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Salaheddine Bensalem

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# Some Aspects of the Generalized Uncertainty Principle Theoretical Development and Applications

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**PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA**

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Department of Physics

**THESIS**

Submitted in partial fulfillment of the requirements for the degree of

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By

SALAHEDDINE BENSALEM

**Theme**

# Some Aspects of the Generalized Uncertainty Principle

*Theoretical Development and Applications*

Defended publicly on November $5^{th}$, 2020 in front of the examination committee

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# SOME ASPECTS OF THE GENERALIZED UNCERTAINTY PRINCIPLE

## THEORETICAL DEVELOPMENT AND APPLICATIONS

By

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Supervisor

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*A Thesis Submitted to the Faculty of Exact and Computer Sciences*

*in partial fulfillment of the requirements for the degree of*

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Department of Physics

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# Dedication

To My Parents

# Acknowledgments

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**Prof. GHARBI Abdelhakim, Prof. MOUMNI Mustapha** and **Prof. ZAIEM Slimane** who agreed to review the work.

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Finally, I am especially grateful to all people who supported me and contributed, from near or from far, to the achievement of this work.

"*Now I think I can safely say that nobody understands quantum mechanics.*"

Richard Phillips Feynman

# Abstract

This thesis investigates some consequences arising from the modification of Heisenberg's Uncertainty Principle (HUP), postulated in quantum theory to introduce limit values in the position and momentum uncertainties. In this context, the HUP is replaced by the so-called Generalized Uncertainty Principle (GUP). Firstly, by focusing on the fundamental physical aspect, we present the arguments of different hypotheses suggesting the existence of upper and lower bounds of some measurable quantities, as well as the related GUPs that have been proposed based on these hypotheses. We focus in particular on three GUPs: the one leading to the existence of a minimal length (suggested in various frameworks, such as quantized space-time theory, string theory and black hole physics), the GUP incorporating a minimal length and a maximal momentum (emerging in doubly special relativity) and the GUP with a maximal length (predicted in cosmology). The formalism of deformed quantum mechanics, which occurs from these GUPs, is studied exhaustively in the second chapter. Especially, we exhibit the modified commutation relations of position and momentum operators, the corresponding Hilbert space representations and the scalar product definition. Moreover, some theoretical developments are discussed by summarizing most important works in the literature.

We consider furthermore some applications in statistical physics by focusing on the recent maximal-length GUP. In fact, three systems are investigated, namely, an ideal gas, an ensemble of harmonic oscillators and a relativistic gas. In this framework, the thermodynamic properties of these systems are studied within the canonical ensemble via the quantum and semiclassical approaches. The comparison with the results obtained in the context of the minimal-length GUPs indicates that the maximum length may induce new effects, which become important at high temperatures and for large volumes. In particular, a modified equation of state for ideal gases emerges in the scope of this new formalism. By analyzing some experimental data, we argue that the maximal length might be viewed as a macroscopic scale associated with the system under study.

**Keywords:** Generalized uncertainty principle, modified dispersion relation, maximal length, minimal length, ideal gas, harmonic oscillator, relativistic gas, partition function, equation of state.

Quelques aspects du principe d'incertitude généralisé:
Développement théorique et applications

# Résumé

Dans cette thèse on étudie les conséquences de certaines modifications du Principe d'Incertitude de Heisenberg (HUP), postulées en théorie quantique afin de tenir compte de l'existence de valeurs limites de longueurs et de quantités de mouvement. Dans ce contexte, le HUP est remplacé par ce qui est appelé Principe d'Incertitude Généralisé (GUP). En premier lieu, en se basant sur des considérations physiques fondamentales, les arguments des différentes hypothèses suggérant l'existence de bornes supérieures et inférieures de certaines quantités mesurables, ainsi que les GUPs qui en découlent sont préentés. Nous nous sommes intéressés en particulier en trois GUPs: celui conduisant à l'existence d'une longueur minimale (suggérée dans divers contextes, tels que la théorie de l'espace-temps quantifié, la théorie des cordes et la physique des trous noirs), le GUP incluant une quantité de mouvement maximale (qui émerge en relativité doublement restreinte) et le GUP avec une longueur maximale (prédite en cosmologie). Le formalisme de la mécanique quantique déformée, basé sur ces GUPs, est étudié de manière exhaustive dans le deuxième chapitre. En particulier, nous exposons les relations de commutation déformées entre les opérateurs de position et d'impulsion, découlant de ces GUPs, les représentations correspondantes dans l'espace de Hilbert et la definition du produit scalar. Par ailleurs, certains développements théoriques sont abordés tout en s'appuyant sur les travaux les plus importants dans la littérature.

Nous considérons en outre quelques applications en physique statistique en nous concentrant sur le récent GUP avec une longueur maximale. En effet, trois systèmes sont étudiés, à savoir, un gaz parfait, un ensemble d'oscillateurs harmoniques et un gaz relativiste. Dans ce cadre, les propriétés thermodynamiques de ces systèmes sont étudiées dans l'ensemble canonique via les deux approches quantique et semi-classique. La comparaison avec les résultats obtenus dans le cadre des GUPs avec une longueur minimale montre que la longueur maximale peut induire de nouveaux effets, qui deviennent importants aux hautes températures et pour des grands volumes. Notamment, une équation d'état modifiée pour les gaz parfaits émerge dans le contexte de ce nouveau formalisme. En analysant certaines données expérimentales, nous soutenons que la longueur maximale pourrait être considérée comme une échelle macroscopique associée au système étudié.

**Mots-clefs:** Principe d'incertitude généralisé, relation de dispersion modifiée, longueur maximale, longueur minimale, gaz idéal, oscillateur harmonique, gaz relativiste, fonction de partition, équation d'état.

# Publications included in this thesis

### Published papers

- **S. Bensalem** and D. Bouaziz, *Statistical description of an ideal gas in maximum length quantum mechanics*, Physica A **523**, 583 (2019).
- **S. Bensalem** and D. Bouaziz, *On the thermodynamics of relativistic ideal gases in the presence of a maximal length*, Phys. Lett. A **384**, 126911 (2020).

### Submitted

- **S. Bensalem** and D. Bouaziz, *Thermostatistics in deformed space with maximal length*, ArXiv201002203 Cond-Mat Physicsquant-Ph. (2020).

# Contents

# Introduction

The modification of the Heisenberg Uncertainty Principle (HUP) seems to be an invaluable idea in modern theoretical physics [1–12]. This strategy is adopted to account for the existence of upper and lower bounds in the position and momentum measurements, involved in several contexts. For instance, a minimum length is a common prediction of various approaches to quantum gravity such as string theory [13, 14] and loop-quantum gravity [15, 16], and black hole physics [1, 17], a maximum momentum appears in Doubly Special Relativity (DSR) [18] and a maximum length arises naturally in cosmology due to the presence of cosmological horizons [19]. These fundamental proposals result in replacing the HUP by the so-called Generalized Uncertainty Principle (GUP), for which several variants have been proposed in the literature [5, 8–12].

The most studied GUP is that including an elementary length [5, 20], introduced as a minimal uncertainty in position measurement. This GUP has first been discussed by Kempf and coworkers in several papers [4, 5, 20–23], focusing especially on the implications of such a GUP on the mathematical aspect of quantum mechanics (QM). Thereafter, diverse topics have been addressed within this new form of QM: the hydrogen atom [24], harmonic oscillator [25, 26], gravitational quantum well [27], potentials with inverse square interaction [28–32] and quantum wells [33]. It has been shown that this new formalism produces correction terms for all quantum mechanical systems [34–36], and most interestingly, can regularize singularities due to ultraviolet divergences [28, 29]. The minimal length GUP was furthermore introduced in the Klein-Gordon equation [37, 38], the Dirac equation [39–41] and the Schrödinger-Newton equation [42]. In addition, topics related to quantum field theory [43, 44], black hole physics [45–48] and cosmology [49] are investigated in the framework of this formalism. Finally, special attention was given to the study of the thermodynamics of various physical systems, including ideal gases and harmonic oscillators in Refs. [50–54].

