad \left. + \frac{\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n\right) R_0.
\end{aligned}$$

$$R_n = dJ_{n+1}R_{-3} + (cJ_{n+1} + dJ_n)\,R_{-2} + (J_{n+3} - aJ_{n+2})\,R_{-1} + J_{n+2}R_0.$$

The proof of the other cases is similar and will be omitted.

Let $A := -a$, $B := b$, $C := -c$ and $D := d$ then equation (1.63) takes the form of (1.62) and the equation (1.64) takes the form of (1.43). Then analogous to the formula of (1.62) we obtain

$$S_n = Dj_{n+1}S_{-3} + (Cj_{n+1} + Dj_n)\, S_{-2} + (j_{n+3} - Aj_{n+2})\, S_{-1} + j_{n+2}S_0.$$

Using the fact that $j_n = (-1)^n J_n$, $A = -a$ and $C := -c$ we get

$$S_n = (-1)^{n+1} \left[dJ_{n+1}S_{-3} - (cJ_{n+1} + dJ_n)\, S_{-2} + (J_{n+3} - aJ_{n+2})\, S_{-1} - J_{n+2}S_0\right].$$

□

Proof of the Main Theorem.

Putting

$$x_n = \frac{u_n}{v_{n-1}}, \quad y_n = \frac{v_n}{u_{n-1}}, \; n \geq -2. \tag{1.70}$$

we get the following linear forth order system of difference equations

$$u_{n+1} = av_n + bu_{n-1} + cv_{n-2} + du_{n-3}, \quad v_{n+1} = au_n + bv_{n-1} + cu_{n-2} + dv_{n-3}, \quad n \in \mathbb{N}_0, \tag{1.71}$$

where the initial values $u_{-3}, u_{-2}, u_{-1}, u_0, v_{-3}, v_{-2}, v_{-1}, v_0$ are nonzero real numbers.
From(1.71) we have for $n \in \mathbb{N}_0$,

$$\begin{cases} u_{n+1} + v_{n+1} = a(v_n + u_n) + b(u_{n-1} + v_{n-1}) + c(v_{n-2} + u_{n-2}) + d(u_{n-3} + v_{n-3}), \\ u_{n+1} - v_{n+1} = a(v_n - u_n) + b(u_{n-1} - v_{n-1}) + c(v_{n-2} - u_{n-2}) + d(u_{n-3} - v_{n-3}). \end{cases}$$

Putting again

$$R_n = u_n + v_n, \quad S_n = u_n - v_n, \; n \geq -2, \tag{1.72}$$

we obtain two homogeneous linear difference equations of forth order:

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2} + dR_{n-3}, \; n \in \mathbb{N}_0,$$

and

$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2} + dS_{n-3}, \; n \in \mathbb{N}_0. \tag{1.73}$$

Using (1.72), we get for $n \geq -3$,

$$u_n = \frac{1}{2}(R_n + S_n), \; v_n = \frac{1}{2}(R_n - S_n).$$

From Lemma 1.20 we obtain,

$$\begin{cases} u_{2n-1} = & \frac{1}{2}\left[dJ_{2n}(R_{-3}+S_{-3})+(cJ_{2n}+dJ_{2n-1})(R_{-2}-S_{-2})+(J_{2n+2}-aJ_{2n+1})(R_{-1}+S_{-1})\right. \\ & \left.+J_{2n+1}(R_0-S_0)\right], n\in\mathbb{N}, \\ u_{2n} = & \frac{1}{2}\left[dJ_{2n+1}(R_{-3}-S_{-3})+(cJ_{2n+1}+dJ_{2n})(R_{-2}+S_{-2})+(J_{2n+3}-aJ_{2n+2})(R_{-1}-S_{-1})\right. \\ & \left.+J_{2n+2}(R_0+S_0)\right], n\in\mathbb{N}_0, \end{cases} \tag{1.74}$$

$$\begin{cases} v_{2n-1} = & \frac{1}{2}\left[dJ_{2n}(R_{-3}-S_{-3})+(cJ_{2n}+dJ_{2n-1})(R_{-2}+S_{-2})+(J_{2n+2}-aJ_{2n+1})(R_{-1}-S_{-1})\right. \\ & \left.+J_{2n+1}(R_0+S_0)\right], n\in\mathbb{N}, \\ v_{2n} = & \frac{1}{2}\left[dJ_{2n+1}(R_{-3}+S_{-3})+(cJ_{2n+1}+dJ_{2n})(R_{-2}-S_{-2})+(J_{2n+3}-aJ_{2n+2})(R_{-1}+S_{-1})\right. \\ & \left.+J_{2n+2}(R_0-S_0)\right], n\in\mathbb{N}_0, \end{cases} \tag{1.75}$$

so

$$u_{2n-1} = dJ_{2n}u_{-3}+(cJ_{2n}+dJ_{2n-1})v_{-2}+(J_{2n+2}-aJ_{2n+1})u_{-1}+J_{2n+1}v_0,\ n\in\mathbb{N}, \tag{1.76}$$

$$u_{2n} = dJ_{2n+1}v_{-3}+(cJ_{2n+1}+dJ_{2n})u_{-2}+(J_{2n+3}-aJ_{2n+2})v_{-1}+J_{2n+2}u_0,\ n\in\mathbb{N}_0, \tag{1.77}$$

$$v_{2n-1} = dJ_{2n}v_{-3}+(cJ_{2n}+dJ_{2n-1})u_{-2}+(J_{2n+2}-aJ_{2n+1})v_{-1}+J_{2n+1}u_0,\ n\in\mathbb{N}, \tag{1.78}$$

$$v_{2n} = dJ_{2n+1}u_{-3}+(cJ_{2n+1}+dJ_{2n})v_{-2}+(J_{2n+3}-aJ_{2n+2})u_{-1}+J_{2n+2}v_0,\ n\in\mathbb{N}_0. \tag{1.79}$$

Substituting (1.74) and (1.75) in (1.70), we get for $n\in\mathbb{N}_0$,

$$x_{2n+1} = \frac{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\frac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}+(J_{2n+4}-aJ_{2n+3})\frac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}}+J_{2n+3}\frac{R_0-S_0}{R_{-3}+S_{-3}}}{dJ_{2n+1}+(cJ_{2n+1}+dJ_{2n})\frac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}+(J_{2n+3}-aJ_{2n+2})\frac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}}+J_{2n+2}\frac{R_0-S_0}{R_{-3}+S_{-3}}}, \tag{1.80}$$

$$x_{2n+2} = \frac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})\frac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}+(J_{2n+5}-aJ_{2n+4})\frac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}}+J_{2n+4}\frac{R_0+S_0}{R_{-3}-S_{-3}}}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\frac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}+(J_{2n+4}-aJ_{2n+3})\frac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}}+J_{2n+3}\frac{R_0+S_0}{R_{-3}-S_{-3}}}, \tag{1.81}$$

$$y_{2n+1}=\frac{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}+(J_{2n+4}-aJ_{2n+3})\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}}+J_{2n+3}\dfrac{R_0+S_0}{R_{-3}-S_{-3}}}{dJ_{2n+1}+(cJ_{2n+1}+dJ_{2n})\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}+(J_{2n+3}-aJ_{2n+2})\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}}+J_{2n+2}\dfrac{R_0+S_0}{R_{-3}-S_{-3}}}, \tag{1.82}$$

and

$$y_{2n+2}=\frac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})\dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}+(J_{2n+5}-aJ_{2n+4})\dfrac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}}+J_{2n+4}\dfrac{R_0-S_0}{R_{-3}+S_{-3}}}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}+(J_{2n+4}-aJ_{2n+3})\dfrac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}}+J_{2n+3}\dfrac{R_0-S_0}{R_{-3}+S_{-3}}}. \tag{1.83}$$

We have

$$x_{-2}=\frac{u_{-2}}{v_{-3}}=\frac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}},\ x_{-1}=\frac{u_{-1}}{v_{-2}}=\frac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}},\ x_0=\frac{u_0}{v_{-1}}=\frac{R_0+S_0}{R_{-1}-S_{-1}}, \tag{1.84}$$

$$y_{-2}=\frac{v_{-2}}{u_{-3}}=\frac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}},\ y_{-1}=\frac{v_{-1}}{u_{-2}}=\frac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}},\ y_0=\frac{v_0}{u_{-1}}=\frac{R_0-S_0}{R_{-1}+S_{-1}} \tag{1.85}$$

From (1.84), (1.85) we get,

$$\begin{cases}\dfrac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}}=\dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}}\times\dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}=x_{-1}y_{-2}\\ \dfrac{R_0-S_0}{R_{-3}+S_{-3}}=\dfrac{R_0-S_0}{R_{-1}+S_{-1}}\times\dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}}\times\dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}=y_0x_{-1}y_{-2}\end{cases} \tag{1.86}$$

$$\begin{cases}\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}}=\dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}}\times\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}=y_{-1}x_{-2}\\ \dfrac{R_0+S_0}{R_{-3}-S_{-3}}=\dfrac{R_0+S_0}{R_{-1}-S_{-1}}\times\dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}}\times\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}=x_0y_{-1}x_{-2}\end{cases} \tag{1.87}$$

Using (1.80), (1.81), (1.82), (1.83), (1.86) and (1.87), we obtain the closed form of the solutions of the system (1.41), that is for $n\in\mathbb{N}_0$, we have

$$\begin{cases}x_{2n+1}=\dfrac{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,y_{-2}+(J_{2n+4}-aJ_{2n+3})\,x_{-1}y_{-2}+J_{2n+3}y_0x_{-1}y_{-2}}{dJ_{2n+1}+(cJ_{2n+1}+dJ_{2n})\,y_{-2}+(J_{2n+3}-aJ_{2n+2})\,x_{-1}y_{-2}+J_{2n+2}y_0x_{-1}y_{-2}},\\ x_{2n+2}=\dfrac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})\,x_{-2}+(J_{2n+5}-aJ_{2n+4})\,y_{-1}x_{-2}+J_{2n+4}x_0y_{-1}x_{-2}}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,x_{-2}+(J_{2n+4}-aJ_{2n+3})\,y_{-1}x_{-2}+J_{2n+3}x_0y_{-1}x_{-2}},\end{cases}$$

$$\begin{cases}y_{2n+1}=\dfrac{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,x_{-2}+(J_{2n+4}-aJ_{2n+3})\,y_{-1}x_{-2}+J_{2n+3}x_0y_{-1}x_{-2}}{dJ_{2n+1}+(cJ_{2n+1}+dJ_{2n})\,x_{-2}+(J_{2n+3}-aJ_{2n+2})\,y_{-1}x_{-2}+J_{2n+2}x_0y_{-1}x_{-2}},\\ y_{2n+2}=\dfrac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})\,y_{-2}+(J_{2n+5}-aJ_{2n+4})\,x_{-1}y_{-2}+J_{2n+4}y_0x_{-1}y_{-2}}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,y_{-2}+(J_{2n+4}-aJ_{2n+3})\,x_{-1}y_{-2}+J_{2n+3}y_0x_{-1}y_{-2}}.\end{cases}$$

1.4 Representations of solutions to two general classes of nonlinear systems of difference equations

In the present section, we continue our interest in solvable difference equations, more precisely, we will solve the following two general systems of difference equations

$$x_{n+1} = f^{-1}\left(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3})\right),$$

$$y_{n+1} = g^{-1}\left(af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3})\right),$$

and

$$x_{n+1} = f^{-1}\left(a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})} + \frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})}\right),$$

$$y_{n+1} = g^{-1}\left(a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})} + \frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})}\right),$$

where $n \in \mathbb{N}_0$, $f, g : D \longrightarrow \mathbb{R}$ are one to one ("$1-1$") continuous functions on $D \subseteq \mathbb{R}$, the initial values x_{-i}, y_{-i}, $i = 0, 1, 2, 3$ are arbitrary real numbers in D and the parameters a, b, c and d are arbitrary real numbers.

In our study, we are inspired and motivated by the ideas, the equations and the systems of some recent published papers. The papers, [2, 3] and especially [96] are our main motivation in the present work. The obtained results considerably generalize some existing results in the literature, see [2, 3, 7, 8, 44, 65, 66, 67, 92, 96, 95, 104, 105, 124].

1.4.1 First class of systems

In this part, we will focus our interest on our first general system of difference equations, that is the system

$$\begin{cases} x_{n+1} = f^{-1}\left(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3})\right), \\ y_{n+1} = g^{-1}\left(af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3})\right), \end{cases} \tag{1.88}$$

where $n \in \mathbb{N}_0$, f, $g : D \longrightarrow \mathbb{R}$ are continuous functions, with $D = D_f = D_g$, that is f and g have the same domain, and it is also assumed that f, g are "$1-1$" on $D \subseteq \mathbb{R}$, the initial values x_{-i}, y_{-i}, $i = 0, 1, 2, 3$ are arbitrary real numbers in D and the parameters a, b, c and d are arbitrary real numbers.

1.4.1.1 Explicit formulas of solutions of system (1.88) with $d \neq 0$

In the following result, we solve in closed form (1.88).

Definition 1.2. *A solution $\{x_n, y_n\}_{n\geq -3}$ of system (1.88), is said to be well-defined if for all $n \in \mathbb{N}_0$, we have*

$$ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3}) \in D_{f^{-1}},$$

and

$$af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3}) \in D_{g^{-1}}.$$

Theorem 1.21. *Let $\{x_n, y_n\}_{n\geq -3}$ be a well-defined solution of the system (1.88), then we have the following representation*

$$\begin{cases} x_{2n-1} &= f^{-1}\left(dJ_{2n}f(x_{-3}) + \left(cJ_{2n} + dJ_{2n-1}\right)g(y_{-2}) + \left(J_{2n+2} - aJ_{2n+1}\right)f(x_{-1})\right. \\ &\left.+J_{2n+1}g(y_0)\right), n \in \mathbb{N}, \\ x_{2n} &= f^{-1}\left(dJ_{2n+1}g(y_{-3}) + \left(cJ_{2n+1} + dJ_{2n}\right)f(x_{-2}) + \left(J_{2n+3} - aJ_{2n+2}\right)g(y_{-1})\right. \\ &\left.+J_{2n+2}f(x_0)\right), n \in \mathbb{N}_0, \end{cases} \tag{1.89}$$

$$\begin{cases} y_{2n-1} &= g^{-1}\left(dJ_{2n}g(y_{-3}) + \left(cJ_{2n} + dJ_{2n-1}\right)f(x_{-2}) + \left(J_{2n+2} - aJ_{2n+1}\right)g(y_{-1})\right. \\ &\left.+J_{2n+1}f(x_0)\right), n \in \mathbb{N}, \\ y_{2n} &= g^{-1}\left(dJ_{2n+1}f(x_{-3}) + \left(cJ_{2n+1} + dJ_{2n}\right)g(y_{-2}) + \left(J_{2n+3} - aJ_{2n+2}\right)f(x_{-1})\right. \\ &\left.+J_{2n+2}g(y_0)\right), n \in \mathbb{N}_0. \end{cases} \tag{1.90}$$

Proof. Since the functions f, g are "$1-1$", then from (1.88) we get

$$\begin{cases} f(x_{n+1}) = ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3}), \\ g(y_{n+1}) = af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3}), \quad n \in \mathbb{N}_0. \end{cases} \tag{1.91}$$

By the change of variables

$$X_n = f(x_n), \qquad Y_n = g(y_n),\ n \geq -3, \tag{1.92}$$

system (1.91) is transformed to the following one

$$\begin{cases} X_{n+1} = aY_n + bX_{n-1} + cY_{n-2} + dX_{n-3}, \\ Y_{n+1} = aX_n + bY_{n-1} + cX_{n-2} + dY_{n-3}, \quad n \in \mathbb{N}_0. \end{cases} \tag{1.93}$$

Clearly (1.93) is in the form of system (1.71), by (1.76), (1.77), (1.78) and (1.79), we obtain the following representation of solutions

$$\begin{cases} X_{2n-1} = dJ_{2n}X_{-3} + (cJ_{2n} + dJ_{2n-1})\,Y_{-2} + (J_{2n+2} - aJ_{2n+1})\,X_{-1} + J_{2n+1}Y_0, n \in \mathbb{N}, \\ X_{2n} = dJ_{2n+1}Y_{-3} + (cJ_{2n+1} + dJ_{2n})\,X_{-2} + (J_{2n+3} - aJ_{2n+2})\,Y_{-1} + J_{2n+2}X_0, n \in \mathbb{N}_0, \end{cases} \tag{1.94}$$

$$\begin{cases} Y_{2n-1} = dJ_{2n}Y_{-3} + (cJ_{2n} + dJ_{2n-1})\,X_{-2} + (J_{2n+2} - aJ_{2n+1})\,Y_{-1} + J_{2n+1}X_0, n \in \mathbb{N}, \\ Y_{2n} = dJ_{2n+1}X_{-3} + (cJ_{2n+1} + dJ_{2n})\,Y_{-2} + (J_{2n+3} - aJ_{2n+2})\,X_{-1} + J_{2n+2}Y_0, n \in \mathbb{N}_0. \end{cases} \tag{1.95}$$

Now, by (1.92) we get that

$$\begin{cases} x_{2n-1} &= f^{-1}\left[dJ_{2n}f(x_{-3}) + (cJ_{2n} + dJ_{2n-1})\,g(y_{-2}) + (J_{2n+2} - aJ_{2n+1})\,f(x_{-1})\right. \\ &\left. +J_{2n+1}g(y_0)\right], n \in \mathbb{N}, \\ x_{2n} &= f^{-1}\left[dJ_{2n+1}g(y_{-3}) + (cJ_{2n+1} + dJ_{2n})\,f(x_{-2}) + (J_{2n+3} - aJ_{2n+2})\,g(y_{-1})\right. \\ &\left. +J_{2n+2}f(x_0)\right], n \in \mathbb{N}_0, \end{cases} \tag{1.96}$$

$$\begin{cases} y_{2n-1} &= g^{-1}\left[dJ_{2n}g(y_{-3}) + (cJ_{2n} + dJ_{2n-1})\,f(x_{-2}) + (J_{2n+2} - aJ_{2n+1})\,g(y_{-1})\right. \\ &\left. +J_{2n+1}f(x_0)\right], n \in \mathbb{N}, \\ y_{2n} &= g^{-1}\left[dJ_{2n+1}f(x_{-3}) + (cJ_{2n+1} + dJ_{2n})\,g(y_{-2}) + (J_{2n+3} - aJ_{2n+2})\,f(x_{-1})\right. \\ &\left. +J_{2n+2}g(y_0)\right], n \in \mathbb{N}_0. \end{cases} \tag{1.97}$$

□

Remark 1.4.1. Moreover, if $g \equiv f$ and $y_{-i} = x_{-i}$, $i = \overline{0,3}$ then, the system (1.88) will be the equation

$$x_{n+1} = f^{-1}\left(af(x_n) + bf(x_{n-1}) + cf(x_{n-2}) + df(x_{n-3})\right) \tag{1.98}$$

and then the representation of the well-defined solutions are given by

$$\begin{cases} x_{2n-1} &= f^{-1}\left[dJ_{2n}f(x_{-3}) + (cJ_{2n} + dJ_{2n-1})\,f(x_{-2}) + (J_{2n+2} - aJ_{2n+1})\,f(x_{-1})\right. \\ &\left. +J_{2n+1}f(x_0)\right], n \in \mathbb{N}, \\ x_{2n} &= f^{-1}\left[dJ_{2n+1}f(x_{-3}) + (cJ_{2n+1} + dJ_{2n})\,f(x_{-2}) + (J_{2n+3} - aJ_{2n+2})\,f(x_{-1})\right. \\ &\left. +J_{2n+2}f(x_0)\right], n \in \mathbb{N}_0. \end{cases} \tag{1.99}$$

In [96], Stevic studied the equation (1.98).

Now as applications of Theorem 1.21, we give the following examples.

Example 1.4.1. Let

$$f(t)=t^{2j+1},\ g(t)=t^{2k+1},\quad j,\,k\in\mathbb{N}_0. \tag{1.100}$$

Then, $D_f=D_g=\mathbb{R}$, clearly the functions f and g are "$1-1$" continuous functions on $\mathbb{R}$ and the system (1.88) becomes

$$\begin{cases} x_{n+1}=\left[ay_n^{2k+1}+bx_{n-1}^{2j+1}+cy_{n-2}^{2k+1}+dx_{n-3}^{2j+1}\right]^{\frac{1}{2j+1}}, \\ y_{n+1}=\left[ax_n^{2j+1}+by_{n-1}^{2k+1}+cx_{n-2}^{2j+1}+dy_{n-3}^{2k+1}\right]^{\frac{1}{2k+1}}, \quad n\in\mathbb{N}_0. \end{cases} \tag{1.101}$$

Then from (1.89) and (1.90), we obtain that general solution of the equation (1.101) is

$$\begin{cases} x_{2n-1} & =\Big[dJ_{2n}x_{-3}^{2j+1}+\left(cJ_{2n}+dJ_{2n-1}\right)y_{-2}^{2k+1}+\left(J_{2n+2}-aJ_{2n+1}\right)x_{-1}^{2j+1} \\ & +J_{2n+1}y_0^{2k+1}\Big]^{\frac{1}{2j+1}},n\in\mathbb{N}, \\ x_{2n} & =\Big[dJ_{2n+1}y_{-3}^{2k+1}+\left(cJ_{2n+1}+dJ_{2n}\right)x_{-2}^{2j+1}+\left(J_{2n+3}-aJ_{2n+2}\right)y_{-1}^{2k+1} \\ & +J_{2n+2}x_0^{2j+1}\Big]^{\frac{1}{2j+1}},n\in\mathbb{N}_0, \end{cases} \tag{1.102}$$

$$\begin{cases} y_{2n-1} & =\Big[dJ_{2n}y_{-3}^{2k+1}+\left(cJ_{2n}+dJ_{2n-1}\right)x_{-2}^{2j+1}+\left(J_{2n+2}-aJ_{2n+1}\right)y_{-1}^{2k+1} \\ & +J_{2n+1}x_0^{2j+1}\Big]^{\frac{1}{2k+1}},n\in\mathbb{N}, \\ y_{2n} & =\Big[dJ_{2n+1}x_{-3}^{2j+1}+\left(cJ_{2n+1}+dJ_{2n}\right)y_{-2}^{2k+1}+\left(J_{2n+3}-aJ_{2n+2}\right)x_{-1}^{2j+1} \\ & +J_{2n+2}y_0^{2k+1}\Big]^{\frac{1}{2k+1}},n\in\mathbb{N}_0. \end{cases} \tag{1.103}$$

Example 1.4.2. Let

$$f(t)=\frac{1}{t^{2j+1}},\ g(t)=\frac{1}{t^{2k+1}},\quad j,\,k\in\mathbb{N}_0. \tag{1.104}$$

Then, $D_f=D_g=\mathbb{R}-\{0\}$, clearly the functions f and g are "$1-1$" continuous functions on $\mathbb{R}-\{0\}$ and the system (1.88) becomes

$$\begin{cases} x_{n+1}=\left[\frac{a}{y_n^{2k+1}}+\frac{b}{x_{n-1}^{2j+1}}+\frac{c}{y_{n-2}^{2k+1}}+\frac{d}{x_{n-3}^{2j+1}}\right]^{\frac{-1}{2j+1}}, \\ y_{n+1}=\left[\frac{a}{x_n^{2j+1}}+\frac{b}{y_{n-1}^{2k+1}}+\frac{c}{x_{n-2}^{2j+1}}+\frac{d}{y_{n-3}^{2k+1}}\right]^{\frac{-1}{2k+1}}, \quad n\in\mathbb{N}_0, \end{cases} \tag{1.105}$$

or equivalently

$$\begin{cases} x_{n+1}=\left[\frac{y_n^{2k+1}x_{n-1}^{2j+1}y_{n-2}^{2k+1}x_{n-3}^{2j+1}}{ax_{n-1}^{2j+1}y_{n-2}^{2k+1}x_{n-3}^{2j+1}+by_n^{2k+1}y_{n-2}^{2k+1}x_{n-3}^{2j+1}+cy_n^{2k+1}x_{n-1}^{2j+1}x_{n-3}^{2j+1}+dy_n^{2k+1}x_{n-1}^{2j+1}y_{n-2}^{2k+1}}\right]^{\frac{1}{2j+1}}, \\ y_{n+1}=\left[\frac{x_n^{2j+1}y_{n-1}^{2k+1}x_{n-2}^{2j+1}y_{n-3}^{2k+1}}{ay_{n-1}^{2k+1}x_{n-2}^{2j+1}y_{n-3}^{2k+1}+bx_n^{2j+1}x_{n-2}^{2j+1}y_{n-3}^{2k+1}+cx_n^{2j+1}y_{n-1}^{2k+1}y_{n-3}^{2k+1}+dx_n^{2j+1}y_{n-1}^{2k+1}x_{n-2}^{2j+1}}\right]^{\frac{1}{2k+1}}, \quad n\in\mathbb{N}_0. \end{cases} \tag{1.106}$$

Then from (1.89) and (1.90), we obtain that general solution of system (1.106) is

$$\begin{cases} x_{2n-1} & = \Big[dJ_{2n}x_{-3}^{-(2j+1)} + (cJ_{2n} + dJ_{2n-1})\,y_{-2}^{-(2k+1)} + (J_{2n+2} - aJ_{2n+1})\,x_{-1}^{-(2j+1)} \\ & \quad +J_{2n+1}y_0^{-(2k+1)}\Big]^{-\frac{1}{2j+1}}, n \in \mathbb{N}, \\ x_{2n} & = \Big[dJ_{2n+1}y_{-3}^{-(2k+1)} + (cJ_{2n+1} + dJ_{2n})\,x_{-2}^{-(2j+1)} + (J_{2n+3} - aJ_{2n+2})\,y_{-1}^{-(2k+1)} \\ & \quad +J_{2n+2}x_0^{-(2j+1)}\Big]^{-\frac{1}{2j+1}}, n \in \mathbb{N}_0, \end{cases} \tag{1.107}$$

$$\begin{cases} y_{2n-1} & = \Big[dJ_{2n}y_{-3}^{-(2k+1)} + (cJ_{2n} + dJ_{2n-1})\,x_{-2}^{-(2j+1)} + (J_{2n+2} - aJ_{2n+1})\,y_{-1}^{-(2k+1)} \\ & \quad +J_{2n+1}x_0^{-(2j+1)}\Big]^{-\frac{1}{2k+1}}, n \in \mathbb{N}, \\ y_{2n} & = \Big[dJ_{2n+1}x_{-3}^{-(2j+1)} + (cJ_{2n+1} + dJ_{2n})\,y_{-2}^{-(2k+1)} + (J_{2n+3} - aJ_{2n+2})\,x_{-1}^{-(2j+1)} \\ & \quad +J_{2n+2}y_0^{-(2k+1)}\Big]^{-\frac{1}{2k+1}}, n \in \mathbb{N}_0. \end{cases} \tag{1.108}$$

If $j = k = 0$, then system (1.105) becomes

$$\begin{cases} x_{n+1} = \frac{y_n x_{n-1} y_{n-2} x_{n-3}}{a x_{n-1} y_{n-2} x_{n-3} + b y_n y_{n-2} x_{n-3} + c y_n x_{n-1} x_{n-3} + d y_n x_{n-1} y_{n-2}}, \\ y_{n+1} = \frac{x_n y_{n-1} x_{n-2} y_{n-3}}{a y_{n-1} x_{n-2} y_{n-3} + b x_n x_{n-2} y_{n-3} + c x_n y_{n-1} y_{n-3} + d x_n y_{n-1} x_{n-2}}, \quad n \in \mathbb{N}_0. \end{cases} \tag{1.109}$$

The form of the well-defined solutions of (1.109), can be obtained by putting $j = k = 0$, in the formulas of the solutions of (1.105). The solutions of the equation, see [96]

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-2} x_{n-3}}{a x_{n-1} x_{n-2} x_{n-3} + b x_n x_{n-2} x_{n-3} + c x_n x_{n-1} x_{n-3} + d x_n x_{n-1} x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.110}$$

can be obtained from the solutions of (1.109) by taking $y_{-i} = x_{-i}$, $i = 0, 1, 2, 3$.

1.4.1.2 Particular cases of system (1.88)

1.4.1.2.1 The case $d = 0$ and $c \neq 0$ In this case the system (1.88) takes the form:

$$\begin{cases} x_{n+1} = f^{-1}\left(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2})\right), \\ y_{n+1} = g^{-1}\left(af(x_n) + bg(y_{n-1}) + cf(x_{n-2})\right), \quad n \in \mathbb{N}_0. \end{cases} \tag{1.111}$$

Using the change of variables (1.92), with $n \geq -2$, we get the third order linear system

$$X_{n+1} = aY_n + bX_{n-1} + cY_{n-2},\ Y_{n+1} = aX_n + bY_{n-1} + cX_{n-2},\ n \geq -2. \tag{1.112}$$

Consider the sequence $\left(\widetilde{J}_n\right)_{n\geq 0}$ defined by

$$\widetilde{J}_{n+3} = a\widetilde{J}_{n+2} + b\widetilde{J}_{n+1} + c\widetilde{J}_n, \quad n \in \mathbb{N}_0, \tag{1.113}$$

and the special initial values

$$\widetilde{J}_0 = 0,\ \widetilde{J}_1 = 1,\ \widetilde{J}_2 = a.$$

The sequence $\left(\widetilde{J}_n\right)_{n\geq 0}$ is obtained from the sequence $(J_n)_{n\geq 0}$ defined by (1.43):

$$J_{n+4} = aJ_{n+3} + bJ_{n+2} + cJ_{n+1} + dJ_n,\ J_0 = 0, \quad J_1 = 0, \quad J_2 = 1 \text{ and } J_3 = a,\ n \in \mathbb{N}_0.$$

For $d = 0$, we obtain

$$J_{n+4} = aJ_{n+3} + bJ_{n+2} + cJ_{n+1}.$$

Putting

$$\widetilde{J}_n = J_{n+1},\ n \in \mathbb{N}_0,$$

we get the sequence (1.113). Noting that in this case, the corresponding sequences $(R_n)_{n\geq 0}$, $(S_n)_{n\geq 0}$ will be

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2},\ S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2},\ n \in \mathbb{N}_0,$$

with the initial values $R_0, R_{-1},\ R_{-2},\ S_0, S_{-1},\ S_{-2}$. The formulas of the solutions of these equations are expressed using the sequence $\left(\widetilde{J}_n\right)_{n\geq 0}$, see [3].
The formulas of the solutions of (1.112) and (1.111), can be obtaining from those of solutions of (1.71) and solutions of (1.88) by changing J_n by $\widetilde{J}_{n-1}$.

In summary we have the following result.

Corollary 1.22. *Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution of system (1.111), then*

$$x_{2n-1} = f^{-1}\left[c\widetilde{J}_{2n-1}g(y_{-2}) + \left(\widetilde{J}_{2n+1} - a\widetilde{J}_{2n}\right)f(x_{-1}) + \widetilde{J}_{2n}g(y_0)\right], n \in \mathbb{N},$$

$$x_{2n} = f^{-1}\left[c\widetilde{J}_{2n}f(x_{-2}) + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right)g(y_{-1}) + \widetilde{J}_{2n+1}f(x_0)\right], n \in \mathbb{N}_0,$$

$$y_{2n-1} = g^{-1}\left[c\widetilde{J}_{2n-1}f(x_{-2}) + \left(\widetilde{J}_{2n+1} - a\widetilde{J}_{2n}\right)g(y_{-1}) + \widetilde{J}_{2n}f(x_0)\right], n \in \mathbb{N},$$

$$y_{2n} = g^{-1}\left[c\widetilde{J}_{2n}g(y_{-2}) + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right)f(x_{-1}) + \widetilde{J}_{2n+1}g(y_0)\right], n \in \mathbb{N}_0.$$

Remark 1.4.2. If $g \equiv f$ and $y_{-i} = x_{-i}$, $i = 0, 1, 2$ then, system (1.111) becomes

$$x_{n+1} = f^{-1}\left[af(x_n) + bf(x_{n-1}) + cf(x_{n-2})\right] \tag{1.114}$$

and by Corollary 1.22, the every well defined solution is given by

$$\begin{cases} x_{2n-1} = f^{-1}\left[c\widetilde{J}_{2n-1}f(x_{-2}) + \left(\widetilde{J}_{2n+1} - a\widetilde{J}_{2n}\right)f(x_{-1}) + \widetilde{J}_{2n}f(x_0)\right], n \in \mathbb{N}, \\ x_{2n} = f^{-1}\left[c\widetilde{J}_{2n}f(x_{-2}) + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right)f(x_{-1}) + \widetilde{J}_{2n+1}f(x_0)\right], n \in \mathbb{N}_0, \end{cases} \tag{1.115}$$

which can written in a unified form as

$$x_n = f^{-1}\left[c\widetilde{J}_n f(x_{-2}) + \left(\widetilde{J}_{n+2} - a\widetilde{J}_{n+1}\right)f(x_{-1}) + \widetilde{J}_{n+1}f(x_0)\right], \; n \in \mathbb{N}_0. \tag{1.116}$$

Noting again that this equation, was studied by Stevic in [96].

1.4.1.2.2 Case $d = 0, c \neq 0$ and $a = 0$ In this case we get the system

$$\begin{cases} x_{n+1} = f^{-1}\left[bf(x_{n-1}) + cg(y_{n-2})\right], \\ y_{n+1} = g^{-1}\left[bg(y_{n-1}) + cf(x_{n-2})\right], \quad n \in \mathbb{N}_0. \end{cases} \tag{1.117}$$

Here, $\left(\widetilde{J}_n\right)_{n\geq 0}$ will be the sequence defined by

$$\mathcal{P}_{n+3} = b\mathcal{P}_{n+1} + c\mathcal{P}_n, \quad n \in \mathbb{N}_0, \tag{1.118}$$

and the special initial values

$$\mathcal{P}_0 = 0, \qquad \mathcal{P}_1 = 1 \text{ and } \mathcal{P}_2 = 0, \tag{1.119}$$

so, the solutions are expressed in terms of $(\mathcal{P})_{n\geq 0}$ and are given by

$$x_{2n-1} = f^{-1}\left[c\mathcal{P}_{2n-1}g(y_{-2}) + \mathcal{P}_{2n+1}f(x_{-1}) + \mathcal{P}_{2n}g(y_0)\right], n \in \mathbb{N},$$

$$x_{2n} = f^{-1}\left[c\mathcal{P}_{2n}f(x_{-2}) + \mathcal{P}_{2n+2}g(y_{-1}) + \mathcal{P}_{2n+1}f(x_0)\right], n \in \mathbb{N}_0,$$

$$y_{2n-1} = g^{-1}\left[c\mathcal{P}_{2n-1}f(x_{-2}) + \mathcal{P}_{2n+1}g(y_{-1}) + \mathcal{P}_{2n}f(x_0)\right], n \in \mathbb{N},$$

$$y_{2n} = g^{-1}\left[c\mathcal{P}_{2n}g(y_{-2}) + \mathcal{P}_{2n+2}f(x_{-1}) + \mathcal{P}_{2n+1}g(y_0)\right], n \in \mathbb{N}_0,$$

for the system (1.117) and by

$$x_n = f^{-1}\left[c\mathcal{P}_n f(x_{-2}) + \mathcal{P}_{n+2}f(x_{-1}) + \mathcal{P}_{n+1}f(x_0)\right], \; n \in \mathbb{N}_0,$$

for its one dimensional version, that is the equation

$$x_{n+1} = f^{-1}\left(bf(x_{n-1}) + cf(x_{n-2})\right).$$

If $b \neq 0$, $(\mathcal{P})_{n\geq 0}$ will be a generalized Padovn sequence and if $b = c = 1$, then $(\mathcal{P}_n)_{n\geq 0}$ will be the famous Padovan sequence.

Noting that system (1.117) generalize for example the works of [44] and [124].

1.4.1.2.3 Case $c = d = 0$ and $b \neq 0$ In this case, we get the system

$$\begin{cases} x_{n+1} = f^{-1}\left[ag(y_n) + bf(x_{n-1})\right], \\ y_{n+1} = g^{-1}\left[af(x_n) + bg(y_{n-1})\right], \quad n \in \mathbb{N}_0. \end{cases} \tag{1.120}$$

By the same philosophy, we obtain the sequence $\left(\widetilde{F}_n\right)_{n\geq 0} = \left(\widetilde{J}_{n+1}\right)_{n\geq 0}$, defined by

$$\widetilde{F}_{n+2} = a\widetilde{F}_{n+1} + b\widetilde{F}_n,\ \widetilde{F}_0 = 1,\ \widetilde{F}_1 = a,\ n \in \mathbb{N}_0,$$

and the solutions of (1.120) and its one dimensional version, are obtained from the solutions of (1.111) and (1.114), by writing $\widetilde{F}_{n-1}$ instead of $\widetilde{J}_n$. If $a \neq 0$, $\left(\widetilde{F}_n\right)_{n\geq 0}$ is a generalized Fibonacci sequence and if $a = b = 1$, $\left(\widetilde{F}_n\right)_{n\geq 0}$ will be the famous Fibonacci sequence.

System (1.120) and its one dimensional versions, generalized for example the works of [104, 105].

1.4.1.2.4 Case $b = c = d = 0$ and $a \neq 0$ In this case, we get the system

$$x_{n+1} = f^{-1}\left(ag(y_n)\right),\ y_{n+1} = g^{-1}\left(af(x_n)\right),\ n \in \mathbb{N}_0. \tag{1.121}$$

Using the fact that f, g are one to one functions, and the change of variables

$$X_n = f(x_n),\ y_n = g(y_n),\ n \geq 0$$

the system (1.121), will be

$$X_{n+1} = aY_n,\ Y_{n+1} = aX_n,\ n \in \mathbb{N}_0.$$

So,

$$X_{2n} = a^{2n}X_0,\ Y_{2n} = a^{2n}Y_0,\ X_{2n+1} = a^{2n+1}Y_0,\ Y_{2n+1} = a^{2n+1}X_0,\ n \in \mathbb{N}_0.$$

Hence

$$x_{2n} = f^{-1}(a^{2n}f(x_0)),\ y_{2n} = g^{-1}(a^{2n}g(y_0)),\ n \in \mathbb{N}_0$$

and

$$x_{2n+1} = f^{-1}(a^{2n+1}g(y_0)),\ y_{2n+1} = g^{-1}(a^{2n+1}f(x_0)),\ n \in \mathbb{N}_0.$$

1.4.2 Second class of systems

In this part, we are interested in the following system of difference equations given by

$$\begin{cases} x_{n+1} = f^{-1}\left(a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})} + \frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})}\right), \\ y_{n+1} = g^{-1}\left(a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})} + \frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})}\right), \end{cases} \tag{1.122}$$

where $n \in \mathbb{N}_0$, f, $g : D \longrightarrow \mathbb{R}$ are continuous functions, with $D = D_f = D_g$, that is f and g have the same domain, in addition we assume that f, g are "$1-1$" on $D \subseteq \mathbb{R}$, the initial values x_{-i}, y_{-i}, $i = 0, 1, 2$, are arbitrary real numbers in D and the parameters a, b, c and d are arbitrary real numbers.

Definition 1.3. *A solution $\{x_n, y_n\}_{n\geq -2}$ of system (1.122), is said to be well-defined if for all $n \in \mathbb{N}_0$, we have*

$$a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})} + \frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})} \in D_{f^{-1}},$$

and

$$a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})} + \frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})} \in D_{g^{-1}}.$$

We solve in closed form (1.122) and we investigated particular cases of it. The philosophy, is the same as in the previous section (1.4.1), so we will brief in presenting our formulas of the solutions.

1.4.2.1 Explicit formulas of solutions of system (1.122) with $d \neq 0$

The following result is devoted to the formulas of well-defined solutions of (1.122).

Theorem 1.23. *Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution of system (1.122). Then, for all $n \in \mathbb{N}_0$ we have*

$$x_{2n+1} = f^{-1}\left[\frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\,g(y_{-2}) + (J_{2n+4} - aJ_{2n+3})\,f(x_{-1})g(y_{-2}) + J_{2n+3}g(y_0)f(x_{-1})g(y_{-2})}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\,g(y_{-2}) + (J_{2n+3} - aJ_{2n+2})\,f(x_{-1})g(y_{-2}) + J_{2n+2}g(y_0)f(x_{-1})g(y_{-2})}\right],$$

$$x_{2n+2} = f^{-1}\left[\frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\,f(x_{-2}) + (J_{2n+5} - aJ_{2n+4})\,g(y_{-1})f(x_{-2}) + J_{2n+4}f(x_0)g(y_{-1})f(x_{-2})}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\,f(x_{-2}) + (J_{2n+4} - aJ_{2n+3})\,g(y_{-1})f(x_{-2}) + J_{2n+3}f(x_0)g(y_{-1})f(x_{-2})}\right],$$

$$y_{2n+1} = g^{-1}\left[\frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\,f(x_{-2}) + (J_{2n+4} - aJ_{2n+3})\,g(y_{-1})f(x_{-2}) + J_{2n+3}f(x_0)g(y_{-1})f(x_{-2})}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\,f(x_{-2}) + (J_{2n+3} - aJ_{2n+2})\,g(y_{-1})f(x_{-2}) + J_{2n+2}f(x_0)g(y_{-1})f(x_{-2})}\right],$$

$$y_{2n+2} = g^{-1}\left[\frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\,g(y_{-2}) + (J_{2n+5} - aJ_{2n+4})\,f(x_{-1})g(y_{-2}) + J_{2n+4}g(y_0)f(x_{-1})g(y_{-2})}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\,g(y_{-2}) + (J_{2n+4} - aJ_{2n+3})\,f(x_{-1})g(y_{-2}) + J_{2n+3}g(y_0)f(x_{-1})g(y_{-2})}\right].$$

Proof. Using the fact that the functions f, g are one to one and using the change of variables (1.92), with $n \geq -2$, the system (1.122) becomes

$$\begin{cases} X_{n+1} = a + \dfrac{b}{Y_n} + \dfrac{c}{Y_n X_{n-1}} + \dfrac{d}{Y_n X_{n-1} Y_{n-2}}, \\ Y_{n+1} = a + \dfrac{b}{X_n} + \dfrac{c}{X_n Y_{n-1}} + \dfrac{d}{X_n Y_{n-1} X_{n-2}}, \quad n \in \mathbb{N}_0, \end{cases} \tag{1.123}$$

or equivalently,

$$\begin{cases} X_{n+1} = \dfrac{a Y_n X_{n-1} Y_{n-2} + b X_{n-1} Y_{n-2} + c Y_{n-2} + d}{Y_n X_{n-1} Y_{n-2}}, \\ Y_{n+1} = \dfrac{a X_n Y_{n-1} X_{n-2} + b Y_{n-1} X_{n-2} + c X_{n-2} + d}{X_n Y_{n-1} X_{n-2}}, \quad n \in \mathbb{N}_0. \end{cases} \tag{1.124}$$

This system was solved in [3], and for $n \in \mathbb{N}_0$, the solutions of (1.124) takes the form

$$X_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1}) Y_{-2} + (J_{2n+4} - aJ_{2n+3}) X_{-1} Y_{-2} + J_{2n+3} Y_0 X_{-1} Y_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n}) Y_{-2} + (J_{2n+3} - aJ_{2n+2}) X_{-1} Y_{-2} + J_{2n+2} Y_0 X_{-1} Y_{-2}}, \tag{1.125}$$

$$X_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2}) X_{-2} + (J_{2n+5} - aJ_{2n+4}) Y_{-1} X_{-2} + J_{2n+4} X_0 Y_{-1} X_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1}) X_{-2} + (J_{2n+4} - aJ_{2n+3}) Y_{-1} X_{-2} + J_{2n+3} X_0 Y_{-1} X_{-2}}, \tag{1.126}$$

$$Y_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1}) X_{-2} + (J_{2n+4} - aJ_{2n+3}) Y_{-1} X_{-2} + J_{2n+3} X_0 Y_{-1} X_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n}) X_{-2} + (J_{2n+3} - aJ_{2n+2}) Y_{-1} X_{-2} + J_{2n+2} X_0 Y_{-1} X_{-2}}, \tag{1.127}$$

$$Y_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2}) Y_{-2} + (J_{2n+5} - aJ_{2n+4}) X_{-1} Y_{-2} + J_{2n+4} Y_0 X_{-1} Y_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1}) Y_{-2} + (J_{2n+4} - aJ_{2n+3}) X_{-1} Y_{-2} + J_{2n+3} Y_0 X_{-1} Y_{-2}}, \tag{1.128}$$

where $(J_n)_{n\in\mathbb{N}_0}$ is the sequence defined by (1.43).

Using (1.92), (1.125), (1.126), (1.127) and (1.128), we get that for $n \in \mathbb{N}_0$, every well-defined solution of system (1.122) has the following representation

$$x_{2n+1} = f^{-1}\left[\frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, g(y_{-2}) + (J_{2n+4} - aJ_{2n+3})\, f(x_{-1}) g(y_{-2}) + J_{2n+3} g(y_0) f(x_{-1}) g(y_{-2})}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\, g(y_{-2}) + (J_{2n+3} - aJ_{2n+2})\, f(x_{-1}) g(y_{-2}) + J_{2n+2} g(y_0) f(x_{-1}) g(y_{-2})}\right], \tag{1.129}$$

$$x_{2n+2} = f^{-1}\left[\frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\, f(x_{-2}) + (J_{2n+5} - aJ_{2n+4})\, g(y_{-1}) f(x_{-2}) + J_{2n+4} f(x_0) g(y_{-1}) f(x_{-2})}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, f(x_{-2}) + (J_{2n+4} - aJ_{2n+3})\, g(y_{-1}) f(x_{-2}) + J_{2n+3} f(x_0) g(y_{-1}) f(x_{-2})}\right], \tag{1.130}$$

$$y_{2n+1}=g^{-1}\left[\frac{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,f(x_{-2})+(J_{2n+4}-aJ_{2n+3})\,g(y_{-1})f(x_{-2})+J_{2n+3}f(x_0)g(y_{-1})f(x_{-2})}{dJ_{2n+1}+(cJ_{2n+1}+dJ_{2n})\,f(x_{-2})+(J_{2n+3}-aJ_{2n+2})\,g(y_{-1})f(x_{-2})+J_{2n+2}f(x_0)g(y_{-1})f(x_{-2})}\right], \tag{1.131}$$

$$y_{2n+2}=g^{-1}\left[\frac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})\,g(y_{-2})+(J_{2n+5}-aJ_{2n+4})\,f(x_{-1})g(y_{-2})+J_{2n+4}g(y_0)f(x_{-1})g(y_{-2})}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,g(y_{-2})+(J_{2n+4}-aJ_{2n+3})\,f(x_{-1})g(y_{-2})+J_{2n+3}g(y_0)f(x_{-1})g(y_{-2})}\right]. \tag{1.132}$$

□

Remark 1.4.3. 1. In [3], to solve the system (1.124), the authors used the change of variables

$$X_n=\frac{u_n}{v_{n-1}},\ Y_n=\frac{v_n}{u_{n-1}},\ n\geq -2,$$

to obtain the forth linear system (1.71).

2. When $g\equiv f$ and $y_{-i}=x_{-i}$, $i=\overline{0,2}$ then system (1.122) becomes the equation

$$x_{n+1}=f^{-1}\left[a+\frac{b}{f(x_n)}+\frac{c}{f(x_n)f(x_{n-1})}+\frac{d}{f(x_n)f(x_{n-1})f(x_{n-2})}\right],\ n\in\mathbb{N}_0, \tag{1.133}$$

and the form of every well-defined solution of (1.133) is given by

$$x_{2n+1}=f^{-1}\left[\frac{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,f(x_{-2})+(J_{2n+4}-aJ_{2n+3})\,f(x_{-1})f(x_{-2})+J_{2n+3}f(x_0)f(x_{-1})f(x_{-2})}{dJ_{2n+1}+(cJ_{2n+1}+dJ_{2n})\,f(x_{-2})+(J_{2n+3}-aJ_{2n+2})\,f(x_{-1})f(x_{-2})+J_{2n+2}f(x_0)f(x_{-1})f(x_{-2})}\right], \tag{1.134}$$

$$x_{2n+2}=f^{-1}\left[\frac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})\,f(x_{-2})+(J_{2n+5}-aJ_{2n+4})\,f(x_{-1})f(x_{-2})+J_{2n+4}f(x_0)f(x_{-1})f(x_{-2})}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,f(x_{-2})+(J_{2n+4}-aJ_{2n+3})\,f(x_{-1})f(x_{-2})+J_{2n+3}f(x_0)f(x_{-1})f(x_{-2})}\right], \tag{1.135}$$

which can be represented in the unified form

$$x_{n+1}=f^{-1}\left[\frac{dJ_{n+2}+(cJ_{n+2}+dJ_{n+1})\,f(x_{-2})+(J_{n+4}-aJ_{n+3})\,f(x_{-1})f(x_{-2})+J_{n+3}f(x_0)f(x_{-1})f(x_{-2})}{dJ_{n+1}+(cJ_{n+1}+dJ_{n})\,f(x_{-2})+(J_{n+3}-aJ_{n+2})\,f(x_{-1})f(x_{-2})+J_{n+2}f(x_0)f(x_{-1})f(x_{-2})}\right] \tag{1.136}$$

Now we give some applications of Theorem (1.23).

