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Defended on: **//2019 in front of the Examination Committee:**

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The secret is comprised in three words: Work, finish, publish.

Michael Faraday

Ere many generations pass, our machinery will be driven by a power obtainable at any point of the universe. This idea is not novel. Men have been led to it long ago by instinct or reason; it has been expressed in many ways, and in many places, in the history of old and new. We find it in the delightful myth of Antheus, who derives power from the earth; we find it among the subtle speculations of one of your splendid mathematicians and in many hints and statements of thinkers of the present time. Throughout space there is energy. Is this energy static or kinetic? If static our hopes are in vain; if kinetic - and this we know it is, for certain - then it is a mere question of time when men will succeed in attaching their machinery to the very wheelwork of nature.

Nikola Tesla, "Experiments with alternate currents of high potential and high frequency", February 1892.

My Publications

Journal Papers

1. Boubellouta, A. and Boulkroune, A. (2018). Intelligent fractional-order control-based projective synchronization for chaotic optical systems.Soft Comput . https://doi.org/10.1007/s00500-018-3490-5.
2. A.Boubellouta, F. Zouari , A. Boulkroune,"Intelligent fuzzy Controller For chaos synchronization of uncertain fractional-order chaotic systems with input nonlinearities", International Journal of General Systems, https://doi.org/10.1080/03081079.2019.1566231
3. A. Boubellouta, A. Boulkroune, "Chaos Synchronization of two different PMSMs Via a fractional order sliding mode controller ", IJCAT 60(2): 165-174 (2019)
4. A. Boulkroune, A. Boubellouta, "Fuzzy Adaptive Backstepping Control Based-Synchronization of Fractional-Order Chaotic Systems", Submitted to the Chaos, Solitons & Fractals Journal.

Conferences Proceedings

1. A. Boubellouta, A. Boulkroune," Adaptive Synchronization Of Uncertain Fractional-Order Chaotic Triangular Systems Via Fuzzy Backstepping Control". 6th International Conference on Control, Decision and Information Technologies, CoDIT 2019, Paris, France. 2004-2009.
2. A. Boubellouta, A. Boulkroune, "Chaos synchronization of two different PMSM using a fractional order sliding mode controller", In : Modelling, Identification and Control (ICMIC), 2016 8th International Conference on.IEEE,2016. p. 995-1001.
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4. A. Boubellouta, A. Boulkroune, "Synchronization of incommensurate fractional-order chaotic systems with input nonlinearities using a fuzzy variablestructure control", In Proc. of the 3th International Conference on Electrical Engineering And Control Applications ICEECA'17, Constantine, Algeria,2017.

Book Chapters

1. Farouk Zouari, Amina Boubellouta. Adaptive Neural Control for nonlinear time-delay fractional-order systems with input saturation. In : Advanced Synchronization Control and Bifurcation of Chaotic Fractional-Order Systems. IGI Global, May, 2018.

2. Amina Boubellouta. Backstepping Control for Synchronizing FractionalOrder Chaotic Systems. In : Advanced Synchronization Control and Bifurcation of Chaotic Fractional-Order Systems. IGI Global, May, 2018.
3. Abdesselem Boulkroune, Amina Boubellouta. Chaos synchronization of optical systems via a fractional-order sliding mode controller. In : Advanced Synchronization Control and Bifurcation of Chaotic Fractional-Order Systems. IGI Global, May, 2018.
4. Abdesselem Boulkroune, Amina Boubellouta. Fuzzy control-based synchronization of fractional-order chaotic systems with input nonlinearities. In Advanced Synchronization Control and Bifurcation of Chaotic FractionalOrder Systems. IGI Global, May, 2018.
5. Farouk Zouari , Amina Boubellouta. Neural Approximation-Based Adaptive Control for fractional-order systems with Output Constraints and Actuator Nonlinearities. In : Advanced Synchronization Control and Bifurcation of Chaotic Fractional-Order Systems. IGI Global, May, 2018.
6. Boubellouta, A. and Boulkroune, A. (2018). Synchronization of Incommensurate Fractional-Order Chaotic Systems with Input Nonlinearities Using a Fuzzy Variable-Structure Control. Advanced Control Engineering Methods in Electrical Engineering Systems, pp.128-142

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Abstract-

This thesis is mainly focused on the design of synchronization systems based on fuzzy adaptive control for three different classes of fractional-order chaotic systems. In all proposed control schemes, the fuzzy systems are employed to approximate unknown nonlinear functions. A Lyapunov approach is exploited to derive the adaptation laws and prove the boundedness and convergence of all signals involved in the closed-loop system. Some numerical simulation results are presented to verify the feasibility and effectiveness of the proposed synchronization schemes.

Key-words:

Chaos synchronization, adaptive fuzzy control, dead-zone, Lyapunov stability, Fractional-order systems**.**

Résumé-

Cette thèse porte principalement sur la conception de systèmes de synchronisation basés sur la commande floue adaptative pour trois classes différentes de systèmes chaotiques d'ordre fractionnel. Dans tous les schémas de commande proposés, les systèmes flous sont utilisés pour approcher des fonctions non linéaires inconnues. Une approche de Lyapunov est exploitée pour dériver les lois d'adaptation et prouver la bornitude et la convergence de tous les signaux impliqués dans le système en boucle fermée. Quelques résultats de simulations numériques sont présentés pour vérifier la faisabilité et l'efficacité des schémas de synchronisation proposés.

Mots-clés:

Synchronisation du chaos, commande floue adaptative, zone-morte, stabilité de Lyapunov.,Systèmes d'ordre fractionnaire.

ملخص

تركز هذه الرسالة بشكل أساسي على تصميم أنظمة التزامن القائمة على التحكم التكيفي الغامض لثلاث فئات مختلفة من أنظمة الفوضى الكسرية. في جميع مخططات التحكم المقترحة ، يتم استخدام الأنظمة الغامضة لتقريب الوظائف غير الخطية غير المعروفة. يتم استغلال نهج ليابونوف لاشتقاق قوانين التكيف وإثبات تقارب وتداخل جميع الإشارات المشاركة في نظام الحلقة المغلقة. يتم تقديم بعض نتائج المحاكاة العددية للتحقق من جدوى وفعالية أنظمة التزامن المقترحة.

كلمات مفتاحية

تزامن الفوضى, تحكم تكيفي غامض, منطقة ميتة, استقرار ليابونوف , الأنظمة الجزئية.

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Intelligent fuzzy controller for chaos synchronization of uncertain fractional-order chaotic systems with input nonlinearities

Chapter4

Adaptive Synchronization of Uncertain Fractional-Order Chaotic triangular Systems via Fuzzy Backstepping Control

Nomenclature

Acronyms

FS	Fuzzy systems
FBF	Fuzzy Basis Function
ML	Mittag-Leffler
RL	Riemann-Liouville
GL	Grünwald-Letnikov
LT	Laplace Transform
BIBO	Bounded-Input Bounded-Output
UUB	Uniformly ultimately bounded
CS	Complete Synchronization
AS	Anti-Synchronization
PS	Projective Synchronization
GPS	Generalized Projective Synchronization
GFPS	Generalized Function Projective Synchronization
FPS	Function Projective Synchronization
MPS	Modified Projective Synchronization
FO	Fractional-Order
IC	Initial Conditions

General Introduction

1. Background and motivation

Chaotic (hyperchaotic) phenomena arise ubiquitously in natural systems and in man-made devices. In the 1960s, an MIT meteorology professor named Ed Lorenz, while working on computer models of weather, realized that teeny tiny, almost immeasurably small differences in the numbers he input into his computer ended up significantly changing the long-term weather patterns. Or put another way, his findings suggested that a butterfly flapping its wings in Brazil could cause a tornado in Texas. As with several terms in science, there exists no standard or unified definition of chaos. But, the typical features of chaos generally include [PRI97]:

- Non-linearity. If the system is linear, it cannot be chaotic.
- Determinism. A chaotic system has deterministic (rather than probabilistic) underlying rules that every its future state must follow.
- Sensitivity to initial conditions. Small changes in the initial state can lead to radically different behavior in its final state.
- Unpredictability. Long-term prediction (but not control!) is mostly impossible due to sensitivity to initial conditions, which can be known only to a finite degree of precision.

LM Pecora and TL Carroll [PEC90], being the first, have studied how identical, or almost identical chaotic systems can be synchronized by a chaotic reference signal so that the two systems follow the same chaotic orbit. One possible use of the capacity to ***synchronize chaotic systems*** is in secure communications. The idea is to mask the information-bearing signal to be transmitted with a chaotic signal that exhibits broadband characteristics. This is an alternative to more classical noise-masking approaches, in which one uses a purely stochastic signal to mask the information to be transmitted. Since the seminal works of LM Pecora and TL Carroll [PEC90], synchronization of chaotic systems has attracted considerable attention due to its great potential applications in various disciplines (especially in secure communications). To achieve chaos synchronization, many control methods, including variable structure control [HAM16], active control [AHM15], adaptive control [VAR2015], sliding mode control [ROO10] and fuzzy adaptive control [HAM16], have been employed. Most of the chaos synchronization schemes consist of master system, slave system and synchronization controller to achieve an appropriate synchronization between two systems.

On the other hand, dynamical systems modeled by using ***fractional-order derivatives*** and integrals have been recently studied with renewed interests owing to their attractive features as well as potential applications in many various areas, such as heat diffusion systems, batteries, neurons, viscoelastic systems, dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, and so on. [VAN50, DAV51, DAR97, BAT00, SER01, COI00]. Unlike conventional integer-order systems, fractional-order (FO) systems, which can be considered as a generalization of the integer-order, can accurately and faithfully describe many physical problems [POD99, LI09, AGH14, DUA15]. Indeed, it has been well established in the literature that the fractional-order model can powerfully describe the data memory as well as heredity, as it has the assets of a history independence as well as a long range correlation [POD99, LI09, AGH14, DUA15]. Likewise, in particular, by employing the fractional-order control systems, one can expand the freedom of the controlled system and obtain better control performances. Motivated by this control performances improvement, many recent works combine the fractional calculus with traditional controllers, observers and synchronization systems [BOU15a, BOU16a, BOU15b, BOU16b, LIU15a, KON17, BOR12, LI15a, HUA14, BOU16c, BOU16d, BOU15e, KHA18, PIS10, BOU18, JAK16, AGH12, HAJ18], such as FO observer [KON17, BOR12], FO fuzzy adaptive control [LI15a, HUA14, BOU16c, BOU16d] , FO fuzzy adaptive synchronization [BOU16e] , FO disturbance observer-based sliding mode control [KHA18], FO sliding mode control [PIS10, BOU18, JAK16, AGH12] , and FO PID control [HAJ18].

Unlike the sliding mode control, ***standard adaptive control*** offers the advantage that the system uncertainties' bounds are not required to be known, since in fact these uncertainties can be approximated (and even compensated for) online in an adaptive manner.

In an adaptive control scheme, control parameters are online updated by using the system signals that are available for measurement. However, the standard adaptive control schemes [SLO91, KRS95, KOK01, BOU05, BOU06, BOU07] are limited to nonlinear systems that can be linearly parameterized (i.e., uncertain nonlinearities of the system or the control law can be expressed as a product of a nonlinear function known by a vector of unknown parameters). Unfortunately, it is often very difficult, if even impossible, to obtain this form affine in unknown parameters, particularly for uncertain complex physical systems.

Fuzzy systems offer a potential solution to the problem of conventional adaptive control and an appropriate way to parameterize the unknown systems nonlinearities. They are considered as universal approximators [WAN92, WAN94]. In fact, they can uniformly approximate any smooth unknown nonlinear function defined on a compact set to any precision. In this context, the difficulty of building the base of fuzzy rules for complex dynamical systems and the need to improve the quality of approximation have motivated LX Wang to introduce adaptive fuzzy systemin the control schemes [WAN94, WAN93]. In these schemes, the universal approximation property of fuzzy systems has been fully exploited, and stability and robustness have been studied by the Lyapunov approach. Since the seminal works of LX Wang, many ***fuzzy adaptive control*** schemes have been developed for uncertain fractional-order nonlinear (or chaotic) systems [BOU14a, LIN11a, CHE12].

Thus, this work presented in this dissertation generally fit into this logic. It fundamentally focuses on the design of new adaptive fuzzy control laws to achieve projective synchronization of fractional-order uncertain chaotic systems with disturbances.

2. Contributions of this Dissertation

This dissertation contributes to the theory of projective synchronization based on fuzzy intelligent control for three different classes of uncertain chaotic systems:

- Fractional-order optical systems with uncertain dynamics and matched disturbances,
- Fractional-order chaotic systems with input nonlinearities (dead-zone and sector nonlinearities)

- Fractional-order chaotic systems with a triangular structure and subject to matched and unmatched disturbances.

3. Outline of the Dissertation

This dissertation is the compilation of my PhD research works. The chapters are based on journal or conference articles, which are either already published or currently under review. Because of this, there can exist some partial overlap between the chapters (especially in the first section of each chapter (i.e. Introduction) where states of the art are made).

This dissertation work is organized as follows:

Chapter 1will recall some basic notions of the fuzzy logic systems, fractional-order systems, chaotic systems as well as some synchronization methods. Moreover, some definitions, lemmas and theorems, which essential in the design of the control and synchronization schemes of fractional-order chaotic systems, will be also given.

Chapter 2 will investigate the problem of projective synchronization based on fractional-order intelligent sliding mode control approach for a class of fractional-order chaotic optical systems with unknown dynamics and matched disturbances. Two fuzzy sliding-mode controllers will be derived based on fractional-order sliding surfaces. Fuzzy systems will be used to online estimate the uncertain nonlinear functions. The stability analysis of the closed-loop system is rigorously performed by using a fractional Lyapunov theory. Finally, some illustrative simulation examples will be given to demonstrate the applicability and effectiveness of the proposed controllers.

Chapter 3 will address the design of a novel fuzzy adaptive control to achieve a generalized projective synchronization of fractional-order chaotic systems with input nonlinearities (dead-zone together with sector nonlinearities). The master-slave systems under consideration are assumed to be with distinct models, different fractional-orders, unknown dynamics, and bounded external disturbances. In this proposed control law, there are two main terms for different purposes, namely a fuzzy adaptive control term for appropriately approximating the uncertainties, and a fractional-order variable-structure control term for robustly dealing with dead-zone and sector nonlinearities. A Lyapunov methodology will be exploited to rigorously derive the updated laws and to prove the stability of the closed-loop control system. At last, a set of computer simulations will be carried out to illustrate and further validate the theoretical findings.

Chapter 4 will propose a novel fractional fuzzy adaptive backstepping control scheme guaranteeing an adequate projective-synchronizing a class of uncertain chaotic master-slave systems. These systems are assumed to be with fractional-order and subject to uncertainties and (matched and unmatched) external disturbances. In the design process, the uncertain nonlinear functions will be online modeled via adaptive fuzzy systems, and the virtual control terms will be systematically and recursively determined using the fractional Lyapunov stability. The proposed fuzzy adaptive backstepping controller ensures the stability of the closed-loop system as well as the convergence of the underlying synchronization error to a small but adjustable neighborhood of the zero. A set of numerical simulations will be given to illustrate the performances of the proposed approach.

A General Conclusion is drawn at the end of this dissertation as an essential report of this present study and of some perspectives attempting to explore the aspects not covered in this dissertation.

Chapter 1

Background on Fuzzy Approximator, Fractional-Order Systems and Chaos

1.1 Introduction

In this chapter, one reviews some basic results on the fuzzy systems, fractional-order systems, and chaotic systems and its synchronization. This review is not intended to be exhaustive; its only purpose is to provide the reader with the necessary background for the results presented throughout subsequent sections. Since most of the results given in this chapter are standard in the literature of the fuzzy systems, fractional-order systems and chaos synchronization, they are given sometimes here without proofs. For detailed proofs, the reader can be referred to classical books in these fields as [WAN94, WAN 97, SLO91, KHA02, CAP10, CHE17, CHE99, CHE98].

1.2 Fuzzy systems

In this thesis, the unknown nonlinear functions will be online approximated by the fuzzy logic systems (FLS). The common configuration of FLS contains four main modules: The fuzzifier, the rule base, the inference model and the defuzzifier, as shown in Figure 1.1. The fuzzy inference model employs the fuzzy If-Then rules to establish a mapping from an input vector $\underline{x}^T = [x_1, \ldots, x_n] \in R^n$ to an output $\hat{f} \in R$. The *ith* fuzzy logic rule is stated as follows:

$$R^{(i)}\text{: if } x_1 \text{ is } A_1^i \text{ and} \ldots \text{and } x_n \text{ is } A_n^i \text{ then } \hat{f} \text{ is } f^i \tag{1.1}$$

where $A_1^i, \dots,$ and $A_n^i$are fuzzy sets and f^i is a fuzzy singleton for the output in the *ith* fuzzy logic rule.

Considering a common method combination which uses the product inference system, singleton fuzzifier and center-average defuzzifier, the mathematical model of this fuzzy system can expressed as follows:

$$\hat{f}(\underline{x}) = \frac{\sum_{i=1}^{M} f^i\left(\prod_{j=1}^{n} \mu_{A_j^i}(x_j)\right)}{\sum_{i=1}^{M}\left(\prod_{j=1}^{n} \mu_{A_j^i}(x_j)\right)} = \theta^T \psi(\underline{x}) \tag{1.2}$$

with $\mu_{A_j^i}(x_j)$ being the degree of membership of x_j to A_j^i, M standing for the number of fuzzy logic rules, $\theta^T = [f^1, \dots, f^M]$ being a vector containing all the values to be online adjusted, and $\psi^T = [\psi^1\psi^2 \dots \psi^M]$ being a vector containing all the fixed values in the fuzzy logic system. The expression (1.2) will be very useful later to introduce the function approximation property of fuzzy systems.

Each element of ψ is given by:

$$\psi^i(\underline{x}) = \frac{\prod_{j=1}^{n} \mu_{A_j^i}(x_j)}{\sum_{i=1}^{M}\left(\prod_{j=1}^{n} \mu_{A_j^i}(x_j)\right)} \tag{1.3}$$

$\psi^i(\underline{x})$ is generally called "fuzzy basis function" (FBF), [WAN94]. These FBFs are selected by the designer such that the following expression is always valid:

$$\sum_{i=1}^{M}\left(\prod_{j=1}^{n} \mu_{A_j^i}(x_j)\right) > 0 \tag{1.4}$$

The fuzzy system (1.2) is commonly used in control applications [BOU08a]. Following the universal approximation results [WAN94], this fuzzy system can approximate any nonlinear smooth function $f(\underline{x})$ on a compact operating space to any degree of accuracy. Most importantly, it is assumed that the structure of the fuzzy system and the membership function parameters are properly specified beforehand by the designer. This means that the designer decision is needed to determine the structure of the fuzzy system, namely the pertinent inputs, the number of membership functions for each input, the membership function parameters and the number of rules. However, the consequent parameters' vector, i.e. θ, will be determined via some learning algorithms.

1.2.1 Universal Function approximation property

The universal function approximation property of the fuzzy systems has been extensively studied [WAN97, WAN 94], and it has proved to be of fundamental importance in control and observation applications. The basic results of the universal approximation of fuzzy systems says that any smooth $f(x)$ can be adequately approximated on a compact set using a fuzzy system with appropriate number of fuzzy rules.

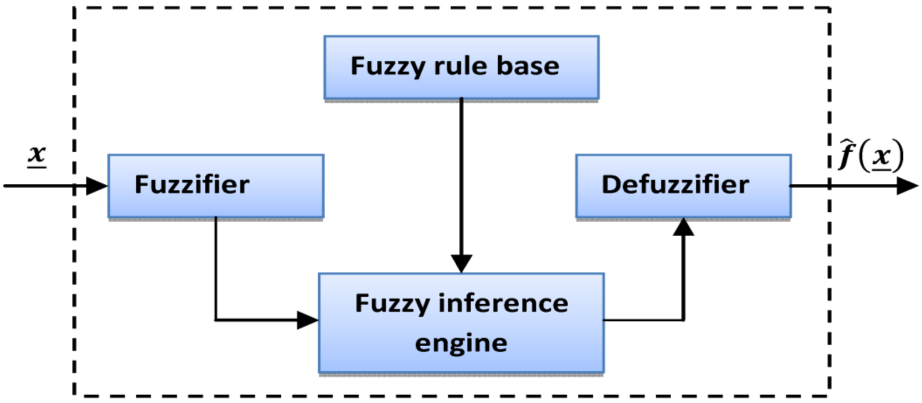


Figure 1.1: *Fuzzy system used for the functions' approximation.*

Theorem1.1: *Let be $f(\underline{x})$is a smooth (nonlinear) function defined on a compact set Ω_x, and for any positive constant ε, there is a fuzzy system $y(\underline{x}) = \theta^T\psi(\underline{x})$ such as:*

$$sup_{x\in\Omega_x}\left|f(\underline{x}) - \theta^T\psi(\underline{x})\right| < \varepsilon \tag{1.5}$$

A detailed proof of this theorem is given in [WAN94].

1.3 Stability of non-fractional order systems

1.3.1 Stability of Lyapunov

The notion of stability of a dynamic system characterizes the behavior of its trajectories around equilibrium points. The stability of an equilibrium point determines whether or not solutions nearby the equilibrium point remain nearby, get closer, or get further away [ZEM07].

In all control or synchronization systems, stability of the closed-loop systems is the primary requirement. One of the most widely used stability concepts in control theory is that of *Lyapunov stability*, which we employ throughout the thesis. In this section, we briefly review basic facts from Lyapuonv's stability theory.

Let consider the class of nonlinear systems described by the following dynamical equations:

$$\dot{x}(t) = f(x(t), t), \quad x(t_0) = x_0 \tag{1.6}$$

where $x(t) \in R^n$ and $f: R^n \times R^+ \to R^n$is continuous. One denotes by $x_e$the equilibrium point of (1.6) such that $f(x_e, t) = 0, \ \forall t \geq t_0$ and by $x(t, t_0, x_0)$ the solution at time $t \geq t_0$ of the system (1.6) initialized at time t_0 with x_0.

Roughly speaking, we say that a system is stable if when displaced slightly from its equilibrium position, it tends to come back to its original position. On the other hand, it is instable if it tends to move away from its equilibrium position.

Mathematically speaking, this is translated into the following definitions. We assume that the system (1.6) has a unique equilibrium point $x_e = 0$.

Definition 1.1 (Stability): *The origin is a stable equilibrium point in the Lyapunov sense for the system (1.6), if* $\forall \varepsilon > 0, \forall t_0 \geq 0$, *there exists a positive scalar* $\delta(\varepsilon, t_0)$ *such that:*

$$\|x_0\| < \delta(\varepsilon, t_0) \Rightarrow \|x(t, t_0, x_0)\| < \varepsilon, \qquad \forall t \geq t_0.$$

It is said that the origin is unstable in the opposite case.

Definition 1.2 (Uniform stability): *The origin is a uniformly stable equilibrium point of the system (1.6), if* $\forall \varepsilon > 0$, *there exists a positive scalar* $\delta(\varepsilon)$ *such that:*

$$\|x_0\| < \delta(\varepsilon) \Rightarrow \|x(t, t_0, x_0)\| < \varepsilon, \qquad \forall t \geq t_0$$

Definition 1.3 (Attractiveness): *The origin is an attractive equilibrium point for the system (1.6), if* $\forall \varepsilon > 0$, *there exists a positive scalar* $\delta(t_0)$ *such that:*

$$\|x_0\| < \delta(t_0) \Rightarrow \lim_{t \to \infty} x(t, t_0, x_0) = 0\,, \qquad \forall t \geq t_0$$

When $\delta(t_0) = +\infty$, *we say that the origin is globally attractive.*

Definition 1.4 (Asymptotic stability): *The origin is an asymptotically (or globally asymptotically) stable equilibrium point for (1.6), if it is stable and attractive (or globally attractive, respectively).*

Definition 1.5 (Exponential Stability): *The origin is a locally exponentially stable equilibrium point for the system (1.6), if there are two strictly positive constants, α and β, such that:*

$$\|x(t, t_0, x_0)\| \leq \alpha \exp(-\beta(t - t_0)), \ \ \forall t \geq t_0, \forall x_0 \in B_r.$$

When $B_r = R^n$, *we say that the origin is globally exponentially stable.*

Definition 1.6 (Uniformly ultimately bounded (UUB)):

-*The solutions of (1.6) are UUB with ultimate boundb if there exist positive constants* b *and* c, *independent of* t_0, *and for every* $a \in [0,\ c]$, *there is* $T = T(a, b) \geq 0$, *independent of* t_0, *such that*

$$\|x_0\| < a \ \Rightarrow \|x(t)\| < b, \qquad \forall t \geq t_0 + T$$

-The solutions of (1.6) are globally UUB, if the previous expression holds for arbitrarily large a.

-In the case of autonomous systems, the word "uniformly" may be removed.

Remark 1.1:

1. The difference between stable and asymptotically stable is that a small perturbation on the initial state around a stable equilibrium point x_e *might lead to small sustained oscillations, whereas these oscillations are dampened in time in the case of asymptotically stable equilibrium point.*

2. For linear systems, all the above definitions are equivalent (except for stable and asymptotically stable). However, for nonlinear systems, stability does not imply attractiveness, attractiveness does not imply stability, asymptotical stability does not imply exponential stability but exponential stability implies asymptotical stability.

3. The use of the previous definitions to demonstrate the stability of the nonlinear differential equations (e.g. the system (1.6)) around its equilibrium point requires the explicit resolution of these differential equations, which is often very difficult or even impossible in most cases. Thereby, Lyapunov's direct method allows to circumvent this obstacle. This method consits in defining a particular function whose existence guarantees stability.

4. Unlike the nonlinear systems, the utilization of the above definitions to demonstrate the stability of the linear systems around its equilibrium point is systematic.

1.3.2 Lyapunov's Direct method

The Lyapunov's direct method allows to draw some conclusions about the stability of nonlinear systems (i.e. nonlinear differential equations (1.6)), without knowing their analytical solutions. This method is based on the existence of a particular function (so-called Lyapunov function) which provides information about the stability of the system.

To properly analyze the Lyapunov functions, some definitions are presented [SLO91, KHA02].

Definition 1.7: *Let $V\,(x,t): R^n \times R^+ \rightarrow R^+$a continuous function. V is proper and positive definite if:*

1. $\forall t \in R^+,\ \forall x \in R^n,\ x \neq 0: V(x,t) > 0;$
2. $\forall t \in R^+,\ \ V(x,t) = 0 \Rightarrow x = 0;$
3. $\forall t \in R^+,\ \lim_{\|x\|\rightarrow\infty} V(x,t) = \infty.$

Definition 1.8 (Lyapunov Function): *The* function *$V(x,t)$ of class C^1 is a local (respectively, global) large Lyapunov function for the system (1.6), if it is proper and positive definite and if there exists a neighborhood V_0 of the origin such that $\forall x \in V_0$ (respectively, $\forall x \in V_0 R^n$):*

$$\dot{V}(x,t) = \frac{\partial V(x,t)}{\partial t} + \left(\frac{\partial V(x,t)}{\partial x}\right) f(x(t),t) \leq 0$$

If $\dot{V}(x,t) < 0$, then V is called the strict Lyapunov function.

Theorem 1.2 (Lyapunov direct method): *If the system (1.6) admits a local large Lyapunov function (respectively, local strict Lyapunov function), then the origin is a locally stable (respectively, asymptotically stable) equilibrium point.*

This result can be validated globally, in the case $\forall x \in R^n$.

Theorem 1.3 (Exponential stability): *The origin of the system (1.6) is locally exponentially stable, if there are some constants* $\alpha, \beta, \gamma > 0, p \geq 0$ *and a function* $V(x,t) : \vartheta_0 \times R^+ \rightarrow R^+$ *of class* C^1 *such that,* $\forall x \in \vartheta_0$:

1. $\alpha\|x\|^p \leq V(x,t) \leq \beta\|x\|^p$
2. $\dot{V}(x,t) < -\gamma V(x,t)$

If the ball ϑ_0 *can be extended to the whole state-space, then the equilibrium at the origin is globally and exponentially stable.*

Remark 1.2: *By choosing a quadratic Lyapunov function as* $V(x,t) = x^T P x,\ with\ P = P^T > 0$, *the linear system* $\dot{x}(t) = Ax(t)$ *is globally exponentially stable at the origin, if and only if the matrix* P *is the solution of the matrix equation* $A^T P + PA = -Q$, *for a matrix* Q *positive definite.*

Remark 1.3:

1. *The criteria for stability and asymptotic stability presented in Theorems 1.2 and 1.3 are easy to utilize in the control design. However, they do not give any information on how to build this Lyapunov function. In reality, there does not exist any universal method for the construction of Lyapunov functions except for some particular classes of nonlinear systems (e.g. linear systems).*

2. *These theorems give sufficient conditions in the sense that if for a certain Lyapunov function* V, *the conditions on* $\dot{V}$ *are not satisfied, this does not imply that the considered system is unstable (perhaps with another function one can prove the system stability).*

3. *Unlike Lyapunov functions which guarantee the stability of the equilibrium points, there are functions, so-called Chetaev functions, that ensure the instability of the equilibrium points. It is worth noting that it is more difficult to demonstrate the instability rather than stability.*

1.4 Models of input nonlinearities

When dealing with real control and synchronization problems, the designer is unavoidably led to face some difficulties (e.g. non-smooth nonlinear constraints) intrinsically tied to real physical systems. In particular, actuators used in practice almost always contain some static nonlinearities (as dead-zone, saturation) or dynamic nonlinearities (as backlash, hysteresis), whose parameters are uncertain and can vary with time. Input nonlinearities (dead-zone, backlash, hysteresis and saturation) exist in mechanical, hydraulic, and magnetic systems and in electronic components and circuits (e.g. op amp). It is worthy pointing out that the presence of these input nonlinearities may result in poor control or synchronization performances or even instability of the closed-loop system. It is so more desirable to consider the effects of such nonlinearities in one way or another in the designing phase.

In this section, we describe some types of nonlinearities that can be found at the input of a nonlinear or chaotic system, namely: saturation, dead zone and/or nonlinearities of the sector. They are generally due to actuators (for electro-mechanical systems) or to some electronic components (for electronic systems).

a) Saturation

Saturation, as a static and hard nonsmooth nonlinearity, is always a potential problem for actuators of control systems as all actuators do saturate at some level. Actuator saturation affects the transient performance and even leads to system instability. Discounting their existence may lead to serious performance degradation and even instability in some cases.

The saturation model (shown in Figure 1.2) with an input v_i and an output $u_i$is defined as follows:

$$u_i = Sat_i(v_i) = \begin{cases} v_i\,, & for|v_i| \leq \bar{u}_i \\ \bar{u}_i sign(v_i), & for|v_i| > \bar{u}_i \end{cases} \tag{1.7}$$

where $\bar{u}_i$ is the saturation bound.

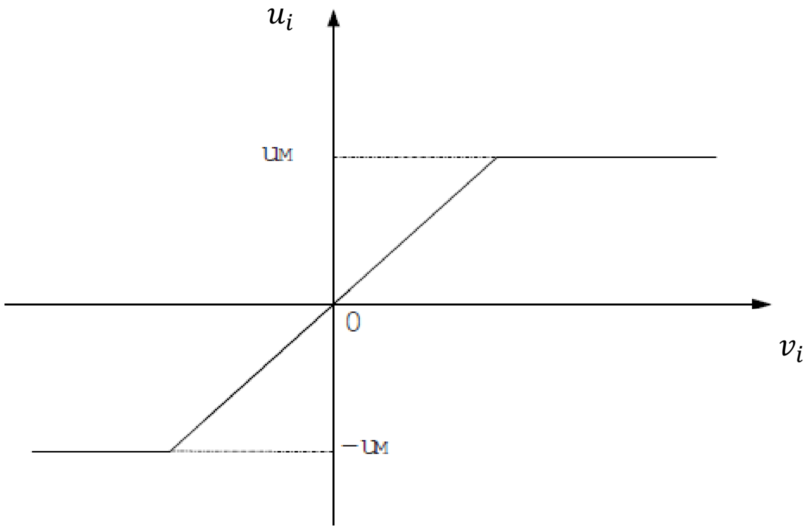


Figure 1.2: *Saturation.*

b) Dead-zone

Dead-zone is a static input-output relationship which, in an input values range, gives a zero signal at the output. The analytical expression of the dead-zone nonlinearity is [BOU08a]:

$$u_i = DZ_i(v_i) = \begin{cases} m_{ri}(v_i - b_{ri}), & for \quad v_i \geq b_{ri} \\ 0 & for \quad b_{li} < v_i < b_{ri} \\ m_{li}(v_i - b_{li}), & for \quad v_i \leq b_{li} \end{cases} \tag{1.8}$$

where v_i and u_i are the input and output of the dead-zone model, respectively.$b_{ri} > 0$, $b_{li} < 0$ and $m_{ri} > 0$, $m_{li} > 0$ are respectively the parameters and slopes of the dead zone model (shown in Figure 1.3).

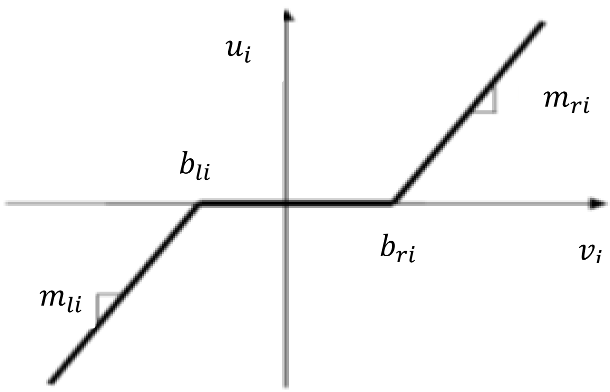


Figure 1.3: *Dead-zone.*

c) Dead zone with sector nonlinearities

This model is mathematically defined as follows [BOU08b, SHY05, BOU11, BOU14b]:

$$\varphi_i(u_i) = \begin{cases} \varphi_{i+}(u_i)(u_i - u_{i+}), & u_i > u_{i+} \\ 0, & -u_{i-} \le u_i \le u_{i+} \\ \varphi_{i-}(u_i)(u_i + u_{i-}), & u_i < -u_{i-} \end{cases} \tag{1.9}$$

where $\varphi_{i+}(u_i) > 0$ and $\varphi_{i-}(u_i) > 0$ are nonlinear functions of u_i, and $u_{i+} > 0$ and $u_{i-} > 0$. This model is given in Figure (1.4). The nonlinearity $\varphi_i(u_i)$ satisfies the following useful properties:

$$(u_i - u_{i+})\varphi_i(u_i) \ \ge m_{i+}^*(u_i - u_{i+})^2, \ \ u_i > u_{i+}$$

$$(u_i + u_{i-})\varphi_i(u_i) \ \ge m_{i-}^*(u_i + u_{i-})^2, \ \ u_i < -u_{i-}\,, \tag{1.10}$$

where m_{i+}^* and m_{i-}^* are strictly positive constants called *gain reduction tolerances* [SHY05, BOU11, BOU14b]. This model will be used in Chapter 3 to describe a class of input nonlinearities.

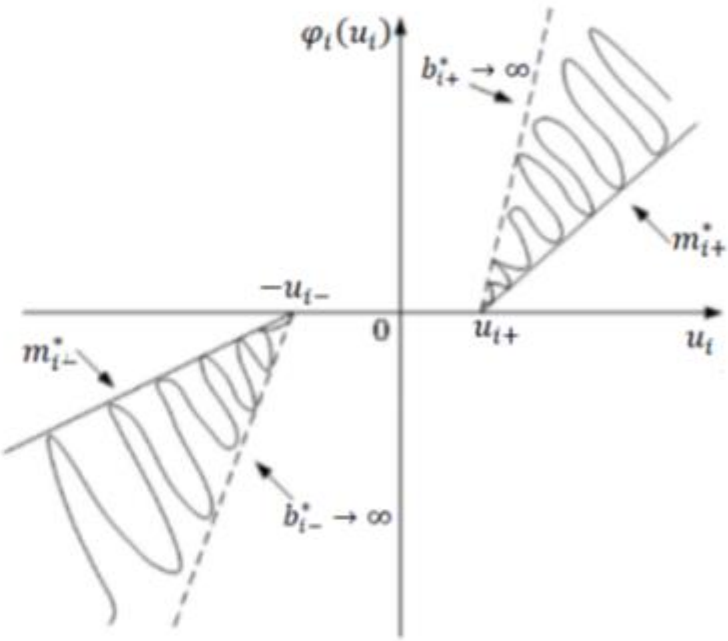


Figure 1.4: *Dead zone with sector nonlinearity*.

1.5 Fractional systems

1.5.1 Some special useful functions

In this section, we give some information on the Euler's gamma, the Mittag-Leffler and error functions which play a very important role in the theory of fractional calculus.

1.5.1.1 Gamma function

The Euler's gamma function, $\Gamma(z)$, is one of the basic functions of the fractional calculus . It can generalize the fractional $n!$ and allows n to take also non-integer and even complex values. That is why it is usually treated as the factorial of non-integer numbers. Its integral form can be written as:

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt \tag{1.11}$$

which converges in the right half of the complex plane $\mathrm{Re}(z) > 0$.

One of the basic properties of this function $\Gamma(z)$ is that it satisfies the following recursive relation:

$$\Gamma(z+1) = z\Gamma(z) \tag{1.12}$$

which can be effortlessly demonstrated just by integrating by parts

$$\Gamma(z+1) = \int_0^\infty e^{-t}\, t^z dt$$

$$= [-e^{-t}t^z]_0^\infty + \int_0^\infty ze^{-t}\, t^{z-1}dt$$

$$= \int_0^\infty ze^{-t}\, t^{z-1}dt = z\Gamma(z) \tag{1.13}$$

There are also some useful properties for this function, namely: $\Gamma(1) = 1$; $\Gamma(1/2) = \sqrt{\pi}$; $\Gamma(n+1) = n!\,(for\;\; n = 0,1,2,\dots)$.

Similarly to integer-order calculus, the fractional order derivative $\frac{d^\alpha}{dy^\alpha}$ of a variable with a power as the fractional-order is a constant:

$$\frac{d^\alpha}{dy^\alpha} y^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\alpha+1)} y^{\alpha-\alpha} = \Gamma(\alpha+1)$$

The alternative definition for the gamma function (1.11) is

$$\Gamma(x) = \lim_{n\to 0} \frac{n! n^x}{x(x+1)(x+2)\dots(x+n)} \tag{1.14}$$

which is equivalent to the previous integral version for $x > 0$.

This definition permits to use negative value of x.One can notice that this function, for $x \leq 0$ (or for $\mathrm{Re}(z) \leq 0$), is discontinuous. This gamma function plays an important role in the calculation of the Laplace transformation.