On the other hand, other GUPs have been suggested to include both a minimal length and a maximal momentum, as required by DSR theories [8–12]. In this framework, numerous works have been devoted to investigate the implications of such GUPs in many physical problems: the effect of this GUP on the structure of space is discussed in Refs. [8–10, 55], black hole physics was probed in Refs. [56–61] and cosmology in Refs. [62, 63], where it has been shown in particular that the Big Bang singularity can be avoided in this scenario; this important finding is also obtained recently in the context of Snyder non-commutative space [64]. Other topics with interest have also been addressed, see, for instance Refs. [65–67]. Finally, in Ref. [68], the authors investigated the thermodynamic properties of ideal and photon gases within the GUP of Refs. [8, 9]. Furthermore, within the higher-order GUP of Refs. [11, 12], the thermostatistics of an ideal gas has been discussed in Refs. [69, 70].

More recently, Perivolaropoulos [19] proposed a new GUP, which includes a maximum length, $l_{\max}$, and has the form $\Delta X \Delta P \geq \frac{\hbar}{2}\frac{1}{1-\alpha(\Delta X)^2}$ where $\alpha = l_{\max}^{-2}$. This GUP with infrared cutoff is motivated by the fact that at large scales, the nonlocal behavior of QM leads to quantum effects. Moreover, a maximum measurable length appears naturally in the context of either cosmological particle horizons or nontrivial cosmic topology. For more details, see Ref. [19] and references therein.

In the framework of this recent version of QM, Perivolaropoulos [19] studied numerically the harmonic oscillator and discussed its new features that appear in this formalism. In order to investigate further the implications of introducing such a maximum length, it has been of interest to consider other physical problems. Then, we have chosen to study some statistical systems in the context of this new GUP. The work is motivated by the fact that in such considered systems, the incorporation of a maximal length in quantum theory would be more relevant than in the treatment of individual microscopic systems [71].

In this thesis, we review different variants of GUPs existing in the literature by focusing our attention on the last mentioned GUP, which includes a maximal length. We present the fundamental tools of the formalism that follows from each GUP. Then, we extended the one-dimensional equations of the maximal length GUP, given in Ref. [19], to arbitrary dimensions [72]. Furthermore, to compare the consequences of the maximal length assumption and those of the minimal length, we consider some important applications in statistical mechanics.

Mainly the layout of this manuscript is divided into two parts: the first part deals with the theoretical development of this new version of QM, and the second part is dedicated to the applications of this formalism in statistical physics.

The first part comprises two chapters. In the first chapter, the motivations and first attempts that consist to deform Heisenberg's algebra are addressed, including, the quantized space-time theory, string theory, black hole physics, doubly special relativity and the maximum length or the cosmological particle horizon. The second chapter deals with the different established formalisms of the GUP, namely, the GUP with minimal length, GUP with minimal length and maximal momentum, higher-order GUP approach and the recent GUP with maximal length. Especially, the review study focuses on the representation of the operators within these deformed algebras. The main objective of this survey study is to give a better understanding of the different representations corresponding to the deformed algebras, proposed in the literature, which have contributed to the development of a coherent formalism of this new form of QM. Furthermore, for the minimal length GUP model of Kempf *et al.* [5], we establish the corresponding Modified Dispersion Relation (MDR) and its consequences, namely, the varying speed of light and the generalized de Broglie wavelength.

The second part comprises three chapters. The third chapter deals with the statistical physics and the GUP scenario, in which we present the two main approaches that may be used to study a given statistical system, namely, the quantum approach and the semiclassical one.

In the fourth chapter, a summary of some studies focusing on the statistical applications of the minimal length GUPs is given. For the sake of comparison with the maximal length effects, this chapter serves to provide the obtained results of these studies concerning the minimal length effects on the considered statistical systems.

In the fifth chapter, the statistical applications of the maximal length GUP is exhibited. Firstly, our contribution on the statistical description of an ideal gas within the maximum length formalism is presented; considering the canonical and microcanonical ensembles. In this regard, we study the problem of a nonrelativistic quantum particle in an infinite square-well potential by including a maximum length; the Schrödinger equation is analytically solved and then the energy spectrum is obtained. Based on the obtained deformed spectrum, the generalized canonical partition function, the density of states, as well as the corresponding thermodynamic properties are derived; moreover, a modified equation of state is established for the ideal gas. Then, the ensemble of $N$ harmonic oscillators system is treated within the canonical ensemble in $1D$ and $3D$ cases, by employing the semiclassical approach.

Furthermore, for the sake of simplicity, the corresponding modified thermodynamic properties are probed for the $1D$ case. Finally, the thermodynamics of a relativistic gas is investigated using the semiclassical treatment.

The thesis is summarized with a conclusion of the main obtained results and some envisaged perspectives.

# Part I

# Theoretical development

# Chapter 1

# Generalized Uncertainty Principle: First attempts

## 1.1 Introduction

In this chapter we present some proposals in fundamental physics that suggest the existence of optimum measurable quantities, as well as the related GUPs that have been proposed based on these proposals. In fact, a minimum length is a common prediction of various approaches to quantum gravity such as string theory [13, 14] and loop quantum gravity [15, 16], black hole physics [1, 17], and quantized spacetime [73, 74], a maximum momentum appears in doubly special relativity [18] and a maximum length arises naturally in cosmology due to the presence of cosmological particle horizon [19]. The attempts of implementation of these optimum quantities on QM -in order to make it in harmony with the existence of optimum measurable quantities- gave rise to the generalization of the HUP to the so-called GUP. However, it is recommended to introduce even briefly the HUP, which may be considered the heart of quantum theory regarding to its scientific and philosophical implications.

Because it led to a reevaluation of the ideas concerning the process of measurement, and the relation between theory and experiment [75].

## 1.2 Heisenberg Uncertainty Principle

Heisenberg's Uncertainty Principle (HUP) is the primordial pillar of QM [75–79]. In QM, the physical observables are described by operators in Hilbert space. Let us consider an observable $A$ and its canonical conjugate $B$ define the operators $\hat{A}$ and $\hat{B}$, respectively. By denoting the uncertainty in the measurement of these observables $\Delta A$ and $\Delta B$. Accordingly, the HUP states [78]

$$\Delta A \Delta B \geq \frac{1}{2} \left| \left\langle \left[ \hat{A}, \hat{B} \right] \right\rangle \right|, \tag{1.1}$$

where $\left[\hat{A}, \hat{B}\right]$ corresponds to Heisenberg's algebra, which stands for the Canonical Commutation Relation (CCR) between the operators $\hat{A}$ and $\hat{B}$.

In the framework of Heisenberg's algebra, considering the Cartesian coordinates, in one-dimensional case, the position $\hat{x}$ and momentum $\hat{p}$ operators satisfy the following CCR:

$$[\hat{x}, \hat{p}] = i\hbar. \tag{1.2}$$

As a consequence, for the uncertainties in the measurement of the position $(\Delta x)$ and momentum $(\Delta p)$, the Heisenberg uncertainty relation reads

$$\Delta x \Delta p \geq \tfrac{\hbar}{2}. \tag{1.3}$$

Two main variants to represent the operators, $\hat{x}$ and $\hat{p}$, that satisfying the Heisenberg algebra defined through the CCR (1.2) are available, namely, the position and momentum representations. In position space, one has the following expressions:

$$\hat{x} = x, \qquad \hat{p} = -i\hbar\tfrac{d}{dx}, \tag{1.4}$$

and in momentum space

$$\hat{x} = i\hbar\tfrac{d}{dp}, \qquad \hat{p} = p. \tag{1.5}$$

In three dimensions the HUP is expressed by the inequalities

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}, \quad \Delta y \Delta p_y \geq \frac{\hbar}{2}, \quad \Delta z \Delta p_z \geq \frac{\hbar}{2}, \tag{1.6}$$

and the corresponding algebra

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \qquad i, j = 1, 2, 3 \equiv x, y, z. \tag{1.7}$$

The generalization to three dimensions case of the relations (1.4) and (1.5) are respectively

$$\hat{\mathbf{r}} = \mathbf{r} \equiv (x, y, z)\,, \qquad \hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla} \equiv \left(\hat{p}_x = -i\hbar\frac{\partial}{\partial x}, \quad \hat{p}_y = -i\hbar\frac{\partial}{\partial y}, \quad \hat{p}_z = -i\hbar\frac{\partial}{\partial z}\right), \tag{1.8}$$

and

$$\hat{\mathbf{r}} \equiv \left(\hat{x} = i\hbar\frac{\partial}{\partial p_x}, \quad \hat{y} = i\hbar\frac{\partial}{\partial p_y}, \quad \hat{z} = i\hbar\frac{\partial}{\partial p_z}\right), \qquad \hat{\mathbf{p}} = \mathbf{p} \equiv (p_x, p_y, p_z)\,. \tag{1.9}$$

As mentioned above, the emergence of optimum measurable quantities in fundamental physics suggest the adjustment of the HUP. In fact, those optimum quantities are related to a hypothesized scale called Planck scale, which will be brought out in the next section.