Example 1.4.3. Let

$$f(t)=t^{2j+1},\ g(t)=t^{2k+1},\ j,\ k\in\mathbb{N}_0.$$

We have the functions f and g are one to one continuous functions on $\mathbb{R}=D_f=D_g$. In this case, system (1.122) becomes

$$\begin{cases} x_{n+1}=\left[a+\frac{b}{y_n^{2k+1}}+\frac{c}{y_n^{2k+1}x_{n-1}^{2j+1}}+\frac{d}{y_n^{2k+1}x_{n-1}^{2j+1}y_{n-2}^{2k+1}}\right]^{\frac{1}{2j+1}}, \\ y_{n+1}=\left[a+\frac{b}{x_n^{2j+1}}+\frac{c}{x_n^{2j+1}y_{n-1}^{2k+1}}+\frac{d}{x_n^{2j+1}y_{n-1}^{2k+1}x_{n-2}^{2j+1}}\right]^{\frac{1}{2k+1}},\quad n\in\mathbb{N}_0. \end{cases} \tag{1.137}$$

Then by Theorem 1.23, we obtain that the solutions of system (1.137) have the following form

$$x_{2n+1}=\left[\frac{dJ_{2n+2}+\left(cJ_{2n+2}+dJ_{2n+1}\right)y_{-2}^{2k+1}+\left(J_{2n+4}-aJ_{2n+3}\right)x_{-1}^{2j+1}y_{-2}^{2k+1}+J_{2n+3}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}}{dJ_{2n+1}+\left(cJ_{2n+1}+dJ_{2n}\right)y_{-2}^{2k+1}+\left(J_{2n+3}-aJ_{2n+2}\right)x_{-1}^{2j+1}y_{-2}^{2k+1}+J_{2n+2}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}}\right]^{\frac{1}{2j+1}}, \tag{1.138}$$

$$x_{2n+2}=\left[\frac{dJ_{2n+3}+\left(cJ_{2n+3}+dJ_{2n+2}\right)x_{-2}^{2j+1}+\left(J_{2n+5}-aJ_{2n+4}\right)y_{-1}^{2k+1}x_{-2}^{2j+1}+J_{2n+4}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}}{dJ_{2n+2}+\left(cJ_{2n+2}+dJ_{2n+1}\right)x_{-2}^{2j+1}+\left(J_{2n+4}-aJ_{2n+3}\right)y_{-1}^{2k+1}x_{-2}^{2j+1}+J_{2n+3}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}}\right]^{\frac{1}{2j+1}}, \tag{1.139}$$

$$y_{2n+1}=\left[\frac{dJ_{2n+2}+\left(cJ_{2n+2}+dJ_{2n+1}\right)x_{-2}^{2j+1}+\left(J_{2n+4}-aJ_{2n+3}\right)y_{-1}^{2k+1}x_{-2}^{2j+1}+J_{2n+3}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}}{dJ_{2n+1}+\left(cJ_{2n+1}+dJ_{2n}\right)x_{-2}^{2j+1}+\left(J_{2n+3}-aJ_{2n+2}\right)y_{-1}^{2k+1}x_{-2}^{2j+1}+J_{2n+2}x_0^{2j+1})y_{-1}^{2k+1}x_{-2}^{2j+1}}\right]^{\frac{1}{2k+1}}, \tag{1.140}$$

$$y_{2n+2}=\left[\frac{dJ_{2n+3}+\left(cJ_{2n+3}+dJ_{2n+2}\right)y_{-2}^{2k+1}+\left(J_{2n+5}-aJ_{2n+4}\right)x_{-1}^{2j+1}y_{-2}^{2k+1}+J_{2n+4}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}}{dJ_{2n+2}+\left(cJ_{2n+2}+dJ_{2n+1}\right)y_{-2}^{2k+1}+\left(J_{2n+4}-aJ_{2n+3}\right)x_{-1}^{2j+1}y_{-2}^{2k+1}+J_{2n+3}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}}\right]^{\frac{1}{2k+1}}. \tag{1.141}$$

Example 1.4.4. Let

$$f(t)=\frac{1}{t^{2j+1}},\ g(t)=\frac{1}{t^{2k+1}},\ j,\ k\in\mathbb{N}_0.$$

We have the functions f and g are one to one continuous functions on $\mathbb{R}-\{0\}=D_f=D_g$. Then, system (1.122) becomes

$$\begin{cases} x_{n+1}=\left[\frac{1}{a+by_n^{2k+1}+cy_n^{2k+1}x_{n-1}^{2j+1}+dy_n^{2k+1}x_{n-1}^{2j+1}y_{n-2}^{2k+1}}\right]^{\frac{1}{2j+1}}, \\ y_{n+1}=\left[\frac{1}{a+bx_n^{2j+1}+cx_n^{2j+1}y_{n-1}^{2k+1}+dx_n^{2j+1}y_{n-1}^{2k+1}x_{n-2}^{2j+1}}\right]^{\frac{1}{2k+1}}, \quad n\in\mathbb{N}_0. \end{cases} \tag{1.142}$$

Then by Theorem 1.23, the solutions of system (1.142) have the following representation

$$x_{2n+1}=\left[\frac{dJ_{2n+1}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}+\left(cJ_{2n+1}+dJ_{2n}\right)y_0^{2k+1}x_{-1}^{2j+1}+\left(J_{2n+3}-aJ_{2n+2}\right)y_0^{2k+1}+J_{2n+2}}{dJ_{2n+2}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}+\left(cJ_{2n+2}+dJ_{2n+1}\right)y_0^{2k+1}x_{-1}^{2j+1}+\left(J_{2n+4}-aJ_{2n+3}\right)y_0^{2k+1}+J_{2n+3}}\right]^{\frac{1}{2j+1}}, \tag{1.143}$$

$$x_{2n+2}=\left[\frac{dJ_{2n+2}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}+\left(cJ_{2n+2}+dJ_{2n+1}\right)x_0^{2j+1}y_{-1}^{2k+1}+\left(J_{2n+4}-aJ_{2n+3}\right)x_0^{2j+1}+J_{2n+3}}{dJ_{2n+3}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}+\left(cJ_{2n+3}+dJ_{2n+2}\right)x_0^{2j+1}y_{-1}^{2k+1}+\left(J_{2n+5}-aJ_{2n+4}\right)x_0^{2j+1}+J_{2n+4}}\right]^{\frac{1}{2j+1}}, \tag{1.144}$$

$$y_{2n+1}=\left[\frac{dJ_{2n+1}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}+(cJ_{2n+1}+dJ_{2n})\,x_0^{2j+1}y_{-1}^{2k+1}+(J_{2n+3}-aJ_{2n+2})\,x_0^{2j+1}+J_{2n+2}}{dJ_{2n+2}x_0^{2j+1}y_{-1}^{2k+1}x_{-2}^{2j+1}+(cJ_{2n+2}+dJ_{2n+1})\,x_0^{2j+1}y_{-1}^{2k+1}+(J_{2n+4}-aJ_{2n+3})\,x_0^{2j+1}+J_{2n+3}}\right]^{\frac{1}{2k+1}}, \tag{1.145}$$

$$y_{2n+2}=\left[\frac{dJ_{2n+2}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}+(cJ_{2n+2}+dJ_{2n+1})\,y_0^{2k+1}x_{-1}^{2j+1}+(J_{2n+4}-aJ_{2n+3})\,y_0^{2k+1}+J_{2n+3}}{dJ_{2n+3}y_0^{2k+1}x_{-1}^{2j+1}y_{-2}^{2k+1}+(cJ_{2n+3}+dJ_{2n+2})\,y_0^{2k+1}x_{-1}^{2j+1}+(J_{2n+5}-aJ_{2n+4})\,y_0^{2k+1}+J_{2n+4}}\right]^{\frac{1}{2k+1}}. \tag{1.146}$$

1.4.2.1.1 The system $x_{n+1}=\frac{1}{a+by_n+cy_nx_{n-1}+dy_nx_{n-1}y_{n-2}}$, $y_{n+1}=\frac{1}{a+bx_n+cx_ny_{n-1}+dx_ny_{n-1}x_{n-2}}$

Here we will focus our study on the system of difference equations

$$\begin{cases} x_{n+1}=\dfrac{1}{a+by_n+cy_nx_{n-1}+dy_nx_{n-1}y_{n-2}}, \\ y_{n+1}=\dfrac{1}{a+bx_n+cx_ny_{n-1}+dx_ny_{n-1}x_{n-2}}, \end{cases} \quad n\in\mathbb{N}_0, \tag{1.147}$$

which is a particular case of system (1.142) with $j=k=0$. Noting that system (1.147), generalize the studies in [65, 66, 67]. Then, putting $j=k=0$ in the formulas of well-defined solutions of system (1.142), we obtain the following result.

Corollary 1.24. *Let $\{x_n, y,\}_{n\geq -2}$ be a well-defined solution of (1.147), then for $n\in\mathbb{N}_0$, we have*

$$x_{2n+1}=\frac{dJ_{2n+1}y_0x_{-1}y_{-2}+(cJ_{2n+1}+dJ_{2n})\,y_0x_{-1}+(J_{2n+3}-aJ_{2n+2})\,y_0+J_{2n+2}}{dJ_{2n+2}y_0x_{-1}y_{-2}+(cJ_{2n+2}+dJ_{2n+1})\,y_0x_{-1}+(J_{2n+4}-aJ_{2n+3})\,y_0+J_{2n+3}}, \tag{1.148}$$

$$x_{2n+2}=\frac{dJ_{2n+2}x_0y_{-1}x_{-2}+(cJ_{2n+2}+dJ_{2n+1})\,x_0y_{-1}+(J_{2n+4}-aJ_{2n+3})\,x_0+J_{2n+3}}{dJ_{2n+3}x_0y_{-1}x_{-2}+(cJ_{2n+3}+dJ_{2n+2})\,x_0y_{-1}+(J_{2n+5}-aJ_{2n+4})\,x_0+J_{2n+4}}, \tag{1.149}$$

$$y_{2n+1}=\frac{dJ_{2n+1}x_0y_{-1}x_{-2}+(cJ_{2n+1}+dJ_{2n})\,x_0y_{-1}+(J_{2n+3}-aJ_{2n+2})\,x_0+J_{2n+2}}{dJ_{2n+2}x_0y_{-1}x_{-2}+(cJ_{2n+2}+dJ_{2n+1})\,x_0y_{-1}+(J_{2n+4}-aJ_{2n+3})\,x_0+J_{2n+3}}, \tag{1.150}$$

$$y_{2n+2}=\frac{dJ_{2n+2}y_0x_{-1}y_{-2}+(cJ_{2n+2}+dJ_{2n+1})\,y_0x_{-1}+(J_{2n+4}-aJ_{2n+3})\,y_0+J_{2n+3}}{dJ_{2n+3}y_0x_{-1}y_{-2}+(cJ_{2n+3}+dJ_{2n+2})\,y_0x_{-1}+(J_{2n+5}-aJ_{2n+4})\,y_0+J_{2n+4}}. \tag{1.151}$$

Moreover, if we choose $a=b=c=d=1$, the sequence $(J_n)_{n=0}^{+\infty}$ will be nothing other than the famous Tetranacci sequence defined for $n\in\mathbb{N}_0$ by

$$T_{n+4}=T_{n+3}+T_{n+2}+T_{n+1}+T_n,\; T_0=T_1=0, \quad T_2=T_3=1, \tag{1.152}$$

and for this choice of the parameters, the solutions takes the form

$$x_{2n+1} = \frac{T_{2n+1}y_0x_{-1}y_{-2} + (T_{2n+1} + T_{2n})\,y_0x_{-1} + (T_{2n+3} - T_{2n+2})\,y_0 + T_{2n+2}}{T_{2n+2}y_0x_{-1}y_{-2} + (T_{2n+2} + T_{2n+1})\,y_0x_{-1} + (T_{2n+4} - T_{2n+3})\,y_0 + T_{2n+3}}, \qquad (1.153)$$

$$x_{2n+2} = \frac{T_{2n+2}x_0y_{-1}x_{-2} + (T_{2n+2} + T_{2n+1})\,x_0y_{-1} + (T_{2n+4} - T_{2n+3})\,x_0 + T_{2n+3}}{T_{2n+3}x_0y_{-1}x_{-2} + (T_{2n+3} + T_{2n+2})\,x_0y_{-1} + (T_{2n+5} - T_{2n+4})\,x_0 + T_{2n+4}}, \qquad (1.154)$$

$$y_{2n+1} = \frac{T_{2n+1}x_0y_{-1}x_{-2} + (T_{2n+1} + T_{2n})\,x_0y_{-1} + (T_{2n+3} - T_{2n+2})\,x_0 + T_{2n+2}}{T_{2n+2}x_0y_{-1}x_{-2} + (T_{2n+2} + T_{2n+1})\,x_0y_{-1} + (T_{2n+4} - T_{2n+3})\,x_0 + T_{2n+3}}, \qquad (1.155)$$

$$y_{2n+2} = \frac{T_{2n+2}y_0x_{-1}y_{-2} + (T_{2n+2} + T_{2n+1})\,y_0x_{-1} + (T_{2n+4} - T_{2n+3})\,y_0 + T_{2n+3}}{T_{2n+3}y_0x_{-1}y_{-2} + (T_{2n+3} + T_{2n+2})\,y_0x_{-1} + (T_{2n+5} - T_{2n+4})\,y_0 + T_{2n+4}}. \qquad (1.156)$$

Now, we will study the stability of the equilibrium points of system (1.147) with $a = b = c = d = 1$, that is the system

$$\begin{cases} x_{n+1} = f_1(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) = \frac{1}{1+y_n+x_{n-1}y_n+y_{n-2}x_{n-1}y_n}, \\ y_{n+1} = f_2(x_n, x_{n-1}, x_{n-2}, y_n, y_{n-1}, y_{n-2}) = \frac{1}{1+x_n+y_{n-1}x_n+x_{n-2}y_{n-1}x_n}. \end{cases} \qquad (1.157)$$

For the stability of the equilibrium points, we assume that the initial values are positive real numbers.

The points $(\frac{1}{\alpha}, \frac{1}{\alpha})$, $(\frac{1}{\beta}, \frac{1}{\beta})$, $(\frac{1}{\gamma, \gamma})$ and $(\frac{1}{\delta}, \frac{1}{\delta})$ are solutions of the of system

$$\begin{cases} x = \dfrac{1}{1 + y + xy + xy^2}, \\ y = \dfrac{1}{1 + x + yx + x^2y}, \end{cases}$$

where α, β, γ and δ are roots of polynomial characteristic associated to the equation (1.152), see [3]. It follows that $(\overline{x}, \overline{y}) = (\frac{1}{\alpha}, \frac{1}{\alpha})$ is the only equilibrium point for system (1.157) in $(0, +\infty)^2$.

For the equilibrium point $(\overline{x}, \overline{y}) = (\frac{1}{\alpha}, \frac{1}{\alpha})$, we have the following result.

Theorem 1.25. *The equilibrium point* $(\overline{x}, \overline{y}) = (\frac{1}{\alpha}, \frac{1}{\alpha})$ *is globally asymptotically stable.*

Proof. The Jacobian matrix associated to the system (1.157) around the equilibrium point $(\overline{x},\overline{y})=(\frac{1}{\alpha},\frac{1}{\alpha})$, is given by

$$A=\begin{pmatrix} 0 & -\frac{\alpha+1}{\alpha^4} & 0 & -\frac{\alpha-1}{\alpha} & 0 & -\frac{1}{\alpha^4} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\alpha-1}{\alpha} & 0 & -\frac{1}{\alpha^4} & 0 & -\frac{\alpha+1}{\alpha^4} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of A is

$$P(\lambda)=-h(\lambda)h(-\lambda),\ \text{where } h(\lambda)=\lambda^3-\frac{\alpha-1}{\alpha}\lambda^2+\frac{\alpha+1}{\alpha^4}\lambda-\frac{1}{\alpha^4}. \tag{1.158}$$

Consider the two functions

$$h_1(\lambda)=\lambda^3,\ h_2(\lambda)=\frac{\alpha-1}{\alpha}\lambda^2-\frac{\alpha+1}{\alpha^4}\lambda+\frac{1}{\alpha^4}.$$

We have

$$|h_2(\lambda)|\le\left|\frac{\alpha-1}{\alpha}\right|+\left|\frac{\alpha+1}{\alpha^4}\right|+\left|\frac{1}{\alpha^4}\right|<1=|h_1(\lambda)|,\forall\lambda\in\mathbb{C}:|\lambda|=1.$$

It follows by Rouché's theorem, that all the roots of $h(\lambda)$ lie in the open unit disk, and so it is for the roots of $P(\lambda)$. Thus the equilibrium point $(\frac{1}{\alpha},\frac{1}{\alpha})$ is locally asymptotically stable.

It remains to prove that

$$\lim_{n\to+\infty}x_n=\lim_{n\to+\infty}y_n=\frac{1}{\alpha}.$$

To this end we will, use the fact

$$\lim_{n\to\infty}\frac{T_{n+k}}{T_n}=\alpha^k\ ,\quad\forall k\in\mathbb{N}. \tag{1.159}$$

Using the formula of the solutions, we have

$$\begin{aligned}\lim_{n\to\infty}x_{2n+1} &= \lim_{n\to\infty}\frac{T_{2n+1}y_0x_{-1}y_{-2}+\left(T_{2n+1}+T_{2n}\right)y_0x_{-1}+\left(T_{2n+3}-T_{2n+2}\right)y_0+T_{2n+2}}{T_{2n+2}y_0x_{-1}y_{-2}+\left(T_{2n+2}+T_{2n+1}\right)y_0x_{-1}+\left(T_{2n+4}-T_{2n+3}\right)y_0+T_{2n+3}}\\ &= \lim_{n\to\infty}\frac{\frac{T_{2n+1}}{T_{2n}}y_0x_{-1}y_{-2}+\left(\frac{T_{2n+1}}{T_{2n}}+\frac{T_{2n}}{T_{2n}}\right)y_0x_{-1}+\left(\frac{T_{2n+3}}{T_{2n}}-\frac{T_{2n+2}}{T_{2n}}\right)y_0+\frac{T_{2n+2}}{T_{2n}}}{\frac{T_{2n+2}}{T_{2n}}y_0x_{-1}y_{-2}+\left(\frac{T_{2n+2}}{T_{2n}}+\frac{T_{2n+1}}{T_{2n}}\right)y_0x_{-1}+\left(\frac{T_{2n+4}}{T_{2n}}-\frac{T_{2n+3}}{T_{2n}}\right)y_0+\frac{T_{2n+3}}{T_{2n}}}\\ &= \frac{\alpha y_0x_{-1}y_{-2}+\left(\alpha+1\right)y_0x_{-1}+\left(\alpha^3-\alpha^2\right)y_0+\alpha^2}{\alpha^2y_0x_{-1}y_{-2}+\left(\alpha^2+\alpha^1\right)y_0x_{-1}+\left(\alpha^4-\alpha^3\right)y_0+\alpha^3}\\ &= \frac{1}{\alpha}.\end{aligned}$$

Similarly, we show that

$$\lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} y_{2n+2} = \frac{1}{\alpha},$$

that is

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \frac{1}{\alpha}. \tag{1.160}$$

This complete the proof. □

As a consequence of Corollary 1.24, when choosing the initial values to satisfies $y_{-i} = x_{-i}$, $i = 0, 1, 2$, we get the formulas of the well-defined solutions of the equation

$$x_{n+1} = \frac{1}{a + bx_n + cx_{n-1}x_n + dx_{n-2}x_{n-1}x_n}, \ n \in \mathbb{N}_0. \tag{1.161}$$

These formulas are given in the next result.

Corollary 1.26. *For all $n \in \mathbb{N}_0$, the form of every well-defined solution of equation (1.161) is given by*

$$x_{2n+1} = \frac{dJ_{2n+1}x_0x_{-1}x_{-2} + (cJ_{2n+1} + dJ_{2n})\, x_0x_{-1} + (J_{2n+3} - aJ_{2n+2})\, x_0 + J_{2n+2}}{dJ_{2n+2}x_0x_{-1}x_{-2} + (cJ_{2n+2} + dJ_{2n+1})\, x_0x_{-1} + (J_{2n+4} - aJ_{2n+3})\, x_0 + J_{2n+3}}, \tag{1.162}$$

$$x_{2n+2} = \frac{dJ_{2n+2}x_0x_{-1}x_{-2} + (cJ_{2n+2} + dJ_{2n+1})\, x_0x_{-1} + (J_{2n+4} - aJ_{2n+3})\, x_0 + J_{2n+3}}{dJ_{2n+3}x_0x_{-1}x_{-2} + (cJ_{2n+3} + dJ_{2n+2})\, x_0x_{-1} + (J_{2n+5} - aJ_{2n+4})\, x_0 + J_{2n+4}}, \tag{1.163}$$

which can represented in a unified form as

$$x_{n+1} = \frac{dJ_{n+1}x_0x_{-1}x_{-2} + (cJ_{n+1} + dJ_n)\, x_0x_{-1} + (J_{n+3} - aJ_{n+2})\, x_0 + J_{n+2}}{dJ_{n+2}x_0x_{-1}x_{-2} + (cJ_{n+2} + dJ_{n+1})\, x_0x_{-1} + (J_{n+4} - aJ_{n+3})\, x_0 + J_{n+3}}. \tag{1.164}$$

Moreover, if $a = b = c = d = 1$, then (1.161) becomes

$$x_{n+1} = \frac{1}{1 + x_n + x_{n-1}x_n + x_{n-2}x_{n-1}x_n} \tag{1.165}$$

and the solutions are expressed in terms of Tetranacci numbers as follows

$$x_{2n+1} = \frac{T_{2n+1}x_0x_{-1}x_{-2} + (T_{2n+1} + T_{2n})\, x_0x_{-1} + (T_{2n+3} - T_{2n+2})\, x_0 + T_{2n+2}}{T_{2n+2}x_0x_{-1}x_{-2} + (T_{2n+2} + T_{2n+1})\, x_0x_{-1} + (T_{2n+4} - T_{2n+3})\, x_0 + T_{2n+3}}, \tag{1.166}$$

$$x_{2n+2} = \frac{T_{2n+2}x_0x_{-1}x_{-2} + (T_{2n+2} + T_{2n+1})\, x_0x_{-1} + (T_{2n+4} - T_{2n+3})\, x_0 + T_{2n+3}}{T_{2n+3}x_0x_{-1}x_{-2} + (T_{2n+3} + T_{2n+2})\, x_0x_{-1} + (T_{2n+5} - T_{2n+4})\, x_0 + T_{2n+4}}, \tag{1.167}$$

or equivalently in the unified form,

$$x_{n+1} = \frac{T_{n+1}x_0x_{-1}x_{-2} + (T_{n+1} + T_n)\, x_0x_{-1} + (T_{n+3} - T_{n+2})\, x_0 + T_{n+2}}{T_{n+2}x_0x_{-1}x_{-2} + (T_{n+2} + T_{n+1})\, x_0x_{-1} + (T_{n+4} - T_{n+3})\, x_0 + T_{n+3}}. \tag{1.168}$$

Remark 1.4.4. It is not hard to see that $\overline{x} = \dfrac{1}{\alpha}$ is the unique equilibrium point for (1.165) in $(0, +\infty)$ when taking the initial values positive real numbers.
The linearized equation of (1.165) about the equilibrium point $\overline{x} = \dfrac{1}{\alpha}$ is

$$w_{n+1} = -\frac{\alpha-1}{\alpha}w_n - \frac{\alpha+1}{\alpha^4}w_{n-1} - \frac{1}{\alpha^4}w_{n-2}. \tag{1.169}$$

We have $\frac{\alpha-1}{\alpha}$, $\frac{\alpha+1}{\alpha^4}$, $\frac{1}{\alpha^4}$ are real numbers and

$$\left|\frac{\alpha-1}{\alpha}\right| + \left|\frac{\alpha+1}{\alpha^4}\right| + \left|\frac{1}{\alpha^4}\right| < 1,$$

thus, $\overline{x} = \frac{1}{\alpha}$ is locally asymptotically stable.

Now, using (1.159) and (1.168), we get

$$\begin{aligned}
\lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} \frac{T_{n+1}x_0x_{-1}x_{-2} + (T_{n+1}+T_n)\,x_0x_{-1} + (T_{n+3}-T_{n+2})\,x_0 + T_{n+2}}{T_{n+2}x_0x_{-1}x_{-2} + (T_{n+2}+T_{n+1})\,x_0x_{-1} + (T_{n+4}-T_{n+3})\,x_0 + T_{n+3}} \\
&= \lim_{n\to\infty} \frac{\frac{T_{n+1}}{T_n}x_0x_{-1}x_{-2} + \left(\frac{T_{n+1}}{T_n}+\frac{T_n}{T_n}\right)x_0x_{-1} + \left(\frac{T_{n+3}}{T_n}-\frac{T_{n+2}}{T_n}\right)x_0 + \frac{T_{n+2}}{T_n}}{\frac{T_{n+2}}{T_n}x_0x_{-1}x_{-2} + \left(\frac{T_{n+2}}{T_n}+\frac{T_{n+1}}{T_n}\right)x_0x_{-1} + \left(\frac{T_{n+4}}{T_n}-\frac{T_{n+3}}{T_n}\right)x_0 + \frac{T_{n+3}}{T_n}} \\
&= \frac{\alpha x_0x_{-1}x_{-2} + (\alpha+1)\,x_0x_{-1} + (\alpha^3-\alpha^2)\,x_0 + \alpha^2}{\alpha^2x_0x_{-1}x_{-2} + (\alpha^2+\alpha)\,x_0x_{-1} + (\alpha^4-\alpha^3)\,x_0 + \alpha^3} \\
&= \frac{1}{\alpha}.
\end{aligned}$$

In summary, the equilibrium point $\overline{x} = \frac{1}{\alpha}$ is globally asymptotically stable.

1.4.2.2 Particular cases of system (1.122)

1.4.2.2.1 The case $d = 0$ and $c \neq 0$ In this case the system (1.122) take the form:

$$\begin{cases} x_{n+1} = f^{-1}\left[a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})}\right], \\ y_{n+1} = g^{-1}\left[a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})}\right], \quad n \in \mathbb{N}_0. \end{cases} \tag{1.170}$$

Corollary 1.27. *The formulas of well-defined solutions of system (1.170), are given, for all $n \in \mathbb{N}_0$, by*

$$x_{2n+1} = f^{-1}\left[\frac{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right)f(x_{-1}) + \widetilde{J}_{2n+2}g(y_0)f(x_{-1})}{c\widetilde{J}_{2n} + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right)f(x_{-1}) + \widetilde{J}_{2n+1}g(y_0)f(x_{-1})}\right], \tag{1.171}$$

$$x_{2n+2} = f^{-1}\left[\frac{c\widetilde{J}_{2n+2} + \left(\widetilde{J}_{2n+4} - a\widetilde{J}_{2n+3}\right)g(y_{-1}) + \widetilde{J}_{2n+3}f(x_0)g(y_{-1})}{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right)g(y_{-1}) + \widetilde{J}_{2n+2}f(x_0)g(y_{-1})}\right], \tag{1.172}$$

$$y_{2n+1} = g^{-1}\left[\frac{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right) g(y_{-1}) + \widetilde{J}_{2n+2} f(x_0) g(y_{-1})}{c\widetilde{J}_{2n} + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right) g(y_{-1}) + \widetilde{J}_{2n+1} f(x_0) g(y_{-1})}\right], \tag{1.173}$$

$$y_{2n+2} = g^{-1}\left[\frac{c\widetilde{J}_{2n+2} + \left(\widetilde{J}_{2n+4} - a\widetilde{J}_{2n+3}\right) f(x_{-1}) + \widetilde{J}_{2n+3} g(y_0) f(x_{-1})}{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right) f(x_{-1}) + \widetilde{J}_{2n+2} g(y_0) f(x_{-1})}\right]. \tag{1.174}$$

If $g = f$ and $y_{-i} = x_{-i}$, $i = 0, 1, 2$ then, system (1.170) becomes

$$x_{n+1} = f^{-1}\left[a + \frac{b}{f(x_n)} + \frac{c}{f(x_n) f(x_{n-1})}\right], \quad n \in \mathbb{N}_0, \tag{1.175}$$

hence, every well-defined solution of equation (1.175) is given by

$$x_{2n+1} = f^{-1}\left[\frac{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right) f(x_{-1}) + \widetilde{J}_{2n+2} f(x_0) f(x_{-1})}{c\widetilde{J}_{2n} + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right) f(x_{-1}) + \widetilde{J}_{2n+1} f(x_0) f(x_{-1})}\right], \tag{1.176}$$

$$x_{2n+2} = f^{-1}\left[\frac{c\widetilde{J}_{2n+2} + \left(\widetilde{J}_{2n+4} - a\widetilde{J}_{2n+3}\right) f(x_{-1}) + \widetilde{J}_{2n+3} f(x_0) f(x_{-1})}{c\widetilde{J}_{2n+1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right) f(x_{-1}) + \widetilde{J}_{2n+2} f(x_0) f(x_{-1})}\right], \tag{1.177}$$

or in the unified form

$$x_{n+1} = f^{-1}\left[\frac{c\widetilde{J}_{n+1} + \left(\widetilde{J}_{n+3} - a\widetilde{J}_{n+2}\right) f(x_{-1}) + \widetilde{J}_{n+2} f(x_0) f(x_{-1})}{c\widetilde{J}_{n} + \left(\widetilde{J}_{n+2} - a\widetilde{J}_{n+1}\right) f(x_{-1}) + \widetilde{J}_{n+1} f(x_0) f(x_{-1})}\right], \; n \in \mathbb{N}_0, \tag{1.178}$$

where $\left(\widetilde{J}_n\right)_{n\geq 0}$ is the sequence defined by

$$\widetilde{J}_{n+3} = a\widetilde{J}_{n+2} + b\widetilde{J}_{n+1} + c\widetilde{J}_n, \; \widetilde{J}_0 = 0, \; \widetilde{J}_1 = 1, \; \widetilde{J}_2 = a, \; n \in \mathbb{N}_0.$$

1.4.2.2.2 The system $x_{n+1} = \frac{1}{a+by_n+cx_{n-1}y_n}$, $y_{n+1} = \frac{1}{a+bx_n+cy_{n-1}x_n}$ In the following, we are interested in a particular system of (1.170) and some particular equations of its one dimensional version.

Let $f(t) = \frac{1}{t}$ and $g(t) = \frac{1}{t}$, then, system (1.170) becomes

$$x_{n+1} = \frac{1}{a + by_n + cx_{n-1}y_n}, \; y_{n+1} = \frac{1}{a + bx_n + cy_{n-1}x_n}. \tag{1.179}$$

So, by Corollary 1.27, we get the following result.

Corollary 1.28. *For all* $n \in \mathbb{N}_0$*, the representation of well-defined solutions of system* (1.179) *is*

$$x_{2n+1} = \frac{c\widetilde{J}_{2n}y_0x_{-1} + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right)y_0 + \widetilde{J}_{2n+1}}{c\widetilde{J}_{2n+1}y_0x_{-1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right)y_0 + \widetilde{J}_{2n+2}}, \tag{1.180}$$

$$x_{2n+2} = \frac{c\widetilde{J}_{2n+1}x_0y_{-1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right)x_0 + \widetilde{J}_{2n+2}}{c\widetilde{J}_{2n+2}x_0y_{-1} + \left(\widetilde{J}_{2n+4} - a\widetilde{J}_{2n+3}\right)x_0 + \widetilde{J}_{2n+3}}, \tag{1.181}$$

$$y_{2n+1} = \frac{c\widetilde{J}_{2n}x_0y_{-1} + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right)x_0 + \widetilde{J}_{2n+1}}{c\widetilde{J}_{2n+1}x_0y_{-1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right)x_0 + \widetilde{J}_{2n+2}}, \tag{1.182}$$

$$y_{2n+2} = \frac{c\widetilde{J}_{2n+1}y_0x_{-1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right)y_0 + \widetilde{J}_{2n+2}}{c\widetilde{J}_{2n+2}y_0x_{-1} + \left(\widetilde{J}_{2n+4} - a\widetilde{J}_{2n+3}\right)y_0 + \widetilde{J}_{2n+3}}. \tag{1.183}$$

Moreover, the solutions of the following equation

$$x_{n+1} = \frac{1}{a + bx_n + cx_{n-1}x_n} \tag{1.184}$$

follows directly from those of the system (1.179) *by taking* $y_{-i} = x_{-i}$*,* $i = 0, 1$*. Then, representation of well-defined solutions is*

$$x_{2n+1} = \frac{c\widetilde{J}_{2n}x_0x_{-1} + \left(\widetilde{J}_{2n+2} - a\widetilde{J}_{2n+1}\right)x_0 + \widetilde{J}_{2n+1}}{c\widetilde{J}_{2n+1}x_0x_{-1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right)x_0 + \widetilde{J}_{2n+2}}, \tag{1.185}$$

$$x_{2n+2} = \frac{c\widetilde{J}_{2n+1}x_0x_{-1} + \left(\widetilde{J}_{2n+3} - a\widetilde{J}_{2n+2}\right)x_0 + \widetilde{J}_{2n+2}}{c\widetilde{J}_{2n+2}x_0x_{-1} + \left(\widetilde{J}_{2n+4} - a\widetilde{J}_{2n+3}\right)x_0 + \widetilde{J}_{2n+3}}. \tag{1.186}$$

Recently in [65], the authors studied the particular difference equations

$$x_{n+1} = \frac{1}{-1 + x_n + x_{n-1}x_n}, \tag{1.187}$$

$$x_{n+1} = \frac{1}{1 + x_n - x_{n-1}x_n}, \tag{1.188}$$

$$x_{n+1} = \frac{1}{1 - x_n + x_{n-1}x_n}, \tag{1.189}$$

$$x_{n+1} = \frac{1}{-1 - x_n - x_{n-1}x_n}, \tag{1.190}$$

and as a generalization of (1.187), (1.188), (1.189), (1.190) the authors studied again in [66], the equation

$$x_{n+1} = \frac{1}{\frac{\beta}{\gamma} + \frac{\alpha}{\gamma}x_n + \frac{1}{\gamma}x_{n-1}x_n}, \ \gamma \neq 0. \tag{1.191}$$

Clearly these equations are particular cases of equation (1.184).

In fact to obtain equation (1.187) it suffices to take $a = -1$, $b = 1$ and $c = 1$ in (1.184), when choosing $a = 1$, $b = 1$ and $c = -1$ in (1.184) we get equation (1.188). Equation (1.189) follows from (1.184) for the choice $a = 1$, $b = -1$ and $c = 1$. Again if we take $a = -1$, $b = -1$ and $c = -1$ in (1.184) we get the equation (1.190), finally for the choice $a = \frac{\beta}{\gamma}$, $b = \frac{\alpha}{\gamma}$ and $c = \frac{1}{\gamma}$ in (1.184) we find the equation (1.191).

Noting, that the authors studied in [67] the following systems

$$\begin{cases} x_{n+1} = \dfrac{1}{1 + y_n + x_{n-1}y_n}, \\ y_{n+1} = \dfrac{1}{1 + x_n + y_{n-1}x_n}, \end{cases} \tag{1.192}$$

$$\begin{cases} x_{n+1} = \dfrac{1}{-1 + y_n - x_{n-1}y_n}, \\ y_{n+1} = \dfrac{1}{-1 + x_n - y_{n-1}x_n}. \end{cases} \tag{1.193}$$

Systems (1.192) and (1.193) are particular cases of system (1.179) for the choices $a = b = c = 1$ and $a = -1, b = 1, c = -1$ respectively.

Now, we will investigate each of these cases separately.

1. Consider $a = -1$, $b = 1$, $c = 1$, equation (1.184) becomes (1.187). For this choice of the parameters, we get the sequence $\left(\widetilde{J}_n\right)_{n=0}^{\infty}$ defined by:

$$\widetilde{J}_{n+3} = -\widetilde{J}_{n+2} + \widetilde{J}_{n+1} + \widetilde{J}_n,\ \widetilde{J}_0 = 0,\ \widetilde{J}_1 = 1,\ \widetilde{J}_2 = -1.$$

It is easy to see that

$$\widetilde{J}_n = -\left(\frac{1}{4} + \frac{1}{2}n\right)(-1)^n + \frac{1}{4}, \quad n \in \mathbb{N}_0.$$

From this it follows that

$$\widetilde{J}_{2n} = -n,\ \widetilde{J}_{2n+1} = n + 1.$$

Replacing in (1.185) and and (1.186), we obtain

$$x_{2n+1} = \frac{-nx_{-1}x_0 + n + 1}{(n+1)x_{-1}x_0 + x_0 - (n+1)},$$

$$x_{2n+2} = \frac{(n+1)x_0x_{-1} + x_0 - (n+1)}{-(n+1)x_0x_{-1} + n + 2},$$

and these are the formulas given in [65] for the solutions of (1.187).

2. Consider $a = 1$, $b = 1$, $c = -1$, equation (1.184) becomes (1.188). For this choice of the parameters, we get the sequence $\left(\widetilde{J}_n\right)_{n=0}^{\infty}$ defined by:

$$\widetilde{J}_{n+3} = \widetilde{J}_{n+2} + \widetilde{J}_{n+1} - \widetilde{J}_n,\ \widetilde{J}_0 = 0,\ \widetilde{J}_1 = 1,\ \widetilde{J}_2 = 1.$$

It is easy to see that

$$\widetilde{J}_n = \frac{1}{4} + \frac{1}{2}n - \frac{1}{4}(-1)^n, \quad n \in \mathbb{N}_0.$$

From this it follows that

$$\widetilde{J}_{2n} = n,\ \widetilde{J}_{2n+1} = n + 1.$$

Replacing in (1.185) and (1.186), we get

$$x_{2n+1} = \frac{nx_{-1}x_0 - (n+1)}{(n+1)x_{-1}x_0 - x_0 - (n+1)},$$

$$x_{2n+2} = \frac{(n+1)x_0x_{-1} - x_0 - (n+1)}{(n+1)x_0x_{-1} - (n+2)},$$

and these are the formulas given in [65] for the solutions of (1.188).

3. Consider $a = 1$, $b = -1$, $c = 1$, equation (1.184) becomes (1.189). For this choice of the parameters, we get the sequence $\left(\widetilde{J}_n\right)_{n=0}^{\infty}$ defined by:

$$\widetilde{J}_{n+3} = \widetilde{J}_{n+2} - \widetilde{J}_{n+1} + \widetilde{J}_n,\ \widetilde{J}_0 = 0,\ \widetilde{J}_1 = 1,\ \widetilde{J}_2 = 1.$$

We have

$$\begin{aligned}\widetilde{J}_n &= \frac{1}{2} + \frac{1}{2(i-1)}(i)^n + \frac{-1}{2(i+1)}(-i)^n \\ &= \frac{1}{2} + \left(\frac{-1}{4} - \frac{1}{4}i\right)(i)^n + \left(\frac{-1}{4} + \frac{1}{4}i\right)(-i)^n, \quad n \in \mathbb{N}_0.\end{aligned}$$

From this it follows that

$$\widetilde{J}_{2n} = \frac{1}{2} - \frac{1}{2}(-1)^n,\ \widetilde{J}_{2n+1} = \frac{1}{2} + \frac{1}{2}(-1)^n.$$

depending on n is even or odd, we get

$$\widetilde{J}_{4n} = 0,\ \widetilde{J}_{4n+1} = 1,\ \widetilde{J}_{4n+2} = 1,\ \widetilde{J}_{4n+3} = 0.$$

Replacing in (1.185) and (1.186), we obtain

$$x_{4n+1} = \frac{1}{x_{-1}x_0 - x_0 + 1},$$

$$x_{4n+2} = \frac{x_{-1}x_0 - x_0 + 1}{x_{-1}x_0},$$

$$x_{4n+3} = x_{-1},$$

$$x_{4n+4} = x_0,$$

and these are the formulas given in [65] for the solutions of (1.188).

4. Consider $a = b = c = -1$, equation (1.184) becomes (1.190). For this choice of the parameters, we get the sequence $\left(\widetilde{J}_n\right)_{n=0}^{\infty}$ defined by:

$$\widetilde{J}_{n+3} = -\widetilde{J}_{n+2} - \widetilde{J}_{n+1} - \widetilde{J}_n,\ \widetilde{J}_0 = 0,\ \widetilde{J}_1 = 1,\ \widetilde{J}_2 = -1.$$

We have

$$\begin{aligned}\widetilde{J}_n &= \frac{-1}{2}(-1)^n + \frac{1}{2(1+i)}(i)^n + \frac{1}{2(1-i)}(-i)^n \\ &= \frac{-1}{2}(-1)^n + \left(\frac{1}{4} - \frac{1}{4}i\right)(i)^n + \left(\frac{1}{4} + \frac{1}{4}i\right)(-i)^n, \quad n \in \mathbb{N}_0.\end{aligned}$$

From this it follows that

$$\widetilde{J}_{2n} = \frac{-1}{2} + \frac{1}{2}(-1)^n,\ \widetilde{J}_{2n+1} = \frac{1}{2} + \frac{1}{2}(-1)^n,$$

depending on n is even or odd, we get

$$\widetilde{J}_{4n} = 0,\ \widetilde{J}_{4n+1} = 1,\ \widetilde{J}_{4n+2} = -1,\ \widetilde{J}_{4n+3} = 0.$$

Replacing in (1.185) and (1.186), we get

$$x_{4n+1} = \frac{1}{-x_{-1}x_0 - x_0 - 1},$$

$$x_{4n+2} = \frac{-x_{-1}x_0 - x_0 - 1}{x_{-1}x_0},$$

$$x_{4n+3} = x_{-1},$$

$$x_{4n+4} = x_0,$$

and these are the formulas given in [65] for the solutions of (1.190).

5. Consider $a = b = c = 1$, system (1.179) becomes (1.192). For this choice of the parameters, the sequence $\left(\widetilde{J}_n\right)_{n=0}^{+\infty}$ will be nothing other than the famous Tribonacci sequence defined for $n \in \mathbb{N}_0$ by

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n,\ T_0 = 0, \quad T_1 = T_2 = 1. \tag{1.194}$$

The solutions are given by

$$\begin{cases} x_{2n+1} = \dfrac{T_{2n}y_0x_{-1} + \left(T_{2n+2} - T_{2n+1}\right)y_0 + T_{2n+1}}{T_{2n+1}y_0x_{-1} + \left(T_{2n+3} - T_{2n+2}\right)y_0 + T_{2n+2}}, \\ x_{2n+2} = \dfrac{T_{2n+1}x_0y_{-1} + \left(T_{2n+3} - T_{2n+2}\right)x_0 + T_{2n+2}}{T_{2n+2}x_0y_{-1} + \left(T_{2n+4} - T_{2n+3}\right)x_0 + T_{2n+3}}, \\ y_{2n+1} = \dfrac{T_{2n}x_0y_{-1} + \left(T_{2n+2} - T_{2n+1}\right)x_0 + T_{2n+1}}{T_{2n+1}x_0y_{-1} + \left(T_{2n+3} - T_{2n+2}\right)x_0 + T_{2n+2}}, \\ y_{2n+2} = \dfrac{T_{2n+1}y_0x_{-1} + \left(T_{2n+3} - T_{2n+2}\right)y_0 + T_{2n+2}}{T_{2n+2}y_0x_{-1} + \left(T_{2n+4} - T_{2n+3}\right)y_0 + T_{2n+3}}. \end{cases} \tag{1.195}$$

6. Consider $a = -1$, $b = 1$, $c = -1$, system (1.179) becomes (1.193). For this choice of the parameters, we get the sequence $\left(\widetilde{J}_n\right)_{n=0}^{\infty}$ defined by:

$$\widetilde{J}_{n+3} = -\widetilde{J}_{n+2} + \widetilde{J}_{n+1} - \widetilde{J}_n,\ \widetilde{J}_0 = 0,\ \widetilde{J}_1 = 1,\ \widetilde{J}_2 = -1.$$

Using the fact that $\widetilde{J}_n = (-1)^{n+1}T_n$ ([2]), we get from (1.28) that the representation of well-defined solutions of system (1.193) is

$$\begin{cases} x_{2n+1} = \dfrac{-\left(T_{2n}y_0x_{-1} + \left(T_{2n+1} - T_{2n+2}\right)y_0 + T_{2n+1}\right)}{T_{2n+1}y_0x_{-1} - \left(T_{2n+1} + T_{2n}\right)y_0 + T_{2n+2}}, \\ x_{2n+2} = \dfrac{-\left(T_{2n+1}x_0y_{-1} + \left(T_{2n+2} - T_{2n+3}\right)x_0 + T_{2n+2}\right)}{T_{2n+2}x_0y_{-1} - \left(T_{2n+2} + T_{2n+1}\right)x_0 + T_{2n+3}}, \\ y_{2n+1} = \dfrac{-\left(T_{2n}x_0y_{-1} + \left(T_{2n+1} - T_{2n+2}\right)x_0 + T_{2n+1}\right)}{T_{2n+1}x_0y_{-1} - \left(T_{2n+1} + T_{2n}\right)x_0 + T_{2n+2}}, \\ y_{2n+2} = \dfrac{-\left(T_{2n+1}y_0x_{-1} + \left(T_{2n+2} - T_{2n+3}\right)y_0 + T_{2n+2}\right)}{T_{2n+2}y_0x_{-1} - \left(T_{2n+2} + T_{2n+1}\right)y_0 + T_{2n+3}}, \end{cases} \tag{1.196}$$

and these are the formulas given in [67] for the solutions of (1.193).

1.4.2.2.3 Case $d = 0, c \neq 0$ and $a = 0$ In this case, we obtain the system

$$\begin{cases} x_{n+1} = f^{-1}\left[\frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})}\right], \\ y_{n+1} = g^{-1}\left[\frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})}\right], \quad n \in \mathbb{N}_0. \end{cases} \tag{1.197}$$

Here, $\left(\widetilde{J}_n\right)_{n\geq 0}$ will be the sequence $(\mathcal{P}_n)_{n\geq 0}$ defined by

$$\mathcal{P}_{n+3} = b\mathcal{P}_{n+1} + c\mathcal{P}_n,\ \mathcal{P}_0 = 0,\ \mathcal{P}_1 = 1,\ \mathcal{P}_2 = 0, n \in \mathbb{N}_0. \tag{1.198}$$

So, well-defined solutions are expressed in terms of $(\mathcal{P}_n)_{n\geq 0}$ and takes the form

$$x_{2n+1} = f^{-1}\left[\frac{c\mathcal{P}_{2n+1} + \mathcal{P}_{2n+3}f(x_{-1}) + \mathcal{P}_{2n+2}g(y_0)f(x_{-1})}{c\mathcal{P}_{2n} + \mathcal{P}_{2n+2}f(x_{-1}) + \mathcal{P}_{2n+1}g(y_0)f(x_{-1})}\right], n \in \mathbb{N}_0, \tag{1.199}$$

$$x_{2n+2} = f^{-1}\left[\frac{c\mathcal{P}_{2n+2} + \mathcal{P}_{2n+4}g(y_{-1}) + \mathcal{P}_{2n+3}f(x_0)g(y_{-1})}{c\mathcal{P}_{2n+1} + \mathcal{P}_{2n+3}g(y_{-1}) + \mathcal{P}_{2n+2}f(x_0)g(y_{-1})}\right], n \in \mathbb{N}_0, \tag{1.200}$$

$$y_{2n+1} = g^{-1}\left[\frac{c\mathcal{P}_{2n+1} + \mathcal{P}_{2n+3}g(y_{-1}) + \mathcal{P}_{2n+2}f(x_0)g(y_{-1})}{c\mathcal{P}_{2n} + \mathcal{P}_{2n+2}g(y_{-1}) + \mathcal{P}_{2n+1}f(x_0)g(y_{-1})}\right], n \in \mathbb{N}_0, \tag{1.201}$$

$$y_{2n+2} = g^{-1}\left[\frac{c\mathcal{P}_{2n+2} + \mathcal{P}_{2n+4}f(x_{-1}) + \mathcal{P}_{2n+3}g(y_0)f(x_{-1})}{c\mathcal{P}_{2n+1} + \mathcal{P}_{2n+3}f(x_{-1}) + \mathcal{P}_{2n+2}g(y_0)f(x_{-1})}\right], n \in \mathbb{N}_0. \tag{1.202}$$

for system (1.197) and by

$$x_{n+1} = f^{-1}\left[\frac{c\mathcal{P}_{n+1} + \mathcal{P}_{n+3}f(x_{-1}) + \mathcal{P}_{n+2}f(x_0)f(x_{-1})}{c\mathcal{P}_n + \mathcal{P}_{n+2}f(x_{-1}) + \mathcal{P}_{n+1}f(x_0)f(x_{-1})}\right], \ n \in \mathbb{N}_0,$$

for its one dimensional version, that is the equation

$$x_{n+1} = f^{-1}\left[\frac{b}{f(x_n)} + \frac{c}{f(x_n)f(x_{n-1})}\right], \quad n \in \mathbb{N}_0.$$

Also, (1.197) generalize some works in the literature, see, for example [44] and [124].

1.4.2.2.4 Case $c = d = 0$, $b \neq 0$ In this case, we get the system

$$x_{n+1} = f^{-1}\left[a + \frac{b}{g(y_n)}\right], y_{n+1} = g^{-1}\left[a + \frac{b}{f(x_n)}\right], \quad n \in \mathbb{N}_0. \tag{1.203}$$

In this case, the well-defined solutions will be expressed using terms of the sequence $\left(\widetilde{F}_n\right)_{n\geq 0} = \left(\widetilde{J}_{n+1}\right)_{n\geq 0}$, defined by

$$\widetilde{F}_{n+2} = a\widetilde{F}_{n+1} + b\widetilde{F}_n,\ \widetilde{F}_0 = 1,\ \widetilde{F}_1 = a,\ n \in \mathbb{N}_0.$$

and the solutions for (1.203) and its one dimensional version, are obtained from Corollary 1.27, by writing $\widetilde{F}_{n-1}$ instead of $\widetilde{J}_n$.
System (1.197) and its one dimensional version, generalized some existing works, for example [104, 105].

Remark 1.4.5. If, $b = c = d = 0$, $a \neq 0$, we get

$$x_n = f^{-1}(a),\ y_n = g^{-1}(a),\ n = 1, 2, \cdots .$$

Chapter 2

On a homogeneous system of difference equations of second order

2.1 Introduction

A lot of studies are devoted to the subject of difference equations, mainly, in two directions. The goal of the first one, is to find explicit formulas for well defined solutions, and then using these formulas to deduce the behavior of the solutions. The second direction is concerned by studying the stability of the corresponding equilibrium points and this is done by using the Lyapunov stability theory, we can consult [2, 12, 13, 14, 30, 31, 32, 35, 41, 45, 48, 68, 91, 92, 111, 108, 118, 116, 127, 126]. Noting also that a huge number of models of difference equations investigated by researchers are defined by homogeneous functions of different order see for example [6, 15, 27, 39].

In the present chapter, we will study the following general system of difference equations defined by

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(z_n, z_{n-1}),\ z_{n+1} = h(x_n, x_{n-1}) \tag{2.1}$$

where $n \in \mathbb{N}_0$, the initial values x_{-1}, x_0, y_{-1}, y_0, z_{-1} and z_0 are positive real numbers, the functions f, g, $h :\ (0, +\infty)^2 \to (0, +\infty)$ are continuous and homogeneous of degree zero. Clearly if we take $z_{-i} = x_{-i}$, $i = 1, 2$, and $h \equiv g$, then the system (2.1), will be

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(x_n, x_{n-1}) \tag{2.2}$$

Noting also that if we choose $z_{-i} = y_{-i} = x_{-i}$, $i = 1, 2$, and $h \equiv g \equiv f$, then system (2.1), will be

$$x_{n+1} = f(x_n, x_{n-1}). \tag{2.3}$$

In [114], the behavior of the solutions of system (2.2) has been investigated. System (2.2) is a generalization of equation (2.3), studied in [62]. The present system (2.1) is the three dimensional generalization of system (2.2).

Now we recall some known definitions and results, which will be very useful for the sequel, for more details we can consult for example the following references [11, 16, 29, 58].

Definition 2.1. *A function* $\Phi : (0, +\infty)^2 \to (0, +\infty)$ *is said to be homogeneous of degree* $m \in \mathbb{R}$ *if for all* $(u, v) \in (0, +\infty)^2$ *and for all* $\lambda > 0$ *we have,*

$$\Phi(\lambda u, \lambda v) = \lambda^m \Phi(u, v).$$

Theorem 2.1. *Let* $\Phi : (0, +\infty)^2 \to (0, +\infty)$ *be a* C^1 *function on* $(0, +\infty)^2$.