1.5.1.2 Mittag-Leffler function

The Mittag-Leffler (ML) function can be seen as a generalization of the usual exponential function which plays an essential role in the solution of fractional-order differential equations just like the exponential function does in the integer-order (ordinary) differential equations. It has four forms, and the most used forms are the 1-parameter and 2-parameter representations [LI01, POD99, LI09]:

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)} \qquad (\alpha > 0) \tag{1.15}$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)} \qquad (\alpha > 0, \beta > 0) \tag{1.16}$$

Below, some special forms of ML function are presented:

For $\beta = 1,\ \alpha > 0,\ E_{\alpha,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)} = E_\alpha(x)$

For $\beta = \alpha = 1,\quad E_{1,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$

For $\beta = 2,\ \alpha = 1,\ E_{1,2}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)} = \frac{1}{x}\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} = \frac{e^x-1}{x}$

For $\beta = 3,\ \alpha = 1,\ E_{1,3}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{x^k}{(k+2)!} = \frac{1}{x^2}\sum_{k=0}^{\infty} \frac{x^{k+2}}{(k+2)!} = \frac{e^x-x-1}{x^2}$

For $\beta = 1,\ \alpha = 0.5,\ \ E_{1,3}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(0.5k+1)} = e^{x^2}\mathrm{erfc}(-x)$

where the function $erfc$ will be defined below.

For $\beta = 1,\ \alpha = 2,\ \ E_{1,3}(x^2) = \sum_{k=0}^{\infty} \frac{x^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = \cosh(x).$

For $\beta = 2,\ \alpha = 2,\ \ E_{1,3}(x^2) = \sum_{k=0}^{\infty} \frac{x^{2k}}{\Gamma(2k+2)} = \frac{1}{x}\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \frac{Sinh(x)}{x}.$

1.5.1.3 Error function

The error function is a special function of the "*S*" shape, and is defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du, \qquad -\infty < x < \infty \tag{1.17}$$

The error function has the following properties,

$$erf(0) = 0$$
$$erf(\infty) = 1$$
$$erf(x) + erfc(x) = 1,$$

where $erfc(x)$ is the so-called complementary error function.

1.5.2 Fractional-order operators and their useful properties

Up to now, there exist more than 10 types of (operator) definitions for fractional-order integrals and derivations. In this section, we will show only the most important and used operators (or definitions), namely Grünwald-Letnikov (GL), Riemann-Liouville (RL) and Caputo (C) definitions.

Recall that the fractional calculus is a generalization of integration and differentiation to non-integer order operator ${}_aD_t^{\alpha}$, where a and t denote the limits of the operation and α denotes the fractional order such that [DAV36]:

$${}_aD_t^{\alpha} = \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}}, & Re(\alpha) > 0 \\ 1, & Re(\alpha) = 0 \\ \int_a^t (d\tau)^{-\alpha}, & Re(\alpha) < 0 \end{cases} \tag{1.18}$$

where it is generally assumed that $\alpha \in R$, but it may be also a complex number. In this work, we consider only the former case [DIE04, DIE02].

1.5.2.1 Definition of Grünwald-Letnikov

The GL derivative of a continuous function can be obtained intuitively from the definitions of the usual derivative (i.e. the derivative of integer order):

$$f^{(n)}(t) = \frac{d^n}{dt^n} f(t) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t-kh) \tag{1.19}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}$ is the usual binomial coefficients. A generalization of the backward difference by allowing the derivative order to be an arbitrary positive real was proposed by Grünwald-Letnikov

$${}_{a}^{GL}D_t^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh), \alpha > 0 \tag{1.20}$$

where $\left[\frac{t-a}{h}\right]$ denotes the integer part and $\binom{\alpha}{j}$ represents binomial coefficients generalized to real numbers, namely [OLD74, POD99]:

$$\binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j+1)\Gamma(j+1)} \tag{1.21}$$

It is worth noting that the GL derivative can be equal to the Riemann-Liouville derivative under the assumption of continuity and a sufficient number of continuous derivatives of $f(t)$. Generally, the GL derivative has been extensively employed for obtaining the numerical solution of the fractional differential equations because of its implementation convenience. For the numerical approximation, the following useful formula can be used.

$${}_{a}^{GL}D_t^{\alpha} f(t) \approx \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} \frac{(-1)^j \Gamma(\alpha+1)}{\Gamma(\alpha-j+1)\Gamma(j+1)} f(t-jh), \tag{1.22}$$

where $[p]$ denotes the integer part of p.

1.5.2.2 Definition of Riemann-Liouville

1.5.2.2.1 RL fractional integral

According to RL approach, the notation of fractional integral of order α $(\alpha > 0)$ is a natural consequence of the well known formula (usually attributed to Cauchy) that reduces the calculation of the $n-$fold integral of a function $f(t)$ to a single integral of convolution type [POD99].

Then, one defines the $\alpha th - order$ RL fractional integration of a function $f(t)$ as follows [POD99]:

$$J_0^\alpha f(t) = {}_0^{RL}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \ \alpha > 0 \tag{1.23}$$

where $\Gamma(\alpha)$ is the so-called Gamma function. When the initial integral limit changes from 0 to any arbitrary point a, this definition can be generalized to that of Weyl [POD99]:

$$J_a^\alpha f(t) = {}_a^{RL}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > a, \ \alpha > 0 \tag{1.24}$$

Example 1.1: Let $f(t) = (t-a)^\beta$ for a fixed $\beta > -1$ and $\alpha > 0$, one has:

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-a)^\beta (t-\tau)^{\alpha-1} d\tau$$

Using the substitution $\tau = a + s(t-a)$ and the Beta function, one gets:

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} (t-a)^{\beta+\alpha} \int_a^1 s^\beta (1-s)^{\alpha-1} ds = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t-a)^{\beta+\alpha}$$

For $\alpha = 1$ and $\beta = n \in N$, the usual result is so recovered, namely

$$J_a^1 (t-a)^n = \frac{\Gamma(n+1)}{\Gamma(n+2)} (t-a)^{n+1} = \frac{n!}{(n+1)!} (t-a)^{n+1} = \frac{(t-a)^{n+1}}{n+1}$$

Some basic properties

a) If $f(t)$ is a continuous function for $t \geq a$ then, we have

$$\lim_{\alpha \to 0} J_a^\alpha f(t) = f(t)$$

So one can put

$$J_a^0 f(t) = f(t) \tag{1.25}$$

b) Let $f(t)$ be a continuous function for $t \geq a$ then, we get

$$J_a^\alpha(J_a^\beta f(t)) = J_a^{\alpha+\beta} f(t) \text{ with } \beta, \alpha > 0 \tag{1.26}$$

The proof of this property is straight forward.

c) If $F(s)$ is the Laplace transform of the function $f(t)$, then the Laplace transform of the RL fractional integral, $J_a^\alpha f(t)$, is given by:

$$\boldsymbol{LT}\{J_a^\alpha f(t)\} = \frac{F(s)}{s^\alpha} \tag{1.27}$$

The proof of this property is straightforward, as the RL fractional integration (1.23) can be seen as a convolution of the two functions, namely $\Phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $f(t)$, with $\alpha > 0$ and $\Phi_\alpha(t) = 0$ for $t \leq 0$.

1.5.2.2.2 RL fractional derivative

The RL definition of fractional derivative of order α is based on that of the fractional-order integral and integer-order derivative [POD99, LI01, LI09, MAT96]:

$${}_a^{RL}D_t^\alpha = D^m J_a^{m-\alpha} f \tag{1.28}$$

where $\alpha > 0,$ and $m = \lceil\alpha\rceil$ (the smallest integer that exceeds α).

Equivalently, one has

$${}_a^{RL}D_t^\alpha = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau, & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases} \tag{1.29}$$

Remark. 1.4:

1- For $\alpha = 0$, we set ${}_a^{RL}D_t^\alpha = I$, the identity operator, and whenever $\alpha \in N$ the new operator ${}_a^{RL}D_t^\alpha$ coincides with the classical differential operator D^α.

2- The RL fractional operator is not local.

Example 1.2: Let $f(t) = c$, where c is a real constant, then:

$${}_a^{RL}D_t^\alpha = D^m J_a^{m-\alpha} f(t)$$

$$= \frac{c}{\Gamma(m-\alpha)} D^m \left(\int_a^1 (t-s)^{m-\alpha-1} ds\right)$$

$$= \frac{c(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \tag{1.30}$$

Example 1.3: Consider $f(t) = (t-a)^{\beta}$ for a fixed $\beta > -1$ and $\alpha > 0$. Then, in view of Example 1.2, one has

$${}_a^{RL}D_t^{\alpha} f(t) = D^m J_a^{m-\alpha} f(t)$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(m-\alpha+\beta+1)} D^m (x-t)^{m-\alpha+\beta} \tag{1.31}$$

One can identify two cases, namely,

- $if(\alpha-\beta) \in N,$ then the right-hand side is the *m-th* derivative of a polynomial of degree $m-(\alpha-\beta),$ this implies that: ${}_a^{RL}D_t^{\alpha} f(t) = 0.$
- But $if(\alpha-\beta) \notin N,$ then ${}_a^{RL}D_t^{\alpha} f(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-s)^{\beta-\alpha}.$

Some useful properties

a) The law of exponents: In general, unlike the RL fractional integration, the rule of composition for RL fractional derivatives does not hold, i.e. ${}_a^{RL}D_t^{\alpha}\left({}_a^{RL}D_t^{\beta}(f(t))\right) \neq {}_a^{RL}D_t^{\alpha+\beta}(f(t))$

Example 1.4: Consider $f(t) = t^{1/2}, \alpha = \frac{1}{2}$ and $\beta = \frac{3}{2},$ then

$${}_a^{RL}D_t^{\alpha}\big(f(t)\big) = \frac{1}{2}\sqrt{\pi},$$

$${}_a^{RL}D_t^{\beta}\big(f(t)\big) = 0,$$

$${}_a^{RL}D_t^{\alpha}\left({}_a^{RL}D_t^{\beta}(f(t))\right) = 0$$

$${}_a^{RL}D_t^{\beta}({}_a^{RL}D_t^{\alpha}(f(t))) = -\frac{1}{4}t^{-3/2}$$

$${}_a^{RL}D_t^{\alpha+\beta}\big(f(t)\big) = -\frac{1}{4}t^{-3/2}$$

Clearly, in this example, one can see that:

$$ {}_{a}^{RL}D_t^{\alpha}\left({}_{a}^{RL}D_t^{\beta}(f(t))\right) \neq {}_{a}^{RL}D_t^{\alpha+\beta}\left(f(t)\right), \ \ and $$

$$ {}_{a}^{RL}D_t^{\alpha}\left({}_{a}^{RL}D_t^{\beta}(f(t))\right) \neq {}_{a}^{RL}D_t^{\beta}({}_{a}^{RL}D_t^{\alpha}(f(t))) $$

In the following cases, one shall state, precisely, some conditions under which the law of exponents holds:

b) Composition with integer-order derivatives:

Case1: Let us consider the *n-th* derivative of the RL fractional derivative of real order α, one has

$$ D^n({}_{a}^{RL}D_t^{\alpha}(f(t))) = \frac{1}{\Gamma(m-\alpha)} D^{n+m}\left(\int_a^t (t-s)^{m-\alpha-1} f(s)ds\right) $$

$$ = \frac{1}{\Gamma(n+m-(n+\alpha))} D^{n+m}\left(\int_a^t (t-s)^{n+m-(n+\alpha)-1} f(s)ds\right) $$

$$ = {}_{a}^{RL}D_t^{n+\alpha} \tag{1.32} $$

Case2: To consider the RL fractional derivative of the *n-th* integer derivative, we recall the following relationships

$$ {}_{a}D_t^{-n} f^{(n)}(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f^{(n)}(s)ds $$

$$ = f(t) - \sum_{j=1}^{n-1} \frac{f^{(j)}(a)(x-a)^j}{\Gamma(j+1)} \tag{1.33} $$

and

$$ {}_{a}^{RL}D_t^{\alpha} g(t) = {}_{a}^{RL}D_t^{\alpha+n}\left({}_{a}D_t^{-n} g(t)\right) $$

Using the previous relation, one gets

$$ {}_{a}^{RL}D_t^{\alpha}(f^{(n)}(t)) = {}_{a}^{RL}D_t^{\alpha+n}\left({}_{a}D_t^{-n} f^{(n)}(t)\right) $$

$$= {}_{a}^{RL}D_t^{\alpha+n}\left(f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)}\right)$$

$$= {}_{a}^{RL}D_t^{\alpha+n}f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(a)(t-a)^{j-n-\alpha}}{\Gamma(j-n-\alpha+1)} \tag{1.34}$$

From this result, one can see that the RL fractional derivative operator ${}_{a}^{RL}D_t^{\alpha}$ commutes with the integer operator D^n only if $f^{(j)}$ vanishes in the lower terminal a for all $j = 0,1,2,\dots,n-1$.

c) Composition with fractional-order derivatives:

Consider now the composition of two fractional RL operators ${}_{a}^{RL}D_t^{\alpha}$ and ${}_{a}^{RL}D_t^{\beta}$, one sets $m = \lceil\alpha\rceil$ and $n = \lceil\beta\rceil$ then:

$${}_{a}^{RL}D_t^{\alpha}\left({}_{a}^{RL}D_t^{\beta}f(t)\right) = D^m\left({}_{a}^{RL}D_t^{-(m-\alpha)}\left({}_{a}^{RL}D_t^{\beta}f(t)\right)\right)$$

$$= D^m\left({}_{a}^{RL}D_t^{\alpha+\beta-m}f(t) - \sum_{j=1}^{n}\left[{}_{a}^{RL}D_t^{\beta-j}f(t)\right]_{t=a}\frac{(t-a)^{m-\alpha-j}}{\Gamma(m-\alpha-j+1)}\right)$$

$$= {}_{a}^{RL}D_t^{\alpha+\beta}f(t) - \sum_{j=1}^{n}\left[{}_{a}^{RL}D_t^{\beta-j}f(t)\right]_{t=a}\frac{(t-a)^{-\alpha-j}}{\Gamma(-\alpha-j+1)} \tag{1.35}$$

And

$${}_{a}^{RL}D_t^{\beta}\left({}_{a}^{RL}D_t^{\alpha}f(t)\right) = {}_{a}^{RL}D_t^{\alpha+\beta}f(t) - \sum_{j=1}^{m}\left[{}_{a}^{RL}D_t^{\alpha-j}f(t)\right]_{t=a}\frac{(t-a)^{-\beta-j}}{\Gamma(-\beta-j+1)} \tag{1.36}$$

From this relationship, one can deduces that in general case the RL fractional operators ${}_{a}^{RL}D_t^{\alpha}$ and ${}_{a}^{RL}D_t^{\beta}$ dot not commute, except where $\alpha = \beta$. When $\alpha \neq \beta$, it commutes only if both sums in the right-hand sides (1.35) and (1.36) vanish, that is if $f^{(j)}(a) = 0$, for all $j = 0,1,2,\dots,\max(n-1, m-1)$.

d) The Laplace transform:

The Laplace transform of the operator ${}_{0}^{RL}D_t^{\alpha}f(t)$is

$$\boldsymbol{LT}\{{}_{0}^{RL}D_t^{\alpha}f(t)\} = s^{\alpha}F(s) - \sum_{j=1}^{n}s^k\left[{}_{0}^{RL}D_t^{\alpha-k-1}f(t)\right]_{t=0} \tag{1.37}$$

The practical application of this Laplace transform is limited by the absence of physical interpretation of the limit values of fractional derivative at the lower terminal$t = 0$.

e) Link to the GL method

Let us assume that the derivatives $f^{(k)}(t)$, for $k = 1,2,\dots,m$, are all continuous in the interval $[a,\ T]$, where $m = \lceil\alpha\rceil$, then we can rewrite (1.20) as follows:

$$ {}_{0}^{GL}D_t^{\alpha} f(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds, \tag{1.38}$$

One can rewrite also the right hand of (1.38) as

$$\frac{d^m}{dt^m}\left\{\sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{m-\alpha+k}}{\Gamma(m-\alpha+k+1)} + \frac{1}{\Gamma(2m-\alpha)} \int_a^t (t-s)^{2m-\alpha-1} f^{(m)}(s)ds\right\} \tag{1.39}$$

And after m integration by part, one gets the relation of the RL derivative:

$$\frac{d^m}{dt^m}\left\{\frac{1}{\Gamma(m-\alpha)}\int_a^t (t-s)^{m-\alpha-1} f(t)dt\right\} = \frac{d^m}{dt^m}\left\{{}_{0}^{RL}D_t^{-(m-\alpha)} f(t)\right\}$$

$$= {}_{0}^{RL}D_t^{\alpha} f(t) \tag{1.40}$$

So, under the aforementioned assumptions one has

$${}_{0}^{RL}D_t^{\alpha} f(t) = {}_{0}^{GL}D_t^{\alpha} f(t) \tag{1.41}$$

So, the properties that one has already seen in the RL definition for the fractional derivative stay valid for the GL definition, under some appropriate assumptions.

Remark 1.5: *The RL definition of the fractional integral and derivative is suitable to find the analytic solution for some quite simple functions. Conversely, the GL definition is generally adopted for numerical computations.*

1.5.2.3 Caputo fractional derivative

As it is already mentioned in the previous section, the Laplace transform of the RL fractional derivative depends on the limit values of fractional derivative at the lower terminal $t = 0$ (or $t = a$ in the general case). Therefore, the initial conditions being necessary for the solution of fractional order differential equations are themselves of fractional-order. In the physical world, these properties of RL definition present a serious problem (as these initial conditions do not have any physical interpretations). Moreover, the fractional derivative of a constant is not a zero.

The Italian mathematician Caputo suggested a solution to this problem in 1967. By presenting a new definition, in which he attempts to find a link between what is possible and what is practical. The key of this minor modification of the concept of fractional derivative is to allow the use of integer order initial conditions in the solution of fractional differential equations. In addition, with this Caputo definition, the fractional order derivative of a constant is 0, as we will see below. In order to realize this objective, Caputo suggests the same operations as in RL definition but in the reverse order, namely to obtain the RL definition derivative of order $\alpha > 0$, of a function $f(t)$, first one shall integrate it by the fractional order $m-\alpha$, after that, one shall differentiate the resultant function by the integer order m. While in the Caputo method, first one shall differentiate $f(t)$ by the integer order m, then one shall integrate $f^{(m)}$ by the fractional order $m-\alpha$.

Definition 1.9: *Let $\alpha \geq 0$, and $m = \lceil \alpha \rceil$. Then, one defines the Caputo's fractional operator ${}_{0}^{C}D_t^{\alpha} f(t)$ by*

$$
\begin{aligned}
{}_{a}^{C}D_t^{\alpha} &= J_a^{m-\alpha} D^m f, \\
&= \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-\tau)^{m-\alpha-1} f^{(m)} d\tau,
\end{aligned}
\qquad (1.42)
$$

whenever $f^{(m)} \in L_1[a,b]$.

Remark 1.6: *Just as in the case of the RL operators, one can see that the Caputo derivative is not also local (i.e. it is with memory).*

Some useful properties

a) Linearity

Let $\nu, \mu \in R$. From the definition of the Caputo's derivative it follows directly that

$$
{}_{a}^{C}D_t^{\alpha}\big(\nu f(t) + \mu g(t)\big) = \nu\, {}_{a}^{C}D_t^{\alpha} f(t) + \mu\, {}_{a}^{C}D_t^{\alpha} g(t), \qquad (1.43)
$$

Notably, ${}_{a}^{C}D_t^{\alpha} f(t) = {}_{a}^{C}D_t^{\alpha}(f(t)+0) = {}_{a}^{C}D_t^{\alpha} f(t) + {}_{a}^{C}D_t^{\alpha} 0$, then one has ${}_{a}^{C}D_t^{\alpha} 0 = 0$.

b) Interpolation

When $\alpha \in N$, one has $\alpha = m$, then the Definition 1.9 implies that

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = J_{a}^{m-m}\left(\frac{d^{m}}{dt^{m}} f(t)\right) = J_{a}^{0}\left(\frac{d^{m}}{dt^{m}} f(t)\right) = \frac{d^{m}}{dt^{m}} f(t) \tag{1.44} $$

This means that: likewise to the RL and GL definitions, the Caputo approach offers also an interpolation between the integer-order derivatives.

c) Composition

Let $n \in R$ and $m = \lceil \alpha \rceil$, one has

$$ {}_{a}^{C}D_{t}^{\alpha}\left({}_{a}^{C}D_{t}^{n} f(t)\right) = {}_{a}^{C}D_{t}^{\alpha+n} f(t) \tag{1.45} $$

Namely,

$$ \begin{aligned} {}_{a}^{C}D_{t}^{\alpha}\left({}_{a}^{C}D_{t}^{n} f(t)\right) &= {}_{a}D_{t}^{-(m-\alpha)}\ {}_{a}D_{t}^{m}\left({}_{a}D_{t}^{n} f(t)\right) \\ &= {}_{a}D_{t}^{-(m-\alpha)}\ {}_{a}D_{t}^{m+n} f(t) \\ &= {}_{a}D_{t}^{-(m+n-(\alpha+n))}\ {}_{a}D_{t}^{m+n} f(t) \\ &= {}_{a}^{C}D_{t}^{\alpha+n} f(t) \end{aligned} $$

d) Laplace transform

We can write the Caputo’s derivative as follows:

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = J^{m-\alpha} g(t), \tag{1.46} $$

with $g(t) = f^{(m)}(t), \quad m = \lceil \alpha \rceil$.

Using the formula for the Laplace transform of RL fractional integral (1.27), and that of integer-order derivative (1.3) one obtains:

$$ \boldsymbol{TL}\{{}_{a}^{C}D_{t}^{\alpha} f(t)\}(s) = s^{-(m-\alpha)} G(s) = s^{\alpha} F(s) - \sum_{j=0}^{m-1} s^{\alpha-j-1} f^{(j)}(0) \tag{1.47} $$

Unlike the RL fractional derivative, clearly from this relation, the Laplace transform of the Caputo's fractional derivative involves only the values of $f(x)$ and its integer-order derivatives at $x = 0$. Therefore, a certain physical interpretation exists for this Caputo derivative definition, hence one thinks that this Caputo derivative definition can be valuable for resolving applied problems.

e) Link to the RL method

Let $\alpha > 0$ and f a function having continuous derivatives $f^{(k)}(t)$, for $k = 1,2,\dots,m,$ in the interval $[a,\ T]$, where $m = \lceil \alpha \rceil$, then from (1.40), one has:

$$ {}_{0}^{RL}D_t^{\alpha} f(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds, $$

$$ = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + {}_{0}^{C}D_t^{\alpha} f(t) \tag{1.48} $$

Clearly, if $f^{(k)}(a) = 0,\ (k = 1,2,\dots,m-1)$ then

$$ {}_{0}^{RL}D_t^{\alpha} f(t) = {}_{0}^{C}D_t^{\alpha} f(t) \tag{1.49} $$

1.5.3 Some useful lemmas and theorems

In this part, we briefly review some basic lemmas and theorems related to the stability of fractional systems.

Definition 1.10 [LI01]: *The solution of the system* $D^{\alpha} x = g(x,t)$ *is ML stable if*

$$ \|x(t)\| \le \left\{ k((x(t_0)) E_{\alpha}(-\lambda (t-t_0)^{\alpha}) \right\}^{b} \tag{1.50} $$

where t_0 *is the initial time,* $\lambda > 0,\ b > 0$, *and* $k(x) \ge 0$ *is a locally Lipschitz function and with* $k(0) = 0$.

Theorem 1.4 [LI09]: *Let* $x = 0$ *be an equilibrium point for the following non-autonomous fractional-order system*

$$ D^{\alpha} x = g(x,t) \tag{1.51} $$

where $g(x,t)$ *is a Lipschitzian function and* $\alpha \in (0,\ 1)$. *Suppose that there exists a candidate Lyapunov function* $V(x,t)$*and class-K functions* β_1, β_2 and β_3 *fulfilling the following conditions:*

$$\beta_1(\|x\|) \le V(x,t) \le \beta_2(\|x\|) \tag{1.52}$$

$$D^{\beta}V(x,t) \le -\beta_3(\|x\|) \tag{1.53}$$

with $\beta \in (0,\ 1)$. *Hence, the equilibrium point of (1.51) is asymptotically (or ML) stable.*

Theorem.1.5 [MAT96, AGH14]: *Consider the following autonomous linear fractional-order system* $D^{\alpha}x = Ax$, $x(0) = \mathrm{x}_0$ with $0 < \alpha < 1$, $x \in R^n$ *and* $A \in R^{n\times n}$. *This system is asymptotically (or ML)-stable iff* $|\arg(eig\ A)| > \alpha\pi/2$. *Then, each element of the states decays toward 0 like* $t^{-\alpha}$.

Lemma 1.1 [LI15**]:** *For the following fractional-order dynamics*

$$D^{\alpha}V(t) \le -\eta V(t) + \vartheta \tag{1.54}$$

where $V(t)$ *is a positive define function with* $V(0) = 0$, η *and* ϑ *are some strictly positive constants, there is a time* $t_1 > 0$, *such that for all* $t \in (t_1, \infty)$, *one has*

$$V(t) \le \frac{2\vartheta}{\eta} \tag{1.55}$$

Lemma 1.2 [AGU14**]:** *If* $x(t)$ *is a derivable function, for the Caputo's derivative operator, the following relation holds:*

$$\frac{1}{2}D^{\alpha}(x^T(t)x(t)) \le x^T(t)D^{\alpha}x(t) \quad (\text{for } 0 < \alpha \le 1) \tag{1.56}$$

Lemma 1.3 [MAT96**]:** *Let us consider a fractional linear system with inner dimension n, given by the following form of state space*:

$$\begin{cases} {}_{t_0}D_t^q x = Ax + Bu \\ y = Cx \end{cases}, x(0) = x_0 \tag{1.57}$$

with $0 < q < 1$, $x \in R^n$, $y \in R^p$ *and* $A \in R^{n\times n}$. *One assumes that the triplet (A,B,C) is minimal. This above system is bounded-input bounded-output (BIBO) stable iff* $\left|arg\left(eig(A)\right)\right| > \frac{q\pi}{2}$. *When this linear system is externally stable, each part of its impulse response behaves as* t^{-1-q} *at infinity.*

Remark 1.7: *Lemma 1.3 indicates the main difference between ordinary differential equations and fractional order equations. It is worth noting that the stability of the fractional order differential equations is not exponential. In fact, for a linear fractional order system, when all the eigenvalues are situated outside the sector defined by* $|arg(eig(A))| \leq \frac{q\pi}{2}$, *the response to the conditions decays as* t^{-q} *but the impulse response decays as* t^{-1-q}.*This behavior does not take place in the linear system described by ordinary differential systems* ***[****POD99****]****.*

1.6 Chaotic systems

1.6.1 Definition of chaos

It is very difficult to define what a chaotic system is, as there does not exist an exact and precise definition. Usually, from a practical point of view, it can be defined as bounded steady state behavior which does not fall into the categories of the other three steady state behaviors, i.e. equilibrium point, periodic solutions and quasi-periodic solutions. Chaos embodies three important characterizes:

1. *Aperiodic time-asymptotic behavior*: This implies the existence of phase-space trajectories which do not settle down to fixed points or periodic orbits. For practical reasons, we also require the trajectories to be *bounded*: *i.e.,* they should not go off to infinity.

2. *Deterministic*: This implies that the equations of motion of the system possess no random inputs. In other words, the irregular behavior of the system arises from non-linear dynamics and not from noisy driving forces.

3. *Sensitive dependence on initial conditions*: This implies that the initially close trajectories in phase-space separate exponentially fast in time: *i.e.*, the system has a positive Lyapunov exponent.

There exist two important mathematical principles explaining the chaotic behavior, that of Li-Yorke [LI01] and that of Devaney [LI09].

1.6.1.1 Li-Yorke's Chaos

Li and Yorke has introduced the first mathematical definition of chaos [LI75]. They have established a very simple criterion, "*the presence of three periods involves chaos*". This criterion plays a very important role in the analysis of chaotic dynamical systems.

Consider the one-dimensional discrete dynamical system:

$$x(i+1) = f\big(x(i)\big),\ \ x \in \Re \ \text{ with } x(0) = x_0,\ \ i = 1,2,\ldots. \tag{1.58}$$

Li-Yorke Theorem: *Let I be an interval in R and $f: I \to I$ a continuous map. Assume that there is a point $a \in I$, for which the points $b = f(a), c = f^2(a)$ and $d = f^3(a)$ satisfy*

$$d \leq a < b < c \quad (\text{or } d \geq a > b > c) \tag{1.59}$$

then

1. *For every $k = 1,2,\ldots$ there is a k-periodic point of period (discrete equivalent of a periodic solution with length k,i.e. $f^k(x^*) = x^*$) in I.*
2. *There is an uncountable set $S \subset I$, containing no periodic points, which satisfying the following conditions:*
   1) *for every $x, y \in S$ with $x \neq y$.*

$$lim_{n\to\infty}\, sup|f^n(x) - f^n(y)| > 0 \tag{1.60}$$

$$lim_{n\to\infty}\, inf|f^n(x) - f^n(y)| = 0 \tag{1.61}$$

   2) *for each $x \in S$ and periodic point $y_{per} \in I$, with $x \neq y_{per}$*

$$lim_{n\to\infty}\, sup\big|f^n(x) - f^n(y_{per})\big| = 0 \tag{1.62}$$

This theorem can be generalized by assumption that $f: \mathcal{R} \to \mathcal{R}$ and that $f(I) \subset I$. In addition, the function f shouldsatisfy:

$$f(I) \cap I \neq \emptyset \tag{1.63}$$

So that it contains the well-defined points $a, b, c,$ and d.

1.6.1.2 Devaney's Chaos

Let S be a set in a topological space (typically, $\mathcal{R}$), and let f^m be the mth-order iteration of a map $f: S \longrightarrow S$, namely, $f^m := f(f^{m-1})$, $m = 1,2,\ldots$ with $f^0 = identity$.

Definition 1.11: *A point $\bar{x} \in S$ is called a periodic point of a period m (or m −period point) if $\bar{x} = f^m(\bar{x})$but $\bar{x} \neq f^k(\bar{x})$ for $1 \leq k < m$. If $m = 1$, that is $\bar{x} = f(\bar{x})$, then $\bar{x}$ is called a fixed point. The point $\bar{x}$ is called periodic, or is named a period point, if it is an m-periodic point for some $m \geq 1$.*

Definition 1.12 (Devaney Definition): *A map $f: S \longrightarrow S$ is chaotic if*

1) *the map f has sensitive dependence on initial conditions, in the sense that for all $x \in S$ and any neighborhood $N$of x in S, there exists $\delta > 0$ such that $|f^m(x) - f^m(y)| > \delta$ for some $y \in N$ and some $m \geq 0$;*
2) *the map f is topologically transitive, in the sense that for any pair of nonempty open subsets $U, V \subset S$, there exists an integer $m > 0$ such that $f^m(U) \cap V \neq \phi$;*
3) *the periodic points of the map f are dense in S.*

1.7 Synchronization of chaotic systems

Synchronization is a process of coupling of two or more chaotic dynamical systems, whereby they are found to acquire a tendency to follow closely related motion. Chaos synchronization in dynamical systems is actually a method of controlling chaos. Its principle is presented by the block diagram of the Figure 1.5 [ODE07]. Synchronization between two identical or non-identical systems (master system and slave system) can broadly be classified into the following types.

In the following sub-section, one will briefly present some methods of synchronization such as closed-loop synchronization, observer-based synchronization, and sliding mode method-based synchronization. Then, some main types of synchronization are discussed namely: complete synchronization, anti-synchronization, lag-synchronization, generalized function projective synchronization, and so on.

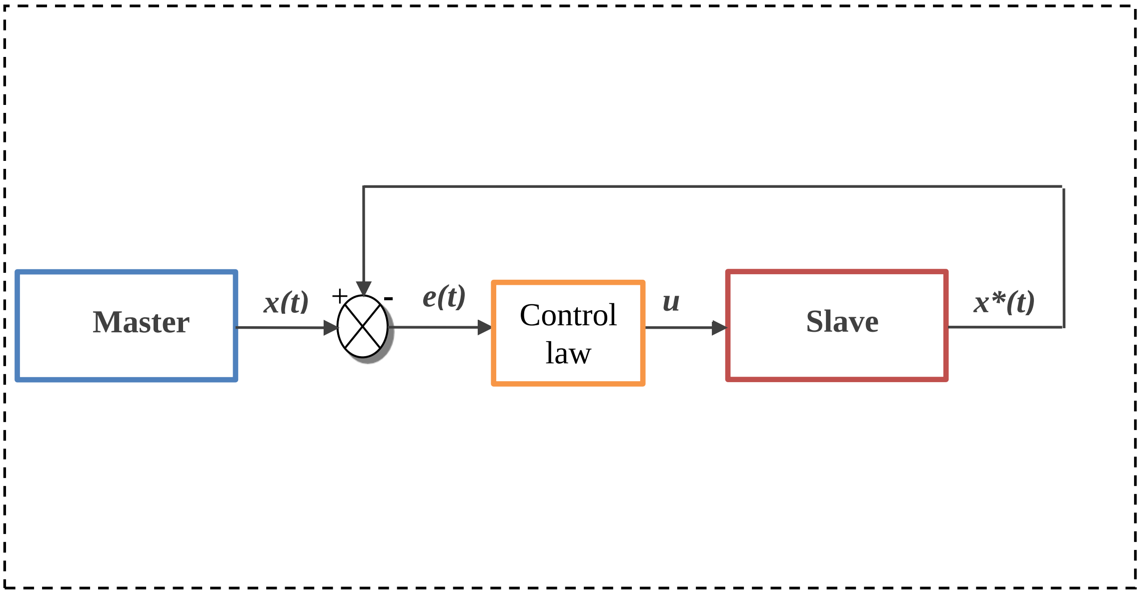


Figure 1.5: *Principle of chaos synchronization.*

1.7.1 Synchronization methods

1.7.1.1 Synchronization by the closed-loop

Closed loop-based synchronization is illustrated in Figure (1.6), where the synchronization error between the master system and the slave one is used to correct the slave behavior in order to realize an appropriate synchronization [ZHE06].

Assume that the transmitter (master system) is described by:

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases} \tag{1. 64}$$

and its corresponding receiver can be designed by:

$$\begin{cases} \dot{\hat{x}} = f(\hat{x}) + g(y - \hat{y}) \\ \hat{y} = h(\hat{x}) \end{cases} \tag{1.65}$$

where g is a function of the synchronization error $(y - \hat{y})$. This function is selected to ensure an adequate chaos synchronization between the transmitter system and the receiver one. In fact, in some cases, the receiver can be designed as a state observer for the master system.

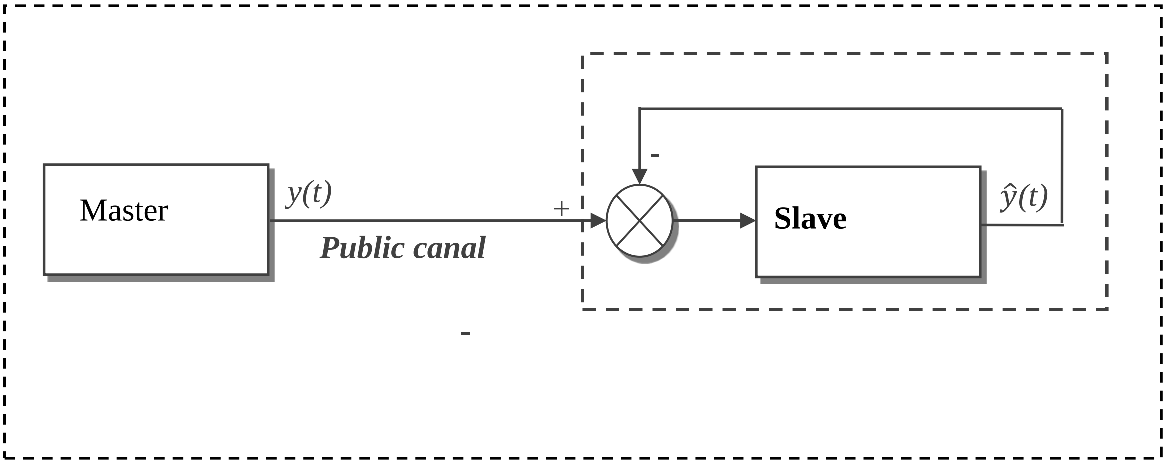


Figure 1.6: *Synchronization by closed-loop control*

1.7.1.2 Observer-based synchronization

The chaos synchronization can also be achieved by designing a state observer (as slave system). The models of chaotic systems can be generally separated into two main parts, namely: a linear part and a non-linear part.

Consider the master system [ZHE06]:

$$\begin{cases} \dot{x} = Ax + f(x) \\ y = Cx \end{cases} \tag{1.66}$$

with $x \in R^n$ and $y \in R$ represent respectively the master state vector and its output, A and C are two constant matrices with appropriate sizes, $f: R^n \to R^n$ is a real vector function, and the pair (C, A) is assumed to be observable.

Its corresponding slave system can be designed as a state observer:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + f(\hat{x}) + K(y - \hat{y}) \\ \hat{y} = C\hat{x} \end{cases} \tag{1.67}$$

with $\hat{x}$ the state vector associated to the slave system, $K \in R^n$ is the observer gain which should be adequately designed, and y corresponds to the slave output. The term $K(y - \hat{y})$ is generally named the observer correction term.

By defining the synchronization error as $e = x - \hat{x}$, we can get:

$$\dot{e} = (A - KC)e + f(x) - f(\hat{x}) \tag{1.68}$$

In this case, from (1.68), it is clear that the synchronization problem becomes that of the error (e) stabilization. If the function $f(x)$ is Lipchitz, and if one can find a suitable gain K which can make the matrix $A - KC$to be Hurwitz, then the synchronization between the transmitter and the receiver can be realized.

1.7.1.3 Synchronization based on sliding mode control

Consider two chaotic systems (master system and slave system, respectively) of dimension **[**IKH11**]**:

$$\dot{x}(t) = Ax(t) + f(x(t)) \tag{1.69}$$

and

$$\dot{y}(t) = Ay(t) + f\big(y(t)\big) + u(t) \tag{1.70}$$

where $A \in R^{n \times n}$ is a constant matrix,$f: R^n \rightarrow R^n$ is a continuous nonlinear function, and $u \in R^n$ is the control input.

The synchronization error can be defined as follows:

$$e(t) = y(t) - x(t) \tag{1.71}$$

Denote also

$$E_{xy} = f(y) - f(x) \tag{1.72}$$

The dynamics of the synchronization error can be written as follows:

$$\dot{e}(t) = Ae(t) + E_{xy} + u(t) \tag{1.73}$$

From (1.73), it is clear that the control law $u(t)$ can be designed as follows:

$$u(t) = Bv(t) - E_{xy} \tag{1.74}$$

where $B$is a constant vector which should be determined so that the pair (A, B) is controllable.

$$\dot{e}(t) = Ae(t) + Bv(t) \tag{1.75}$$

The design of the control law is often based on the method given by Hung and Gao [HUN93].

Let $S(e)$ be the sliding surface:

$$S(e) = Ce(t) \tag{1.76}$$

where C is a constant vector.