## 1.3 The Planck scale

In particle physics and cosmology, Planck units are a set of units of measurement defined in terms of four universal physical constants. In fact, these four physical constants take on the numerical value of 1 when expressed in terms of these units. The Planck units system is proposed in 1899 by the German physicist Max Planck. The four universal constants that Planck units normalize to 1 are

- The speed of light in vacuum, $c$,
- The gravitational constant, $G$,
- The reduced Planck constant, $\hbar$ ($\hbar = h/2\pi$ with $h$ being the Planck constant),

- The Boltzmann constant, $k_B$.

The Planck length, $l_p$, is defined as

$$l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35} \text{ m.} \tag{1.10}$$

The Planck time, $t_p$, is given by

$$t_p = \sqrt{\frac{\hbar G}{c^5}} \approx 10^{-44} \text{ s.} \tag{1.11}$$

The Planck energy, $E_p$, is

$$E_p = \sqrt{\frac{\hbar c^5}{G}} \approx 10^{19} \text{ GeV.} \tag{1.12}$$

The Planck temperature, $T_p$, is defined as

$$T_p = \sqrt{\frac{\hbar c^5}{G k_B^2}} \approx 10^{32} \text{ K.} \tag{1.13}$$

It is to note that the latest generation of particle accelerators produce energies of ~TeV. Therefore, the effects of the Planck scale are so far from current experimental technology. Even that Planck's scale remains a theoretical proposal, it has important implications in physical theories.

Quantum gravity effects are expected to become important at Planck's scale. Physics at Planckian energies represents a common challenge between the interests of particle physics and those of general relativity. At the Planck-scale energies, large gravitational fields are generated and felt by the interacting particles, which can then be used as sources and probes for quantum gravity effects. Therefore, Planck scale physics is a key for understanding the fundamental laws of nature [80].

## 1.4 Quantization of space-time

A theory for quantized space-time was proposed by Snyder [73, 74]. The aim was to remove the infinities problem in the early stages of the development of quantum field theory [81]. The author considered a de-Sitter space with real coordinates ($\eta_0$,

$\eta_1$, $\eta_2$, $\eta_3$, $\eta_4$), which satisfies

$$-\eta^2 = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2. \tag{1.14}$$

The position $\hat{X}_i$ and time $\hat{T}$ operators, which act on functions of variables ($\eta_0$, $\eta_1$, $\eta_2$, $\eta_3$, $\eta_4$), are defined respectively as [73]

$$\hat{X}_i = ia\left(\eta_4 \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_4}\right), \tag{1.15}$$

$$\hat{T} = \frac{ia}{c}\left(\eta_4 \frac{\partial}{\partial \eta_0} + \eta_0 \frac{\partial}{\partial \eta_4}\right), \tag{1.16}$$

where $i = 1, 2, 3$, $a$ is a natural unit of length and $c$ is the speed of light. Furthermore, the energy and momentum operators read [73, 74]

$$\hat{P}_T = \frac{\hbar c}{a}\frac{\eta_0}{\eta_4}, \tag{1.17}$$

$$\hat{P}_i = \frac{\hbar}{a}\frac{\eta_i}{\eta_4}. \tag{1.18}$$

Therefore, one has the following modified Heisenberg algebra between positions and momenta:

$$\left[\hat{X}_i, \hat{P}_j\right] = i\hbar\left(\delta_{ij} + \left(\frac{a}{\hbar}\right)^2 \hat{P}_i \hat{P}_j\right), \tag{1.19}$$

this deformed algebra represents an essential milestone on the construction of a theoretical framework for the GUP.

## 1.5 String theory

The impossibility of resolving arbitrarily small structures with an object of a finite extension has been observed in string theory [80]. The ansatz that the string not to interact at distances smaller than its size, leads to the existence of a minimum length on the resolution of the measurement process. This elementary length is implemented into QM as minimal uncertainty in the position measurement. Thus, the existence of a minimum length in the framework of string theory leads to the generalization of

Heisenberg's uncertainty principle. A string approach of the GUP was proposed by Amati et *al.* [13]. The characteristic string length, $\lambda_s$, is given by

$$\lambda_s = \sqrt{\hbar\mu}, \tag{1.20}$$

where $\mu$ is the string tension. The suggested modification of the uncertainty relation takes the form [13]

$$\Delta X \sim \frac{\hbar}{\Delta P} + Y\mu\Delta P, \tag{1.21}$$

where $Y$ is a suitable constant. Consequently, the existence of a minimal observable length of the order of string size $\sim \lambda_s\sqrt{Y}$ is likely [13].

## 1.6 Black hole physics

Hawking's radiation is the main physical quantity that may characterize the behavior of black holes [82]. Based on the hypothesis that a black hole emits detecting Hawking radiation turns out to be possible to capture a black hole image[1]. Besides, by measuring the direction of the propagating photons that are emitted at different angles and tracing them back, one can locate the position of the black hole center [82]. In such a way, the radius $R_h$ of the apparent horizon of a black hole will be measured. Thought experiments have been proposed to measure the area of the apparent horizon of a black hole [1]. Apparently, this measurement has two sources of uncertainty [1]

$$\Delta X^{(1)} \sim \frac{\lambda}{\sin\theta}, \tag{1.22}$$

and

$$\Delta X^{(2)} \sim \frac{\hbar G}{c^3\lambda} = \frac{l_p^2}{\lambda}, \tag{1.23}$$

[1]This was realized recently on April $10^{th}$ 2019, where scientists captured the first image of a black hole, using Event Horizon Telescope observations of the center of the galaxy M87. For more details, consult the following website: https://eventhorizontelescope.org/

where $\lambda$ is photon's wavelength and $\theta$ is the scattering angle [1]. By combining $\Delta X^{(1)}$ and $\Delta X^{(2)}$ linearly and using the trivial inequality

$$\frac{\lambda}{\sin\theta} \geq \lambda, \tag{1.24}$$

one obtains

$$\Delta X \gtrsim \lambda + \nu \frac{l_p^2}{\lambda}, \tag{1.25}$$

where $\nu$ is a constant. Eq. (1.25) implies that there exists a minimum error $\Delta X_{\min} \sim \nu l_p$.

In terms of $\Delta P \sim h\frac{\sin\theta}{\lambda}$, one has

$$\Delta X^{(1)} \sim \frac{\hbar}{\Delta P}. \tag{1.26}$$

By using the trivial inequality

$$\frac{\Delta P}{\sin\theta} \geq \Delta P, \tag{1.27}$$

one can obtain

$$\Delta X^{(2)} \sim Const. \frac{G}{c^3} \Delta P. \tag{1.28}$$

Therefore, the following generalized uncertainty relation is established [1]

$$\Delta X \gtrsim \frac{\hbar}{\Delta P} + Const. \frac{G}{c^3} \Delta P. \tag{1.29}$$

Consequently, in the context of black hole physics, a generalization of HUP has been deduced, which agrees well with the one obtained in the framework of string theory.

## 1.7 Doubly special relativity (DSR)

The Planck length, $l_p$, and time, $t_p$, are related to invariant universal constants, hence, they should have the same values in all inertial frames. On the other hand, according to the principle of special relativity, if these length and time are observable as physical length and interval, respectively in moving frames; thus they obey to the length contraction and time dilation phenomena. Consequently, it appears that modifications on special relativity are needed. A theory that does so is called Doubly Special Relativity (DSR), suggested in 2002 by Giovanni Amelino-Camelia [18].