1. *Then,* Φ *is homogeneous of degree* m *if and only if*

$$u\frac{\partial \Phi}{\partial u}(u, v) + v\frac{\partial \Phi}{\partial v}(u, v) = m\Phi(u, v),\ (u, v) \in (0, +\infty)^2.$$

   *(This statement, is usually called **Euler's Theorem**).*

2. *If* Φ *is homogeneous of degree* m *on* $(0, +\infty)^2$*, then* $\frac{\partial \Phi}{\partial u}$ *and* $\frac{\partial \Phi}{\partial v}$ *are homogeneous of degree* $m - 1$ *on* $(0, +\infty)^2$.

2.2 Local and global stability of the unique equilibrium points

A point $(\overline{x}, \overline{y}, \overline{z}) \in (0, +\infty)^3$ is an equilibrium point of system (2.1) if it is a solution of the following system

$$\overline{x} = f(\overline{y}, \overline{y}),\ \overline{y} = g(\overline{z}, \overline{z}),\ \overline{z} = h(\overline{x}, \overline{x}).$$

Using the fact that f, g and h are homogeneous of degree zero, we get that

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1, 1), g(1, 1), h(1, 1))$$

is the unique equilibrium point of our system (2.1).

Let $F : (0,+\infty)^6 \to (0,+\infty)^6$ be the function defined by

$$F(W) = (f_1(W), f_2(W), g_1(W), g_2(W), h_1(W), h_2(W)), \; W = (u, v, w, t, r, s)$$

with

$$f_1(W) = f(w,t), \; f_2(W) = u, \; g_1(W) = g(r,s), \; g_2(W) = w, \; h_1(W) = h(u,v), \; g_2(W) = r.$$

Then, system (2.1) can be written as follows

$$W_{n+1} = F(W_n), \; W_n = (x_n, x_{n-1}, y_n, y_{n-1}, z_n, z_{n-1})^t, \; n \in \mathbb{N}_0.$$

So, $(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1))$ is an equilibrium point of system (2.1) if and only if

$$\overline{W} = (\overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{z}, \overline{z}) = (f(1,1), f(1,1), g(1,1), g(1,1), h(1,1), h(1,1))$$

is an equilibrium point of $W_{n+1} = F(W_n)$.

Assume that the functions f, g and h are C^1 on $(0,+\infty)^2$. To system (2.1), we associate about the equilibrium point $\overline{W}$ the following linear system

$$X_{n+1} = J_F X_n, \; n \in \mathbb{N}_0$$

where J_F is the Jacobian matrix associated to the function F evaluated at

$$\overline{W} = (f(1,1), f(1,1), g(1,1), g(1,1), h(1,1), h(1,1)).$$

We have

$$J_F = \begin{pmatrix} 0 & 0 & \frac{\partial f}{\partial w}(\overline{y},\overline{y}) & \frac{\partial f}{\partial t}(\overline{y},\overline{y}) & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial g}{\partial r}(\overline{z},\overline{z}) & \frac{\partial g}{\partial s}(\overline{z},\overline{z}) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial h}{\partial u}(\overline{x},\overline{x}) & \frac{\partial h}{\partial v}(\overline{x},\overline{x}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

As f, g and h are homogeneous of degree 0, then using Part 1. of Theorem 2.1, we get

$$\overline{y}\frac{\partial f}{\partial w}(\overline{y},\overline{y}) + \overline{y}\frac{\partial f}{\partial t}(\overline{y},\overline{y}) = 0$$

which implies

$$\frac{\partial f}{\partial t}(\overline{y},\overline{y}) = -\frac{\partial f}{\partial w}(\overline{y},\overline{y}).$$

Similarly we get

$$\frac{\partial g}{\partial s}(\overline{z},\overline{z}) = -\frac{\partial g}{\partial r}(\overline{z},\overline{z}),\ \frac{\partial h}{\partial v}(\overline{x},\overline{x}) = -\frac{\partial h}{\partial u}(\overline{x},\overline{x}).$$

It follows that J_F takes the form:

$$J_F = \begin{pmatrix} 0 & 0 & \frac{\partial f}{\partial w}(\overline{y},\overline{y}) & -\frac{\partial f}{\partial w}(\overline{y},\overline{y}) & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial g}{\partial r}(\overline{z},\overline{z}) & -\frac{\partial g}{\partial r}(\overline{z},\overline{z}) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\partial h}{\partial u}(\overline{x},\overline{x}) & -\frac{\partial h}{\partial u}(\overline{x},\overline{x}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of the matrix J_F is given by

$$\begin{aligned} P(\lambda) &= \lambda^6 - \frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda^3 + 3\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda^2 \\ &\quad -3\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda + \frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y}). \end{aligned}$$

Now assume that

$$\left|\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\right| < \frac{1}{8}$$

and consider the two functions

$$\Phi(\lambda) = \lambda^6,$$

$$\begin{aligned} \Psi(\lambda) &= -\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda^3 + 3\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda^2 \\ &\quad -3\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\lambda + \frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y}). \end{aligned}$$

We have

$$|\Psi(\lambda)| \leq 8\left|\frac{\partial h}{\partial u}(\overline{x},\overline{x})\frac{\partial g}{\partial r}(\overline{z},\overline{z})\frac{\partial f}{\partial w}(\overline{y},\overline{y})\right| < 1 = |\Phi(\lambda)|,\ \forall\,\lambda\in\mathbb{C}:\ |\lambda| = 1.$$

So, by Rouché's theorem it follows that all roots of $P(\lambda)$ lie inside the unit disk.
Hence by Theorem 1.1, we deduce that the equilibrium point $(\overline{x},\overline{y},\overline{z}) = (f(1,1), g(1,1), h(1,1))$ is locally asymptotically stable.
Using Part 2. of Theorem 2.1 and using the fact that f, g and h are homogeneous of degree zero, we get that $\frac{\partial f}{\partial w}$, $\frac{\partial g}{\partial r}$ and $\frac{\partial h}{\partial u}$ are homogeneous of degree -1. So, it follows that

$$\frac{\partial f}{\partial w}(\overline{y},\overline{y}) = \frac{\frac{\partial f}{\partial w}(1,1)}{\overline{y}},\ \frac{\partial g}{\partial r}(\overline{z},\overline{z}) = \frac{\frac{\partial g}{\partial r}(1,1)}{\overline{z}},\ \frac{\partial h}{\partial u}(\overline{x},\overline{x}) = \frac{\frac{\partial h}{\partial u}(1,1)}{\overline{x}}.$$

In summary, we have proved the following result.

Theorem 2.2. *Assume that* $f(u,v)$, $g(u,v)$ *and* $h(u,v)$ *are* C^1 *on* $(0,+\infty)^2$. *The equilibrium point*

$$(\overline{x},\overline{y},\overline{z}) = (f(1,1), g(1,1), h(1,1))$$

of system (2.1) *is locally asymptotically stable if*

$$\left|\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial u}(1,1)\frac{\partial h}{\partial u}(1,1)\right| < \frac{f(1,1)g(1,1)h(1,1)}{8}.$$

In order to achieve our results on the stability of the equilibrium point $(\overline{x},\overline{y},\overline{z}) = (f(1,1), g(1,1), h(1,1))$ we need to prove that this equilibrium point is a global attractor. To this goal, we will prove the following general convergence theorems.

Theorem 2.3. *Consider system* (2.1). *Assume that the following statements are trues:*

1. H_1: *There exist* a, b, α, β, λ, $\gamma \in (0,+\infty)$ *such that*

$$a \le f(u,v) \le b,\ \alpha \le g(u,v) \le \beta,\ \lambda \le h(u,v) \le \gamma,\ \forall (u,v) \in (0,+\infty)^2.$$

2. H_2: $f(u,v)$, $g(u,v)$, $h(u,v)$ *are increasing in* u *for all* v *and decreasing in* v *for all* u.

3. H_3: *If* $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ *is a solution of the system*

$$m_1 = f(m_2, M_2),\ M_1 = f(M_2, m_2),\ m_2 = g(m_3, M_3),\ M_2 = g(M_3, m_3),\ m_3 = h(m_1, M_1),$$
$$M_3 = h(M_1, m_1),$$

then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of system (2.1) *converges to the unique equilibrium point*

$$(\overline{x},\overline{y},\overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$m_1^{i+1} := f(m_2^i, M_2^i),\ M_1^{i+1} := f(M_2^i, m_2^i),$$

$$m_2^{i+1} := g(m_3^i, M_3^i),\ M_2^{i+1} := g(M_3^i, m_3^i),$$

$$m_3^{i+1} := h(m_1^i, M_1^i),\ M_3^{i+1} := h(M_1^i, m_1^i).$$

We have

$$a \le f(\alpha, \beta) \le f(\beta, \alpha) \le b,$$

$$\alpha \le g(\lambda, \gamma) \le g(\gamma, \lambda) \le \beta,$$

$$\lambda \le h(a, b) \le h(b, a) \le \gamma,$$

and so,

$$m_1^0 = a \le f(m_2^0, M_2^0) \le f(M_2^0, m_2^0) \le b = M_1^0,$$

$$m_2^0 = \alpha \le g(m_3^0, M_3^0) \le g(M_3^0, m_3^0) \le \beta = M_2^0,$$

and

$$m_3^0 = \lambda \le h(m_1^0, M_1^0) \le g(M_1^0, m_1^0) \le \gamma = M_3^0$$

Hence,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$m_1^1 = f(m_2^0, M_2^0) \le f(m_2^1, M_2^1) = m_1^2 \le f(M_2^1, m_2^1) = M_1^2 \le f(M_2^0, m_2^0) = M_1^1,$$

$$m_2^1 = g(m_3^0, M_3^0) \le g(m_3^1, M_3^1) = m_2^2 \le g(M_3^1, m_3^1) = M_2^2 \le g(M_3^0, m_3^0) = M_2^1,$$

$$m_3^1 = h(m_1^0, M_1^0) \le h(m_1^1, M_1^1) = m_3^2 \le h(M_1^1, m_1^1) = M_3^2 \le h(M_1^0, m_1^0) = M_2^1,$$

and it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$a = m_1^0 \leq m_1^1 \leq ... \leq m_1^{i-1} \leq m_1^i \leq M_1^i \leq M_1^{i-1} \leq ... \leq M_1^1 \leq M_1^0 = b,$$

$$\alpha = m_2^0 \leq m_2^1 \leq ... \leq m_2^{i-1} \leq m_2^i \leq M_2^i \leq M_2^{i-1} \leq ... \leq M_2^1 \leq M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \leq m_3^1 \leq ... \leq m_3^{i-1} \leq m_3^i \leq M_3^i \leq M_3^{i-1} \leq ... \leq M_3^1 \leq M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ m_2 = \lim_{i\to+\infty} m_2^i, m_3 = \lim_{i\to+\infty} m_3^i,$$
$$M_1 = \lim_{i\to+\infty} M_1^i,\ M_2 = \lim_{i\to+\infty} M_2^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \leq m_1 \leq M_1 \leq b,\ \alpha \leq m_2 \leq M_2 \leq \beta,\ \lambda \leq m_3 \leq M_3 \leq \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(m_2^i, M_2^i),\ M_1^{i+1} = f(M_2^i, m_2^i),$$

$$m_2^{i+1} = g(m_3^i, M_3^i),\ M_2^{i+1} = g(M_3^i, m_3^i),$$

$$m_3^{i+1} = h(m_1^i, M_1^i),\ M_3^{i+1} = h(M_1^i, m_1^i),$$

and using the continuity of f, g and h we obtain

$$m_1 = f(m_2, M_2),\ m_2 = f(M_2, m_2),\ m_2 = g(m_3, M_3),\ M_2 = g(M_3, m_3),\ m_3 = h(m_1, M_1),$$
$$M_3 = h(M_1, m_1)$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, we get

$$m_1^0 = a \leq x_n \leq b = M_1^0,\ m_2^0 = \alpha \leq y_n \leq \beta = M_2^0,\ m_3^0 = \lambda \leq z_n \leq \gamma = M_3^0,\ n = 1, 2, \cdots .$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(m_2^0, M_2^0) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(M_2^0, m_2^0) = M_1^1,$$

$$m_2^1 = g(m_3^0, M_3^0) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(M_3^0, m_3^0) = M_3^1,$$

and

$$m_3^1 = h(m_1^0, M_1^0) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^0, m_1^0) = M_3^1$$

that is

$$m_1^1 \leq x_n \leq M_1^1,\ m_2^1 \leq y_n \leq M_2^1,\ m_3^1 \leq z_n \leq M_3^1,\ n = 3, 4, \cdots .$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(m_2^1, M_2^1) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(M_2^1, m_2^1) = M_1^2,$$

and

$$m_2^2 = g(m_3^1, M_3^1) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(M_3^1, m_3^1) = M_2^2,$$

$$m_3^2 = h(m_1^1, M_1^1) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^1, m_1^1) = M_3^2$$

that is

$$m_1^2 \leq x_n \leq M_1^2,\ m_2^2 \leq y_n \leq M_2^2,\ m_3^2 \leq z_n \leq M_3^2,\ n = 5, 6, \cdots .$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(m_2^2, M_2^2) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(M_2^2, m_2^2) = M_1^3,$$

$$m_2^3 = g(m_3^2, M_3^2) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(M_3^2, m_3^2) = M_2^3,$$

and

$$m_3^3 = h(m_1^2, M_1^2) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^2, m_1^2) = M_3^3$$

that is

$$m_1^3 \leq x_n \leq M_1^3,\ m_2^3 \leq y_n \leq M_2^3,\ m_3^3 \leq z_n \leq M_3^3,\ n = 7, 8, \cdots .$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \leq x_n \leq M_1^i,\ m_2^i \leq y_n \leq M_2^i,\ m_3^i \leq z_n \leq M_3^i,\ n \geq 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ ,\ \lim_{n\to+\infty} z_n = M_3.$$

From (2.1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1, 1),\ M_2 = g(M_3, M_3) = g(1, 1),\ M_3 = h(M_1, M_1) = h(1, 1).$$

□

Theorem 2.4. *Consider system (2.1). Assume that the following statements are trues:*

1. H_1: *There exist* a, b, α, β, λ, $\gamma \in (0, +\infty)$ *such that*

$$a \leq f(u,v) \leq b,\ \alpha \leq g(u,v) \leq \beta,\ \lambda \leq h(u,v) \leq \gamma,\ \forall (u,v) \in (0,+\infty)^2.$$

2. H_2: $f(u,v)$, $g(u,v)$ *are increasing in* u *for all* v *and decreasing in* v *for all* u *and* $h(u,v)$ *is decreasing in* u *for all* v *and increasing in* v *for all* u.

3. H_3: *If* $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ *is a solution of the system*

$m_1 = f(m_2, M_2)$, $M_1 = f(M_2, m_2)$, $m_2 = g(m_3, M_3)$, $M_2 = g(M_3, m_3)$, $m_3 = h(M_1, m_1)$, $M_3 = h(m_1, M_1)$

then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of system (2.1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$m_1^{i+1} := f(m_2^i, M_2^i),\ M_1^{i+1} := f(M_2^i, m_2^i),$$

$$m_2^{i+1} := g(m_3^i, M_3^i),\ M_2^{i+1} := g(M_3^i, m_3^i),$$

$$m_3^{i+1} := h(M_1^i, m_1^i),\ M_3^{i+1} := h(m_1^i, M_1^i).$$

We have

$$a \leq f(\alpha, \beta) \leq f(\beta, \alpha) \leq b,$$

$$\alpha \leq g(\lambda, \gamma) \leq g(\gamma, \lambda) \leq \beta,$$

$$\lambda \leq h(b, a) \leq h(a, b) \leq \gamma,$$

and so,

$$m_1^0 = a \leq f(m_2^0, M_2^0) \leq f(M_2^0, m_2^0) \leq b = M_1^0,$$

$$m_2^0 = \alpha \leq g(m_3^0, M_3^0) \leq g(M_3^0, m_3^0) \leq \beta = M_2^0,$$

and

$$m_3^0 = \lambda \le h(M_1^0, m_1^0) \le h(m_1^0, M_1^0) \le \gamma = M_3^0$$

Hence,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$m_1^1 = f(m_2^0, M_2^0) \le f(m_2^1, M_2^1) = m_1^2 \le f(M_2^1, m_2^1) = M_1^2 \le f(M_2^0, m_2^0) = M_1^1,$$

$$m_2^1 = g(m_3^0, M_3^0) \le g(m_3^1, M_3^1) = m_2^2 \le g(M_3^1, m_3^1) = M_2^2 \le g(M_3^0, m_3^0) = M_2^1,$$

$$m_3^1 = h(M_1^0, m_1^0) \le h(M_1^1, m_1^1) = m_3^2 \le h(m_1^1, M_1^1) = M_3^2 \le h(m_1^0, M_1^0) = M_2^1,$$

and it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b,$$

$$\alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ m_2 = \lim_{i\to+\infty} m_2^i, m_3 = \lim_{i\to+\infty} m_3^i,$$

$$M_1 = \lim_{i\to+\infty} M_1^i,\ M_2 = \lim_{i\to+\infty} M_2^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(m_2^i, M_2^i),\ M_1^{i+1} = f(M_2^i, m_2^i),$$

$$m_2^{i+1} = g(m_3^i, M_3^i),\ M_2^{i+1} = g(M_3^i, m_3^i),$$

$$m_3^{i+1} = h(M_1^i, m_1^i),\ M_3^{i+1} = h(m_1^i, M_1^i),$$

and using the continuity of f, g and h we obtain

$$m_1 = f(m_2, M_2),\ m_2 = f(M_2, m_2),\ m_2 = g(m_3, M_3),\ M_2 = g(M_3, m_3),\ m_3 = h(M_1, m_1),$$
$$M_3 = h(m_1, M_1)$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0,\ n = 1, 2, \cdots .$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(m_2^0, M_2^0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^0, m_2^0) = M_1^1,$$

$$m_2^1 = g(m_3^0, M_3^0) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^0, m_3^0) = M_3^1,$$

and

$$m_3^1 = h(M_1^0, m_1^0) \le z_{n+1} = h(x_n, x_{n-1}) \le h(m_1^0, M_1^0) = M_3^1$$

that is

$$m_1^1 \le x_n \le M_1^1,\ m_2^1 \le y_n \le M_2^1,\ m_3^1 \le z_n \le M_3^1,\ n = 3, 4, \cdots .$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(m_2^1, M_2^1) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^1, m_2^1) = M_1^2,$$

and

$$m_2^2 = g(m_3^1, M_3^1) \le y_{n+1} = g(z_n, z_{n-1}) \le g(M_3^1, m_3^1) = M_2^2,$$

$$m_3^2 = h(M_1^1, m_1^1) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^1, M_1^1) = M_3^2$$

that is

$$m_1^2 \leq x_n \leq M_1^2,\ m_2^2 \leq y_n \leq M_2^2,\ m_3^2 \leq z_n \leq M_3^2,\ n = 5, 6, \cdots .$$

Similarly, for $n = 6, 7, ...,$ we have

$$m_1^3 = f(m_2^2, M_2^2) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(M_2^2, m_2^2) = M_1^3,$$

$$m_2^3 = g(m_3^2, M_3^2) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(M_3^2, m_3^2) = M_2^3,$$

and

$$m_3^3 = h(M_1^2, m_1^2) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^2, M_1^2) = M_3^3$$

that is

$$m_1^3 \leq x_n \leq M_1^3,\ m_2^3 \leq y_n \leq M_2^3,\ m_3^3 \leq z_n \leq M_3^3,\ n = 7, 8, \cdots .$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \leq x_n \leq M_1^i,\ m_2^i \leq y_n \leq M_2^i,\ m_3^i \leq z_n \leq M_3^i,\ n \geq 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ ,\ \lim_{n\to+\infty} z_n = M_3.$$

From (2.1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1, 1),\ M_2 = g(M_3, M_3) = g(1, 1),\ M_3 = h(M_1, M_1) = h(1, 1).$$

□

Theorem 2.5. *Consider system (2.1). Assume that the following statements are trues:*

1. H_1*: There exist* a, b, α, β, λ, $\gamma \in (0, +\infty)$ *such that*

$$a \leq f(u, v) \leq b,\ \alpha \leq g(u, v) \leq \beta,\ \lambda \leq h(u, v) \leq \gamma,\ \forall (u, v) \in (0, +\infty)^2 .$$

2. H_2*:* $f(u, v)$ *is increasing in* u *for all* v *and decreasing in* v *for all* u *and* $g(u, v)$, $h(u, v)$ *are decreasing in* u *for all* v *and increasing in* v *for all* u.

3. *H_3: If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ is a solution of the system*

$m_1 = f(m_2, M_2),\ M_1 = f(M_2, m_2),\ m_2 = g(M_3, m_3),\ M_2 = g(m_3, M_3),\ m_3 = h(M_1, m_1),$
$M_3 = h(m_1, M_1)$

then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of system (2.1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$m_1^{i+1} := f(m_2^i, M_2^i),\ M_1^{i+1} := f(M_2^i, m_2^i),$$

$$m_2^{i+1} := g(M_3^i, m_3^i),\ M_2^{i+1} := g(m_3^i, M_3^i),$$

$$m_3^{i+1} := h(M_1^i, m_1^i),\ M_3^{i+1} := h(m_1^i, M_1^i).$$

We have

$$a \le f(\alpha, \beta) \le f(\beta, \alpha) \le b,$$

$$\alpha \le g(\gamma, \lambda) \le g(\lambda, \gamma) \le \beta,$$

$$\lambda \le h(b, a) \le h(a, b) \le \gamma,$$

and so,

$$m_1^0 = a \le f(m_2^0, M_2^0) \le f(M_2^0, m_2^0) \le b = M_1^0,$$

$$m_2^0 = \alpha \le g(M_3^0, m_3^0) \le g(m_3^0, M_3^0) \le \beta = M_2^0,$$

and

$$m_3^0 = \lambda \le h(M_1^0, m_1^0) \le h(m_1^0, M_1^0) \le \gamma = M_3^0$$

Hence,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \leq m_3^1 \leq M_3^1 \leq M_3^0.$$

Now, we have

$$m_1^1 = f(m_2^0, M_2^0) \leq f(m_2^1, M_2^1) = m_1^2 \leq f(M_2^1, m_2^1) = M_1^2 \leq f(M_2^0, m_2^0) = M_1^1,$$

$$m_2^1 = g(M_3^0, m_3^0) \leq g(M_3^1, m_3^1) = m_2^2 \leq g(m_3^1, M_3^1) = M_2^2 \leq g(m_3^0, M_3^0) = M_2^1,$$

$$m_3^1 = h(M_1^0, m_1^0) \leq h(M_1^1, m_1^1) = m_3^2 \leq h(m_1^1, M_1^1) = M_3^2 \leq h(m_1^0, M_1^0) = M_2^1,$$

and it follows that

$$m_1^0 \leq m_1^1 \leq m_1^2 \leq M_1^2 \leq M_1^1 \leq M_1^0,$$

$$m_2^0 \leq m_2^1 \leq m_2^2 \leq M_2^2 \leq M_2^1 \leq M_2^0,$$

and

$$m_3^0 \leq m_3^1 \leq m_3^2 \leq M_3^2 \leq M_3^1 \leq M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$a = m_1^0 \leq m_1^1 \leq ... \leq m_1^{i-1} \leq m_1^i \leq M_1^i \leq M_1^{i-1} \leq ... \leq M_1^1 \leq M_1^0 = b,$$

$$\alpha = m_2^0 \leq m_2^1 \leq ... \leq m_2^{i-1} \leq m_2^i \leq M_2^i \leq M_2^{i-1} \leq ... \leq M_2^1 \leq M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \leq m_3^1 \leq ... \leq m_3^{i-1} \leq m_3^i \leq M_3^i \leq M_3^{i-1} \leq ... \leq M_3^1 \leq M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ m_2 = \lim_{i\to+\infty} m_2^i, m_3 = \lim_{i\to+\infty} m_3^i,$$

$$M_1 = \lim_{i\to+\infty} M_1^i,\ M_2 = \lim_{i\to+\infty} M_2^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \leq m_1 \leq M_1 \leq b,\ \alpha \leq m_2 \leq M_2 \leq \beta,\ \lambda \leq m_3 \leq M_3 \leq \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(m_2^i, M_2^i),\ M_1^{i+1} = f(M_2^i, m_2^i),$$

$$m_2^{i+1} = g(M_3^i, m_3^i),\ M_2^{i+1} = g(m_3^i, M_3^i),$$

$$m_3^{i+1} = h(M_1^i, m_1^i),\ M_3^{i+1} = h(m_1^i, M_1^i),$$

and using the continuity of f, g and h we obtain

$$m_1 = f(m_2, M_2),\ m_2 = f(M_2, m_2),\ m_2 = g(M_3, m_3),\ M_2 = g(m_3, M_3),\ m_3 = h(M_1, m_1),$$
$$M_3 = h(m_1, M_1)$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, we get

$$m_1^0 = a \leq x_n \leq b = M_1^0,\ m_2^0 = \alpha \leq y_n \leq \beta = M_2^0,\ m_3^0 = \lambda \leq z_n \leq \gamma = M_3^0,\ n = 1, 2, \cdots.$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(m_2^0, M_2^0) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(M_2^0, m_2^0) = M_1^1,$$

$$m_2^1 = g(M_3^0, m_3^0) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^0, M_3^0) = M_3^1,$$

and

$$m_3^1 = h(M_1^0, m_1^0) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^0, M_1^0) = M_3^1$$

that is

$$m_1^1 \leq x_n \leq M_1^1,\ m_2^1 \leq y_n \leq M_2^1,\ m_3^1 \leq z_n \leq M_3^1,\ n = 3, 4, \cdots.$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(m_2^1, M_2^1) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(M_2^1, m_2^1) = M_1^2,$$

and

$$m_2^2 = g(M_3^1, m_3^1) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^1, M_3^1) = M_2^2,$$

$$m_3^2 = h(M_1^1, m_1^1) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^1, M_1^1) = M_3^2$$

that is

$$m_1^2 \leq x_n \leq M_1^2,\ m_2^2 \leq y_n \leq M_2^2,\ m_3^2 \leq z_n \leq M_3^2,\ n = 5, 6, \cdots.$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(m_2^2, M_2^2) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(M_2^2, m_2^2) = M_1^3,$$

$$m_2^3 = g(M_3^2, m_3^2) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^2, M_3^2) = M_2^3,$$

and

$$m_3^3 = h(M_1^2, m_1^2) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^2, M_1^2) = M_3^3$$

that is

$$m_1^3 \leq x_n \leq M_1^3,\ m_2^3 \leq y_n \leq M_2^3,\ m_3^3 \leq z_n \leq M_3^3,\ n = 7, 8, \cdots .$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \leq x_n \leq M_1^i,\ m_2^i \leq y_n \leq M_2^i,\ m_3^i \leq z_n \leq M_3^i,\ n \geq 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ ,\ \lim_{n\to+\infty} z_n = M_3.$$

From (2.1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1, 1),\ M_2 = g(M_3, M_3) = g(1, 1),\ M_3 = h(M_1, M_1) = h(1, 1).$$

□

Theorem 2.6. *Consider system (2.1). Assume that the following statements are trues:*

1. *H_1: There exist a, b, α, β, λ, $\gamma \in (0, +\infty)$ such that*

$$a \leq f(u, v) \leq b,\ \alpha \leq g(u, v) \leq \beta,\ \lambda \leq h(u, v) \leq \gamma,\ \forall (u, v) \in (0, +\infty)^2 .$$

2. *H_2: $f(u, v)$, $h(u, v)$ are increasing in u for all v and decreasing in v for all u and $g(u, v)$ is decreasing in u for all v and increasing in v for all u.*

3. *H_3: If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a, b]^2 \times [\alpha, \beta]^2 \times [\lambda, \gamma]^2$ is a solution of the system*

   $m_1 = f(m_2, M_2)$, $M_1 = f(M_2, m_2)$, $m_2 = g(M_3, m_3)$, $M_2 = g(m_3, M_3)$, $m_3 = h(m_1, M_1)$, $M_3 = h(M_1, m_1)$

   then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of system (2.1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1, 1), g(1, 1), h(1, 1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$m_1^{i+1} := f(m_2^i, M_2^i),\ M_1^{i+1} := f(M_2^i, m_2^i),$$

$$m_2^{i+1} := g(M_3^i, m_3^i),\ M_2^{i+1} := g(m_3^i, M_3^i),$$

$$m_3^{i+1} := h(m_1^i, M_1^i),\ M_3^{i+1} := h(M_1^i, m_1^i).$$

We have

$$a \le f(\alpha, \beta) \le f(\beta, \alpha) \le b,$$

$$\alpha \le g(\gamma, \lambda) \le g(\lambda, \gamma) \le \beta,$$

$$\lambda \le h(a, b) \le h(b, a) \le \gamma,$$

and so,

$$m_1^0 = a \le f(m_2^0, M_2^0) \le f(M_2^0, m_2^0) \le b = M_1^0,$$

$$m_2^0 = \alpha \le g(M_3^0, m_3^0) \le g(m_3^0, M_3^0) \le \beta = M_2^0,$$

and

$$m_3^0 = \lambda \le h(m_1^0, M_1^0) \le h(M_1^0, m_1^0) \le \gamma = M_3^0$$

Hence,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$m_1^1 = f(m_2^0, M_2^0) \le f(m_2^1, M_2^1) = m_1^2 \le f(M_2^1, m_2^1) = M_1^2 \le f(M_2^0, m_2^0) = M_1^1,$$

$$m_2^1 = g(M_3^0, m_3^0) \le g(M_3^1, m_3^1) = m_2^2 \le g(m_3^1, M_3^1) = M_2^2 \le g(m_3^0, M_3^0) = M_2^1,$$

$$m_3^1 = h(m_1^0, M_1^0) \le h(m_1^1, M_1^1) = m_3^2 \le h(M_1^1, m_1^1) = M_3^2 \le h(M_1^0, m_1^0) = M_2^1,$$

and it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b,$$

$$\alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ m_2 = \lim_{i\to+\infty} m_2^i, m_3 = \lim_{i\to+\infty} m_3^i,$$

$$M_1 = \lim_{i\to+\infty} M_1^i,\ M_2 = \lim_{i\to+\infty} M_2^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(m_2^i, M_2^i),\ M_1^{i+1} = f(M_2^i, m_2^i),$$

$$m_2^{i+1} = g(M_3^i, m_3^i),\ M_2^{i+1} = g(m_3^i, M_3^i),$$

$$m_3^{i+1} = h(m_1^i, M_1^i),\ M_3^{i+1} = h(M_1^i, m_1^i),$$

and using the continuity of f, g and h we obtain

$$m_1 = f(m_2, M_2),\ m_2 = f(M_2, m_2),\ m_2 = g(M_3, m_3),\ M_2 = g(m_3, M_3),\ m_3 = h(m_1, M_1),$$
$$M_3 = h(M_1, m_1)$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0,\ n = 1, 2, \cdots .$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(m_2^0, M_2^0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^0, m_2^0) = M_1^1,$$

$$m_2^1 = g(M_3^0, m_3^0) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^0, M_3^0) = M_3^1,$$

and

$$m_3^1 = h(m_1^0, M_1^0) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^0, m_1^0) = M_3^1$$

that is

$$m_1^1 \le x_n \le M_1^1,\ m_2^1 \le y_n \le M_2^1,\ m_3^1 \le z_n \le M_3^1,\ n = 3, 4, \cdots .$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(m_2^1, M_2^1) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^1, m_2^1) = M_1^2,$$

and

$$m_2^2 = g(M_3^1, m_3^1) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^1, M_3^1) = M_2^2,$$

$$m_3^2 = h(m_1^1, M_1^1) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^1, m_1^1) = M_3^2$$

that is

$$m_1^2 \le x_n \le M_1^2,\ m_2^2 \le y_n \le M_2^2,\ m_3^2 \le z_n \le M_3^2,\ n = 5, 6, \cdots .$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(m_2^2, M_2^2) \le x_{n+1} = f(y_n, y_{n-1}) \le f(M_2^2, m_2^2) = M_1^3,$$

$$m_2^3 = g(M_3^2, m_3^2) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^2, M_3^2) = M_2^3,$$

and

$$m_3^3 = h(m_1^2, M_1^2) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^2, m_1^2) = M_3^3$$

that is

$$m_1^3 \le x_n \le M_1^3,\ m_2^3 \le y_n \le M_2^3,\ m_3^3 \le z_n \le M_3^3,\ n = 7, 8, \cdots .$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \le x_n \le M_1^i,\ m_2^i \le y_n \le M_2^i,\ m_3^i \le z_n \le M_3^i,\ n \ge 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ ,\ \lim_{n\to+\infty} z_n = M_3.$$

From (2.1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1,1),\ M_2 = g(M_3, M_3) = g(1,1),\ M_3 = h(M_1, M_1) = h(1,1).$$

□

Theorem 2.7. *Consider system (2.1). Assume that the following statements are true:*

1. H_1*: There exist* $a,\ b,\ \alpha,\ \beta, \lambda,\ \gamma \in (0,+\infty)$ *such that*

$$a \le f(u,v) \le b,\ \alpha \le g(u,v) \le \beta,\ \lambda \le h(u,v) \le \gamma,\ \forall (u,v) \in (0,+\infty)^2 .$$

2. H_2*:* $f(u,v),\ g(u,v),\ h(u,v)$ *are decreasing in* u *for all* v *and increasing in* v *for all* u*.*

3. H_3*: If* $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ *is a solution of the system*

$$m_1 = f(M_2, m_2),\ M_1 = f(m_2, M_2),\ m_2 = g(M_3, m_3),\ M_2 = g(m_3, M_3),\ m_3 = h(M_1, m_1),$$
$$M_3 = h(m_1, M_1)$$

then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of system (2.1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$m_1^{i+1} := f(M_2^i, m_2^i),\ M_1^{i+1} := f(m_2^i, M_2^i),$$

$$m_2^{i+1} := g(M_3^i, m_3^i),\ M_2^{i+1} := g(m_3^i, M_3^i),$$

$$m_3^{i+1} := h(M_1^i, m_1^i),\ M_3^{i+1} := h(m_1^i, M_1^i).$$

We have

$$a \leq f(\beta, \alpha) \leq f(\alpha, \beta) \leq b,$$

$$\alpha \leq g(\gamma, \lambda) \leq g(\lambda, \gamma) \leq \beta,$$

$$\lambda \leq h(b, a) \leq h(a, b) \leq \gamma$$

and so,

$$m_1^0 = a \leq f(M_2^0, m_2^0) \leq f(m_2^0, M_2^0) \leq b = M_1^0,$$

$$m_2^0 = \alpha \leq g(M_3^0, m_3^0) \leq g(m_3^0, M_3^0) \leq \beta = M_2^0,$$

$$m_3^0 = \lambda \leq h(M_1^0, m_1^0) \leq h(m_1^0, M_1^0) \leq \gamma = M_3^0.$$

Hence,

$$m_1^0 \leq m_1^1 \leq M_1^1 \leq M_1^0,$$

$$m_2^0 \leq m_2^1 \leq M_2^1 \leq M_2^0,$$

and

$$m_3^0 \leq m_3^1 \leq M_3^1 \leq M_3^0.$$

Now, we have

$$m_1^1 = f(M_2^0, m_2^0) \leq f(M_2^1, m_2^1) = m_1^2 \leq f(m_2^1, M_2^1) = M_1^2 \leq f(m_2^0, M_2^0) = M_1^1,$$

$$m_2^1 = g(M_3^0, m_3^0) \leq g(M_3^1, m_3^1) = m_2^2 \leq g(m_3^1, M_3^1) = M_2^2 \leq g(m_3^0, M_3^0) = M_2^1,$$

$$m_3^1 = h(M_1^0, m_1^0) \leq h(M_1^1, m_1^1) = m_3^2 \leq h(m_1^1, M_1^1) = M_3^2 \leq h(m_1^0, M_1^0) = M_3^1$$

and it follows that

$$m_1^0 \leq m_1^1 \leq m_1^2 \leq M_1^2 \leq M_1^1 \leq M_1^0,$$

$$m_2^0 \leq m_2^1 \leq m_2^2 \leq M_2^2 \leq M_2^1 \leq M_2^0,$$

$$m_3^0 \leq m_3^1 \leq m_3^2 \leq M_3^2 \leq M_3^1 \leq M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$a = m_1^0 \leq m_1^1 \leq ... \leq m_1^{i-1} \leq m_1^i \leq M_1^i \leq M_1^{i-1} \leq ... \leq M_1^1 \leq M_1^0 = b,$$

$$\alpha = m_2^0 \leq m_2^1 \leq ... \leq m_2^{i-1} \leq m_2^i \leq M_2^i \leq M_2^{i-1} \leq ... \leq M_2^1 \leq M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \leq m_3^1 \leq ... \leq m_3^{i-1} \leq m_3^i \leq M_3^i \leq M_3^{i-1} \leq ... \leq M_3^1 \leq M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$ $(m_3^i)_{i\in\mathbb{N}_0}$(resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ m_2 = \lim_{i\to+\infty} m_2^i,\ m_3 = \lim_{i\to+\infty} m_3^i,$$
$$M_1 = \lim_{i\to+\infty} M_1^i,\ M_2 = \lim_{i\to+\infty} M_2^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \leq m_1 \leq M_1 \leq b,\ \alpha \leq m_2 \leq M_2 \leq \beta,\ \lambda \leq m_3 \leq M_3 \leq \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(M_2^i, m_2^i),\ M_1^{i+1} = f(m_2^i, M_2^i),$$

$$m_2^{i+1} = g(M_3^i, m_3^i),\ M_2^{i+1} = g(m_3^i, M_3^i),$$

$$m_3^{i+1} = h(M_1^i, m_1^i),\ M_3^{i+1} = h(m_1^i, M_1^i)$$

and using the continuity of f, g, and h we obtain

$$m_1 = f(M_2, m_2),\ M_1 = f(m_2, M_2),\ m_2 = g(M_3, m_3),\ M_2 = g(m_3, M_3),\ m_3 = h(M_1, m_1),$$
$$M_3 = h(m_1, M_1)$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, we get

$$m_1^0 = a \leq x_n \leq b = M_1^0,\ m_2^0 = \alpha \leq y_n \leq \beta = M_2^0,\ m_3^0 = \lambda \leq z_n \leq \gamma = M_3^0,\ n = 1, 2, \cdots .$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(M_2^0, m_2^0) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^0, M_2^0) = M_1^1,$$

$$m_2^1 = g(M_3^0, m_3^0) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^0, M_3^0) = M_2^1,$$

$$m_3^1 = h(M_1^0, m_1^0) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^0, M_1^0) = M_3^1$$

that is

$$m_1^1 \leq x_n \leq M_1^1,\ m_2^1 \leq y_n \leq M_2^1,\ m_3^1 \leq z_n \leq M_3^1,\ n = 3, 4, \cdots .$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(M_2^1, m_2^1) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^1, M_2^1) = M_1^2,$$

$$m_2^2 = g(M_3^1, m_1^3) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^1, M_1^3) = M_2^2,$$

$$m_3^2 = h(M_1^1, m_1^1) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^1, M_1^1) = M_3^2$$

that is

$$m_1^2 \leq x_n \leq M_1^2,\ m_2^2 \leq y_n \leq M_2^2,\ m_3^2 \leq z_n \leq M_3^2,\ n = 5, 6, \cdots .$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(M_2^2, m_2^2) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^2, M_2^2) = M_1^3,$$

$$m_2^3 = g(M_3^2, m_3^2) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(m_3^2, M_3^2) = M_2^3,$$

$$m_3^3 = h(M_1^2, m_1^2) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^2, M_1^2) = M_2^3$$

that is

$$m_1^3 \leq x_n \leq M_1^3,\ m_2^3 \leq y_n \leq M_2^3,\ m_3^3 \leq z_n \leq M_3^3,\ n = 7, 8, \cdots .$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \leq x_n \leq M_1^i,\ m_2^i \leq y_n \leq M_2^i,\ m_3^i \leq z_n \leq M_3^i,\ n \geq 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ \lim_{n\to+\infty} z_n = M_3.$$

From (2.1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1, 1),\ M_2 = g(M_3, M_3) = g(1, 1),\ M_3 = h(M_1, M_1) = h(1, 1).$$

□

The following theorems can be proved similarly.

Theorem 2.8. *Consider system* (2.1). *Assume that the following statements are true:*

1. *H_1: There exist a, b, α, β, λ, $\gamma \in (0, +\infty)$ such that*

$$a \leq f(u, v) \leq b,\ \alpha \leq g(u, v) \leq \beta,\ \lambda \leq h(u, v) \leq \gamma,\ \forall (u, v) \in (0, +\infty)^2 .$$

2. *H_2: $f(u, v)$, $g(u, v)$ are decreasing in u for all v and increasing in v for all u, however $h(u, v)$ is increasing in u for all v and decreasing in v for all u .*

3. *H_3: If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a, b]^2 \times [\alpha, \beta]^2 \times [\lambda, \gamma]^2$ is a solution of the system*

   $m_1 = f(M_2, m_2),\ M_1 = f(m_2, M_2),\ m_2 = g(M_3, m_3),\ M_2 = g(m_3, M_3),\ m_3 = h(m_1, M_1),$
   $M_3 = h(M_1, m_1)$

   then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of system (2.1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1, 1), g(1, 1), h(1, 1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$m_1^{i+1} := f(M_2^i, m_2^i),\ M_1^{i+1} := f(m_2^i, M_2^i),$$

$$m_2^{i+1} := g(M_3^i, m_3^i),\ M_2^{i+1} := g(m_3^i, M_3^i),$$

$$m_3^{i+1} := h(m_1^i, M_1^i),\ M_3^{i+1} := h(M_1^i, m_1^i).$$

We have

$$a \leq f(\beta, \alpha) \leq f(\alpha, \beta) \leq b,$$

$$\alpha \leq g(\gamma, \lambda) \leq g(\lambda, \gamma) \leq \beta,$$

$$\lambda \leq h(a, b) \leq h(b, a) \leq \gamma,$$

and so,

$$m_1^0 = a \leq f(M_2^0, m_2^0) \leq f(m_2^0, M_2^0) \leq b = M_1^0,$$

$$m_2^0 = \alpha \leq g(M_3^0, m_3^0) \leq g(m_3^0, M_3^0) \leq \beta = M_2^0,$$

and

$$m_3^0 = \lambda \leq h(m_1^0, M_1^0) \leq h(M_1^0, m_1^0) \leq \gamma = M_3^0$$

Hence,

$$m_1^0 \leq m_1^1 \leq M_1^1 \leq M_1^0,$$

$$m_2^0 \leq m_2^1 \leq M_2^1 \leq M_2^0,$$

and

$$m_3^0 \leq m_3^1 \leq M_3^1 \leq M_3^0.$$

Now, we have

$$m_1^1 = f(M_2^0, m_2^0) \leq f(M_2^1, m_2^1) = m_1^2 \leq f(m_2^1, M_2^1) = M_1^2 \leq f(m_2^0, M_2^0) = M_1^1,$$

$$m_2^1 = g(M_3^0, m_3^0) \leq g(M_3^1, m_3^1) = m_2^2 \leq g(m_3^1, M_3^1) = M_2^2 \leq g(m_3^0, M_3^0) = M_2^1,$$

$$m_3^1 = h(m_1^0, M_1^0) \leq h(m_1^1, M_1^1) = m_3^2 \leq h(M_1^1, m_1^1) = M_3^2 \leq h(M_1^0, m_1^0) = M_2^1,$$

and it follows that

$$m_1^0 \leq m_1^1 \leq m_1^2 \leq M_1^2 \leq M_1^1 \leq M_1^0,$$

$$m_2^0 \leq m_2^1 \leq m_2^2 \leq M_2^2 \leq M_2^1 \leq M_2^0,$$

and

$$m_3^0 \leq m_3^1 \leq m_3^2 \leq M_3^2 \leq M_3^1 \leq M_3^0.$$

By induction, we get for $i = 0, 1, ...$, that

$$a = m_1^0 \leq m_1^1 \leq ... \leq m_1^{i-1} \leq m_1^i \leq M_1^i \leq M_1^{i-1} \leq ... \leq M_1^1 \leq M_1^0 = b,$$

$$\alpha = m_2^0 \leq m_2^1 \leq ... \leq m_2^{i-1} \leq m_2^i \leq M_2^i \leq M_2^{i-1} \leq ... \leq M_2^1 \leq M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \leq m_3^1 \leq ... \leq m_3^{i-1} \leq m_3^i \leq M_3^i \leq M_3^{i-1} \leq ... \leq M_3^1 \leq M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$ $(m_3^i)_{i\in\mathbb{N}_0}$(resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ m_2 = \lim_{i\to+\infty} m_2^i,\ m_3 = \lim_{i\to+\infty} m_3^i,$$
$$M_1 = \lim_{i\to+\infty} M_1^i,\ M_2 = \lim_{i\to+\infty} M_2^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(M_2^i, m_2^i),\ M_1^{i+1} = f(m_2^i, M_2^i),$$

$$m_2^{i+1} = g(M_3^i, m_3^i),\ M_2^{i+1} = g(m_3^i, M_3^i),$$

$$m_3^{i+1} = h(m_1^i, M_1^i),\ M_3^{i+1} = h(M_1^i, m_1^i),$$

and using the continuity of f, g, and h we obtain

$$m_1 = f(M_2, m_2),\ M_1 = f(m_2, M_2),\ m_2 = g(M_3, m_3),\ M_2 = g(m_3, M_3),\ m_3 = h(m_1, M_1),$$
$$M_3 = h(M_1, m_1)$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0,\ n = 1, 2, \cdots .$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(M_2^0, m_2^0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^0, M_2^0) = M_1^1,$$

$$m_2^1 = g(M_3^0, m_3^0) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^0, M_3^0) = M_2^1,$$

and

$$m_3^1 = h(m_1^0, M_1^0) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^0, m_1^0) = M_3^1$$

that is

$$m_1^1 \le x_n \le M_1^1,\ m_2^1 \le y_n \le M_2^1,\ m_3^1 \le z_n \le M_3^1,\ n = 3, 4, \cdots .$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(M_2^1, m_2^1) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^1, M_2^1) = M_1^2,$$

$$m_2^2 = g(M_3^1, m_1^3) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^1, M_1^3) = M_2^2,$$

and

$$m_3^2 = h(m_1^1, M_1^1) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^1, m_1^1) = M_3^2$$

that is

$$m_1^2 \le x_n \le M_1^2,\ m_2^2 \le y_n \le M_2^2,\ m_3^2 \le z_n \le M_3^2,\ n = 5, 6, \cdots.$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(M_2^2, m_2^2) \le x_{n+1} = f(y_n, y_{n-1}) \le f(m_2^2, M_2^2) = M_1^3,$$

$$m_2^3 = g(M_3^2, m_3^2) \le y_{n+1} = g(z_n, z_{n-1}) \le g(m_3^2, M_3^2) = M_2^3,$$

and

$$m_3^3 = h(m_1^2, M_1^2) \le z_{n+1} = h(x_n, x_{n-1}) \le h(M_1^2, m_1^2) = M_3^3$$

that is

$$m_1^3 \le x_n \le M_1^3,\ m_2^3 \le y_n \le M_2^3,\ m_3^3 \le z_n \le M_3^3,\ n = 7, 8, \cdots.$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \le x_n \le M_1^i,\ m_2^i \le y_n \le M_2^i,\ m_3^i \le z_n \le M_3^i,\ n \ge 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ \lim_{n\to+\infty} z_n = M_3.$$

From (2.1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1, 1),\ M_2 = g(M_3, M_3) = g(1, 1),\ M_3 = h(M_1, M_1) = h(1, 1).$$

□

Theorem 2.9. *Consider system (2.1). Assume that the following statements are true:*

1. H_1: *There exist* a, b, α, β, λ, $\gamma \in (0, +\infty)$ *such that*

$$a \le f(u, v) \le b,\ \alpha \le g(u, v) \le \beta,\ \lambda \le h(u, v) \le \gamma,\ \forall (u, v) \in (0, +\infty)^2.$$

2. H_2: $f(u, v)$ *is decreasing in* u *for all* v *and increasing in* v *for all* u*, however* $g(u, v)$, $h(u, v)$ *are increasing in* u *for all* v *and decreasing in* v *for all* u *.*

3. *H_3: If $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ is a solution of the system*

   $m_1 = f(M_2, m_2),\ M_1 = f(m_2, M_2),\ m_2 = g(m_3, M_3),\ M_2 = g(M_3, m_3),\ m_3 = h(m_1, M_1),$
   $M_3 = h(M_1, m_1)$

   then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of system (2.1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$m_1^{i+1} := f(M_2^i, m_2^i),\ M_1^{i+1} := f(m_2^i, M_2^i),$$

$$m_2^{i+1} := g(m_3^i, M_3^i),\ M_2^{i+1} := g(M_3^i, m_3^i),$$

$$m_3^{i+1} := h(m_1^i, M_1^i),\ M_3^{i+1} := h(M_1^i, m_1^i).$$

We have

$$a \le f(\beta, \alpha) \le f(\alpha, \beta) \le b,$$

$$\alpha \le g(\lambda, \gamma) \le g(\gamma, \lambda) \le \beta,$$

$$\lambda \le h(a, b) \le h(b, a) \le \gamma,$$

and so,

$$m_1^0 = a \le f(M_2^0, m_2^0) \le f(m_2^0, M_2^0) \le b = M_1^0,$$

$$m_2^0 = \alpha \le g(m_3^0, M_3^0) \le g(M_3^0, m_3^0) \le \beta = M_2^0,$$

and

$$m_3^0 = \lambda \le h(m_1^0, M_1^0) \le g(M_1^0, m_1^0) \le \gamma = M_3^0$$

Hence,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$m_1^1 = f(M_2^0, m_2^0) \le f(M_2^1, m_2^1) = m_1^2 \le f(m_2^1, M_2^1) = M_1^2 \le f(m_2^0, M_2^0) = M_1^1,$$

$$m_2^1 = g(m_3^0, M_3^0) \le g(m_3^1, M_3^1) = m_2^2 \le g(M_3^1, m_3^1) = M_2^2 \le g(M_3^0, m_3^0) = M_2^1,$$

$$m_3^1 = h(m_1^0, M_1^0) \le h(m_1^1, M_1^1) = m_3^2 \le h(M_1^1, m_1^1) = M_3^2 \le h(M_1^0, m_1^0) = M_2^1,$$

and it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b,$$

$$\alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ m_2 = \lim_{i\to+\infty} m_2^i, m_3 = \lim_{i\to+\infty} m_3^i,$$

$$M_1 = \lim_{i\to+\infty} M_1^i,\ M_2 = \lim_{i\to+\infty} M_2^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(M_2^i, m_2^i),\ M_1^{i+1} = f(m_2^i, M_2^i),$$

$$m_2^{i+1} = g(m_3^i, M_3^i),\ M_2^{i+1} = g(M_3^i, m_3^i),$$

$$m_3^{i+1} = h(m_1^i, M_1^i),\ M_3^{i+1} = h(M_1^i, m_1^i),$$

and using the continuity of f, g and h we obtain

$$m_1 = f(M_2, m_2),\ M_1 = f(m_2, M_2),\ m_2 = g(m_3, M_3),\ M_2 = g(M_3, m_3),\ m_3 = h(m_1, M_1),$$
$$M_3 = h(M_1, m_1)$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, we get

$$m_1^0 = a \leq x_n \leq b = M_1^0,\ m_2^0 = \alpha \leq y_n \leq \beta = M_2^0,\ m_3^0 = \lambda \leq z_n \leq \gamma = M_3^0,\ n = 1, 2, \cdots .$$

For $n = 2, 3, ...$, we have

$$m_1^1 = f(M_2^0, m_2^0) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^0, M_2^0) = M_1^1,$$

$$m_2^1 = g(m_3^0, M_3^0) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(M_3^0, m_3^0) = M_3^1,$$

and

$$m_3^1 = h(m_1^0, M_1^0) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^0, m_1^0) = M_3^1$$

that is

$$m_1^1 \leq x_n \leq M_1^1,\ m_2^1 \leq y_n \leq M_2^1,\ m_3^1 \leq z_n \leq M_3^1,\ n = 3, 4, \cdots .$$

Now, for $n = 4, 5, ...$, we have

$$m_1^2 = f(M_2^1, m_2^1) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^1, M_2^1) = M_1^2,$$

$$m_2^2 = g(m_3^1, M_3^1) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(M_3^1, m_3^1) = M_2^2,$$

and

$$m_3^2 = h(m_1^1, M_1^1) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^1, m_1^1) = M_3^2$$

that is

$$m_1^2 \leq x_n \leq M_1^2,\ m_2^2 \leq y_n \leq M_2^2,\ m_3^2 \leq z_n \leq M_3^2,\ n = 5, 6, \cdots .$$

Similarly, for $n = 6, 7, ...$, we have

$$m_1^3 = f(M_2^2, m_2^2) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^2, M_2^2) = M_1^3,$$

$$m_2^3 = g(m_3^2, M_3^2) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(M_3^2, m_3^2) = M_2^3,$$

and

$$m_3^3 = h(m_1^2, M_1^2) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(M_1^2, m_1^2) = M_3^3$$

that is

$$m_1^3 \leq x_n \leq M_1^3,\ m_2^3 \leq y_n \leq M_2^3,\ m_3^3 \leq z_n \leq M_3^3,\ n = 7, 8, \cdots.$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \leq x_n \leq M_1^i,\ m_2^i \leq y_n \leq M_2^i,\ m_3^i \leq z_n \leq M_3^i,\ n \geq 2i+1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2,\ ,\ \lim_{n\to+\infty} z_n = M_3.$$

From (2.1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1,1),\ M_2 = g(M_3, M_3) = g(1,1),\ M_3 = h(M_1, M_1) = h(1,1).$$

□

Theorem 2.10. *Consider system (2.1). Assume that the following statements are true:*

1. H_1: *There exist* a, b, α, β, λ, $\gamma \in (0, +\infty)$ *such that*

$$a \leq f(u,v) \leq b,\ \alpha \leq g(u,v) \leq \beta,\ \lambda \leq h(u,v) \leq \gamma,\ \forall (u,v) \in (0,+\infty)^2.$$

2. H_2: $f(u,v)$, $h(u,v)$ *are decreasing in* u *for all* v *and increasing in* v *for all* u, *however* $g(u,v)$ *is increasing in* u *for all* v *and decreasing in* v *for all* u .