The sliding surface should satisfy:

$$S(e) = 0, \;\; \dot{S}(e) = 0 \tag{1.77}$$

Therefore, one can write the sliding mode surface dynamics as follows

$$\dot{S}(e) = C\big(Ae(t) + Bv(t)\big) = 0 \tag{1.78}$$

From (1.78), the equivalent control law can be determined as

$$v(t) = -(CB)^{-1}CAe(t\,) \tag{1.79}$$

By substituting (1.79) into (1.75), the dynamics of the synchronization error become:

$$\dot{e}(t) = [I - B(CB)^{-1}C]Ae(t) \tag{1.80}$$

To ensure an asymptotic stability of the dynamics (1.80), the matrix Cshould be chosen so that the real parts of the eigenvalues of $[I - B(CB)^{-1}C]A$ are all negative. In [YOU99], Young et al. have proposed a sliding-mode controller which allows having the following closed-loop dynamics:

$$\dot{s} = -qsgn(s) - ks \tag{1.81}$$

where $sgn(.)$ is the usual sign function, and $q > 0$ and $k > 0$ are design constants. The control law in this case should be designed as:

$$v(t) = -(CB)^{-1}[C(kI + A)e(t) + qsgn(s)] \tag{1.82}$$

To study the stability of the synchronization error dynamics, let us consider the following Lyapunov candidate function:

$$V = \frac{s^2}{2} \tag{1.83}$$

Its time derivative is

$$\dot{V} = -ks^2 - q|s| \tag{1.84}$$

The derivative is always negative as long as $s \neq 0$ and $q,\ k > 0$. Therefore, the signal s is stable and converges to zero in finite time.

1.7.2 Types of chaos synchronization

There are many types of synchronization, namely: complete synchronization, generalized synchronization, phase synchronization, lag-synchronization, projective synchronization and anticipatory synchronization.

Consider a master system represented by the following equations:

$$\dot{Y} = H_1(Y) \tag{1.85}$$

And its slave system:

$$\dot{X} = H_2(X, u) \tag{1.86}$$

where $X \in R^n$ *and* $Y \in R^n$ are the states of the master and slave systems respectively, $H_1(Y)$ *and* $H_2(X, u)$ are the continuous nonlinear functions, and $u \in R^n$ is the control input.

1) ***Complete synchronization.*** *The complete synchronization (CS) is carried out if:*

$$lim_{t\to\infty}\|X - Y\| = 0 \tag{1.87}$$

2) Anti-synchronization. *The anti-synchronization (AS) is realized if:*

$$lim_{t\to\infty}\|X+Y\|=0 \tag{1.88}$$

3) Lag-synchronization. This type is also called ***delayed synchronization***. It can be realized if $X(t) \approx Y(t-\tau)$, where τ is a very small positive real.

4) Generalized function projective Synchronization (GFPS). It can be considered as a generalization of the other types of synchronization (namely CS, AS,...). It manifests itself through a functional relationship between two coupled chaotic systems, the complexity of which gives rise to the difficulty of chaotic decoding. We define the GFPS error as follows:

$$E = AX - B(Y)Y \tag{1.89}$$

where $B(Y) = diag\big(b_1(Y), \dots, b_n(Y)\big)$ and $b_i(Y): R^n \to R$ are bounded differentiable functions, and A is a constant matrix with an appropriate dimension. In particular, one has:

- If $B(Y) = bI$ and $A = I$ with $b\epsilon R$, the problem of GFPS will be reduced to that of the projective synchronization (PS).
- If $b_1(Y) = b_2(Y) = \dots = b_n(Y)$ and $a_1 = a_2 = \dots = a_n = 1$, the GFPS will be simplified to function projective synchronization (FPS).
- If $B(Y) = diag(c_1, c_2, \dots, c_n)$ and $A = I$, a modified projective synchronization (MPS) will take place.
- In the case where $B = 0I$ and $A = I$, this synchronization problem becomes that of controlling the chaos.

1.8 Conclusion

In this chapter, some basic results on the fuzzy systems, fractional-order systems, and chaotic systems and its synchronization have been briefly reviewed. The fuzzy systems will be used in the following chapters to online approximate the unknown nonlinear functions. The chaotic systems under-consideration later will be assumed to be described by fractional–order model.

Chapter 2

Intelligent Fractional-order Control-Based Projective Synchronization for Chaotic Optical Systems

2.1 Introduction

Chaos synchronization of the (fractional-order or integer-order) chaotic systems has recently attracted widespread interest, because its potential applications in many fields of sciences and engineering including information processing, biological systems, chemical science, secure communications, control theory, parameter estimation from time series, optimization problems, and so on [CHE98, PYR92, MAY99, ZHA16, AZI05, BOU15a, BOU16a]. Nowadays, several control methods have been investigated to deal with chaos synchronization problem such as: feedback control method [PAN11], active control technique [BHA10], fractional PID control method [MAH10], adaptive impulsive control [XI14], optimized active control [BEH15], LMI-based linear control [FAI14], adaptive sliding mode control [YAN06].

Most of the above works generally focus on control and synchronization of fractional-order (FO) chaotic systems without disturbances and uncertainties (i.e. the models of the master-slave system is almost known), however, in practical situations, owing to the physical devices limitations and the unavoidable interferences (including noise, temperature, and so on),

uncertainties and disturbances are always exist. To robustly deal with the bounded uncertainties and disturbances, the sliding mode control methodology can be used. This effective approach has numerous attractive features, including finite-time and fast convergence, strong robustness with respect to the bounded unmodelled dynamics and external disturbances. On the other hand, recently, the fractional-order calculus has been incorporated in the sliding mode control methodologies, in order to search for better performances [EFE08a, SI09, PIS10, WAN12]. Also, in recent researches, the fuzzy systems are used in the sliding mode control in order to cope with the main problems related to the sliding mode control, including the high-gain authority and chattering. This hybridization can smoothen the sliding mode control in diverse ways, and can also successfully approximate online the model, uncertainties and disturbances being present in the system [BOU15a, BOU16a, LI14, LIN11c, LIN11b].

By using a fuzzy adaptive sliding mode control strategy, a generalized projective synchronization of fractional order chaotic systems has been proposed in [LI14]. In [LIN11b], a fuzzy adaptive control has been investigated for the synchronization of uncertain fractional-order chaotic delayed systems. In [LIN11c], a fuzzy adaptive controller has been constructed to realize an H_∞ synchronizing for uncertain fractional-order chaotic systems. Nevertheless, the fundamental results of [LIN11b, LIN11c] are already questionable, because the stability analysis has not been derived rigorously in mathematics, as stated in [TAV12, AGH12a]. In [BOU15a, BOU16a, BOU16b, BOU16c], some adaptive fuzzy controllers have been designed in a sliding mode frame-work for realizing an appropriate synchronization of fractional-order chaotic master-slave systems. In these synchronization approaches, the fuzzy systems have been employed to online estimate the unknown nonlinear functions. Although these schemes can guarantee the satisfactory performances, they already suffer from some intrinsic limitations, including: a) These schemes generally focus on asymptotic stability leading thereby to infinite-time synchronization. However, in practical situations, the finite-time synchronization is more helpful than infinite-time synchronization. b) The most previous (fractional-order or integer-order) sliding mode control laws suffer from the chattering phenomenon being caused by the sign function in control input. c) The number of adaptive parameters in these controllers is very important, which can make their practical implementation somewhat difficult.

Motivated by the above discussion, in this paper, one proposes two novel alternative control approaches, which use fractional dynamic sliding surfaces together with a fuzzy system to online model the uncertain nonlinear functions. The designed fractional intelligent sliding-mode controllers are applied for projective-synchronizing a class of uncertain fractional-order chaotic systems with bounded external disturbances. These controllers can ensure the existence of the sliding motion in the finite-time. A fractional Lyapunov stability theory is employed to bear out the stability of the corresponding closed-loop system. The validity of these proposed synchronization schemes are confirmed via some numerical simulation experiments. The main contributions of this work are as follows:

1) Design of two new fractional dynamic sliding manifolds for FO chaotic systems.

2) Utilization of the fuzzy systems to online model the uncertainties and the fractional-order variable structure control to robustly deal with the external disturbances and the fuzzy approximation errors.

3) The stability analysis of the corresponding closed-loop system is rigorously performed by means of a fractional Lyapunov theory.

4) One of those proposed control laws (i.e. the second control law) can significantly reduce the chattering effect.

5) The proposed controllers are very simple and have few fuzzy design parameters. So, they are of practical significant importance.

6) These controllers can ensure a projective synchronization in the finite-time.

7) The efficiency and validity of these investigated controllers are tested, in an appropriate numerical simulation frame-work, on two chaotic optical systems.

2. 2 Problem statement

Consider a class of fractional order chaotic models given by:

$$\begin{cases} D^{\alpha} x_i = x_{i+1}, \quad \text{for } i = 1,\dots,n-1 \\ D^{\alpha} x_n = f_m(x), \end{cases} \tag{2.1}$$

where D^{α} with $0 < \alpha < 1$, is the fractional derivative-order, $x = [x_1,\dots,x_n]^T \in R^n$ is the over-all pseudo-state vector of the master system, and $f_m(x) \in R$ is a continuous uncertain nonlinear function.

Let model (2.1) be considered as a master system, then its associated and controlled slave system can be given by

$$\begin{cases} D^{\alpha} y_i = y_{i+1}, & \text{for } i = 1,\ldots,n-1 \\ D^{\alpha} y_n = f_s(y) + d(y,t) + u(t) \end{cases} \tag{2.2}$$

with $y = [y_1,\ldots,y_n]^T \in R^n$ is the overall pseudo-state vector of this slave system, and $f_m(y) \in R$ is a continuous uncertain nonlinear function. In the rest of this chapter, D^{α} stands for RL derivative's operator.

The following mild assumptions are required in the sequel.

Assumption 2.1: *The disturbance is bounded as follows:*

$$|d(y,t)| \le \bar{d}_1 \tag{2.3}$$

where $\bar{d}_1$ *is an unknown positive constant.*

The projective synchronization (PS) error is quantified as follows:

$$e_i = y_i(t) - \lambda x_i(t) \tag{2.4}$$

where λ is a scaling factor.

Our objective consists to design two appropriate FO fuzzy sliding mode controllers for achieving an asymptotical PS between the master system (2.1) and its slave system (2.2).

Because the presence of unknown functions (as $f_m(y)$ and $f_s(y)$) and bounded external disturbances, in the subsequent section, one will use:

1) a fuzzy system to estimate the uncertain functions, and
2) a sliding mode control term to effectively deal with disturbances and fuzzy approximation errors (being due to the functions approximation by the fuzzy systems).

Remark 2.1: *It is worth noting that the RL derivative definition will used as a modeling operator in the rest of this chapter. From the RL definition and its properties (given in chapter1), one has:*

- $D^{\beta_i}\left(D_t^{-\beta_i}g(t)\right) = D^{\beta_i}\left(J_0^{\beta_i}g(t)\right) = g(t).$
- $D^{\beta_i}g_1(t) = g_2(t) \implies g_1(t) = D^{-\beta_i}g_2(t) = J_0^{\beta_i}g_2(t), \quad with\ g_1(0) = 0.$

2.3 Design of FO Fuzzy sliding-mode controllers

In this section, two FO fuzzy sliding mode controllers are designed, namely

- a FO fuzzy sliding mode control (first controller), and
- a FO fuzzy **dynamic** sliding mode control (this second controller is free of chattering).

2.3.1 First controller

2.3.1.1 Choice of an appropriate sliding surface

From (2.1), (2.2) and (2.4), one can obtain the synchronization errors' dynamics as follows:

$$\begin{cases} D^{\alpha}e_i = e_{i+1}, \quad \text{for for } i = 1,\ldots,n-1 \\ D^{\alpha}e_n = f_s(y) + d(y,t) + u(t) - \lambda f_m(x) \end{cases} \tag{2.5}$$

Now, by putting $e_0 = D^{-\alpha}e_1$, dynamics (2.5) can be extended as

$$\begin{cases} D^{\alpha}e_0 = e_1 \\ D^{\alpha}e_1 = e_2 \\ \vdots \\ D^{\alpha}e_n = H_1(x,y) + d(y,t) + u(t) \end{cases} \tag{2.6}$$

with

$$H_1(x,y) = f_s(y) - \lambda f_m(x) \tag{2.7}$$

Let a FO integral-type sliding surface be defined as follows:

$$s(t) = e_n + D^{-\alpha}\sum_{i=0}^{n} k_{1i}e_i \tag{2.8}$$

where $k_{1i} > 0$ are design parameters to be defined later.

Once the system reaches in the sliding surface $s(t)=0$, one has

$$e_n = -D^{-\alpha}\sum_{i=0}^{n} k_{1i} e_i \tag{2.9}$$

Using Remark 2.1, (2.9) can be rewritten as:

$$D^{\alpha} e_n = -\sum_{i=0}^{n} k_{1i} e_i \tag{2.10}$$

The state space representation of (2.10) can be given by

$$\begin{cases} D^{\alpha} e_0 = e_1 \\ D^{\alpha} e_1 = e_2 \\ \vdots \\ D^{\alpha} e_{n-1} = e_n \\ D^{\alpha} e_n = -\sum_{i=0}^{n} k_{1i} e_i \end{cases} \Rightarrow \underbrace{\begin{bmatrix} D^{\alpha} e_0 \\ D^{\alpha} e_1 \\ \vdots \\ D^{\alpha} e_{n-1} \\ D^{\alpha} e_n \end{bmatrix}}_{D^{\alpha} E_1} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \\ -\sum_{i=0}^{n} k_{1i} e_i \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_{10} & -k_{11} & -k_{12} & \dots & -k_{1n} \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \\ e_n \end{bmatrix}}_{E_1} = A_1 E_1$$

(2.11)

Or equivalently

$$D^{\alpha} E_1 = A_1 E_1 \tag{2.12}$$

where A_1 is a $(n+1)\times(n+1)$ constant matrix. The design parameters k_{1i} are chosen to be strictly positive such that the eigen-values of A_1 satisfy the following well-known stability conditions related to the linear FO systems, i.e. $|\arg(eig\ A_1)| > \alpha\pi/2$. So, if the system state reaches the sliding surface (i.e. if one has $s(t)=0$), one can establish that the sliding mode dynamics (2.9) are ML (or asymptotic) stable and, then the synchronization errors will vanish at zero.

Once a suitable sliding surface is determined, the next step is to establish a control signal $u(\mathrm{t})$ ensuring that the synchronization errors' trajectories reach to the sliding surface $s(\mathrm{t})=0$ and stay on it forever.

2.3.1.2 Design of the first controller

Considering (2.6) and (2.8), the fractional-order dynamics of $s(t)$ are given by:

$$D^{\alpha}s(t) = D^{\alpha}e_n + \sum_{i=0}^{n} k_{1i}e_i = H_1(x,y) + d(y,t) + u(t) + \sum_{i=0}^{n} k_{1i}e_i \tag{2.13}$$

Now**,** since the nonlinear function $H_1(x,y)$ is unknown, the design of an adequate control system to asymptotically stabilize the fractional-order dynamics (2.13) is not easy. Next, to resolve such a problem, a fuzzy system will be used to model this uncertain function, $H_1(x,y)$.

According to fuzzy approximation theorem [WAN94], the unknown function $H_1(x,y)$ can be optimally approximated by the linearly parameterized fuzzy system (1.2), as follows [BEN16, HAM16, ZOU16, RIG16, BOU16d, BOU15b]:

$$H_1(x,y) = \theta_1^{*T}\psi_1(y) + \varepsilon_1(x,y) \tag{2.14}$$

with $\psi_1(y)$ being the vector of FBF, $\varepsilon_1(x,y)$ is the fuzzy approximation error and θ_1^* is the unknown optimal parameter vector which is defined as:

$$\theta_1^* = \operatorname{argmin}_{\theta}\left[\sup_{y\in\Omega_y}\left|H_1(x,y) - \theta_1^T\psi_1(y)\right|\right] \tag{2.15}$$

According to [WAN94, BEN16, HAM16, ZOU16, RIG16, BOU16d, BOU15b], the fuzzy approximation error $\varepsilon_1(x,y)$ can be bounded as follows:

$$\left|\varepsilon_1(x,y)\right| \le \bar{\varepsilon}_1 \tag{2.16}$$

where $\bar{\varepsilon}_1$ is an unknown positive constant.

For the master-slave system (2.1)-(2.2), one can consider the following fuzzy sliding-mode controller:

$$u(t) = -\left(\frac{\gamma_0}{2}\left\|\psi(y)\right\|^2 + \gamma_1 + \gamma_2\right) sign(s) - \sum_{i=0}^{n} k_{1i}e_i - \gamma_3 s \tag{2.17}$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3 > 0$ are design constants.

Theorem 2.1: *Consider the master and slave systems described by (2.1) and (2.2) with Assumption 2.1. By applying the control law (2.17) to the slave system, the pseudo-states of the dynamics (2.6) asymptotically converges to the sliding surface s=0.*

Proof of Theorem 2.1:

First, motivated by [EFE8b], one can write:

$$\left|\sum_{i=1}^{\infty}\frac{\Gamma(1+\alpha)}{\Gamma(1+i)\Gamma(1-i+\alpha)}D^{i}sD^{\alpha-i}s\right|\le v|s| \tag{2.18}$$

with v being a positive constant.

Consider the following quadratic Lyapunov function candidate

$$V=s^{2} \tag{2.19}$$

Its fractional-order derivative is

$$D^{\alpha}V=sD^{\alpha}s+\sum_{i=1}^{\infty}\frac{\Gamma(1+\alpha)}{\Gamma(1+i)\Gamma(1-i+\alpha)}D^{i}sD^{\alpha-i}s\le sD^{\alpha}s+v|s|$$

Considering the dynamics (2.13) leads to

$$D^{\alpha}V\le sD^{\alpha}s+v|s|\le sH_{1}\left(x,y\right)+sd(y,t)+su(t)+v|s|+s\sum_{i=0}^{n}k_{1i}e_{i} \tag{2.20}$$

By using (2.3), (2.13) and (2.14), one gets

$$D^{\alpha}V\le s\theta_{1}^{*T}\psi_{1}\left(y\right)+s\varepsilon_{1}(x,y)+sd(y,t)+su(t)+v|s|+s\sum_{i=0}^{n}k_{1i}e_{i}$$

$$\le\frac{\gamma_{0}}{2}\left\|\psi\left(y\right)\right\|^{2}|s|+\left(\bar{\varepsilon}_{1}+\bar{d}_{1}+v+\frac{1}{2\gamma_{0}}\left\|\theta_{1}^{*}\right\|^{2}\right)|s|+su(t)+s\sum_{i=0}^{n}k_{1i}e_{i} \tag{2.21}$$

Replacing the control law (2.17) into (2.21) yields

$$D^{\alpha}V \leq -\gamma_3 s^2 - \gamma_2 |s| \tag{2.22}$$

where $\gamma_1 \geq \bar{\varepsilon}_1 + \bar{d}_1 + v + \frac{1}{2\gamma_0}\left\|\theta_1^*\right\|^2$.

Then, according to Theorem 1.4, the signal s asymptotically converges to the origin. Then, by Theorem 1.5 and because s converges to zero (i.e. by (2.12)), the synchronization errors e_i also asymptotically vanish at the origin.

Now it is clear from (2.22), that we can proceed in the same way as in [AGH14a] to proof that the signal s also converges to the origin in finite time.

2.3.2 Second controller

From (2.5), one can extend the synchronization errors' dynamics as follows:

$$\begin{cases} D^{\alpha}e_i = e_{i+1}, \quad \text{for for } i = 1,\dots,n-1 \\ D^{\alpha}e_n = e_{n+1} \\ D^{\alpha}e_{n+1} = D^{\alpha}\left(f_s(y) + d(y,t) + u(t) - \lambda f_m(x)\right) \end{cases} \tag{2.23}$$

Or

$$\begin{cases} D^{\alpha}e_i = e_{i+1}, \quad \text{for for } i = 1,\dots,n-1 \\ D^{\alpha}e_n = e_{n+1} \\ D^{\alpha}e_{n+1} = D^{\alpha}\left(H_1(x,y) + d(y,t) + u(t)\right) \end{cases} \tag{2.24}$$

with $H_1(x,y) = f_s(y) - \lambda f_m(x)$.

The following mild assumptions will be useful to derive our main result:

Assumption 2.2: *One assumes that:*

1) $D^{\alpha}(H_1(x,y)) \leq \bar{H}_2(x,y)$
2) $D^{\alpha}(d(y,t)) \leq \bar{d}_2$

where $\bar{H}_2(x,y)$ is an unknown positive function and $\bar{d}_2$ is a given positive constant.

2.3.2.1 Choice of an appropriate sliding surface

The sliding surface can be determined as:

$$s(t) = e_{n+1} + D^{-\alpha} \sum_{i=1}^{n+1} k_{2i} e_i \tag{2.25}$$

where $k_{2i} > 0$, for $i = 1, \ldots, n+1$, are design parameters to be defined later.

If $s(t) = 0$, one has

$$e_{n+1} = -D^{-\alpha} \sum_{i=1}^{n+1} k_{2i} e_i \tag{2.26}$$

Using Remark 2.1, (2.26) can be rewritten as:

$$D^{\alpha} e_{n+1} = -\sum_{i=0}^{n+1} k_{2i} e_i \tag{2.27}$$

The state space representation of (2.27) can be given by:

$$\begin{cases} D^{\alpha} e_1 = e_2 \\ D^{\alpha} e_2 = e_3 \\ \vdots \\ D^{\alpha} e_n = e_{n+1} \\ D^{\alpha} e_{n+1} = -\sum_{i=1}^{n+1} k_{2i} e_i \end{cases} \Rightarrow \underbrace{\begin{bmatrix} D^{\alpha} e_1 \\ D^{\alpha} e_2 \\ \vdots \\ D^{\alpha} e_n \\ D^{\alpha} e_{n+1} \end{bmatrix}}_{D^{\alpha} E_2} = \begin{bmatrix} e_2 \\ e_3 \\ \vdots \\ e_{n+1} \\ -\sum_{i=1}^{n+1} k_{2i} e_i \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_{21} & -k_{22} & -k_{23} & \ldots & -k_{2(n+1)} \end{bmatrix}}_{A_2} \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \\ e_{n+1} \end{bmatrix}}_{E_2} = A_2 E_2 \tag{2.28}$$

Or

$$D^{\alpha} E_2 = A_2 E_2 \tag{2.29}$$

where A_2 is a $(n+1) \times (n+1)$ constant matrix. The design parameters k_{2i} are chosen to be strictly positive such that the eigen-values of A_2 satisfy the following well-known stability conditions, related to the linear FO systems, i.e. $|\arg(eig\ A_2)| > \alpha\pi/2$. So, if the system state reaches the sliding surface (i.e. if one has $s(t)=0$), one can establish that the sliding mode dynamics (2.27) are ML (or asymptotic) stable and, then synchronization errors will vanish at the origin.

2.3.2.2 Design of the second controller

By (2.24) and (2.25), one gets

$$D^{\alpha}s(t)=D^{\alpha}e_{n+1}+\sum_{i=1}^{n+1}k_{2i}e_i=D^{\alpha}\left(H_1\left(x,y\right)+d(y,t)+u(t)\right)+\sum_{i=1}^{n+1}k_{2i}e_i$$

$$=D^{\alpha}H_1\left(x,y\right)+D^{\alpha}d(y,t)+D^{\alpha}u(t)+\sum_{i=1}^{n+1}k_{2i}e_i \tag{2.30}$$

Now, since the nonlinear function $D^{\alpha}H_1\left(x,y\right)$ is unknown, one will use a fuzzy system to online model its uncertain upper bound function, $\bar{H}_2(x,y)$.

According to [WAN94], the uncertain function $\bar{H}_2(x,y)$ can be optimally approximated by a linearly parameterized fuzzy system (1.2), as follows [WAN94, BEN16, HAM16, ZOU16, RIG16, BOU16d, BOU15b]:

$$\bar{H}_2(x,y)=\theta_2^{*T}\psi_2\left(y\right)+\varepsilon_2(x,y) \tag{2.31}$$

with $\psi_2\left(y\right)$ being the vector of FBF (assumed to be chosen a priori by the designer), $\varepsilon_2(x,y)$ is the fuzzy approximation error and θ_2^* is the unknown optimal parameter vector which is defined as:

$$\theta^*=\underset{\theta}{\operatorname{argmin}}\left[\sup_{y\in\Omega_y}\left|\bar{H}_2(x,y)-\theta_2^T\psi_2\left(y\right)\right|\right]$$

According to [WAN94, BEN16, HAM16, ZOU16, RIG16, BOU16d, BOU15b], the fuzzy approximation error $\varepsilon_2(x,y)$ can be bounded as follows:

$$\left|\varepsilon_2(x,y)\right|\le\bar{\varepsilon}_2 \tag{2.32}$$

where $\bar{\varepsilon}_2$ is an unknown positive constant.

For the master-slave system (2.1)-(2.2), one can consider the following fuzzy dynamical sliding-mode controller:

$$u(t) = -D^{-\alpha}\left(\left(\frac{\gamma_0}{2}\|\psi_2(y)\|^2 + \gamma_1 + \gamma_2\right)sign(s) + \sum_{i=1}^{n+1} k_{2i}e_i + \gamma_3 s\right) \tag{2.33}$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3 > 0$ are design constants.

Theorem 2.2: *Consider the master and slave systems described by (2.1) and (2.2) with Assumption 2.2. By applying the control law (2.33) to the slave system, the pseudo-states of the dynamics (2.24) asymptotically converges to the sliding surface s=0.*

Proof of Theorem 2.2:

According to [EFE08b], one can write

$$\left|\sum_{i=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+i)\Gamma(1-i+\alpha)} D^i s D^{\alpha-i} s\right| \le v|s| \tag{2.34}$$

with v being a positive constant.

Consider the following quadratic Lyapunov function candidate

$$V = s^2 \tag{2.35}$$

Using (2.34) and a similar way to the proof of Theorem 2.1, the fractional-order derivative of

V can be given by

$$D^{\alpha}V = sD^{\alpha}s + \sum_{i=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+i)\Gamma(1-i+\alpha)} D^i s D^{\alpha-i} s \le sD^{\alpha}s + v|s|$$

By considering (2.30), $D^{\alpha}V$ becomes

$$D^{\alpha}V \le sD^{\alpha}s + v|s| \le |s|\bar{H}_2(x,y) + |s|\bar{d}_2 + sD^{\alpha}u(t) + v|s| + s\sum_{i=1}^{n+1} k_{2i}e_i \tag{2.36}$$

Using Assumption 2.2 and (2.31)-(2.33), one gets

$$D^{\alpha}V \le |s|\bar{H}_2(x,y) + |s|\bar{d}_2 + sD^{\alpha}u(t) + v|s| + s\sum_{i=1}^{n+1} k_{2i}e_i$$

$$\le |s|\theta_2^{*T}\psi_2(y) + |s|\bar{\varepsilon}_2 + |s|\bar{d}_2 + sD^{\alpha}u(t) + v|s| + s\sum_{i=1}^{n+1} k_{2i}e_i$$

$$\le \frac{\gamma_0}{2}\left\|\psi_2(y)\right\|^2 |s| + \left(\bar{\varepsilon}_2 + \bar{d}_2 + v + \frac{1}{2\gamma_0}\left\|\theta_2^*\right\|^2\right)|s| + sD^{\alpha}u(t) + s\sum_{i=1}^{n+1} k_{2i}e_i \quad (2.37)$$

where γ_0 is a design positive constant.

Replacing the control law (2.33) into (2.37) yields

$$D^{\alpha}V \le -\gamma_3 s^2 - \gamma_2|s| \quad (2.38)$$

where $\gamma_1 > \bar{\varepsilon}_2 + \bar{d}_2 + v + \frac{1}{2\gamma_0}\left\|\theta_2^*\right\|^2$.

Then, according to Theorem 1.4, the error s asymptotically converges to the origin. Then, by Theorem 1.5 and because s converges to zero (i.e. by (2.29)), the synchronization errors e_i also asymptotically vanish at the origin.

Now it is clear from (2.38), that we can proceed in the same way as in [AGH14a] to show that the error s can also converge to the origin in finite-time.

2.4 Simulation Study

Two simulation examples are presented in this section to show the effectiveness of the theoretical results obtained.

2.4.1 Example 1 (FO hybrid optical system)

In this subsection, one will consider the projective synchronization of two fractional-order hybrid optical system [MIT84, ABD14]. The dynamic equations of this system are given by:

$$Master\begin{cases} D^{\alpha}x_1 = x_2 \\ D^{\alpha}x_2 = x_3 \\ D^{\alpha}x_3 = -\beta x_3 - x_2 + f\,(\mathrm{x}_1) \end{cases} \tag{2.39}$$

where $\beta = 0.326$, and f is the so-called logistic function given by $f\,(x_1) = \mu x_1(1-x_1)$, with $\mu = 0.761$.

The system (2.39) here is used as a master system. Its controlled slave system is given by

$$Slave\begin{cases} D^{\alpha}y_1 = y_2 \\ D^{\alpha}y_2 = y_3 \\ D^{\alpha}y_3 = -\beta y_3 - y_2 + f\,(\mathrm{y}_1) + d(\mathrm{y},t) + u \end{cases} \tag{2.40}$$

The no-controlled system (2.39) behaves chaotically when the fractional order $\alpha =$ 0.99 [ABD14].

The simulation was performed using the following initial conditions $(x_1(0), x_2(0), x_3(0)) = (1.5, 0.01, 0.02)$, $(y_1(0), y_2(0), y_3(0)) = (0.5, -0.03, -0.01)$.

Figure 2.1 shows the chaotic attractor of the system (2.39). The Lyapunov exponents of the master system (2.39) are depicted in Figure 2.2. From this figure, it is clear that the sum of these Lyapunov exponents is always negative. This confirms that the system (2.39) is dissipative.

without loss of generality, the uncertainties and external dynamic disturbances are chosen as:

$$d(y,t) = -0.1\beta y_3 + 0.4\cos(t) + 0.8\sin(t) + 1.1.$$

The design parameters of the *First controller* are selected are follows: $k_{10} = 2$, $k_{1i} = 1$ (for i=1,2,3), $\gamma_0 = 2,\ \gamma_1 = 1.5,\ \gamma_2 = 2.5,$ and $\gamma_3 = 10.$ Those of the *second controller* are: $k_{24} = 1,\ k_{2i} = 1$ (for i=1,2,3), $\gamma_0 = 2, \gamma_1 = 1.5, \gamma_2 = 2,$ and $\gamma_3 = 12.$

To test our proposed FO fuzzy sliding-mode controllers for the master-slave system (2.39) and (2.40), four cases are considered here:

1) ***Case 1: Complete synchronization without uncertainties and external disturbances*** $(\lambda = 1)$

For both controllers, the simulation results of this complete synchronization ($\lambda = 1$) and the corresponding control inputs are presented in Figure 2.3 and Figure 2.4. It can be seen that both controllers efficiently synchronize the slave system (2.40) with the master one (2.39). But, unlike the first controller, the chattering effect is notably attenuated in the simulation results obtained by applying the second controller (i.e. those in Figure 2.4).

2) ***Case 2: Complete synchronization with uncertainties and external disturbances*** $(\lambda = 1)$

In presence of uncertainties and external disturbances, the simulation results for both controllers are respectively depicted in Figure 2.5 and Figure 2.6. From these figures, it is clear that the state trajectories of the slave system (2.40) are quickly synchronized with those of the master system (2.39) despite uncertainties and disturbances. In addition, the chattering effect is considerably eliminated in the results obtained by the second controller. This is not surprising for us, because the second control is of dynamic type.

3) ***Case 3: Anti-phase synchronization without uncertainties and external disturbances*** $(\lambda = -1)$

Simulation results, obtained by both proposed controllers, are respectively given in Figure 2.7 and Figure 2.8. These figures show that both controllers performs well. However, in the first controller results, the chattering phenomenon is clearly present in the control signal, and this is due to the discontinuous *Sign* function.

4) ***Case 4: Anti-phase synchronization with uncertainties and external disturbances*** $(\lambda = -1)$

For both controllers, the obtained simulation results for this case are given in Figure 2.9 and Figure 2.10. It can be clearly seen that the slave trajectories are effectively anti-phase-synchronized with those of the master system, despite the presence of bounded uncertainties and external disturbances. In addition, in contrast to the first proposed controller, the control signal generated by the second proposed controller is free of chattering effect.

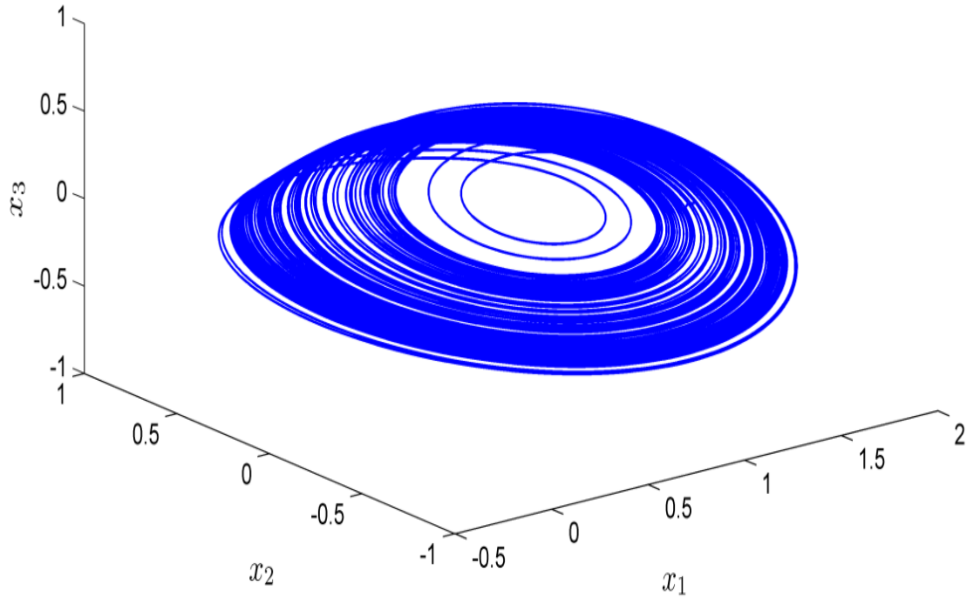


Figure 2.1: *Chaotic attractor of the system (2.39).*

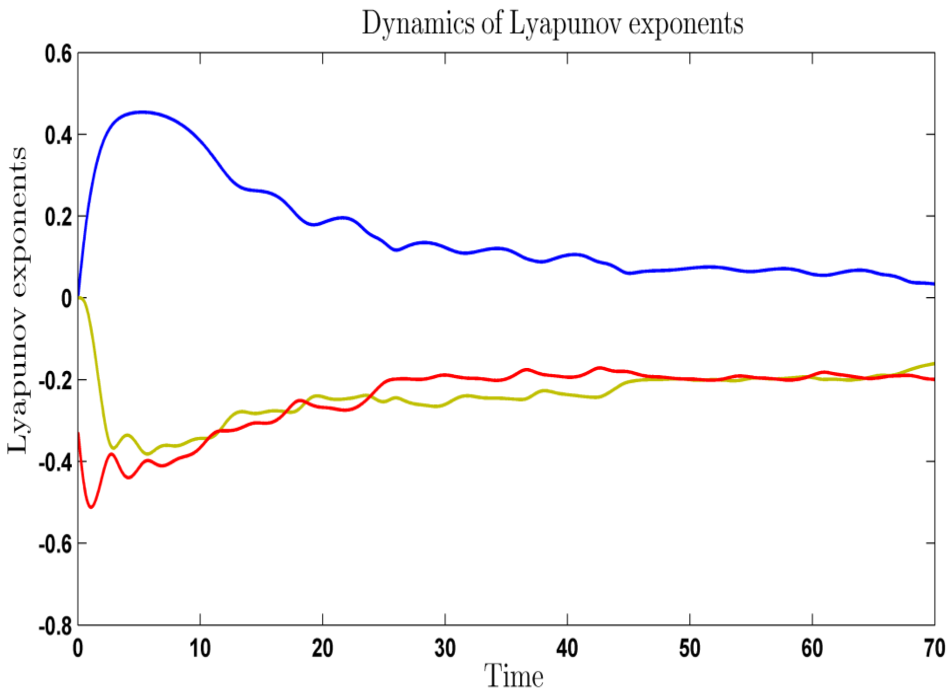
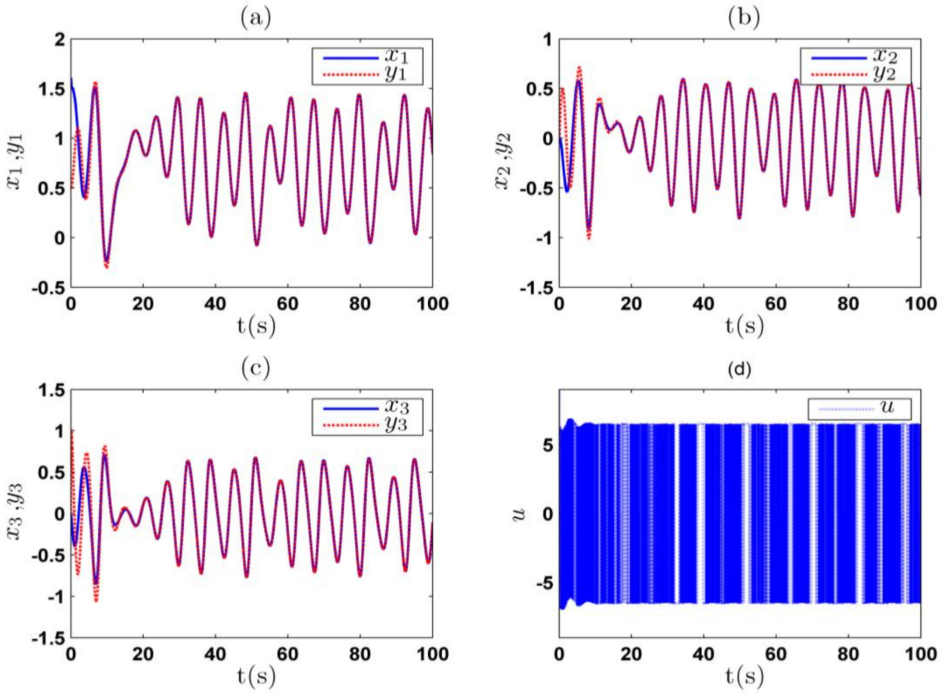


Figure 2.2: *Lyapunov exponents.*



d

Figure 2.3: *Synchronization trajectories of master-slave system and control signal, obtained by using the first controller an without uncertainties and disturbances.*