The DSR theories are groups of transformations with two Lorentzian invariants, the constant speed of light and an invariant energy scale [83,84]. By parameterization with respect to an invariant length $l$, a nonlinear realization of Lorentz transformations in energy-momentum $(E, \mathbf{P})$ space was proposed [84]. In fact, the auxiliary-linearly transforming variables $\epsilon$, and $\pi$, respectively, read[2] [84]

$$\epsilon = Ef\left(lE, l^2\mathbf{P}^2\right), \tag{1.30}$$

$$\pi_i = P_i g\left(lE, l^2\mathbf{P}^2\right). \tag{1.31}$$

The two functions $f$ and $g$ parameterize the nonlinear realization of Lorentz transformations. An example of a nonlinear realization of Lorentz transformations corresponds to the following choice of functions [84]

$$f = \frac{1}{2}\left[\left(1 + l^2\mathbf{P}^2\right)\frac{e^{lE}}{lE} - \frac{e^{-lE}}{lE}\right], \tag{1.32}$$

$$g = e^{lE}. \tag{1.33}$$

For a particle of mass $m$, the energy and momentum are related to each other by [84]

$$\left(1 - l^2\mathbf{P}^2\right)e^{lE} + e^{-lE} = e^{lm} + e^{-lm}, \tag{1.34}$$

so one gets

$$e^{lE} = \frac{\cosh(lm) + \sqrt{\cosh^2(lm) - (1 - l^2\mathbf{P}^2)}}{(1 - l^2\mathbf{P}^2)}. \tag{1.35}$$

From this relation of the energy as a function of momentum, one has an upper bound on the momentum given by [84]

$$\mathbf{P}^2_{\max} < \frac{1}{l^2}, \tag{1.36}$$

this means that the existence of a minimal measurable length restricting the mo-

[2] In this section, the convention $c = \hbar = 1$ is used.

mentum to take any arbitrary value. At the Planck scale, i.e., $l \sim l_p$ this leads to a maximum momentum due to the fundamental structure of space-time [84].

Based on the above defined functions $f$ and $g$, the following modified commutation relation (MCR) was established in Ref. [84]

$$\left[\hat{X}_i, \hat{P}_j\right] = i\hbar \left[e^{-lE}\delta_{ij} + \frac{l^2}{\cosh(lm)}\hat{P}_i\hat{P}_j\right]. \tag{1.37}$$

For massless particles, one can obtain [84]

$$e^{lE} = \frac{1}{1 - l\,|\mathbf{P}|}, \tag{1.38}$$

which results in the following MCR [84]:

$$\left[\hat{X}_i, \hat{P}_j\right] = i\hbar\left((1 - l\,|\mathbf{P}|)\,\delta_{ij} + l^2 P_i P_j\right). \tag{1.39}$$

As an important result, a modified commutation relation may be occurring from DSR formalism.

## 1.8 Cosmological horizons

In the context of either cosmological particle horizons [85, 86] or nontrivial cosmic topology [87], a maximum measurable length emerges naturally. This maximum measurable length in the Universe, which corresponds to the cosmological particle horizon [19] is the length scale of the boundary between the observable and the unobservable regions of the Universe. This scale at any time defines the size of the observable Universe, and at the cosmic time $t$ is given by [19]

$$l_{\max}(t) = a(t)\int_0^t \frac{cdt}{a(t)}, \tag{1.40}$$

where $a(t)$ is the cosmic scale factor. For the best-fit $\Lambda$CDM[3] cosmic background at the present time $t_0$, one has[4] [19]

$$l_{\max}(t_0) \simeq 14 \text{ Gpc} \simeq 10^{26} \text{ m}. \tag{1.41}$$

[3] The $\Lambda$CDM is the abbreviation of Lambda Cold Dark Matter, which is the standard cosmological model.

[4] In this equation Gpc means Gigaparsec; 1 Gpc= 3.26 billion light-year (1 light-year= $9.46\times10^{15}$ m).

The existence of such a maximum observable length scale implies a modification of the HUP. In Ref. [19], the author proposed a new kind of the GUP that is consistent with the presence of a maximal length. In the next chapter, the one-dimensional maximal length GUP, as well as the extension of the formalism to arbitrary dimensions will be presented.

# Chapter 2

# Generalized Uncertainty Principle: Formalism

## 2.1 Introduction

This second chapter of the manuscript deals with a literature survey, focusing on the different formulations of the GUP. The corresponding representations are reported considering both position and momentum spaces.

## 2.2 GUP with minimal length

In order to make QM consistent with the minimum length hypothesis, modifications on the usual HUP have been suggested [1]. In fact, this elementary length is translated quantum mechanically as a nonzero minimal uncertainty in position measurements and leads to the GUP with a minimal length. The first systematic study, which opened this area of research, has been performed by Kempf and coworkers in their renowned paper [5]. In one dimension, the authors proposed the following uncertainty relation:

$$\Delta X \Delta P \geq \tfrac{\hbar}{2}\left(1+\gamma\left(\Delta P\right)^2+\gamma\left\langle \hat{P}\right\rangle^2\right), \tag{2.1}$$

where $\gamma=\gamma_0 l_p^2/\hbar^2$ is the GUP parameter; where $\gamma_0>0$ is a dimensionless parameter. This GUP implies the existence of a minimal length corresponding to a minimal uncertainty in position measurement, which is given by

$$\Delta X_{\min}=\hbar\sqrt{\gamma}. \tag{2.2}$$

The GUP (2.1) is originated from the MCR

$$\left[\hat{X},\hat{P}\right]=i\hbar\left(1+\gamma\hat{P}^2\right). \tag{2.3}$$

In the framework of the modified algebra (2.3), several representations were proposed, in which the generalized position and momentum operators obeying the MCR (2.3) are expressed in terms of the ordinary operators $\hat{x}$ and $\hat{p}$.

In Ref. [5], the authors proposed the following momentum space representation:

$$\hat{P}=p, \qquad \hat{X}=i\hbar\left(1+\gamma p^2\right)\frac{d}{dp}, \tag{2.4}$$

where $\hat{X}$ and $\hat{P}$ are symmetric[1] operators on the dense domain $S_\infty$ with respect to the following scalar product:

$$\left\langle\psi\right|\left.\phi\right\rangle=\int_{-\infty}^{+\infty}\tfrac{dp}{1+\gamma p^2}\psi^*\left(p\right)\phi\left(p\right), \tag{2.5}$$

also

$$\int_{-\infty}^{+\infty}\tfrac{dp}{1+\gamma p^2}\left|p\right\rangle\left\langle p\right|=1, \tag{2.6}$$

and

$$\left\langle p\right|\left.p'\right\rangle=\left(1+\gamma p^2\right)\delta\left(p-p'\right). \tag{2.7}$$

The authors also proposed the so-called quasiposition representation [5]

$$\hat{X}=\hat{x}+\hbar\sqrt{\gamma}\tan\left(\sqrt{\gamma}\hat{p}\right), \qquad \hat{P}=\tfrac{\tan\left(\sqrt{\gamma}\hat{p}\right)}{\sqrt{\gamma}}. \tag{2.8}$$

This representation results in complicated generalized wave-equations for which the search for solutions, even for simple potentials, would not be an easy task [88]. To

[1] That means $\left(\left\langle\psi\right|\hat{X}\right)\left|\phi\right\rangle=\left\langle\psi\right|\left(\hat{X}\left|\phi\right\rangle\right)$.

overcome this difficulty, it has been proposed the following generalized representation [20, 88]:

$$\hat{X} = \hat{x}, \qquad \hat{P} = \hat{p}\left(1 + \tfrac{\gamma}{3}\hat{p}^2\right). \tag{2.9}$$

In representation (2.4), the Hamiltonian eigenvalue problems may usually be expressed as a relatively simpler second-order ordinary differential equation in momentum space. However, representation (2.9) is more suitable for perturbative analysis of quantum systems.

In three dimensions, the GUP including a minimal length can be expressed as [28, 30]

$$\Delta X_i \Delta P_i \geq \frac{\hbar}{2}\left(1 + \gamma \sum_{j=1}^{3}\left[(\Delta P_j)^2 + \left\langle \hat{P}_j \right\rangle^2\right] + \gamma' \left[(\Delta P_i)^2 + \left\langle \hat{P}_i \right\rangle^2\right]\right), \tag{2.10}$$

where $\gamma, \gamma' > 0$ are considered as small parameters. This GUP model implies the existence of a minimal uncertainty for each coordinate given by $(\Delta X_i)_{\min} = \hbar\sqrt{3\gamma + \gamma'}, \forall i$. Furthermore, the GUP (2.10) arises from the following deformed Heisenberg algebra [28, 30]:

$$\begin{cases} \left[\hat{X}_i, \hat{P}_j\right] = i\hbar\left[\left(1 + \gamma \hat{\mathbf{P}}^2\right)\delta_{ij} + \gamma' \hat{P}_i \hat{P}_j\right], \\ \left[\hat{P}_i, \hat{P}_j\right] = 0, \\ \left[\hat{X}_i, \hat{X}_j\right] = i\hbar \dfrac{2\gamma - \gamma' + \gamma\left(2\gamma + \gamma'\right)\hat{\mathbf{P}}^2}{1 + \gamma \hat{\mathbf{P}}^2}\left(\hat{P}_i \hat{X}_j - \hat{P}_j \hat{X}_i\right), \end{cases} \tag{2.11}$$

where $\hat{\mathbf{P}}^2 = \sum_{j=1}^{3} \hat{P}_i \hat{P}_i$ .