3. H_3: *If* $(m_1, M_1, m_2, M_2, m_3, M_3) \in [a,b]^2 \times [\alpha,\beta]^2 \times [\lambda,\gamma]^2$ *is a solution of the system*

$m_1 = f(M_2, m_2)$, $M_1 = f(m_2, M_2)$, $m_2 = g(m_3, M_3)$, $M_2 = g(M_3, m_3)$, $m_3 = h(M_1, m_1)$, $M_3 = h(m_1, M_1)$

then

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

Then every solution of system (2.1) converges to the unique equilibrium point

$$(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1)).$$

Proof. Let

$$m_1^0 := a,\ M_1^0 := b,\ m_2^0 := \alpha,\ M_2^0 := \beta,\ m_3^0 := \lambda,\ M_3^0 := \gamma$$

and for each $i = 0, 1, ...,$

$$m_1^{i+1} := f(M_2^i, m_2^i),\ M_1^{i+1} := f(m_2^i, M_2^i),$$

$$m_2^{i+1} := g(m_3^i, M_3^i),\ M_2^{i+1} := g(M_3^i, m_3^i),$$

$$m_3^{i+1} := h(M_1^i, m_1^i),\ M_3^{i+1} := h(m_1^i, M_1^i).$$

We have

$$a \le f(\beta, \alpha) \le f(\alpha, \beta) \le b,$$

$$\alpha \le g(\lambda, \gamma) \le g(\gamma, \lambda) \le \beta,$$

$$\lambda \le h(b, a) \le h(a, b) \le \gamma,$$

and so,

$$m_1^0 = a \le f(M_2^0, m_2^0) \le f(m_2^0, M_2^0) \le b = M_1^0,$$

$$m_2^0 = \alpha \le g(m_3^0, M_3^0) \le g(M_3^0, m_3^0) \le \beta = M_2^0,$$

and

$$m_3^0 = \lambda \le h(M_1^0, m_1^0) \le h(m_1^0, M_1^0) \le \gamma = M_3^0$$

Hence,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Now, we have

$$m_1^1 = f(M_2^0, m_2^0) \le f(M_2^1, m_2^1) = m_1^2 \le f(m_2^1, M_2^1) = M_1^2 \le f(m_2^0, M_2^0) = M_1^1,$$

$$m_2^1 = g(m_3^0, M_3^0) \le g(m_3^1, M_3^1) = m_2^2 \le g(M_3^1, m_3^1) = M_2^2 \le g(M_3^0, m_3^0) = M_2^1,$$

$$m_3^1 = h(M_1^0, m_1^0) \le h(M_1^1, m_1^1) = m_3^2 \le h(m_1^1, M_1^1) = M_3^2 \le h(m_1^0, M_1^0) = M_2^1,$$

and it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

and

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we get for $i = 0, 1, ...,$ that

$$a = m_1^0 \le m_1^1 \le ... \le m_1^{i-1} \le m_1^i \le M_1^i \le M_1^{i-1} \le ... \le M_1^1 \le M_1^0 = b,$$

$$\alpha = m_2^0 \le m_2^1 \le ... \le m_2^{i-1} \le m_2^i \le M_2^i \le M_2^{i-1} \le ... \le M_2^1 \le M_2^0 = \beta,$$

and

$$\lambda = m_3^0 \le m_3^1 \le ... \le m_3^{i-1} \le m_3^i \le M_3^i \le M_3^{i-1} \le ... \le M_3^1 \le M_3^0 = \gamma.$$

It follows that the sequences $(m_1^i)_{i\in\mathbb{N}_0}$, $(m_2^i)_{i\in\mathbb{N}_0}$, $(m_3^i)_{i\in\mathbb{N}_0}$ (resp. $(M_1^i)_{i\in\mathbb{N}_0}$, $(M_2^i)_{i\in\mathbb{N}_0}$, $(M_3^i)_{i\in\mathbb{N}_0}$) are increasing (resp. decreasing) and also bounded, so convergent. Let

$$m_1 = \lim_{i\to+\infty} m_1^i,\ m_2 = \lim_{i\to+\infty} m_2^i, m_3 = \lim_{i\to+\infty} m_3^i,$$

$$M_1 = \lim_{i\to+\infty} M_1^i,\ M_2 = \lim_{i\to+\infty} M_2^i,\ M_3 = \lim_{i\to+\infty} M_3^i.$$

Then

$$a \le m_1 \le M_1 \le b,\ \alpha \le m_2 \le M_2 \le \beta,\ \lambda \le m_3 \le M_3 \le \gamma.$$

By taking limits in the following equalities

$$m_1^{i+1} = f(M_2^i, m_2^i),\ M_1^{i+1} = f(m_2^i, M_2^i),$$

$$m_2^{i+1} = g(m_3^i, M_3^i),\ M_2^{i+1} = g(M_3^i, m_3^i),$$

$$m_3^{i+1} = h(M_1^i, m_1^i),\ M_3^{i+1} = h(m_1^i, M_1^i),$$

and using the continuity of f, g and h we obtain

$$m_1 = f(M_2, m_2),\ M_1 = f(m_2, M_2),\ m_2 = g(m_3, M_3),\ M_2 = g(M_3, m_3),\ m_3 = h(M_1, m_1),$$
$$M_3 = h(m_1, M_1)$$

so it follows from H_3 that

$$m_1 = M_1,\ m_2 = M_2,\ m_3 = M_3.$$

From H_1, we get

$$m_1^0 = a \le x_n \le b = M_1^0,\ m_2^0 = \alpha \le y_n \le \beta = M_2^0,\ m_3^0 = \lambda \le z_n \le \gamma = M_3^0,\ n = 1, 2, \cdots.$$

For $n = 2, 3, ...,$ we have

$$m_1^1 = f(M_2^0, m_2^0) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^0, M_2^0) = M_1^1,$$

$$m_2^1 = g(m_3^0, M_3^0) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(M_3^0, m_3^0) = M_3^1,$$

and

$$m_3^1 = h(M_1^0, m_1^0) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^0, M_1^0) = M_3^1$$

that is

$$m_1^1 \leq x_n \leq M_1^1,\ m_2^1 \leq y_n \leq M_2^1,\ m_3^1 \leq z_n \leq M_3^1,\ n = 3, 4, \cdots .$$

Now, for $n = 4, 5, ...,$ we have

$$m_1^2 = f(M_2^1, m_2^1) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^1, M_2^1) = M_1^2,$$

$$m_2^2 = g(m_3^1, M_3^1) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(M_3^1, m_3^1) = M_2^2,$$

and

$$m_3^2 = h(M_1^1, m_1^1) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^1, M_1^1) = M_3^2$$

that is

$$m_1^2 \leq x_n \leq M_1^2,\ m_2^2 \leq y_n \leq M_2^2,\ m_3^2 \leq z_n \leq M_3^2,\ n = 5, 6, \cdots .$$

Similarly, for $n = 6, 7, ...,$ we have

$$m_1^3 = f(M_2^2, m_2^2) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(m_2^2, M_2^2) = M_1^3,$$

$$m_2^3 = g(m_3^2, M_3^2) \leq y_{n+1} = g(z_n, z_{n-1}) \leq g(M_3^2, m_3^2) = M_2^3,$$

and

$$m_3^3 = h(M_1^2, m_1^2) \leq z_{n+1} = h(x_n, x_{n-1}) \leq h(m_1^2, M_1^2) = M_3^3$$

that is

$$m_1^3 \leq x_n \leq M_1^3,\ m_2^3 \leq y_n \leq M_2^3,\ m_3^3 \leq z_n \leq M_3^3,\ n = 7, 8, \cdots .$$

It follows by induction that for $i = 0, 1, ...$ we get

$$m_1^i \leq x_n \leq M_1^i,\ m_2^i \leq y_n \leq M_2^i,\ m_3^i \leq z_n \leq M_3^i,\ n \geq 2i + 1.$$

Using the fact that $i \to +\infty$ implies $n \to +\infty$ and $m_1 = M_1$, $m_2 = M_2$, $m_3 = M_3$, we obtain that

$$\lim_{n\to+\infty} x_n = M_1,\ \lim_{n\to+\infty} y_n = M_2, ,\ \lim_{n\to+\infty} z_n = M_3.$$

From (2.1) and using the fact that f, g and h are continuous and homogeneous of degree zero, we get

$$M_1 = f(M_2, M_2) = f(1,1),\ M_2 = g(M_3, M_3) = g(1,1),\ M_3 = h(M_1, M_1) = h(1,1).$$

□

The following theorem is devoted to global stability of the equilibrium point.

Theorem 2.11. *Under the hypotheses of Theorem 2.2 and one of Theorems 2.3–2.10, the equilibrium point* $(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1))$ *is globally stable.*

2.2.1 Applications

As application of the results of this section we consider the following systems of difference equations

$$x_{n+1} = a_1 + \frac{y_n}{b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = a_2 + \frac{z_n}{b_2 z_n + c_2 z_{n-1}},\ z_{n+1} = a_3 + \frac{x_n}{b_3 x_n + c_3 x_{n-1}},\ n \in \mathbb{N}_0, \tag{2.4}$$

$$x_{n+1} = a_1 + \frac{y_n}{b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = a_2 + \frac{z_n}{b_2 z_n + c_2 z_{n-1}},\ z_{n+1} = a_3 + \frac{x_{n-1}}{b_3 x_n + c_3 x_{n-1}},\ n \in \mathbb{N}_0, \tag{2.5}$$

$$x_{n+1} = a_1 + \frac{y_n}{b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = a_2 + \frac{z_{n-1}}{b_2 z_n + c_2 z_{n-1}},\ z_{n+1} = a_3 + \frac{x_{n-1}}{b_3 x_n + c_3 x_{n-1}},\ n \in \mathbb{N}_0, \tag{2.6}$$

$$x_{n+1} = a_1 + \frac{y_n}{b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = a_2 + \frac{z_{n-1}}{b_2 z_n + c_2 z_{n-1}},\ z_{n+1} = a_3 + \frac{x_n}{b_3 x_n + c_3 x_{n-1}},\ n \in \mathbb{N}_0, \tag{2.7}$$

$$x_{n+1} = a_1 + \frac{y_{n-1}}{b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = a_2 + \frac{z_{n-1}}{b_2 z_n + c_2 z_{n-1}},\ z_{n+1} = a_3 + \frac{x_{n-1}}{b_3 x_n + c_3 x_{n-1}},\ n \in \mathbb{N}_0, \tag{2.8}$$

$$x_{n+1} = a_1 + \frac{y_{n-1}}{b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = a_2 + \frac{z_{n-1}}{b_2 z_n + c_2 z_{n-1}},\ z_{n+1} = a_3 + \frac{x_n}{b_3 x_n + c_3 x_{n-1}},\ n \in \mathbb{N}_0, \tag{2.9}$$

$$x_{n+1} = a_1 + \frac{y_{n-1}}{b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = a_2 + \frac{z_n}{b_2 z_n + c_2 z_{n-1}},\ z_{n+1} = a_3 + \frac{x_n}{b_3 x_n + c_3 x_{n-1}},\ n \in \mathbb{N}_0, \tag{2.10}$$

$$x_{n+1} = a_1 + \frac{y_{n-1}}{b_1 y_n + c_1 y_{n-1}},\ y_{n+1} = a_2 + \frac{z_n}{b_2 z_n + c_2 z_{n-1}},\ z_{n+1} = a_3 + \frac{x_{n-1}}{b_3 x_n + c_3 x_{n-1}},\ n \in \mathbb{N}_0, \tag{2.11}$$

where $x_{-i},\ y_{-i},\ i = 0, 1,\ a_j,\ b_j,\ c_j,\ j = 0, 1, 2$ are positive real numbers.

Let $f_1,\ f_2,\ g_1,\ g_2,\ h_1,\ h_2 : (0,+\infty)^2 \to (0,+\infty)$ be the functions defined by

$$f_1(u,v) = a_1 + \frac{u}{b_1u + c_1v},\ f_2(u,v) = a_1 + \frac{v}{b_1u + c_1v},$$

$$g_1(u,v) = a_2 + \frac{u}{b_2u + c_2v},\ g_2(u,v) = a_2 + \frac{v}{b_2u + c_2v}$$

and

$$h_1(u,v) = a_3 + \frac{u}{b_3u + c_3v},\ h_2(u,v) = a_3 + \frac{v}{b_3u + c_3v}.$$

Then,
system (2.4) will be

$$x_{n+1} = f_1(y_n, y_{n-1}),\ y_{n+1} = g_1(z_n, z_{n-1}),\ z_{n+1} = h_1(x_n, x_{n-1}),$$

system (2.5) will be

$$x_{n+1} = f_1(y_n, y_{n-1}),\ y_{n+1} = g_1(z_n, z_{n-1}),\ z_{n+1} = h_2(x_n, x_{n-1}),$$

system (2.6) will be

$$x_{n+1} = f_1(y_n, y_{n-1}),\ y_{n+1} = g_2(z_n, z_{n-1}),\ z_{n+1} = h_2(x_n, x_{n-1}),$$

system (2.7) will be

$$x_{n+1} = f_1(y_n, y_{n-1}),\ y_{n+1} = g_2(z_n, z_{n-1}),\ z_{n+1} = h_1(x_n, x_{n-1}),$$

system (2.8) will be

$$x_{n+1} = f_2(y_n, y_{n-1}),\ y_{n+1} = g_2(z_n, z_{n-1}),\ z_{n+1} = h_2(x_n, x_{n-1}),$$

system (2.9) will be

$$x_{n+1} = f_2(y_n, y_{n-1}),\ y_{n+1} = g_2(z_n, z_{n-1}),\ z_{n+1} = h_1(x_n, x_{n-1}),$$

system (2.10) will be

$$x_{n+1} = f_2(y_n, y_{n-1}),\ y_{n+1} = g_1(z_n, z_{n-1}),\ z_{n+1} = h_1(x_n, x_{n-1}),$$

system (2.11) will be

$$x_{n+1} = f_2(y_n, y_{n-1}),\ y_{n+1} = g_1(z_n, z_{n-1}),\ z_{n+1} = h_2(x_n, x_{n-1}).$$

Clearly f_1, f_2, g_1, g_2, h_1, h_2 are continuous and homogeneous of degree zero. Also it is not hard to see that:

$$\begin{aligned}
&\frac{\partial f_1}{\partial u}(u,v) = \frac{c_1 v}{(b_1 u + c_1 v)^2} > 0,\ \frac{\partial f_1}{\partial v}(u,v) = -\frac{c_1 u}{(b_1 u + c_1 v)^2} < 0,\\
&\frac{\partial g_1}{\partial u}(u,v) = \frac{c_2 v}{(b_2 u + c_2 v)^2} > 0,\ \frac{\partial g_1}{\partial v}(u,v) = -\frac{c_2 u}{(b_2 u + c_2 v)^2} < 0,\\
&\frac{\partial h_1}{\partial u}(u,v) = \frac{c_3 v}{(b_3 u + c_3 v)^2} > 0,\ \frac{\partial h_1}{\partial v}(u,v) = -\frac{c_3 u}{(b_3 u + c_3 v)^2} < 0,\\
&\frac{\partial f_2}{\partial u}(u,v) = -\frac{b_1 v}{(b_1 u + c_1 v)^2} < 0,\ \frac{\partial f_2}{\partial v}(u,v) = \frac{b_1 u}{(b_1 u + c_1 v)^2} > 0,\\
&\frac{\partial g_2}{\partial u}(u,v) = -\frac{b_2 v}{(b_2 u + c_2 v)^2} < 0,\ \frac{\partial g_2}{\partial v}(u,v) = \frac{b_2 u}{(b_2 u + c_2 v)^2} > 0\\
&\frac{\partial h_2}{\partial u}(u,v) = -\frac{b_3 v}{(b_3 u + c_3 v)^2} < 0,\ \frac{\partial h_2}{\partial v}(u,v) = \frac{b_3 u}{(b_3 u + c_3 v)^2} > 0.
\end{aligned}$$

We have

$$a_1 \le f_1(u,v) \le a_1 + \frac{1}{b_1},\ a_2 \le g_1(u,v) \le a_2 + \frac{1}{b_2},\ a_3 \le h_1(u,v) \le a_3 + \frac{1}{b_3},\ \forall u, v \in (0, +\infty),$$

$$a_1 \le f_2(u,v) \le a_1 + \frac{1}{c_1},\ a_2 \le g_2(u,v) \le a_2 + \frac{1}{c_2},\ a_3 \le h_2(u,v) \le a_3 + \frac{1}{c_3},\ \forall u, v \in (0, +\infty).$$

System (2.4) has the unique equilibrium point

$$(\overline{x}_1, \overline{y}_1, \overline{z}_1) = (f_1(1,1), g_1(1,1), h_1(1,1)) = \left(a_1 + \frac{1}{b_1 + c_1}, a_2 + \frac{1}{b_2 + c_2}, a_3 + \frac{1}{b_3 + c_3}\right).$$

System (2.5) has the unique equilibrium point

$$(\overline{x}_2, \overline{y}_2, \overline{z}_2) = (f_1(1,1), g_1(1,1), h_2(1,1)) = \left(a_1 + \frac{1}{b_1 + c_1}, a_2 + \frac{1}{b_2 + c_2}, a_3 + \frac{1}{b_3 + c_3}\right).$$

System (2.6) has the unique equilibrium point

$$(\overline{x}_3, \overline{y}_3, \overline{z}_3) = (f_1(1,1), g_2(1,1), h_2(1,1)) = \left(a_1 + \frac{1}{b_1 + c_1}, a_2 + \frac{1}{b_2 + c_2}, a_3 + \frac{1}{b_3 + c_3}\right).$$

System (2.7) has the unique equilibrium point

$$(\overline{x}_4, \overline{y}_4, \overline{z}_4) = (f_1(1,1), g_2(1,1), h_1(1,1)) = \left(a_1 + \frac{1}{b_1 + c_1}, a_2 + \frac{1}{b_2 + c_2}, a_3 + \frac{1}{b_3 + c_3}\right).$$

System (2.8) has the unique equilibrium point

$$(\overline{x}_5, \overline{y}_5, \overline{z}_5) = (f_2(1,1), g_2(1,1), h_2(1,1)) = \left(a_1 + \frac{1}{b_1 + c_1}, a_2 + \frac{1}{b_2 + c_2}, a_3 + \frac{1}{b_3 + c_3}\right).$$

System (2.9) has the unique equilibrium point

$$(\overline{x}_6, \overline{y}_6, \overline{z}_6) = (f_2(1,1), g_2(1,1), h_1(1,1)) = \left(a_1 + \frac{1}{b_1 + c_1}, a_2 + \frac{1}{b_2 + c_2}, a_3 + \frac{1}{b_3 + c_3}\right).$$

System (2.10) has the unique equilibrium point

$$(\overline{x}_7, \overline{y}_7, \overline{z}_7) = (f_2(1,1), g_1(1,1), h_1(1,1)) = \left(a_1 + \frac{1}{b_1 + c_1}, a_2 + \frac{1}{b_2 + c_2}, a_3 + \frac{1}{b_3 + c_3}\right).$$

System (2.11) has the unique equilibrium point

$$(\overline{x}_8, \overline{y}_8, \overline{z}_8) = (f_2(1,1), g_1(1,1), h_2(1,1)) = \left(a_1 + \frac{1}{b_1 + c_1}, a_2 + \frac{1}{b_2 + c_2}, a_3 + \frac{1}{b_3 + c_3}\right).$$

2.2.1.1 Stability of the equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1)$

Consider system (2.4) and its equilibrium point $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = \left(a_1 + \frac{1}{b_1+c_1}, a_2 + \frac{1}{b_2+c_2}, a_3 + \frac{1}{b_3+c_3}\right)$. We have the following results

Corollary 2.12. *The equilibrium point* $(\overline{x}_1, \overline{y}_1, \overline{z}_1) = \left(a_1 + \frac{1}{b_1+c_1}, a_2 + \frac{1}{b_2+c_2}, a_3 + \frac{1}{b_3+c_3}\right)$ *of system* (2.4) *is locally asymptotically stable if*

$$(a_1(b_1 + c_1) + 1)\,(a_2(b_2 + c_2) + 1)\,(a_3(b_3 + c_3) + 1)\,(b_1 + c_1)(b_2 + c_2)(b_3 + c_3) - 8c_1c_2c_3 > 0.$$

Proof. We have

$$f_1(1,1) = a_1 + \frac{1}{b_1 + c_1}, \quad \frac{\partial f_1}{\partial u}(1,1) = \frac{c_1}{(b_1 + c_1)^2},$$
$$g_1(1,1) = a_2 + \frac{1}{b_2 + c_2}, \quad \frac{\partial g_1}{\partial u}(1,1) = \frac{c_2}{(b_2 + c_2)^2},$$
$$h_1(1,1) = a_3 + \frac{1}{b_3 + c_3}, \quad \frac{\partial h_1}{\partial u}(1,1) = \frac{c_3}{(b_3 + c_3)^2}.$$

From Theorem 2.2 $(\overline{x}_1, \overline{y}_1, \overline{z}_1)$ is asymptotically stable if

$$\left|\frac{\partial f_1}{\partial u}(1,1)\frac{\partial g_1}{\partial u}(1,1)\frac{\partial h_1}{\partial u}(1,1)\right| < \frac{f_1(1,1)g_1(1,1)h_1(1,1)}{8},$$

that is

$$\left|\frac{c_1}{(b_1 + c_1)^2} \cdot \frac{c_2}{(b_2 + c_2)^2} \cdot \frac{c_3}{(b_3 + c_3)^2}\right| < \frac{1}{8}\left(a_1 + \frac{1}{b_1 + c_1}\right)\left(a_2 + \frac{1}{b_2 + c_2}\right)\left(a_3 + \frac{1}{b_3 + c_3}\right),$$

or

$$(a_1(b_1 + c_1) + 1)\,(a_2(b_2 + c_2) + 1)\,(a_3(b_3 + c_3) + 1)\,(b_1 + c_1)(b_2 + c_2)(b_3 + c_3) - 8c_1c_2c_3 > 0.$$

□

Theorem 2.13. *Assume that*

$$8c_1c_2c_3\left(a_1+\frac{1}{b_1}\right)\left(a_2+\frac{1}{b_2}\right)\left(a_3+\frac{1}{b_3}\right)<a_1^2a_2^2a_3^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2.$$

Then, the equilibrium point $(\overline{x}_1,\overline{y}_1,\overline{z}_1)$ *of system* (2.4) *is globally attractive.*

Proof. To prove the global attractivity of the equilibrium point we will use Theorem 2.3. From above we have

$$a:=a_1\leq f_1(u,v)\leq b:=a_1+\frac{1}{b_1},\ \alpha:=a_2\leq g_1(u,v)\leq\beta:=a_2+\frac{1}{b_2},$$
$$\lambda:=a_3\leq h_1(u,v)\leq\gamma:=a_3+\frac{1}{b_3},\ \forall u,v\in(0,+\infty),$$

and

$$\frac{\partial f_1}{\partial u}(u,v)>0,\ \frac{\partial f_1}{\partial v}(u,v)<0,\ \frac{\partial g_1}{\partial u}(u,v)>0,\ \frac{\partial g_1}{\partial v}(u,v)<0,\ \frac{\partial h_1}{\partial u}(u,v)>0,\ \frac{\partial h_1}{\partial v}(u,v)<0.$$

So, it follows that conditions H_1 and H_2 are satisfied. It remains to check condition H_3. To this end, let

$$(m_1,M_1,m_2,M_2,m_3,M_3)\in[a,b]^2\times[\alpha,\beta]^2\times[\lambda,\gamma]^2$$

be a solution of the system

$$m_1=a_1+\frac{m_2}{b_1m_2+c_1M_2},\ M_1=a_1+\frac{M_2}{b_1M_2+c_1m_2},\ m_2=a_2+\frac{m_3}{b_2m_3+c_2M_3},\ M_2=a_2+\frac{M_3}{b_2M_3+c_2m_3},$$
$$m_3=a_3+\frac{m_1}{b_3m_1+c_3M_1},\ M_3=a_3+\frac{M_1}{b_3M_1+c_3m_1}. \tag{2.12}$$

From (2.12), we get

$$m_1-M_1=\frac{c_1(m_2-M_2)(m_2+M_2)}{(b_1m_2+c_1M_2)(b_1M_2+c_1m_2)}, \tag{2.13}$$

$$m_2-M_2=\frac{c_2(m_3-M_3)(m_3+M_3)}{(b_2m_3+c_2M_3)(b_2M_3+c_2m_3)}, \tag{2.14}$$

and

$$m_3-M_3=\frac{c_3(m_1-M_1)(m_1+M_1)}{(b_3m_1+c_3M_1)(b_3M_1+c_3m_1)}. \tag{2.15}$$

From (2.13), (2.14) and (2.15), we obtain

$$(m_1-M_1)(m_2-M_2)(m_3-M_3)=\frac{c_1c_2c_3(m_1-M_1)(m_2-M_2)(m_3-M_3)(m_1+M_1)(m_2+M_2)(m_3+M_3)}{(b_1m_2+c_1M_2)(b_1M_2+c_1m_2)(b_2m_3+c_2M_3)(b_2M_3+c_2m_3)(b_3m_1+c_3M_1)(b_3M_1+c_3m_1)}.$$

So it follows that

$$(m_1-M_1)(m_2-M_2)(m_3-M_3)=0$$

or

$$\frac{c_1c_2c_3(m_1+M_1)(m_2+M_2)(m_3+M_3)}{(b_1m_2+c_1M_2)(b_1M_2+c_1m_2)(b_2m_3+c_2M_3)(b_2M_3+c_2m_3)(b_3m_1+c_3M_1)(b_3M_1+c_3m_1)}=1.$$

We will show that

$$h(m_1,M_1,m_2,M_2,m_3,M_3)=\frac{c_1c_2c_3(m_1+M_1)(m_2+M_2)(m_3+M_3)}{(b_1m_2+c_1M_2)(b_1M_2+c_1m_2)(b_2m_3+c_2M_3)(b_2M_3+c_2m_3)(b_3m_1+c_3M_1)(b_3M_1+c_3m_1)}\neq 1.$$

In fact, we have

$$\frac{8c_1c_2c_3a\alpha\lambda}{b^2\beta^2\gamma^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2}\leq h(m_1,M_1,m_2,M_2,m_3,M_3)\leq\frac{8c_1c_2c_3b\beta\gamma}{a^2\alpha^2\lambda^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2}.$$

Noting that

$$8c_1c_2c_3a\alpha\lambda=8c_1c_2c_3a_1a_2a_3,$$

and

$$b^2\beta^2\gamma^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2>\left(\frac{2a_1}{b_1}\right)\left(\frac{2a_2}{b_2}\right)\left(\frac{2a_3}{b_3}\right)(2b_1c_1)(2b_2c_2)(2b_3c_3).$$

So it follows that

$$\frac{8c_1c_2c_3a\alpha\lambda}{b^2\beta^2\gamma^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2}<\frac{8c_1c_2c_3a_1a_2a_3}{\left(\frac{2a_1}{b_1}\right)\left(\frac{2a_2}{b_2}\right)\left(\frac{2a_3}{b_3}\right)(2b_1c_1)(2b_2c_2)(2b_3c_3)}=\frac{1}{8}.$$

Using the fact that

$$8c_1c_2c_3b\beta\gamma<a^2\alpha^2\lambda^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2,$$

we get

$$\frac{8c_1c_2c_3b\beta\gamma}{a^2\alpha^2\lambda^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2}<1.$$

Hence,

$$h(m_1,M_1,m_2,M_2,m_3,M_3)\neq 1,\ (m_1-M_1)(m_2-M_2)(m_3-M_3)=0$$

which implies with (2.13), (2.14) and (2.15) that

$$m_1=M_1,\ m_2=M_2,\ m_3=M_3$$

and so condition H_3 is satisfied and then the equilibrium point $(\overline{x}_1,\overline{y}_1,\overline{z}_1)$ is globally attractive. □

Theorem 2.14. *Assume that*

$$\left(a_1(b_1+c_1)+1\right)\left(a_2(b_2+c_2)+1\right)\left(a_3(b_3+c_3)+1\right)(b_1+c_1)(b_2+c_2)(b_3+c_3)-8c_1c_2c_3>0.$$

and

$$8c_1c_2c_3\left(a_1+\frac{1}{b_1}\right)\left(a_2+\frac{1}{b_2}\right)\left(a_3+\frac{1}{b_3}\right)<a_1^2a_2^2a_3^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2.$$

Then the equilibrium point $(\overline{x}_1,\overline{y}_1,\overline{z}_1)$ is globally asymptotically stable

Proof. It follows from Corollary 2.12 and Theorem 2.13. □

In the same way, we can prove similar results for the remaining systems. So, for each system we give a theorem that includes conditions for global stability of the the corresponding equilibrium point.

2.2.1.2 Stability of the equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$

Consider system (2.5) and its equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2) = \left(a_1 + \frac{1}{b_1+c_1}, a_2 + \frac{1}{b_2+c_2}, a_3 + \frac{1}{b_3+c_3}\right)$. We have the following result.

Theorem 2.15. *Assume that*

1. *Local stability condition:*

$$(a_1(b_1 + c_1) + 1)\,(a_2(b_2 + c_2) + 1)\,(a_3(b_3 + c_3) + 1)\,(b_1+c_1)(b_2+c_2)(b_3+c_3) - 8c_1c_2b_3 > 0.$$

2. *Global attractivity condition:*

$$8c_1c_2b_3\left(a_1 + \frac{1}{b_1}\right)\left(a_2 + \frac{1}{b_2}\right)\left(a_3 + \frac{1}{c_3}\right) < a_1^2a_2^2a_3^2(b_1 + c_1)^2(b_2 + c_2)^2(b_3 + c_3)^2.$$

Then the equilibrium point $(\overline{x}_2, \overline{y}_2, \overline{z}_2)$ is globally asymptotically stable.

2.2.1.3 Stability of the equilibrium point $(\overline{x}_3, \overline{y}_3, \overline{z}_3)$

Consider system (2.6) and its equilibrium point $(\overline{x}_3, \overline{y}_3, \overline{z}_3) = \left(a_1 + \frac{1}{b_1+c_1}, a_2 + \frac{1}{b_2+c_2}, a_3 + \frac{1}{b_3+c_3}\right)$. We have the following result.

Theorem 2.16. *Assume that*

1. *Local stability condition:*

$$(a_1(b_1 + c_1) + 1)\,(a_2(b_2 + c_2) + 1)\,(a_3(b_3 + c_3) + 1)\,(b_1+c_1)(b_2+c_2)(b_3+c_3) - 8c_1b_2b_3 > 0.$$

2. *Global attractivity condition:*

$$8c_1b_2b_3\left(a_1 + \frac{1}{b_1}\right)\left(a_2 + \frac{1}{c_2}\right)\left(a_3 + \frac{1}{c_3}\right) < a_1^2a_2^2a_3^2(b_1 + c_1)^2(b_2 + c_2)^2(b_3 + c_3)^2.$$

Then the equilibrium point $(\overline{x}_3, \overline{y}_3, \overline{z}_3)$ is globally asymptotically stable.

2.2.1.4 Stability of the equilibrium point $(\overline{x}_4, \overline{y}_4, \overline{z}_4)$

Consider system (2.7) and its equilibrium point $(\overline{x}_4, \overline{y}_4, \overline{z}_4) = \left(a_1 + \frac{1}{b_1+c_1}, a_2 + \frac{1}{b_2+c_2}, a_3 + \frac{1}{b_3+c_3}\right)$. We have the following result.

Theorem 2.17. *Assume that*

1. *Local stability condition:*

$$(a_1(b_1+c_1)+1)(a_2(b_2+c_2)+1)(a_3(b_3+c_3)+1)(b_1+c_1)(b_2+c_2)(b_3+c_3) - 8c_1b_2c_3 > 0.$$

2. *Global attractivity condition:*

$$8c_1b_2c_3\left(a_1 + \frac{1}{b_1}\right)\left(a_2 + \frac{1}{c_2}\right)\left(a_3 + \frac{1}{b_3}\right) < a_1^2a_2^2a_3^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2.$$

Then the equilibrium point $(\overline{x}_4, \overline{y}_4, \overline{z}_4)$ is globally asymptotically stable.

2.2.1.5 Stability of the equilibrium point $(\overline{x}_5, \overline{y}_5, \overline{z}_5)$

Consider system (2.8) and its equilibrium point $(\overline{x}_5, \overline{y}_5, \overline{z}_5) = \left(a_1 + \frac{1}{b_1+c_1}, a_2 + \frac{1}{b_2+c_2}, a_3 + \frac{1}{b_3+c_3}\right)$. We have the following result.

Theorem 2.18. *Assume that*

1. *Local stability condition:*

$$(a_1(b_1+c_1)+1)(a_2(b_2+c_2)+1)(a_3(b_3+c_3)+1)(b_1+c_1)(b_2+c_2)(b_3+c_3) - 8b_1b_2b_3 > 0.$$

2. *Global attractivity condition:*

$$8b_1b_2b_3\left(a_1 + \frac{1}{c_1}\right)\left(a_2 + \frac{1}{c_2}\right)\left(a_3 + \frac{1}{c_3}\right) < a_1^2a_2^2a_3^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2.$$

Then the equilibrium point $(\overline{x}_5, \overline{y}_5, \overline{z}_5)$ is globally asymptotically stable.

2.2.1.6 Stability of the equilibrium point $(\overline{x}_6, \overline{y}_6, \overline{z}_6)$

Consider system (2.9) and its equilibrium point $(\overline{x}_6, \overline{y}_6, \overline{z}_6) = \left(a_1 + \frac{1}{b_1+c_1}, a_2 + \frac{1}{b_2+c_2}, a_3 + \frac{1}{b_3+c_3}\right)$. We have the following result.

Theorem 2.19. *Assume that*

1. *Local stability condition:*

$$(a_1(b_1+c_1)+1)(a_2(b_2+c_2)+1)(a_3(b_3+c_3)+1)(b_1+c_1)(b_2+c_2)(b_3+c_3)-8b_1b_2c_3>0.$$

2. *Global attractivity condition:*

$$8b_1b_2c_3\left(a_1+\frac{1}{c_1}\right)\left(a_2+\frac{1}{c_2}\right)\left(a_3+\frac{1}{b_3}\right)<a_1^2a_2^2a_3^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2.$$

Then the equilibrium point $(\overline{x}_6,\overline{y}_6,\overline{z}_6)$ is globally asymptotically stable.

2.2.1.7 Stability of the equilibrium point $(\overline{x}_7,\overline{y}_7,\overline{z}_7)$

Consider system (2.10) and its equilibrium point $(\overline{x}_7,\overline{y}_7,\overline{z}_7)=\left(a_1+\frac{1}{b_1+c_1},a_2+\frac{1}{b_2+c_2},a_3+\frac{1}{b_3+c_3}\right)$. We have the following result.

Theorem 2.20. *Assume that*

1. *Local stability condition:*

$$(a_1(b_1+c_1)+1)(a_2(b_2+c_2)+1)(a_3(b_3+c_3)+1)(b_1+c_1)(b_2+c_2)(b_3+c_3)-8b_1c_2c_3>0.$$

2. *Global attractivity condition:*

$$8b_1c_2c_3\left(a_1+\frac{1}{c_1}\right)\left(a_2+\frac{1}{b_2}\right)\left(a_3+\frac{1}{b_3}\right)<a_1^2a_2^2a_3^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2.$$

Then the equilibrium point $(\overline{x}_7,\overline{y}_7,\overline{z}_7)$ is globally asymptotically stable.

2.2.1.8 Stability of the equilibrium point $(\overline{x}_8,\overline{y}_8,\overline{z}_8)$

Consider system (2.11) and its equilibrium point $(\overline{x}_8,\overline{y}_8,\overline{z}_8)=\left(a_1+\frac{1}{b_1+c_1},a_2+\frac{1}{b_2+c_2},a_3+\frac{1}{b_3+c_3}\right)$. We have the following result.

Theorem 2.21. *Assume that*

1. *Local stability condition:*

$$(a_1(b_1+c_1)+1)(a_2(b_2+c_2)+1)(a_3(b_3+c_3)+1)(b_1+c_1)(b_2+c_2)(b_3+c_3)-8b_1c_2b_3>0.$$

2. *Global attractivity condition:*

$$8b_1c_2b_3\left(a_1+\frac{1}{c_1}\right)\left(a_2+\frac{1}{b_2}\right)\left(a_3+\frac{1}{c_3}\right)<a_1^2a_2^2a_3^2(b_1+c_1)^2(b_2+c_2)^2(b_3+c_3)^2.$$

Then the equilibrium point $(\overline{x}_8,\overline{y}_8,\overline{z}_8)$ is globally asymptotically stable.

2.3 Existence of periodic solutions

Here, we are interested in existence of periodic solutions for system (2.1).

Definition 2.2. *A solution* $(x_n, y_n, z_n)_{n=-1,0,...}$ *of system* (2.1) *is said to be periodic of period* $p \in \mathbb{N}$ *if*

$$x_{n+p} = x_n,\ y_{n+p} = y_n,\ z_{n+p} = z_n,\ n = -1, 0, \cdots .$$

In the following result we will established a necessary and sufficient condition for which there exist prime period two solutions for system (2.1).

Theorem 2.22. *Assume that* α, β *and* γ *are positive real numbers such that* $(\alpha-1)(\beta-1)(\gamma-1) \neq 0$. *Then, system* (2.1) *have a prime period two solution*

$$..., (\alpha p, \beta q, \gamma r),\ (p, q, r), (\alpha p, \beta q, \gamma r),\ (p, q, r), ...$$

if and only if

$$f(1, \beta) = \alpha f(\beta, 1),\ g(1, \gamma) = \beta g(\gamma, 1),\ h(1, \alpha) = \gamma h(\alpha, 1),$$

where

$$p = f(\beta, 1),\ q = g(\gamma, 1),\ r = h(\alpha, 1).$$

Proof. 1. Let α, β, γ be positive real numbers and assume that

$$..., (\alpha p, \beta q, \gamma r),\ (p, q, r), (\alpha p, \beta q, \gamma r),\ (p, q, r), ...$$

is a solution for system (2.1). Then, we have

$$\alpha p = f(q, \beta q) = f(1, \beta) \tag{2.16}$$

$$p = f(\beta q, q) = f(\beta, 1) \tag{2.17}$$

$$\beta q = g(r, \gamma r) = g(1, \gamma) \tag{2.18}$$

$$q = g(\gamma r, r) = g(\gamma, 1) \tag{2.19}$$

$$\gamma r = h(p, \alpha p) = h(1, \alpha) \tag{2.20}$$

$$r = h(\alpha p, p) = h(\alpha, 1). \tag{2.21}$$

From (2.16)-(2.21), it follows that

$$f(1, \beta) = \alpha f(\beta, 1),\ g(1, \gamma) = \beta g(\gamma, 1),\ h(1, \alpha) = \gamma h(\alpha, 1).$$

2. Now, assume that

$$f(1,\beta)=\alpha f(\beta,1),\ g(1,\gamma)=\beta g(\gamma,1),\ h(1,\alpha)=\gamma h(\alpha,1).$$

and let

$$x_0=f(\beta,1),\ x_{-1}=f(1,\beta),\ y_0=g(\gamma,1),\ y_{-1}=g(1,\gamma),\ z_0=h(\alpha,1),\ z_{-1}=h(1,\alpha).$$

We have

$$x_1=f(y_0,y_{-1})=f(g(\gamma,1),g(1,\gamma))=f(g(\gamma,1),\beta g(\gamma,1))=f(1,\beta)=x_{-1},$$

$$y_1=g(z_0,z_{-1})=g(h(\alpha,1),h(1,\alpha))=g(h(\alpha,1),\gamma h(\alpha,1))=g(1,\gamma)=y_{-1},$$

$$z_1=h(x_0,x_{-1})=h(f(\beta,1),f(1,\beta))=h(f(\beta,1),\alpha f(\beta,1))=h(1,\alpha)=z_{-1},$$

$$x_2=f(y_1,y_0)=f(g(1,\gamma),g(\gamma,1))=f(\beta g(\gamma,1),g(\gamma,1))=f(\beta,1)=x_0,$$

$$y_2=g(z_1,z_0)=g(h(1,\alpha),h(\alpha,1))=g(\gamma h(\alpha,1),h(\alpha,1))=g(1,\gamma)=y_0,$$

$$z_2=h(x_1,x_0)=h(f(1,\beta),f(\beta,1))=h(\alpha f(\beta,1),f(\beta,1))=h(1,\alpha)=z_0.$$

By induction we get

$$x_{2n-1}=x_{-1},\ x_{2n}=x_0,\ y_{2n-1}=y_{-1},\ y_{2n}=y_0,\ z_{2n-1}=z_{-1},\ z_{2n}=z_0,\ n\in\mathbb{N}_0.$$

□

Now, we will applied our result in finding prime period two solutions of some special cases of system (2.1).

2.3.1 The first special system

In [27], the author investigated the existence of periodic solutions of the equation

$$x_{n+1}=a_1+b_1\frac{x_n}{x_{n-1}}+c_1\frac{x_{n-1}}{x_n}. \tag{2.22}$$

The author of [60], make some additional remarks and results on the same equation. Here, as a generalization of equation (2.22) and the system

$$x_{n+1}=a_1+b_1\frac{y_n}{y_{n-1}}+c_1\frac{y_{n-1}}{y_n},\ y_{n+1}=a_2+b_2\frac{x_n}{x_{n-1}}+c_2\frac{x_{n-1}}{x_n},$$

studied by [114], we consider the three dimensional system

$$x_{n+1} = a_1+b_1\frac{y_n}{y_{n-1}}+c_1\frac{y_{n-1}}{y_n},\ y_{n+1} = a_2+b_2\frac{z_n}{z_{n-1}}+c_2\frac{z_{n-1}}{z_n},\ z_{n+1} = a_3+b_3\frac{x_n}{x_{n-1}}+c_3\frac{x_{n-1}}{x_n},\ n \in \mathbb{N}_0, \quad (2.23)$$

where the initial values x_{-i}, y_{-i}, z_{-i}, $i = 0, 1$ and a_j, b_j, c_j, $j = 1, 2, 3$ are positive real numbers. For this system we have the following result.

Corollary 2.23. *Assume that* $(\alpha - 1)(\beta - 1)(\gamma - 1) \neq 0$, *then system* (2.23) *had prime period two solution of the form*

$$\ldots,(\alpha f(\beta, 1), \beta g(\gamma, 1), \gamma h(\alpha, 1)),\ (f(\beta, 1), g(\gamma, 1), h(\alpha, 1)),\ (\alpha f(\beta, 1), \beta g(\gamma, 1), \gamma h(\alpha, 1)),$$
$$(f(\beta, 1), g(\gamma, 1), h(\alpha, 1)), \ldots$$

if and only if

$$\begin{aligned}(b_1\alpha - c_1)\beta^2 + a_1\beta(\alpha - 1) + c_1\alpha - b_1 &= 0,\ (b_2\beta - c_2)\gamma^2 + a_2\gamma(\beta - 1) + c_2\beta - b_2 = 0,\\ (b_3\gamma - c_3)\alpha^2 + a_3\alpha(\gamma - 1) + c_3\gamma - b_3 &= 0. \end{aligned} \quad (2.24)$$

Proof. System (2.23) can be written as

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(z_n, z_{n-1}),\ z_{n+1} = h(x_n, x_{n-1}),$$

where

$$f(u, v) = a_1 + b_1\frac{u}{v} + c_1\frac{v}{u},\ g(u, v) = a_2 + b_2\frac{u}{v} + c_2\frac{v}{u},\ h(u, v) = a_3 + b_3\frac{u}{v} + c_3\frac{v}{u}.$$

So, from Theorem 2.22,

$$\ldots,(\alpha f(\beta, 1), \beta g(\gamma, 1), \gamma h(\alpha, 1)),\ (f(\beta, 1), g(\gamma, 1), h(\alpha, 1)),\ (\alpha f(\beta, 1), \beta g(\gamma, 1), \gamma h(\alpha, 1)),$$
$$(f(\beta, 1), g(\gamma, 1), h(\alpha, 1)), \ldots$$

will be a period prime two solution of system (2.23) if and only if

$$f(1, \beta) = \alpha f(\beta, 1),\ g(1, \gamma) = \beta g(\gamma, 1),\ h(1, \alpha) = \gamma h(\alpha, 1).$$

Clearly this condition is equivalent to

$$(b_1\alpha - c_1)\beta^2 + a_1\beta(\alpha - 1) + c_1\alpha - b_1 = 0,\ (b_2\beta - c_2)\gamma^2 + a_2\gamma(\beta - 1) + c_2\beta - b_2 = 0,$$
$$(b_3\gamma - c_3)\alpha^2 + a_3\alpha(\gamma - 1) + c_3\gamma - b_3 = 0.$$

□

Example 2.3.1. If we choose $\alpha = 2$, $\beta = 3$, $\gamma = \frac{1}{2}$, the condition (2.24) will be

$$3a_1 + 17b_1 - 7c_1 = 0,\ 4a_2 - b_2 + 11c_2 = 0,\ -2a_3 + 2b_3 - 7c_3 = 0.$$

The last condition is satisfied for the choice

$$a_1 = \frac{4}{3},\ b_1 = 1,\ c_1 = 3,\ a_2 = \frac{1}{4},\ b_2 = 2,\ c_2 = \frac{1}{11},\ a_3 = \frac{1}{2},\ b_3 = 4,\ c_3 = 1$$

of the parameters.

The corresponding prime period two solutions, will be

$$x_{2n-1} = x_{-1} = \alpha f(\beta, 1) = \frac{32}{3},\ y_{2n-1} = y_{-1} = \beta g(\gamma, 1) = \frac{189}{44},\ z_{2n-1} = z_{-1} = \gamma h(\alpha, 1) = \frac{9}{2},$$

and

$$x_{2n} = x_0 = f(\beta, 1) = \frac{16}{3},\ y_{2n} = y_0 = g(\gamma, 1) = \frac{63}{44},\ z_{2n} = z_0 = h(\alpha, 1) = 9$$

that is

$$\left\{\left(\frac{32}{3}, \frac{189}{44}, \frac{9}{2}\right), \left(\frac{16}{3}, \frac{63}{44}, 9\right), \left(\frac{32}{3}, \frac{189}{44}, \frac{9}{2}\right), \cdots\right\}.$$

2.3.2 The second special system

Consider the system

$$x_{n+1} = a_1 + b_1 \frac{y_n}{y_{n-1}} + c_1 \left(\frac{y_{n-1}}{y_n}\right)^2, y_{n+1} = a_2 + b_2 \frac{z_{n-1}}{z_n} + c_2 \left(\frac{z_{n-1}}{z_n}\right)^2, z_{n+1} = a_3 + b_3 \frac{x_{n-1}}{x_n} + 3_2 \left(\frac{x_{n-1}}{x_n}\right)^2, n \in \mathbb{N}_0, \tag{2.25}$$

where the initial values x_{-i}, y_{-i}, z_{-i}, $i = 0, 1$ and a_j, b_j, c_j, $j = 1, 2, 3$ are positive real numbers. System (2.25) is a modification of system (2.23) and we have the following result.