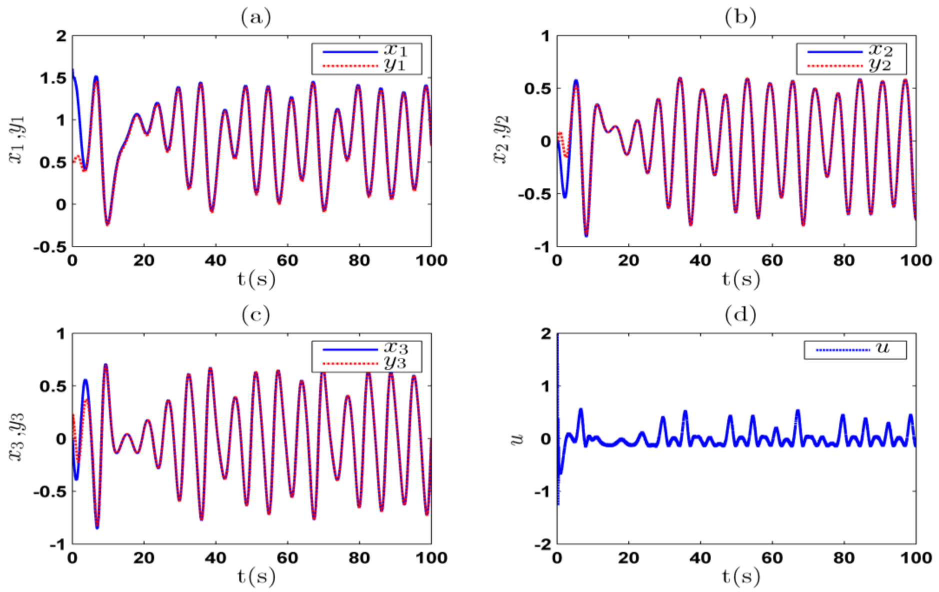


Figure 2.4: *Synchronization trajectories of master-slave system and control signal, obtained by using the second controller and without uncertainties and disturbances.*

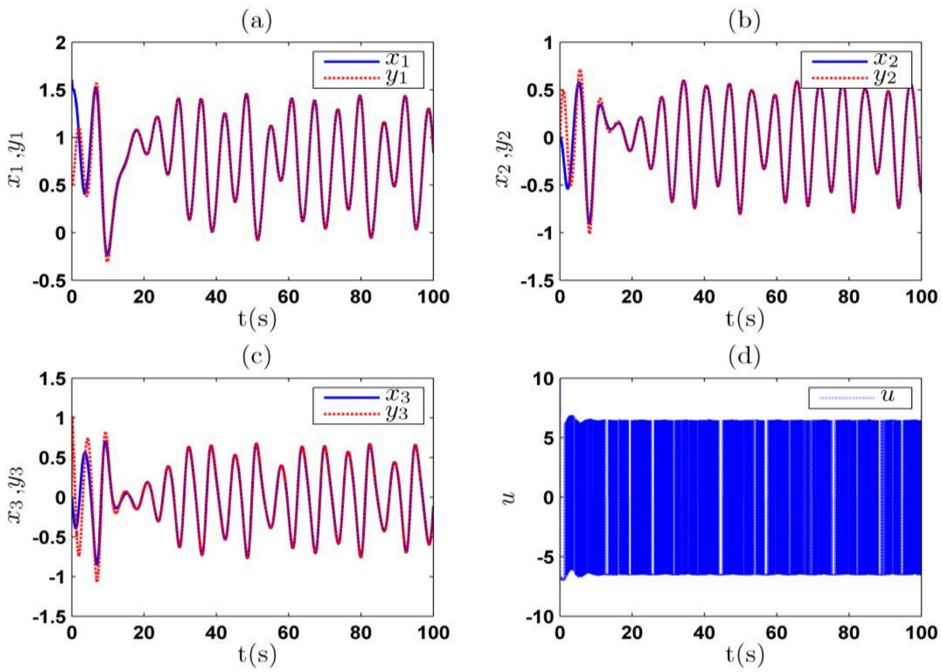


Figure 2.5: *Synchronization trajectories of master-slave system and control signal, obtained by using the first controller and with uncertainties and disturbances.*

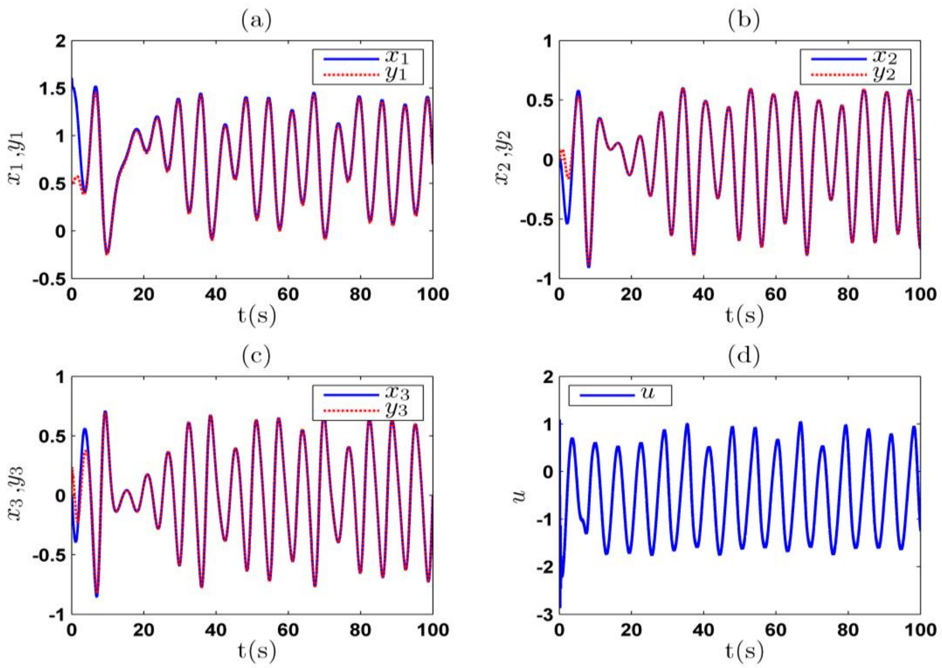


Figure 2.6: *Synchronization trajectories of master-slave system and control signal, obtained by using the second controller and with uncertainties and disturbances.*

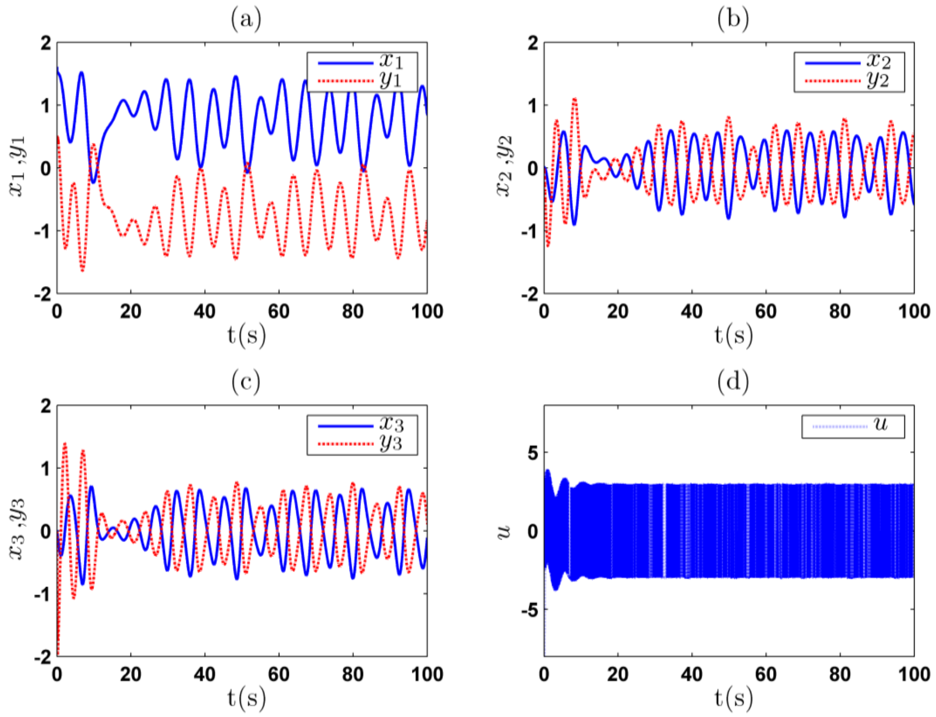


Figure 2.7: *Anti-phase synchronization trajectories of master-slave system and control signal, obtained by using the first controller and without uncertainties and disturbances.*

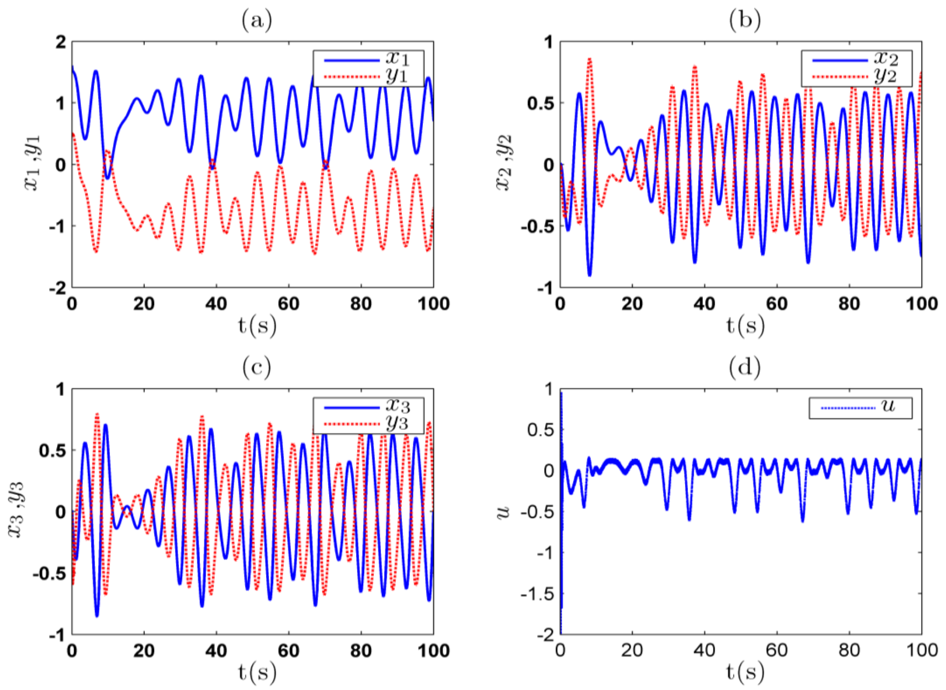


Figure 2.8: *Anti-phase synchronization trajectories of master-slave system and control signal, obtained by using the second controller and without uncertainties and disturbances.*

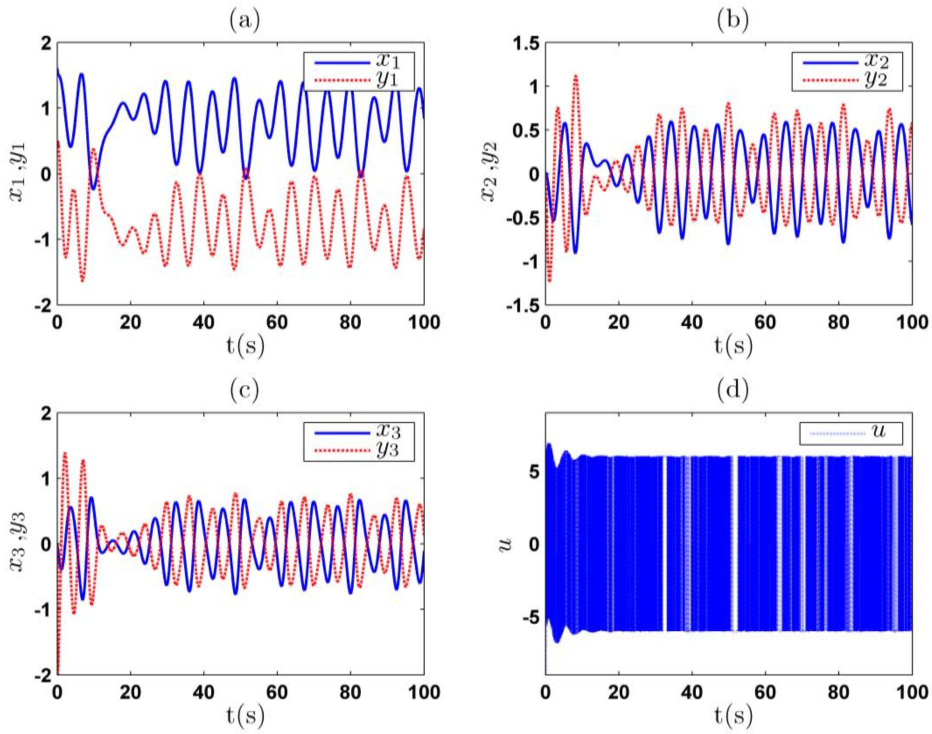


Figure 2.9: *Anti-phase synchronization trajectories of master-slave system and control signal, obtained by using the first controller and with uncertainties and disturbances.*

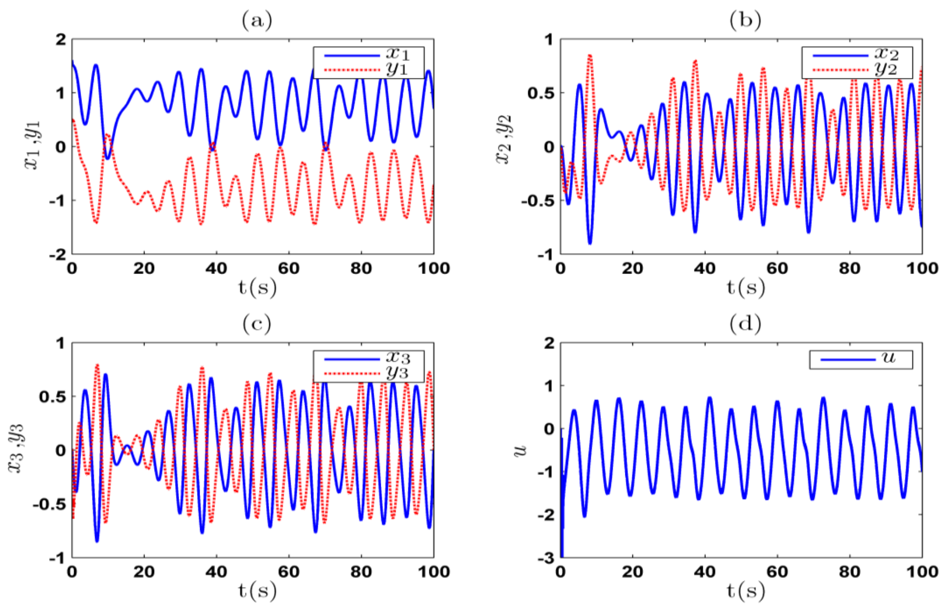


Figure 2.10: *Anti-phase synchronization trajectories of master-slave system and control signal, obtained by using the second controller and with uncertainties and disturbances.*

2.4.2 Example 2 (FO Single mode laser Lorenz)

The well-known single mode laser Lorenz system is described by [LI11]:

$$\begin{cases} \dot{x}_1 = \delta(\mathrm{x}_2 - \mathrm{x}_1) \\ \dot{x}_2 = (a - x_3)x_1 - x_2 \\ \dot{x}_3 = x_1 x_2 - r x_3 \end{cases} \tag{2.41}$$

where x_1, x_2 and x_3 are state variables, and symbols δ, a and r are its parameters.

When parameters are taken as: $\delta = 10$, $a = 28$ and $r = 8/3$, equations (2.41) exhibits chaotic behavior [LI11]. In [WU09], a fractional generalization of the laser Lorenz system has been introduced. It is given by:

$$Master \begin{cases} D^{\alpha} x_1 = \delta(x_2 - x_1) \\ D^{\beta} x_2 = (a - x_3)x_1 - x_2 \\ D^{\gamma} x_3 = x_1 x_2 - r x_3 \end{cases} \tag{2.42}$$

where α, β and γ are the fractional order, $0 < \alpha \le 1, 0 < \beta \le 1, 0 < \gamma \le 1.$

In this simulation, one chooses as in [WU09]: $\alpha = 1, \beta = 0.99, \gamma = 0.98$. Note that, with these parameters, the system (2.42) behaves chaotically.

Figure 2.11 depicts the chaotic attractor of (2.42) in three-dimensional space and its projection on two-dimensional plan.

The Lyapunov exponents of the system (2.42) are depicted in Figure 2.12. From this figure, one can remark the presence of a positive Lyapunov exponent which confirms well the chaotic behavior of this system.

Now, let's take the system (2.42) as a master system. Then, its corresponding slave system can be given by:

$$Slave \begin{cases} D^{\alpha} y_1 = \delta(y_2 - y_1) + d_1(y,t) + u_1 \\ D^{\beta} y_2 = (a - y_3)y_1 - y_2 + d_2(y,t) + u_2 \\ D^{\gamma} y_3 = y_1 y_2 - r y_3 + d_3(y,t) + u_3 \end{cases} \tag{2.43}$$

Without loss of generality, the uncertainties and the external dynamic disturbances are chosen as: $d_i(y,t) = 1.1 + 10\cos(0.5t)$.

The design parameters of the **First controller** are selected as follows: $k_{11} = 1$, $k_{10} = 1$, $\gamma_0 = 1$, $\gamma_1 = 5$, $\gamma_2 = 15$, and $\gamma_3 = 20$. But those of the **second controller** are chosen as: $\gamma_0 = 0.1$, $\gamma_1 = 1$, $\gamma_2 = 0.2$, and $\gamma_3 = 1$, $k_{21} = 1$, $k_{22} = 1$.

The initial conditions of the master and slave system are chosen as follows: $(x_1(0), x_2(0), x_3(0)) = (-2, -3.2, 10.9)$, $(y_1(0), y_2(0), y_3(0)) = (-2.9, -3.9, 11.5)$.

To test the proposed controllers for projective synchronizing the master-slave system (2.42) and (2.43), two cases are considered here:

1) Case 1: Synchronization with uncertainties and external disturbances:

In presence of uncertainties and external disturbances, the simulation results of a complete synchronization (i.e. for $\lambda_i = 1$) and the corresponding control signals generated by both controllers are respectively presented in Figure 2.13 and Figure 2.14. From these figures, one can see that the state trajectories of the slave system are quickly synchronized with those of the master system, despite uncertainties and unknown external disturbances. In addition, the chattering effect is significantly reduced in the results obtained by the second controller.

2) Case 2: Synchronization with uncertainties and external disturbances:

The simulation results obtained by using both controllers for this case are given in Figure 2.15 and Figure 2.16. It can be clearly seen that the trajectories of the slave system are effectively synchronized with those of the master system, despite uncertainties and external disturbances. In addition, the control signals generated by the second controller are free of chattering effect.

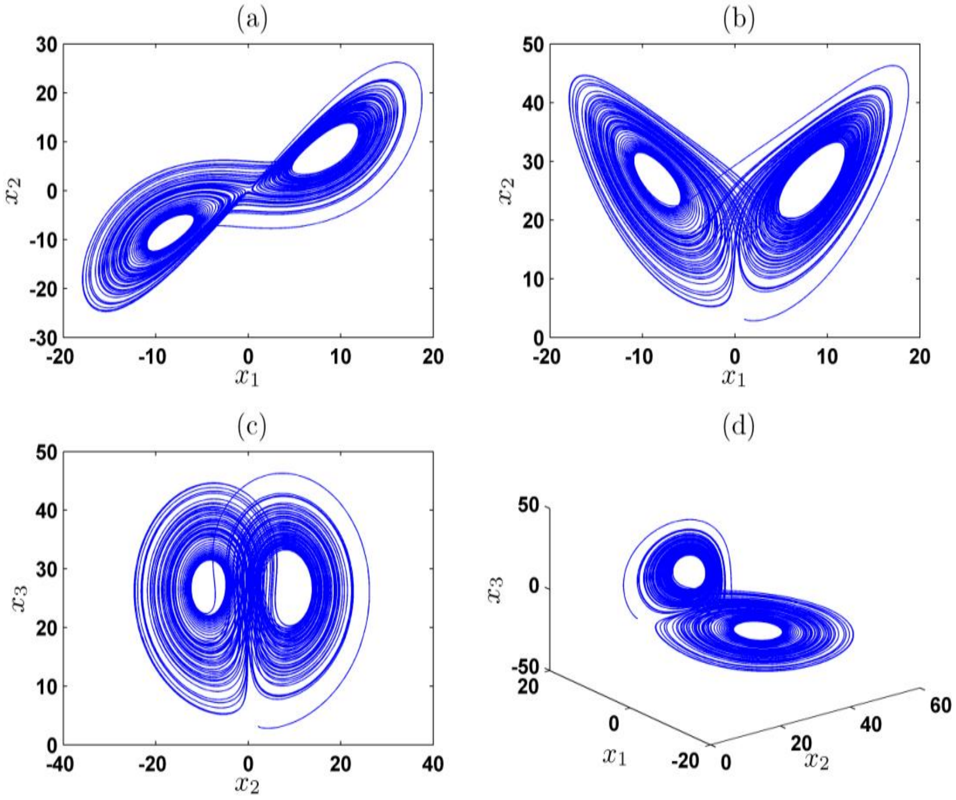


Figure 2.11: *Chaotic attractors of (fractional-order) single mode laser Lorenz system.*

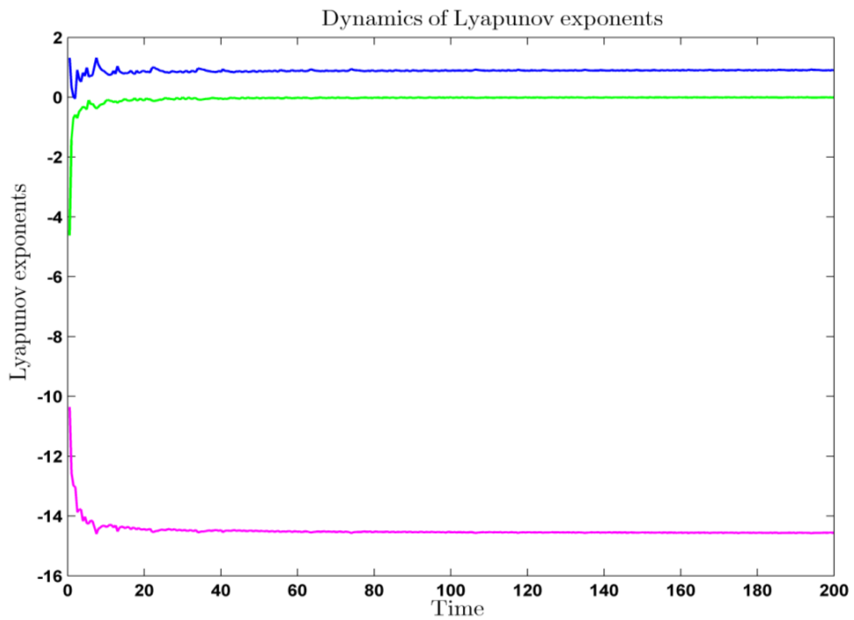


Figure 2.12: *Lyapunov exponents of FO single mode laser Lorenz system.*

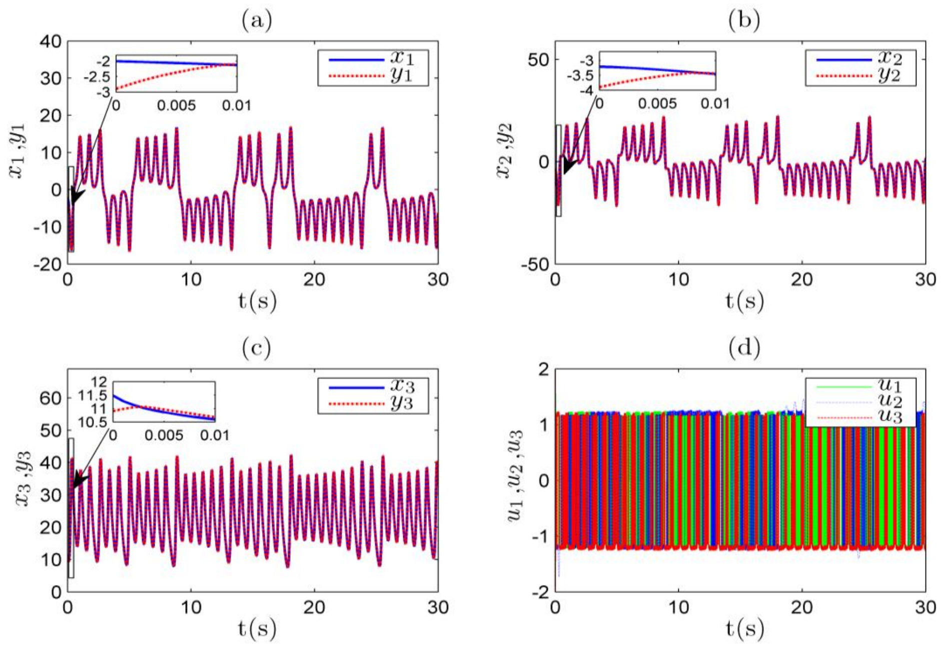


Figure 2.13: *Complete synchronization trajectories of master-slave system and control signals, obtained by using the first controller and without uncertainties and disturbances (example 2).*

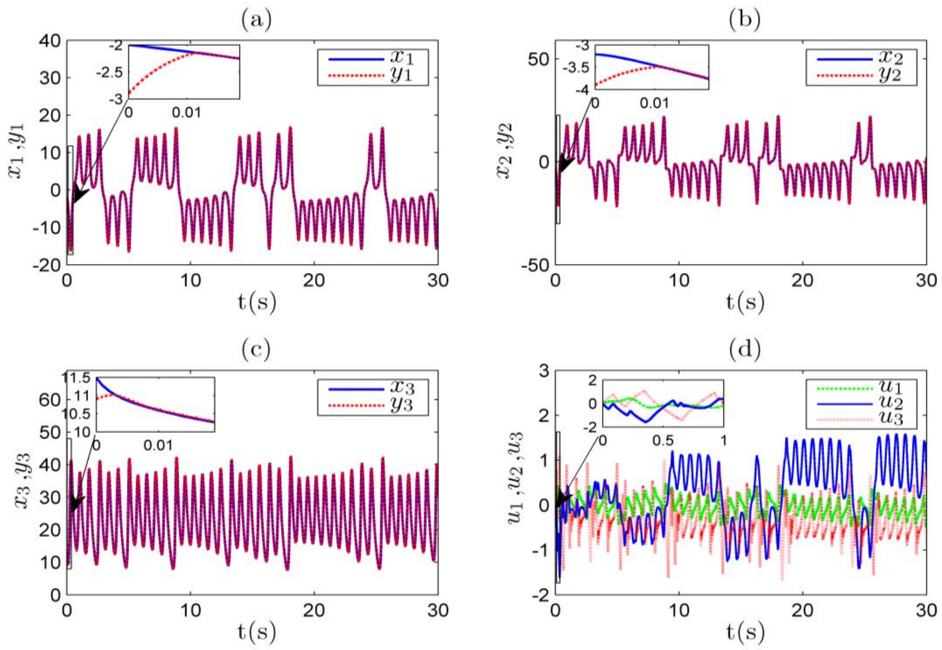


Figure 2.14: *Complete synchronization trajectories of master-slave system and control signals, obtained by using the second controller and without uncertainties and disturbances (example 2).*

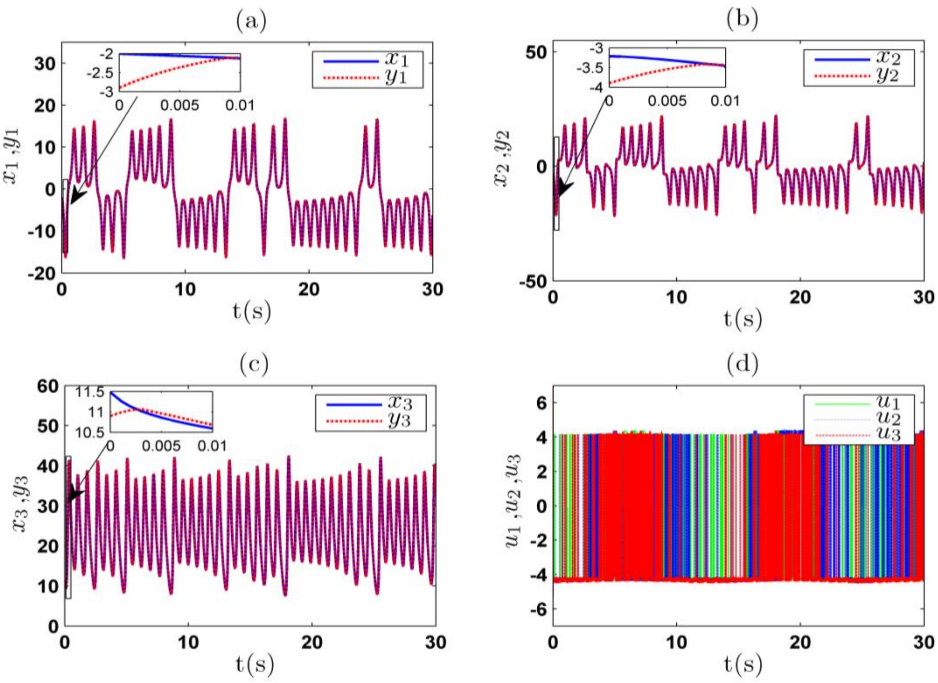


Figure 2.15: *Complete synchronization trajectories of master-slave system and generated control signals, obtained by using the first controller and with uncertainties and disturbances (example 2).*

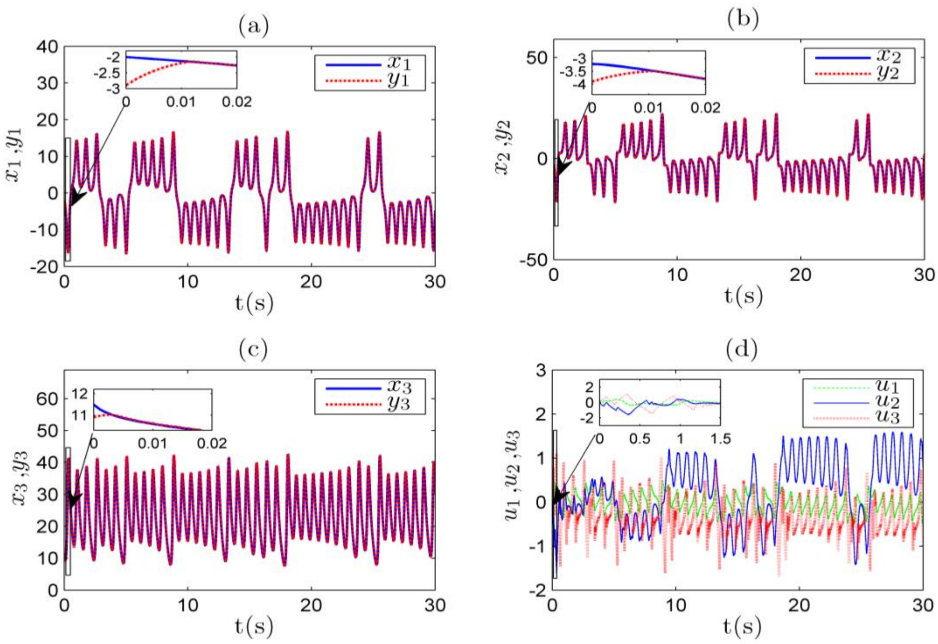


Figure 2.16: *Complete synchronization trajectories of master-slave system and generated control signals, obtained by using the second controller and with uncertainties and disturbances (example 2).*

2.5 Conclusion

In this chapter, two fractional fuzzy sliding mode controls have been investigated to achieve an appropriate projective synchronization for a class of uncertain chaotic optical systems. Indeed, two simple but effective fractional dynamic sliding surfaces with some desired stability features have been introduced. The fuzzy systems have been employed to online approximate the uncertain functions in the master-slave models. The stability analysis of the corresponding closed-loop system has been rigorously performed by means of a fractional Lyapunov theory. Numerical simulations carried out on some optical chaotic systems have demonstrated the feasibility and effectiveness of the proposed control methodologies.

Chapter 3

Intelligent fuzzy controller for chaos synchronization of uncertain fractional-order chaotic systems with input nonlinearities

3.1 Introduction

Throughout the last decades, ***fractional-order plants*** have been studied by several researchers in many branches of engineering and sciences [POD 99, HIL20]. It has turned out that several plants, in interdisciplinary research areas, may present fractional-order dynamics including: fluid mechanics, spectral densities of music, transmission lines, cardiac rhythm, electromagnetic waves, viscoelastic systems, dielectric polarization, heat diffusion systems, electrode-electrolyte polarization, and many others [BAL10, BOU16a, BOU16b, SAB07].

Chaotic systems have nonlinear and deterministic behavior. These systems are, besides, typified by the self-similarity of the extreme sensitivity to initial conditions (IC) and strange attractor, which are respectively quantified by the existence of a positive Lyapunov-exponent and a fractal dimension [BOU16a, BOU16b]. In recent works, it has been shown that some fractional-order systems can chaotically behave, e.g. the fractional-order systems of Lü [DEN05], Arneodo [LU05], Lorenz [GRI03], and Rössler [LI04], and so on.

The sliding-mode control methodology is an effective tool to synthesize robust and fast controllers for nonlinear systems subject to bounded disturbances and uncertainties [BAN15, BOS17, KAR15, UTK09]. The latter has several attractive advantages, including finite-time and rapid convergence, and an enhanced robustness against unmodeled dynamics, parameter variations and bounded external disturbances [BAN15, UTK09]. The purpose of the sliding-mode is extremely straightforward: *it consists to force the system trajectories (or states) to reach in finite-time an appropriate sliding manifold or surface, by using a switching control* *approach.* Recently, the fractional-order calculus has been incorporated in the sliding mode controller, in order to enhance control performances [CAL06, EFE08, PIS10, SIA09]. It is worthy to note that the choice of the sliding *manifold or surface* for systems with fractional-orders is a hard task in general. On the other hand, in numerous recent researches, the fuzzy logic system has been combined with the sliding-mode control in order to eliminate the main drawbacks of the sliding-mode control, including the high-gain authority and chattering problem in the control signal. This hybridization can smoothen the control signal in different ways, and may online approximate successfully the system's model, the uncertainties and the disturbances [BOU16a, BOU16b].

In recent years, ***the synchronization*** of fractional-order systems has become one of the most attractive research themes [AGH12a, BOU16a, BOU16b, BOU16c,BOU16e LI14, LIN11b, LIN11c, TAV12, WAN12]. In [WAN12], a modified projective synchronization of two distinct fractional-order systems has been developed through active sliding-mode control. By using a fuzzy adaptive sliding-mode control methodology, a generalized projective synchronization of a class of uncertain fractional-order chaotic systems has been studied in [LI14, LIN11]. A fuzzy adaptive controller has been proposed, for synchronizing of uncertain fractional-order chaotic delayed systems, in [LIN11]. H_∞-synchronization system based on fuzzy adaptive control has also been studied for uncertain fractional-order chaotic systems in [LIN11b]. Nevertheless, the theoretical findings of [LIN11b, LIN11c] are already questionable, since the stability analysis has not been rigorously derived in mathematics [AGH12a, TAV12]. Some adaptive fuzzy approach-based sliding-mode controllers have been proposed in [BOU16a, BOU16b, BOU16c], for ensuring an appropriate synchronization for a class of fractional-order chaotic master-slave systems. In such synchronization methods, the fuzzy logic systems have been employed to online model uncertainties. Although those methods can often provide good synchronization performances, the issue of the input nonlinearities (dead-zone together with sector nonlinearities) has not been taken into account in the designing phase of these control systems. This is by no means the case in the real world life, as the chaotic physical systems commonly involve input hysteresis, dead-zones, saturation, and so on. The presence of these input nonlinearities may result in poor synchronization performances or even instability of the system. It is so more desirable to consider the effects of such nonlinearities in one way or another in the designing phase.

In this chapter, motivated by the above considerations, we will propose a new generalized projective synchronization (GPS) scheme based on fuzzy adaptive variable-structure controller, for a class of incommensurate fractional-order chaotic systems subject to dynamic external disturbances, uncertainties and input nonlinearities. Adaptive fuzzy systems are used to jointly approximate upper bounds of the uncertainties. These uncertainties essentially consist of unknown nonlinear functions and external disturbances. The control law is designed in a variable-structure frame-work by introducing a fractional-order sliding surface. A Lyapunov stability theorem is used to derive the training laws for the unknown fuzzy parameters and to establish the stability of the over-all system. The validity of the proposed synchronization methodology is confirmed by means of computer simulation results.

The novelty and added value of the proposed control-based synchronization approach are listed as follows:

1) It is the first time that a GPS problem of uncertain incommensurate fractional-order chaotic systems with both input constraints and uncertain dynamic disturbances is solved. The input constraints (i.e. input dead-zone and sector nonlinearities) are considered in the control design, which makes the presented fuzzy control strategy more general for application in engineering.

2) The considered class of fractional-order chaotic systems is relatively general, as it is characterized by:

    - Uncertain models for the master and slave systems and their different structures.
    - Incommensurate fractional-orders.
    - Uncertain dynamic disturbances.
    - Physical input constraints (i.e. Input dead-zone and sector nonlinearities).

3) The fuzzy adaptive systems which improve on and bridge the gaps of some existing methods are applied to approximate unknown system dynamics and uncertain disturbances, [DEN05, LU05, PIS10, SI09, WAN12]. Thereby, the proposed synchronization method does not depend on the master-slave models.

3.2 Problem statement

Consider the following class of uncertain fractional-order chaotic master systems

$$D_t^{\alpha_i} x_i = f_{mi}(x), \ \text{ for } \quad i = 1, \dots, n \tag{3.1}$$

where $D_t^{\alpha_i} = \frac{d^{\alpha_i}}{dt^{\alpha_i}}$, $0 < \alpha_i < 1$ is the fractional-order of the system, $x = [x_1, \dots, \ x_n]^T \in R^n$ is the measurable pseudo-state vector, and $f_{mi}(x)$ is an unknown smooth function.

Its slave system with input nonlinearity is given by

$$D_t^{\beta_i} y_i = f_{si}(y) + \varphi_i(u_i) + d_i(t, y), \ \text{ for } \quad i = 1, \dots, n \tag{3.2}$$

where $0 < \beta_i < 1$ is the fractional-orderof the slave system, $f_{si}(y)$ is an unknown smooth function, $y = [y_1, \dots, y_n]^T \in R^n$ is its measurable pseudo-state vector. u_i represents the control input which will be determined later, $\varphi_i(u_i)$ is the input nonlinearity (i.e. sector nonlinearities with dead-zone) and $d_i(t, y)$ is the unknown dynamical external disturbance.

Remark 3.1: *Various fractional-order chaotic systems can be described as (3.1) or (3.2), like: fractional-order unified chaotic system, fractional-order Lorenz system, fractional-order Lu system, fractional-order Chen system, and so on [BOU16a, BOU16b].*

The objective of this paper is to synthesize a fuzzy adaptive variable-structure control law u_i, with $i = 1, \dots, n,$ for robustly projective-synchronizing the master system (3.1) with the slave one (3.2), and for guaranteeing the boundedness of all signals in the closed-loop system, despite the attendance of dynamical external disturbances, uncertainties, together with input nonlinearities.

Figure 3.1 illustrates the proposed synchronization scheme based on fuzzy adaptive variable-structure control. In the rest of this chapter, D_t^{α} *stands for Caputo's derivative*.