In coordinate space, up to $O(\gamma)$ and in the special case $\gamma' = 2\gamma$; the generalized operators $\hat{X}_i$ and $\hat{P}_i$ may be represented as [31, 89]

$$\hat{X}_i = x_i, \qquad \hat{P}_i = \hat{p}_i\left(1 + \gamma \hat{\mathbf{p}}^2\right), \tag{2.12}$$

with $\hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}$ is the ordinary momentum operator and $\hat{\mathbf{p}}^2 = \sum_{j=1}^{3} \hat{p}_i \hat{p}_i$.

The representation (2.12) has firstly been used to study the hydrogen atom in

Ref. [89]. In this special case, the deformed algebra (2.11) reads

$$\begin{cases} \left[\hat{X}_i, \hat{P}_j\right] = i\hbar \left[\left(1+\gamma \hat{\mathbf{P}}^2\right)\delta_{ij} + 2\gamma \hat{P}_i \hat{P}_j\right], \\ \left[\hat{P}_i, \hat{P}_j\right] = 0, \\ \left[\hat{X}_i, \hat{X}_j\right] = 0. \end{cases} \tag{2.13}$$

Within this deformed algebra, we establish the corresponding Modified Dispersion Relation (MDR) and its consequences, namely, the varying speed of light and the generalized de Broglie wavelength. The details of the study are given at the end of this chapter.

## 2.3 GUP with minimal length from modified de Broglie's relation

It is worth to mention that the GUP (2.1) has been reestablished by Hossenfelder et *al.* [6] by employing another approach. In fact, the authors addressed the modification of de Broglie's relation.

In order to implement the notion of a minimal length $l_m$, the authors assumed that when the momentum $P$ increases arbitrarily, the wavenumber $k$ must not exceed an upper bound of the order of $(1/l_m)$. This effect will show up when $P$ approaches a certain scale $(\hbar/l_m)$. The physical interpretation of this is that particles could not possess arbitrarily small Compton wavelengths, $\lambda = 2\pi/k$, and that arbitrarily small scales could not be resolved anymore [6].

The authors assumed a relation between $k$ and $P$ of the form $P = f(k)$. This function is an odd function (because of parity) and the inverse function must approach asymptotically some upper limit, which is proportional to a minimal length, $l_m^{-1}$, when $P$ goes to infinity [6, 90].

Several forms of the function $f(k)$ can be found [91–93], for instance, Hossenfelder made the choice [6]

$$P = \frac{\hbar}{l_m} \tanh^{-1}\left(l_m k\right). \tag{2.14}$$

However, Bouaziz opted for the following choice [90]:

$$P = \frac{\hbar}{l_m} \tan\left(l_m k\right). \tag{2.15}$$

By using the expansion for small arguments

$$\tan y = y + \frac{y^3}{3} + \cdots , \tag{2.16}$$

up to second order in $l_m$, $P$ is written

$$P = \hbar\left(k + l_m^2 \frac{k^3}{3} + \cdots\right). \tag{2.17}$$

Assuming that the commutator between $\hat{X}$ and $\hat{k}$ keeps the standard form, i.e., $\left[\hat{X}_i, \hat{k}_j\right] = i\delta_{ij}$, and using the general relationship

$$\left[\hat{X}, \hat{A}\left(\hat{k}\right)\right] = i\frac{\partial \hat{A}}{\partial \hat{k}}, \tag{2.18}$$

one obtains the commutation relation defining the modified Heisenberg algebra

$$\left[\hat{X}, \hat{P}\left(\hat{k}\right)\right] = i\frac{\partial \hat{P}}{\partial \hat{k}}. \tag{2.19}$$

From relation (2.17), one gets

$$i\frac{\partial \hat{P}}{\partial \hat{k}} = i\hbar\left(1 + l_m^2 \hat{k}^2 + \cdots\right). \tag{2.20}$$

Now, one has

$$l_m^2 \hat{k}^2 \simeq \frac{l_m^2}{\hbar^2}\hat{P}^2 + O\left(l_m^4\right). \tag{2.21}$$

Then one finds

$$\left[\hat{X}, \hat{P}\right] = i\hbar\left(1 + \left(\frac{l_m}{\hbar}\right)^2 \hat{P}^2 + \cdots\right). \tag{2.22}$$

Let us introduce a parameter $\gamma$, linked to the minimal length by

$$\gamma = \left(\frac{l_m}{\hbar}\right)^2, \text{ whether } l_m = \hbar\sqrt{\gamma}. \tag{2.23}$$

One obtains the following form of the MCR:

$$\left[\hat{X}, \hat{P}\right] = i\hbar\left(1 + \gamma\hat{P}^2 + \cdots\right). \tag{2.24}$$

In QM, the commutation relation is directly connected to the uncertainty relation through the formula [78, 90]

$$\Delta A \Delta B \geq \frac{1}{2}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|, \tag{2.25}$$

which yields

$$\Delta X \Delta P \geq \frac{1}{2}\left|\left\langle\left[\hat{X},\hat{P}\right]\right\rangle\right|. \tag{2.26}$$

Up to first order of the parameter $\gamma$, the generalized uncertainty relation takes the form

$$\Delta X \Delta P \geq \tfrac{\hbar}{2}\left(1+\gamma\left\langle\hat{P}^2\right\rangle\right). \tag{2.27}$$

By using the definition of the mean-square deviation

$$(\Delta P)^2 = \left\langle\hat{P}^2\right\rangle - \left\langle\hat{P}\right\rangle^2, \tag{2.28}$$

one can write

$$\Delta X \Delta P \geq \tfrac{\hbar}{2}\left(1+\gamma\left[(\Delta P)^2+\left\langle\hat{P}\right\rangle^2\right]\right). \tag{2.29}$$

Obviously, this GUP is identical to the that of Kempf et *al.* [5], see Eq. (2.1).

## 2.4 GUP with minimal length and maximal momentum

To incorporate the idea of the maximal momentum of the DSR, Ali, Das and Vagenas (ADV) proposed the following modified commutation relations [8, 9]:

$$\begin{cases} \left[\hat{X}_i,\hat{P}_j\right] = i\hbar\left(\delta_{ij}-\kappa\left(\hat{\mathbf{P}}\delta_{ij}+\frac{\hat{P}_i\hat{P}_j}{\hat{\mathbf{P}}}\right)+\kappa^2\left(\hat{\mathbf{P}}^2\delta_{ij}+3\hat{P}_i\hat{P}_j\right)\right), \\ \left[\hat{X}_i,\hat{X}_j\right] = 0, \\ \left[\hat{P}_i,\hat{P}_j\right] = 0, \end{cases} \tag{2.30}$$

where $\kappa = \dfrac{\kappa_0}{M_p c} = \dfrac{\kappa_0 l_p}{\hbar}$ is the GUP parameter with $\kappa_0$ is a dimensionless parameter, $M_p$ is the Planck mass and $M_p c^2 \sim 10^{19}$ GeV is the Planck energy, and $\hat{\mathbf{P}}^2 = \sum_{i=1}^{3}\hat{P}_i\hat{P}_i$.

In the framework of the deformed algebra (2.30), the following representation is proposed in coordinate space

$$\hat{X}_i = x_i, \qquad \hat{P}_i = \hat{p}_i \left(1 - \kappa \hat{\mathbf{p}} + 2\kappa^2 \hat{\mathbf{p}}^2\right), \tag{2.31}$$

where $\hat{\mathbf{p}}^2 = \sum_{i=1}^{3} \hat{p}_i \hat{p}_i$.

However, in momentum space, one has

$$\hat{P}_i = p_i, \qquad \hat{X}_i = \left(1 - \kappa \left|\mathbf{p}\right| + 2\kappa^2 \left|\mathbf{p}\right|^2\right) \hat{x}_i. \tag{2.32}$$

where $|\mathbf{p}|$ is the magnitude of the vector $\mathbf{p}$, i.e., $|\mathbf{p}| = \sqrt{\mathbf{p}^2}$.