Corollary 2.24. *Assume that* $(\alpha - 1)(\beta - 1)(\gamma - 1) \neq 0$, *then system* (2.25) *had prime period two solution of the form*

$$...,(\alpha f(\beta, 1), \beta g(\gamma, 1), \gamma h(\alpha, 1)),\ (f(\beta, 1), g(\gamma, 1), h(\alpha, 1)),\ (\alpha f(\beta, 1), \beta g(\gamma, 1), \gamma h(\alpha, 1)),$$
$$(f(\beta, 1), g(\gamma, 1), h(\alpha, 1)), ...$$

if and only if

$$a_1\beta^2(\alpha - 1) + b_1\beta(\alpha\beta^2 - 1) + c_1(\alpha - \beta^4) = 0,\ a_2\gamma^2(\beta - 1) + b_2\gamma(\beta\gamma^2 - 1) + c_2(\beta - \gamma^4) = 0,$$
$$a_3\alpha^2(\gamma - 1) + b_3\alpha(\gamma\alpha^2 - 1) + c_3(\gamma - \alpha^4) = 0. \tag{2.26}$$

Proof. System (2.25) can be written as

$$x_{n+1} = f(y_n, y_{n-1}),\ y_{n+1} = g(z_n, z_{n-1}),\ z_{n+1} = h(x_n, x_{n-1}),$$

where

$$f(u,v) = a_1 + b_1\frac{u}{v} + c_1\left(\frac{v}{u}\right)^2,\ g(u,v) = a_2 + b_2\frac{v}{u} + c_2\left(\frac{v}{u}\right)^2,\ h(u,v) = a_3 + b_3\frac{v}{u} + c_3\left(\frac{v}{u}\right)^2.$$

So, from Theorem 2.22,

$$\ldots,(\alpha f(\beta,1), \beta g(\gamma,1), \gamma h(\alpha,1)),\ (f(\beta,1), g(\gamma,1), h(\alpha,1)),\ (\alpha f(\beta,1), \beta g(\gamma,1), \gamma h(\alpha,1)),$$
$$(f(\beta,1), g(\gamma,1), h(\alpha,1)), \ldots$$

will be a period prime two solution of system (2.23) if and only if

$$f(1,\beta) = \alpha f(\beta,1),\ g(1,\gamma) = \beta g(\gamma,1),\ h(1,\alpha) = \gamma h(\alpha,1).$$

Clearly this condition is equivalent to

$$a_1\beta^2(\alpha-1) + b_1\beta(\alpha\beta^2-1) + c_1(\alpha-\beta^4) = 0,\ a_2\gamma^2(\beta-1) + b_2\gamma(\beta\gamma^2-1) + c_2(\beta-\gamma^4) = 0,$$
$$a_3\alpha^2(\gamma-1) + b_3\alpha(\gamma\alpha^2-1) + c_3(\gamma-\alpha^4) = 0.$$

□

Example 2.3.2. For $\alpha = 3,\ \beta = 2,\ \gamma = \frac{1}{3}$, the condition (2.26) will be

$$8a_1 + 22b_1 - 13c_1 = 0,\ 9a_2 - 21b_2 + 161c_2 = 0,\ 18a_3 - 18b_3 + 242c_3 = 0.$$

The last condition is satisfied for the choice

$$a_1 = 1,\ b_1 = 1,\ c_1 = \frac{30}{13},\ a_2 = \frac{19}{9},\ b_2 = 2,\ c_2 = \frac{1}{7},\ a_3 = \frac{1}{6},\ b_3 = \frac{7}{9},\ c_3 = \frac{1}{22}$$

of the parameters.

The corresponding prime period two solutions, will be

$$x_{2n-1} = x_{-1} = 3f(2,1) = \frac{297}{26},\ y_{2n-1} = y_{-1} = 2g(\frac{1}{3},1) = \frac{512}{63},\ z_{2n-1} = z_{-1} = \frac{1}{3}h(3,1) = \frac{248}{297},$$

and

$$x_{2n} = x_0 = f(2,1) = \frac{93}{26},\ y_{2n} = y_0 = g(\frac{1}{3},1) = \frac{256}{63},\ z_{2n} = z_0 = h(3,1) = \frac{744}{297}$$

that is

$$\left\{\left(\frac{297}{26},\frac{512}{63},\frac{248}{297}\right),\left(\frac{93}{26},\frac{256}{63},\frac{744}{297}\right),\left(\frac{297}{26},\frac{512}{63},\frac{248}{297}\right),\ldots\right\}.$$

2.4 Existence of oscillatory Solutions

Here, we are interested in the oscillation of the solutions of system (2.1) about the equilibrium point $(\overline{x}, \overline{y}, \overline{z}) = (f(1,1), g(1,1), h(1,1))$.

Definition 2.3. *Let $(x_n, y_n, z_n)_{n\geq -1}$ be a solution of system (2.1). We say that the sequence $(x_n)_{n\geq -1}$ (resp. $(y_n)_{n\geq -1}$, $(z_n)_{n\geq -1}$) oscillate about $\overline{x}$ (resp. $\overline{y}$, $\overline{z}$) with a semi-cycle of length one if:*
$(x_n - \overline{x})(x_{n+1} - \overline{x}) < 0$, $n \geq -1$ (resp. $(y_n - \overline{y})(y_{n+1} - \overline{y}) < 0$, $n \geq -1$, $(z_n - \overline{z})(y_{n+1} - \overline{y}) < 0$, $n \geq -1$).

Remark 2.4.1. For every term x_{n_0} of the sequence $(x_n)_{n\geq -1}$, the notation "+" means $x_{n_0} - \overline{x} > 0$ and the notation "-" means $x_{n_0} - \overline{x} < 0$. The same notations will be used for the terms of the sequences $(y_n)_{n\geq -1}$ and $(z_n)_{n\geq -1}$.

Theorem 2.25. *Let $(x_n, y_n, z_n)_{n\geq -1}$ be a solution of system (2.1) and assume that $f(x,y)$, $g(x,y)$ $h(x,y)$ are decreasing in x for all y and are increasing in y for all x.*

1. *If*

$$x_0 < \overline{x},\ x_{-1} > \overline{x},\ y_0 < \overline{y},\ y_{-1} > \overline{y},\ z_0 < \overline{z},\ z_{-1} > \overline{z},$$

   then we get

$$x_{2n} < \overline{x},\ x_{2n-1} > \overline{x},\ y_{2n} < \overline{y},\ y_{2n-1} > \overline{y},\ z_{2n} < \overline{z},\ z_{2n-1} > \overline{z},\ n \in \mathbb{N}.$$

   That is for both $(x_n)_{n\geq -1}$, $(y_n)_{n\geq -1}$ and $(z_n)_{n\geq -1}$ we have semi-cycles of length one of the form

$$+ - + - + - \cdots .$$

2. *If*

$$x_0 > \overline{x},\ x_{-1} < \overline{x},\ y_0 > \overline{y},\ y_{-1} < \overline{y},\ z_0 > \overline{z},\ z_{-1} < \overline{z},$$

   then we get

$$x_{2n} > \overline{x},\ x_{2n-1} < \overline{x},\ y_{2n} > \overline{y},\ y_{2n-1} < \overline{y},\ z_{2n} > \overline{z},\ z_{2n-1} < \overline{z},\ n \in \mathbb{N}.$$

   That is for both $(x_n)_{n\geq -1}$, $(y_n)_{n\geq -1}$ and $(z_n)_{n\geq -1}$ we have semi-cycles of length one of the form

$$- + - + - + \cdots .$$

Proof. 1. Assume that

$$x_0 < \overline{x},\ x_{-1} > \overline{x},\ y_0 < \overline{y},\ y_{-1} > \overline{y},\ z_0 < \overline{z},\ z_{-1} > \overline{z}.$$

We have

$$x_1 = f(y_0, y_{-1}) > f(\overline{y}, y_{-1}) > f(\overline{y}, \overline{y}) = f(1,1) = \overline{x},$$

$$y_1 = g(z_0, z_{-1}) > g(\overline{z}, z_{-1}) > g(\overline{z}, \overline{z}) = g(1,1) = \overline{y},$$

$$z_1 = h(x_0, x_{-1}) > h(\overline{x}, x_{-1}) > h(\overline{x}, \overline{x}) = h(1,1) = \overline{z},$$

$$x_2 = f(y_1, y_0) < f(\overline{y}, y_0) < f(\overline{y}, \overline{y}) = f(1,1) = \overline{x},$$

$$y_2 = g(z_1, z_0) < g(\overline{z}, z_0) < g(\overline{z}, \overline{z}) = g(1,1) = \overline{y}.$$

$$z_2 = h(x_1, x_0) < h(\overline{x}, x_0) < h(\overline{x}, \overline{x}) = h(1,1) = \overline{z}.$$

By induction, we get

$$x_{2n} < \overline{x},\ x_{2n-1} > \overline{x},\ y_{2n} < \overline{y},\ y_{2n-1} > \overline{y},\ z_{2n} < \overline{z},\ z_{2n-1} > \overline{z},\ n \in \mathbb{N}.$$

2. Assume that

$$x_0 > \overline{x},\ x_{-1} < \overline{x},\ y_0 > \overline{y},\ y_{-1} < \overline{y},\ z_0 > \overline{z},\ z_{-1} < \overline{z}.$$

We have

$$x_1 = f(y_0, y_{-1}) < f(\overline{y}, y_{-1}) < f(\overline{y}, \overline{y}) = f(1,1) = \overline{x},$$

$$y_1 = g(z_0, z_{-1}) < g(\overline{z}, z_{-1}) < g(\overline{z}, \overline{z}) = g(1,1) = \overline{y},$$

$$z_1 = h(x_0, x_{-1}) < h(\overline{x}, x_{-1}) < h(\overline{x}, \overline{x}) = h(1,1) = \overline{z},$$

$$x_2 = f(y_1, y_0) > f(\overline{y}, y_0) > f(\overline{y}, \overline{y}) = f(1,1) = \overline{x},$$

$$y_2 = g(z_1, z_0) > g(\overline{z}, z_0) > g(\overline{z}, \overline{z}) = g(1,1) = \overline{y}.$$

$$z_2 = h(x_1, x_0) > h(\overline{x}, x_0) > h(\overline{x}, \overline{x}) = h(1,1) = \overline{y}.$$

By induction, we get

$$x_{2n} > \overline{x},\ x_{2n-1} < \overline{x},\ y_{2n} > \overline{y},\ y_{2n-1} < \overline{y},\ z_{2n} > \overline{z},\ z_{2n-1} < \overline{z},\ n \in \mathbb{N}.$$

□

Now in order to confirm the results of this section, we consider the following particular system.

Example 2.4.1. Consider the system

$$x_{n+1} = a_1 + b_1 \left(\frac{y_{n-1}}{y_n}\right)^p, \; y_{n+1} = a_2 + b_2 \left(\frac{z_{n-1}}{z_n}\right)^q, \; z_{n+1} = a_3 + b_3 \left(\frac{x_{n-1}}{x_n}\right)^k, \; n \in \mathbb{N}_0, \quad (2.27)$$

where $p, q, k \in \mathbb{N}$, x_{-i}, y_{-i}, z_{-i}, $i = 0, 1$, a_j, b_j, $i = 1, 2, 3$ are positive real numbers.

Let f, g and h be the functions defined by

$$f(u, v) = a_1 + b_1 \left(\frac{v}{u}\right)^p, \; g(u, v) = a_2 + b_2 \left(\frac{v}{u}\right)^q, \; h(u, v) = a_3 + b_3 \left(\frac{v}{u}\right)^k, \; u, v \in (0, +\infty).$$

It is not hard to see that

$$\frac{\partial f}{\partial u}(u, v) < 0, \; \frac{\partial f}{\partial v}(u, v) > 0, \; \frac{\partial g}{\partial u}(u, v) < 0, \; \frac{\partial g}{\partial v}(u, v) > 0 \; \frac{\partial h}{\partial u}(u, v) < 0, \; \frac{\partial h}{\partial v}(u, v) > 0.$$

System (2.27) has the unique equilibrium point $(\overline{x}, \overline{y}, \overline{z}) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

Corollary 2.26. *Let $(x_n, y_n, z_n)_{n=-1,0,\ldots}$ be a solution of system (2.27). The following statements holds true:*

1. *Let*

$$x_0 < \overline{x}, \; x_{-1} > \overline{x}, \; y_0 < \overline{y}, \; y_{-1} > \overline{y}, \; z_0 < \overline{z}, \; z_{-1} > \overline{z}.$$

Then the sequence $(x_n)_n$ (resp. $(y_n)_n$, $(z_n)_n$) oscillate about $\overline{x}$ (resp. about $\overline{y}$, $\overline{z}$) with semi-cycle of length one and every semi-cycle is in the form

$$+ - + - + - \cdots.$$

2. *Let*

$$x_0 > \overline{x}, \; x_{-1} < \overline{x}, \; y_0 > \overline{y}, \; y_{-1} < \overline{y}, \; z_0 > \overline{z}, \; z_{-1} < \overline{z}.$$

Then the sequence $(x_n)_n$ (resp. $(y_n)_n$, $(z_n)_n$) oscillate about $\overline{x}$ (resp. about $\overline{y}$, $\overline{z}$) with semi-cycle of length one and every semi-cycle is in the form

$$+ - + - + - \cdots.$$

Proof. 1. Let

$$x_0 < \overline{x}, \; x_{-1} > \overline{x}, \; y_0 < \overline{y}, \; y_{-1} > \overline{y}, \; z_0 < \overline{z}, \; z_{-1} > \overline{z}.$$

We have

$$\frac{y_{-1}}{y_0} > \frac{\overline{y}}{\overline{y}} = 1,$$

which implies that

$$x_1 = a_1 + b_1 \left(\frac{y_{-1}}{y_0}\right)^p > a_1 + b_1 = \overline{x}.$$

Using the fact that

$$\frac{z_{-1}}{z_0} > \frac{\overline{z}}{\overline{z}} = 1,$$

we get

$$y_1 = a_2 + b_2 \left(\frac{z_{-1}}{z_0}\right)^q > a_2 + b_2 = \overline{y}.$$

Also,

$$\frac{x_{-1}}{x_0} > \frac{\overline{x}}{\overline{x}} = 1,$$

we get

$$z_1 = a_3 + b_3 \left(\frac{x_{-1}}{x_0}\right)^k > a_3 + b_3 = \overline{z}.$$

Now, as

$$\frac{y_0}{y_1} < \frac{\overline{y}}{\overline{y}} = 1,$$

we obtain

$$x_2 = a_1 + b_1 \left(\frac{y_0}{y_1}\right)^p < a_1 + b_1 = \overline{x}.$$

Similarly,

$$\frac{z_0}{z_1} < \frac{\overline{z}}{\overline{z}} = 1 \Rightarrow y_2 = a_2 + b_2 \left(\frac{z_0}{z_1}\right)^q < a_2 + b_2 = \overline{y},$$

and

$$\frac{x_0}{x_1} < \frac{\overline{x}}{\overline{x}} = 1 \Rightarrow z_2 = a_3 + b_3 \left(\frac{x_0}{x_1}\right)^k < a_3 + b_3 = \overline{z},$$

and by induction we get that

$$x_{2n}-\overline{x} < 0,\ y_{2n}-\overline{y} < 0,\ z_{2n}-\overline{z} < 0,\ x_{2n-1}-\overline{x} > 0,\ y_{2n-1}-\overline{y} > 0,\ z_{2n-1}-\overline{z} > 0,\ n \in \mathbb{N}_0,$$

that is the sequences $(x_n)_n$ (resp. $(y_n)_n$, $(z_n)_n$) oscillate about $\overline{x}$ (resp. about $\overline{y}$, $\overline{z}$) with semi-cycle of length one and every semi-cycle is in the form

$$- + - + - + \cdots .$$

and this confirm Part 1. of Theorem 2.25.

2. Let

$$x_0 > \overline{x},\ x_{-1} < \overline{x},\ y_0 > \overline{y},\ y_{-1} < \overline{y},\ z_0 > \overline{z},\ z_{-1} < \overline{z}.$$

We have

$$\frac{y_{-1}}{y_0} < \frac{\overline{y}}{\overline{y}} = 1,$$

which implies that

$$x_1 = a_1 + b_1 \left(\frac{y_{-1}}{y_0}\right)^p < a_1 + b_1 = \overline{x}.$$

Using the fact that

$$\frac{z_{-1}}{z_0} < \frac{\overline{z}}{\overline{z}} = 1,$$

we get

$$y_1 = a_2 + b_2 \left(\frac{z_{-1}}{z_0}\right)^q < a_2 + b_2 = \overline{y}.$$

Also,

$$\frac{x_{-1}}{x_0} < \frac{\overline{x}}{\overline{x}} = 1,$$

we get

$$z_1 = a_3 + b_3 \left(\frac{x_{-1}}{x_0}\right)^k < a_3 + b_3 = \overline{z}.$$

Now, as

$$\frac{y_0}{y_1} > \frac{\overline{y}}{\overline{y}} = 1,$$

we obtain

$$x_2 = a_1 + b_1 \left(\frac{y_0}{y_1}\right)^p > a_1 + b_1 = \overline{x}.$$

Similarly,

$$\frac{z_0}{z_1} > \frac{\overline{z}}{\overline{z}} = 1 \Rightarrow y_2 = a_2 + b_2 \left(\frac{z_0}{z_1}\right)^q > a_2 + b_2 = \overline{y},$$

and

$$\frac{x_0}{x_1} > \frac{\overline{x}}{\overline{x}} = 1 \Rightarrow z_2 = a_3 + b_3 \left(\frac{x_0}{x_1}\right)^k > a_3 + b_3 = \overline{z}.$$

Thus, by induction we get that

$$x_{2n} - \overline{x} > 0,\ y_{2n} - \overline{y} > 0,\ z_{2n} - \overline{z} > 0,\ x_{2n-1} - \overline{x} < 0,\ y_{2n-1} - \overline{y} < 0,\ z_{2n-1} - \overline{z} < 0,\ n \in \mathbb{N}_0,$$

that is the sequences $(x_n)_n$ (resp. $(y_n)_n$, $(z_n)_n$) oscillate about $\overline{x}$ (resp. about $\overline{y}$, $\overline{z}$) with semi-cycle of length one and every semi-cycle is in the form

$$+ - + - + - \cdots .$$

and this confirm Part 2. of Theorem 2.25.

□

Chapter 3

Formulas and behavior of solutions of a three dimensional system of difference equations

3.1 Introduction

One of the basic nonlinear difference equation is

$$x_{n+1} = \frac{x_{n-1}x_n}{x_n + x_{n-2}}, \; n \in N_0. \tag{3.1}$$

First time, equation (3.1) is solved by Elabbasy et al. in [17]. Then, Stevic differently expressed general solution to equation (3.1) in [94]. In addition, Elabbasy et al. showed that the following difference equation

$$x_{n+1} = \frac{x_{n-1}x_n}{x_n - x_{n-2}}, \; n \in N_0. \tag{3.2}$$

is solvable in [18].

In [35], the authors presented the solutions of the two-dimensional system of difference equations which extended of equation (3.1) and equation (3.2)

$$x_{n+1} = \frac{x^p_{n-k+1}y_n}{ay^p_{n-k} + by_n}, \; y_{n+1} = \frac{y^p_{n-k+1}x_n}{\alpha x^p_{n-k} + \beta x_n}, \; n \in \mathbb{N}_0, \; p, k \in \mathbb{N}. \tag{3.3}$$

The authors of [112] studied the case $k = 2$, $p = 1$, in system (3.3) with a special choices of a, b, α, β. In addition, for the case $k = 3$, $p = 1$, with a special choices of a, b, α, β in system (3.3) is investigated in [19]. Further, in [5, 20] authors obtained the solutions of other two-dimensional system of difference equations which is related to equation (3.1) and equation (3.2).

In [126], Yazlik et all. presented the solutions of following three-dimensional system of difference equations which generalized both equations (3.1)-(3.2) and systems are given [5, 19, 20, 112],

$$x_{n+1}=\frac{x_ny_{n-1}}{a_0x_n+b_0y_{n-2}},\ y_{n+1}=\frac{y_nz_{n-1}}{a_1y_n+b_1z_{n-2}},\ z_{n+1}=\frac{z_nx_{n-1}}{a_2z_n+b_2x_{n-2}},\ n\in\mathbb{N}_0, \tag{3.4}$$

where the parameters a_i, b_i and the initial values x_{-i}, y_{-i}, z_{-i} $(i=0,1,2)$ are real numbers.

A natural question is to study both three-dimensional form of equations (3.1)-(3.2), system (3.3) and more general system of (3.4) solvable in explicit-form. Here we study such a system. That is, we deal with the following system of difference equations

$$x_{n+1}=\frac{x^p_{n-k+1}y_n}{\alpha y^p_{n-k}+\beta y_n},\ y_{n+1}=\frac{y^p_{n-k+1}z_n}{az^p_{n-k}+bz_n},\ z_{n+1}=\frac{z^p_{n-k+1}x_n}{Ax^p_{n-k}+Bx_n},\ n\in\mathbb{N}_0,\ p,k\in\mathbb{N}. \tag{3.5}$$

3.2 Form of the solutions

To solve system (3.5) we need to use the following lemma.

Lemma 3.1. *For $a, b\in\mathbb{R}$, consider the linear difference equation*

$$y_{n+3}=ay_n+b,\ n\in\mathbb{N}_0.$$

Then,

$$\forall n\in\mathbb{N}_0,\ y_{3n+i}=\begin{cases} y_i+bn, & a=1,\\ a^ny_i+\left(\dfrac{a^n-1}{a-1}\right)b, & otherwise,\end{cases}\quad for\ i=0,1,2.$$

Through the rest of the chapter by a solution of (3.5), we mean a well defined solution, that is a solution such that

$$\left(\alpha y^p_{n-k}+\beta y_n\right)\left(az^p_{n-k}+bz_n\right)\left(Ax^p_{n-k}+Bx_n\right)\neq 0,\ n\in\mathbb{N}_0.$$

Rearrange system (3.5) as follows

$$\frac{x^p_{n-k+1}}{x_{n+1}}=\alpha\frac{y^p_{n-k}}{y_n}+\beta,\ \frac{y^p_{n-k+1}}{y_{n+1}}=a\frac{z^p_{n-k}}{z_n}+b,\ \frac{z^p_{n-k+1}}{z_{n+1}}=A\frac{x^p_{n-k}}{x_n}+B.$$

Putting

$$u_n=\frac{x^p_{n-k}}{x_n},\ v_n=\frac{y^p_{n-k}}{y_n},\ w_n=\frac{z^p_{n-k}}{z_n},\ \forall n\in\mathbb{N}_0, \tag{3.6}$$

we get

$$u_{n+1} = \alpha v_n + \beta, \; v_{n+1} = a w_n + b, \; w_{n+1} = A u_n + B, \; \forall n \in \mathbb{N}_0. \tag{3.7}$$

So, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} u_{n+3} &= \alpha v_{n+2} + \beta = \alpha \left[a w_{n+1} + b \right] + \beta = \alpha \left[a \left(A u_n + B \right) + b \right] + \beta, \\ &= \alpha a A u_n + \alpha a B + \alpha b + \beta. \end{aligned}$$

$$\begin{aligned} v_{n+3} &= a w_{n+2} + b = a \left[A u_{n+1} + B \right] + b = a \left[A \left(\alpha v_n + \beta \right) + B \right] + b, \\ &= \alpha a A v_n + a A \beta + a B + b. \end{aligned}$$

$$\begin{aligned} w_{n+3} &= A u_{n+2} + B = A \left[\alpha v_{n+1} + \beta \right] + B = A \left[\alpha \left(a w_n + b \right) + \beta \right] + B, \\ &= \alpha a A w_n + \alpha A b + A \beta + B. \end{aligned}$$

From this, we get, for all $n \in \mathbb{N}_0$, the following linear first order nonhomogeneous difference equations,

$$\begin{aligned} u_{3(n+1)} &= \alpha a A u_{3n} + \alpha a B + \alpha b + \beta, \\ u_{3(n+1)+1} &= \alpha a A u_{3n+1} + \alpha a B + \alpha b + \beta, \\ u_{3(n+1)+2} &= \alpha a A u_{3n+2} + \alpha a B + \alpha b + \beta, \end{aligned}$$

$$\begin{aligned} v_{3(n+1)} &= \alpha a A v_{3n} + a A \beta + a B + b, \\ v_{3(n+1)+1} &= \alpha a A v_{3n+1} + a A \beta + a B + b, \\ v_{3(n+1)+2} &= \alpha a A v_{3n+2} + a A \beta + a B + b, \end{aligned}$$

$$\begin{aligned} w_{3(n+1)} &= \alpha a A w_{3n} + \alpha A b + A \beta + B, \\ w_{3(n+1)+1} &= \alpha a A w_{3n+1} + \alpha A b + A \beta + B, \\ w_{3(n+1)+2} &= \alpha a A w_{3n+2} + \alpha A b + A \beta + B, \end{aligned}$$

Then, we get for all $n \in \mathbb{N}_0$,

$$u_{3n+i} = \begin{cases} u_i + (\alpha a B + \alpha b + \beta) n, & \alpha a A = 1, \\ (\alpha a A)^n u_i + \left(\dfrac{(\alpha a A)^n - 1}{\alpha a A - 1} \right) (\alpha a B + \alpha b + \beta), & \textit{otherwise}, \end{cases} \text{ for } i = 0, 1, 2; \tag{3.8}$$

$$v_{3n+i} = \begin{cases} v_i + (aA\beta + aB + b)n, & \alpha aA = 1, \\ (\alpha aA)^n v_i + \left(\dfrac{(\alpha aA)^n - 1}{\alpha aA - 1}\right)(aA\beta + aB + b), & otherwise, \end{cases} \quad \text{for } i = 0,1,2; \tag{3.9}$$

$$w_{3n+i} = \begin{cases} w_i + (\alpha Ab + A\beta + B)n, & \alpha aA = 1, \\ (\alpha aA)^n w_i + \left(\dfrac{(\alpha aA)^n - 1}{\alpha aA - 1}\right)(\alpha Ab + A\beta + B), & otherwise, \end{cases} \quad \text{for } i = 0,1,2. \tag{3.10}$$

From (3.6) and equations (3.8), (3.9) and (3.10), it follows that for all $n \in \mathbb{N}_0$,

$$u_{3n} = \begin{cases} \dfrac{x^p_{-k} + (\alpha aB + \alpha b + \beta)nx_0}{x_0}, & \alpha aA = 1, \\ \dfrac{(\alpha aA)^n x^p_{-k} + \left(\dfrac{(\alpha aA)^n - 1}{\alpha aA - 1}\right)(\alpha aB + \alpha b + \beta)x_0}{x_0}, & otherwise, \end{cases} \tag{3.11}$$

$$u_{3n+1} = \begin{cases} \dfrac{\alpha y^p_{-k} + \beta y_0 + (\alpha aB + \alpha b + \beta)ny_0}{y_0}, & \alpha aA = 1, \\ \dfrac{(\alpha aA)^n \left(\alpha y^p_{-k} + \beta y_0\right) + \left(\dfrac{(\alpha aA)^n - 1}{\alpha aA - 1}\right)(\alpha aB + \alpha b + \beta)y_0}{y_0}, & otherwise, \end{cases} \tag{3.12}$$

$$u_{3n+2} = \begin{cases} \dfrac{\alpha a z^p_{-k} + (\alpha b + \beta)z_0 + (\alpha aB + \alpha b + \beta)nz_0}{z_0}, & \alpha aA = 1, \\ \dfrac{(\alpha aA)^n \left(\alpha a z^p_{-k} + (\alpha b + \beta)z_0\right) + \left(\dfrac{(\alpha aA)^n - 1}{\alpha aA - 1}\right)(\alpha aB + \alpha b + \beta)z_0}{z_0}, & otherwise, \end{cases} \tag{3.13}$$

$$v_{3n} = \begin{cases} \dfrac{y^p_{-k} + (aA\beta + aB + b)ny_0}{y_0}, & \alpha aA = 1, \\ \dfrac{(\alpha aA)^n y^p_{-k} + \left(\dfrac{(\alpha aA)^n - 1}{\alpha aA - 1}\right)(aA\beta + aB + b)y_0}{y_0}, & otherwise, \end{cases} \tag{3.14}$$

$$v_{3n+1} = \begin{cases} \dfrac{az^p_{-k} + b + (aA\beta + aB + b)nz_0}{z_0}, & \alpha aA = 1, \\ \dfrac{(\alpha aA)^n \left(az^p_{-k} + bz_0\right) + \left(\dfrac{(\alpha aA)^n - 1}{\alpha aA - 1}\right)(aA\beta + aB + b)z_0}{z_0}, & otherwise, \end{cases} \tag{3.15}$$

$$v_{3n+2} = \begin{cases} \dfrac{aAx^p_{-k} + (aB + b)x_0 + (aA\beta + aB + b)nx_0}{x_0}, & \alpha aA = 1, \\ \dfrac{(\alpha aA)^n \left(aAx^p_{-k} + (aB + b)x_0\right) + \left(\dfrac{(\alpha aA)^n - 1}{\alpha aA - 1}\right)(aA\beta + aB + b)x_0}{x_0}, & otherwise, \end{cases} \tag{3.16}$$

$$w_{3n}=\begin{cases}\dfrac{z_{-k}^p+(\alpha Ab+A\beta+B)nz_0}{z_0}, & \alpha aA=1,\\ \dfrac{(\alpha aA)^n z_{-k}^p+\left(\dfrac{(\alpha aA)^n-1}{\alpha aA-1}\right)(\alpha Ab+A\beta+B)z_0}{z_0}, & otherwise,\end{cases} \tag{3.17}$$

$$w_{3n+1}=\begin{cases}\dfrac{Ax_{-k}^p+Bx_0+(\alpha Ab+A\beta+B)nx_0}{x_0}, & \alpha aA=1,\\ \dfrac{(\alpha aA)^n\left(Ax_{-k}^p+Bx_0\right)+\left(\dfrac{(\alpha aA)^n-1}{\alpha aA-1}\right)(\alpha Ab+A\beta+B)x_0}{x_0}, & otherwise,\end{cases} \tag{3.18}$$

$$w_{3n+2}=\begin{cases}\dfrac{A\alpha y_{-k}^p+(A\beta+B)y_0+(\alpha Ab+A\beta+B)ny_0}{y_0}, & \alpha aA=1,\\ \dfrac{(\alpha aA)^n\left(A\alpha y_{-k}^p+(A\beta+B)y_0\right)+\left(\dfrac{(\alpha aA)^n-1}{\alpha aA-1}\right)(\alpha Ab+A\beta+B)y_0}{y_0}, & otherwise,\end{cases} \tag{3.19}$$

Now, by rearranging equation (3.6), we have

$$x_n=\frac{x_{n-k}^p}{u_n},\ y_n=\frac{y_{n-k}^p}{v_n},\ z_n=\frac{z_{n-k}^p}{w_n},\ \forall n\in\mathbb{N}_0. \tag{3.20}$$

Replacing n by $kn+r$, for $r=0,1,...,k-1$, we get

$$x_{kn+r}=\frac{x_{k(n-1)+r}^p}{u_{kn+r}},\ y_{kn+r}=\frac{y_{k(n-1)+r}^p}{v_{kn+r}},\ z_{kn+r}=\frac{z_{k(n-1)+r}^p}{w_{kn+r}},\ r=\overline{0,k-1},\ n\in\mathbb{N}_0. \tag{3.21}$$

Iterating the right-hand side of the aforementioned equations, we get

$$x_{kn+r}=\frac{x_{r-k}^{p^{n+1}}}{\prod_{i=0}^{n}u_{ki+r}^{p^{(n-i)}}},\ y_{kn+r}=\frac{y_{r-k}^{p^{n+1}}}{\prod_{i=0}^{n}v_{ki+r}^{p^{(n-i)}}},\ z_{kn+r}=\frac{z_{r-k}^{p^{n+1}}}{\prod_{i=0}^{n}w_{ki+r}^{p^{(n-i)}}},\ \forall r=\overline{0,k-1},n\in\mathbb{N}_0. \tag{3.22}$$

We consider three cases: $k\equiv 0(mod3)$, $k\equiv 1(mod3)$ and $k\equiv 2(mod3)$.

Case1 ($k\equiv 0(mod3)$): Suppose $k=3l,(l=1,2,...)$. Then, from (3.22) and depending on the value of r modulo 3, we have

$$\forall r=\overline{0,l-1},\ n\in\mathbb{N}_0:\begin{cases}x_{3(ln+r)+j}=\dfrac{x_{3(r-l)+j}^{p^{n+1}}}{\prod_{i=0}^{n}u_{3(li+r)+j}^{p^{(n-i)}}},\\ y_{3(ln+r)+j}=\dfrac{y_{3(r-l)+j}^{p^{n+1}}}{\prod_{i=0}^{n}v_{3(li+r)+j}^{p^{(n-i)}}},\\ z_{3(ln+r)+j}=\dfrac{z_{3(r-l)+j}^{p^{n+1}}}{\prod_{i=0}^{n}w_{3(li+r)+j}^{p^{(n-i)}}},\end{cases}\quad j=0,1,2. \tag{3.23}$$

Case2 ($k \equiv 1(mod3)$): Suppose $k = 3l+1, (l = 0, 1, 2, ...)$. Then, from (3.22) and depending on the value of n modulo 3, we have $\forall r = \overline{0, 3l}, n \in \mathbb{N}_0$:

$$\begin{cases} x_{(3l+1)(3n)+r} = \dfrac{x_{r-3l-1}^{p^{3n+1}}}{\prod\limits_{i=0}^{3n} u_{(3l+1)i+r}^{p^{(3n-i)}}}, \\ x_{(3l+1)(3n+1)+r} = \dfrac{x_{r-3l-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{3n+1} u_{(3l+1)i+r}^{p^{(3n+1-i)}}}, \\ x_{(3l+1)(3n+2)+r} = \dfrac{x_{r-3l-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{3n+2} u_{(3l+1)i+r}^{p^{(3n+2-i)}}}, \end{cases} \tag{3.24}$$

$$\begin{cases} x_{(3l+1)(3n)+r} = \dfrac{x_{r-3l-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+1)(3i)+r}^{p^{(3n-3i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{(3l+1)(3i+1)+r}^{p^{(3n-(3i+1))}}\right)\left(\prod\limits_{i=0}^{n-1} u_{(3l+1)(3i+2)+r}^{p^{(3n-(3i+2))}}\right)}, \\ x_{(3l+1)(3n+1)+r} = \dfrac{x_{r-3l-1}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+1)(3i)+r}^{p^{(3n+1-(3i))}}\right)\left(\prod\limits_{i=0}^{n} u_{(3l+1)(3i+1)+r}^{p^{(3n+1-(3i+1))}}\right)\left(\prod\limits_{i=0}^{n-1} u_{(3l+1)(3i+2)+r}^{p^{(3n+1-(3i+2))}}\right)}, \\ x_{(3l+1)(3n+2)+r} = \dfrac{x_{r-3l-1}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+1)(3i)+r}^{p^{(3n+2-(3i))}}\right)\left(\prod\limits_{i=0}^{n} u_{(3l+1)(3i+1)+r}^{p^{(3n+2-(3i+1))}}\right)\left(\prod\limits_{i=0}^{n} u_{(3l+1)(3i+2)+r}^{p^{(3n+2-(3i+2))}}\right)}, \end{cases} \tag{3.25}$$

$$\begin{cases} x_{(3l+1)(3n)+r} = \dfrac{x_{r-3l-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+1)(3i)+r}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(u_{(3l+1)(3i+1)+r}^{p^{(3(n-i)-1)}} u_{(3l+1)(3i+2)+r}^{p^{(3(n-i)-2)}}\right)}, \\ x_{(3l+1)(3n+1)+r} = \dfrac{x_{r-3l-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(u_{(3l+1)(3i)+r}^{p^{(3(n-i)+1)}} u_{(3l+1)(3i+1)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{(3l+1)(3i+2)+r}^{p^{(3(n-i)-1)}}\right)}, \\ x_{(3l+1)(3n+2)+r} = \dfrac{x_{r-3l-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(u_{(3l+1)(3i)+r}^{p^{(3(n-i)+2)}} u_{(3l+1)(3i+1)+r}^{p^{(3(n-i)+1)}} u_{(3l+1)(3i+2)+r}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.26}$$

$$\begin{cases} y_{(3l+1)(3n)+r} = \dfrac{y_{r-3l-1}^{p^{3n+1}}}{\prod\limits_{i=0}^{3n} v_{(3l+1)i+r}^{p^{(3n-i)}}}, \\ y_{(3l+1)(3n+1)+r} = \dfrac{y_{r-3l-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{3n+1} v_{(3l+1)i+r}^{p^{(3n+1-i)}}}, \\ y_{(3l+1)(3n+2)+r} = \dfrac{y_{r-3l-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{3n+2} v_{(3l+1)i+r}^{p^{(3n+2-i)}}}, \end{cases} \tag{3.27}$$

$$\begin{cases} y_{(3l+1)(3n)+r} = \dfrac{y_{r-3l-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} v_{(3l+1)(3i)+r}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(v_{(3l+1)(3i+1)+r}^{p^{(3(n-i)-1)}} v_{(3l+1)(3i+2)+r}^{p^{(3(n-i)-2)}}\right)}, \\ y_{(3l+1)(3n+1)+r} = \dfrac{y_{r-3l-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(v_{(3l+1)(3i)+r}^{p^{(3(n-i)+1)}} v_{(3l+1)(3i+1)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{(3l+1)(3i+2)+r}^{p^{(3(n-i)-1)}}\right)}, \\ y_{(3l+1)(3n+2)+r} = \dfrac{y_{r-3l-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(v_{(3l+1)(3i)+r}^{p^{(3(n-i)+2)}} v_{(3l+1)(3i+1)+r}^{p^{(3(n-i)+1)}} v_{(3l+1)(3i+2)+r}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.28}$$

$$\begin{cases} z_{(3l+1)(3n)+r} = \dfrac{z_{r-3l-1}^{p^{3n+1}}}{\prod\limits_{i=0}^{3n} w_{(3l+1)i+r}^{p^{(3n-i)}}}, \\ z_{(3l+1)(3n+1)+r} = \dfrac{z_{r-3l-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{3n+1} w_{(3l+1)i+r}^{p^{(3n+1-i)}}}, \\ z_{(3l+1)(3n+2)+r} = \dfrac{z_{r-3l-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{3n+2} w_{(3l+1)i+r}^{p^{(3n+2-i)}}}, \end{cases} \tag{3.29}$$

$$\begin{cases} z_{(3l+1)(3n)+r} = \dfrac{z_{r-3l-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} w_{(3l+1)(3i)+r}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(w_{(3l+1)(3i+1)+r}^{p^{(3(n-i)-1)}} w_{(3l+1)(3i+2)+r}^{p^{(3(n-i)-2)}}\right)}, \\ z_{(3l+1)(3n+1)+r} = \dfrac{z_{r-3l-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(w_{(3l+1)(3i)+r}^{p^{(3(n-i)+1)}} w_{(3l+1)(3i+1)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{(3l+1)(3i+2)+r}^{p^{(3(n-i)-1)}}\right)}, \\ z_{(3l+1)(3n+2)+r} = \dfrac{z_{r-3l-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(w_{(3l+1)(3i)+r}^{p^{(3(n-i)+2)}} w_{(3l+1)(3i+1)+r}^{p^{(3(n-i)+1)}} w_{(3l+1)(3i+2)+r}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.30}$$

Here we consider two sub-cases:

Sub-case2.1 ($l \neq 0$): From (3.26), (3.28), (3.30) and depending on the value of r modulo 3 , we get, for all $n \in \mathbb{N}_0$, the following expressions:

If $r \equiv 0(mod3)$. Put $3r$ instead r, we obtain, $\forall r = \overline{0, l}, n \in \mathbb{N}_0$:

$$\begin{cases} x_{(3l+1)(3n)+3r} = \dfrac{x_{3r-3l-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+1)(3i)+3r}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(u_{(3l+1)(3i+1)+3r}^{p^{(3(n-i)-1)}} u_{(3l+1)(3i+2)+3r}^{p^{(3(n-i)-2)}}\right)}, \\ x_{(3l+1)(3n+1)+3r} = \dfrac{x_{3r-3l-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(u_{(3l+1)(3i)+3r}^{p^{(3(n-i)+1)}} u_{(3l+1)(3i+1)+3r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{(3l+1)(3i+2)+3r}^{p^{(3(n-i)-1)}}\right)}, \\ x_{(3l+1)(3n+2)+3r} = \dfrac{x_{3r-3l-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(u_{(3l+1)(3i)+3r}^{p^{(3(n-i)+2)}} u_{(3l+1)(3i+1)+3r}^{p^{(3(n-i)+1)}} u_{(3l+1)(3i+2)+3r}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.31}$$

Thus, $\forall r=\overline{0,l}, n\in\mathbb{N}_0$:

$$\begin{cases} x_{3((3l+1)n+r)}=\dfrac{x_{3(r-l)-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n}u_{3((3l+1)i+r)}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(u_{3((3l+1)i+l+r)+1}^{p^{(3(n-i)-1)}}u_{3((3l+1)i+2l+r)+2}^{p^{(3(n-i)-2)}}\right)},\\ x_{3((3l+1)n+l+r)+1}=\dfrac{x_{3(r-l)-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(u_{3((3l+1)i+r)}^{p^{(3(n-i)+1)}}u_{3((3l+1)i+l+r)+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1}u_{3((3l+1)i+2l+r)+2}^{p^{(3(n-i)-1)}}\right)},\\ x_{3((3l+1)n+2l+r)+2}=\dfrac{x_{3(r-l)-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(u_{3((3l+1)i+r)}^{p^{(3(n-i)+2)}}u_{3((3l+1)i+l+r)+1}^{p^{(3(n-i)+1)}}u_{3((3l+1)i+2l+r))+2}^{p^{3(n-i)}}\right)}. \end{cases} \tag{3.32}$$

$$\begin{cases} y_{3((3l+1)n+r)}=\dfrac{y_{3(r-l)-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n}v_{3((3l+1)i+r)}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(v_{3((3l+1)i+l+r)+1}^{p^{(3(n-i)-1)}}v_{3((3l+1)i+2l+r)+2}^{p^{(3(n-i)-2)}}\right)},\\ y_{3((3l+1)n+l+r)+1}=\dfrac{y_{3(r-l)-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(v_{3((3l+1)i+r)}^{p^{(3(n-i)+1)}}v_{3((3l+1)i+l+r)+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1}v_{3((3l+1)i+2l+r)+2}^{p^{(3(n-i)-1)}}\right)},\\ y_{3((3l+1)n+2l+r)+2}=\dfrac{y_{3(r-l)-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(v_{3((3l+1)i+r)}^{p^{(3(n-i)+2)}}v_{3((3l+1)i+l+r)+1}^{p^{(3(n-i)+1)}}v_{3((3l+1)i+2l+r)+2}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.33}$$

$$\begin{cases} z_{3((3l+1)n+r)}=\dfrac{z_{3(r-l)-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n}w_{3((3l+1)i+r)}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(w_{3((3l+1)i+l+r)+1}^{p^{(3(n-i)-1)}}w_{3((3l+1)i+2l+r)+2}^{p^{(3(n-i)-2)}}\right)},\\ z_{3((3l+1)n+l+r)+1}=\dfrac{z_{3(r-l)-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(w_{3((3l+1)i+r)}^{p^{(3(n-i)+1)}}w_{3((3l+1)i+l+r)+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1}w_{3((3l+1)i+2l+r)+2}^{p^{(3(n-i)-1)}}\right)},\\ z_{3((3l+1)n+2l+r)+2}=\dfrac{z_{3(r-l)-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(w_{3((3l+1)i+r)}^{p^{(3(n-i)+2)}}w_{3((3l+1)i+l+r)+1}^{p^{(3(n-i)+1)}}w_{3((3l+1)i+2l+r)+2}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.34}$$

If $r\equiv 1(mod3)$. Put $3r+1$ instead r, we obtain, $\forall r=\overline{0,l-1}, n\in\mathbb{N}_0$:

$$\begin{cases} x_{3((3l+1)n+r)+1}=\dfrac{x_{3(r-l)}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n}u_{3((3l+1)i+r)+1}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(u_{3((3l+1)i+l+r)+2}^{p^{(3(n-i)-1)}}u_{3((3l+1)i+2l+r+1)}^{p^{(3(n-i)-2)}}\right)},\\ x_{3((3l+1)n+l+r)+2}=\dfrac{x_{3(r-l)}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(u_{3((3l+1)i+r)+1}^{p^{(3(n-i)+1)}}u_{3((3l+1)i+l+r)+2}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1}u_{3((3l+1)i+2l+r+1)}^{p^{(3(n-i)-1)}}\right)},\\ x_{3((3l+1)n+2l+r+1)}=\dfrac{x_{3(r-l)}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(u_{3((3l+1)i+r)+1}^{p^{(3(n-i)+2)}}u_{3((3l+1)i+l+r)+2}^{p^{(3(n-i)+1)}}u_{3((3l+1)i+2l+r+1)}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.35}$$

$$\begin{cases} y_{3((3l+1)n+r)+1} = \dfrac{y_{3(r-l)}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+1)i+r)+1}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(v_{3((3l+1)i+l+r)+2}^{p^{(3(n-i)-1)}} v_{3((3l+1)i+2l+r+1)}^{p^{(3(n-i)-2)}}\right)}, \\ y_{3((3l+1)n+l+r)+2} = \dfrac{y_{3(r-l)}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(v_{3((3l+1)i+r)+1}^{p^{(3(n-i)+1)}} v_{3((3l+1)i+l+r)+2}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3((3l+1)i+2l+r+1)}^{p^{(3(n-i)-1)}}\right)}, \\ y_{3((3l+1)n+2l+r+1)} = \dfrac{y_{3(r-l)}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(v_{3((3l+1)i+r)+1}^{p^{(3(n-i)+2)}} v_{3((3l+1)i+l+r)+2}^{p^{(3(n-i)+1)}} v_{3((3l+1)i+2l+r+1)}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.36}$$

$$\begin{cases} z_{3((3l+1)n+r)+1} = \dfrac{z_{3(r-l)}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} w_{3((3l+1)i+r)+1}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(w_{3((3l+1)i+l+r)+2}^{p^{(3(n-i)-1)}} w_{3((3l+1)i+2l+r+1)}^{p^{(3(n-i)-2)}}\right)}, \\ z_{3((3l+1)n+l+r)+2} = \dfrac{z_{3(r-l)}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(w_{3((3l+1)i+r)+1}^{p^{(3(n-i)+1)}} w_{3((3l+1)i+l+r)+2}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3((3l+1)i+2l+r+1)}^{p^{(3(n-i)-1)}}\right)}, \\ z_{3((3l+1)n+2l+r+1)} = \dfrac{z_{3(r-l)}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(w_{3((3l+1)i+r)+1}^{p^{(3(n-i)+2)}} w_{3((3l+1)i+l+r)+2}^{p^{(3(n-i)+1)}} w_{3((3l+1)i+2l+r+1)}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.37}$$

If $r \equiv 2(mod3)$. Put $3r+2$ instead r, we obtain, $\forall r = \overline{0, l-1}, n \in \mathbb{N}_0$:

$$\begin{cases} x_{3((3l+1)n+r)+2} = \dfrac{x_{3(r-l)+1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+1)i+r)+2}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(u_{3((3l+1)i+l+r+1)}^{p^{(3(n-i)-1)}} u_{3((3l+1)i+2l+r+1)+1}^{p^{(3(n-i)-2)}}\right)}, \\ x_{3((3l+1)n+l+r+1)} = \dfrac{x_{3(r-l)+1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(u_{3((3l+1)i+r)+2}^{p^{(3(n-i)+1)}} u_{3((3l+1)i+l+r+1)}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3((3l+1)i+2l+r+1)+1}^{p^{(3(n-i)-1)}}\right)}, \\ x_{3((3l+1)n+2l+r+1)+1} = \dfrac{x_{3(r-l)+1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(u_{3((3l+1)i+r)+2}^{p^{(3(n-i)+2)}} u_{3((3l+1)i+l+r+1)}^{p^{(3(n-i)+1)}} u_{3((3l+1)i+2l+r+1)+1}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.38}$$

$$\begin{cases} y_{3((3l+1)n+r)+2} = \dfrac{y_{3(r-l)+1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+1)i+r)+2}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(v_{3((3l+1)i+l+r+1)}^{p^{(3(n-i)-1)}} v_{3((3l+1)i+2l+r+1)+1}^{p^{(3(n-i)-2)}}\right)}, \\ y_{3((3l+1)n+l+r+1)} = \dfrac{y_{3(r-l)+1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(v_{3((3l+1)i+r)+2}^{p^{(3(n-i)+1)}} v_{3((3l+1)i+l+r+1)}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3((3l+1)i+2l+r+1)+1}^{p^{(3(n-i)-1)}}\right)}, \\ y_{3((3l+1)n+2l+r+1)+1} = \dfrac{y_{3(r-l)+1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(v_{3((3l+1)i+r)+2}^{p^{(3(n-i)+2)}} v_{3((3l+1)i+l+r+1)}^{p^{(3(n-i)+1)}} v_{3((3l+1)i+2l+r+1)+1}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.39}$$

$$\begin{cases} z_{3((3l+1)n+r)+2} = \dfrac{z_{3(r-l)+1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+1)i+r)+2}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(u_{3((3l+1)i+l+r+1)}^{p^{(3(n-i)-1)}} u_{3((3l+1)i+2l+r+1)+1}^{p^{(3(n-i)-2)}}\right)}, \\ z_{3((3l+1)n+l+r+1)} = \dfrac{z_{3(r-l)+1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(w_{3((3l+1)i+r)+2}^{p^{(3(n-i)+1)}} w_{3((3l+1)i+l+r+1)}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3((3l+1)i+2l+r+1)+1}^{p^{(3(n-i)-1)}}\right)}, \\ z_{3((3l+1)n+2l+r+1)+1} = \dfrac{z_{3(r-l)+1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(w_{3((3l+1)i+r)+2}^{p^{(3(n-i)+2)}} w_{3((3l+1)i+l+r+1)}^{p^{(3(n-i)+1)}} w_{3((3l+1)i+2l+r+1)+1}^{p^{3(n-i)}}\right)}. \end{cases} \tag{3.40}$$

Sub-case2.2 ($l = 0$): Using the fact that in this case $r = 0$, $k = 1$, we get from (3.26), (3.28) and (3.30) system (3.5) become, for all $n \in \mathbb{N}_0$

$$x_{n+1} = \frac{x_n^p y_n}{\alpha y_{n-1}^p + \beta y_n}, \quad y_{n+1} = \frac{y_n^p z_n}{a z_{n-1}^p + b z_n}, \quad z_{n+1} = \frac{z_n^p x_n}{A x_{n-1}^p + B x_n}. \tag{3.41}$$