The GPS error variables are defined as:

$$e_i = y_i - \lambda_i x_i, \quad \text{ for } \ i = 1, \dots, n \tag{3.3}$$

where λ_i is a scaling factor.

Let us describe a fractional-order sliding surface as follows:

$$S_i = D_t^{\beta_i - 1} e_i + k_{0i} \int_0^t e_i d\tau, \quad \text{for } i = 1,2, \dots, n \tag{3.4}$$

with $k_{0i} > 0$ being a design parameter which determines the sliding speed. In order to make GPS errors operate in the sliding mode, the following conditions should be satisfied: $S_i = \dot{S}_i = 0$.

Thus, from $\dot{S}_i = 0$, one obtains

$$D_t^{\beta_i} e_i + k_{0i} e_i = 0, \qquad \text{for } i = 1,2, \dots, n \tag{3.5}$$

Because k_{0i} is positive and $0 < \beta_i < 1$, it is clear that the sliding-mode dynamics (3.5) are always stable [BOU16, POD99]. In other words, the following FO stability condition is always verified.

$$|Arg(-k_{0i})| > \beta_i \pi/2, \qquad \text{for } i = 1, \dots, n \tag{3.6}$$

Using the useful Caputo's property $D_t^1 \left(D_t^{-(1-\beta_i)} f(t) \right) = D_t^1 \left(J_a^{(1-\beta_i)} f(t) \right) = D_t^{\beta_i} f(t)$, and the expressions (3.1)-(3.4), the dynamics of S_i can be given by

$$\dot{S}_i = D_t^{\beta_i} e_i + k_{0i} e_i = k_{0i} e_i + f_{si}(y) + \varphi_i(u_i) - \lambda_i D_t^{\beta_i} x_i + d_i(t, y) \tag{3.7}$$

or

$$\dot{S}_i = H_i(x, y, d_i) + \varphi_i(u_i) \tag{3.8}$$

where

$$H_i(x, y, d_i) = k_{0i} e_i + f_{si}(y) - \lambda_i D_t^{\beta_i} x_i + d_i(t, y) \tag{3.9}$$

3.3 Input Nonlinearity

The model of the input nonlinearity including both dead-zones and sector nonlinearities can be given by [BOU11, SHY05]

$$\varphi_i(u_i) = \begin{cases} \varphi_{i+}(u_i)(u_i - u_{i+}), & u_i > u_{i+} \\ 0, & -u_{i-} \leq u_i \leq u_{i+} \\ \varphi_{i-}(u_i)(u_i + u_{i-}), & u_i < -u_{i-} \end{cases} \tag{3.10}$$

with $\varphi_{i+}(u_i) > 0$ and $\varphi_{i-}(u_i) > 0$ being nonlinear functions of u_i, and $u_{i+} > 0$ and $u_{i-} > 0$.

The functions $\varphi_i(u_i)$ should satisfy:

$$(u_i - u_{i+})\varphi_i(u_i) \geq m_{i+}^*(u_i - u_{i+})^2, \; u_i > u_{i+}$$

$$(u_i + u_{i-})\varphi_i(u_i) \geq m_{i-}^*(u_i + u_{i-})^2, \; u_i < -u_{i-} \tag{3.11}$$

with m_{i+}^* and m_{i-}^*, being strictly positive constants so-called "*gain-reduction tolerances*" [BOU11, SHY05].

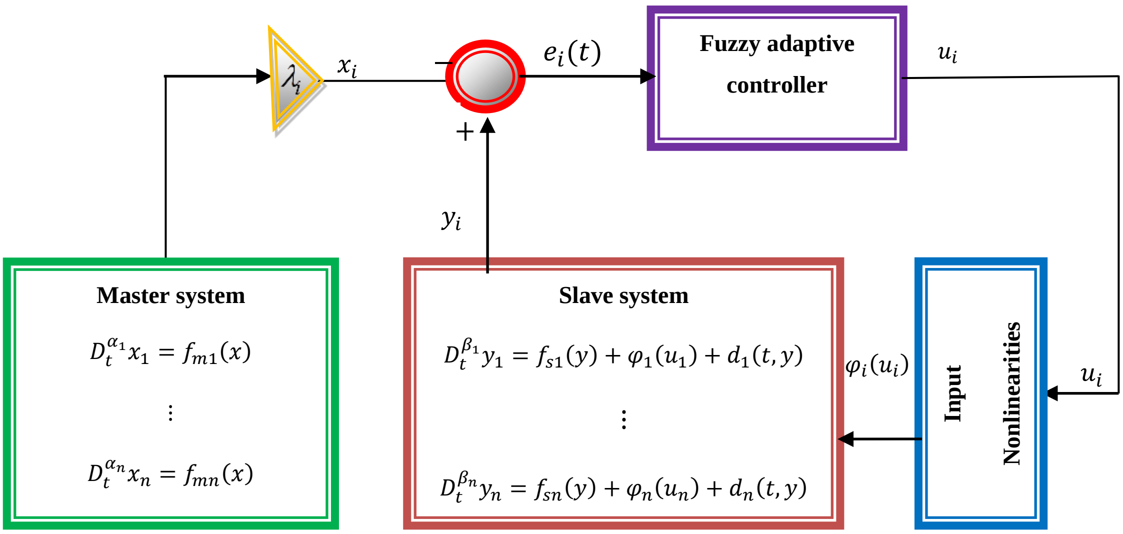


Figure 3.1: *Proposed synchronization scheme.*

Assumption 3.1: *The constants m_{i+}^* and m_{i-}^* and the functions $\varphi_{i+}(u_i)$ and $\varphi_{i-}(u_i)$ are uncertain. Nevertheless, u_{i+} and u_{i-} are strictly positive and known.*

3.4 Design of adaptive fuzzy control

The next mild assumptions will be helpful to develop the main result:

Assumption 3.2: *The possible disturbances should satisfy*

$$|d_i(t,y)| \leq \bar{d}_i(y) \tag{3.12}$$

with $\bar{d}_i(y)$ being an unknown smooth positive function.

Assumption 3.3: *The nonlinear function $H_i(x, y, d_i)$ can be bounded by:*

$$|H_i(x, y, d_i)\,)| \leq \eta \bar{h}_i(y), \text{ for } i = 1, \dots, n \tag{3.13}$$

with $\eta = min_i\{m_{i+}^*, m_{i-}^*\}$, for $i = 1, \dots, n$.

Remark 3.2: *Assumptions 2 and 3 are commonly employed in the open control literature, e.g. [BOU16a, BOU16b]. They can be seen as mild assumptions, as the bounds $\bar{d}_i(y)$ and $\bar{h}_i(y)$ are uncertain functions and the state vector x of the non-controlled chaotic system (3.1) is always bounded.*

Over a compact set Ω_y, the unknown smooth function $\bar{h}_i(y)$ can be modeled by the linearly parameterized fuzzy systems (1.2) as follows [BEN16, HAM16, WAN94, ZOU17]:

$$\hat{\bar{h}}_i(y, \theta_i) = \theta_i^T \psi_i(y), \text{ with } i = 1, \dots, n \tag{3.14}$$

where θ_i is the adjustable parameters vector. But, the FBF vector, $\psi_i(y)$, is supposed to be determined a priori.

Without loss of generality, one assumes that there exists an optimal fuzzy system with m fuzzy rules that can model the smooth function $\bar{h}_i(y)$ with a minimal fuzzy approximation error, i.e.

$$\bar{h}_i(y) = \hat{\bar{h}}_i(y, \theta_i^*) + \delta_i(y) = \theta_i^{*T} \psi_i(y) + \delta_i(y) \tag{3.15}$$

where the minimal fuzzy approximation error, $\delta_i(y)$, is generally assumed to be bounded for all $y \epsilon \Omega_y$, i.e. $|\delta_i(y)| \leq \bar{\delta}_i$, with $\bar{\delta}_i$ is an unknown constant [BOU16a, BOU15b, BOU15c, BOU08c, LIU17a, PRE14, RIG16], and

$$\theta_i^* = \arg\min_{\theta_i} \left[\sup_{y \epsilon \Omega_y} \left| \bar{h}_i(y) - \hat{\bar{h}}_i(y, \theta_i) \right| \right] \tag{3.16}$$

Notice that θ_i^* is the ideal value of θ_i and essentially introduced for theoretical analysis purposes.

From the aforementioned analysis, one can write:

$$\hat{\bar{h}}_i(y, \theta_i) - \bar{h}_i(y) = \hat{\bar{h}}_i(y, \theta_i) - \hat{\bar{h}}_i(y, \theta_i^*) + \hat{\bar{h}}_i(y, \theta_i^*) - \bar{h}_i(y),$$

$$= \theta_i^T \psi_i(y) - \theta_i^{*T} \psi_i(y) - \delta_i(y),$$

$$= \tilde{\theta}_i^T \psi_i(y) - \delta_i(y) \tag{3.17}$$

with $\tilde{\theta}_i = \theta_i - \theta_i^*$, for $i = 1, \dots, n$.

To reach our goal, one can construct the following fuzzy variable-structure controller:

$$u_i = \begin{cases} -\rho_i(t) sign(S_i) - u_{i-}\,, & S_i > 0 \\ 0\,, & S_i = 0 \\ -\rho_i(t) sign(S_i) + u_{i+}\,, & S_i < 0 \end{cases} \tag{3.18}$$

with

$$\rho_i(t) = k_{1i} + k_{2i} + k_{3i}|S_i| + \theta_i^T \psi_i(y)\,, \ \forall i = 1, \dots, n \tag{3.19}$$

Its adaptation laws can be designed as follows

$$\dot{\theta}_i = \gamma_{\theta i}(|S_i|\psi_i(y) - \sigma_{\theta i}|S_i|\theta_i), \text{with } \theta_{ij}(0) > 0 \tag{3.20}$$

$$\dot{k}_{1i} = \gamma_{ki}(\,|S_i| - \sigma_{ki}|S_i|k_{1i}), \text{with } \ k_{1i}(0) > 0 \tag{3.21}$$

with $\gamma_{\theta i}, \gamma_{ki}, \sigma_{\theta i}, \sigma_{ki}, k_{2i}$ and k_{3i} being strictly positive design constants. Note that k_{1i} is the estimate of the unknown term $k_{1i}^* = \overline{\delta}_i + \frac{\sigma_{\theta_i}}{2} \|\theta_i^*\|^2$.

Using Assumption 3.3 and dynamics (3.8), one gets

$$\frac{1}{\eta} S_i \dot{S}_i = \frac{1}{\eta} S_i H_i(x, y, d_i) + \frac{1}{\eta} S_i \varphi_i(u_i) \ \le |S_i|\bar{h}_i(y) + \frac{1}{\eta} S_i \varphi_i(u_i) + \rho_i|S_i| - \rho_i|S_i| \tag{3.22}$$

By (3.17) and replacing (3.19) into (3.22) leads to

$$\frac{1}{\eta} S_i \dot{S}_i \le \frac{1}{\eta} S_i \varphi_i(u_i) + \rho_i|S_i| - \left(k_{1i} + k_{2i} + k_{3i}|S_i| + \tilde{\theta}_i^T \psi_i(y)\right)|S_i| + \delta_i(y) S_i \tag{3.23}$$

Now, we announce a useful technical lemma that will be used in the stability analysis.

Lemma 3.1. *Assume that the input constraint $\varphi_i(u_i)$, defined by (3.10), meets assumption 3.1 and satisfies the useful property (3.11), and if the control law is designed as (3.18), one ha:* $\frac{S_i \varphi_i(u_i)}{\eta} \le -\rho_i(t)|S_i|$, *with* $\eta = min_i\{m_{i+}^*, m_{i-}^*\}$, *for* $i = 1, \dots, n$.

Proof of Lemma 3.1: From (3.11) and (3.18), one can write

If $u_i < -u_{i-}$, one should have $S_i > 0$, and hence

$$(u_i + u_{i-})\varphi_i(u_i) \geq m_{i-}^*(u_i + u_{i-})^2 \geq \eta(u_i + u_{i-})^2 \quad (3.24)$$

If $u_i > u_{i+}$, one should have $S_i < 0$, and therefore

$$(u_i - u_{i+})\varphi_i(u_i) \geq m_{i+}^*(u_i - u_{i+})^2 \geq \eta(u_i - u_{i+})^2 \quad (3.25)$$

By considering (3.18) again, one can establish

For $S_i > 0 \Rightarrow$

$$(u_i + u_{i-})\varphi_i(u_i) = -\rho_i(t)sign(S_i)\varphi_i(u_i) \geq \ m_{i-}^*\rho_i^2(t)[sign(S_i)]^2\eta\rho_i^2(t) \quad (3.26)$$

And for $S_i < 0 \Rightarrow$

$$(u_i - u_{i+})\varphi_i(u_i) = -\rho_i(t)sign(S_i)\varphi_i(u_i) \geq m_{i+}^*\rho_i^2(t)[sign(S_i)]^2 \geq \eta\rho_i^2(t) \quad (3.27)$$

Therefore, for $S_i > 0$ and $S_i < 0$, one obtains

$$-\rho_i(t)sign(S_i)\varphi_i(u_i) \geq \eta\rho_i^2(t) \quad (3.28)$$

Because $S_i sign(S_i) = |S_i|$, (3.28) can be written as follows:

$$-\rho_i(t){S_i}^2 sign(S_i)\varphi_i(u_i) \geq \eta\rho_i^2(t){S_i}^2 = \eta\rho_i^2(t)|S_i|^2 \quad (3.29)$$

Finally, because $\rho_i(t) > 0$, the following inequality holds (for all S_i):

$$\frac{S_i\varphi_i(u_i)}{\eta} \leq -\rho_i(t)|S_i| \quad (3.30)$$

Thus, this ends the proof of Lemma 3.1.

From the concepts mentioned above, one can state the following theorem.

Theorem 3.1. *Consider the master-slave systems (3.1) and (3.2), with Assumptions 3.1. Therefore, the control law, given by (3.18)-(3.21), can guarantee the following properties:*

- *All signals involved in the closed-loop system remain uniformly bounded.*

- *Variable errors,S_i, asymptotically vanish to zero.*

Proof of Theorem 3.1.

Evoking Lemma 3.1, one can write (3.23) as follows:

$$\frac{1}{\eta}S_i\dot{S}_i \leq -\left(k_{1i} + k_{2i} + k_{3i}|S_i| + \tilde{\theta}_i^T\psi_i(y)\right)|S_i| + \delta_i(y) \tag{3.31}$$

Consider the following continuous differentiable function as a Lyapunov function candidate for the subsystem *i*:

$$V_i = \frac{1}{2}{S_i}^2 + \frac{1}{2\gamma_{\theta i}}\left\|\tilde{\theta}_i\right\|^2 + \frac{1}{2\gamma_{ki}}\tilde{k}_{1i}^2. \text{ For} i\text{=1,...,}n \tag{3.32}$$

with $\tilde{k}_{1i} = k_{1i} - k_{1i}^*$, where k_{1i}^*=$\bar{\delta}_i + \frac{\sigma_{\theta i}}{2}\|\theta_i^*\|^2$.

Derivative of V_i is

$$\dot{V}_i = S_i\dot{S}_i + \frac{1}{\gamma_{\theta i}}\tilde{\theta}_i^T\dot{\theta}_i + \frac{1}{\gamma_{ki}}\tilde{k}_{1i}\dot{k}_{1i} \tag{3.33}$$

Considering (3.18)-(3.21), (3.33) becomes

$$\dot{V}_i \leq -k_{3i}S_i^2 - k_{2i}|S_i| - k_{1i}|S_i| - \frac{\sigma_{\theta_i}}{2}\|\theta_i^*\|^2|S_i| + \frac{\sigma_{\theta_i}}{2}\|\theta_i^*\|^2|S_i| + \bar{\delta}_i|S_i| - \sigma_{\theta_i}|S_i|\tilde{\theta}_i^T\theta_i + \frac{1}{\gamma_{ki}}\tilde{k}_{1i}\dot{k}_{1i}$$

$$= -k_{3i}S_i^2 - k_{2i}|S_i| - \frac{\sigma_{\theta_i}}{2}\|\theta_i^*\|^2|S_i| - \sigma_{\theta_i}|S_i|\tilde{\theta}_i^T\theta_i - \tilde{k}_{1i}|S_i| + \frac{1}{\gamma_{ki}}\tilde{k}_{1i}\dot{k}_{1i}$$

$$= -k_{3i}S_i^2 - k_{2i}|S_i| - \frac{\sigma_{\theta_i}}{2}\|\theta_i^*\|^2|S_i| - \sigma_{\theta_i}|S_i|\tilde{\theta}_i^T\theta_i - \sigma_{k_i}|S_i|k_{1i}\tilde{k}_{1i}. \tag{3.34}$$

It is obvious that

$$-\sigma_{\theta_i}|S_i|\tilde{\theta}_i^T\theta_i \leq -\frac{\sigma_{\theta_i}}{2}|S_i|\left\|\tilde{\theta}_i\right\|^2 + \frac{\sigma_{\theta_i}}{2}|S_i|\|\theta_i^*\|^2 \tag{3.35}$$

$$-\sigma_{k_i}|S_i|k_{1i}\tilde{k}_{1i} \leq -\frac{\sigma_{k_i}}{2}|S_i|k_{1i}^2 + \frac{\sigma_{k_i}}{2}|S_i|k_{1i}^{*2} \tag{3.36}$$

Substituting (3.35) and (3.36) into (3.34) yields

$$\dot{V}_i \leq -k_{3i}S_i^2 \tag{3.37}$$

where k_{2i} is selected such as $k_{2i} \geq \frac{\sigma_{k_i}}{2} k_{1i}^{*2}$.

Let $V(t) = \sum_1^n V_i(t)$ be the Lyapunov candidate function of over-all subsystems. Its time derivative $V(t)$ is :

$$\dot{V} = \sum_1^n \dot{V}_i \leq -\sum_1^n k_{3i} S_i^2 \tag{3.38}$$

Hence, all closed-loop signals are bounded. By using the Barbalat’s lemma, one can deduce the asymptotic vanishing of the signal S_i at the origin.

Thus, this ends the proof of Theorem 3.1.

Remark 3.3: *In this chapter, the control law is designed in a variable-structure control framework and augmented by a fuzzy adaptive system to estimate and compensate for the disturbances and uncertain functions, thereby improving robust performance of the proposed control scheme.*

Remark 3.4: *A novel adaptive fuzzy variable-structure control-based GPS for a class of fractional-order chaotic master-slave systems is investigated in this chapter. These chaotic master-slave systems are characterized by unknown models, nonlinear dead-zone at input, and incommensurate fractional-orders, dynamical disturbances and different structures. To the best of authors' knowledge, the GPS problem for this class of chaotic systems has received little attention in the open control literature. Compared to [DEN05, HAM16, LIU18, LU05, MOH17, NOG15, PIS10, SI09, WAN12], the contribution of this paper can be listed as following:*

1) Unlike [DEN05, HAM16, LIU18, LU05, MOH17, NOG15, PIS10, SI09, WAN12], the proposed adaptive control approach does not depend on the master-slave model, as it uses fuzzy systems to model the uncertainties.

2) Unlike the recent and close-related works [LIN11], the stability analysis associated to our control system is rigorously proven. In fact, as stated by [AGH12a, TAV12], the works of [LIN11] are already questionable.

3) Compared to works in [HAM16, LIU18, MOH17, NOG15], the proposed controller has some added advantages that it does not require the restrictive assumption on the model of the input nonlinearities, fractional-order degrees and the external disturbances.

Remark 3.5: *When* $u_{i+} = u_{i-} = u_{i0}$*, (3.18) can be simply rewritten as*

$$u_i = -(\rho_i(t) + u_{i0})Sign(S_i) \tag{3.39}$$

where $\rho_i(t) = k_{1i} + k_{2i} + k_{3i}|S_i| + \theta_i^T \psi_i(y)$ *,* $\forall i = 1, \dots, n$.

To cope with the undesirable chattering effects, one can change the Sign function by an equivalent smooth function (e.g. Tanh: tangent hyperbolic function). In this case, (3.39) becomes:

$$u_i = -(\rho_i(t) + u_{i0})Tanh(S_i/\varepsilon_i) \tag{3.40}$$

with ε_i *being a small positive design constant.*

Remark 3.6: *Theorem 3.1 does not apply the convergence of the adaptive fuzzy parameters* θ_i *to the corresponding ideal vector* θ_i^**.*

Remark 3.7: *A modification of Adams-Bashforth-Moulton algorithm proposed in [DIE02a, DIE02b], implemented in Matlab, is used to numerically solve the fractional-order differential equations (FDE) of the master-slave system (3.1)-(3.2) and to simulate the fractional-order sliding mode surface (3.4). Recall that these FDE and sliding surface are assumed to be modeled by the Caputo operator.*

3.5 Simulation Study

In this section, one will give two simulation examples to show the effectiveness and validity of the above theoretical finding.

3.5.1 Example 1

Consider the synchronization of two fractional-order coupled dynamos systems [WAN09]:

The master system

$$\begin{cases} D_t^{\alpha} x_1 = -\mu x_1 + x_2(x_3 + \gamma), \\ D_t^{\alpha} x_2 = -\mu x_2 + x_1(x_3 - \gamma), \\ D_t^{\alpha} x_3 = x_3 - x_1 x_2, \end{cases} \tag{3.41}$$

The slave system:

$$\begin{cases} D_t^{\beta} y_1 = -\mu y_1 + y_2(y_3 + \gamma) + \varphi_1(u_1) + d_1(t), \\ D_t^{\beta} y_2 = -\mu y_2 + y_1(y_3 - \gamma) + \varphi_2(u_2) + d_2(t), \\ D_t^{\beta} y_3 = y_3 - y_1 y_2 + \varphi_3(u_3) + d_3(t), \end{cases} \tag{3.42}$$

When the fractional order $\alpha \geq 0.87$ and for the parameters $\mu = 2$ and $\gamma = 1$, the system (3.41) can chaotically behave [WAN09].

The simulation was performed using the following initial conditions: $y(0) = [y_1(0), y_2(0), y_3(0)]^T = [-1, -2, 0]^T$ and $x(0) = [x_1(0), x_2(0), x_3(0)]^T = [1,2, -3]$. These initial conditions are chosen randomly, it is worth noting that the proposed control remains valid for any initial conditions given for the master and slave systems.

The input nonlinearities, $\varphi_i(u_i)$ for $i = 1,2,3,$ are supposed to be modeled by:

$$\varphi_i(u_i) = \begin{cases} (u_i - 2)\left(1.5 - 0.3\mathrm{e}^{0.3|\sin(u_i)|}\right), & u_i > 2 \\ 0, & -2 \leq u_i \leq 2 \\ (u_i + 2)\left(1.5 - 0.3\mathrm{e}^{0.3|\sin(u_i)|}\right), & u_i < -2 \end{cases} \tag{3.43}$$

The dynamic disturbances are selected as follows: $d_1(t) = d_2(t) = d_3(t) = 0.25cos(6t) - 0.15sin(t)$.

Three adaptive fuzzy systems, $\theta_i^T \psi_i(y)$ with *i* = 1,2,3, are designed to online approximate the uncertainties. As in [BOU08c], for each input y_i of those fuzzy systems, one can define three membership functions uniformly distributed on a normalized universe of discourse, [-10, 10].

The design parameters are selected as follows: $k_{21} = k_{22} = k_{23} = 0.5, k_{31} = k_{32} = k_{33} = 2,\ \gamma_{\theta 1} = \gamma_{\theta 2} = \gamma_{\theta 3} = 50, \sigma_{\theta 1} = \sigma_{\theta 2} = \sigma_{\theta 3} = 0.0001,\ \gamma_{k1} = \gamma_{k2} = \gamma_{k3} = 5, \sigma_{k1} = \sigma_{k2} = \sigma_{k3} = 0.0001$. The initial conditions for the adaptive parameters are chosen in a random way: $\theta_{1j}(0) = \theta_{2j}(0) = \theta_{3j}(0) = 0.0$ and $k_{11}(0) = k_{12}(0) = k_{13}(0) = 0.1$.

In the following simulation scenarios, one tries to change the fractional-orders ($\alpha\ and\ \beta$) of the master system and the slave one, in order to show that the proposed synchronization system is well functional for any fractional value between 0 and 1. To do this, six simulation cases are considered here, for $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda = 1$.

Case.1: Complete synchronization of two chaotic systems with identical fractional orders by applying the no-smooth controller (i.e. with Sign function):

When $\alpha = \beta = 0.97$ and when the controller (3.39) is employed, the simulation results of this complete synchronization are shown in Figure 3.2. From this figure, despite the attendance of external disturbances, uncertainties and input nonlinearities, a satisfactory complete synchronization between both systems (3.41) and (3.42) is realized. Nevertheless, one can clearly observe the no-smoothness of the control signals and the chattering phenomenon. The latter is caused by the control switches (i.e. by the Sign function in (3.39) or (3.18)) when the system states cross the sliding surface. This chattering is undesirable, as it increases control effort and excites the so-called high-frequency modes of the system. To reduce chattering, a practical solution has been discussed in Remark 3.5, which consists in replacing this Sign function by an equivalent smooth function (e.g. *Tanh*: tangent hyperbolic function). The smooth version of this control law is given by (3.40). In what follows, one discusses the simulation results obtained by this smooth version.

Case.2: Complete synchronization of two chaotic systems with identical fractional orders by applying the smooth controller (i.e. with Tanh function):

In the case where $\alpha = \beta = 0.97$ and the smooth controller (3.40) is used, the obtained simulation results are depicted in Figure 3.3, it is clear from this figure that the slave system effectively follows its master system. The control efforts, displayed by in Figure 3.3, have no chattering, as the responses are smooth. Thus, the control law (3.40) is feasible to be implemented on a real-world system.

Case.3: Complete synchronization of two chaotic systems with different fractional orders by applying the no-smooth controller (3.39):

When $\alpha = 0.91$ and $\beta = 0.97$, and when the controller (3.39) is used, the obtained simulation results are shown in Figure 3.4. From those results, it can be deduced that a complete synchronization between the master system and the slave one is effectively achieved, despite uncertainties, disturbances and input dead-zone. However, the corresponding control input signals suffer from chattering, which is due the high-frequency nature of the proposed variable structure approach (3.39). To overcome this drawback, an equivalent smooth control is given in (3.40). In what follows, one discusses the simulation results obtained by this smooth control law.

Case.4: Complete synchronization of two chaotic systems with different fractional-orders by applying the smooth controller (3.40):

The simulation results, obtained when $\alpha = 0.91$ and $\beta = 0.97$ and when the controller (3.40) is applied, are given in Figure 3.5. It is obvious from this figure that the slave system and the master one are successfully synchronized, and the chattering effect is noticeably reduced. Thus, the control law (3.40) is feasible to be implemented in the practice.

Case. 5: Complete synchronization of an integer-order chaotic system with a fractional-order one by applying the no-smooth controller (3.39):

For $\alpha = 1$ and $\beta = 0.97$, the obtained simulation results are presented in Figure 3.6. This figure shows that, under the proposed no-smooth controller (3.39), the slave state practically converges to that of the master system, despite the attendance of input constraint, dynamic disturbances and uncertainties. However, the chattering phenomenon is clearly present in the control signals.

Case. 6: Synchronization of an integer-order chaotic system with a fractional-order one by applying the smooth controller (3.40):

The simulation results, obtained for $\alpha = 1$ and $\beta = 0.97$ and when the no-smooth controller (3.40) is used, are presented in Figure 3.7 in this case, the closed-loop responses converge to the corresponding desired trajectories without chattering.

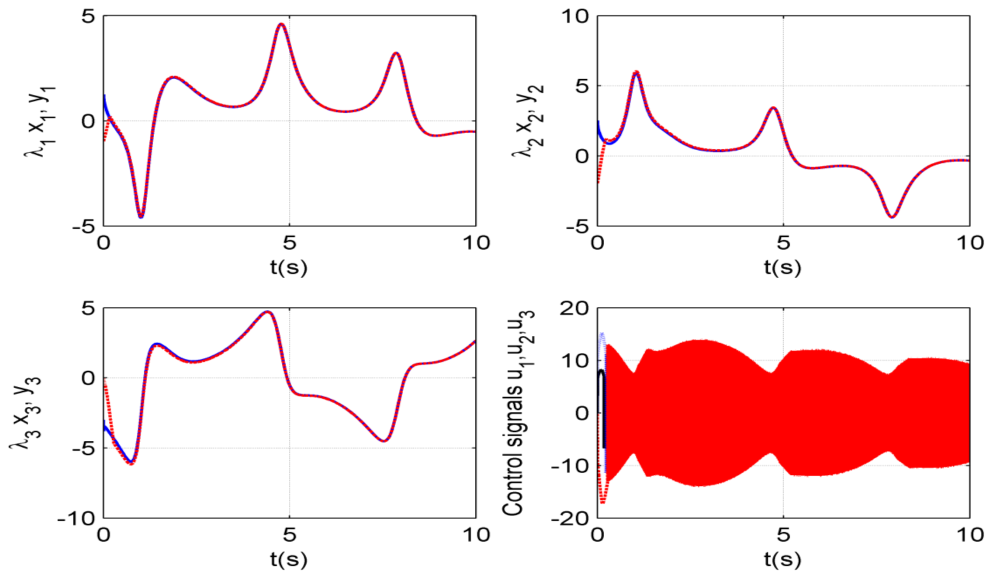


Figure 3.2: *Simulation results with SIGN function for example 1 (when* $\alpha_i = \beta_i = 0.97$ *,* $\lambda_i = 1$ *): (a)* $\lambda_1 x_1$ *(solid line) and* y_1 *(dashed line). (b)* $\lambda_2 x_2$*(solid line) and* y_2 *(dashed line). (c)*$\lambda_3 x_3$ *(solid line) and* y_3 *(dashed line). (d)* u_1 *(solid line),* u_2 *(dotted line) and* u_3 *(dashed line).*

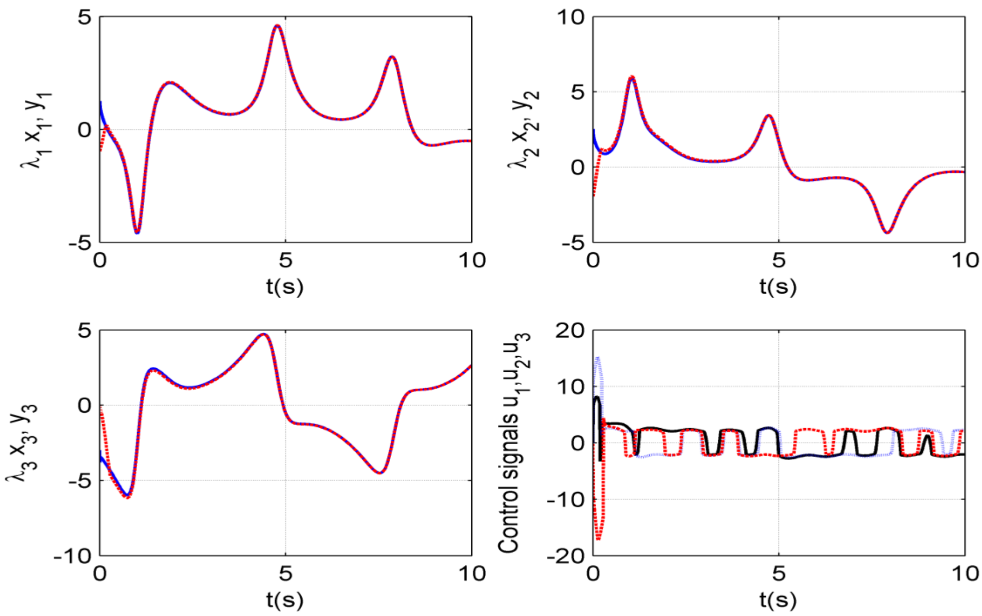


Figure 3.3: *Simulation results with TANH function for example 1 (when* $\alpha_i = \beta_i = 0.97$ *,* $\lambda_i = 1$ *): (a)* $\lambda_1 x_1$ *(solid line) and* y_1 *(dashed line). (b)* $\lambda_2 x_2$*(solid line) and* y_2 *(dashed line). (c)*$\lambda_3 x_3$ *(solid line) and* y_3 *(dashed line). (d)* u_1 *(solid line),* u_2 *(dotted line) and* u_3 *(dashed line).*

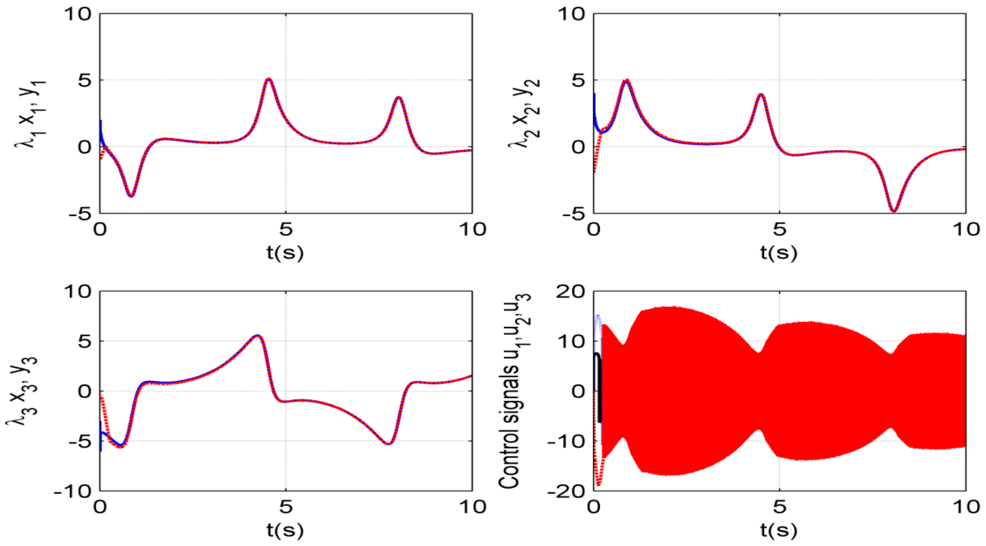


Figure 3.4: *Simulation results with SIGN function for example 1 ($\alpha_i = 0.91$, $\beta_i = 0.97$ and $\lambda_i = 1$): (a) $\lambda_1 x_1$ (solid line) and y_1 (dashed line). (b) $\lambda_2 x_2$(solid line) and y_2 (dashed line). (c)$\lambda_3 x_3$ (solid line) and y_3 (dashed line). (d) u_1 (solid line), u_2 (dotted line) and u_3 (dashed line).*

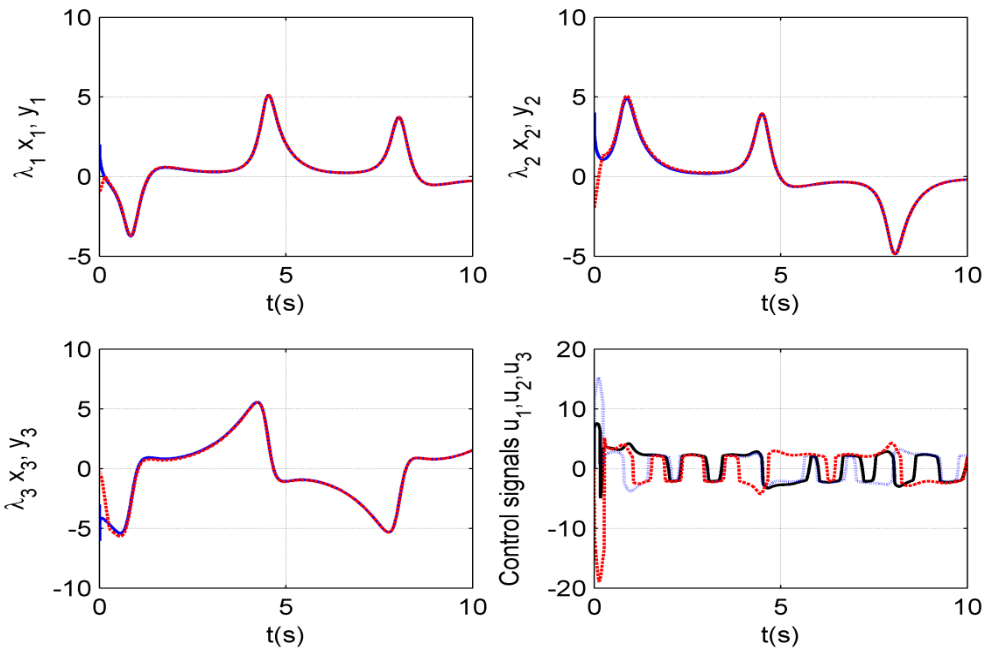


Figure 3.5: *Simulation results with TANH function for example 1 ($\alpha_i = 0.91$, $\beta_i = 0.97$ and $\lambda_i = 1$): (a) $\lambda_1 x_1$ (solid line) and y_1 (dashed line). (b) $\lambda_2 x_2$(solid line) and y_2 (dashed line). (c)$\lambda_3 x_3$ (solid line) and y_3 (dashed line). (d) u_1 (solid line), u_2 (dotted line) and u_3 (dashed line).*

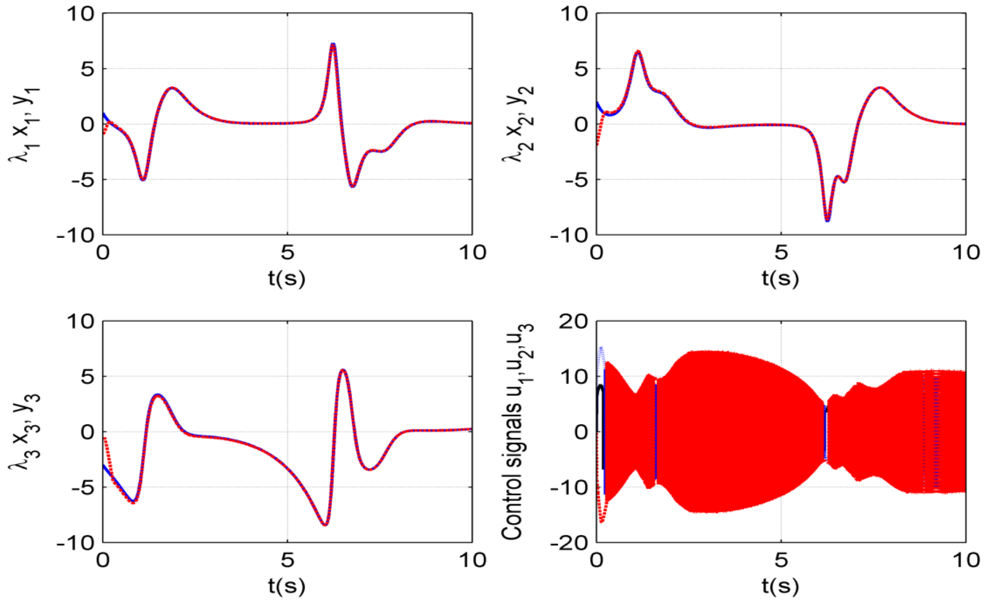


Figure 3.6: *Simulation results with SIGN function for example 1 (when* $\alpha_i = 1, \beta_i = 0.91$ *and* $\lambda_i = 1$*): (a)* $\lambda_1 x_1$ *(solid line) and* y_1 *(dashed line). (b)* $\lambda_2 x_2$*(solid line) and* y_2 *(dashed line). (c)* $\lambda_3 x_3$ *(solid line) and* y_3 *(dashed line). (d)* u_1 *(solid line),* u_2 *(dotted line) and* u_3 *(dashed line).*

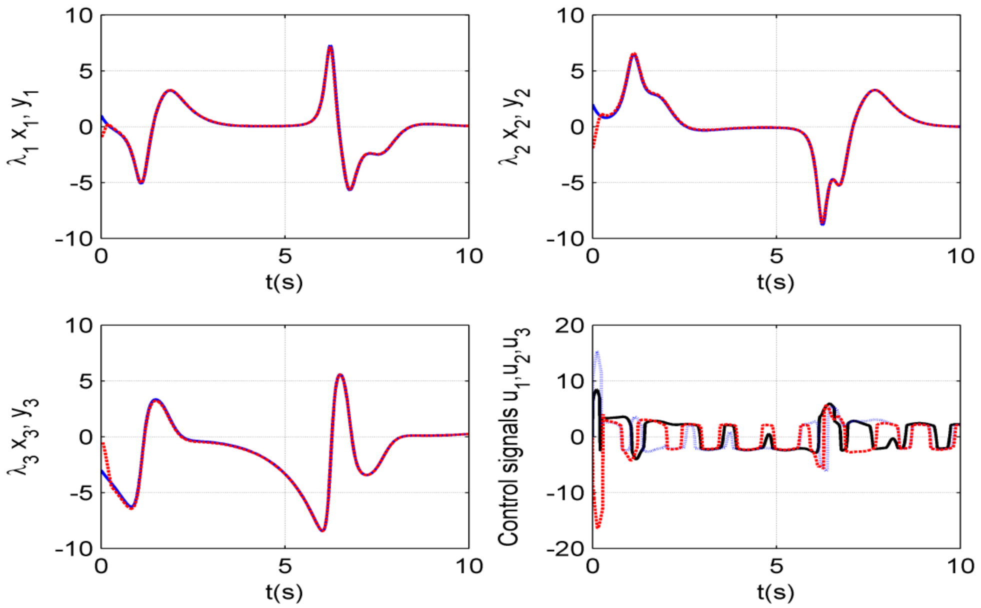


Figure 3.7: *Simulation results with TANH function for example 1 (when* $\alpha_i = 1, \beta_i = 0.91$ *and* $\lambda_i = 1$*): (a)* $\lambda_1 x_1$ *(solid line) and* y_1 *(dashed line). (b)* $\lambda_2 x_2$*(solid line) and* y_2 *(dashed line). (c)* $\lambda_3 x_3$ *(solid line) and* y_3 *(dashed line). (d)* u_1 *(solid line),* u_2 *(dotted line) and* u_3 *(dashed line).*

3.5.2 Example 2

Now, one considers the GPS of two different chaotic systems. According to the values of λ_i and the controller applied (smooth or no-smooth controller), four simulation cases will be made.