Within the ADV approach, in one-dimensional case, the following GUP has been established [8, 9]:

$$\Delta X \Delta P \geq \tfrac{\hbar}{2} \left(1 - 2\kappa \left\langle \hat{P} \right\rangle + 4\kappa^2 \left\langle \hat{P}^2 \right\rangle\right). \tag{2.33}$$

Since

$$(\Delta P)^2 = \left\langle \hat{P}^2 \right\rangle - \left\langle \hat{P} \right\rangle^2, \tag{2.34}$$

therefore

$$\Delta X \Delta P \geq \tfrac{\hbar}{2} \left(1 + \left(\frac{\kappa}{\sqrt{\left\langle \hat{P}^2 \right\rangle}} + 4\kappa^2\right) \Delta P^2 + 4\kappa^2 \left\langle \hat{P} \right\rangle^2 - 2\kappa \sqrt{\left\langle \hat{P}^2 \right\rangle}\right). \tag{2.35}$$

This form of GUP implies both a minimal length uncertainty and a maximal momentum uncertainty, namely [8]

$$\Delta X \geq \Delta X_{\min} \approx \kappa_0 l_p \equiv \hbar \kappa, \tag{2.36}$$

$$\Delta P \leq \Delta P_{\max} \approx \frac{M_p c}{\kappa_0} \equiv \frac{1}{\kappa}. \tag{2.37}$$

The corresponding MCR reads

$$\left[\hat{X}, \hat{P}\right] = i\hbar \left(1 - \kappa \hat{P} + 2\kappa^2 \hat{P}^2\right). \tag{2.38}$$

In the position space we have

$$\hat{X} = x, \qquad \hat{P} = \left(1 - \kappa \hat{p} + 2\kappa^2 \hat{p}^2\right) \hat{p}, \tag{2.39}$$

and in momentum space

$$\hat{X} = \left(1 - \kappa p + 2\kappa^2 p^2\right) \hat{x}, \qquad P = p. \tag{2.40}$$

In the framework of this representation, the scalar product should be modified due to the presence of the additional factor $(1 - \kappa p + 2\kappa^2 p^2)$, and the maximal momentum. The integrals are calculated between $-P_p$ and $P_p$, Planck momentum[2]

$$\langle \phi | \, \psi \rangle = \int_{-P_p}^{P_p} \frac{\phi^* (p) \, \psi (p)}{1 - \kappa p + 2\kappa^2 p^2} dp. \tag{2.41}$$

The identity operator is given by

$$\int_{-P_p}^{P_p} \frac{|p\rangle \, \langle p|}{1 - \kappa p + 2\kappa^2 p^2} dp = 1. \tag{2.42}$$

The scalar product of the momentum eigenstates is expressed by

$$\langle p | \, p' \rangle = \left(1 - \kappa p + 2\kappa^2 p^2\right) \delta \left(p - p'\right). \tag{2.43}$$

## 2.5 Higher-order GUP approach

Higher-order modifications on the HUP have been proposed to solve some problems that appeared when applying either linear[3] or quadratic[4] GUP models. According to Pedram the formalism of ADV [8] has the following difficulties [11]:

- It is only valid for small values of the GUP parameter, i.e., it is a perturbative formalism.
- The maximal momentum uncertainty differs from the maximal momentum required in DSR theories. In fact, Eq. (2.37) puts an upper bound on the uncertainty of the momentum measurement, not on the value of the observed momentum.
- It does not imply noncommutative geometry, see Eq. (2.30); $[\hat{X}_i, \hat{X}_j] = 0$.

[2] The Planck momentum is $P_p = M_p c$ with $M_p = \sqrt{\frac{\hbar c}{G}}$ is the Planck mass.
[3] Linear with respect to $\hat{P}$, like that of Ali et *al.* [8]. See Eq. (2.38) up to $O(\kappa)$.
[4] Quadratic with respect to $\hat{P}$, like that of Kempf et *al.* [5]. See Eq. (2.3).

To overcome these problems, Pedram proposed the following higher-order GUP [11]:

$$\Delta X \Delta P \geq \frac{\hbar}{2} \frac{1}{1-\gamma\left(\Delta P\right)^2}, \tag{2.44}$$

which implies both the minimal length uncertainty

$$\Delta X_{\min} = \frac{3\sqrt{3}}{4}\hbar\sqrt{\gamma}, \tag{2.45}$$

and the maximal observable momentum

$$\Delta P_{\max} = P_{\max} = \frac{1}{\sqrt{\gamma}}. \tag{2.46}$$

This means that the presence of an upper bound on the momentum properly agrees with DSR theories.

The corresponding MCR reads

$$\left[\hat{X}, \hat{P}\right] = \frac{i\hbar}{1-\gamma\hat{P}^2}, \tag{2.47}$$

within this MCR, in the momentum space representation, one can write the momentum and position operators as

$$\hat{P} = p, \qquad \hat{X} = \frac{i\hbar}{1-\gamma p^2}\frac{d}{dp}. \tag{2.48}$$

By relying on the symmetricity condition of the position operator, the modified completeness relation and scalar product can be written as follows

$$\langle\psi|\,\phi\rangle = \int_{-1/\sqrt{\gamma}}^{+1/\sqrt{\gamma}} dp\left(1-\gamma p^2\right)\psi^*\left(p\right)\phi\left(p\right), \tag{2.49}$$

$$\langle p|\,p'\rangle = \frac{\delta\left(p-p'\right)}{1-\gamma p^2}. \tag{2.50}$$

A generalization to $D$-dimensions of the one-dimensional commutation relation (2.47) that preserves the rotational symmetry is [12]

$$\left[\hat{X}_i, \hat{P}_j\right] = \frac{i\hbar\delta_{ij}}{1-\gamma\hat{\mathbf{P}}^2}, \tag{2.51}$$

where $\hat{\mathbf{P}}^2 = \sum_{i=1}^{D}\hat{P}_i\hat{P}_i$. This relation implies a nonzero minimal uncertainty and a maximal observable momentum in each position coordinate. If the components of the momentum operator are assumed to be commutative, i.e.,

$$\left[\hat{P}_i, \hat{P}_j\right] = 0, \tag{2.52}$$

then the Jacobi identity[5] determines the commutation relations between the components of the position operator as

$$\left[\hat{X}_i, \hat{X}_j\right] = \frac{2i\hbar\gamma}{\left(1-\gamma\mathbf{P}^2\right)^2}\left(\hat{P}_i\hat{X}_j - \hat{P}_j\hat{X}_i\right), \tag{2.53}$$

which results in a noncommutative geometric generalization of position space. To exactly satisfy these commutation relations, the momentum and position operators in the momentum space representation can be written as follows [12]:

$$\hat{P}_i = p_i, \qquad \hat{X}_i = \frac{i\hbar}{1-\gamma\mathbf{p}^2}\frac{\partial}{\partial p_i}. \tag{2.54}$$

where $\mathbf{p}^2 = \sum_{i=1}^{D} p_i p_i$.

The operators $\hat{X}_i$ and $\hat{P}_j$ are now symmetric operator on the domain $S_\infty$ with respect to the scalar product

$$\langle\psi|\,\phi\rangle = \int_{-1/\sqrt{\gamma}}^{+1/\sqrt{\gamma}} d^D p\left(1-\gamma\mathbf{p}^2\right)\psi^*(\mathbf{p})\,\phi(\mathbf{p}). \tag{2.55}$$

The identity operator is given by

$$1 = \int_{-1/\sqrt{\gamma}}^{+1/\sqrt{\gamma}} \frac{d^D p}{\left(1-\gamma\mathbf{p}^2\right)}\,|p\rangle\langle p|, \tag{2.56}$$

and the scalar product of momentum eigenstates is

$$\langle p|\,p'\rangle = \frac{\delta^D(p-p')}{1-\gamma\mathbf{p}^2}. \tag{2.57}$$

## 2.6 GUP with maximal length

### 2.6.1 One-dimensional case

The GUP of Eq (2.44) can be further generalized to include explicit maxima and minima in both position and momentum uncertainties. In Ref. [19], the author proposed such a GUP of the form

$$\Delta X \Delta P \geq \frac{\hbar}{2}\frac{1}{1-\gamma\left(\Delta P\right)^2}\frac{1}{1-\alpha\left(\Delta X\right)^2}. \tag{2.58}$$

Concerning the GUP incorporating a maximal length, $l_{\max}$, it has been proposed

[5] $\left[\left[\hat{X}_i, \hat{X}_j\right], \hat{P}_k\right] + \left[\left[\hat{P}_k, \hat{X}_i\right], \hat{X}_j\right] + \left[\left[\hat{X}_j, \hat{P}_k\right], \hat{X}_i\right] = 0.$

in $1D$ in the form [19]

$$\Delta X \Delta P \geq \frac{\hbar}{2} \frac{1}{1-\alpha\left(\Delta X\right)^{2}}, \tag{2.59}$$

where $\alpha = l_{\max}^{-2} = \alpha_0 \left(\frac{H_0}{c}\right)^2$ with $H_0$ is the Hubble constant, $c$ is the speed of light and $\alpha_0$ is a dimensionless parameter.