Thus,

$$n \in \mathbb{N}_0 : \begin{cases} x_{3n} = \dfrac{x_{-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{3i}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(u_{3i+1}^{p^{(3(n-i)-1)}} u_{3i+2}^{p^{(3(n-i)-2)}}\right)}, \\ x_{3n+1} = \dfrac{x_{-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(u_{3i}^{p^{(3(n-i)+1)}} u_{3i+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3i+2}^{p^{(3(n-i)-1)}}\right)}, \\ x_{3n+2} = \dfrac{x_{-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(u_{3i}^{p^{(3(n-i)+2)}} u_{3i+1}^{p^{(3(n-i)+1)}} u_{3i+2}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.42}$$

$$n \in \mathbb{N}_0 : \begin{cases} y_{3n} = \dfrac{y_{-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} v_{3i}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(v_{3i+1}^{p^{(3(n-i)-1)}} v_{3i+2}^{p^{(3(n-i)-2)}}\right)}, \\ y_{3n+1} = \dfrac{y_{-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(v_{3i}^{p^{(3(n-i)+1)}} v_{3i+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3i+2}^{p^{(3(n-i)-1)}}\right)}, \\ y_{3n+2} = \dfrac{y_{-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(v_{3i}^{p^{(3(n-i)+2)}} v_{3i+1}^{p^{(3(n-i)+1)}} v_{3i+2}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.43}$$

$$n \in \mathbb{N}_0 : \begin{cases} z_{3n} = \dfrac{z_{-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} w_{3i}^{p^{3(n-i)}}\right)\prod\limits_{i=0}^{n-1}\left(w_{3i+1}^{p^{(3(n-i)-1)}} w_{3i+2}^{p^{(3(n-i)-2)}}\right)}, \\ z_{3n+1} = \dfrac{z_{-1}^{p^{3n+2}}}{\prod\limits_{i=0}^{n}\left(w_{3i}^{p^{(3(n-i)+1)}} w_{3i+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3i+2}^{p^{(3(n-i)-1)}}\right)}, \\ z_{3n+2} = \dfrac{z_{-1}^{p^{3n+3}}}{\prod\limits_{i=0}^{n}\left(w_{3i}^{p^{(3(n-i)+2)}} w_{3i+1}^{p^{(3(n-i)+1)}} w_{3i+2}^{p^{3(n-i)}}\right)}. \end{cases} \tag{3.44}$$

Case3 ($k \equiv 2(mod3)$): Suppose $k = 3l+2, (l = 0, 1, 2, ...)$. Then, from (3.22) and depending on the value of n modulo 3, we have, $\forall r = \overline{0, 3l+1}$, $n \in \mathbb{N}_0$:

$$\begin{cases} x_{(3l+2)(3n)+r} = \dfrac{x_{r-(3l+2)}^{p^{3n+1}}}{\prod\limits_{i=0}^{3n} u_{(3l+2)i+r}^{p^{(3n-i)}}}, \\ x_{(3l+2)(3n+1)+r} = \dfrac{x_{r-(3l+2)}^{p^{3n+2}}}{\prod\limits_{i=0}^{3n+1} u_{(3l+2)i+r}^{p^{(3n+1-i)}}}, \\ x_{(3l+2)(3n+2)+r} = \dfrac{x_{r-(3l+2)}^{p^{3n+3}}}{\prod\limits_{i=0}^{3n+2} u_{(3l+2)i+r}^{p^{(3n+2-i)}}}, \end{cases} \tag{3.45}$$

$$\begin{cases} x_{(3l+2)(3n)+r} = \dfrac{x_{r-(3l+2)}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+2)(3i)+r}^{p^{(3n-3i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{(3l+2)(3i+1)+r}^{p^{(3n-(3i+1))}}\right)\left(\prod\limits_{i=0}^{n-1} u_{(3l+2)(3i+2)+r}^{p^{(3n-(3i+2))}}\right)}, \\ x_{(3l+2)(3n+1)+r} = \dfrac{x_{r-(3l+2)}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+2)(3i)+r}^{p^{(3n+1-(3i))}}\right)\left(\prod\limits_{i=0}^{n} u_{(3l+2)(3i+1)+r}^{p^{(3n+1-(3i+1))}}\right)\left(\prod\limits_{i=0}^{n-1} u_{(3l+2)(3i+2)+r}^{p^{(3n+1-(3i+2))}}\right)}, \\ x_{(3l+2)(3n+2)+r} = \dfrac{x_{r-(3l+2)}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+2)(3i)+r}^{p^{(3n+2-(3i))}}\right)\left(\prod\limits_{i=0}^{n} u_{(3l+2)(3i+1)+r}^{p^{(3n+2-(3i+1))}}\right)\left(\prod\limits_{i=0}^{n} u_{(3l+2)(3i+2)+r}^{p^{(3n+2-(3i+2))}}\right)}, \end{cases} \tag{3.46}$$

$$\begin{cases} x_{(3l+2)(3n)+r} = \dfrac{x_{r-3l-2}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+2)(3i)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{(3l+2)(3i+1)+r}^{p^{(3(n-i)-1)}} u_{(3l+2)(3i+2)+r}^{p^{(3(n-i)-2)}}\right)}, \\ x_{(3l+2)(3n+1)+r} = \dfrac{x_{r-3l-2}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+2)(3i)+r}^{p^{(3(n-i)+1)}} u_{(3l+2)(3i+1)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{(3l+2)(3i+2)+r}^{p^{(3(n-i)-1)}}\right)}, \\ x_{(3l+2)(3n+2)+r} = \dfrac{x_{r-3l-2}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} u_{(3l+2)(3i)+r}^{p^{(3(n-i)+2)}} u_{(3l+2)(3i+1)+r}^{p^{(3(n-i)+1)}} u_{(3l+2)(3i+2)+r}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.47}$$

$$\begin{cases} y_{(3l+2)(3n)+r} = \dfrac{y_{r-(3l+2)}^{p^{3n+1}}}{\prod\limits_{i=0}^{3n} v_{(3l+2)i+r}^{p^{(3n-i)}}}, \\ y_{(3l+2)(3n+1)+r} = \dfrac{y_{r-(3l+2)}^{p^{3n+2}}}{\prod\limits_{i=0}^{3n+1} v_{(3l+2)i+r}^{p^{(3n+1-i)}}}, \\ y_{(3l+2)(3n+2)+r} = \dfrac{y_{r-(3l+2)}^{p^{3n+3}}}{\prod\limits_{i=0}^{3n+2} v_{(3l+2)i+r}^{p^{(3n+2-i)}}}, \end{cases} \tag{3.48}$$

$$\begin{cases} y_{(3l+2)(3n)+r} = \dfrac{y_{r-3l-2}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} v_{(3l+2)(3i)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{(3l+2)(3i+1)+r}^{p^{(3(n-i)-1)}} v_{(3l+2)(3i+2)+r}^{p^{(3(n-i)-2)}}\right)}, \\ y_{(3l+2)(3n+1)+r} = \dfrac{y_{r-3l-2}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} v_{(3l+2)(3i)+r}^{p^{(3(n-i)+1)}} v_{(3l+2)(3i+1)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{(3l+2)(3i+2)+r}^{p^{(3(n-i)-1)}}\right)}, \\ y_{(3l+2)(3n+2)+r} = \dfrac{y_{r-3l-2}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} v_{(3l+2)(3i)+r}^{p^{(3(n-i)+2)}} v_{(3l+2)(3i+1)+r}^{p^{(3(n-i)+1)}} v_{(3l+2)(3i+2)+r}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.49}$$

$$\begin{cases} z_{(3l+2)(3n)+r} = \dfrac{z_{r-(3l+2)}^{p^{3n+1}}}{\prod\limits_{i=0}^{3n} w_{(3l+2)i+r}^{p^{(3n-i)}}}, \\ z_{(3l+2)(3n+1)+r} = \dfrac{z_{r-(3l+2)}^{p^{3n+2}}}{\prod\limits_{i=0}^{3n+1} w_{(3l+2)i+r}^{p^{(3n+1-i)}}}, \\ z_{(3l+2)(3n+2)+r} = \dfrac{z_{r-(3l+2)}^{p^{3n+3}}}{\prod\limits_{i=0}^{3n+2} w_{(3l+2)i+r}^{p^{(3n+2-i)}}}, \end{cases} \tag{3.50}$$

$$\begin{cases} z_{(3l+2)(3n)+r} = \dfrac{z_{r-3l-2}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} w_{(3l+2)(3i)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{(3l+2)(3i+1)+r}^{p^{(3(n-i)-1)}} w_{(3l+2)(3i+2)+r}^{p^{(3(n-i)-2)}}\right)}, \\ z_{(3l+2)(3n+1)+r} = \dfrac{z_{r-3l-2}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} w_{(3l+2)(3i)+r}^{p^{(3(n-i)+1)}} w_{(3l+2)(3i+1)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{(3l+2)(3i+2)+r}^{p^{(3(n-i)-1)}}\right)}, \\ z_{(3l+2)(3n+2)+r} = \dfrac{z_{r-3l-2}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} w_{(3l+2)(3i)+r}^{p^{(3(n-i)+2)}} w_{(3l+2)(3i+1)+r}^{p^{(3(n-i)+1)}} w_{(3l+2)(3i+2)+r}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.51}$$

Here we consider two sub-cases:

Sub-case3.1 ($l \neq 0$): From (3.47), (3.49), (3.51) and depending on the value of r modulo 3 , we get, for all $n \in \mathbb{N}_0$, the following expressions:

If $r \equiv 0 (mod 3)$. Put $3r$ instead r, we obtain, $\forall r = \overline{0, l}, n \in \mathbb{N}_0$:

$$\begin{cases} x_{3((3l+2)n+r)} = \dfrac{x_{3(r-l)-2}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+2)i+r)}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3((3l+2)i+l+r)+2}^{p^{(3(n-i)-1)}} u_{3((3l+2)i+2l+r+1)+1}^{p^{(3(n-i)-2)}}\right)}, \\ x_{3((3l+2)n+l+r)+2} = \dfrac{x_{3(r-l)-2}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+2)i+r)}^{p^{(3(n-i)+1)}} u_{3((3l+2)i+l+r)+2}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3((3l+2)i+2l+r+1)+1}^{p^{(3(n-i)-1)}}\right)}, \\ x_{3((3l+2)n+2l+r+1)+1} = \dfrac{x_{3(r-l)-2}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+2)i+r)}^{p^{(3(n-i)+2)}} u_{3((3l+2)i+l+r)+2}^{p^{(3(n-i)+1)}} u_{3((3l+2)i+2l+r+1)+1}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.52}$$

$$\begin{cases} y_{3((3l+2)n+r)} = \dfrac{y_{3(r-l)-2}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+2)i+r)}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3((3l+2)i+l+r)+2}^{p^{3(n-i)-1}} v_{3((3l+2)i+2l+r+1)+1}^{p^{3(n-i)-2}}\right)}, \\ y_{3((3l+2)n+l+r)+2} = \dfrac{y_{3(r-l)-2}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+2)i+r)}^{p^{3(n-i)+1}} v_{3((3l+2)i+l+r)+2}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3((3l+2)i+2l+r+1)+1}^{p^{3(n-i)-1}}\right)}, \\ y_{3((3l+2)n+2l+r+1)+1} = \dfrac{y_{3(r-l)-2}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+2)i+r)}^{p^{3(n-i)+2}} v_{3((3l+2)i+l+r)+2}^{p^{3(n-i)+1}} v_{3((3l+2)i+2l+r+1)+1}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.53}$$

$$\begin{cases} z_{3((3l+2)n+r)} = \dfrac{z_{3(r-l)-2}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} w_{3((3l+2)i+r)}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3((3l+2)i+l+r)+2}^{p^{3(n-i)-1}} w_{3((3l+2)i+2l+r+1)+1}^{p^{3(n-i)-2}}\right)}, \\ z_{3((3l+2)n+l+r)+2} = \dfrac{z_{3(r-l)-2}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} w_{3((3l+2)i+r)}^{p^{3(n-i)+1}} w_{3((3l+2)i+l+r)+2}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3((3l+2)i+2l+r+1)+1}^{p^{3(n-i)-1}}\right)}, \\ z_{3((3l+2)n+2l+r+1)+1} = \dfrac{z_{3(r-l)-2}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} w_{3((3l+2)i+r)}^{p^{3(n-i)+2}} w_{3((3l+2)i+l+r)+2}^{p^{3(n-i)+1}} w_{3((3l+2)i+2l+r+1)+1}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.54}$$

If $r \equiv 1(mod3)$. Put $3r+1$ instead r, we obtain, $\forall r = \overline{0,l}, n \in \mathbb{N}_0$:

$$\begin{cases} x_{3((3l+2)n+r)+1} = \dfrac{x_{3(r-l)-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+2)i+r)+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3((3l+2)i+l+r+1)}^{p^{3(n-i)-1}} u_{3((3l+2)i+2l+r+1)+2}^{p^{3(n-i)-2}}\right)}, \\ x_{3((3l+2)n+l+r+1)} = \dfrac{x_{3(r-l)-1}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+2)i+r)+1}^{p^{3(n-i)+1}} u_{3((3l+2)i+l+r+1)}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3((3l+2)i+2l+r+1)+2}^{p^{3(n-i)-1}}\right)}, \\ x_{3((3l+2)n+2l+r+1)+2} = \dfrac{x_{3(r-l)-1}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+2)i+r)+1}^{p^{3(n-i)+2}} u_{3((3l+2)i+l+r+1)}^{p^{3(n-i)+1}} u_{3((3l+2)i+2l+r+1)+2}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.55}$$

$$\begin{cases} y_{3((3l+2)n+r)+1} = \dfrac{y_{3(r-l)-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+2)i+r)+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3((3l+2)i+l+r+1)}^{p^{3(n-i)-1}} v_{3((3l+2)i+2l+r+1)+2}^{p^{3(n-i)-2}}\right)}, \\ y_{3((3l+2)n+l+r+1)} = \dfrac{y_{3(r-l)-1}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+2)i+r)+1}^{p^{3(n-i)+1}} v_{3((3l+2)i+l+r+1)}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3((3l+2)i+2l+r+1)+2}^{p^{3(n-i)-1}}\right)}, \\ y_{3((3l+2)n+2l+r+1)+2} = \dfrac{y_{3(r-l)-1}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+2)i+r)+1}^{p^{3(n-i)+2}} v_{3((3l+2)i+l+r+1)}^{p^{3(n-i)+1}} v_{3((3l+2)i+2l+r+1)+2}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.56}$$

$$\begin{cases} z_{3((3l+2)n+r)+1} = \dfrac{z_{3(r-l)-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} w_{3((3l+2)i+r)+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3((3l+2)i+l+r+1)}^{p^{(3(n-i)-1)}} w_{3((3l+2)i+2l+r+1)+2}^{p^{(3(n-i)-2)}}\right)}, \\ z_{3((3l+2)n+l+r+1)} = \dfrac{z_{3(r-l)-1}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} w_{3((3l+2)i+r)+1}^{p^{(3(n-i)+1)}} w_{3((3l+2)i+l+r+1)}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3((3l+2)i+2l+r+1)+2}^{p^{(3(n-i)-1)}}\right)}, \\ z_{3((3l+2)n+2l+r+1)+2} = \dfrac{z_{3(r-l)-1}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} w_{3((3l+2)i+r)+1}^{p^{(3(n-i)+2)}} w_{3((3l+2)i+l+r+1)}^{p^{(3(n-i)+1)}} w_{3((3l+2)i+2l+r+1)+2}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.57}$$

<u>If $r \equiv 2(mod3)$</u>. Put $3r+2$ instead r, we obtain, $\forall r = \overline{0, l-1}, n \in \mathbb{N}_0$:

$$\begin{cases} x_{3((3l+2)n+r)+2} = \dfrac{x_{3(r-l)}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+2)i+r)+2}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3((3l+2)i+l+r+1)+1}^{p^{(3(n-i)-1)}} u_{3((3l+2)i+2l+r+2)}^{p^{(3(n-i)-2)}}\right)}, \\ x_{3((3l+2)n+l+r+1)+1} = \dfrac{x_{3(r-l)}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+2)i+r)+2}^{p^{(3(n-i)+1)}} u_{3((3l+2)i+l+r+1)+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3((3l+2)i+2l+r+2)}^{p^{(3(n-i)-1)}}\right)}, \\ x_{3((3l+2)n+2l+r+2)} = \dfrac{x_{3(r-l)}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} u_{3((3l+2)i+r)+2}^{p^{(3(n-i)+2)}} u_{3((3l+2)i+l+r+1)+1}^{p^{(3(n-i)+1)}} u_{3((3l+2)i+2l+r+2)}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.58}$$

$$\begin{cases} y_{3((3l+2)n+r)+2} = \dfrac{y_{3(r-l)}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+2)i+r)+2}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3((3l+2)i+l+r+1)+1}^{p^{(3(n-i)-1)}} v_{3((3l+2)i+2l+r+2)}^{p^{(3(n-i)-2)}}\right)}, \\ y_{3((3l+2)n+l+r+1)+1} = \dfrac{y_{3(r-l)}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+2)i+r)+2}^{p^{(3(n-i)+1)}} v_{3((3l+2)i+l+r+1)+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3((3l+2)i+2l+r+2)}^{p^{(3(n-i)-1)}}\right)}, \\ y_{3((3l+2)n+2l+r+2)} = \dfrac{y_{3(r-l)}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} v_{3((3l+2)i+r)+2}^{p^{(3(n-i)+2)}} v_{3((3l+2)i+l+r+1)+1}^{p^{(3(n-i)+1)}} v_{3((3l+2)i+2l+r+2)}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.59}$$

$$\begin{cases} z_{3((3l+2)n+r)+2} = \dfrac{z_{3(r-l)}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} w_{3((3l+2)i+r)+2}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3((3l+2)i+l+r+1)+1}^{p^{(3(n-i)-1)}} w_{3((3l+2)i+2l+r+2)}^{p^{(3(n-i)-2)}}\right)}, \\ z_{3((3l+2)n+l+r+1)+1} = \dfrac{z_{3(r-l)}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} w_{3((3l+2)i+r)+2}^{p^{(3(n-i)+1)}} w_{3((3l+2)i+l+r+1)+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3((3l+2)i+2l+r+2)}^{p^{(3(n-i)-1)}}\right)}, \\ z_{3((3l+2)n+2l+r+2)} = \dfrac{z_{3(r-l)}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} w_{3((3l+2)i+r)+2}^{p^{(3(n-i)+2)}} w_{3((3l+2)i+l+r+1)+1}^{p^{(3(n-i)+1)}} w_{3((3l+2)i+2l+r+2)}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.60}$$

Sub-case3.2 ($l = 0$): Using the fact that in this case $r = 0, 1$, $k = 2$, we get from (3.47), (3.49) and (3.51), for all $n \in \mathbb{N}_0$:

$$\begin{cases} x_{3(2n)} = \dfrac{x_{-2}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{3(2i)}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3(2i)+2}^{p^{(3(n-i)-1)}} u_{3(2i+1)+1}^{p^{(3(n-i)-2)}}\right)}, \\ x_{3(2n)+2} = \dfrac{x_{-2}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} u_{3(2i)}^{p^{(3(n-i)+1)}} u_{3(2i)+2}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3(2i+1)+1}^{p^{(3(n-i)-1)}}\right)}, \\ x_{3(2n)+4} = \dfrac{x_{-2}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} u_{3(2i)}^{p^{(3(n-i)+2)}} u_{3(2i)+2}^{p^{(3(n-i)+1)}} u_{3(2i+1)+1}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.61}$$

$$\begin{cases} x_{3(2n)+1} = \dfrac{x_{-1}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{3(2i)+1}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3(2i)+3}^{p^{(3(n-i)-1)}} u_{3(2i+1)+2}^{p^{(3(n-i)-2)}}\right)}, \\ x_{3(2n)+3} = \dfrac{x_{-1}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} u_{3(2i)+1}^{p^{(3(n-i)+1)}} u_{3(2i)+3}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3(2i+1)+2}^{p^{(3(n-i)-1)}}\right)}, \\ x_{3(2n)+5} = \dfrac{x_{-1}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} u_{3(2i)+1}^{p^{(3(n-i)+2)}} u_{3(2i)+3}^{p^{(3(n-i)+1)}} u_{3(2i+1)+2}^{p^{3(n-i)}}\right)}, \end{cases} \tag{3.62}$$

Finally,

$$\begin{cases} x_{3(2n)+r} = \dfrac{x_{r-2}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} u_{3(2i)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3(2i)+2+r}^{p^{(3(n-i)-1)}} u_{3(2i+1)+1+r}^{p^{(3(n-i)-2)}}\right)}, \\ x_{3(2n)+2+r} = \dfrac{x_{r-2}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} u_{3(2i)+r}^{p^{(3(n-i)+1)}} u_{3(2i)+2+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} u_{3(2i+1)+1+r}^{p^{(3(n-i)-1)}}\right)}, \\ x_{3(2n+1)+1+r} = \dfrac{x_{r-2}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} u_{3(2i)+r}^{p^{(3(n-i)+2)}} u_{3(2i)+2+r}^{p^{(3(n-i)+1)}} u_{3(2i+1)+1+r}^{p^{3(n-i)}}\right)}, \end{cases} \quad , \; r = 0, 1 \tag{3.63}$$

$$\begin{cases} y_{3(2n)+r} = \dfrac{y_{r-2}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} v_{3(2i)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3(2i)+2+r}^{p^{(3(n-i)-1)}} v_{3(2i+1)+1+r}^{p^{(3(n-i)-2)}}\right)}, \\ y_{3(2n)+2+r} = \dfrac{y_{r-2}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} v_{3(2i)+r}^{p^{(3(n-i)+1)}} v_{3(2i)+2+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} v_{3(2i+1)+1+r}^{p^{(3(n-i)-1)}}\right)}, \\ y_{3(2n+1)+1+r} = \dfrac{y_{r-2}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} v_{3(2i)+r}^{p^{(3(n-i)+2)}} v_{3(2i)+2+r}^{p^{(3(n-i)+1)}} v_{3(2i+1)+1+r}^{p^{3(n-i)}}\right)}, \end{cases} \quad , \; r = 0, 1 \tag{3.64}$$

$$\begin{cases} z_{3(2n)+r} = \dfrac{z_{r-2}^{p^{3n+1}}}{\left(\prod\limits_{i=0}^{n} w_{3(2i)+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3(2i)+2+r}^{p^{(3(n-i)-1)}} w_{3(2i+1)+1+r}^{p^{(3(n-i)-2)}}\right)}, \\ z_{3(2n)+2+r} = \dfrac{z_{r-2}^{p^{3n+2}}}{\left(\prod\limits_{i=0}^{n} w_{3(2i)+r}^{p^{(3(n-i)+1)}} w_{3(2i)+2+r}^{p^{3(n-i)}}\right)\left(\prod\limits_{i=0}^{n-1} w_{3(2i+1)+1+r}^{p^{(3(n-i)-1)}}\right)}, \\ z_{3(2n+1)+1+r} = \dfrac{z_{r-2}^{p^{3n+3}}}{\left(\prod\limits_{i=0}^{n} w_{3(2i)+r}^{p^{(3(n-i)+2)}} w_{3(2i)+2+r}^{p^{(3(n-i)+1)}} w_{3(2i+1)+1+r}^{p^{3(n-i)}}\right)}, \end{cases} \quad ,r = 0, 1. \qquad (3.65)$$

The following theorem summarizes our previous discussion.

Theorem 3.2. *Consider system (3.5), where the parameters α, β, a, b, A, B and the initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{0, 1, \ldots, k\}$ are non-zero real numbers. Then, the following statements hold:*

(a) If $k = 3l$, $(l = 1, 2, \ldots)$, then for all $n \in \mathbb{N}_0$, the solution of system (3.5) is given by (3.23).

(b) If $k = 3l + 1$, $(l = 1, 2, \ldots)$, then for all $n \in \mathbb{N}_0$, the solution of system (3.5) is given by (3.32), (3.33), (3.34), (3.35), (3.36), (3.37), (3.38), (3.39) and (3.40).

(c) If $k = 3l + 2$, $(l = 1, 2, \ldots)$, then for all $n \in \mathbb{N}_0$, the solution of system (3.5) is given by (3.52), (3.53), (3.54), (3.55), (3.56), (3.57), (3.58), (3.59) and (3.60).

(d) If $k = 1$, then for all $n \in \mathbb{N}_0$, the solution of system (3.5) is given by (3.42), (3.43) and (3.44).

(e) If $k = 2$, then for all $n \in \mathbb{N}_0$, the solution of system (3.5) is given by (3.63), (3.64) and (3.65).

Where the terms of the sequences u_n, v_n and w_n modulo 3 in formulas of the solutions are given by (3.8), (3.9) and (3.10).

Remark 3.2.1.

If we take $\alpha = a = A$ and $\beta = b = B$ and choose initial values such that $x_{-i} = y_{-i} = z_{-i}$, $i = 0, 1, ..., k$, then system (3.5) will reduced to the nonlinear difference equation

$$x_{n+1} = \frac{x_{n-k+1}^{p} x_n}{\alpha x_{n-k}^{p} + \beta x_n}, \; n \in \mathbb{N}_0, \; p, k \in \mathbb{N}.$$

3.3 Behavior of the solutions a particular case

In this section, we focus our attention on a special case of system (3.5). In particular, we examine the boundedness, the asymptotic behavior, and periodicity of solutions of system (3.5) with $p = 1$, that is, the system

$$x_{n+1} = \frac{x_{n-k+1}y_n}{\alpha y_{n-k} + \beta y_n},\ y_{n+1} = \frac{y_{n-k+1}z_n}{az_{n-k} + bz_n},\ z_{n+1} = \frac{z_{n-k+1}x_n}{Ax_{n-k} + Bx_n},\ n \in \mathbb{N}_0,\ k \in \mathbb{N}. \quad (3.66)$$

Throughout this section, we also assume that the parameters α, β, a, b , A, B and the initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{0, 1, \ldots, k\}$ are positive. We start with the following theorem concerning the boundedness of solutions of system (3.66).

Theorem 3.3. *Consider the system (3.66) such that*

(1) $\min(\beta, b, B) \geq 1$ *; or*

(2) $\min(\alpha, a, A) \geq 1$*,* $az_{-k} \geq z_0$*,* $\alpha y_{-k} \geq y_0$ *and* $Ax_{-k} \geq x_0$*.*

Then, every positive solution is bounded.

Proof. Let $\{x_n, y_n, z_n\}_{n\geq -k}$ be a solution of (3.66).
Hypothesis **(1)** is satisfied. Suppose that $\min(\beta, b, B) \geq 1$, then it follows from system (3.66) that for all $n \in \mathbb{N}_0$,

$$x_{n+1} \leq \frac{x_{n-k+1}}{\beta} \leq x_{n-k+1},\ y_{n+1} \leq \frac{y_{n-k+1}}{b} \leq y_{n-k+1} \text{ and } z_{n+1} \leq \frac{z_{n-k+1}}{B} \leq z_{n-k+1},$$

and so the subsequences $\{x_{kn-i}\}_{n\geq 0}$, $\{y_{kn-i}\}_{n\geq 0}$ and $\{z_{kn-i}\}_{n\geq 0}$, $i = 0, \ldots, k-1$ are decreasing. Moreover, we have for all $n \in \mathbb{N}_0$,

$$x_n \leq \max_{i=0,\ldots,k-1}\left\{\frac{x_{-i}}{\beta}\right\},\ y_n \leq \max_{i=0,\ldots,k-1}\left\{\frac{y_{-i}}{b}\right\} \text{ and } z_n \leq \max_{i=0,\ldots,k-1}\left\{\frac{z_{-i}}{B}\right\}.$$

Thus, the solution is bounded. Hypothesis **(2)** is satisfied. If, on the other hand, $\min(\alpha, a, A) \geq 1$, $az_{-k} \geq z_0$, $\alpha y_{-k} \geq y_0$ and $Ax_{-k} \geq x_0$, then it follows from (3.66) that for $n = 0$,

$$x_1 \leq \frac{x_{-k+1}y_0}{\alpha y_{-k}} \leq x_{-k+1},\ y_1 \leq \frac{y_{-k+1}z_0}{az_{-k}} \leq y_{-k+1},\ \text{and } z_1 \leq \frac{z_{-k+1}x_0}{Ax_{-k}} \leq z_{-k+1},$$

and from this, together with the assumption that $\min(\alpha, a, A) \geq 1$, we get for $n = 0$,

$$x_2 \leq \frac{x_{-k+2}y_1}{\alpha y_{-k+1}} \leq x_{-k+2},\ y_2 \leq \frac{y_{-k+2}z_1}{az_{-k+1}} \leq y_{-k+2},\ \text{and } z_2 \leq \frac{z_{-k+2}x_1}{Ax_{-k+1}} \leq z_{-k+2}.$$

Continuing the process, we obtain, for $n = k-1$,

$$x_k \leq \frac{x_0 y_{k-1}}{\alpha y_{-1}} \leq x_0, \; y_k \leq \frac{y_0 z_{k-1}}{a z_{-1}} \leq y_0, \text{ and } z_k \leq \frac{z_0 x_{k-1}}{A x_{-1}} \leq z_0.$$

It follows by induction that the subsequences $\{x_{kn-i}\}_{n\geq 0}$, $\{y_{kn-i}\}_{n\geq 0}$ and $\{z_{kn-i}\}_{n\geq 0}$, $i = 0, ..., k-1$, are decreasing. Furthermore, we have for all $n \in \mathbb{N}_0$,

$$x_n \leq \max_{i=0,\ldots,k-1} \{x_{-i}\}, \; y_n \leq \max_{i=0,\ldots,k-1} \{y_{-i}\} \text{ and } z_n \leq \max_{i=0,\ldots,k-1} \{z_{-i}\}.$$

Hence, in this case, the solution is also bounded. This completes the proof of the theorem. □

In the next theorem, we give the necessary and sufficient conditions for the solutions of system (3.66) to be periodic of period k (not necessary prime).

Theorem 3.4. *Let $\{x_n, y_n, z_n\}_{n\geq -k}$ be a solution of (3.66). Then, $(x_n, y_n, z_n) = (x_{n-k}, y_{n-k}, z_{n-k})$ for all $n \in \mathbb{N}_0$, if and only if $(x_0, y_0, z_0) = (x_{-k}, y_{-k}, z_{-k})$ and $\alpha + \beta = a + b = A + B = 1$.*

Proof. First, assume that $(x_n, y_n, z_n) = (x_{n-k}, y_{n-k}, z_{n-k})$ for all $n \in \mathbb{N}_0$, Particularly, we have $(x_0, y_0, z_0) = (x_{-k}, y_{-k}, z_{-k})$ and

$$x_{-k+1} = x_1 = \frac{x_{-k+1} y_0}{\alpha y_{-k} + \beta y_0}, \; y_{-k+1} = y_1 = \frac{y_{-k+1} z_0}{a z_{-k} + b z_0}, \text{ and } z_{-k+1} = z_1 = \frac{z_{-k+1} x_0}{A x_{-k} + B x_0}.$$

These equations imply that

$$\frac{1}{\alpha + \beta} = \frac{1}{a+b} = \frac{1}{A+B} = 1 \text{ or equivalently, } \alpha + \beta = a + b = A + B = 1.$$

Conversely, suppose that $(x_0, y_0, z_0) = (x_{-k}, y_{-k}, z_{-k})$ and $\alpha + \beta = a + b = A + B = 1$. Then, from (3.66), we get

$$\begin{aligned} x_1 &= \frac{x_{-k+1} y_0}{\alpha y_{-k} + \beta y_0} = \frac{x_{-k+1}}{\alpha + \beta} = x_{-k+1}, \\ y_1 &= \frac{y_{-k+1} z_0}{a z_{-k} + b z_0} = \frac{y_{-k+1}}{a + b} = y_{-k+1}, \\ z_1 &= \frac{z_{-k+1} x_0}{A x_{-k} + B x_0} = \frac{z_{-k+1}}{A + B} = z_{-k+1}. \end{aligned}$$

Again, from (3.66) and using the aforementioned relation, we get

$$\begin{aligned} x_2 &= \frac{x_{-k+2} y_1}{\alpha y_{-k+1} + \beta y_1} = \frac{x_{-k+2}}{\alpha + \beta} = x_{-k+2}, \\ y_2 &= \frac{y_{-k+2} z_1}{a z_{-k+1} + b z_1} = \frac{y_{-k+2}}{a + b} = y_{-k+2}, \\ z_2 &= \frac{z_{-k+2} x_1}{A x_{-k+1} + B x_1} = \frac{z_{-k+2}}{A + B} = z_{-k+2}. \end{aligned}$$

Continuing the process and by principle of induction, we arrive at the desired result. □

The next result provides the limiting properties of solutions of system (3.66).

Theorem 3.5. *Let* $\{x_n, y_n, z_n\}_{n\geq -k}$ *be a solution of* (3.66). *Then, the following statements hold.*

(a) *If* $\alpha a A > 1$, *then* $\lim_{n\to\infty}(x_n, y_n, z_n) = (0,0,0)$.

(b) *If* $\alpha a A = 1$, *then* $\lim_{n\to\infty}(x_n, y_n, z_n) = (0,0,0)$.

(c) *If* $\alpha a A < 1$, *then*

$$\lim_{n\to\infty} x_n = \begin{cases} 0, & R > 1. \\ \infty, & R < 1 \end{cases}, \lim_{n\to\infty} y_n = \begin{cases} 0, & S > 1. \\ \infty, & S < 1 \end{cases}, \lim_{n\to\infty} z_n = \begin{cases} 0, & T > 1 \\ \infty, & T < 1 \end{cases}$$

where: $R := \dfrac{\alpha a B + \alpha b + \beta}{1 - \alpha a A}$, $S := \dfrac{aA\beta + aB + b}{1 - \alpha a A}$ *and* $T := \dfrac{\alpha A b + A\beta + B}{1 - \alpha a A}$.

Proof. We will only prove detailed properties **(a)**, **(b)**, and **(c)** for the limits of x_n. The limits of y_n and z_n follows a similar inductive lines. First, note that from (3.22) the limit of x_{kn+r} as $n \to \infty$ depends on the limit of u_{kn+r} as $n \to \infty$, which, on the other hand, depends on the value of $\alpha a A$.

(a) When $\alpha a A > 1$, $\dfrac{(\alpha a A)^n - 1}{\alpha a A - 1} \to \infty$ as $n \to \infty$. So, from (3.11), (3.12) and (3.13), we have $u_n \to \infty$ as $n \to \infty$. Then, in view of (3.22), $x_n \to 0$ as $n \to \infty$. Similarly, we obtain $y_n \to 0$ and $z_n \to 0$ as $n \to \infty$.

(b) When $\alpha a A = 1$, then from (3.11), (3.12) and (3.13) we get

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{3n} = \lim_{n\to\infty} u_{3n+1} = \lim_{n\to\infty} u_{3n+2} = \lim_{n\to\infty}(\alpha a B + \alpha b + \beta)n = \infty.$$

Hence, from (3.22), we have $x_n \to 0$ as $n \to \infty$. Similarly, we have $y_n \to 0$ and $z_n \to 0$ as $n \to \infty$.

(c) When $\alpha a A < 1$, then $(\alpha a A)^n \to 0$ as$n \to \infty$. So, in reference to (3.11), (3.12), (3.13), (3.14), (3.15), (3.16), (3.17), (3.18) and (3.19), we have

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{3n} = \lim_{n\to\infty} u_{3n+1} = \lim_{n\to\infty} u_{3n+2} = \frac{\alpha a B + \alpha b + \beta}{1 - \alpha a A} = R,$$
$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} v_{3n} = \lim_{n\to\infty} v_{3n+1} = \lim_{n\to\infty} v_{3n+2} = \frac{aA\beta + aB + b}{1 - \alpha a A} = S,$$
$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} w_{3n} = \lim_{n\to\infty} w_{3n+1} = \lim_{n\to\infty} w_{3n+2} = \frac{\alpha A b + A\beta + B}{1 - \alpha a A} = T.$$

Let $r \in \{0, ..., k-1\}$ be fixed. If $R > 1$, then $\lim_{n\to\infty} \prod_{m=0}^{n} u_{km+r} = \infty$ Therefore, $\lim_{n\to\infty} x_{kn+r} = 0$ or equivalently, $\lim_{n\to\infty} x_n = 0$. If, on the other hand, $R < 1$ then $\lim_{n\to\infty} \frac{1}{u_{kn+r}} = \lim_{n\to\infty} \frac{1}{u_n} = \frac{1}{R} > 1$. Hence, we have the product limit $\lim_{n\to\infty} \prod_{m=0}^{n} \frac{1}{u_{km+r}} = \infty$. Thus, $\lim_{n\to\infty} x_{kn+r} = \lim_{n\to\infty} x_n = \infty$.

With the same steps we prove the limits of y_n and z_n of this case as it was mentioned in the theorem.

□

The next theorems provides the behavior of solutions of system (3.66) for the cases $R = 1$, $S = 1$ and $T = 1$.

Theorem 3.6. *Let $\{x_n, y_n, z_n\}_{n\geq -k}$ be a solution of (3.66), with $k = 3l$ ($l = 1, 2, ...$). Assume that $\alpha a A < 1$ Then, the following statements hold.*

(a) *If $R = 1$ (resp. $S = 1$ and $T = 1$) and $x_{-k} \neq x_0$ (resp. $y_{-k} \neq y_0$ and $z_{-k} \neq z_0$) then the subsequences $\{x_{kn+3r}\}$ (resp. $\{y_{kn+3r}\}$ and $\{z_{kn+3r}\}$), for all $r = \overline{0, \frac{k}{3}}$, are convergent.*

(b) *If $R = 1$ (resp. $S = 1$ and $T = 1$) and $x_{-k} = x_0$ (resp. $y_{-k} = y_0$ and $z_{-k} = z_0$) then $x_{kn+3r} = x_{3r-k}$ (resp. $y_{kn+3r} = y_{3r-k}$ and $z_{kn+3r} = z_{3r-k}$), for all $r = \overline{0, \frac{k}{3}}$.*

(c) *If $R = 1$ (resp. $S = 1$ and $T = 1$) and $\alpha y_{-k} \neq (1-\beta)y_0$ (resp. $az_{-k} \neq (1-b)z_0$ and $Ax_{-k} \neq (1-B)x_0$) then the subsequences $\{x_{kn+3r+1}\}$ (resp. $\{y_{kn+3r+1}\}$ and $\{z_{kn+3r+1}\}$), for all $r = \overline{0, \frac{k}{3}}$, are convergent.*

(d) *If $R = 1$ (resp. $S = 1$ and $T = 1$) and $\alpha a z_{-k} \neq (1-\beta-\alpha b)z_0$ (resp. $aAx_{-k} \neq (1-b-aB)x_0$ and $\alpha A y_{-k} \neq (1-B-A\beta)y_0$) then then the subsequences $\{x_{kn+3r+2}\}$ (resp. $\{y_{kn+3r+2}\}$ and $\{z_{kn+3r+2}\}$), for all $r = \overline{0, \frac{k}{3}}$, are convergent.*

(e) *If $R = 1$ (resp. $S = 1$ and $T = 1$) and $\alpha y_{-k} = (1-\beta)y_0$ (resp. $az_{-k} = (1-b)z_0$ and $Ax_{-k} = (1-B)x_0$) then then $x_{kn+3r+1} = x_{3r-k+1}$ (resp. $y_{kn+3r+1} = y_{3r-k+1}$ and $z_{kn+3r+1} = z_{3r-k+1}$), for all $r = \overline{0, \frac{k}{3}}$.*

(f) *If $R = 1$ (resp. $S = 1$ and $T = 1$) and $\alpha a z_{-k} = (1-\beta-\alpha b)z_0$ (resp. $aAx_{-k} = (1-b-aB)x_0$ and $\alpha A y_{-k} = (1-B-A\beta)y_0$) then $x_{kn+3r+2} = x_{3r-k+2}$ (resp. $y_{kn+3r+2} = y_{3r-k+2}$ and $z_{kn+3r+2} = z_{3r-k+2}$), for all $r = \overline{0, \frac{k}{3}}$.*

Proof. We prove the results for the subsequences $\{x_{kn+3r+i}\}, (i=0,1,2)$. The same lines of proof can be followed respectively to prove the results for the subsequences $\{y_{kn+3r+i}\}, (i=0,1,2)$ and $\{z_{kn+3r+i}\}, (i=0,1,2)$. First, we note that in all cases

$$u_{3n}=\frac{(\alpha aA)^n(x_{-k}-x_0)}{x_0}+1,\ u_{3n+1}=\frac{(\alpha aA)^n\,(\alpha y_{-k}+(\beta-1)y_0)}{y_0}+1,\ u_{3n+2}=\frac{(\alpha aA)^n\,(\alpha a z_{-k}+(\alpha b+\beta-1)z_0)}{z_0}+1,$$

(a) By Theorem 3.2, we have

$$x_{3(ln+r)}=\frac{x_{3(r-l)}}{\prod\limits_{i=0}^{n}u_{3(li+r)}}=\frac{x_{3(r-l)}}{\prod\limits_{i=0}^{n}\left(\frac{(\alpha aA)^{(li+r)}(x_{-k}-x_0)}{x_0}+1\right)}.$$

Here we distinguish two cases:

(i) if $x_{-k}>x_0$: Then,

$$x_{3(ln+r)}=\frac{x_{3(r-l)}}{\exp\left[\sum\limits_{i=0}^{n}\ln\left(\frac{(\alpha aA)^{(li+r)}(x_{-k}-x_0)}{x_0}+1\right)\right]}.$$

Using a property of logarithms, we have

$$\ln\left(\frac{(\alpha aA)^n(x_{-k}-x_0)}{x_0}+1\right)\sim_{\infty}\frac{(\alpha aA)^n(x_{-k}-x_0)}{x_0}$$

Now, because $\sum\limits_{i=0}^{n}(\alpha aA)^n$ is a geometric sum, with $\alpha aA<1$, then the sum

$$\sum_{i=0}^{n}\frac{(x_{-k}-x_0)}{x_0}(\alpha aA)^{(li+r)}$$

is convergent.

(ii) if $x_{-k}<x_0$: Then,

$$\begin{aligned}
x_{3(ln+r)} &= \frac{x_{3(r-l)}}{\prod\limits_{i=0}^{n}\left(\frac{(\alpha aA)^{(li+r)}(x_{-k}-x_0)}{x_0}+1\right)}=x_{3(r-l)}\prod_{i=0}^{n}\left(\frac{x_0}{(\alpha aA)^{(li+r)}(x_{-k}-x_0)+x_0}\right)\\
&= x_{3(r-l)}\prod_{i=0}^{n}\left(1+\frac{-(\alpha aA)^{(li+r)}(x_{-k}-x_0)}{(\alpha aA)^{(li+r)}(x_{-k}-x_0)+x_0}\right)=x_{3(r-l)}\prod_{i=0}^{n}\left(1+\frac{(\alpha aA)^{(li+r)}(x_0-x_{-k})}{(\alpha aA)^{(li+r)}(x_{-k}-x_0)+x_0}\right)\\
&= x_{3(r-l)}\exp\left[\sum_{i=0}^{n}\ln\left(1+\frac{(\alpha aA)^{(li+r)}(x_0-x_{-k})}{(\alpha aA)^{(li+r)}(x_{-k}-x_0)+x_0}\right)\right].
\end{aligned}$$

Using again property of logarithm, we have

$$\ln\left(1+\frac{(\alpha aA)^n(x_0-x_{-k})}{(\alpha aA)^n(x_{-k}-x_0)+x_0}\right)\sim_{+\infty}\frac{(\alpha aA)^n(x_0-x_{-k})}{(\alpha aA)^n(x_{-k}-x_0)+x_0},$$

and

$$\frac{(\alpha aA)^n(x_0 - x_{-k})}{(\alpha aA)^n(x_{-k} - x_0) + x_0} \sim_{+\infty} \frac{(\alpha aA)^n(x_0 - x_{-k})}{x_0},$$

as above, because $\sum_{i=0}^{n}(\alpha aA)^n$ is a geometric sum, with $\alpha aA < 1$, then the sum

$$\sum_{i=0}^{n} \frac{(x_0 - x_{-k})}{x_0}(\alpha aA)^{(li+r)},$$

is convergent, so it is for the sum

$$\sum_{i=0}^{n} \frac{(\alpha aA)^{(li+r)}(x_0 - x_{-k})}{(\alpha aA)^{(li+r)}(x_{-k} - x_0) + x_0}.$$

Thus the desired result follows.

(b) The result is immediate because $u_{3n} = 1$ in this case.

(c) The proof is similar to item **(a)**. That is, by Theorem 3.2, we have

$$x_{3(ln+r)+1} = \frac{x_{3(r-l)+1}}{\prod_{i=0}^{n} u_{3(li+r)+1}} = \frac{x_{3(r-l)+1}}{\prod_{i=0}^{n}\left(\frac{(\alpha aA)^{li+r}\left(\alpha y_{-k} + (\beta - 1)y_0\right)}{y_0} + 1\right)}.$$

Also, we distinguish two cases:
(i) if $\alpha y_{-k} > (1 - \beta)y_0$, then

$$x_{3(ln+r)+1} = \frac{x_{3(r-l)+1}}{\exp\left[\sum_{i=0}^{n} \ln\left(\frac{(\alpha aA)^{li+r}\left(\alpha y_{-k} + (\beta - 1)y_0\right)}{y_0} + 1\right)\right]}.$$

Using a property of logarithm, we have

$$\ln\left(\frac{(\alpha aA)^{li+r}\left(\alpha y_{-k} + (\beta - 1)y_0\right)}{y_0} + 1\right) \sim_{\infty} \frac{(\alpha aA)^{li+r}\left(\alpha y_{-k} + (\beta - 1)y_0\right)}{y_0}$$

Because $\sum_{i=0}^{n}(\alpha aA)^n$ is a geometric sum, with $\alpha aA < 1$, then the sum

$$\sum_{i=0}^{n} \frac{(\alpha aA)^{li+r}\left(\alpha y_{-k} + (\beta - 1)y_0\right)}{y_0}$$

is convergent.

(ii) if $\alpha y_{-k} < (1-\beta)y_0$, then

$$\begin{aligned} x_{3(ln+r)+1} &= \frac{x_{3(r-l)+1}}{\prod_{i=0}^{n}\left(\frac{(\alpha aA)^{li+r}(\alpha y_{-k}+(\beta-1)y_0)}{y_0}+1\right)} \\ &= x_{3(r-l)+1}\prod_{i=0}^{n}\left(\frac{y_0}{(\alpha aA)^{li+r}(\alpha y_{-k}+(\beta-1)y_0)+y_0}\right) \\ &= x_{3(r-l)+1}\prod_{i=0}^{n}\left(1+\frac{-(\alpha aA)^{li+r}(\alpha y_{-k}+(\beta-1)y_0)}{(\alpha aA)^{li+r}(\alpha y_{-k}+(\beta-1)y_0)+y_0}\right) \\ &= x_{3(r-l)+1}\exp\left[\sum_{i=0}^{n}\ln\left(1+\frac{(\alpha aA)^{li+r}((1-\beta)y_0-\alpha y_{-k})}{(\alpha aA)^{li+r}(\alpha y_{-k}+(\beta-1)y_0)+y_0}\right)\right]. \end{aligned}$$

We have,

$$\ln\left(1+\frac{(\alpha aA)^{n}((1-\beta)y_0-\alpha y_{-k})}{(\alpha aA)^{n}(\alpha y_{-k}+(\beta-1)y_0)+y_0}\right)\sim_{+\infty}\frac{(\alpha aA)^{n}((1-\beta)y_0-\alpha y_{-k})}{(\alpha aA)^{n}(\alpha y_{-k}+(\beta-1)y_0)+y_0},$$

and

$$\frac{(\alpha aA)^{n}((1-\beta)y_0-\alpha y_{-k})}{(\alpha aA)^{n}(\alpha y_{-k}+(\beta-1)y_0)+y_0}\sim_{+\infty}\frac{(\alpha aA)^{n}((1-\beta)y_0-\alpha y_{-k})}{y_0}.$$

The sum

$$\sum_{i=0}^{n}\frac{(\alpha aA)^{li+r}((1-\beta)y_0-\alpha y_{-k})}{y_0},$$

is convergent, so it is for the sum

$$\sum_{i=0}^{n}\frac{(\alpha aA)^{li+r}((1-\beta)y_0-\alpha y_{-k})}{(\alpha aA)^{li+r}(\alpha y_{-k}+(\beta-1)y_0)+y_0}.$$

Hence, conclusion follows.

(d) The proof is similar to item **(c)** so, we omit it.

(e) (f) As in items **(c)** and **(d)**, the results are immediate because, in these cases, $u_{3n+1}=1$, $u_{3n+2}=1$ respectively . □

The following theorem extend the results obtained in theorem 3.6.

Theorem 3.7. *Let $\{x_n, y_n, z_n\}_{n\geq -k}$ be a solution of (3.66). Assume that $\alpha aA < 1$, $R=1$ (resp. $S=1$ and $T=1$) then, for all $k=3l+j$, $(l=0,1,\cdots)$, $(j=1,2)$ we have: If $x_{-k}=x_0$ (resp. $y_{-k}=y_0$ and $z_{-k}=z_0$), $\alpha y_{-k}=(1-\beta)y_0$ (resp. $az_{-k}=(1-b)z_0$ and $Ax_{-k}=(1-B)x_0$) and $\alpha az_{-k}=(1-\beta-\alpha b)z_0$ (resp. $aAx_{-k}=(1-b-aB)x_0$ and $\alpha Ay_{-k}=(1-B-A\beta)y_0$), then the subsequences of $\{x_n\}$ (resp. $\{y_n\}$ and $\{z_n\}$) mentioned by their relations in Theorem 3.2 are periodic with period k, otherwise they are convergent.*

Proof. Not that in the proof we use the same method and techniques as in proof of Theorem 3.6 to prove the following theorem, so we will do it for some cases and others are similar.

Consider $k = 1$, $(l = 0, j = 1)$, (we mean by a sequence periodic with period $k = 1$ a constant sequence). By Theorem 3.2 the subsequences of $\{x_n\}$ are $\{x_{3n}\}$, $\{x_{3n+1}\}$ and $\{x_{3n+2}\}$ so the same thing for the sequences $\{y_n\}$ and $\{z_n\}$. We prove the results for the subsequence $\{x_{3n}\}$. The same lines of proof can be followed inductively to prove the results for the subsequences $\{x_{3n+1}\}$, $\{x_{3n+2}\}$ and those of $\{y_n\}$ and $\{z_n\}$. We note that in all cases

$$u_{3n} = \frac{(\alpha a A)^n (x_{-k} - x_0)}{x_0} + 1,\ u_{3n+1} = \frac{(\alpha a A)^n\,(\alpha y_{-k} + (\beta - 1) y_0)}{y_0} + 1,\ u_{3n+2} = \frac{(\alpha a A)^n\,(\alpha a z_{-k} + (\alpha b + \beta - 1) z_0)}{z_0} + 1,$$

It is clear because in this case $u_{3n} = u_{3n+1} = u_{3n+2} = 1$.

Suppose that the condition of Theorem 3.7 is not satisfied, then we distinguish seven possible cases:

(a) $x_{-k} \neq x_0$ (resp. $y_{-k} \neq y_0$ and $z_{-k} \neq z_0$) and $\alpha y_{-k} \neq (1 - \beta) y_0$ (resp. $a z_{-k} \neq (1 - b) z_0$ and $A x_{-k} \neq (1 - B) x_0$) and $\alpha a z_{-k} \neq (1 - \beta - \alpha b) z_0$ (resp. $a A x_{-k} \neq (1 - b - aB) x_0$ and $\alpha A y_{-k} \neq (1 - B - A\beta) y_0$).