The master system is a fractional-order Chua's Oscillator, [ZHU09]:

$$\begin{cases} D_t^{\alpha_1} x_1 = a(x_2 - x_1 - f(x_1)), \\ D_t^{\alpha_2} x_2 = x_1 - x_2 + x_3, \\ D_t^{\alpha_3} x_3 = -bx_2 - cx_3, \end{cases} \tag{3.44}$$

with $f(x_1) = m_1 x_1 + \frac{1}{2}(m_0 - m_1)(|x_1 + 1| - |x_1 - 1|), m_0 = -1.1726,\ m_1 = -0.7872,$ $a = 10{,}725,\ b = 10.593,\ c = 0.268.$ According to [ZHU09], the system (3.44) can chaotically behave, for $\alpha_1 = 0.93, \alpha_2 = 0.99$ and $\alpha_3 = 0.92$.

The slave system is a fractional-order financial system, [BHA14, CHE08]:

$$\begin{cases} D_t^{\beta_1} y_1 = y_3 + (y_2 - a_1) y_1 + \varphi_1(u_1) + d_1(t), \\ D_t^{\beta_2} y_2 = 1 - b_1 y_2 - y_1^2 + \varphi_2(u_2) + d_2(t), \\ D_t^{\beta_3} y_3 = -y_1 - c_1 y_3 + \varphi_3(u_3) + d_3(t), \end{cases} \tag{3.45}$$

where $c_1 = 1$ is the elasticity of demand of commercial markets, $b_1 = 0.1$ is the cost per investment and $a_1 = 3$ is the saving amount, [BHA14, CHE08]. For $u_i = 0$ and $d_i(.)$=0, the system (3.45) can chaotically behave for $\beta_1 = 0.97$, $\beta_2 = 0.90$ and $\beta_3 = 0.96$ [BHA14, CHE08].The external disturbances are : $d_1(t) = d_2(t) = d_3(t) = 0.2sin(3t) + 0.2cos(3t)$. The initial conditions of the master and slave systems are freely selected : $x(0) = [0.2, -0.1, 0.1]$and $y(0) = [2, 1, 2]^T$. The input nonlinearities, $\varphi_i(u_i)$ for = 1,2,3, are modeled as

$$\varphi_i(u_i) = \begin{cases} (u_i - 3)\left(1.5 - 0.3\mathrm{e}^{0.3|\sin(u_i)|}\right), & u_i > 3 \\ 0, & -3 \le u_i \le 3 \\ (u_i + 3)\left(1.5 - 0.3\mathrm{e}^{0.3|\sin(u_i)|}\right), & u_i < -3 \end{cases} \tag{3.46}$$

Three adaptive fuzzy systems, $\theta_i^T \psi_i(y)$ with *i* =1,2,3, are designed to online approximate the uncertainties. As in [BOU08c], for each input y_i of those fuzzy systems, one can define three membership functions uniformly distributed on a normalized universe of discourse, [-2, 2]. The design parameters are selected as follows: $k_{21} = k_{22} = k_{23} = 2, k_{31} = k_{32} = k_{33} =$

10, $\gamma_{\theta 1} = \gamma_{\theta 2} = \gamma_{\theta 3} = 500$, $\sigma_{\theta 1} = \sigma_{\theta 2} = \sigma_{\theta 3} = 0.0001$, $\gamma_{k1} = \gamma_{k2} = \gamma_{k3} = 5$, $\sigma_{k1} = \sigma_{k2} = \sigma_{k3} = 0.0001$. The initial conditions for the adaptive parameters are chosen in a random way : $\theta_{1j}(0) = \theta_{2j}(0) = \theta_{3j}(0) = 0.0$ and $k_{11}(0) = k_{12}(0) = k_{13}(0) = 0.1$. Now, one tries to change the scaling factor λ_i, in order to validate the proposed adaptive fuzzy control methodology for the different types of synchronization, namely: anti-phase projective synchronization and generalized projective synchronization. Four simulation experiments are considered here.

Case.1: Anti-phase projective synchronization ($\boldsymbol{\lambda_i = -0.55}$) by applying the no-smooth controller (3.39):

The results of this numerical simulation are presented in Figure 3.8. It is seen that the slave state trajectories (y_1, y_2, y_3) quickly track those of the master $(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$, despite the presence of uncertainties, possible external disturbances and input nonlinearities. Hence, a projective anti-phase synchronization between both systems is successfully realized. The control input signals are presented in Figure 3.8(d). It is clear from this figure that the control signals are no-smooth and with chattering.

Case.2: Anti-phase projective synchronization ($\boldsymbol{\lambda_i = -0.55}$) by applying the smooth controller (3.40):

For $\lambda_i = -0.55$ and when this smooth controller is used, the obtained simulation results are illustrated in Figure 3.9. It is obvious that the slave state trajectories (y_1, y_2, y_3) effectively track those of the master $(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$ and the control inputs are smooth. The chattering effect present in the previous case, excited by the switching gain due to high activity of the control input, is considerably reduced here by using the smooth equivalent law (3.40).

Case. 3: Generalized projective synchronization ($\boldsymbol{\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 0.1}$), by applying the no-smooth controller (3.39):

The simulation results of this case are depicted in Figure 3.10. It is clear from those results that the proposed controller (3.39) effectively synchronizes the slave system with the master one.

Case. 4: Generalized projective synchronization ($\boldsymbol{\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 0.1}$), by applying the smooth controller (3.40):

For the smooth controller (3.40) and $(\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0.1)$, the obtained simulation results are given in Figure 3.11. In this case, the closed-loop responses converge to the corresponding desired trajectories without chattering.

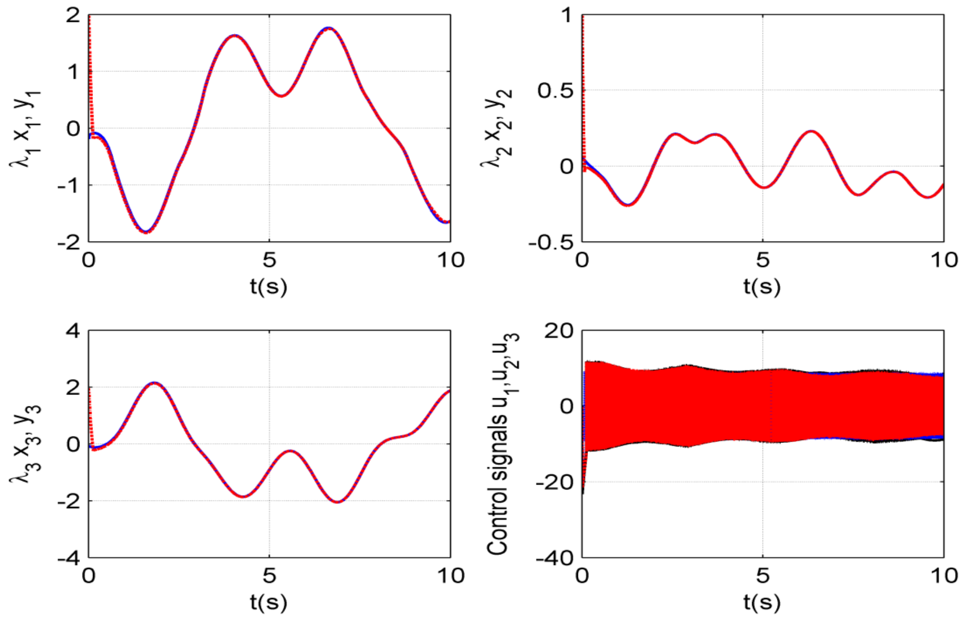


Figure 3.8: *Simulation results with SIGN function for example 2 (when $\lambda_i = -0.55$): (a)$\lambda_1 x_1$ (solid line) and y_1 (dashed line). (b) $\lambda_2 x_2$ (solid line) and y_2 (dashed line). (c) $\lambda_3 x_3$ (solid line) and y_3 (dashed line). (d) u_1 (solid line), u_2 (dotted line) and u_3 (dashed line).*

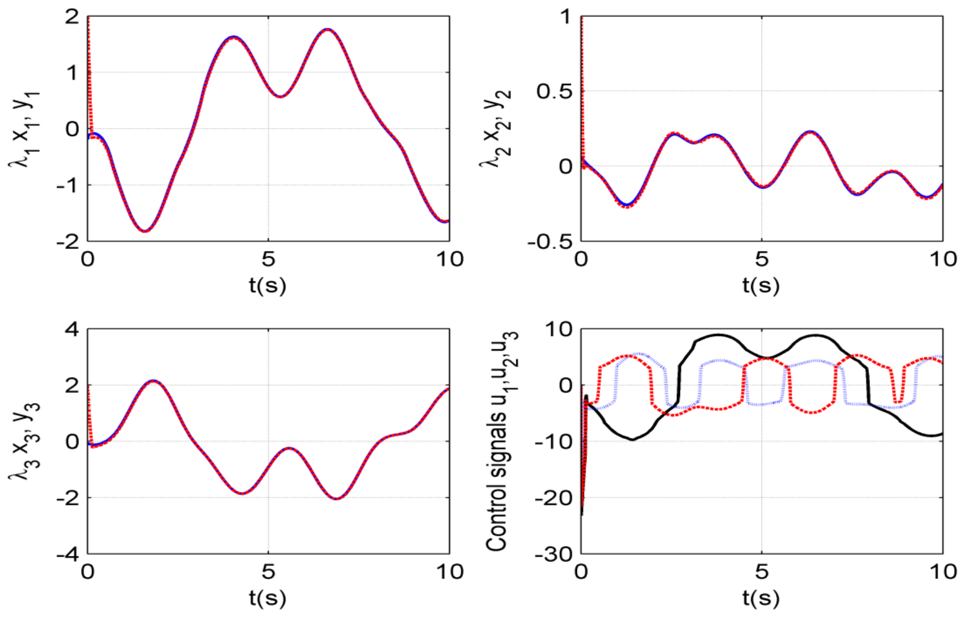


Figure 3.9: *Simulation results with TANH function for example 2 (when $\lambda_i = -0.55$): (a)$\lambda_1 x_1$ (solid line) and y_1 (dashed line). (b) $\lambda_2 x_2$ (solid line) and y_2 (dashed line). (c) $\lambda_3 x_3$ (solid line) and y_3 (dashed line). (d) u_1 (solid line), u_2 (dotted line) and u_3 (dashed line).*

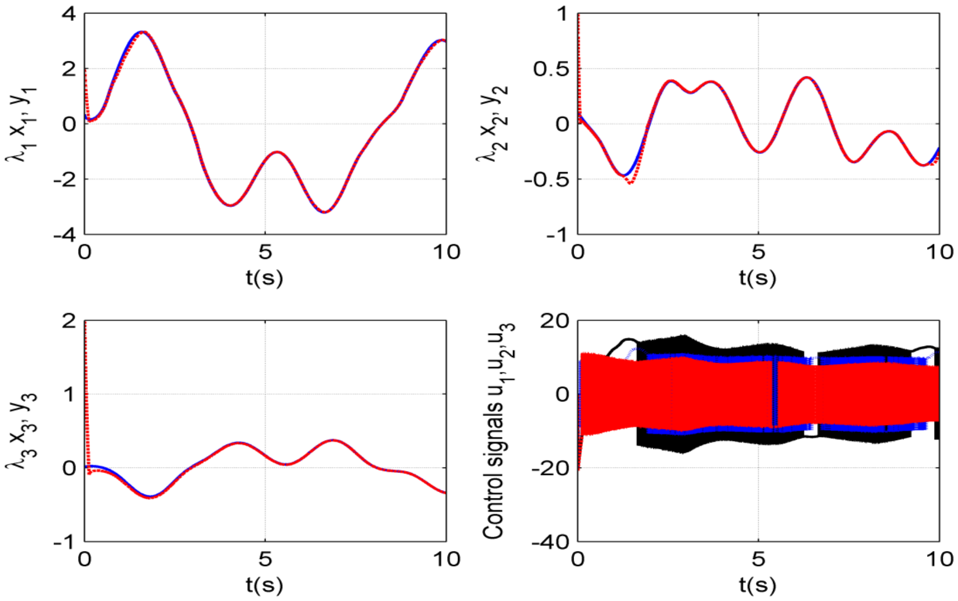


Figure 3.10: *Simulation results with SIGN function for example 2 ($\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0.1$): (a) $\lambda_1 x_1$ (solid line) and y_1 (dashed line). (b) $\lambda_2 x_2$ (solid line) and y_2 (dashed line). (c) $\lambda_3 x_3$ (solid line) and y_3 (dashed line). (d) u_1 (solid line), u_2 (dotted line) and u_3 (dashed line).*

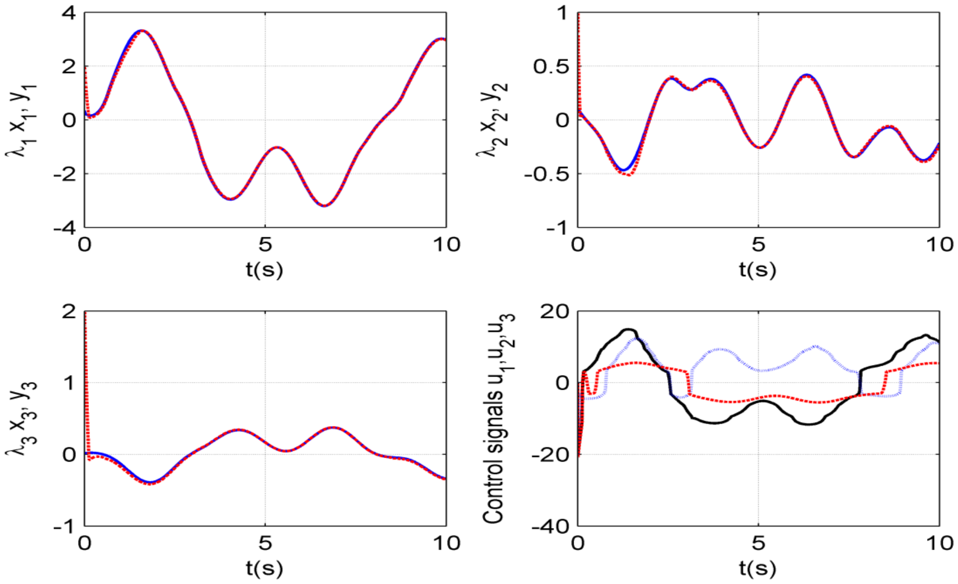


Figure 3.11: *Simulation results with TANH function for example 2 ($\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0.1$): (a) $\lambda_1 x_1$ (solid line) and y_1 (dashed line). (b) $\lambda_2 x_2$(solid line) and y_2 (dashed line). (c)$\lambda_3 x_3$ (solid line) and y_3 (dashed line). (d) u_1 (solid line), u_2 (dotted line) and u_3 (dashed line).*

3.6 Conclusion

This chapter has investigated to resolve the problem of GPS of uncertain incommensurate fractional-order chaotic systems with input constraint (dead-zone together with sector nonlinearities). The proposed control method uses the adaptive control concept to estimate the uncertain parameters, the variable-structure control framework to robustly deal with the input nonlinearities and dynamic disturbances, and the fuzzy systems to model nonlinear uncertain functions of the slave-slave model. Under some mild assumptions, the stability and the convergence of the synchronization errors have been rigorously demonstrated. Some simulation experiments have been presented in order to verify the validity of the proposed GPS system and its associated adaptive fuzzy control.

Chapter 4

Adaptive Synchronization of Uncertain Fractional-Order Chaotic triangular Systems via Fuzzy Backstepping Control

4.1 Introduction

Backstepping control design for triangular systems is based on the additive property of Lyapunov functions. In the triangular systems, the control input variables appear in the last stage (or in the last subsystem) of the entire systems [KRS95, KWA20, PAN16]. Backstepping control design aims at the tracking (or stabilizing around the origin) of the output in the first stage (or the first subsystem). It is virtually an output tracking control design procedure. Through a series of intermediate virtual control (fictitious control or stabilizing function) designs starting out with the first the subsystem, the backstepping concept eventually turns out the specification of the control variable as a function of the system states. The virtual control (fictitious control) terms should be determined according to the Lyapunov function of the corresponding subsystem and, at the same time, be differentiable up to a required order depending on the steps in the backstepping concept. The classical backstepping control methods [KRS95, KWA20, PAN16], which are just applicable for integer-order nonlinear systems, cannot be directly extended for fractional-order nonlinear systems, since the fractional-order differentiation of a quadratic Lyapunov function is already difficult. Some authors have attempted to utilize the backstepping procedure to design control laws for a class of fractional-order systems [EFE11, BAL11, DIN14]. Some adaptive backstepping control systems have been proposed in [SHE17, WEI15, WIE16], for uncertain triangular nonlinear systems with fractional-order models. In order to study the stability of the associated closed-loop system, an integer-order Lyapunov method has been utilized. In [LIU17b, BIG17, NIK16, SHU17a, SHU17b, SHU18], via a fractional-order Lyapunov stability approach, some adaptive backstepping control methodologies for fractional-order nonlinear systems have been designed. However, these fractional backstepping control schemes suffer following shortcomings:

- The control laws in [LIU17b, NIK16, SHU17a] have not been rigorously developed and analyzed, as the virtual control terms have been designed as discontinuous (therefore not differentiable) functions.
- Triangular models under-consideration are assumed to be known, e.g. [SHE17, WEI15, WIE16, NIK16, SHU17a, SHU17b, SHU18].
- Unmatched disturbances were not taken into account in the model [SHE17, WEI15, WIE16, LIU17c, BIG17, NIK16, SHU17a, SHU17b, SHU18].

Motivated by above-cited drawbacks in the previous backstepping control schemes [SHE17, WEI15, WIE16, LIU17c, BIG17, NIK16, SHU17a, SHU17b, SHU18], this chapter focuses on the design of a novel fuzzy adaptive backstepping control to attain an accurate projective synchronization of chaotic fractional-order master-slave systems. The chaotic systems under-consideration are assumed to be with a lower triangular structure, fractional-order and uncertain dynamics, and subject to both matched and unmatched external disturbances. In each backstepping step, the unknown nonlinear functions are online modeled via fuzzy logic systems, and the so-called virtual control laws are properly constructed based on the fractional Lyapunov-stability. A fractional fuzzy adaptive control, guaranteeing the convergence of the synchronization errors as well as the stability of the closed-loop system, is designed at the backstepping last step. Make a comparison with the recently available results [SHE17, WEI15, WIE16, LIU17c, BIG17, NIK16, SHU17a, SHU17b, SHU18], the most important advantages and contributions of our proposed synchronization system based on fractional backstepping control can be summarized as follows:

1) A novel continuous adaptive fuzzy backstepping control method is investigated to deal with a projective synchronization problem of uncertain fractional-order chaotic master-slave systems. Note that the design of a fuzzy backstepping control law for this class of systems is already challenging, as the fractional-order differentiation of a Lyapunov function (even for a quadratic simple function) is in general difficult.

2) Via a fractional Lyapunov method, the stability of the closed-loop system is carefully analyzed.

3) The uncertain dynamics are online approximated by using adaptive fuzzy systems. Notably, their update are designed as fractional order ones.

4) Unlike the previous works [SHE17, WEI15, WIE16, LIU17c, BIG17, NIK16, SHU17a, SHU17b, SHU18], both matched and unmatched disturbances are considered in the system model.

4.2 Adaptive Synchronization Based on Fuzzy Backstepping Control

4.2.1 Problem Statement

Consider the following class of FO chaotic **master systems**:

$$Master\begin{cases} D^{\alpha}x_i = f_{mi}(\underline{x}_i),\ i=1,...,n-1 \\ D^{\alpha}x_n = f_{mn}(\underline{x}_n) \end{cases} \tag{4.1}$$

where $0<\alpha\le 1$ is the fractional-order, $\underline{x}_n = x = [x_1, x_2, \ldots, x_n]^T \in R^n$ is the pseudo-state vector of the master system, $\underline{x}_i = [x_1, x_2, \ldots, x_i]^T \in R^i$, $f_{mi}(\underline{x}_i) \in R$, for $i=1,\ldots,n$, are uncertain nonlinear smooth functions.

The slave system, having a different structure from the master one, is defined as:

$$Slave\begin{cases} D^{\alpha}y_i = f_{si}(\underline{y}_i) + g_{si}y_{i+1} + d_{si}(t),\ i=1,...,n-1 \\ D^{\alpha}y_n = f_{sn}(\underline{y}_n) + g_{sn}u + d_{sn}(t) \end{cases} \tag{4.2}$$

where $0<\alpha\le 1$ is the fractional-order, $y(t) = [y_1, y_2, \ldots, y_n]^T \in R^n$ is the pseudo-state vector of the slave system, $\underline{y}_i = [y_1, y_2, \ldots, y_i]^T \in R^i$, $f_{si}(\underline{x}_i) \in R$, for $i=1,\ldots,n$, are uncertain nonlinear smooth functions. $g_{si} \neq 0$, $i=1,\ldots,n$, are known real constants. u is the control input. d_{si} are the unknown external disturbances which are assumed to be bounded by an unknown constant as follows: $|d_{si}(t)| \le \bar{d}_{si}$, for $i=1,\ldots,n$.

In the rest of this chapter, D^{α} stands for Caputo's derivative.

Objective of this chapter is to design a fuzzy backstepping controller u ensuring a practical projective synchronization between the systems (4.1) and (4.2), such that: (i) All signals in the closed-loop control system are ultimately stable. (ii) The synchronization errors remain in a small but adjustable neighborhood of zero.

Remark 4.1: *The master-slave systems under consideration are characterized by the following important properties:*

1- The slave system is only controlled via a scalar control input. Note that this feature is already of practical significance and importance.

2- Many FO chaotic systems can be modeled as (4.1) and (4.2), namely: FO Arneodo's system, FO Duffing's systems, FO unified chaotic system, tonamefew.

3- Unlike many previous works dealing with the backstepping control problem for fractional-order systems [SHE17, WEI15, WIE16, NIK16, SHU17a, SHU17b, SHU18], the master-slave model is uncertain (almost unknown) in this chapter.

4- Unlike the early works in [SHE17, WEI15, WIE16, LIU17C, BIG17, NIK16, SHU17A, SHU17B, SHU18], the slave model is assumed to be subject to matched and unmatched external disturbances. Note the control design problem for triangular chaotic systems subject to unmatched disturbances is quite challenge.

The following lemma is useful for the subsequent developments.

Lemma 4.1 [BOU15c, POL96]**.** *The following inequality holds for any* $\tau > 0$ *and for any* $X \in R$ *:*

$$X \in R,\ 0 \le |X| - X\,Tanh(\mathrm{X}/\tau) \le \bar{\tau} = \kappa\tau \tag{4.3}$$

where Tanh(.) denotes the hyperbolic tangent usual function, $\kappa > 0$ *is a small constant that should satisfy the following nonlinear equation* $\kappa = e^{-(1+\kappa)}$, *whose solution is* $\kappa = 0.2785$.

The synchronization errors are defined as

$$e_i = y_i(t) - \lambda_i x_i(t),\ \ i = 1,...,n, \tag{4.4}$$

where λ_i, the scaling factor, is a real number.

From (4.1), (4.2) and (4.4), the dynamics of the synchronization can be written as:

$$\begin{cases} D^{\alpha}e_i = f_{si}(\underline{y}_i) + g_{si}e_{i+1} - \lambda_i f_{mi}(\underline{x}_i) + \lambda_{i+1}g_{si}x_{i+1} + d_{si}, \quad i = 1,\dots,n-1 \\ D^{\alpha}e_n = f_{sn}(\underline{y}_n) + g_{sn}u - \lambda_n f_{mn}(\underline{x}_n) + d_{sn} \end{cases} \tag{4.5}$$

or equivalently

$$\begin{cases} D^{\alpha}e_i = h_i(\underline{y}_i, \underline{x}_{i+1}) + g_{si}e_{i+1} + d_{si}, \quad i = 1,\dots,n-1 \\ D^{\alpha}e_n = h_n(\underline{y}_n, \underline{x}_n) + g_{sn}u + d_{sn} \end{cases} \tag{4.6}$$

where $h_i(\underline{y}_i, \underline{x}_{i+1}) = f_{si}(\underline{y}_i) - \lambda_i f_{mi}(\underline{x}_i) + \lambda_{i+1}g_{si}x_{i+1}$, for $i = 1,\dots,n-1$ and $h_n(\underline{y}_n, \underline{x}_n) = f_{sn}(\underline{y}_n) - \lambda_n f_{mn}(\underline{x}_n)$.

4.2.2 Design of the fuzzy adaptive backstepping control

The control design method, that is based on the well-known backstepping concept, consists in *n* steps.

Step.1 (Stabilization of the first subsystem in (4.6)): The uncertain smooth function $h_1(\underline{y}_1, \underline{x}_2)$ can be online modeled by the fuzzy system (1.2), on a compact set $\Omega_{\underline{y}_1}$, as follows:

$$h_1 = \theta_1^T \psi_1\left(\underline{y}_1\right) \tag{4.7}$$

where $\psi_1\left(\underline{y}_1\right)$ is the FBF vector which is supposed to be adequately determined by the designer, and θ_1 is the adjustable parameter vector [LIU15b, TON13, LIU14, LI15B, LIN17, LI13, BOU08d].

The best value of θ_1 can be defined by:

$$\theta_1^* = \arg\min_{\theta_1}\left[\sup_{\underline{y}_1 \in \Omega_{\underline{y}_1}} \left| h_1(\underline{y}_1, \underline{x}_2) - \theta_1^T \psi_1\left(\underline{y}_1\right)\right|\right] \tag{4.8}$$

Denote also

$$\tilde{\theta}_1 = \theta_1 - \theta_1^* \text{ and } \varepsilon_1\left(\underline{y}_1, \underline{x}_2\right) = h_1(\underline{y}_1, \underline{x}_2) - \theta_1^{*T}\psi_1\left(\underline{y}_1\right) \tag{4.9}$$

as the parameter error vector and the approximation error, respectively.

This approximation error, $\varepsilon_1\left(\underline{y}_1, \underline{x}_2\right)$, should satisfy [LIU15b, TON13, LIU14, LI15B, LIN17, LI13, BOU08d] :

$$\left|\varepsilon_1\left(\underline{y}_1, \underline{x}_2\right)\right| \le \bar{\varepsilon}_1, \quad \forall \underline{x}_2 \in \Omega_{\underline{x}_2}, \underline{y}_1 \in \Omega_{\underline{y}_1} \tag{4.10}$$

where $\bar{\varepsilon}_1$ is an unknown positive constant. Since the fuzzy logic systems are universal approximtors, this approximation error may be made arbitrary small by choosing properly this fuzzy system (of course this may need an arbitrary large number of rules).

From (4.7) and (4.9), one can get

$$h_1(\underline{y}_1, \underline{x}_2) - \theta_1^T \psi_1\left(\underline{y}_1\right) = h_1(\underline{y}_1, \underline{x}_2) - \theta_1^{*T} \psi_1\left(\underline{y}_1\right) + \theta_1^{*T} \psi_1\left(\underline{y}_1\right) - \theta_1^T \psi_1\left(\underline{y}_1\right)$$

$$= -\tilde{\theta}_1^T \psi_1\left(\underline{y}_1\right) + \varepsilon_1\left(\underline{y}_1, \underline{x}_2\right) \tag{4.11}$$

Denote $\omega_1 = e_1$. Thus, from (4.6) and (4.11), we can write

$$D^\alpha \omega_1 = h_1(\underline{y}_1, \underline{x}_2) + g_{s1} e_2 + d_{s1}$$

$$= -\tilde{\theta}_1^T \psi_1\left(\underline{y}_1\right) + \theta_1^T \psi_1\left(\underline{y}_1\right) + g_{s1} e_2 + d_{s1} + \varepsilon_1\left(\underline{y}_1, \underline{x}_2\right) \tag{4.12}$$

Design the first virtual control input α_1 and its update laws as follows

$$\alpha_1 = g_{s1}^{-1}\left(-\theta_1^T \psi_1\left(\underline{y}_1\right) Tanh\left(\frac{\omega_1 \theta_1^T \psi_1\left(\underline{y}_1\right)}{\zeta_1}\right) - k_1 \omega_1 - \rho_1 Tanh(\omega_1 / \tau_1)\right) \tag{4.13}$$

$$D^\alpha \theta_1 = \gamma_{\theta 1} \omega_1 \psi_1\left(\underline{y}_1\right) - \gamma_{\theta 1} \sigma_{\theta 1} \theta_1 \tag{4.14}$$

$$D^\alpha \rho_1 = \gamma_{\rho 1} \omega_1 Tanh(\omega_1 / \tau_1) - \gamma_{\rho 1} \sigma_{\rho 1} \rho_1 \tag{4.15}$$

with k_1, $\gamma_{\theta 1}$, and $\gamma_{\rho 1}$ being strictly positive design parameters. $\sigma_{\theta 1}$, $\sigma_{\rho 1}, \zeta_1$, and τ_1 are some small strictly positive design parameters.

Note that ρ_1 is the estimate the uncertain term $\rho_1^* = \bar{d}_{s1} + \bar{\varepsilon}_1$.

Remark 4.2: *The virtual control input (4.13) is smooth and essentially comprised of three main terms, namely:*

- ***a fuzzy term,*** $\theta_1^T \psi_1\left(\underline{y}_1\right) Tanh\left(\dfrac{\omega_1 \theta_1^T \psi_1\left(\underline{y}_1\right)}{\zeta_1}\right)$*, employed to model the uncertain functions* $h_1(\underline{y}_1, \underline{x}_2)$*,*
- ***a linear control term****,* $k_1\omega_1$*, introduced for stability purposes.*
- ***a robust-like control term,*** $\rho_1 Tanh(\omega_1 / \tau_1)$**,** *used to compensate for the fuzzy approximation errors and the disturbances.*

Remark 4.3: *The update laws (4.14) and (4.15) contain two main terms, namely:*

- σ***-modification terms****,* $\gamma_{\theta 1}\sigma_{\theta 1}\theta_1$ *and* $\gamma_{\rho 1}\sigma_{\rho 1}\rho_1$*, guarantee the boundedness of the estimated parameters, and*
- ***Gradient terms,*** $\gamma_{\theta 1}\omega_1\psi_1\left(\underline{y}_1\right)$ *and* $\gamma_{\rho 1}\omega_1 Tanh(\omega_1 / \tau_1)$*, are employed to cancel the terms produced by fractional-differentiating of a quadratic Lyapunov function.*

Since the Caputo fractional derivative of a constant is zero, (4.14) and (4.15) can be simply expressed as follows:

$$D^{\alpha}\tilde{\theta}_1 = \gamma_{\theta 1}\omega_1\psi_1\left(\underline{y}_1\right) - \gamma_{\theta 1}\sigma_{\theta 1}\theta_1 \tag{4.16}$$

$$D^{\alpha}\tilde{\rho}_1 = \gamma_{\rho 1}\omega_1 Tanh(\omega_1 / \tau_1) - \gamma_{\rho 1}\sigma_{\rho 1}\rho_1 \tag{4.17}$$

where $\tilde{\theta}_1 = \theta_1 - \theta_1^*$ and $\tilde{\rho}_1 = \rho_1 - \rho_1^*$.

Let's denote $\omega_2 = e_2 - \alpha_1$. From (4.12) and (4.13), the fractional dynamics of ω_1 can be written as:

$$D^{\alpha}\omega_1 = -\tilde{\theta}_1^T\psi_1\left(\underline{y}_1\right) + \theta_1^T\psi_1\left(\underline{y}_1\right) + g_{s1}e_2 + d_{s1} + \varepsilon_1\left(\underline{y}_1, \underline{x}_2\right)$$

$$=-\tilde{\theta}_1^T\psi_1\left(\underline{y}_1\right)+\theta_1^T\psi_1\left(\underline{y}_1\right)+g_{s1}\left(e_2-\alpha_1\right)+g_{s1}\alpha_1+d_{s1}+\varepsilon_1\left(\underline{y}_1,\underline{x}_2\right)$$

$$\begin{aligned}&=-\tilde{\theta}_1^T\psi_1\left(\underline{y}_1\right)+\theta_1^T\psi_1\left(\underline{y}_1\right)+g_{s1}\omega_2-\theta_1^T\psi_1\left(\underline{y}_1\right)Tanh\left(\frac{\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)}{\zeta_1}\right)\\&\quad-k_1\omega_1-\rho_1Tanh(\omega_1/\tau_1)+d_{s1}+\varepsilon_1\left(\underline{y}_1,\underline{x}_2\right)\end{aligned}\tag{4.18}$$

Multiplying both sides of (4.18) by ω_1 yields

$$\begin{aligned}\omega_1D^\alpha\omega_1&=-\omega_1\tilde{\theta}_1^T\psi_1\left(\underline{y}_1\right)+\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)+g_{s1}\omega_1\omega_2-\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)Tanh\left(\frac{\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)}{\zeta_1}\right)-k_1\omega_1^2\\&\quad-\rho_1\omega_1Tanh(\omega_1/\tau_1)+\omega_1d_{s1}+\omega_1\varepsilon_1\left(\underline{y}_1,\underline{x}_2\right)\end{aligned}\tag{4.19}$$

Construct the following Lyapunov function candidate

$$V_1=\frac{1}{2}\omega_1^2+\frac{1}{2\gamma_{\theta1}}\tilde{\theta}_1^T\tilde{\theta}_1+\frac{1}{2\gamma_{\rho1}}\tilde{\rho}_1^2\tag{4.20}$$

Evoking Lemma 1.2, the Caputo fractional derivative of (4.20) is:

$$D^\alpha V_1=\frac{1}{2}D^\alpha\omega_1^2+\frac{1}{2\gamma_{\theta1}}D^\alpha\tilde{\theta}_1^T\tilde{\theta}_1+\frac{1}{2\gamma_{\rho1}}D^\alpha\tilde{\rho}_1^2\le\omega_1D^\alpha\omega_1+\frac{1}{\gamma_{\theta1}}\tilde{\theta}_1^TD^\alpha\tilde{\theta}_1+\frac{1}{\gamma_{\rho1}}\tilde{\rho}_1D^\alpha\tilde{\rho}_1\tag{4.21}$$

Using Lemma 4.1 and substituting (4.16), (4.17) and (4.19) into (4.21) yields

$$\begin{aligned}D^\alpha V_1&\le-\omega_1\tilde{\theta}_1^T\psi_1\left(\underline{y}_1\right)+\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)+g_{s1}\omega_1\omega_2-\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)Tanh\left(\frac{\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)}{\zeta_1}\right)-\rho_1\omega_1Tanh(\omega_1/\tau_1)\\&\quad-k_1\omega_1^2+\frac{1}{\gamma_{\theta1}}\tilde{\theta}_1^T\left(\gamma_{\theta1}\omega_1\psi_1\left(\underline{y}_1\right)-\gamma_{\theta1}\sigma_{\theta1}\theta_1\right)_1+\frac{1}{\gamma_{\rho1}}\tilde{\rho}_1D^\alpha\tilde{\rho}_1+\omega_1d_{s1}+\omega_1\varepsilon_1\left(\underline{y}_1,\underline{x}_2\right)\end{aligned}$$

$$\begin{aligned}&\le-k_1\omega_1^2-\sigma_{\theta1}\tilde{\theta}_1^T\theta_1+g_{s1}\omega_1\omega_2+\frac{1}{\gamma_{\rho1}}\tilde{\rho}_1D^\alpha\tilde{\rho}_1-\rho_1\omega_1Tanh(\omega_1/\tau_1)+\rho_1^*\left|\omega_1\right|+\left|\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)\right|\\&\quad-\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)Tanh\left(\frac{\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)}{\zeta_1}\right)\end{aligned}$$

$$= -k_1\omega_1^2 - \sigma_{\theta 1}\tilde{\theta}_1^T\theta_1 + g_{s1}\omega_1\omega_2 + \frac{1}{\gamma_{\rho 1}}\tilde{\rho}_1 D^\alpha\tilde{\rho}_1 - \tilde{\rho}_1\omega_1 Tanh(\omega_1/\tau_1) + \rho_1^*\left|\omega_1\right| - \rho_1^*\omega_1 Tanh(\omega_1/\tau_1)$$
$$+\left|\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)\right| - \omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)Tanh\left(\frac{\omega_1\theta_1^T\psi_1\left(\underline{y}_1\right)}{\zeta_1}\right)$$

$$\le -k_1\omega_1^2 - \sigma_{\theta 1}\tilde{\theta}_1^T\theta_1 - \sigma_{\rho 1}\tilde{\rho}_1\rho_1 + g_{s1}\omega_1\omega_2 + \rho_1^*\overline{\tau}_1 + \overline{\zeta}_1 \tag{4.22}$$

where $\overline{\tau}_1 = \kappa\tau_1$ and $\overline{\zeta}_1 = \kappa\zeta_1$

Substituting

$$-\sigma_{\theta 1}\tilde{\theta}_1^T\theta_1 \le \frac{\sigma_{\theta 1}}{2}\left\|\theta_1^*\right\|^2 - \frac{\sigma_{\theta 1}}{2}\left\|\tilde{\theta}_1\right\|^2, \text{ and} \tag{4.23}$$

$$-\sigma_{\rho 1}\tilde{\rho}_1\rho_1 \le \frac{\sigma_{\rho 1}}{2}\rho_1^{*2} - \frac{\sigma_{\rho 1}}{2}\tilde{\rho}_1^2, \tag{4.24}$$

into (4.22) yields

$$D^\alpha V_1(\mathrm{t}) \le -k_1\omega_1^2 - \frac{\sigma_{\theta 1}}{2}\left\|\tilde{\theta}_1\right\|^2 - \frac{\sigma_{\rho 1}}{2}\tilde{\rho}_1^2 + g_{s1}\omega_1\omega_2 + \frac{\sigma_{\rho 1}}{2}\rho_1^{*2} + \frac{\sigma_{\theta 1}}{2}\left\|\theta_1^*\right\|^2 + \rho_1^*\overline{\tau}_1 + \overline{\zeta}_1 \tag{4.25}$$

In the right-hand side of (4.25), the first three terms are stable, provided that k_1, $\sigma_{\theta 1}$ and $\sigma_{\rho 1}$ are positive. But the term $g_{s1}\omega_1\omega_2$ will be cancelled in the second step. As to the latest three terms, i.e. $\frac{\sigma_{\rho 1}}{2}\rho_1^{*2} + \frac{\sigma_{\theta 1}}{2}\left\|\theta_1^*\right\|^2 + \rho_1^*\overline{\tau}_1 + \overline{\zeta}_1$, will be considered in the stability analysis.