This GUP implies the existence of a maximum position uncertainty corresponding to a maximum length given by

$$\Delta X_{\max} = l_{\max} = \frac{1}{\sqrt{\alpha}}, \tag{2.60}$$

and a minimum momentum uncertainty

$$\Delta P_{\min} = \frac{3\sqrt{3}}{4}\hbar\sqrt{\alpha}. \tag{2.61}$$

The GUP (2.59) follows from the MCR [19]

$$\left[\hat{X}, \hat{P}\right] = \frac{i\hbar}{1-\alpha\hat{X}^2}. \tag{2.62}$$

The position and momentum operators, satisfying this modified algebra, can be represented as [19]

$$\hat{X} = \hat{x}, \qquad \hat{P} = \frac{1}{1-\alpha\hat{x}^2}\hat{p}, \tag{2.63}$$

where the operators $\hat{x}$ and $\hat{p}$ obey the ordinary commutation relation $[\hat{x}, \hat{p}] = i\hbar$. In coordinate space, the undeformed position and momentum operators are respectively $\hat{x} = x$ and $\hat{p} = -i\hbar\frac{d}{dx}$, which yields the following deformed representation:

$$\hat{X} = x, \qquad \hat{P} = -\frac{i\hbar}{1-\alpha x^2}\frac{d}{dx} \tag{2.64}$$

In order to retain the symmetry of the momentum operator, i.e., $\left(\langle\psi|\,\hat{P}\right)|\phi\rangle = \langle\psi|\left(\hat{P}\,|\phi\rangle\right)$, the scalar product is defined within this formalism as follows [19]:

$$\langle\psi|\,\phi\rangle = \int_{-l_{\max}}^{+l_{\max}} \left(1-\alpha x^2\right)\psi^*\left(x\right)\phi\left(x\right)dp. \tag{2.65}$$

### 2.6.2 Generalization to $D$-dimensions

The generalization of the one-dimensional commutation relation (2.62) that preserves the rotational symmetry is [72, 94]

$$\left[\hat{X}_i, \hat{P}_j\right] = \frac{i\hbar\delta_{ij}}{1-\alpha\hat{\mathbf{X}}^2}, \tag{2.66}$$

where $\hat{\mathbf{X}}^2 = \sum_{i=1}^{D} \hat{X}_i\hat{X}_i$. This Heisenberg deformed algebra implies a maximal observable length and a nonzero minimal momentum uncertainty in each position coordinate.

Assuming that the components of the position operator commutate among them, i.e., keeping a commutative geometry

$$\left[\hat{X}_i, \hat{X}_j\right] = 0, \tag{2.67}$$

then the Jacobi identity[6] determines the commutation relations between the components of the momentum operator as [72, 94]

$$\left[\hat{P}_i, \hat{P}_j\right] = \frac{2i\hbar\alpha}{\left(1-\alpha\hat{\mathbf{X}}^2\right)^2}\left(\hat{X}_j\hat{P}_i - \hat{X}_i\hat{P}_j\right). \tag{2.68}$$

Therefore, one can generalize the representation (2.64) in coordinate space to $D$-dimensions [72, 94]

$$\hat{X}_i = x_i, \qquad \hat{P}_i = -\frac{i\hbar}{1-\alpha\mathbf{x}^2}\frac{\partial}{\partial x_i}. \tag{2.69}$$

where $\mathbf{x}^2 = \sum_{i=1}^{D} x_i x_i$.

The operators $\hat{X}_i$ and $\hat{P}_j$ are now symmetric, but with respect to the scalar product [72, 94]

$$\langle\psi|\,\phi\rangle = \int_{-1/\sqrt{\alpha}}^{+1/\sqrt{\alpha}} \left(1-\alpha\mathbf{x}^2\right)\psi^*\left(\mathbf{x}\right)\phi\left(\mathbf{x}\right)d^Dx. \tag{2.70}$$

[6] $\left[\left[\hat{P}_i, \hat{P}_j\right], \hat{X}_k\right] + \left[\left[\hat{X}_k, \hat{P}_i\right], \hat{P}_j\right] + \left[\left[\hat{P}_j, \hat{X}_k\right], \hat{P}_i\right] = 0.$

## 2.7 GUP and Modified Dispersion Relation (MDR)

Studies in Loop Quantum Gravity (LQG) support that the ordinary energy-momentum dispersion relation $E^2 = m^2c^4 + \mathbf{p}^2c^2$, has to be modified under Planck-scale ($l_p$) effect [95–98].

Given that the modification of the dispersion relation and the Heisenberg uncertainty principle is a consequence of quantum gravitational effects, accordingly, looking for a connection MDR-GUP seems to be natural; this issue has been addressed in Refs. [5, 6, 99].

### 2.7.1 Modified dispersion relation establishment

In this subsection we derive the dispersion relation corresponding to the minimal length GUP (2.10) in the special case $\gamma' = 2\gamma$, by using the generalized representation (2.12). For this purpose, we follow the scheme of Majhi and Vagenas [99].

Let us now start our investigation by considering the gravitational background metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}c^2dt^2 + g_{ij}dx^i dx^j, \tag{2.71}$$

we adopt the familiar convention: small Greek indices, such as $\mu, \nu, \ldots$, take the values $0, 1, 2, 3, \ldots$ and small Latin indices, such as $i, j, \ldots$, denote spatial indices and take the values $1, 2, 3, \ldots$.

The modified four momentum is denoted by $P^\mu$, while the usual four momentum is denoted by $p^\mu$. According to the representation (2.12), $P^\mu$ and $p^\mu$ are related by the following relations:

$$P^0 = p^0, \tag{2.72}$$

$$P^i = p^i\left(1 + \gamma\mathbf{p}^2\right), \tag{2.73}$$

where $\mathbf{p}^2 = g_{ij}p^ip^j$, thus $\mathbf{p} = \sqrt{g_{ij}p^ip^j}$.

For a detailed review on the correspondence between quantum representations and the classical transformations, which links the modified momentum and the ordinary one, we refer the reader to Ref. [100].

The square of the four-momentum within the considered background metric (2.71) reads

$$P^\mu P_\mu = g_{\mu\nu}P^\mu P^\nu = g_{00}\left(P^0\right)^2 + \mathbf{P}^2 = g_{00}\left(P^0\right)^2 + g_{ij}P^iP^j. \tag{2.74}$$

By employing Eqs. (2.72)-(2.73), one gets

$$P^\mu P_\mu = g_{00}\left(p^0\right)^2 + g_{ij}p^ip^j\left(1+\gamma\mathbf{p}^2\right)^2, \tag{2.75}$$

up to $O(\gamma)$ this relation yields

$$P^\mu P_\mu = g_{00}\left(p^0\right)^2 + \mathbf{p}^2 + 2\gamma\mathbf{p}^2\mathbf{p}^2. \tag{2.76}$$

By using the ordinary dispersion relation $p^\mu p_\mu = g_{00}\left(p^0\right)^2 + \mathbf{p}^2 = -m^2c^2$, Eq. (2.76) takes the form

$$P^\mu P_\mu = -m^2c^2 + 2\gamma\mathbf{p}^2\mathbf{p}^2. \tag{2.77}$$

On the other hand, by inverting Eq. (2.73), one obtains up to $O(\gamma)$

$$p^i = P^i\left(1-\gamma\mathbf{p}^2\right), \tag{2.78}$$

therefore up to first order in the deformation parameter $\gamma$, one has

$$\mathbf{p}^2 = \mathbf{P}^2 - 2\gamma\mathbf{P}^2\mathbf{p}^2. \tag{2.79}$$

This expression allows us to express Eq. (2.77) in the first order of $\gamma$ as

$$P^\mu P_\mu = -m^2c^2 + 2\gamma\mathbf{P}^2\mathbf{P}^2. \tag{2.80}$$

Given that [99]

$$E = -g_{00}cP^0. \tag{2.81}$$

From (2.74) and (2.80), one can find the time component of the momentum

$$\left(P^0\right)^2 = -\frac{1}{g_{00}}\left(m^2c^2 + \mathbf{P}^2\left(1-2\gamma\mathbf{P}^2\right)\right). \tag{2.82}$$