By Theorem 3.2, we have, for all $n \in \mathbb{N}_0$:

$$\begin{aligned} x_{3n} &= \frac{x_{-1}}{\left(\prod_{i=0}^{n} u_{3i}\right) \prod_{i=0}^{n-1} (u_{3i+1} u_{3i+2})} \\ &= \frac{x_{-1}}{\left(\prod_{i=0}^{n}\left(\frac{(\alpha a A)^i (x_{-1} - x_0)}{x_0} + 1\right)\right) \prod_{i=0}^{n-1}\left(\left(\frac{(\alpha a A)^i\,(\alpha y_{-1} + (\beta - 1) y_0)}{y_0} + 1\right)\left(\frac{(\alpha a A)^i\,(\alpha a z_{-1} + (\alpha b + \beta - 1) z_0)}{z_0} + 1\right)\right)} \\ &= \frac{x_{-1}}{\prod_{i=0}^{n}\left(\frac{(\alpha a A)^i (x_{-1} - x_0)}{x_0} + 1\right) \prod_{i=0}^{n-1}\left(\frac{(\alpha a A)^i\,(\alpha y_{-1} + (\beta - 1) y_0)}{y_0} + 1\right) \prod_{i=0}^{n-1}\left(\frac{(\alpha a A)^i\,(\alpha a z_{-1} + (\alpha b + \beta - 1) z_0)}{z_0} + 1\right)}. \end{aligned}$$

Here we distinguish eight possible sub-cases:

(i) if $x_{-1} > x_0$, $\alpha y_{-1} > (1 - \beta) y_0$ and $\alpha a z_{-1} > (1 - \beta - \alpha b) z_0$, then

$$x_{3n} = \frac{x_{-1}}{\exp\left[\sum_{i=0}^{n} \ln\left(\frac{(\alpha a A)^i (x_{-1} - x_0)}{x_0} + 1\right) + \sum_{i=0}^{n-1} \ln\left(\frac{(\alpha a A)^i\,(\alpha y_{-1} + (\beta - 1) y_0)}{y_0} + 1\right) + \sum_{i=0}^{n-1} \ln\left(\frac{(\alpha a A)^i\,(\alpha a z_{-1} + (\alpha b + \beta - 1) z_0)}{z_0} + 1\right)\right]}.$$

Using a property of logarithms, we have

$$\ln\left(\frac{(\alpha a A)^n (x_{-1} - x_0)}{x_0} + 1\right) \sim_\infty \frac{(\alpha a A)^n (x_{-1} - x_0)}{x_0}, \tag{3.67}$$

$$\ln\left(\frac{(\alpha a A)^n\,(\alpha y_{-1} + (\beta - 1) y_0)}{y_0} + 1\right) \sim_\infty \frac{(\alpha a A)^n\,(\alpha y_{-1} + (\beta - 1) y_0)}{y_0}, \tag{3.68}$$

$$\ln\left(\frac{(\alpha a A)^n\,(\alpha a z_{-1} + (\alpha b + \beta - 1) z_0)}{z_0} + 1\right) \sim_\infty \frac{(\alpha a A)^n\,(\alpha a z_{-1} + (\alpha b + \beta - 1) z_0)}{z_0}. \tag{3.69}$$

Now, because $\sum_{i=0}^{n}(\alpha a A)^n$ is a geometric sum, with $\alpha a A < 1$, then the sums

$$\sum_{i=0}^{n}\frac{(x_{-1}-x_0)}{x_0}(\alpha a A)^n, \sum_{i=0}^{n}\frac{(\alpha y_{-1}+(\beta-1)y_0)}{y_0}(\alpha a A)^n \text{ and } \sum_{i=0}^{n}\frac{(\alpha a z_{-1}+(\alpha b+\beta-1)z_0)}{z_0}(\alpha a A)^n \tag{3.70}$$

are convergent. The desired result then follows.

(ii) if $x_{-1} > x_0$, $\alpha y_{-1} < (1-\beta)y_0$ and $\alpha a z_{-1} > (1-\beta-\alpha b)z_0$, then

$$\begin{aligned}
x_{3n} &= \frac{x_{-1}}{\exp\left[\sum_{i=0}^{n}\ln\left(\frac{(\alpha aA)^i(x_{-1}-x_0)}{x_0}+1\right)+\sum_{i=0}^{n-1}\ln\left(\frac{(\alpha aA)^i(\alpha a z_{-1}+(\alpha b+\beta-1)z_0)}{z_0}+1\right)\right]}\\
&\quad\times\frac{1}{\prod_{i=0}^{n-1}\left(\frac{(\alpha aA)^i(\alpha y_{-1}+(\beta-1)y_0)}{y_0}+1\right)}\\
&= \frac{x_{-1}}{\exp\left[\sum_{i=0}^{n}\ln\left(\frac{(\alpha aA)^i(x_{-1}-x_0)}{x_0}+1\right)+\sum_{i=0}^{n-1}\ln\left(\frac{(\alpha aA)^i(\alpha a z_{-1}+(\alpha b+\beta-1)z_0)}{z_0}+1\right)\right]}\\
&\quad\times\prod_{i=0}^{n-1}\left(\frac{y_0}{(\alpha aA)^i(\alpha y_{-1}+(\beta-1)y_0)+y_0}\right)\\
&= \frac{x_{-1}}{\exp\left[\sum_{i=0}^{n}\ln\left(\frac{(\alpha aA)^i(x_{-1}-x_0)}{x_0}+1\right)+\sum_{i=0}^{n-1}\ln\left(\frac{(\alpha aA)^i(\alpha a z_{-1}+(\alpha b+\beta-1)z_0)}{z_0}+1\right)\right]}\\
&\quad\times\prod_{i=0}^{n-1}\left(1+\frac{-(\alpha aA)^i(\alpha y_{-1}+(\beta-1)y_0)}{(\alpha aA)^i(\alpha y_{-1}+(\beta-1)y_0)+y_0}\right)\\
&= \frac{x_{-1}}{\exp\left[\sum_{i=0}^{n}\ln\left(\frac{(\alpha aA)^i(x_{-1}-x_0)}{x_0}+1\right)+\sum_{i=0}^{n-1}\ln\left(\frac{(\alpha aA)^i(\alpha a z_{-1}+(\alpha b+\beta-1)z_0)}{z_0}+1\right)\right]}\\
&\quad\times\exp\left[\sum_{i=0}^{n-1}\ln\left(1+\frac{(\alpha aA)^i((1-\beta)y_0-\alpha y_{-1})}{(\alpha aA)^i(\alpha y_{-1}+(\beta-1)y_0)+y_0}\right)\right].
\end{aligned}$$

By (3.67), (3.69), (3.70) and

$$\ln\left(1+\frac{(\alpha aA)^n((1-\beta)y_0-\alpha y_{-1})}{(\alpha aA)^n(\alpha y_{-1}+(\beta-1)y_0)+y_0}\right)\sim_\infty\frac{(\alpha aA)^n((1-\beta)y_0-\alpha y_{-1})}{(\alpha aA)^n(\alpha y_{-1}+(\beta-1)y_0)+y_0},$$

and the fact that

$$\frac{(\alpha aA)^n((1-\beta)y_0-\alpha y_{-1})}{(\alpha aA)^n(\alpha y_{-1}+(\beta-1)y_0)+y_0}\sim_\infty\frac{(\alpha aA)^n((1-\beta)y_0-\alpha y_{-1})}{y_0}$$

Thus we get the desired results.

(iii) if $x_{-1} < x_0$, $\alpha y_{-1} > (1-\beta)y_0$ and $\alpha a z_{-1} > (1-\beta-\alpha b)z_0$, then

$$
\begin{aligned}
x_{3n} &= \frac{x_{-1}}{\prod_{i=0}^{n-1}\left(\frac{(\alpha aA)^i\left(\alpha y_{-1}+(\beta-1)y_0\right)}{y_0}+1\right)\prod_{i=0}^{n-1}\left(\frac{(\alpha aA)^i\left(\alpha a z_{-1}+(\alpha b+\beta-1)z_0\right)}{z_0}+1\right)} \\
&\quad\times\prod_{i=0}^{n}\left(\frac{x_0}{(\alpha aA)^i(x_{-1}-x_0)+x_0}\right) \\
&= \frac{x_{-1}}{\prod_{i=0}^{n-1}\left(\frac{(\alpha aA)^i\left(\alpha y_{-1}+(\beta-1)y_0\right)}{y_0}+1\right)\prod_{i=0}^{n-1}\left(\frac{(\alpha aA)^i\left(\alpha a z_{-1}+(\alpha b+\beta-1)z_0\right)}{z_0}+1\right)} \\
&\quad\times\prod_{i=0}^{n}\left(1+\frac{-(\alpha aA)^i(x_{-1}-x_0)}{(\alpha aA)^i(x_{-1}-x_0)+x_0}\right) \\
&= \frac{x_{-1}}{\exp\left[\sum_{i=0}^{n-1}\ln\left(\frac{(\alpha aA)^i\left(\alpha y_{-1}+(\beta-1)y_0\right)}{y_0}+1\right)+\sum_{i=0}^{n-1}\ln\left(\frac{(\alpha aA)^i\left(\alpha a z_{-1}+(\alpha b+\beta-1)z_0\right)}{z_0}+1\right)\right]} \\
&\quad\times\exp\left[\sum_{i=0}^{n}\ln\left(1+\frac{(\alpha aA)^i(x_0-x_{-1})}{(\alpha aA)^i(x_{-1}-x_0)+x_0}\right)\right].
\end{aligned}
$$

By (3.68), (3.69), (3.70) and

$$\ln\left(1+\frac{(\alpha aA)^n(x_0-x_{-1})}{(\alpha aA)^n(x_{-1}-x_0)+x_0}\right)\sim_{+\infty}\frac{(\alpha aA)^n(x_0-x_{-1})}{(\alpha aA)^n(x_{-1}-x_0)+x_0},$$

the fact that

$$\frac{(\alpha aA)^n(x_0-x_{-1})}{(\alpha aA)^n(x_{-1}-x_0)+x_0}\sim_{+\infty}\frac{(\alpha aA)^n(x_0-x_{-1})}{x_0},$$

we get the result. For the remaining sub-cases:

(iv) $x_{-1}<x_0$, $\alpha y_{-1}<(1-\beta)y_0$ and $\alpha a z_{-1}>(1-\beta-\alpha b)z_0$,

(v) $x_{-1}>x_0$, $\alpha y_{-1}>(1-\beta)y_0$ and $\alpha a z_{-1}<(1-\beta-\alpha b)z_0$,

(vi) $x_{-1}>x_0$, $\alpha y_{-1}<(1-\beta)y_0$ and $\alpha a z_{-1}<(1-\beta-\alpha b)z_0$,

(vii) $x_{-1}<x_0$, $\alpha y_{-1}>(1-\beta)y_0$ and $\alpha a z_{-1}<(1-\beta-\alpha b)z_0$,

(viii) $x_{-1}<x_0$, $\alpha y_{-1}<(1-\beta)y_0$ and $\alpha a z_{-1}<(1-\beta-\alpha b)z_0$,

we follow the same techniques as in **(i)**, **(ii)** and **(iii)**.

For the others items:

(b) $x_{-k}=x_0$ (resp. $y_{-k}=y_0$ and $z_{-k}=z_0$) and $\alpha y_{-k}\neq(1-\beta)y_0$ (resp. $az_{-k}\neq(1-b)z_0$ and $Ax_{-k}\neq(1-B)x_0$) and $\alpha a z_{-k}\neq(1-\beta-\alpha b)z_0$ (resp. $aAx_{-k}\neq(1-b-aB)x_0$ and $\alpha Ay_{-k}\neq(1-B-A\beta)y_0$),

(c) $x_{-k}\neq x_0$ (resp. $y_{-k}\neq y_0$ and $z_{-k}\neq z_0$) and $\alpha y_{-k}=(1-\beta)y_0$ (resp. $az_{-k}=(1-b)z_0$ and $Ax_{-k}=(1-B)x_0$) and $\alpha a z_{-k}\neq(1-\beta-\alpha b)z_0$ (resp. $aAx_{-k}\neq(1-b-aB)x_0$ and $\alpha Ay_{-k}\neq(1-B-A\beta)y_0$),

(d) and $x_{-k} = x_0$ (resp. $y_{-k} = y_0$ and $z_{-k} = z_0$) and $\alpha y_{-1} = (1-\beta)y_0$ (resp. $az_{-k} = (1-b)z_0$ and $Ax_{-k} = (1-B)x_0$) and $\alpha a z_{-k} \neq (1-\beta-\alpha b)z_0$ (resp. $aAx_{-k} \neq (1-b-aB)x_0$ and $\alpha A y_{-k} \neq (1-B-A\beta)y_0$),

(e) $x_{-k} \neq x_0$ (resp. $y_{-k} \neq y_0$ and $z_{-k} \neq z_0$) and $\alpha y_{-k} \neq (1-\beta)y_0$ (resp. $az_{-k} \neq (1-b)z_0$ and $Ax_{-k} \neq (1-B)x_0$) and $\alpha a z_{-k} = (1-\beta-\alpha b)z_0$ (resp. $aAx_{-k} = (1-b-aB)x_0$ and $\alpha A y_{-k} = (1-B-A\beta)y_0$),

(f) $x_{-k} = x_0$ (resp. $y_{-k} = y_0$ and $z_{-k} = z_0$) and $\alpha y_{-k} \neq (1-\beta)y_0$ (resp. $az_{-k} \neq (1-b)z_0$ and $Ax_{-k} \neq (1-B)x_0$) and $\alpha a z_{-k} = (1-\beta-\alpha b)z_0$ (resp. $aAx_{-k} = (1-b-aB)x_0$ and $\alpha A y_{-k} = (1-B-A\beta)y_0$),

(g) $x_{-k} \neq x_0$ (resp. $y_{-1} \neq y_0$ and $z_{-k} \neq z_0$) and $\alpha y_{-k} = (1-\beta)y_0$ (resp. $az_{-k} = (1-b)z_0$ and $Ax_{-k} = (1-B)x_0$) and $\alpha a z_{-k} = (1-\beta-\alpha b)z_0$ (resp. $aAx_{-k} = (1-b-aB)x_0$ and $\alpha A y_{-k} = (1-B-A\beta)y_0$),

the proof of is similar to **(a)**, and the results are immediate because in this cases, each time u_{3n} or u_{3n+1} or u_{3n+2} equal to 1.

□

Chapter 4

A max-type system of difference equations of third order

4.1 Introduction

In this chapter we study a max type system of difference equations. This type of difference equations and systems have been investigated by a lot of authors, see for instance, [10, 21, 22, 23, 49, 51, 52, 69, 70, 73, 74, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 97, 98, 99, 100, 101, 102, 103, 107, 113, 120, 121, 122, 125].

In the same line of the works in the above references, we solve in a closed form the following third order max-type system of difference equations

$$x_{n+1} = \max\left(x_{n-1}, \frac{x_n y_{n-1}}{y_{n-2}}\right),\; y_{n+1} = \max\left(y_{n-1}, \frac{y_n x_{n-1}}{x_{n-2}}\right), \tag{4.1}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the initial values $x_{-i}, y_{-i} \in (0, +\infty)$, $i = 0, 1, 2$.
To do this, we distinguish for cases depending on the relation between the quantities x_{-1} and $\frac{x_0 y_{-1}}{y_{-2}}$, and, y_{-1} and $\frac{y_0 x_{-1}}{x_{-2}}$.

4.2 Main results and closed form of the solutions

4.2.1 The case $x_{-1} \leq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \leq \frac{y_0 x_{-1}}{x_{-2}}$

In the following result, we give the closed form of the solutions of system (4.1) under the assumptions $x_{-1} \leq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \leq \frac{y_0 x_{-1}}{x_{-2}}$.

Theorem 4.1. *Let* $(x_n)_{n\geq -2}$ *and* $(y_n)_{n\geq -2}$ *be a solution of system* (4.1) *such that* $x_{-1}\leq \frac{x_0y_{-1}}{y_{-2}}$ *and* $y_{-1}\leq \frac{y_0x_{-1}}{x_{-2}}$*. Then the following statements hold:*

(H_1) : *If* $\frac{x_0}{x_{-2}}\geq 1$, $\frac{y_0}{y_{-2}}\geq 1$, *then*

$$\begin{cases} x_{4n-1}=x_{-1}\left(\frac{x_0y_0}{x_{-2}y_{-2}}\right)^n, \ n\in\mathbb{N}_0,\\ x_{4n}=x_0\left(\frac{x_0y_0}{x_{-2}y_{-2}}\right)^n, \ n\in\mathbb{N}_0,\\ x_{4n+1}=\frac{x_0y_{-1}}{y_{-2}}\left(\frac{x_0y_0}{x_{-2}y_{-2}}\right)^n, \ n\in\mathbb{N}_0,\\ x_{4n+2}=\frac{x_0y_0}{y_{-2}}\left(\frac{x_0y_0}{x_{-2}y_{-2}}\right)^n, \ n\in\mathbb{N}_0,\end{cases} \quad \begin{cases} y_{4n-1}=y_{-1}\left(\frac{x_0y_0}{x_{-2}y_{-2}}\right)^n, \ n\in\mathbb{N}_0,\\ y_{4n}=y_o\left(\frac{x_0y_0}{x_{-2}y_{-2}}\right)^n, \ n\in\mathbb{N}_0,\\ y_{4n+1}=\frac{y_0x_{-1}}{x_{-2}}\left(\frac{x_0y_0}{x_{-2}y_{-2}}\right)^n, \ n\in\mathbb{N}_0,\\ y_{4n+2}=\frac{x_0y_0}{x_{-2}}\left(\frac{x_0y_0}{x_{-2}y_{-2}}\right)^n, \ n\in\mathbb{N}_0.\end{cases} \tag{4.2}$$

(H_2) : *Let* $\frac{x_0}{x_{-2}}\leq 1$, $\frac{y_0}{y_{-2}}\geq 1$ *(with* $\frac{x_0y_0}{x_{-2}y_{-2}}\geq 1$*). Then*

$$\begin{cases} x_{4n}=x_0\left(\frac{y_0}{y_{-2}}\right)^n, \ n\in\mathbb{N}_0,\\ x_{4n+1}=\frac{x_0y_{-1}}{y_0}\left(\frac{y_0}{y_{-2}}\right)^{n+1}, n\in\mathbb{N}_0\\ x_{4n+2}=x_0\left(\frac{y_0}{y_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0,\\ x_{4n+3}=\frac{x_0x_{-1}}{x_{-2}}\left(\frac{y_0}{y_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0.\end{cases} \quad \begin{cases} y_{4n}=y_0\left(\frac{y_0}{y_{-2}}\right)^n, \ n\in\mathbb{N}_0,\\ y_{4n+1}=\frac{x_{-1}y_{-2}}{x_{-2}}\left(\frac{y_0}{y_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0,\\ y_{4n+2}=y_0\left(\frac{y_0}{y_{-2}}\right)^n, \ n\in\mathbb{N}_0,\\ y_{4n+3}=y_{-1}\left(\frac{y_0}{y_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0,\end{cases} \tag{4.3}$$

(H_3) : *Let* $\frac{x_0}{x_{-2}}\geq 1$, $\frac{y_0}{y_{-2}}\leq 1$ *(with* $\frac{x_0y_0}{x_{-2}y_{-2}}\geq 1$*). Then*

$$\begin{cases} x_{4n}=x_{-2}\left(\frac{x_0}{x_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0,\\ x_{4n+1}=\frac{x_{-2}y_{-1}}{y_{-2}}\left(\frac{x_0}{x_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0,\\ x_{4n+2}=x_{-2}\left(\frac{x_0}{x_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0,\\ x_{4n+3}=x_{-1}\left(\frac{x_0}{x_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0,\end{cases} \quad \begin{cases} y_{4n}=y_0\left(\frac{x_0}{x_{-2}}\right)^n, \ n\in\mathbb{N}_0,\\ y_{4n+1}=\frac{x_{-1}y_0}{x_0}\left(\frac{x_0}{x_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0,\\ y_{4n+2}=y_0\left(\frac{x_0}{x_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0,\\ y_{4n+3}=\frac{y_{-1}y_0}{y_{-2}}\left(\frac{x_0}{x_{-2}}\right)^{n+1}, \ n\in\mathbb{N}_0.\end{cases} \tag{4.4}$$

Proof. From the hypothesis $x_{-1}\leq \frac{x_0y_{-1}}{y_{-2}}$ and $y_{-1}\leq \frac{y_0x_{-1}}{x_{-2}}$, we get

$$x_1=\max\left(x_{-1},\frac{x_0y_{-1}}{y_{-2}}\right)=\frac{x_0y_{-1}}{y_{-2}}, \tag{4.5}$$

$$y_1=\max\left(y_{-1},\frac{y_0x_{-1}}{x_{-2}}\right)=\frac{y_0x_{-1}}{x_{-2}}. \tag{4.6}$$

Using (4.5) and (4.6), we get

$$x_2=x_0\max\left(1,\frac{y_0}{y_{-2}}\right), \ y_2=y_0\max\left(1,\frac{x_0}{x_{-2}}\right). \tag{4.7}$$

Again, it follows from $x_{-1}\leq \frac{x_0y_{-1}}{y_{-2}}$ and $y_{-1}\leq \frac{y_0x_{-1}}{x_{-2}}$ that $\frac{x_0y_0}{x_{-2}y_{-2}}\geq 1$. In fact, we have

$$x_{-1}\leq \frac{x_0y_{-1}}{y_{-2}}\leq \frac{x_0}{y_{-2}}\frac{y_0x_{-1}}{x_{-2}}=\frac{y_0x_0}{x_{-2}y_{-2}}x_{-1},$$

that is, $\frac{x_0y_0}{x_{-2}y_{-2}}\geq 1$. Taking this in mind, we got the following three possibilities.

$(H_1): \frac{x_0}{x_{-2}} \geq 1, \frac{y_0}{y_{-2}} \geq 1.$

$(H_2): \frac{x_0}{x_{-2}} \leq 1, \frac{y_0}{y_{-2}} \geq 1.$

$(H_3): \frac{x_0}{x_{-2}} \geq 1, \frac{y_0}{y_{-2}} \leq 1.$

From hypothesis (H_1) , we obtain:

$$x_2 = \frac{x_0 y_0}{y_{-2}}, \tag{4.8}$$

$$y_2 = \frac{x_0 y_0}{x_{-2}}. \tag{4.9}$$

Using (4.5)-(4.9), we get

$$x_3 = \max\left(\frac{x_0 y_{-1}}{y_{-2}}, \frac{x_0}{y_{-2}} \frac{y_0 x_{-1}}{x_{-2}}\right), \; y_3 = \max\left(\frac{y_0 x_{-1}}{x_{-2}}, \frac{y_0}{x_{-2}} \frac{x_0 y_{-1}}{y_{-2}}\right),$$

we have

$$\frac{y_0 x_{-1}}{x_{-2}} \geq y_{-1} \Rightarrow \frac{x_0}{y_{-2}} \frac{y_0 x_{-1}}{x_{-2}} \geq \frac{x_0}{y_{-2}} y_{-1},$$
$$\frac{x_0 y_{-1}}{y_{-2}} \geq x_{-1} \Rightarrow \frac{y_0}{x_{-2}} \frac{x_0 y_{-1}}{y_{-2}} \geq \frac{y_0}{x_{-2}} x_{-1},$$

so,

$$x_3 = \frac{x_0 y_0}{x_{-2} y_{-2}} x_{-1}, \tag{4.10}$$

$$y_3 = \frac{x_0 y_0}{x_{-2} y_{-2}} y_{-1}. \tag{4.11}$$

Using (4.8)-(4.11), we get

$$x_4 = \frac{x_0 y_0}{y_{-2}} \max\left(1, \frac{x_0}{x_{-2}}\right), \; y_4 = \frac{x_0 y_0}{x_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right),$$

by (H_1) we obtain:

$$x_4 = \frac{x_0 y_0}{x_{-2} y_{-2}} x_0, \tag{4.12}$$

$$y_4 = \frac{x_0 y_0}{x_{-2} y_{-2}} y_0. \tag{4.13}$$

Using (4.10)-(4.13), we have

$$x_5 = \frac{x_0 y_0}{x_{-2} y_{-2}} \max\left(x_{-1}, \frac{x_0 y_{-1}}{y_{-2}}\right) = \frac{x_0 y_{-1}}{y_{-2}} \frac{x_0 y_0}{x_{-2} y_{-2}}, \tag{4.14}$$

$$y_5 = \frac{x_0 y_0}{x_{-2} y_{-2}} \max\left(y_{-1}, \frac{y_0 x_{-1}}{x_{-2}}\right) = \frac{y_0 x_{-1}}{x_{-2}} \frac{x_0 y_0}{x_{-2} y_{-2}}. \tag{4.15}$$

From (4.12)-(4.15), we obtain

$$x_6 = \frac{x_0^2 y_0}{x_{-2}y_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{x_0 y_0}{y_{-2}} \frac{x_0 y_0}{x_{-2}y_{-2}}, \tag{4.16}$$

$$y_6 = \frac{y_0^2 x_0}{x_{-2}y_{-2}} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0 y_0}{x_{-2}} \frac{x_0 y_0}{x_{-2}y_{-2}}. \tag{4.17}$$

Now, from (4.14)-(4.17), we get

$$x_7 = \frac{x_0^2 y_0}{y_{-2}^2 x_{-2}} \max\left(\frac{y_0 x_{-1}}{x_{-2}}, y_{-1}\right) = x_{-1}\left(\frac{x_0 y_0}{x_{-2}y_{-2}}\right)^2, \tag{4.18}$$

$$y_7 = \frac{y_0^2 x_0}{x_{-2}^2 y_{-2}} \max\left(x_{-1}, \frac{x_0 y_{-1}}{y_{-2}}\right) = y_{-1}\left(\frac{x_0 y_0}{x_{-2}y_{-2}}\right)^2. \tag{4.19}$$

Using (4.16)-(4.19), we have

$$x_8 = \frac{x_0^2 y_0^2}{y_{-2}^2 x_{-2}} \max\left(1, \frac{x_0}{x_{-2}}\right) = x_0\left(\frac{x_0 y_0}{x_{-2}y_{-2}}\right)^2, \tag{4.20}$$

$$y_8 = \frac{y_0^2 x_0^2}{x_{-2}^2 y_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right) = y_0\left(\frac{x_0 y_0}{x_{-2}y_{-2}}\right)^2. \tag{4.21}$$

From (4.5)-(4.21) and by induction we obtain the results in (4.2).

Now, let consider the second cases (H_2). We have

$$x_2 = \max\left(x_0, \frac{x_0 y_0}{y_{-2}}\right) = x_0 \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{x_0 y_0}{y_{-2}}, \tag{4.22}$$

$$y_2 = \max\left(y_0, \frac{x_0 y_0}{x_{-2}}\right) = y_0 \max\left(1, \frac{x_0}{x_{-2}}\right) = y_0. \tag{4.23}$$

Using (4.5), (4.6), (4.22) and (4.23) we get

$$x_3 = \max\left(x_1, \frac{x_2 y_1}{y_0}\right) = \frac{x_0}{y_{-2}} \max\left(y_{-1}, \frac{x_{-1} y_0}{x_{-2}}\right) = x_{-1}\frac{x_0 y_0}{x_{-2}y_{-2}}, \tag{4.24}$$

$$y_3 = \max\left(y_1, \frac{y_2 x_1}{x_0}\right) = \frac{y_0}{x_{-2}} \max\left(x_{-1}, \frac{x_{-2} y_{-1}}{y_{-2}}\right) = \frac{y_0 y_{-1}}{y_{-2}}. \tag{4.25}$$

By (4.5), (4.6) (4.22),(4.23), (4.24) and (4.25), we have

$$x_4 = \max\left(x_2, \frac{x_3 y_2}{y_1}\right) = \max\left(\frac{x_0 y_0}{y_{-2}}, \frac{x_0 y_0}{y_{-2}}\right) = \frac{x_0 y_0}{y_{-2}}, \tag{4.26}$$

$$y_4 = \max\left(y_2, \frac{y_3 x_2}{x_1}\right) = y_0 \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{y_0^2}{y_{-2}}. \tag{4.27}$$

From (4.22),(4.23), (4.24)-(4.27), we get

$$x_5 = \max\left(x_3, \frac{x_4 y_3}{y_2}\right) = x_{-1}\frac{x_0 y_0}{x_{-2}y_{-2}} \max\left(1, \frac{x_{-2}y_{-1}}{x_{-1}y_{-2}}\right) = \frac{x_0 y_{-1} y_0}{y_{-2}^2}, \tag{4.28}$$

$$y_5 = \max\left(y_3, \frac{y_4 x_3}{x_2}\right) = \frac{y_0 y_{-1}}{y_{-2}} \max\left(1, \frac{x_{-1}y_0}{x_{-2}y_{-1}}\right) = \frac{x_{-1}y_0^2}{x_{-2}y_{-2}}. \tag{4.29}$$

Using (4.24)-(4.29), we get

$$x_6 = \max\left(x_4, \frac{x_5 y_4}{y_3}\right) = \frac{x_0 y_0}{y_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{x_0 y_0^2}{y_{-2}^2}. \tag{4.30}$$

$$y_6 = \max\left(y_4, \frac{y_5 x_4}{x_3}\right) = \max\left(\frac{y_0^2}{y_{-2}}, \frac{y_0^2}{y_{-2}}\right) = \frac{y_0^2}{y_{-2}}. \tag{4.31}$$

Then, by (4.26)-(4.31), we have

$$x_7 = \max\left(x_5, \frac{x_6 y_5}{y_4}\right) = \frac{x_0 y_{-1} y_0}{y_{-2}^2} \max\left(1, \frac{x_{-1}y_0}{x_{-2}y_{-1}}\right) = \frac{x_0 x_{-1} y_0^2}{x_{-2}y_{-2}^2}, \tag{4.32}$$

$$y_7 = \max\left(y_5, \frac{y_6 x_5}{x_4}\right) = \frac{x_{-1}y_0^2}{x_{-2}y_{-2}} \max\left(1, \frac{x_{-2}y_{-1}}{x_{-1}y_{-2}}\right) = \frac{y_{-1}y_0^2}{y_{-2}^2}. \tag{4.33}$$

So, from (4.28)-(4.33), we have

$$x_8 = \max\left(x_6, \frac{x_7 y_6}{y_5}\right) = \max\left(\frac{x_0 y_0^2}{y_{-2}^2}, \frac{x_0 y_0^2}{y_{-2}^2}\right) = \frac{x_0 y_0^2}{y_{-2}^2}, \tag{4.34}$$

$$y_8 = \max\left(y_6, \frac{y_7 x_6}{x_5}\right) = \frac{y_0^2}{y_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{y_0^3}{y_{-2}^2}. \tag{4.35}$$

By induction we get the formulas in (4.3).

Now, consider the case (H_3), that is: $\frac{x_0}{x_{-2}} \geq 1$ and $\frac{y_0}{y_{-2}} \leq 1$. Using (4.5) and (4.6), we get

$$x_2 = x_0 \max\left(1, \frac{y_0}{y_{-2}}\right) = x_0, \tag{4.36}$$

$$y_2 = y_0 \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0 y_0}{x_{-2}}. \tag{4.37}$$

Using (4.5), (4.6), (4.36) and and (4.37), we get

$$x_3 = \max\left(x_1, \frac{x_2 y_1}{y_0}\right) = \frac{x_0}{y_{-2}} \max\left(y_{-1}, \frac{y_{-2}x_{-1}}{x_{-2}}\right) = x_0\frac{x_{-1}}{x_{-2}}, \tag{4.38}$$

$$y_3 = \max\left(y_1, \frac{y_2 x_1}{x_0}\right) = \frac{y_0}{x_{-2}} \max\left(x_{-1}, \frac{x_0 y_{-1}}{y_{-2}}\right) = \frac{x_0 y_0 y_{-1}}{x_{-2}y_{-2}}. \tag{4.39}$$

From (4.5), (4.6), (4.36)-(4.39), we have

$$x_4 = \max\left(x_2, \frac{x_3 y_2}{y_1}\right) = x_0 \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0^2}{x_{-2}}, \tag{4.40}$$

$$y_4 = \max\left(y_2, \frac{y_3 x_2}{x_1}\right) = \max\left(\frac{x_0 y_0}{x_{-2}}, \frac{x_0 y_0}{x_{-2}}\right) = \frac{x_0 y_0}{x_{-2}}. \tag{4.41}$$

Using (4.36)-(4.41), we get

$$x_5 = \max\left(x_3, \frac{x_4 y_3}{y_2}\right) = \frac{x_0}{x_{-2}} \max\left(x_{-1}, \frac{x_0 y_{-1}}{y_{-2}}\right) = \frac{x_0^2 y_{-1}}{x_{-2} y_{-2}}, \tag{4.42}$$

$$y_5 = \max\left(y_3, \frac{y_4 x_3}{x_2}\right) = \frac{x_0 y_0}{x_{-2}} \max\left(\frac{y_{-1}}{y_{-2}}, \frac{x_{-1}}{x_{-2}}\right) = \frac{x_0 y_0 x_{-1}}{x_{-2}^2}. \tag{4.43}$$

By (4.38)-(4.43), we have

$$x_6 = \max\left(x_4, \frac{x_5 y_4}{y_3}\right) = \max\left(\frac{x_0^2}{x_{-2}}, \frac{x_0^2}{x_{-2}}\right) = \frac{x_0^2}{x_{-2}}, \tag{4.44}$$

$$y_6 = \max\left(y_4, \frac{y_5 x_4}{x_3}\right) = \frac{x_0 y_0}{x_{-2}} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0^2 y_0}{x_{-2}^2}. \tag{4.45}$$

Then, by (4.40)-(4.45), we have

$$x_7 = \max\left(x_5, \frac{x_6 y_5}{y_4}\right) = \frac{x_0^2}{x_{-2}} \max\left(\frac{y_{-1}}{y_{-2}}, \frac{x_{-1}}{x_{-2}}\right) = \frac{x_{-1} x_0^2}{x_{-2}^2}, \tag{4.46}$$

$$y_7 = \max\left(y_5, \frac{y_6 x_5}{x_4}\right) = \frac{x_0 y_0}{x_{-2}^2} \max\left(x_{-1}, \frac{x_0 y_{-1}}{y_{-2}}\right) = \frac{x_0^2 y_0 y_{-1}}{x_{-2}^2 y_{-2}}. \tag{4.47}$$

So, from (4.42)-(4.47), we get

$$x_8 = \max\left(x_6, \frac{x_7 y_6}{y_5}\right) = \frac{x_0^2}{x_{-2}} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0^3}{x_{-2}^2}, \tag{4.48}$$

$$y_8 = \max\left(y_6, \frac{y_7 x_6}{x_5}\right) = \max\left(\frac{x_0^2 y_0}{x_{-2}^2}, \frac{x_0^2 y_0}{x_{-2}^2}\right) = \frac{x_0^2 y_0}{x_{-2}^2}. \tag{4.49}$$

By induction we obtain the results in (4.4).

□

Remark 4.2.1. Assume that $x_0 = x_{-2}$ and $y_0 = y_{-2}$. Using the fact that $x_{-1} \leq \frac{x_0 y_{-1}}{y_{-2}}$, $y_{-1} \leq \frac{y_0 x_{-1}}{x_{-2}}$ it follows that:

$$x_{-1} = \frac{x_0 y_{-1}}{y_{-2}}, \; y_{-1} = \frac{y_0 x_{-1}}{x_{-2}}.$$

The following result, which is a direct consequence of Theorem 4.1 and Remark 4.2.1, is devoted to the existence of periodic solutions of system (4.1).

Corollary 4.2. *Let $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ be a solution of system (4.1) such that $x_{-1} \leq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \leq \frac{y_0 x_{-1}}{x_{-2}}$. Then, If $x_0 = x_{-2}$ and $y_0 = y_{-2}$, we have for all $n \in \mathbb{N}_0$*

$$x_{2n} = x_{-2},\ y_{2n} = y_{-2},$$

$$x_{2n+1} = x_{-1},\ y_{2n+1} = y_{-1}.$$

That is the solutions are periodic with period 2. When $x_0 \neq x_{-2}$ and $y_0 \neq y_{-2}$, the solutions are unbounded, that is

$$(x_n, y_n) \longrightarrow (+\infty, +\infty).$$

4.2.2 The case $x_{-1} \geq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \geq \frac{y_0 x_{-1}}{x_{-2}}$

In the following section, we give the closed form of the solutions of system (4.1) under the assumptions $x_{-1} \geq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \geq \frac{y_0 x_{-1}}{x_{-2}}$.

Theorem 4.3. *Let $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ be a solution of system (4.1) such that $x_{-1} \geq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \geq \frac{y_0 x_{-1}}{x_{-2}}$. Then the following statements hold:*

(H_1) : *If $\frac{x_{-1} y_0}{x_0 y_{-1}} \leq 1$. Then*

$$x_{2n} = \begin{cases} x_0 \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^{\frac{n}{2}}, n = 0, 2, \ldots \\ x_0 \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^{\frac{n-1}{2}}, n = 1, 3, \ldots \end{cases} \qquad x_{2n+1} = \begin{cases} x_{-1} \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^{\frac{n}{2}}, n = 0, 2, \ldots \\ x_{-1} \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^{\frac{n+1}{2}}, n = 1, 3, \ldots \end{cases} \tag{4.50}$$

$$y_{2n} = \begin{cases} y_0 \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^{\frac{n}{2}}, n = 0, 2, \ldots \\ y_0 \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^{\frac{n+1}{2}}, n = 1, 3, \ldots \end{cases} \qquad y_{2n+1} = \begin{cases} y_{-1} \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^{\frac{n}{2}}, n = 0, 2, \ldots \\ y_{-1} \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^{\frac{n+1}{2}}, n = 1, 3, \ldots \end{cases} \tag{4.51}$$

(H_2) : *If $\frac{x_{-1} y_0}{x_0 y_{-1}} \geq 1$. Then*

$$\begin{cases} x_{4n-3} = x_{4n-1} = x_{-1} \left(\frac{x_{-1} y_0}{x_0 y_{-1}}\right)^{n-1}, n \in \mathbb{N}, \\ x_{4n-2} = x_{4n} = x_0 \left(\frac{x_{-1} y_0}{x_0 y_{-1}}\right)^{n}, n \in \mathbb{N}. \end{cases} \qquad \begin{cases} y_{4n-1} = y_{4n+1} = y_{-1} \left(\frac{x_{-1} y_0}{x_0 y_{-1}}\right)^{n}, n \in \mathbb{N}_0, \\ y_{4n} = y_{4n+2} = y_0 \left(\frac{x_{-1} y_0}{x_0 y_{-1}}\right)^{n}, n \in \mathbb{N}_0, \end{cases} \tag{4.52}$$

Proof. From hypothesis $x_{-1} \geq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \geq \frac{y_0 x_{-1}}{x_{-2}}$ of Theorem 4.3. We have

$$x_1 = \max\left(x_{-1}, \frac{x_0 y_{-1}}{y_{-2}}\right) = x_{-1}, \tag{4.53}$$

$$y_1 = \max\left(y_{-1}, \frac{y_0 x_{-1}}{x_{-2}}\right) = y_{-1}. \tag{4.54}$$

Using (4.53) and (4.55), we get

$$x_2 = \max\left(x_0, \frac{x_1 y_0}{y_{-1}}\right) = x_0 \max\left(1, \frac{x_{-1} y_0}{x_0 y_{-1}}\right), \tag{4.55}$$

$$y_2 = \max\left(y_0, \frac{y_1 x_0}{x_{-1}}\right) = y_0 \max\left(1, \frac{y_{-1} x_0}{y_0 x_{-1}}\right), \tag{4.56}$$

so, we have two cases:

(H_1) : $\frac{x_{-1} y_0}{x_0 y_{-1}} \leq 1.$

(H_2) : $\frac{x_{-1} y_0}{x_0 y_{-1}} \geq 1.$

If $\frac{x_{-1} y_0}{x_0 y_{-1}} \leq 1$, then by (4.55) and (4.56) we have

$$x_2 = x_0 \max\left(1, \frac{x_{-1} y_0}{x_0 y_{-1}}\right) = x_0, \tag{4.57}$$

$$y_2 = y_0 \max\left(1, \frac{y_{-1} x_0}{y_0 x_{-1}}\right) = y_0 \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right). \tag{4.58}$$

Using (4.53), (4.54), (4.57) and (4.58), we get

$$x_3 = \max\left(x_1, \frac{x_2 y_1}{y_0}\right) = x_{-1} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = x_{-1} \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right), \tag{4.59}$$

$$y_3 = \max\left(y_1, \frac{y_2 x_1}{x_0}\right) = \max\left(y_{-1}, y_{-1}\right) = y_{-1}. \tag{4.60}$$

From (4.53), (4.54) and (4.57)-(4.60), we have

$$x_4 = \max\left(x_2, \frac{x_3 y_2}{y_1}\right) = x_0 \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = x_0 \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right), \tag{4.61}$$

$$y_4 = \max\left(y_2, \frac{y_3 x_2}{x_1}\right) = \max\left(y_0 \left(\frac{y_{-1} x_0}{x_{-1} y_0}\right), \frac{y_{-1} x_0}{x_{-1}}\right) = y_0 \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right). \tag{4.62}$$

By (4.57)-(4.62), we get

$$x_5 = \max\left(x_3, \frac{x_4 y_3}{y_2}\right) = \max\left(x_{-1} \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right), x_{-1} \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)\right) = x_{-1} \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right), \tag{4.63}$$

$$y_5 = \max\left(y_3, \frac{y_4 x_3}{x_2}\right) = y_{-1} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = y_{-1} \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right). \tag{4.64}$$

Using (4.59)-(4.64), we have

$$x_6 = \max\left(x_4, \frac{x_5 y_4}{y_3}\right) = \max\left(x_0 \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right), x_0 \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)\right) = x_0 \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right), \tag{4.65}$$

$$y_6 = \max\left(y_4, \frac{y_5 x_4}{x_3}\right) = y_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)\max\left(1, \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)\right) = y_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2. \tag{4.66}$$

From (4.61)-(4.66), we get

$$x_7 = \max\left(x_5, \frac{x_6 y_5}{y_4}\right) = x_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)\max\left(1, \left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)\right) = x_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2, \tag{4.67}$$

$$y_7 = \max\left(y_5, \frac{y_6 x_5}{x_4}\right) = \max\left(y_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right), y_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)\right) = y_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right). \tag{4.68}$$

By (4.63)-(4.68), we get

$$x_8 = \max\left(x_6, \frac{x_7 y_6}{y_5}\right) = x_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)\max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = x_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2, \tag{4.69}$$

$$y_8 = \max\left(y_6, \frac{y_7 x_6}{x_5}\right) = \max\left(y_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2, y_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2\right) = y_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2. \tag{4.70}$$

Using (4.65)-(4.70), we get

$$x_9 = \max\left(x_7, \frac{x_8 y_7}{y_6}\right) = \max\left(x_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2, x_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2\right) = x_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2, \tag{4.71}$$

$$y_9 = \max\left(y_7, \frac{y_8 x_7}{x_6}\right) = y_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)\max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = y_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2. \tag{4.72}$$

From (4.67)-(4.72), we get

$$x_{10} = \max\left(x_8, \frac{x_9 y_8}{y_7}\right) = \max\left(x_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2, x_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2\right) = x_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2, \tag{4.73}$$

$$y_{10} = \max\left(y_8, \frac{y_9 x_8}{x_7}\right) = y_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2\max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = y_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^3. \tag{4.74}$$

By (4.69)-(4.74), we have

$$x_{11} = \max\left(x_9, \frac{x_{10} y_9}{y_8}\right) = x_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2\max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = x_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^3, \tag{4.75}$$

$$y_{11} = \max\left(y_9, \frac{y_{10} x_9}{x_8}\right) = \max\left(y_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2, y_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2\right) = y_{-1}\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2. \tag{4.76}$$

From (4.71)-(4.76), we get

$$x_{12} = \max\left(x_{10}, \frac{x_{11} y_{10}}{y_9}\right) = x_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^2\max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = x_0\left(\frac{x_0 y_{-1}}{x_{-1} y_0}\right)^3, \tag{4.77}$$

$$y_{12} = \max\left(y_{10}, \frac{y_{11}x_{10}}{x_9}\right) = \max\left(y_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3, y_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3\right) = y_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3. \quad (4.78)$$

Then, by (4.73)-(4.78), we get

$$x_{13} = \max\left(x_{11}, \frac{x_{12}y_{11}}{y_{10}}\right) = \max\left(x_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3, x_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3\right) = x_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3, \quad (4.79)$$

$$y_{13} = \max\left(y_{11}, \frac{y_{12}x_{11}}{x_{10}}\right) = y_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^2 \max\left(1, \frac{x_0y_{-1}}{x_{-1}y_0}\right) = y_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3. \quad (4.80)$$

So, from (4.75)-(4.80), we have

$$x_{14} = \max\left(x_{12}, \frac{x_{13}y_{12}}{y_{11}}\right) = \max\left(x_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3, x_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3\right) = x_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3, \quad (4.81)$$

$$y_{14} = \max\left(y_{12}, \frac{y_{13}x_{12}}{x_{11}}\right) = y_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^3 \max\left(1, \frac{x_0y_{-1}}{x_{-1}y_0}\right) = y_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^4. \quad (4.82)$$

By induction we obtain the results in (4.50) and (4.51).

If $\frac{x_{-1}y_0}{x_0y_{-1}} \geq 1$. By (4.55) and (4.56) we get

$$x_2 = x_0 \max\left(1, \frac{x_{-1}y_0}{x_0y_{-1}}\right) = x_0\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right), \quad (4.83)$$

$$y_2 = y_0 \max\left(1, \frac{y_{-1}x_0}{y_0x_{-1}}\right) = y_0. \quad (4.84)$$

Using (4.53), (4.54), (4.83) and (4.84), we get

$$x_3 = \max\left(x_1, \frac{x_2y_1}{y_0}\right) = \max\left(x_{-1}, x_{-1}\right) = x_{-1}, \quad (4.85)$$

$$y_3 = \max\left(y_1, \frac{y_2x_1}{x_0}\right) = y_{-1} \max\left(1, \frac{x_{-1}y_0}{x_0y_{-1}}\right) = y_{-1}\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right). \quad (4.86)$$

From (4.53), (4.54) and (4.83)-(4.86), we have

$$x_4 = \max\left(x_2, \frac{x_3y_2}{y_1}\right) = \max\left(x_0\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right), \frac{x_{-1}y_0}{y_{-1}}\right) = x_0\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right), \quad (4.87)$$

$$y_4 = \max\left(y_2, \frac{y_3x_2}{x_1}\right) = y_0 \max\left(1, \frac{x_{-1}y_0}{x_0y_{-1}}\right) = y_0\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right). \quad (4.88)$$

By (4.83)-(4.88), we get

$$x_5 = \max\left(x_3, \frac{x_4y_3}{y_2}\right) = x_{-1} \max\left(1, \frac{x_{-1}y_0}{x_0y_{-1}}\right) = x_{-1}\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right), \quad (4.89)$$

$$y_5 = \max\left(y_3, \frac{y_4 x_3}{x_2}\right) = \max\left(y_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right), y_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)\right) = y_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right). \tag{4.90}$$

From (4.85)-(4.90), we have

$$x_6 = \max\left(x_4, \frac{x_5 y_4}{y_3}\right) = x_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)\max\left(1, \frac{x_{-1}y_0}{x_0 y_{-1}}\right) = x_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2, \tag{4.91}$$

$$y_6 = \max\left(y_4, \frac{y_5 x_4}{x_3}\right) = \max\left(y_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right), y_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)\right) = y_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right). \tag{4.92}$$

Using (4.87)-(4.92), we get

$$x_7 = \max\left(x_5, \frac{x_6 y_5}{y_4}\right) = \max\left(x_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right), x_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)\right) = x_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right), \tag{4.93}$$

$$y_7 = \max\left(y_5, \frac{y_6 x_5}{x_4}\right) = y_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)\max\left(1, \frac{x_{-1}y_0}{x_0 y_{-1}}\right) = y_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2. \tag{4.94}$$

From (4.89)-(4.94), we get

$$x_8 = \max\left(x_6, \frac{x_7 y_6}{y_5}\right) = \max\left(x_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2, x_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2\right) = x_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2, \tag{4.95}$$

$$y_8 = \max\left(y_6, \frac{y_7 x_6}{x_5}\right) = y_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)\max\left(1, \frac{x_{-1}y_0}{x_0 y_{-1}}\right) = y_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2. \tag{4.96}$$

By (4.91)-(4.96), we get

$$x_9 = \max\left(x_7, \frac{x_8 y_7}{y_6}\right) = x_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)\max\left(1, \frac{x_{-1}y_0}{x_0 y_{-1}}\right) = x_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2, \tag{4.97}$$

$$y_9 = \max\left(y_7, \frac{y_8 x_7}{x_6}\right) = \max\left(y_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2, y_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2\right) = y_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2. \tag{4.98}$$

Using (4.93)-(4.98), we have

$$x_{10} = \max\left(x_8, \frac{x_9 y_8}{y_7}\right) = x_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2\max\left(1, \frac{x_{-1}y_0}{x_0 y_{-1}}\right) = x_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^3, \tag{4.99}$$

$$y_{10} = \max\left(y_8, \frac{y_9 x_8}{x_7}\right) = \max\left(y_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2, y_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2\right) = y_0\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2. \tag{4.100}$$

Then, from (4.95)-(4.100), we get

$$x_{11} = \max\left(x_9, \frac{x_{10} y_9}{y_8}\right) = \max\left(x_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2, x_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2\right) = x_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2, \tag{4.101}$$

$$y_{11} = \max\left(y_9, \frac{y_{10} x_9}{x_8}\right) = y_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^2\max\left(1, \frac{x_{-1}y_0}{x_0 y_{-1}}\right) = y_{-1}\left(\frac{x_{-1}y_0}{x_0 y_{-1}}\right)^3. \tag{4.102}$$

So, by (4.95)-(4.100), we have

$$x_{12} = \max\left(x_{10}, \frac{x_{11}y_{10}}{y_9}\right) = \max\left(x_0\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^3, x_0\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^3\right) = x_0\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^3, \quad (4.103)$$

$$y_{12} = \max\left(y_{10}, \frac{y_{11}x_{10}}{x_9}\right) = y_0\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^2 \max\left(1, \frac{x_{-1}y_0}{x_0y_{-1}}\right) = y_0\left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^3. \quad (4.104)$$

By induction we obtain the formulas in (4.52). □

As a direct consequence of Theorem 4.3, in the following result, we show existence of periodic solutions for system (4.1).