Step.i (Stabilization of the i-subsystem in (4.6), with i=2,...,n-1):

Similar to the derivation in Step 1, by defining the novel variable errors, $\omega_i = e_i - \alpha_{i-1}$ with $i = 2,...,n-1$, we get:

$$D^\alpha\omega_i = h_i(\underline{y}_i, \underline{x}_{i+1}) + g_{si}e_{i+1} + d_{si} - D^\alpha\alpha_{i-1}, \text{ for } i = 2,...,n-1 \tag{4.26}$$

The uncertain smooth nonlinear function $h_i(\underline{y}_i, \underline{x}_{i+1})$ can be online approximated, on a compact set $\Omega_{\underline{y}_i}$, by the fuzzy system (1.2), as follows:

$$h_i = \theta_i^T \psi_i\left(\underline{y}_i\right), \quad \text{with}\, i = 2,\ldots,n-1, \tag{4.27}$$

where θ_i is the adaptive fuzzy parameter vector and $\psi_i\left(\underline{y}_i\right)$ is the FBF vector.

The unknown best values of θ_i can be defined as:

$$\theta_i^* = \arg\min_{\theta_i}\left[\sup_{\underline{y}_i \in \Omega_{\underline{y}_i}} \left|h_i(\underline{y}_i, \underline{x}_{i+1}) - \theta_i^T \psi_i\left(\underline{y}_i\right)\right|\right] \tag{4.28}$$

Denote also

$$\tilde{\theta}_i = \theta_i - \theta_i^* \text{ and } \varepsilon_i(\underline{y}_i, \underline{x}_{i+1}) = h_i(\underline{y}_i, \underline{x}_{i+1}) - \theta_i^{*T} \psi_i\left(\underline{y}_i\right) \tag{4.29}$$

as the parameter error vector and the approximation error, respectively.

This approximation error, $\varepsilon_i(\underline{y}_i, \underline{x}_{i+1})$, is commonly assumed to be bounded by an unknown constant as follows [LIU15b, TON13, LIU14, LI15B, LIN17, LI13, BOU08d] :

$$\left|\varepsilon_i(\underline{y}_i, \underline{x}_{i+1})\right| \le \bar{\varepsilon}_i, \quad \forall\, \underline{x}_{i+1} \in \Omega_{\underline{x}_{i+1}},\ \underline{y}_i \in \Omega_{\underline{y}_i} \tag{4.30}$$

From (4.27) and (4.29), one can write

$$\begin{aligned} h_i(\underline{y}_i, \underline{x}_{i+1}) - \theta_i^T \psi_i\left(\underline{y}_i\right) &= h_i(\underline{y}_i, \underline{x}_{i+1}) - \theta_i^{*T} \psi_i\left(\underline{y}_i\right) + \theta_i^{*T} \psi_i\left(\underline{y}_i\right) - \theta_i^T \psi_i\left(\underline{y}_i\right) \\ &= -\tilde{\theta}_i^T \psi_i\left(\underline{y}_i\right) + \varepsilon_i(\underline{y}_i, \underline{x}_{i+1}) \end{aligned} \tag{4.31}$$

Using (4.31), (4.26) becomes

$$\begin{aligned} D^\alpha \omega_i &= h_i(\underline{y}_i, \underline{x}_{i+1}) + g_{si} e_{i+1} + d_{si} - D^\alpha \alpha_{i-1} \\ &= -\tilde{\theta}_i^T \psi_i\left(\underline{y}_i\right) + \theta_i^T \psi_i\left(\underline{y}_i\right) + g_{si} e_{i+1} + d_{si} + \varepsilon_i\left(\underline{y}_i, \underline{x}_{i+1}\right) - D^\alpha \alpha_{i-1} \end{aligned} \tag{4.32}$$

Design the virtual control input α_i and its fractional-order adaptation laws as follows

$$\alpha_i = g_{si}^{-1}\left(-\theta_i^T \psi_i\left(\underline{y}_i\right) Tanh\left(\frac{\theta_i^T \psi_i\left(\underline{y}_i\right)\omega_i}{\zeta_i}\right) - k_i \omega_i - \rho_i Tanh(\omega_i / \tau_i) - g_{si-1}\omega_{i-1} + D^\alpha \alpha_{i-1}\right) \tag{4.33}$$

$$D^{\alpha}\theta_i = \gamma_{\theta i}\omega_i\psi_i\left(\underline{y}_i\right) - \gamma_{\theta i}\sigma_{\theta i}\theta_i \tag{4.34}$$

$$D^{\alpha}\rho_i = \gamma_{\rho i}\omega_i Tanh(\omega_i/\tau_i) - \gamma_{\rho i}\sigma_{\rho i}\rho_i \tag{4.35}$$

with k_i, $\gamma_{\theta i}$, and $\gamma_{\rho i}$ being strictly positive design parameters. $\sigma_{\theta i}$, $\sigma_{\rho i}, \zeta_i$, and τ_i are some small strictly positive design parameters. Note that ρ_i is the estimate the uncertain term $\rho_i^* = \bar{d}_{si} + \bar{\varepsilon}_i$.

Since the Caputo fractional derivative of a constant is zero, (4.34) and (4.35) can be rewritten as:

$$D^{\alpha}\tilde{\theta}_i = \gamma_{\theta i}\omega_i\psi_i\left(\underline{y}_i\right) - \gamma_{\theta i}\sigma_{\theta i}\theta_i \tag{4.36}$$

$$D^{\alpha}\tilde{\rho}_i = \gamma_{\rho i}\omega_i Tanh(\omega_i/\tau_i) - \gamma_{\rho i}\sigma_{\rho i}\rho_i \tag{4.37}$$

where $\tilde{\theta}_i = \theta_i - \theta_i^*$ and $\tilde{\rho}_i = \rho_i - \rho_i^*$.

Denoting $\omega_{i+1} = e_{i+1} - \alpha_i$ and using (4.32) and (4.33), we get

$$\begin{aligned}
D^{\alpha}\omega_i &= -\tilde{\theta}_i^T\psi_i\left(\underline{y}_i\right) + \theta_i^T\psi_i\left(\underline{y}_i\right) + g_{si}e_{i+1} + d_{si} + \varepsilon_i\left(\underline{y}_i, \underline{x}_{i+1}\right) - D^{\alpha}\alpha_{i-1} \\
&= -\tilde{\theta}_i^T\psi_i\left(\underline{y}_i\right) + \theta_i^T\psi_i\left(\underline{y}_i\right) + g_{si}(e_{i+1} - \alpha_i) + g_{si}\alpha_i + d_{si} + \varepsilon_i\left(\underline{y}_i, \underline{x}_{i+1}\right) - D^{\alpha}\alpha_{i-1} \\
&= -\tilde{\theta}_i^T\psi_i\left(\underline{y}_i\right) + \theta_i^T\psi_i\left(\underline{y}_i\right) + g_{si}\omega_{i+1} - \theta_i^T\psi_i\left(\underline{y}_i\right)Tanh\left(\frac{\omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)}{\zeta_i}\right) - k_i\omega_i - \rho_i Tanh(\omega_i/\tau_i) \\
&\quad - g_{si-1}\omega_{i-1} + d_{si} + \varepsilon_i\left(\underline{y}_i, \underline{x}_{i+1}\right)
\end{aligned} \tag{4.38}$$

Multiplying both sides of (4.38) by ω_i yields

$$\begin{aligned}
\omega_i D^{\alpha}\omega_i &= -\omega_i\tilde{\theta}_i^T\psi_i\left(\underline{y}_i\right) + \omega_i\theta_i^T\psi_i\left(\underline{y}_i\right) + g_{si}\omega_i\omega_{i+1} - \omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)Tanh\left(\frac{\omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)}{\zeta_i}\right) - k_i\omega_i^2 \\
&\quad - \rho_i\omega_i Tanh(\omega_i/\tau_i) - g_{si-1}\omega_i\omega_{i-1} + \omega_i d_{si} + \omega_i\varepsilon_i\left(\underline{y}_i, \underline{x}_{i+1}\right)
\end{aligned} \tag{4.39}$$

Construct a Lyapunov function candidate V_i, related to the ith subsystem, as follows

$$V_i = V_{i-1} + \frac{1}{2}\omega_i^2 + \frac{1}{2\gamma_{\theta i}}\tilde{\theta}_i^T\tilde{\theta}_i + \frac{1}{2\gamma_{\rho i}}\tilde{\rho}_i^2 \tag{4.40}$$

Using Lemma 1.2, the fractional time derivative of (4.40) can be obtained as:

$$D^\alpha V_i = D^\alpha V_{i-1} + \frac{1}{2}D^\alpha\omega_i^2 + \frac{1}{2\gamma_{\theta i}}D^\alpha\tilde{\theta}_i^T\tilde{\theta}_i + \frac{1}{2\gamma_{\rho i}}D^\alpha\tilde{\rho}_i^2 \le D^\alpha V_{i-1} + \omega_i D^\alpha\omega_i + \frac{1}{\gamma_{\theta i}}\tilde{\theta}_i^T D^\alpha\tilde{\theta}_i + \frac{1}{\gamma_{\rho i}}\tilde{\rho}_i D^\alpha\tilde{\rho}_i \tag{4.41}$$

Using Lemma 4.1 and substituting (4.36), (4.37) and (4.39) into (4.41) yields

$$\begin{aligned}
D^\alpha V_i \le\; & D^\alpha V_{i-1} - \omega_i\tilde{\theta}_i^T\psi_i\left(\underline{y}_i\right) + \omega_i\theta_i^T\psi_i\left(\underline{y}_i\right) + g_{si}\omega_i\omega_{i+1} - \omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)Tanh\left(\frac{\omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)}{\zeta_i}\right) - k_i\omega_i^2 \\
& -\rho_i\omega_i Tanh(\omega_i/\tau_i) - g_{si-1}\omega_i\omega_{i-1} + \omega_i d_{si} + \omega_i\varepsilon_i\left(\underline{y}_i, \underline{x}_{i+1}\right) + \frac{1}{\gamma_{\theta i}}\tilde{\theta}_i^T\left(\gamma_{\theta i}\omega_i\psi_i\left(\underline{y}_i\right) - \gamma_{\theta i}\sigma_{\theta i}\theta_i\right) + \frac{1}{\gamma_{\rho i}}\tilde{\rho}_i D^\alpha\tilde{\rho}_i
\end{aligned}$$

$$\begin{aligned}
\le\; & D^\alpha V_{i-1} - k_i\omega_i^2 - \sigma_{\theta i}\tilde{\theta}_i^T\theta_i - g_{si-1}\omega_i\omega_{i-1} + g_{si}\omega_i\omega_{i+1} + \frac{1}{\gamma_{\rho i}}\tilde{\rho}_i D^\alpha\tilde{\rho}_i - \rho_i\omega_i Tanh(\omega_i/\tau_i) \\
& + \rho_i^*\left|\omega_i\right| + \omega_i\theta_i^T\psi_i\left(\underline{y}_i\right) + \left|\omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)\right| - \omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)Tanh\left(\frac{\omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)}{\zeta_i}\right)
\end{aligned}$$

$$\begin{aligned}
=\; & D^\alpha V_{i-1} - k_i\omega_i^2 - \sigma_{\theta i}\tilde{\theta}_i^T\theta_i - g_{si-1}\omega_i\omega_{i-1} + g_{si}\omega_i\omega_{i+1} + \left(\frac{1}{\gamma_{\rho i}}\tilde{\rho}_i D^\alpha\tilde{\rho}_i - \tilde{\rho}_i\omega_i Tanh(\omega_i/\tau_i)\right) + \\
& \left(\rho_i^*\left|\omega_i\right| - \rho_i^*\omega_i Tanh(\omega_i/\tau_i)\right) + \left|\omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)\right| - \omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)Tanh\left(\frac{\omega_i\theta_i^T\psi_i\left(\underline{y}_i\right)}{\zeta_i}\right)
\end{aligned}$$

$$\begin{aligned}
\le\; & -\sum_{j=1}^{i}k_j\omega_j^2 - \sum_{j=1}^{i-1}\frac{\sigma_{\theta j}}{2}\left\|\tilde{\theta}_j\right\|^2 - \sum_{j=1}^{i-1}\frac{\sigma_{\rho j}}{2}\tilde{\rho}_j^2 - \sigma_{\theta i}\tilde{\theta}_i^T\theta_i - \sigma_{\rho i}\tilde{\rho}_i\rho_i + \sum_{j=1}^{i-1}\frac{\sigma_{\theta j}}{2}\left\|\theta_j^*\right\|^2 + \sum_{j=1}^{i-1}\frac{\sigma_{\rho j}}{2}\rho_j^{*2} \\
& + \sum_{j=1}^{i}\rho_j^*\bar{\tau}_j + g_{si}\omega_i\omega_{i+1} + \sum_{j=1}^{i}\bar{\zeta}_j
\end{aligned} \tag{4.42}$$

where $\bar{\tau}_j = \kappa\tau_j$ and $\bar{\zeta}_j = \kappa\zeta_j$

It can be shown easily that

$$-\sigma_{\theta i}\tilde{\theta}_i^T\theta_i \le \frac{\sigma_{\theta i}}{2}\left\|\theta_i^*\right\|^2 - \frac{\sigma_{\theta i}}{2}\left\|\tilde{\theta}_i\right\|^2 \tag{4.43}$$

$$-\sigma_{\rho i}\tilde{\rho}_i\,\rho_i \le \frac{\sigma_{\rho i}}{2}\rho_i^{*2} - \frac{\sigma_{\rho i}}{2}\tilde{\rho}_i^2 \tag{4.44}$$

Substituting (4.43) and (4.44) into (4.42), one obtains

$$\begin{aligned} D^\alpha V_i \le &-\sum_{j=1}^{i} k_j\omega_j^2 - \sum_{j=1}^{i}\frac{\sigma_{\theta j}}{2}\left\|\tilde{\theta}_j\right\|^2 - \sum_{j=1}^{i}\frac{\sigma_{\rho j}}{2}\tilde{\rho}_j^2 + \sum_{j=1}^{i}\frac{\sigma_{\theta j}}{2}\left\|\theta_j^*\right\|^2 + \sum_{j=1}^{i}\frac{\sigma_{\rho j}}{2}\rho_j^{*2} + \sum_{j=1}^{i}\rho_j^*\overline{\tau}_j \\ &+\sum_{j=1}^{i}\overline{\zeta}_j + g_{si}\omega_i\omega_{i+1} \end{aligned} \tag{4.45}$$

Step.n (Stabilization of the n-subsystem in (4.6)):

From (4.6), the fractional-order dynamics of $\omega_n = e_n - \alpha_{n-1}$ can be written as:

$$D^\alpha\omega_n = h_n(\underline{y}_n, \underline{x}_n) + g_{sn}u + d_{sn} - D^\alpha\alpha_{n-1} \tag{4.46}$$

The uncertain smooth function $h_n(\underline{y}_n, \underline{x}_n)$ can be online modeled, on a compact set $\Omega_{\underline{y}_n}$, by the fuzzy logic system (1.2), as follows:

$$\hat{h}_n = \theta_n^T\psi_n\left(\underline{y}_n\right) \tag{4.47}$$

where $\psi_n\left(\underline{y}_n\right)$ is the FBF vector, and θ_n is the adjustable fuzzy parameter vector.

The best (ideal) value of θ_n is generally defined as

$$\theta_n^* = \arg\min_{\theta_n}\left[\sup_{\underline{y}_n\in\Omega_{\underline{y}_n}}\left|h_n(\underline{y}_n, \underline{x}_n) - \theta_n^T\psi_n\left(\underline{y}_n\right)\right|\right] \tag{4.48}$$

Notice that θ_n^* is an unknown vector used only for analysis purposes. In fact, its knowledge is not necessary when implementing the control system.

Denote

$$\tilde{\theta}_n = \theta_n - \theta_n^* \quad \text{and} \quad \varepsilon_n(\underline{y}_n, \underline{x}_n) = h_n(\underline{y}_n, \underline{x}_n) - \theta_n^{*T}\psi_n\left(\underline{y}_n\right) \tag{4.49}$$

as the parameter error vector and the approximation error, respectively.

This approximation error, $\varepsilon_n(\underline{y}_n, \underline{x}_n)$, is commonly assumed to be bounded by an unknown constant, as follows [LIU15b, TON13, LIU14, LI15b, LIN17, LI13, BOU08d]:

$$\left|\varepsilon_n(\underline{y}_n, \underline{x}_n)\right| \le \bar{\varepsilon}_n, \quad \forall \underline{x}_n \in \Omega_{\underline{x}_n}, \underline{y}_n \in \Omega_{\underline{y}_n} \tag{4.50}$$

From (4.47) and (4.49), one can obtain

$$h_n(\underline{y}_n, \underline{x}_n) - \theta_n^T\psi_n\left(\underline{y}_n\right) = h_n(\underline{y}_n, \underline{x}_n) - \theta_n^{*T}\psi_n\left(\underline{y}_n\right) + \theta_n^{*T}\psi_n\left(\underline{y}_n\right) - \theta_n^T\psi_n\left(\underline{y}_n\right)$$

$$= -\tilde{\theta}_n^T\psi_n\left(\underline{y}_n\right) + \varepsilon_n(\underline{y}_n, \underline{x}_n) \tag{4.51}$$

It follows from (4.46) and (4.51) that

$$D^\alpha\omega_n = h_n(\underline{y}_n, \underline{x}_n) + g_{sn}u + d_{sn} - D^\alpha\alpha_{n-1}$$

$$= -\tilde{\theta}_n^T\psi_n\left(\underline{y}_n\right) + \theta_n^T\psi_n\left(\underline{y}_n\right) + g_{sn}u + d_{sn} + \varepsilon_n(\underline{y}_n, \underline{x}_n) - D^\alpha\alpha_{n-1} \tag{4.52}$$

We now design our feedback control to stabilize the full system (4.6) and the associated update laws, as follows:

$$u = g_{sn}^{-1}\left(-\theta_n^T\psi_n\left(\underline{y}_n\right)Tanh\left(\frac{\omega_n\theta_n^T\psi_n\left(\underline{y}_n\right)}{\zeta_n}\right) - k_n\omega_n - \rho_n Tanh(\omega_n/\tau_n) - g_{sn-1}\omega_{n-1} + D^\alpha\alpha_{n-1}\right) \tag{4.53}$$

$$D^\alpha\theta_n = \gamma_{\theta n}\omega_n\psi_n\left(\underline{y}_n\right) - \gamma_{\theta n}\sigma_{\theta n}\theta_n \tag{4.54}$$

$$D^\alpha\rho_n = \gamma_{\rho n}\omega_n Tanh(\omega_n/\tau_n) - \gamma_{\rho n}\sigma_{\rho n}\rho_n \tag{4.55}$$

with k_n, $\gamma_{\theta n}$, and $\gamma_{\rho n}$ being strictly positive design parameters. $\sigma_{\theta n}$, $\sigma_{\rho n}$, τ_n and ζ_n are some small strictly positive design parameters. Note that ρ_n is the estimate the uncertain term $\rho_n^* = \bar{d}_{sn} + \bar{\varepsilon}_n$.

From (4.52) and (4.53), one has

$$D^{\alpha}\omega_n = -\tilde{\theta}_n^T\psi_n\left(\underline{y}_n\right) + \theta_n^T\psi_n\left(\underline{y}_n\right) + g_{sn}u + d_{sn} + \varepsilon_n(\underline{y}_n, \underline{x}_n) - D^{\alpha}\alpha_{n-1}$$

$$\begin{aligned} &= -\tilde{\theta}_n^T\psi_n\left(\underline{y}_n\right) + \theta_n^T\psi_n\left(\underline{y}_n\right) - \theta_n^T\psi_n\left(\underline{y}_n\right)Tanh\left(\frac{\omega_n\theta_n^T\psi_n\left(\underline{y}_n\right)}{\zeta_n}\right) \\ &\quad - k_n\omega_n - \rho_n Tanh(\omega_n/\tau_n) - g_{sn-1}\omega_{n-1} + d_{sn} + \varepsilon_n(\underline{y}_n, \underline{x}_n) \end{aligned} \tag{4.56}$$

Multiplying both sides of (4.56) by ω_n, we get

$$\begin{aligned} \omega_n D^{\alpha}\omega_n &= -\omega_n\tilde{\theta}_n^T\psi_n\left(\underline{y}_n\right) + \omega_n\theta_n^T\psi_n\left(\underline{y}_n\right) - \omega_n\theta_n^T\psi_n\left(\underline{y}_n\right)Tanh\left(\frac{\omega_n\theta_n^T\psi_n\left(\underline{y}_n\right)}{\zeta_n}\right) \\ &\quad - k_n\omega_n^2 - \rho_n\omega_n Tanh(\omega_n/\tau_n) - g_{sn-1}\omega_n\omega_{n-1} + \omega_n d_{sn} + \omega_n\varepsilon_n(\underline{y}_n, \underline{x}_n) \end{aligned} \tag{4.57}$$

Construct the following Lyapunov function for the entire system:

$$V_n = V_{n-1} + \frac{1}{2}\omega_n^2 + \frac{1}{2\gamma_{\theta n}}\tilde{\theta}_n^T\tilde{\theta}_n + \frac{1}{2\gamma_{\rho n}}\tilde{\rho}_n^2 \tag{4.58}$$

where $\tilde{\theta}_n = \theta_n - \theta_n^*$ and $\tilde{\rho}_n = \rho_n - \rho_n^*$.

Invoking Lemma 1.2, the fractional derivative of (4.58) is :

$$\begin{aligned} D^{\alpha}V_n &= D^{\alpha}V_{n-1} + \frac{1}{2}D^{\alpha}\omega_n^2 + \frac{1}{2\gamma_{\theta n}}D^{\alpha}\tilde{\theta}_n^T\tilde{\theta}_n + \frac{1}{2\gamma_{\rho n}}D^{\alpha}\tilde{\rho}_n^2 \\ &\le D^{\alpha}V_{n-1} + \omega_n D^{\alpha}\omega_n + \frac{1}{\gamma_{\theta n}}\tilde{\theta}_n^T D^{\alpha}\tilde{\theta}_n + \frac{1}{\gamma_{\rho n}}\tilde{\rho}_n D^{\alpha}\tilde{\rho}_n \end{aligned} \tag{4.59}$$

Using Lemma 4.1, and substituting (4.54), (4.55), and (4.57) into (4.59) yields

$$\begin{aligned} D^{\alpha}V_n &\le D^{\alpha}V_{n-1} - k_n\omega_n^2 - \sigma_{\theta n}\tilde{\theta}_n^T\theta_n - g_{sn-1}\omega_n\omega_{n-1} + \frac{1}{\gamma_{\rho n}}\tilde{\rho}_n D^{\alpha}\tilde{\rho}_n - \rho_n\omega_n Tanh(\omega_n/\tau_n) + \omega_n d_{sn} \\ &\quad + \omega_n\varepsilon_n\left(\underline{y}_n, \underline{x}_n\right) + \omega_n\theta_n^T\psi_n\left(\underline{y}_n\right) - \omega_n\theta_n^T\psi_n\left(\underline{y}_n\right)Tanh\left(\frac{\omega_n\theta_n^T\psi_n\left(\underline{y}_n\right)}{\zeta_n}\right) \end{aligned}$$

$$D^{\alpha}V_n \le D^{\alpha}V_{n-1} - k_n\omega_n^2 - \sigma_{\theta n}\tilde{\theta}_n^T\theta_n - g_{sn-1}\omega_n\omega_{n-1} + \frac{1}{\gamma_{\rho n}}\tilde{\rho}_n D^{\alpha}\tilde{\rho}_n - \rho_n\omega_n Tanh(\omega_n/\tau_n) + \omega_n d_{sn}$$

$$+\omega_n\varepsilon_n\left(\underline{y}_n, \underline{x}_n\right) + \left|\omega_n\theta_n^T\psi_n\left(\underline{y}_n\right)\right| - \omega_n\theta_n^T\psi_n\left(\underline{y}_n\right)Tanh\left(\frac{\omega_n\theta_n^T\psi_n\left(\underline{y}_n\right)}{\zeta_n}\right)$$

$$\le D^{\alpha}V_{n-1} - k_n\omega_n^2 - \sigma_{\theta n}\tilde{\theta}_n^T\theta_n - g_{sn-1}\omega_n\omega_{n-1} + \frac{1}{\gamma_{\rho n}}\tilde{\rho}_n D^{\alpha}\tilde{\rho}_n - \rho_n\omega_n Tanh(\omega_n/\tau_n) + \rho_n^*\left|\omega_n\right| + \bar{\zeta}_n$$

$$= D^{\alpha}V_{n-1} - k_n\omega_n^2 - \sigma_{\theta n}\tilde{\theta}_n^T\theta_n - g_{sn-1}\omega_n\omega_{n-1} + \left(\frac{1}{\gamma_{\rho n}}\tilde{\rho}_n D^{\alpha}\tilde{\rho}_n - \tilde{\rho}_n\omega_n Tanh(\omega_n/\tau_n)\right) +$$

$$\left(\rho_n^*\left|\omega_n\right| - \rho_n^*\omega_n Tanh(\omega_n/\tau_n)\right) + \bar{\zeta}_n$$

$$\le -\sum_{j=1}^{n}k_j\omega_j^2 - \sum_{j=1}^{n-1}\frac{\sigma_{\theta j}}{2}\left\|\tilde{\theta}_j\right\|^2 - \sum_{j=1}^{n-1}\frac{\sigma_{\rho j}}{2}\tilde{\rho}_j^2 - \sigma_{\theta n}\tilde{\theta}_n^T\theta_n - \sigma_{\rho n}\tilde{\rho}_n\rho_n + \sum_{j=1}^{n-1}\frac{\sigma_{\theta j}}{2}\left\|\theta_j^*\right\|^2 + \sum_{j=1}^{n-1}\frac{\sigma_{\rho j}}{2}\rho_j^{*2}$$

$$+\sum_{j=1}^{n}\rho_j^*\bar{\tau}_j + \sum_{j=1}^{n}\bar{\zeta}_j \tag{4.60}$$

It is clear that

$$-\sigma_{\theta n}\tilde{\theta}_n^T\theta_n \le \frac{\sigma_{\theta n}}{2}\left\|\theta_n^*\right\|^2 - \frac{\sigma_{\theta n}}{2}\left\|\tilde{\theta}_n\right\|^2 \tag{4.61}$$

$$-\sigma_{\rho n}\tilde{\rho}_n\rho_n \le \frac{\sigma_{\rho n}}{2}\rho_n^{*2} - \frac{\sigma_{\rho n}}{2}\tilde{\rho}_n^2 \tag{4.62}$$

Substituting (4.61)and (4.62) into (4.60) yields

$$D^{\alpha}V_n \le -\sum_{j=1}^{n}k_j\omega_j^2 - \sum_{j=1}^{n}\frac{\sigma_{\theta j}}{2}\left\|\tilde{\theta}_j\right\|^2 - \sum_{j=1}^{n}\frac{\sigma_{\rho j}}{2}\tilde{\rho}_j^2 + \sum_{j=1}^{n}\frac{\sigma_{\theta j}}{2}\left\|\theta_j^*\right\|^2 + \sum_{j=1}^{n}\frac{\sigma_{\rho j}}{2}\rho_j^{*2} + \sum_{j=1}^{n}\rho_j^*\bar{\tau}_j + \sum_{j=1}^{n}\bar{\zeta}_j \tag{4.63}$$

The stability result of the closed-loop system is given as follows.

Theorem 4.1. *For the master-salve system (4.1)-(4.2), if the control input is designed as (4.13), (4.33) and (4.53), and the adaptation laws are selected as (4.14), (4.15), (4.34), (4.35), (4.54) and (4.55), then the synchronization errors as well as parametric estimation errors are ultimately stable, and they remain in a small but adjustable neighborhood of zero.*

Proof of Theorem 4.1. From (4.63), one has

$$D^{\alpha}V_n \leq -\eta V_n + \vartheta \tag{4.64}$$

where $\vartheta = \sum_{j=1}^{n} \frac{\sigma_{\theta j}}{2} \left\| \theta_j^* \right\|^2 + \sum_{j=1}^{n} \frac{\sigma_{\rho j}}{2} \rho_j^{*2} + \sum_{j=1}^{n} \rho_j^* \overline{\tau}_j + \sum_{j=1}^{n} \overline{\zeta}_j$, and

$\eta = \min\left\{ \min_j \left\{ 2k_j \right\}, \min_j \left\{ \gamma_{\theta j}\sigma_{\theta j} \right\}, \min_j \left\{ \gamma_{\rho j}\sigma_{\rho j} \right\} \right\}$

By giving strictly positive values to all design constants, the synchronization error is bounded. Applying Lemma 1.1 to (4.64), we get:

$$V_n \leq \frac{2\vartheta}{\eta} = \frac{2\left(\sum_{j=1}^{n} \frac{\sigma_{\theta j}}{2} \left\| \theta_j^* \right\|^2 + \sum_{j=1}^{n} \frac{\sigma_{\rho j}}{2} \rho_j^{*2} + \sum_{j=1}^{n} \rho_j^* \overline{\tau}_j + \sum_{j=1}^{n} \overline{\zeta}_j \right)}{\eta} \tag{4.65}$$

Equations (4.65) and (4.58) simply imply that

$$\left\| \omega \right\| \leq \sqrt{\frac{4\left(\sum_{j=1}^{n} \frac{\sigma_{\theta j}}{2} \left\| \theta_j^* \right\|^2 + \sum_{j=1}^{n} \frac{\sigma_{\rho j}}{2} \rho_j^{*2} + \sum_{j=1}^{n} \rho_j^* \overline{\tau}_j + \sum_{j=1}^{n} \overline{\zeta}_j \right)}{\eta}} \tag{4.66}$$

where $\left\| \omega \right\| = \sqrt{\omega_1^2 + \ldots + \omega_n^2}$.

It is clear now, from (4.66), that the underlying synchronization error ω_1 will be ultimately uniformly bounded as $t \to \infty$. Therefore, the closed-loop system is stable and all involved signals remains bounded, based on the useful Lemma 1.1. From (4.66), it is clear that the ultimate bound of $\left\| \omega \right\|$ and V_n is closely related to the design constants, namely k_i, $\gamma_{\theta i}$, $\sigma_{\theta i}$, $\gamma_{\rho i}$, $\sigma_{\rho i}$, and τ_j. From (4.65) and (4.66), therefore, we are able to choose proper design constants such that the synchronization errors ω_1 can converge to a small but adjustable neighborhood of the zero. This ends the proof of Theorem 4.1.

4.3 Simulation Study

Our main results are simulated on three examples to verify the effectiveness of the proposed synchronization approach based on fuzzy adaptive backstepping control.

4.3.1 Example 1

Consider the synchronization of two fractional-order Duffing's systems [XU13, ROS15].

The master system:

$$\begin{cases} D^{\alpha}x_1(t) = x_2 \\ D^{\alpha}x_2(t) = x_1 - x_1^{\ 3} - 0.15x_2 - 0.3\cos(t) \end{cases} \tag{4.67}$$

The slave system :

$$\begin{cases} D^{\alpha}y_1(t) = y_2 + d_1(t) \\ D^{\alpha}y_2(t) = y_1 - y_1^{\ 3} - 0.15y_2 + d_2(t) + u \end{cases} \tag{4.68}$$

The dynamical external disturbances are considered as: $d_1(t) = 1 + 0.1\cos(t),$ and $d_2(t) = 0.1 - 0.4\cos(t).$ Note that the master system (4.67) evolves in a chaotic behavior when $\alpha = 0.97$ [XU13, ROS15].

Figure 4.1 shows the chaotic attractor of the system (4.67). Its Lyapunov exponents are also depicted in Figure 4.2.

The synchronization errors are defined as $e_1 = y_1(t) - \lambda_1 x_1(t),$ $e_2 = y_2(t) - \lambda_2 x_2(t)$, and we make the following coordinate changes: : $\omega_1 = e_1$ and $\omega_2 = e_2 - \alpha_1$.

According to Theorem 4.1, the virtual and actual controls are designed as:

$$\alpha_1 = -\theta_1^T \psi_1\left(\underline{y}_1\right) Tanh\left(\frac{\omega_1 \theta_1^T \psi_1\left(\underline{y}_1\right)}{\zeta_1}\right) - k_1\omega_1 - \rho_1 Tanh(\omega_1 / \tau_1) \tag{4.6}$$

$$u = -\theta_2^T \psi_2\left(\underline{y}_2\right) Tanh\left(\frac{\omega_2 \theta_2^T \psi_2\left(\underline{y}_2\right)}{\zeta_2}\right) - k_2\omega_2 - \omega_1 - \rho_2 Tanh(\omega_2 / \tau_2) - D^{\alpha}\alpha_1 \tag{4.70}$$

Their update laws are

$$D^{\alpha}\theta_i = \gamma_{\theta i}\omega_i\psi_i\left(\underline{y}_i\right) - \gamma_{\theta i}\sigma_{\theta i}\theta_i\,,\ \text{ for } i = 1, 2 \tag{4.71}$$

$$D^{\alpha}\rho_i = \gamma_{\rho i}\omega_i Tanh(\omega_i / \tau_i) - \gamma_{\rho i}\sigma_{\rho i}\rho_i\,,\ \text{ for } i = 1, 2 \tag{4.72}$$

with $\underline{y}_1 = y_1$, $\underline{y}_2 = [y_1, y_2]^T$, $\gamma_{\theta 1} = 100$, $\sigma_{\theta 1} = 0.001$, $\gamma_{\theta 2} = 100$, $\sigma_{\theta 2} = 0.001$, $\gamma_{\rho 1} = 10$, $\sigma_{\rho 1} = 0.01$, $\gamma_{\rho 2} = 10$, $\sigma_{\rho 2} = 0.01$, $k_1 = 3$, $k_2 = 8$, $\tau_1 = \tau_2 = 0.01$ and $\zeta_1 = \zeta_2 = 0.5$.

To construct the FBFs of the fuzzy systems $\theta_1^T\psi_1\left(\underline{y}_1\right)$ and $\theta_2^T\psi_2\left(\underline{y}_2\right)$, three membership functions with triangular and trapezoidal shapes, as in [BOU08d], are defined for each input y_i. They are uniformly distributed over the input universe of discourse which is selected as

[-4, 4].

In this simulation, the initial conditions are set as: $(x_1(0), x_2(0)) = (-0.2, 0.1)$, $(y_1(0), y_2(0)) = (0.3, -0.4)$. $\theta_{1k}(0) = 0$, $\theta_{2j}(0) = 0$ and $\rho_i(0) = 0.1$, for $k = 1, 2, 3$, $j = 1, ..., 9$, and $i = 1, 2$.

Two simulation cases are considered here i.e. for $\lambda_i = 1$ and for $\lambda_i = -1$.

2) Case 1: Complete synchronization (i.e. $\lambda_i = 1$)

The simulation results are depicted in Figure 4.3. It can be seen from this figure, that the state trajectories of the slave system (4.68) rapidly converge to those of the master system (4.67), despite the presence of unknown dynamics and external disturbances. The applied control signal is bounded, continuous as well as free of chattering.

4) Case 2: Anti-phase synchronization (i.e. $\lambda_i = -1$)

The obtained simulation results are plotted in Figure 4.4. It can be clearly observed from this figure that the slave states are successfully anti-phase-synchronized with those of the master one, despite the presence of uncertain dynamics and unknown external disturbances.