Corollary 4.4. *Let $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ be a solution of system (4.1) such that $x_{-1} \geq \frac{x_0y_{-1}}{y_{-2}}$ and $y_{-1} \geq \frac{y_0x_{-1}}{x_{-2}}$. Then, if $x_{-1}y_0 = x_0y_{-1}$, we have for all $n \in \mathbb{N}_0$*

$$x_{2n} = x_0,\ y_{2n} = y_0,$$

$$x_{2n+1} = x_{-1},\ y_{2n+1} = y_{-1}.$$

That is the solutions are periodic with period 2. If in addition $x_0 = x_{-2}$ and $y_0 = y_{-2}$ the solutions will be periodic with of period 2. When $x_{-1}y_0 \neq x_0y_{-1}$, the solutions are unbounded, that is

$$(x_n, y_n) \longrightarrow (+\infty, +\infty).$$

4.2.3 The case $x_{-1} \geq \frac{x_0y_{-1}}{y_{-2}}$ and $y_{-1} \leq \frac{y_0x_{-1}}{x_{-2}}$

The following result deals to give the closed form of the solutions of system (4.1) under the assumptions $x_{-1} \geq \frac{x_0y_{-1}}{y_{-2}}$ and $y_{-1} \leq \frac{y_0x_{-1}}{x_{-2}}$.

Theorem 4.5. *Let $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ be a solution of system (4.1) such that $x_{-1} \geq \frac{x_0y_{-1}}{y_{-2}}$ and $y_{-1} \leq \frac{y_0x_{-1}}{x_{-2}}$. Then the following statements hold:*

(H_1) : *If $\frac{x_0}{x_{-1}} \geq \frac{y_0}{y_{-1}}$, then*

$$\begin{cases} x_{4n-1} = x_{4n+1} = x_{-1}\left(\frac{x_0}{x_{-2}}\right)^n, \ n \in \mathbb{N}_0, \\ x_{4n} = x_{4n+2} = x_0\left(\frac{x_0}{x_{-2}}\right)^n, \ n \in \mathbb{N}_0, \end{cases} \quad \begin{cases} y_{4n-3} = y_{4n-1} = \frac{x_{-1}y_0}{x_0}\left(\frac{x_0}{x_{-2}}\right)^n, \ n \in \mathbb{N}, \\ y_{4n} = y_{4n-2} = y_0\left(\frac{x_0}{x_{-2}}\right)^n, \ n \in \mathbb{N}. \end{cases} \quad (4.105)$$

(H_2) : *Let* $\frac{x_0}{x_{-1}} \leq \frac{y_0}{y_{-1}}$.

$(H_{2.1})$: *If* $x_0 \leq x_{-2}$, *then*

$$\begin{cases} x_{4n-2} = x_0 \left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^n, \ n \in \mathbb{N}, \\ x_{4n-1} = \frac{x_{-1}x_0}{x_{-2}} \left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^n, \ n \in \mathbb{N}, \\ x_{4n} = x_0 \left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^n, \ n \in \mathbb{N}_0, \\ x_{4n+1} = x_{-1} \left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^n, \ n \in \mathbb{N}_0, \end{cases} \quad \begin{cases} y_{4n-1} = y_{-1} \left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^n, \ n \in \mathbb{N}_0, \\ y_{4n} = y_0 \left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^n, \ n \in \mathbb{N}_0, \\ y_{4n+1} = \frac{x_{-1}y_0}{x_{-2}} \left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^n, \ n \in \mathbb{N}_0, \\ y_{4n+2} = y_0 \left(\frac{x_{-1}y_0}{x_0y_{-1}}\right)^n, \ n \in \mathbb{N}_0, \end{cases} \tag{4.106}$$

$(H_{2.2})$: *If* $x_0 \geq x_{-2}$, *then*

$$\begin{cases} x_{4n-2} = x_{-2} \left(\frac{x_{-1}y_0}{x_{-2}y_{-1}}\right)^n, \ n \in \mathbb{N}_0, \\ x_{4n-1} = x_{4n+1} = x_{-1} \left(\frac{x_{-1}y_0}{x_{-2}y_{-1}}\right)^n, \ n \in \mathbb{N}_0, \\ x_{4n} = x_0 \left(\frac{x_{-1}y_0}{x_{-2}y_{-1}}\right)^n, \ n \in \mathbb{N}_0, \end{cases} \quad \begin{cases} y_{4n-3} = y_{-1} \left(\frac{x_{-1}y_0}{x_{-2}y_{-1}}\right)^n, \ n \in \mathbb{N}, \\ y_{4n-2} = \frac{x_0y_{-1}}{x_{-1}} \left(\frac{x_{-1}y_0}{x_{-2}y_{-1}}\right)^n, \ n \in \mathbb{N}, \\ y_{4n-1} = y_{-1} \left(\frac{x_{-1}y_0}{x_{-2}y_{-1}}\right)^n, \ n \in \mathbb{N}_0, \\ y_{4n} = y_0 \left(\frac{x_{-1}y_0}{x_{-2}y_{-1}}\right)^n, \ n \in \mathbb{N}_0. \end{cases} \tag{4.107}$$

Proof. From hypothesis $x_{-1} \geq \frac{x_0y_{-1}}{y_{-2}}$ and $y_{-1} \leq \frac{y_0x_{-1}}{x_{-2}}$. We have

$$x_1 = \max\left(x_{-1}, \frac{x_0y_{-1}}{y_{-2}}\right) = x_{-1}, \tag{4.108}$$

$$y_1 = \max\left(y_{-1}, \frac{y_0x_{-1}}{x_{-2}}\right) = \frac{y_0x_{-1}}{x_{-2}}. \tag{4.109}$$

Using (4.108) and (4.109), we get

$$x_2 = \max\left(x_0, \frac{x_1y_0}{y_{-1}}\right) = x_{-1}\max\left(\frac{x_0}{x_{-1}}, \frac{y_0}{y_{-1}}\right), \tag{4.110}$$

$$y_2 = \max\left(y_0, \frac{y_1x_0}{x_{-1}}\right) = y_0\max\left(1, \frac{x_0}{x_{-2}}\right). \tag{4.111}$$

If we consider

$$\frac{x_0}{x_{-1}} \geq \frac{y_0}{y_{-1}},$$

then, using (4.108) (4.109) and from

$$y_{-1} \leq \frac{y_0x_{-1}}{x_{-2}} \Leftrightarrow \frac{x_{-2}}{x_{-1}} \leq \frac{y_0}{y_{-1}}, \tag{4.112}$$

we get

$$x_0 \geq x_{-2}.$$

So, for this, we have three cases:

$(H_1): \frac{x_0}{x_{-1}} \geq \frac{y_0}{y_{-1}}$.

$(H_{2.1}): \frac{x_0}{x_{-1}} \leq \frac{y_0}{y_{-1}}$ and $x_0 \leq x_{-2}$.

$(H_{2.2}): \frac{x_0}{x_{-1}} \leq \frac{y_0}{y_{-1}}$ and $x_0 \geq x_{-2}$.

If $\frac{x_0}{x_{-1}} \geq \frac{y_0}{y_{-1}}$. We get from (4.108) and (4.111)

$$x_2 = \max\left(x_0, \frac{x_1 y_0}{y_{-1}}\right) = x_{-1} \max\left(\frac{x_0}{x_{-1}}, \frac{y_0}{y_{-1}}\right) = x_0, \tag{4.113}$$

$$y_2 = \max\left(y_0, \frac{y_1 x_0}{x_{-1}}\right) = y_0 \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0 y_0}{x_{-2}}. \tag{4.114}$$

Using (4.108), (4.109), (4.113) and (4.114) we get

$$x_3 = \max\left(x_1, \frac{x_2 y_1}{y_0}\right) = x_{-1} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_{-1} x_0}{x_{-2}}, \tag{4.115}$$

$$y_3 = \max\left(y_1, \frac{y_2 x_1}{x_0}\right) = \max\left(\frac{y_0 x_{-1}}{x_{-2}}, \frac{y_0 x_{-1}}{x_{-2}}\right) = \frac{y_0 x_{-1}}{x_{-2}}. \tag{4.116}$$

By (4.108), (4.109) and (4.113)-(4.116) we have

$$x_4 = \max\left(x_2, \frac{x_3 y_2}{y_1}\right) = x_0 \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0^2}{x_{-2}}, \tag{4.117}$$

$$y_4 = \max\left(y_2, \frac{y_3 x_2}{x_1}\right) = \max\left(\frac{x_0 y_0}{x_{-2}}, \frac{x_0 y_0}{x_{-2}}\right) = \frac{x_0 y_0}{x_{-2}}. \tag{4.118}$$

From (4.113)-(4.118) we get

$$x_5 = \max\left(x_3, \frac{x_4 y_3}{y_2}\right) = \max\left(\frac{x_{-1} x_0}{x_{-2}}, \frac{x_{-1} x_0}{x_{-2}}\right) = \frac{x_{-1} x_0}{x_{-2}}, \tag{4.119}$$

$$y_5 = \max\left(y_3, \frac{y_4 x_3}{x_2}\right) = \frac{y_0 x_{-1}}{x_{-2}} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{y_0 x_0 x_{-1}}{x_{-2}^2}. \tag{4.120}$$

By (4.115)-(4.120) we have

$$x_6 = \max\left(x_4, \frac{x_5 y_4}{y_3}\right) = \max\left(\frac{x_0^2}{x_{-2}}, \frac{x_0^2}{x_{-2}}\right) = \frac{x_0^2}{x_{-2}}, \tag{4.121}$$

$$y_6 = \max\left(y_4, \frac{y_5 x_4}{x_3}\right) = \frac{x_0 y_0}{x_{-2}} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0^2 y_0}{x_{-2}^2}. \tag{4.122}$$

Using (4.117)-(4.122) we get

$$x_7 = \max\left(x_5, \frac{x_6 y_5}{y_4}\right) = \frac{x_{-1} x_0}{x_{-2}} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_{-1} x_0^2}{x_{-2}^2}, \tag{4.123}$$

$$y_7 = \max\left(y_5, \frac{y_6 x_5}{x_4}\right) = \max\left(\frac{y_0 x_0 x_{-1}}{x_{-2}^2}, \frac{y_0 x_0 x_{-1}}{x_{-2}^2}\right) = \frac{y_0 x_0 x_{-1}}{x_{-2}^2}. \tag{4.124}$$

From (4.119)-(4.124) we have

$$x_8 = \max\left(x_6, \frac{x_7 y_6}{y_5}\right) = \frac{x_0^2}{x_{-2}} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0^3}{x_{-2}^2}, \tag{4.125}$$

$$y_8 = \max\left(y_6, \frac{y_7 x_6}{x_5}\right) = \max\left(\frac{x_0^2 y_0}{x_{-2}^2}, \frac{x_0^2 y_0}{x_{-2}^2}\right) = \frac{x_0^2 y_0}{x_{-2}^2}. \tag{4.126}$$

By (4.121)-(4.126) we have

$$x_9 = \max\left(x_7, \frac{x_8 y_7}{y_6}\right) = \max\left(\frac{x_{-1} x_0^2}{x_{-2}^2}, \frac{x_{-1} x_0^2}{x_{-2}^2}\right) = \frac{x_{-1} x_0^2}{x_{-2}^2}, \tag{4.127}$$

$$y_9 = \max\left(y_7, \frac{y_8 x_7}{x_6}\right) = \frac{y_0 x_0 x_{-1}}{x_{-2}^2} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{y_0 x_0^2 x_{-1}}{x_{-2}^3}. \tag{4.128}$$

Then, using (4.123)-(4.128) we get

$$x_{10} = \max\left(x_8, \frac{x_9 y_8}{y_7}\right) = \max\left(\frac{x_0^3}{x_{-2}^2}, (\frac{x_0^3}{x_{-2}^2}\right) = \frac{x_0^3}{x_{-2}^2}, \tag{4.129}$$

$$y_{10} = \max\left(y_8, \frac{y_9 x_8}{x_7}\right) = \frac{x_0^2 y_0}{x_{-2}^2} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0^3 y_0}{x_{-2}^3}. \tag{4.130}$$

So, from (4.125)-(4.130) we have

$$x_{11} = \max\left(x_9, \frac{x_{10} y_9}{y_8}\right) = \frac{x_{-1} x_0^2}{x_{-2}^2} \max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_{-1} x_0^3}{x_{-2}^3}, \tag{4.131}$$

$$y_{11} = \max\left(y_9, \frac{y_{10} x_9}{x_8}\right) = \max\left(\frac{y_0 x_0^2 x_{-1}}{x_{-2}^3}, \frac{y_0 x_0^2 x_{-1}}{x_{-2}^3}\right) = \frac{y_0 x_0^2 x_{-1}}{x_{-2}^3}. \tag{4.132}$$

By induction we obtain the formulas in (4.105).

If we consider $\frac{x_0}{x_{-1}} \leq \frac{y_0}{y_{-1}}$ and $x_0 \leq x_{-2}$. By (4.108) and (4.109), we get

$$x_2 = \max\left(x_0, \frac{x_1 y_0}{y_{-1}}\right) = x_{-1} \max\left(\frac{x_0}{x_{-1}}, \frac{y_0}{y_{-1}}\right) = \frac{x_{-1} y_0}{y_{-1}}, \tag{4.133}$$

$$y_2 = \max\left(y_0, \frac{y_1 x_0}{x_{-1}}\right) = y_0 \max\left(1, \frac{x_0}{x_{-2}}\right) = y_0. \tag{4.134}$$

Using (4.108), (4.109), (4.133) and (4.134) we have

$$x_3 = \max\left(x_1, \frac{x_2 y_1}{y_0}\right) = x_{-1} \max\left(1, \frac{x_{-1} y_0}{y_{-1} x_{-2}}\right) = \frac{x_{-1}^2 y_0}{x_{-2} y_{-1}}, \tag{4.135}$$

$$y_3 = \max\left(y_1, \frac{y_2 x_1}{x_0}\right) = \max\left(\frac{y_0 x_{-1}}{x_{-2}}, \frac{y_0 x_{-1}}{x_0}\right) = \frac{x_{-1} y_0}{x_0}. \tag{4.136}$$

From (4.108), (4.109) and (4.133)-(4.136) we get

$$x_4 = \max\left(x_2, \frac{x_3 y_2}{y_1}\right) = \max\left(\frac{x_{-1}y_0}{y_{-1}}, \frac{x_{-1}y_0}{y_{-1}}\right) = \frac{x_{-1}y_0}{y_{-1}}, \tag{4.137}$$

$$y_4 = \max\left(y_2, \frac{y_3 x_2}{x_1}\right) == y_0 \max\left(1, \frac{x_{-1}y_0}{x_0 y_{-1}}\right) = \frac{x_{-1}y_0^2}{x_0 y_{-1}}. \tag{4.138}$$

Using (4.133)-(4.138) we have

$$x_5 = \max\left(x_3, \frac{x_4 y_3}{y_2}\right) = \frac{x_{-1}^2 y_0}{y_{-1}} \max\left(\frac{1}{x_{-2}}, \frac{1}{x_0}\right) = \frac{x_{-1}^2 y_0}{x_0 y_{-1}}, \tag{4.139}$$

$$y_5 = \max\left(y_3, \frac{y_4 x_3}{x_2}\right) = \frac{x_{-1}y_0}{x_0} \max\left(1, \frac{x_{-1}y_0}{x_{-2}y_{-1}}\right) = \frac{x_{-1}^2 y_0^2}{x_{-2}x_0 y_{-1}}. \tag{4.140}$$

From (4.135)-(4.140) we have

$$x_6 = \max\left(x_4, \frac{x_5 y_4}{y_3}\right) = \frac{x_{-1}y_0}{y_{-1}} \max\left(1, \frac{x_{-1}y_0}{x_0 y_{-1}}\right) = \frac{x_{-1}^2 y_0^2}{x_0 y_{-1}^2}, \tag{4.141}$$

$$y_6 = \max\left(y_4, \frac{y_5 x_4}{x_3}\right) = \max\left(\frac{x_{-1}y_0^2}{x_0 y_{-1}}, \frac{x_{-1}y_0^2}{x_0 y_{-1}}\right) = \frac{x_{-1}y_0^2}{x_0 y_{-1}}. \tag{4.142}$$

By (4.137)-(4.142) we get

$$x_7 = \max\left(x_5, \frac{x_6 y_5}{y_4}\right) = \frac{x_{-1}^2 y_0}{x_0 y_{-1}} \max\left(1, \frac{x_{-1}y_0}{x_{-2}y_{-1}}\right) = \frac{x_{-1}^3 y_0^2}{x_{-2}x_0 y_{-1}^2}, \tag{4.143}$$

$$y_7 = \max\left(y_5, \frac{y_6 x_5}{x_4}\right) = \frac{x_{-1}^2 y_0^2}{x_0 y_{-1}} \max\left(\frac{1}{x_{-2}}, \frac{1}{x_0}\right) = \frac{x_{-1}^2 y_0^2}{x_0^2 y_{-1}}. \tag{4.144}$$

Then, using (4.139)-(4.144) we have

$$x_8 = \max\left(x_6, \frac{x_7 y_6}{y_5}\right) = \max\left(\frac{x_{-1}^2 y_0^2}{x_0 y_{-1}^2}, \frac{x_{-1}^2 y_0^2}{x_0 y_{-1}^2}\right) = \frac{x_{-1}^2 y_0^2}{x_0 y_{-1}^2}, \tag{4.145}$$

$$y_8 = \max\left(y_6, \frac{y_7 x_6}{x_5}\right) = \frac{x_{-1}y_0^2}{x_0 y_{-1}} \max\left(1, \frac{x_{-1}y_0}{x_0 y_{-1}}\right) = \frac{x_{-1}^2 y_0^3}{x_0^2 y_{-1}^2}. \tag{4.146}$$

So, by (4.141)-(4.146) we get

$$x_9 = \max\left(x_7, \frac{x_8 y_7}{y_6}\right) = \frac{x_{-1}^3 y_0^2}{x_0 y_{-1}^2} \max\left(\frac{1}{x_{-2}}, \frac{1}{x_0}\right) = \frac{x_{-1}^3 y_0^2}{x_0^2 y_{-1}^2}, \tag{4.147}$$

$$y_9 = \max\left(y_7, \frac{y_8 x_7}{x_6}\right) = \frac{x_{-1}^2 y_0^2}{x_0^2 y_{-1}} \max\left(1, \frac{x_{-1}y_0}{x_{-2}y_{-1}}\right) = \frac{x_{-1}^3 y_0^3}{x_{-2}x_0^2 y_{-1}^2}. \tag{4.148}$$

By induction we obtain the results in (4.106).

If we have $\frac{x_0}{x_{-1}} \le \frac{y_0}{y_{-1}}$ and $x_0 \ge x_{-2}$. Then, from (4.108) and (4.109) we get

$$x_2 = \max\left(x_0, \frac{x_1 y_0}{y_{-1}}\right) = x_{-1}\max\left(\frac{x_0}{x_{-1}}, \frac{y_0}{y_{-1}}\right) = \frac{x_{-1}y_0}{y_{-1}}, \tag{4.149}$$

$$y_2 = \max\left(y_0, \frac{y_1 x_0}{x_{-1}}\right) = y_0\max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_0 y_0}{x_{-2}}. \tag{4.150}$$

Using (4.108), (4.109), (4.149) and (4.150) we get

$$x_3 = \max\left(x_1, \frac{x_2 y_1}{y_0}\right) = x_{-1}\max\left(1, \frac{x_{-1}y_0}{x_{-2}y_{-1}}\right) = \frac{x_{-1}^2 y_0}{x_{-2}y_{-1}}, \tag{4.151}$$

$$y_3 = \max\left(y_1, \frac{y_2 x_1}{x_0}\right) = \max\left(\frac{x_{-1}y_0}{x_{-2}}, \frac{x_{-1}y_0}{x_{-2}}\right) = \frac{x_{-1}y_0}{x_{-2}}. \tag{4.152}$$

From (4.108), (4.109) and (4.149)-(4.152) we get

$$x_4 = \max\left(x_2, \frac{x_3 y_2}{y_1}\right) = \frac{x_{-1}y_0}{y_{-1}}\max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_{-1}x_0 y_0}{x_{-2}y_{-1}}, \tag{4.153}$$

$$y_4 = \max\left(y_2, \frac{y_3 x_2}{x_1}\right) = \frac{y_0}{x_{-2}}\max\left(x_0, \frac{x_{-1}y_0}{y_{-1}}\right) = \frac{x_{-1}y_0^2}{x_{-2}y_{-1}}. \tag{4.154}$$

By (4.149)-(4.154) we have

$$x_5 = \max\left(x_3, \frac{x_4 y_3}{y_2}\right) = \max\left(\frac{x_{-1}^2 y_0}{x_{-2}y_{-1}}, \frac{x_{-1}^2 y_0}{x_{-2}y_{-1}}\right) = \frac{x_{-1}^2 y_0}{x_{-2}y_{-1}}, \tag{4.155}$$

$$y_5 = \max\left(y_3, \frac{y_4 x_3}{x_2}\right) = \frac{x_{-1}y_0}{x_{-2}}\max\left(1, \frac{x_{-1}y_0}{x_{-2}y_{-1}}\right) = \frac{x_{-1}^2 y_0^2}{x_{-2}^2 y_{-1}}. \tag{4.156}$$

Using (4.151)-(4.156) we get

$$x_6 = \max\left(x_4, \frac{x_5 y_4}{y_3}\right) = \frac{x_{-1}y_0}{x_{-2}y_{-1}}\max\left(x_0, \frac{x_{-1}y_0}{y_{-1}}\right) = \frac{x_{-1}^2 y_0^2}{x_{-2}y_{-1}^2}, \tag{4.157}$$

$$y_6 = \max\left(y_4, \frac{y_5 x_4}{x_3}\right) = \frac{x_{-1}y_0^2}{x_{-2}y_{-1}}\max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_{-1}x_0 y_0^2}{x_{-2}^2 y_{-1}}. \tag{4.158}$$

From (4.153)-(4.158) we have

$$x_7 = \max\left(x_5, \frac{x_6 y_5}{y_4}\right) = \frac{x_{-1}^2 y_0}{x_{-2}y_{-1}}\max\left(1, \frac{x_{-1}y_0}{x_{-2}y_{-1}}\right) = \frac{x_{-1}^3 y_0^2}{x_{-2}^2 y_{-1}^2}, \tag{4.159}$$

$$y_7 = \max\left(y_5, \frac{y_6 x_5}{x_4}\right) = \max\left(\frac{x_{-1}^2 y_0^2}{x_{-2}^2 y_{-1}}, \frac{x_{-1}^2 y_0^2}{x_{-2}^2 y_{-1}}\right) = \frac{x_{-1}^2 y_0^2}{x_{-2}^2 y_{-1}}. \tag{4.160}$$

Then, by (4.155)-(4.160) we get

$$x_8 = \max\left(x_6, \frac{x_7 y_6}{y_5}\right) = \frac{x_{-1}^2 y_0^2}{x_{-2}y_{-1}^2}\max\left(1, \frac{x_0}{x_{-2}}\right) = \frac{x_{-1}^2 x_0 y_0^2}{x_{-2}^2 y_{-1}^2}, \tag{4.161}$$

$$y_8 = \max\left(y_6, \frac{y_7 x_6}{x_5}\right) = \frac{x_{-1}y_0^2}{x_{-2}^2 y_{-1}} \max\left(x_0, \frac{x_{-1}y_0}{y_{-1}}\right) = \frac{x_{-1}^2 y_0^3}{x_{-2}^2 y_{-1}^2}. \quad (4.162)$$

So, using (4.157)-(4.162) we have

$$x_9 = \max\left(x_7, \frac{x_8 y_7}{y_6}\right) = \max\left(\frac{x_{-1}^3 y_0^2}{x_{-2}^2 y_{-1}^2}, \frac{x_{-1}^3 y_0^2}{x_{-2}^2 y_{-1}^2}\right) = \frac{x_{-1}^3 y_0^2}{x_{-2}^2 y_{-1}^2}, \quad (4.163)$$

$$y_9 = \max\left(y_7, \frac{y_8 x_7}{x_6}\right) = \frac{x_{-1}^2 y_0^2}{x_{-2}^2 y_{-1}} \max\left(1, \frac{x_{-1}y_0}{x_{-2}y_{-1}}\right) = \frac{x_{-1}^3 y_0^3}{x_{-2}^3 y_{-1}^2}. \quad (4.164)$$

By induction we obtain the results in (4.107). □

We show in the following result, existence of periodic solutions for system (4.1), which is a direct consequence of Theorem 4.5.

Corollary 4.6. *Let $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ be a solution of system (4.1) such that $x_{-1} \geq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \leq \frac{y_0 x_{-1}}{x_{-2}}$. If $\frac{x_0}{x_{-1}} = \frac{y_0}{y_{-1}}$ and $x_0 = x_{-2}$. Then for all $n \in \mathbb{N}_0$:*

$$x_{2n} = x_{-2},\ y_{2n} = y_0,$$

$$x_{2n+1} = x_{-1},\ y_{2n+1} = y_{-1}.$$

That is the solutions are eventually periodic of period 2. If in addition $y_0 = y_{-2}$, then the solutions will be periodic of period 2. When $\left(\frac{x_0}{x_{-1}} > \frac{y_0}{y_{-1}} \text{ and } x_0 > x_{-2}\right)$ or $\left(\frac{x_0}{x_{-1}} < \frac{y_0}{y_{-1}}\right)$, the solutions are unbounded, that is

$$(x_n, y_n) \longrightarrow (+\infty, +\infty).$$

4.2.4 The case $x_{-1} \leq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \geq \frac{y_0 x_{-1}}{x_{-2}}$

Similarly to the above sections, in the following one, we give also the closed form of the solutions of system (4.1) under the assumptions $x_{-1} \leq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \geq \frac{y_0 x_{-1}}{x_{-2}}$.

Theorem 4.7. *Let $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ be a solution of system (4.1) such that $x_{-1} \leq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \geq \frac{y_0 x_{-1}}{x_{-2}}$. Then the following statements hold:*

(H_1) : *If $\frac{x_0}{x_{-1}} \leq \frac{y_0}{y_{-1}}$. Then*

$$x_{2n} = \begin{cases} x_0 \left(\frac{y_0}{y_{-2}}\right)^{\frac{n}{2}}, & n = 0, 2, \ldots \\ x_0 \left(\frac{y_0}{y_{-2}}\right)^{\frac{n+1}{2}}, & n = 1, 3, \ldots \end{cases} \quad x_{2n+1} = \begin{cases} \frac{x_0 y_{-1}}{y_0} \left(\frac{y_0}{y_{-2}}\right)^{\frac{n}{2}}, & n = 0, 2, \ldots \\ \frac{x_0 y_{-1}}{y_0} \left(\frac{y_0}{y_{-2}}\right)^{\frac{n+1}{2}}, & n = 1, 3, \ldots \end{cases} \quad (4.165)$$

$$y_{2n} = \begin{cases} y_0 \left(\frac{y_0}{y_{-2}}\right)^{\frac{n}{2}}, & n = 0, 2, \ldots \\ y_0 \left(\frac{y_0}{y_{-2}}\right)^{\frac{n-1}{2}}, & n = 1, 3, \ldots \end{cases} \quad y_{2n+1} = \begin{cases} y_{-1} \left(\frac{y_0}{y_{-2}}\right)^{\frac{n}{2}}, & n = 0, 2, \ldots \\ y_{-1} \left(\frac{y_0}{y_{-2}}\right)^{\frac{n+1}{2}}, & n = 1, 3, \ldots \end{cases} \quad (4.166)$$

(H_2) : *Let* $\frac{y_0}{y_{-1}} \leq \frac{x_0}{x_{-1}}$

$(H_{2.1})$: *If* $y_0 \leq y_{-2}$. *Then*

$$\begin{cases} x_{4n-1} = x_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^n, \ n \in \mathbb{N}_0, \\ x_{4n} = x_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^n, \ n \in \mathbb{N}_0, \\ x_{4n+1} = \frac{x_0y_{-1}}{y_{-2}}\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^n, \ n \in \mathbb{N}_0, \\ x_{4n+2} = x_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^n, \ n \in \mathbb{N}_0, \end{cases} \quad \begin{cases} y_{4n-2} = y_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^n, \ n \in \mathbb{N}, \\ y_{4n-1} = \frac{y_{-1}y_0}{y_{-2}}\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^n, \ n \in \mathbb{N}, \\ y_{4n} = y_0\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^n, \ n \in \mathbb{N}_0, \\ y_{4n+1} = y_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_0}\right)^n, \ n \in \mathbb{N}_0, \end{cases} \tag{4.167}$$

$(H_{2.2})$: *If* $y_0 \geq y_{-2}$. *Then*

$$\begin{cases} x_{4n-3} = x_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_{-2}}\right)^n, \ n \in \mathbb{N}, \\ x_{4n-2} = \frac{x_{-1}y_0}{y_{-1}}\left(\frac{x_0y_{-1}}{x_{-1}y_{-2}}\right)^n, \ n \in \mathbb{N}, \\ x_{4n-1} = x_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_{-2}}\right)^n, \ n \in \mathbb{N}_0, \\ x_{4n} = x_0\left(\frac{x_0y_{-1}}{x_{-1}y_{-2}}\right)^n, \ n \in \mathbb{N}_0, \end{cases} \quad \begin{cases} y_{4n-2} = y_{-2}\left(\frac{x_0y_{-1}}{x_{-1}y_{-2}}\right)^n, \ n \in \mathbb{N}_0, \\ y_{4n-1} = y_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_{-2}}\right)^n, \ n \in \mathbb{N}_0, \\ y_{4n} = y_0\left(\frac{x_0y_{-1}}{x_{-1}y_{-2}}\right)^n, \ n \in \mathbb{N}_0, \\ y_{4n+1} = y_{-1}\left(\frac{x_0y_{-1}}{x_{-1}y_{-2}}\right)^n, \ n \in \mathbb{N}_0. \end{cases} \tag{4.168}$$

Proof. From hypothesis $x_{-1} \leq \frac{x_0y_{-1}}{y_{-2}}$ and $y_{-1} \geq \frac{y_0x_{-1}}{x_{-2}}$ of Theorem 4.7. We have

$$x_1 = \max\left(x_{-1}, \frac{x_0y_{-1}}{y_{-2}}\right) = \frac{x_0y_{-1}}{y_{-2}}, \tag{4.169}$$

$$y_1 = \max\left(y_{-1}, \frac{y_0x_{-1}}{x_{-2}}\right) = y_{-1}. \tag{4.170}$$

$$x_2 = \max\left(x_0, \frac{x_1y_0}{y_{-1}}\right) = x_0 \max\left(1, \frac{y_0}{y_{-2}}\right), \tag{4.171}$$

$$y_2 = \max\left(y_0, \frac{y_1x_0}{x_{-1}}\right) = y_{-1} \max\left(\frac{y_0}{y_{-1}}, \frac{x_0}{x_{-1}}\right). \tag{4.172}$$

If we consider

$$\frac{x_0}{x_{-1}} \leq \frac{y_0}{y_{-1}}.$$

By (4.169), (4.170) and from

$$x_{-1} \leq \frac{x_0y_{-1}}{y_{-2}} \Leftrightarrow \frac{y_{-2}}{y_{-1}} \leq \frac{x_0}{x_{-1}}, \tag{4.173}$$

we get

$$y_{-2} \leq y_0.$$

So, we have three cases

(H_1) : $\frac{x_0}{x_{-1}} \leq \frac{y_0}{y_{-1}}$.

$(H_{2.1})$: $\frac{y_0}{y_{-1}} \leq \frac{x_0}{x_{-1}}$ and $y_0 \leq y_{-2}$.

$(H_{2.2})$: $\frac{y_0}{y_{-1}} \leq \frac{x_0}{x_{-1}}$ and $y_0 \geq y_{-2}$.

If we consider the case $\frac{x_0}{x_{-1}} \leq \frac{y_0}{y_{-1}}$, then by (4.169), (4.170), (4.171) and (4.172) we have

$$x_2 = x_0 \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{x_0 y_0}{y_{-2}}, \tag{4.174}$$

$$y_2 = y_{-1} \max\left(\frac{y_0}{y_{-1}}, \frac{x_0}{x_{-1}}\right) = y_0. \tag{4.175}$$

From (4.169), (4.170), (4.174) and (4.175) we have

$$x_3 = \max\left(x_1, \frac{x_2 y_1}{y_0}\right) = \max\left(\frac{x_0 y_{-1}}{y_{-2}}, \frac{x_0 y_{-1}}{y_{-2}}\right) = \frac{x_0 y_{-1}}{y_{-2}}, \tag{4.176}$$

$$y_3 = \max\left(y_1, \frac{y_2 x_1}{x_0}\right) = y_{-1} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{y_{-1} y_0}{y_{-2}}. \tag{4.177}$$

Using (4.169), (4.170), (4.174)-(4.177) we get

$$x_4 = \max\left(x_2, \frac{x_3 y_2}{y_1}\right) = \max\left(\frac{x_0 y_0}{y_{-2}}, \frac{x_0 y_0}{y_{-2}}\right) = \frac{x_0 y_0}{y_{-2}}, \tag{4.178}$$

$$y_4 = \max\left(y_2, \frac{y_3 x_2}{x_1}\right) = y_0 \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{y_0^2}{y_{-2}}. \tag{4.179}$$

By (4.174)-(4.179) we get

$$x_5 = \max\left(x_3, \frac{x_4 y_3}{y_2}\right) = \frac{x_0 y_{-1}}{y_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{x_0 y_{-1} y_0}{y_{-2}^2}, \tag{4.180}$$

$$y_5 = \max\left(y_3, \frac{y_4 x_3}{x_2}\right) = \max\left(\frac{y_{-1} y_0}{y_{-2}}, \frac{y_{-1} y_0}{y_{-2}}\right) = \frac{y_{-1} y_0}{y_{-2}}. \tag{4.181}$$

From (4.176)-(4.181) we have

$$x_6 = \max\left(x_4, \frac{x_5 y_4}{y_3}\right) = \frac{x_0 y_0}{y_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{x_0 y_0^2}{y_{-2}^2}, \tag{4.182}$$

$$y_6 = \max\left(y_4, \frac{y_5 x_4}{x_3}\right) = \max\left(\frac{y_0^2}{y_{-2}}, \frac{y_0^2}{y_{-2}}\right) = \frac{y_0^2}{y_{-2}}. \tag{4.183}$$

Then, using (4.178)-(4.183) we get

$$x_7 = \max\left(x_5, \frac{x_6 y_5}{y_4}\right) = \max\left(\frac{x_0 y_{-1} y_0}{y_{-2}^2}, \frac{x_0 y_{-1} y_0}{y_{-2}^2}\right) = \frac{x_0 y_{-1} y_0}{y_{-2}^2}, \tag{4.184}$$

$$y_7 = \max\left(y_5, \frac{y_6 x_5}{x_4}\right) = \frac{y_{-1} y_0}{y_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{y_{-1} y_0^2}{y_{-2}^2}. \tag{4.185}$$

So, by (4.180)-(4.185) we have

$$x_8 = \max\left(x_6, \frac{x_7 y_6}{y_5}\right) = \max\left(\frac{x_0 y_0^2}{y_{-2}^2}, \frac{x_0 y_0^2}{y_{-2}^2}\right) = \frac{x_0 y_0^2}{y_{-2}^2}, \tag{4.186}$$

$$y_8 = \max\left(y_6, \frac{y_7 x_6}{x_5}\right) = \frac{y_0^2}{y_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{y_0^3}{y_{-2}^2}. \tag{4.187}$$

By induction we obtain the formulas in (4.165) and (4.166).

If $\frac{y_0}{y_{-1}} \leq \frac{x_0}{x_{-1}}$ and $y_0 \leq y_{-2}$. Then by (4.169), (4.170), (4.171) and (4.172) we have

$$x_2 = x_0 \max\left(1, \frac{y_0}{y_{-2}}\right) = x_0, \tag{4.188}$$

$$y_2 = y_{-1} \max\left(\frac{y_0}{y_{-1}}, \frac{x_0}{x_{-1}}\right) = \frac{x_0 y_{-1}}{x_{-1}}. \tag{4.189}$$

From (4.169), (4.170), (4.188) and (4.189) we have

$$x_3 = \max\left(x_1, \frac{x_2 y_1}{y_0}\right) = \max\left(\frac{x_0 y_{-1}}{y_{-2}}, \frac{x_0 y_{-1}}{y_0}\right) = \frac{x_0 y_{-1}}{y_0}, \tag{4.190}$$

$$y_3 = \max\left(y_1, \frac{y_2 x_1}{x_0}\right) = y_{-1} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_{-2}}\right) = \frac{x_0 y_{-1}^2}{x_{-1} y_{-2}}. \tag{4.191}$$

By (4.169), (4.170) and (4.188)-(4.191) we get

$$x_4 = \max\left(x_2, \frac{x_3 y_2}{y_1}\right) = x_0 \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = \frac{x_0^2 y_{-1}}{x_{-1} y_0}, \tag{4.192}$$

$$y_4 = \max\left(y_2, \frac{y_3 x_2}{x_1}\right) = \max\left(\frac{x_0 y_{-1}}{x_{-1}}, \frac{x_0 y_{-1}}{x_{-1}}\right) = \frac{x_0 y_{-1}}{x_{-1}}. \tag{4.193}$$

Using (4.188)-(4.193) we have

$$x_5 = \max\left(x_3, \frac{x_4 y_3}{y_2}\right) = \frac{x_0 y_{-1}}{y_0} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_{-2}}\right) = \frac{x_0^2 y_{-1}^2}{x_{-1} y_{-2} y_0}, \tag{4.194}$$

$$y_5 = \max\left(y_3, \frac{y_4 x_3}{x_2}\right) = \frac{x_0 y_{-1}^2}{x_{-1}} \max\left(\frac{1}{y_{-2}}, \frac{1}{y_0}\right) = \frac{x_0 y_{-1}^2}{x_{-1} y_0}. \tag{4.195}$$

From (4.190)-(4.195) we get

$$x_6 = \max\left(x_4, \frac{x_5 y_4}{y_3}\right) = \max\left(\frac{x_0^2 y_{-1}}{x_{-1} y_0}, \frac{x_0^2 y_{-1}}{x_{-1} y_0}\right) = \frac{x_0^2 y_{-1}}{x_{-1} y_0}, \tag{4.196}$$

$$y_6 = \max\left(y_4, \frac{y_5 x_4}{x_3}\right) = \frac{x_0 y_{-1}}{x_{-1}} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = \frac{x_0^2 y_{-1}^2}{x_{-1}^2 y_0}. \tag{4.197}$$

By (4.192)-(4.197) we have

$$x_7 = \max\left(x_5, \frac{x_6 y_5}{y_4}\right) = \frac{x_0^2 y_{-1}^2}{x_{-1} y_0} \max\left(\frac{1}{y_{-2}}, \frac{1}{y_0}\right) = \frac{x_0^2 y_{-1}^2}{x_{-1} y_0^2}, \tag{4.198}$$

$$y_7 = \max\left(y_5, \frac{y_6 x_5}{x_4}\right) = \frac{x_0 y_{-1}^2}{x_{-1} y_0} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_{-2}}\right) = \frac{x_0^2 y_{-1}^3}{x_{-1}^2 y_{-2} y_0}. \tag{4.199}$$

Then, using (4.194)-(4.199) we get

$$x_8 = \max\left(x_6, \frac{x_7 y_6}{y_5}\right) = \frac{x_0^2 y_{-1}}{x_{-1} y_0} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = \frac{x_0^3 y_{-1}^2}{x_{-1}^2 y_0^2}, \tag{4.200}$$

$$y_8 = \max\left(y_6, \frac{y_7 x_6}{x_5}\right) = \max\left(\frac{x_0^2 y_{-1}^2}{x_{-1}^2 y_0}, \frac{x_0^2 y_{-1}^2}{x_{-1}^2 y_0}\right) = \frac{x_0^2 y_{-1}^2}{x_{-1}^2 y_0}. \tag{4.201}$$

So, by (4.196)-(4.201) we have

$$x_9 = \max\left(x_7, \frac{x_8 y_7}{y_6}\right) = \frac{x_0^2 y_{-1}^2}{x_{-1} y_0^2} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_{-2}}\right) = \frac{x_0^3 y_{-1}^3}{x_{-1}^2 y_{-2} y_0^2}, \tag{4.202}$$

$$y_9 = \max\left(y_7, \frac{y_8 x_7}{x_6}\right) = \frac{x_0^2 y_{-1}^3}{x_{-1}^2 y_0} \max\left(\frac{1}{y_{-2}}, \frac{1}{y_0}\right) = \frac{x_0^2 y_{-1}^3}{x_{-1}^2 y_0^2}. \tag{4.203}$$

By induction we obtain the results in (4.167).

If we consider the case $\frac{y_0}{y_{-1}} \leq \frac{x_0}{x_{-1}}$ and $y_0 \geq y_{-2}$. Then by (4.169), (4.170), (4.171) and (4.172) we have

$$x_2 = \max\left(x_0, \frac{x_1 y_0}{y_{-1}}\right) = x_0 \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{x_0 y_0}{y_{-2}}, \tag{4.204}$$

$$y_2 = \max\left(y_0, \frac{y_1 x_0}{x_{-1}}\right) = y_{-1} \max\left(\frac{y_0}{y_{-1}}, \frac{x_0}{x_{-1}}\right) = \frac{x_0 y_{-1}}{x_{-1}}. \tag{4.205}$$

From (4.169), (4.170), (4.204) and (4.205) we have

$$x_3 = \max\left(x_1, \frac{x_2 y_1}{y_0}\right) = \max\left(\frac{x_0 y_{-1}}{y_{-2}}, \frac{x_0 y_{-1}}{y_{-2}}\right) = \frac{x_0 y_{-1}}{y_{-2}}, \tag{4.206}$$

$$y_3 = \max\left(y_1, \frac{y_2 x_1}{x_0}\right) = y_{-1} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_{-2}}\right) = \frac{x_0 y_{-1}^2}{x_{-1} y_{-2}}. \tag{4.207}$$

By (4.169), (4.170) and (4.204)-(4.207) we get

$$x_4 = \max\left(x_2, \frac{x_3 y_2}{y_1}\right) = \frac{x_0 y_0}{y_{-2}} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_0}\right) = \frac{x_0^2 y_{-1}}{x_{-1} y_{-2}}, \tag{4.208}$$

$$y_4 = \max\left(y_2, \frac{y_3 x_2}{x_1}\right) = \frac{x_0 y_{-1}}{x_{-1}} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{x_0 y_{-1} y_0}{x_{-1} y_{-2}}. \tag{4.209}$$

Using (4.204)-(4.209) we have

$$x_5 = \max\left(x_3, \frac{x_4 y_3}{y_2}\right) = \frac{x_0 y_{-1}}{y_{-2}} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_{-2}}\right) = \frac{x_0^2 y_{-1}^2}{x_{-1} y_{-2}^2}, \tag{4.210}$$

$$y_5 = \max\left(y_3, \frac{y_4 x_3}{x_2}\right) = \max\left(\frac{x_0 y_{-1}^2}{x_{-1} y_{-2}}, \frac{x_0 y_{-1}^2}{x_{-1} y_{-2}}\right) = \frac{x_0 y_{-1}^2}{x_{-1} y_{-2}}. \tag{4.211}$$

From (4.206)-(4.211) we get

$$x_6 = \max\left(x_4, \frac{x_5 y_4}{y_3}\right) = \frac{x_0^2 y_{-1}}{x_{-1} y_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{x_0^2 y_{-1} y_0}{x_{-1} y_{-2}^2}, \tag{4.212}$$

$$y_6 = \max\left(y_4, \frac{y_5 x_4}{x_3}\right) = \frac{x_0 y_{-1}}{x_{-1} y_{-2}} \max\left(y_0, \frac{x_0 y_{-1}}{x_{-1}}\right) = \frac{x_0^2 y_{-1}^2}{x_{-1}^2 y_{-2}}. \tag{4.213}$$

Then, by (4.208)-(4.213) we have

$$x_7 = \max\left(x_5, \frac{x_6 y_5}{y_4}\right) = \max\left(\frac{x_0^2 y_{-1}^2}{x_{-1} y_{-2}^2}, \frac{x_0^2 y_{-1}^2}{x_{-1} y_{-2}^2}\right) = \frac{x_0^2 y_{-1}^2}{x_{-1} y_{-2}^2}, \tag{4.214}$$

$$y_7 = \max\left(y_5, \frac{y_6 x_5}{x_4}\right) = \frac{x_0 y_{-1}^2}{x_{-1} y_{-2}} \max\left(1, \frac{x_0 y_{-1}}{x_{-1} y_{-2}}\right) = \frac{x_0^2 y_{-1}^3}{x_{-1}^2 y_{-2}^2}. \tag{4.215}$$

So, using (4.210)-(4.215) we get

$$x_8 = \max\left(x_6, \frac{x_7 y_6}{y_5}\right) = \frac{x_0^2 y_{-1}}{x_{-1} y_{-2}^2} \max\left(y_0, \frac{x_0 y_{-1}}{x_{-1}}\right) = \frac{x_0^3 y_{-1}^2}{x_{-1}^2 y_{-2}^2}, \tag{4.216}$$

$$y_8 = \max\left(y_6, \frac{y_7 x_6}{x_5}\right) = \frac{x_0^2 y_{-1}^2}{x_{-1}^2 y_{-2}} \max\left(1, \frac{y_0}{y_{-2}}\right) = \frac{x_0^2 y_{-1}^2 y_0}{x_{-1}^2 y_{-2}^2}. \tag{4.217}$$

By induction we obtain the results in (4.168).

□

As a direct consequence of Theorem 4.7, the following result show the existence of periodic solutions for system (4.1).

Corollary 4.8. *Let $(x_n)_{n\geq -2}$ and $(y_n)_{n\geq -2}$ be a solution of system (4.1) such that $x_{-1} \leq \frac{x_0 y_{-1}}{y_{-2}}$ and $y_{-1} \geq \frac{y_0 x_{-1}}{x_{-2}}$. If $\frac{x_0}{x_{-1}} = \frac{y_0}{y_{-1}}$ and $y_0 = y_{-2}$ then for all $n \in \mathbb{N}_0$:*

$$x_{2n} = x_0,\ y_{2n} = y_{-2},$$

$$x_{2n+1} = x_{-1},\ y_{2n+1} = y_{-1}.$$

That is the solutions are eventually periodic of period 2.In addition if $x_0 = x_{-2}$, then the solution will be periodic of period 2 . When $\left(\frac{x_0}{x_{-1}} < \frac{y_0}{y_{-1}} \text{ and } y_0 > y_{-2}\right)$ or $\left(\frac{x_0}{x_{-1}} > \frac{y_0}{y_{-1}}\right)$, the solutions are unbounded, that is

$$(x_n, y_n) \longrightarrow (+\infty, +\infty).$$

Conclusion and perspectives

Our works generalizes a lot of existing works in the literature on solvable difference equations and systems.

In the first chapter we have presented formulas of well-defined solutions of some general systems of difference equations and others defined by one to one functions on a set D of real numbers. Noting that the obtained formulas of the solutions of our systems are expressed using some remarkable sequences, like Fibonacci, Tribonaci, Padovan, Teternacci and their generalizations. Our results, can be used to obtain the formulas of well-defined solutions of other systems, that their solvability, seems for the first sight impossible. So, under appropriate choice of the the set D, we can solve complicated difference equations and systems involving for example functions like tan, ln and others.

In the same context of the works previously studied, it should be noted that it is possible to extend those works to the study of the systems

$$\begin{cases} x_{n+1} = \dfrac{ax_{n-3}y_{n-2}x_{n-1}y_n + bx_{n-1}y_{n-2}x_{n-3} + cy_{n-2}x_{n-3} + dx_{n-3} + e}{x_{n-3}y_{n-2}x_{n-1}y_n}, \\ y_{n+1} = \dfrac{ay_{n-3}x_{n-2}y_{n-1}x_n + by_{n-1}x_{n-2}y_{n-3} + cx_{n-2}y_{n-3} + dy_{n-3} + e}{y_{n-3}x_{n-2}y_{n-1}x_n}, \end{cases}$$

$$\begin{cases} x_{n+1} = f^{-1}\left(ag(y_n) + bf(x_{n-1}) + cg(y_{n-2}) + df(x_{n-3}) + eg(x_{n-4})\right), \\ y_{n+1} = g^{-1}\left(af(x_n) + bg(y_{n-1}) + cf(x_{n-2}) + dg(y_{n-3}) + ef(x_{n-4})\right), \end{cases}$$

and

$$\begin{cases} x_{n+1} = f^{-1}\left(a + \frac{b}{g(y_n)} + \frac{c}{g(y_n)f(x_{n-1})} + \frac{d}{g(y_n)f(x_{n-1})g(y_{n-2})} + \frac{e}{g(y_n)f(x_{n-1})g(y_{n-2})f(x_{n-3})}\right), \\ y_{n+1} = g^{-1}\left(a + \frac{b}{f(x_n)} + \frac{c}{f(x_n)g(y_{n-1})} + \frac{d}{f(x_n)g(y_{n-1})f(x_{n-2})} + \frac{e}{f(x_n)g(y_{n-1})f(x_{n-2})g(y_{n-3})}\right), \end{cases}$$

where $n \in \mathbb{N}_0$ and the parameters a, b, c, d and e are arbitrary real numbers with $e \neq 0$, and their one dimensional version can be solved in a closed form and that the solutions can be expressed using the Pentanacci numbers and their generalizations. It should be noted that in this case the corresponding characteristic equation

$$\lambda^5 - a\lambda^4 - b\lambda^3 - c\lambda^2 - d\lambda - e = 0,$$

generally can't be solved by radicals as it is known in the Galois theory of algebraic equations.

In the second chapter, we have studied a general system of difference equations of second order defined by homogeneous functions. Conditions and some convergence theorems for which the unique equilibrium point of the system is globally asymptotically stable are established. Conditions for the existence of prime period two solutions are also provided. Finally a result on oscillatory solutions is proved. All obtained results are confirmed on particular systems. Noting that our system generalize the equation in [62] and our results can be applied to study new systems and to extend a lot of existing work in literature. For interested readers and as generalization of our system and the equations in [1] and [61], we propose to study the following two systems of difference equations

$$x_{n+1} = f(y_{n-k}, y_{n-m}),\ y_{n+1} = g(z_{n-k}, z_{n-m}),\ z_{n+1} = h(x_n, x_{n-1}),\ n \in \mathbb{N}_0,\ k, m \in \mathbb{N}$$

$$x_{n+1} = f(y_n, y_{n-1}, ..., y_{n-k}),\ y_{n+1} = g(z_n, z_{n-1}, ..., z_{n-k}),\ z_{n+1} = h(x_n, x_{n-1}, ..., x_{n-k}),\ n \in \mathbb{N}_0,\ k \in \mathbb{N}$$

where the initial values are positive real numbers and the functions $f,\ g,\ h : (0, +\infty)^2 \longrightarrow (0, +\infty)$ are continuous and homogeneous of degree zero.

The system (3.5) can be generalized to r-dimensional form of equations and examine the boundedness, the asymptotic behavior, and periodicity of solutions when $p = 1$.
Also, as a natural question, is to study the three-dimensional form of Max-type system (4.1).

As we have said in the introduction that is difficult to determines methods to solve non linear equations and their systems and the famous method is by the help of some change of variables, non linear difference equations or systems are transformed to very simple one, with known form of the solutions.
However, it is fair to point out that there are other ways to solve these equations, using methods of differential equations such as using the Lie symmetries, see for example the works of P. E. Hydon [47].

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