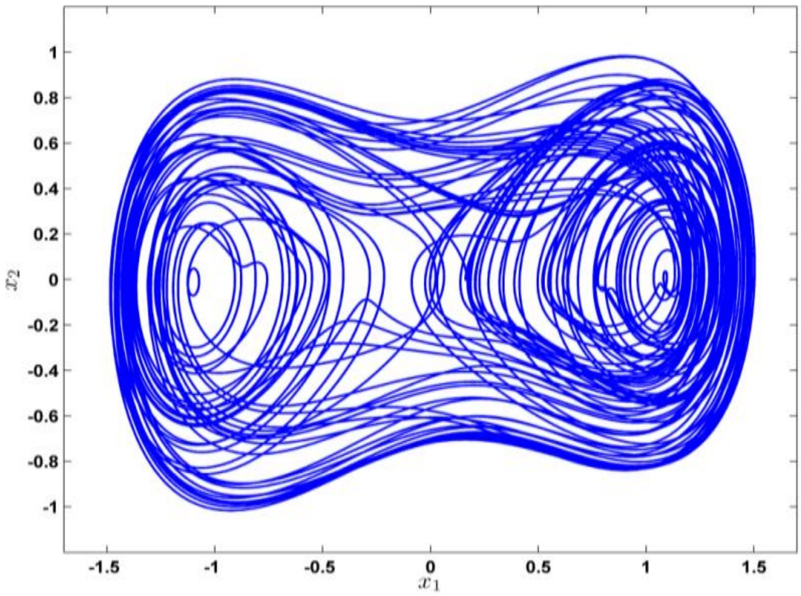


Figure 4.1: *Dynamical behavior of the uncontrolled fractional-order Duffing's system.*

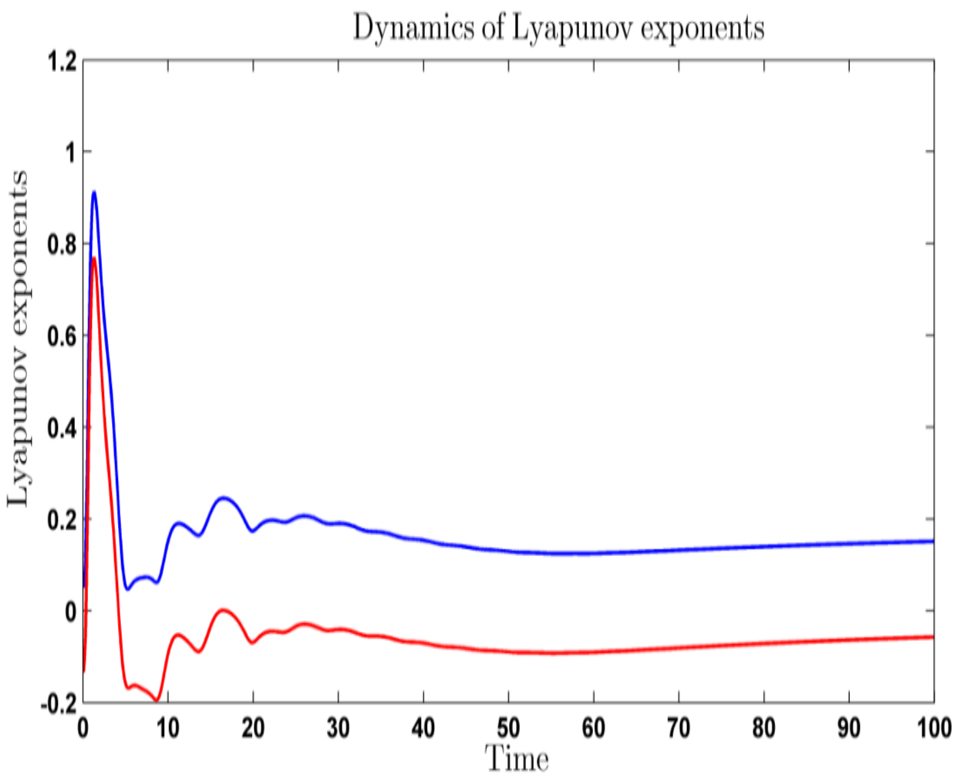


Figure 4.2: *Lyapunov exponents of the uncontrolled fractional-order Duffing's system.*

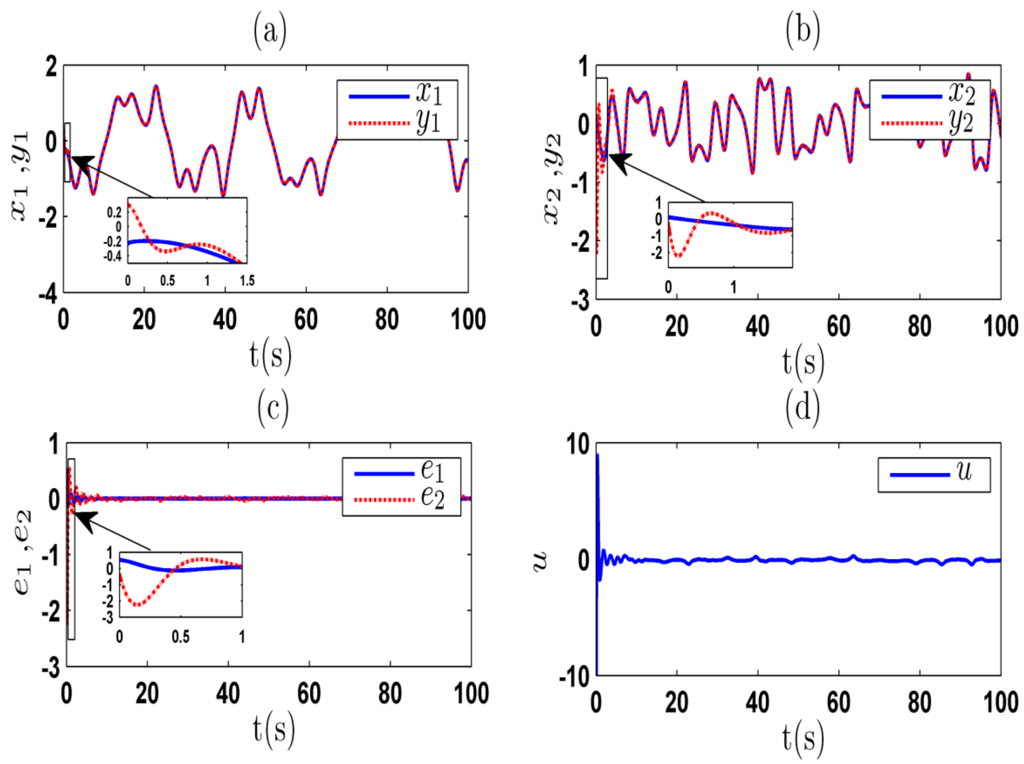


Figure 4.3: *Complete synchronization for example 1.*

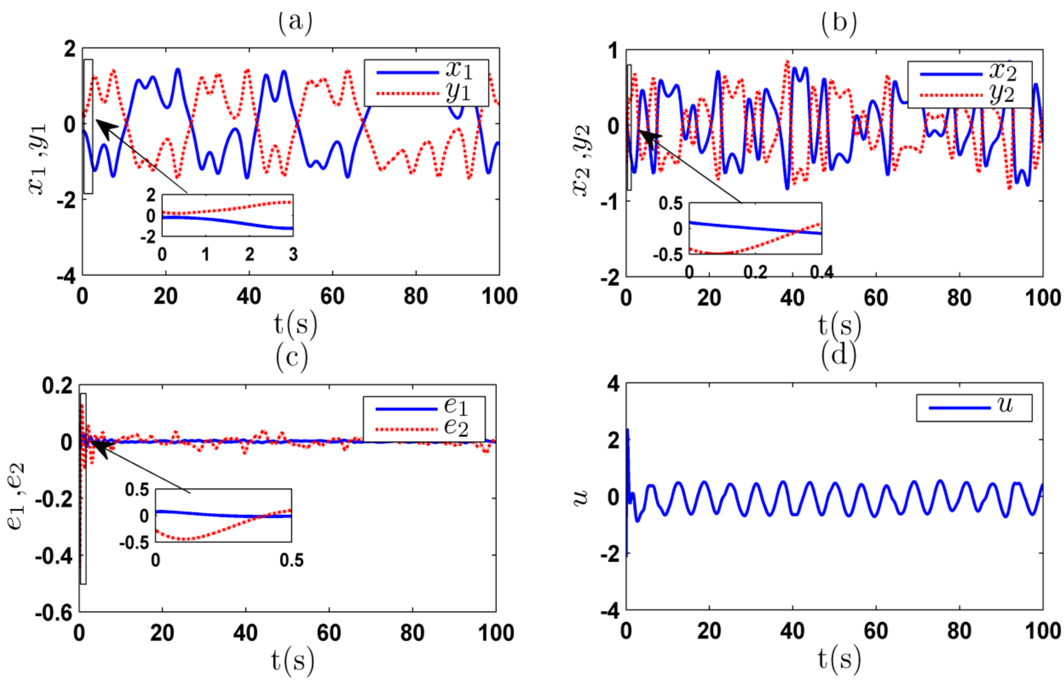


Figure 4.4: *Anti-phase synchronization for example 1.*

4.3.2 Example 2

In this simulation, the effectiveness of the proposed fuzzy adaptive controller is tested by the way of projective synchronizing two fractional-order Genesio–Tesi systems [LOU17].

The master system:

$$\begin{cases} D^{\alpha} x_1(t) = x_2 \\ D^{\alpha} x_2(t) = x_3 \\ D^{\alpha} x_3(t) = -cx_1 - bx_2 - ax_3 + x_1^{\,2} \end{cases} \tag{4.73}$$

with x_i, $i = 1, 2, 3,$ being state variables, a, b and c are the positive real constants satisfying $ab < c$.

If we choose these constants as $a = 1.2,\ b = 2.92,\ c = 6$, then system (4.73) yields a chaotic trajectory for $\alpha = 0.98$ [LOU17].

The slave system :

$$\begin{cases} D^{\alpha} y_1 = y_2 + d_1(t) \\ D^{\alpha} y_2 = y_3 + d_2(t) \\ D^{\alpha} y_3 = -cy_1 - by_2 - ay_3 + y^2{}_1 + d_3(t) + u \end{cases} \tag{4.74}$$

The dynamical external disturbances, without lost of generality, are considered as follows: $d_1(t) = 1,\ d_2(t) = 0.1 + \cos(t),$ and $d_3(t) = 0.1 + 2\cos(3t) + \sin(5t).$

The chaotic attractor of (4.74) is depicted in Figure 4.5 and its Lyapunov exponents are plotted in Figure 4.6. It is obvious from this figure that the sum of the Lyapunov exponents is negative, which confirms that both systems are dissipative.

The synchronization errors are defined as $e_1 = y_1(t) - \lambda_1 x_1(t),$ for $i = 1, 2, 3$, and we make the following coordinate changes: : $\omega_1 = e_1$, $\omega_2 = e_2 - \alpha_1$, $\omega_3 = e_3 - \alpha_2$. According to Theorem 4.1, the virtual and actual controls are designed as:

$$\alpha_1 = -\theta_1^T \psi_1\left(\underline{y}_1\right) Tanh\left(\frac{\omega_1 \theta_1^T \psi_1\left(\underline{y}_1\right)}{\zeta_1}\right) - k_1\omega_1 - \rho_1 Tanh(\omega_1 / \tau_1) \tag{4.75}$$

$$\alpha_2 = -\theta_2^T \psi_2\left(\underline{y}_2\right) Tanh\left(\frac{\omega_2 \theta_2^T \psi_2\left(\underline{y}_2\right)}{\zeta_2}\right) - k_2\omega_2 - \omega_1 - \rho_2 Tanh(\omega_2 / \tau_2) - D^\alpha \alpha_1 \quad (4.76)$$

$$u = -\theta_3^T \psi_3\left(\underline{y}_3\right) Tanh\left(\frac{\omega_3 \theta_3^T \psi_3\left(\underline{y}_3\right)}{\zeta_3}\right) - k_3\omega_3 - \omega_2 - \rho_3 Tanh(\omega_3 / \tau_3) - D^\alpha \alpha_2 \quad (4.77)$$

Their update laws are

$$D^\alpha \theta_i = \gamma_{\theta i}\omega_i \psi_i\left(\underline{y}_i\right) - \gamma_{\theta i}\sigma_{\theta i}\theta_i, \text{ for } i = 1,2,3 \quad (4.78)$$

$$D^\alpha \rho_i = \gamma_{\rho i}\omega_i Tanh(\omega_i / \tau_i) - \gamma_{\rho i}\sigma_{\rho i}\rho_i, \text{ for } i = 1,2,3 \quad (4.79)$$

with $\underline{y}_1 = y_1$, $\underline{y}_2 = [y_1, y_2]^T$, $\underline{y}_3 = [y_1, y_2, y_3]^T$, $k_1 = 3$, $k_2 = 5$, $k_3 = 10$, and $\tau_1 = \tau_2 = \tau_3 = 0.01$, $\gamma_{\theta 3} = 100$, $\sigma_{\theta 3} = 0.001$, $\gamma_{\rho 1} = 8$, $\sigma_{\rho 1} = 0.01$, $\gamma_{\rho 2} = 8$, $\sigma_{\rho 2} = 0.01$, $\gamma_{\rho 3} = 10$, $\sigma_{\rho 3} = 0.01$,

$\gamma_{\theta 1} = 100$, $\sigma_{\theta 1} = 0.001$, $\gamma_{\theta 2} = 100$, $\sigma_{\theta 2} = 0.001$ and $\zeta_1 = \zeta_2 = \zeta_2 = 0.5$.

In this simulation, the initial conditions are set as: $(x_1(0), x_2(0), x_3(0)) = (-1, 2, 3)$, and $(y_1(0), y_2(0), y_3(0)) = (0.5, -1, 2)$, $\theta_{1i}(0) = 0$, $\theta_{2j}(0) = 0$, $\theta_{2k}(0) = 0$ and $\rho_i(0) = 0.1$, for $i = 1, 2, 3,$ $j = 1, ..., 9$, and $k = 1, ..., 27$.

To construct the FBFs for the fuzzy systems $\theta_1^T \psi_1\left(\underline{y}_1\right)$, $\theta_2^T \psi_2\left(\underline{y}_2\right)$ and $\theta_3^T \psi_3\left(\underline{y}_3\right)$, three membership functions with triangular and trapezoidal shapes, as in [BOU08d], are defined for each input y_i. They are uniformly distributed over the input universes of discourse which are selected as [-10, 10].

Two simulation cases are considered here.

1) Case 1: Complete synchronization $(\lambda_i = 1)$

The obtained simulation results are illustrated in Figure 4.7. It can be observed from this figure that the slave system (4.75) is practically synchronized with the master one (4.74), despite the presence of the external disturbances and unknown dynamics. In addition, the applied control signal is admissible as well as continuous.

2) Case 2: Anti-phase synchronization $(\lambda_i = -1)$

The simulation results obtained by employing the proposed fuzzy adaptive backstepping controller are depicted in Figure 4.8. It can be clearly seen from this figure that the states of the slave system are fruitfully anti-phase-synchronized with those of the master one, in spite of the presence of the external disturbances and uncertain dynamics.

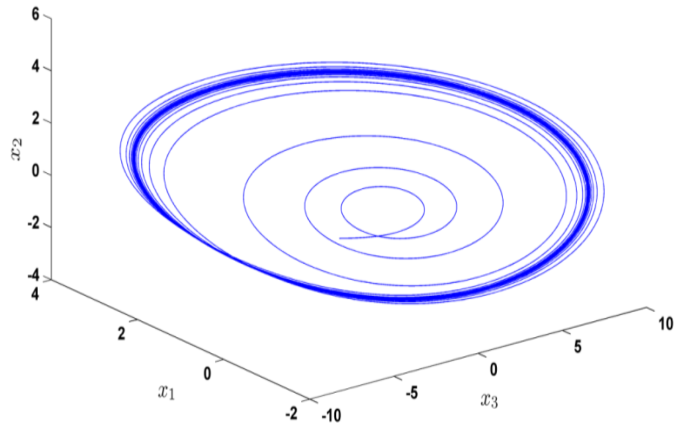


Figure 4.5: *Dynamical behavior of the uncontrolled fractional-order Genesio–Tesi system.*

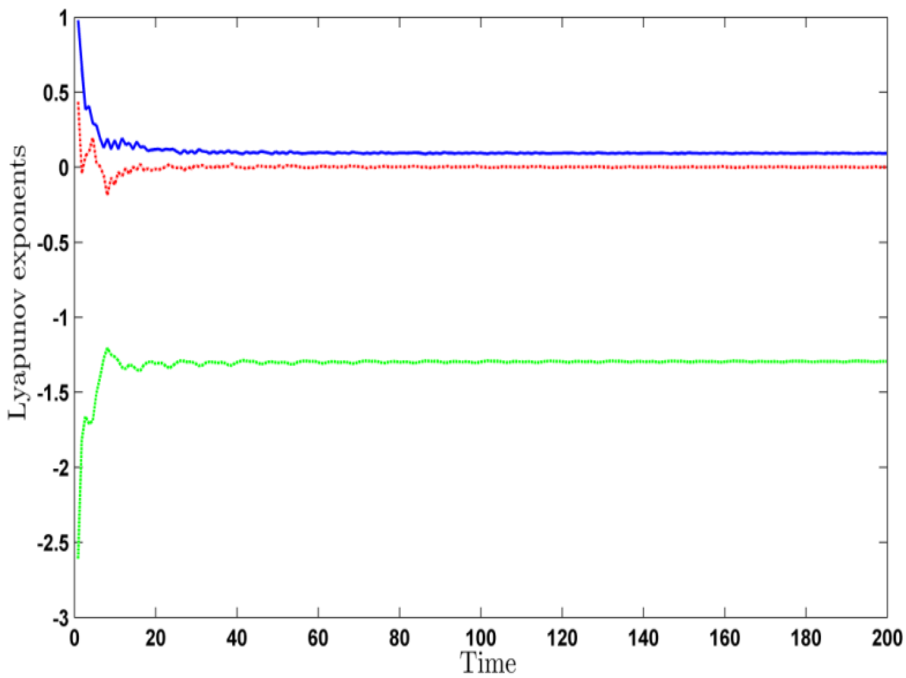


Figure 4.6: *Lyapunov exponents of the uncontrolled fractional-order Genesi–Tesi system.*

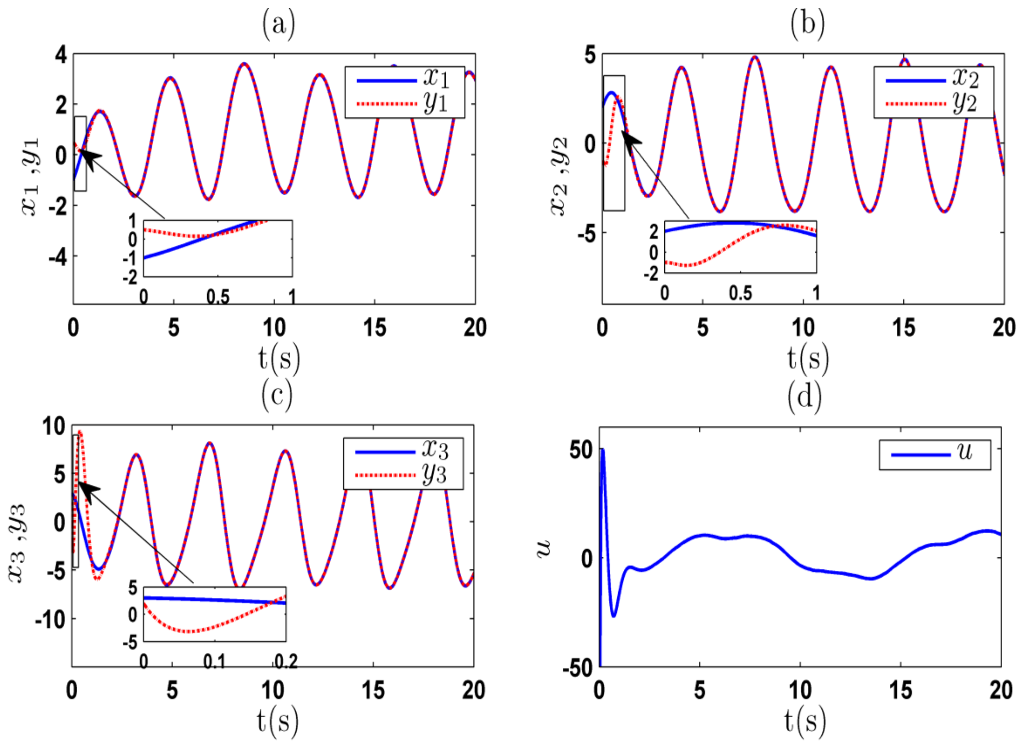


Figure 4.7: *Complete synchronization for example 2.*

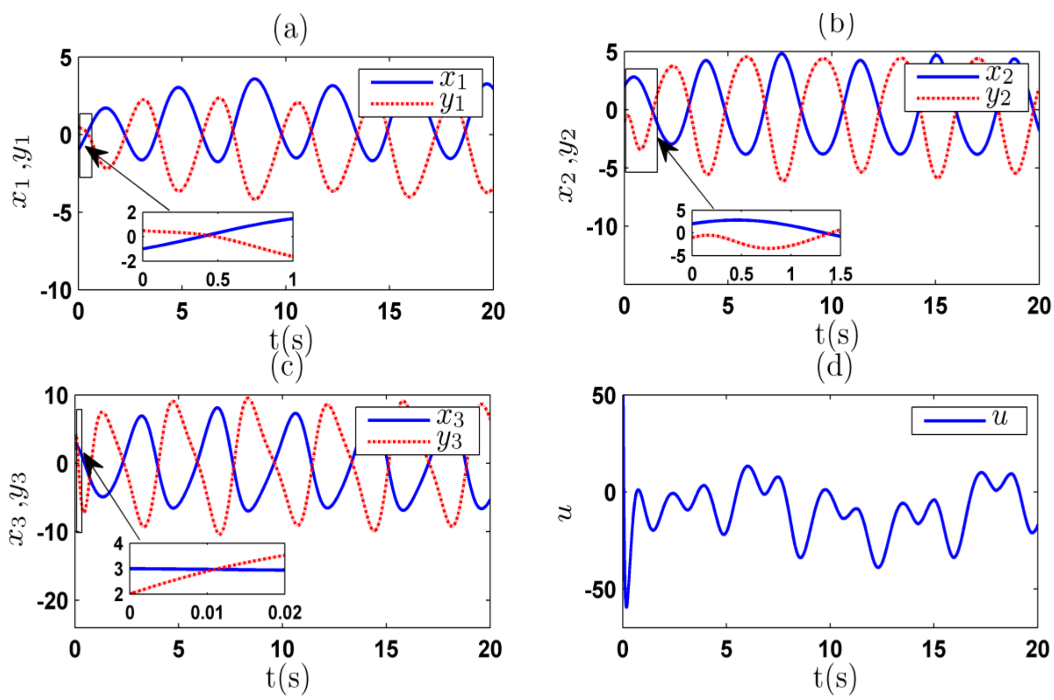


Figure 4.8: *Anti-phase synchronization for example 2.*

4.3.3 Example 3

In this simulation, the effectiveness of the proposed fuzzy adaptive backstepping controller is tested by the way of projective synchronizing of two different fractional-order chaotic systems, namely a Coullet's system (selected as a master system) and a Genesio–Tesisystem (selected as slave system).

The master system[SHA10]:

$$\begin{cases} D^{\alpha} x_1(t) = x_2 \\ D^{\alpha} x_2(t) = x_3 \\ D^{\alpha} x_3(t) = -0.45x_3 - x_2 + 0.8x_1 - x_1^3 \end{cases} \tag{4.80}$$

The slave system [LUO17]:

$$\begin{cases} D^{\alpha} y_1 = y_2 + d_1(t) \\ D^{\alpha} y_2 = y_3 + d_2(t) \\ D^{\alpha} y_3 = -6y_1 - 2.92y_2 - 1.2y_3 + y_1^2 + d_3(t) + u \end{cases} \tag{4.81}$$

$d_1(t) = 1,\ d_2(t) = 0.1 + \cos(t),$ and $d_3(t) = 0.1\cos(2t) + 0.35\sin(\pi t) y_3.$

It has been shown in [LUO17, SHA10] that the slave system (4.80) and master system (4.81) with $d_i(t) = 0$ and $u = 0$ are characterized by a chaotic behavior when $\alpha = 0.98$.

The chaotic attractor of the system (4.80) is depicted in Figure 4.9 and its Lyapunov exponents are plotted in Figure 4.10. It is obvious from this figure that the sum of the Lyapunov exponents is negative, which confirms that this system is dissipative.

The synchronization errors are defined as $e_1 = y_1(t) - \lambda_1 x_1(t)$, for $i = 1,2,3$, and we make the following coordinate changes: : $\omega_1 = e_1$, $\omega_2 = e_2 - \alpha_1$, $\omega_3 = e_3 - \alpha_2$. According to Theorem 4.1, the virtual and actual controls are designed as:

$$\alpha_1 = -\theta_1^T \psi_1\left(\underline{y}_1\right) Tanh\left(\frac{\omega_1 \theta_1^T \psi_1\left(\underline{y}_1\right)}{\zeta_1}\right) - k_1 \omega_1 - \rho_1 Tanh(\omega_1 / \tau_1) \tag{4.82}$$

$$\alpha_2 = -\theta_2^T \psi_2\left(\underline{y}_2\right) Tanh\left(\frac{\omega_2 \theta_2^T \psi_2\left(\underline{y}_2\right)}{\zeta_2}\right) - k_2\omega_2 - \omega_1 - \rho_2 Tanh(\omega_2 / \tau_2) - D^\alpha \alpha_1 \quad (4.83)$$

$$u = -\theta_3^T \psi_3\left(\underline{y}_3\right) Tanh\left(\frac{\omega_3 \theta_3^T \psi_3\left(\underline{y}_3\right)}{\zeta_3}\right) - k_3\omega_3 - \omega_2 - \rho_3 Tanh(\omega_3 / \tau_3) - D^\alpha \alpha_2 \quad (4.84)$$

Theirs update laws are

$$D^\alpha \theta_i = \gamma_{\theta i}\omega_i \psi_i\left(\underline{y}_i\right) - \gamma_{\theta i}\sigma_{\theta i}\theta_i, \text{ for } i = 1, 2, 3 \quad (4.85)$$

$$D^\alpha \rho_i = \gamma_{\rho i}\omega_i Tanh(\omega_i / \tau_i) - \gamma_{\rho i}\sigma_{\rho i}\rho_i, \text{ for } i = 1, 2, 3 \quad (4.86)$$

with

$\underline{y}_1 = y_1$, $\underline{y}_2 = [y_1, y_2]^T$, $\underline{y}_3 = [y_1, y_2, y_3]^T$, $k_1 = 11$, $k_2 = 15$, $k_3 = 25$, $\tau_1 = \tau_2 = \tau_3 = 0.01.$

$\gamma_{\rho 1} = 12$, $\sigma_{\rho 1} = 0.01$, $\gamma_{\rho 2} = 8$, $\sigma_{\rho 2} = 0.01$, $\gamma_{\rho 3} = 14$, $\sigma_{\rho 3} = 0.01$, $\gamma_{\theta 3} = 100$, $\sigma_{\theta 3} = 0.001$, and $\zeta_1 = \zeta_2 = \zeta_2 = 0.5$.

In this simulation, the initial conditions are set as: $(x_1(0), x_2(0), x_3(0)) = (1, -1, 0.1)$, and $(y_1(0), y_2(0), y_3(0)) = (0.5, -0.5, -0.3)$, $\theta_{1i}(0) = 0$, $\theta_{2j}(0) = 0$, $\theta_{2k}(0) = 0$ and $\rho_i(0) = 0.3$, for $i = 1, 2, 3$, $j = 1, \ldots, 9$, and $k = 1, \ldots, 27$.

To construct the FBFs of the fuzzy systems $\theta_1^T \psi_1\left(\underline{y}_1\right)$, $\theta_2^T \psi_2\left(\underline{y}_2\right)$ and $\theta_3^T \psi_3\left(\underline{y}_3\right)$, three membership functions with triangular and trapezoidal shapes, as in [BOU08d], are defined for each input y_i. They are uniformly distributed over the input universes of discourse which are selected as [-4, 4].

Two simulation cases are considered here.

1) Case 1: Complete synchronization $(\lambda_i = 1)$

The obtained simulation results are illustrated in Figure 4.11. It can be observed from this figure that the slave system (4.81) is practically synchronized with the master one (4.80), despite the presence of the external disturbances and unknown dynamics. In addition, the applied control signal is admissible as well as continuous.

2) Case 2: Anti-phase synchronization $(\lambda_i = -1)$

The simulation results obtained by employing the proposed fuzzy adaptive backstepping controller are depicted in Figure 4.12. It can be clearly seen from this figure that the states of the slave system are fruitfully anti-phase-synchronized with those of the master one, in spite of the presence of the external disturbances and uncertain dynamics.

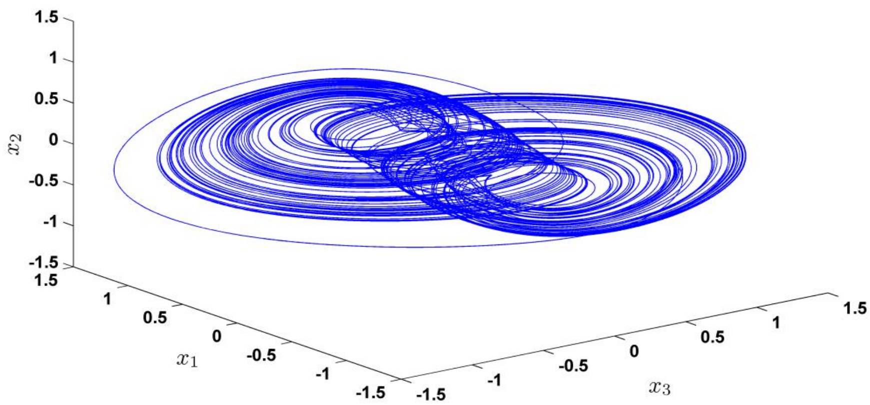


Figure 4.9: *Dynamical behavior of the uncontrolled fractional-order Coullet's system.*

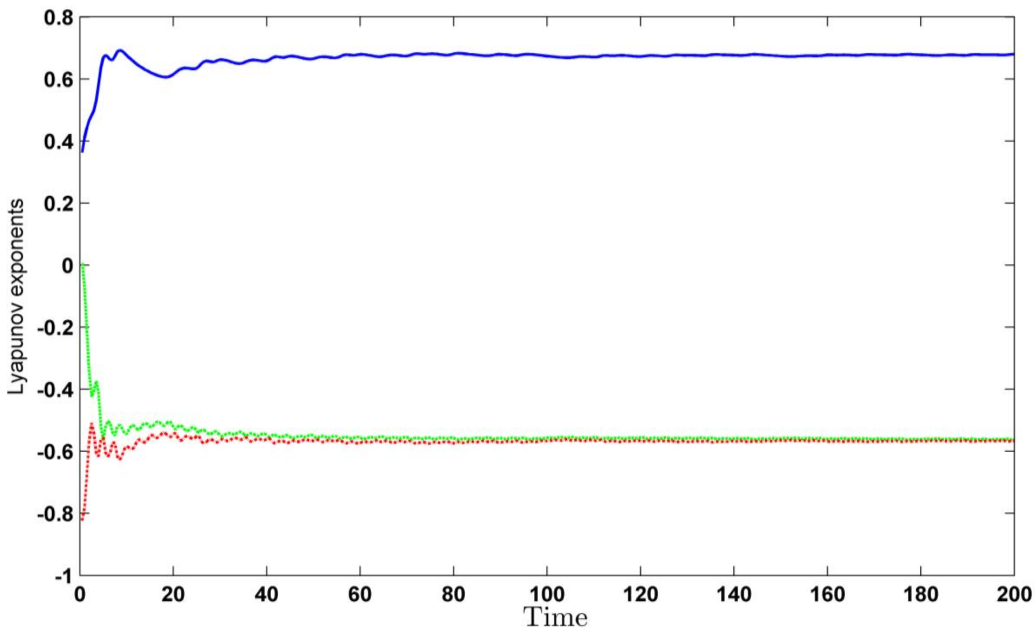


Figure 4.10: *Lyapunov exponents of the uncontrolled fractional-order Coullet's system.*

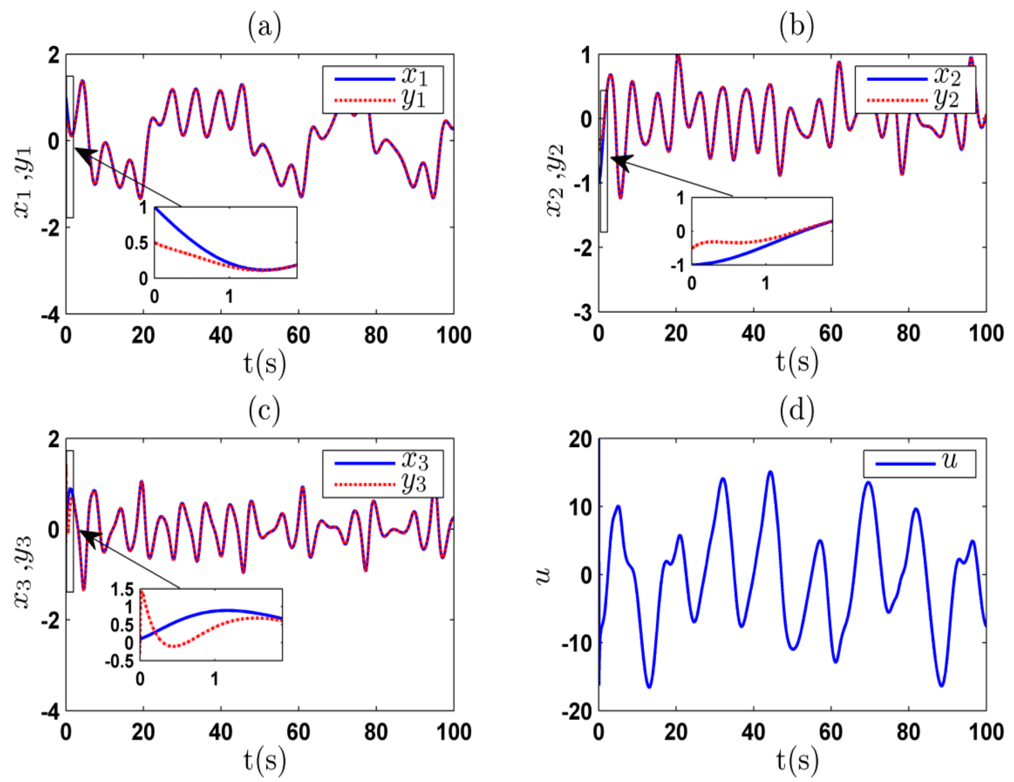


Figure 4.11: *Complete synchronization for example 3.*

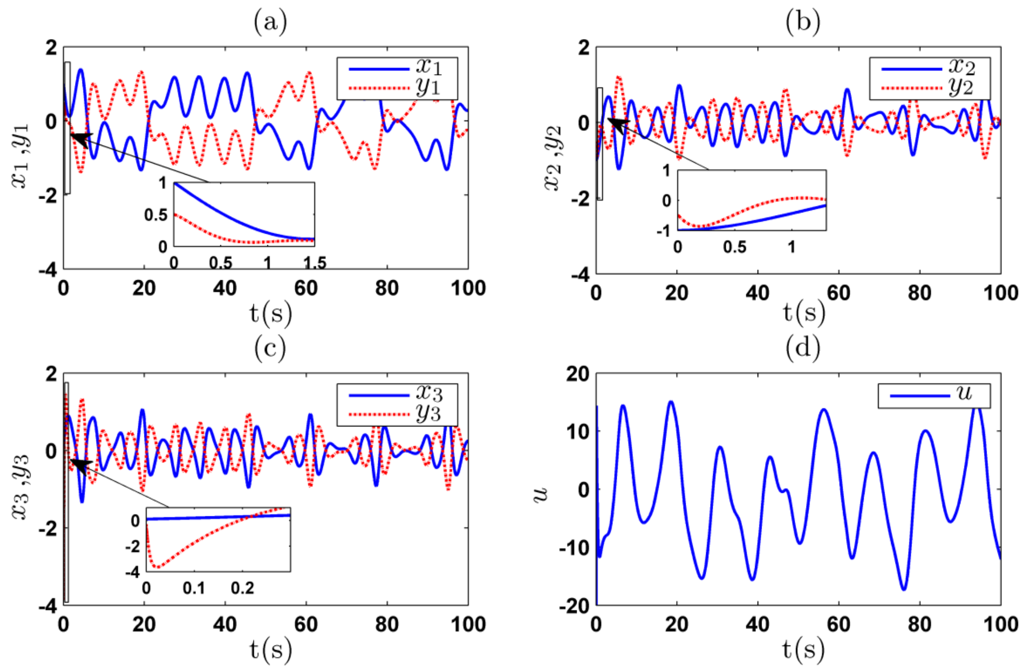


Figure 4.12: *Anti-phase synchronization for example 3.*

4.4 Conclusion

In this chapter, the design of a novel fractional-order adaptive backstepping control system was attempted for achieving a practical but robust projective synchronization of fractional-order chaotic systems subject to both (matched and unmatched) disturbances and uncertainties. The backstepping framework has been employed to systematically derive step by step our main results. The adaptive fuzzy systems have been used to online model the unknown dynamics. Fractional-order updated laws were rigorously derived based on Lyapunov's stability theorem and should ensure system stability. From the numerical simulation results, it can be clearly seen that this proposed control system showed fairly satisfactory synchronization performances.

General conclusion

The research work in this thesis focuses on three major themes:

- Design of a projective synchronization scheme for a class of fractional-order optical systems ***with uncertain dynamics and bounded disturbances.***
- Proposition of a generalized projective synchronization system for a class of fractional-order chaotic systems ***with input constraints*** (dead-zone together with sector nonlinearities).
- Development of a projective synchronization methodology for a class of *fractional-order chaotic systems* **having a lower triangular structure.**

In the first chapter, some basic results on the fuzzy systems, fractional-order systems, and chaotic systems and its synchronization have been briefly reviewed. Some definitions, lemmas and theorems, which are essential for the fractional-order control design and for the stability analysis of the fractional-order systems, have been also recalled.

In the second chapter, we have investigated the problem of chaos synchronization based on fractional-order intelligent sliding-mode control approach for a class of fractional-order chaotic optical systems with unknown dynamics and disturbances. Two simple but effective fractional dynamic sliding surfaces with some desired stability features have been employed to derive two fuzzy sliding-mode controllers. Fuzzy systems have been used to online approximate the uncertain dynamics. The stability analysis of the closed-loop system has been rigorously performed by using a fractional Lyapunov theory. The validity of these proposed synchronization schemes has been confirmed via some numerical simulation experiments.

In the third chapter, a novel fuzzy adaptive control has been designed to achieve a generalized projective synchronization for a class of fractional-order chaotic systems with input nonlinearities (dead-zone together with sector nonlinearities). These master-slave systems under consideration have been supposed to be with distinct models, different fractional-orders, unknown models, and dynamic external disturbances. The proposed control law consists of two main terms, namely: a fuzzy adaptive control term for appropriately approximating the uncertainties and a fractional-order variable-structure control term for robustly dealing with these inherent input nonlinearities. A Lyapunov method has been exploited to derive the updated laws as well as to prove the stability of the closed-loop control system. Some simulation experiments have been carried out to assess the performance of the proposed synchronization scheme.

In the last chapter, a new fractional fuzzy adaptive backstepping control scheme has been investigated for projective-synchronizing of a class of uncertain chaotic master-slave systems having a lower triangular structure. In the design process, the uncertain nonlinear dynamics have been online learned via the use of the fuzzy systems, and the virtual control terms have been determined based on a fractional Lyapunov stability approach. The proposed fuzzy adaptive backstepping controller can guarantee the stability of the closed-loop system as well as the convergence of the underlying synchronization error to a small but adjustable neighborhood of the zero.

Our future works will focus on:

- The implementation of the proposed control laws on real-world chaotic systems.
- The design of an output-feedback backstepping control for a class of uncertain fractional-order systems subject to input constraints (hysteresis, dead-zone, or saturation).

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