One then gets

$$E^2 = -g_{00}\left(m^2c^4 + c^2\mathbf{P}^2\left(1 - 2\gamma\mathbf{P}^2\right)\right). \tag{2.83}$$

This MDR corresponds to the GUP (2.10) proposed by Kempf et *al.* [5] in the special case $\gamma' = 2\gamma$, implemented through the classical transformation corresponding to the generalized representation (2.12) of Ref. [89]. For Minkowski space-time $g_{\mu\nu} = diag\,(-1, 1, 1, 1)$, i.e., $g_{00} = -1$, one has

$$E^2 = m^2c^4 + c^2\mathbf{P}^2\left(1 - 2\gamma\mathbf{P}^2\right). \tag{2.84}$$

Remarkably, Eq. (2.84) is in agreement with other modified dispersion relations existing in the literature [97,101,102], which have been established in different contexts. In fact, phenomenological investigations in quantum gravity led to the following formula [97, 102]:

$$E^2 = m^2c^4 + c^2\mathbf{P}^2\left(1 - \zeta\left(\frac{E}{E_p}\right)^n\right), \tag{2.85}$$

where $E_p$ is the Planck energy, $\zeta$ is a coefficient of order 1, whose precise value depends upon the considered quantum-gravity model and $n$ can be chosen as $n = 1$ or 2. In LQG one might typically expect to obtain $n = 2$ [97].

The concordance between Eqs. (2.84) and (2.85) might be viewed as an implicit indication of a consistency between the two approaches of quantum gravity, as Eq. (2.84) originates from the GUP (2.10), which is initially proposed in connection with string theory.

### 2.7.2 Photon's velocity within minimal length GUP

The result the last subsection (2.7.1) leads to an interesting conclusion, as it will be discussed here. Let us derive a modified photon velocity ($u$) from the MDR (2.83) by relying on the following definition [99]:

$$u = \frac{1}{\sqrt{-g_{00}}}\frac{\partial E}{\partial \mathbf{P}}. \tag{2.86}$$

By setting $m = 0$ in the MDR (2.83) one obtains

$$E^2 = -g_{00}c^2\mathbf{P}^2\left(1 - 2\gamma\mathbf{P}^2\right). \tag{2.87}$$

Therefore, up to the first-order in $\gamma$, one gets

$$u = c\left(1 - 3\gamma\mathbf{P}^2\right). \tag{2.88}$$

One can express the corrective term in this equation as a function of energy. Thus, from Eq. (2.87), in first order of $\gamma$, one can write

$$\mathbf{P}^2 = \left(\frac{E}{c\sqrt{-g_{00}}}\right)^2\left(1 + 2\gamma\left(\frac{E}{c\sqrt{-g_{00}}}\right)^2\right), \tag{2.89}$$

hence, up to $O(\gamma)$, Eqs. (2.88) and (2.89) lead to the following modified photon velocity:

$$u = c\left(1 - 3\gamma\left(\frac{E}{c\sqrt{-g_{00}}}\right)^2\right). \tag{2.90}$$

For the special case of Minkowski space-time, i.e., $g_{00} = -1$, one has

$$u = c\left(1 - 3\gamma\left(\frac{E}{c}\right)^2\right). \tag{2.91}$$

Formula (2.91) indicates that there is a velocity dispersion and that the photon velocities are energy-dependent; this result agrees that of Refs. [97, 103]. Especially, from Eq. (2.85) of Ref. [97], the modified photon velocity in the framework of LQG, i.e., $n = 2$, up to the leading corrective term, may be deduced, so

$$u = c\left(1 - \frac{3}{2}\zeta\left(\frac{E}{E_p}\right)^2\right) = c\left(1 - \frac{3}{2}\zeta\frac{l_p^2}{\hbar^2}\left(\frac{E}{c}\right)^2\right). \tag{2.92}$$

here, the definition of the Planck energy $E_p = \frac{\hbar c}{l_p}$ is used. The comparison between Eqs. (2.91) and (2.92) leads to the following relation between $\zeta$ and the minimal length parameter $\gamma$:

$$\zeta = 2\gamma\frac{\hbar^2}{l_p^2}. \tag{2.93}$$

It is worth mentioning that the modified photon velocity, given by both Eqs. (2.88) and (2.91), shows that photon propagation does not appear tachyonic within the considered GUP model, which includes a minimal length. This result is different from that obtained in Ref. [99], where it has been shown that photons exhibit a superluminal behavior in the context of the GUP model of Refs. [8–10], which incorporates both a minimal length and a maximal momentum.

### 2.7.3 Minimal length assumption and de Broglie wavelength

We now proceed to a heuristic investigation on the de Broglie wavelength form in the presence of a minimal length. To this end, we write the Schrödinger equation for a free particle by using the representation (2.9) in coordinate space; up to first order of $\gamma$, one has

$$\frac{d^2\psi}{dx^2} - \frac{2}{3}\gamma\hbar^2\frac{d^4\psi}{dx^4} + \frac{2mE}{\hbar^2}\psi = 0. \tag{2.94}$$

Equation (2.94) can be transformed to a second order differential equation by performing the following transformation [104]:

$$\psi\left(x\right) = \left(1 + \frac{2}{3}\gamma\hbar^2\frac{d^2}{dx^2}\right)\varphi\left(x\right), \tag{2.95}$$

the new function, $\varphi$, satisfies then the second order differential equation

$$\frac{d^2\varphi}{dx^2} + k^2\varphi = 0, \tag{2.96}$$

where $k^2 = \frac{2mE}{\hbar^2\left(1+\frac{4}{3}m\gamma E\right)}$, the general solution of this equation is given by

$$\varphi\left(x\right) = A\exp\left(ikx\right) + B\exp\left(-ikx\right), \tag{2.97}$$

where $A$ and $B$ are two arbitrary constants.

According to the transformation (2.95), the solution of Eq. (2.94) is

$$\psi\left(x\right) = \left(1 - \frac{2}{3}\gamma\hbar^2k^2\right)\varphi\left(x\right). \tag{2.98}$$

One observes that, with the minimal length assumption, the stationary states of a free particle are still propagating plane waves. However, for their wavelengths one gets in the first order of the deformation parameter $\gamma$

$$\lambda\left(E\right) = \frac{h}{\sqrt{2mE}}\left(1 + \frac{2}{3}\gamma mE\right). \tag{2.99}$$

This formula may be viewed as a generalized de Broglie wavelength including the minimal length effects. The relationship (2.99) is similar to some formulas in the literature, obtained by using other approaches; see, for instance, Refs. [5, 6].

Finally, it is worth to mention that the correction term in Eq. (2.99) prevents the wavelength to vanish even at arbitrary high energies. This finding is in agreement with the claim of Ref. [105]; where a de Broglie wavelength, embodying string corrections, has been established in the framework of UV self-complete theory of quantum gravity.

# Part II

# Statistical applications of GUP

# Chapter 3

# Statistical physics and the GUP scenario

## 3.1 Introduction

In the framework of the above mentioned GUPs, several works addressed different statistical systems, including, ideal gas, harmonic oscillator and relativistic gas. In this part of the manuscript, we consider these statistical systems in the context of the maximal length GUP. To distinguish between the deformed and the undeformed functions, we use small letters for the ordinary quantities and capital letters for the generalized ones.

Let us consider a system consisting of $N$ identical non-interacting particles in the external field $U$. The behavior of each particle may be described by the corresponding deformed Hamiltonian $H = \frac{\mathbf{P}^2}{2m} + U(\mathbf{X})$. To deal with such a statistical system within the GUP scenario, two approaches may be employed, as in the ordinary (undeformed) case, namely, the quantum approach and the semiclassical one.

The quantum method is based on the following definition of the deformed single-particle canonical partition function [51, 52, 71]:

$$Q_1 = \sum_n \exp\left(-\beta E_n\right), \tag{3.1}$$

where $E_n$ are eigenvalues of the deformed Hamiltonian, obtained by solving the corresponding deformed stationary wave equation, and $\beta = \left(k_B T\right)^{-1}$, with $k_B$ is Boltzmann's constant and $T$ is the temperature.

The semiclassical scheme is based on the following deformed single-particle canonical partition function [51]:

$$Q_1 = \frac{1}{h^D} \int \frac{(dX)\,(dP)}{J} \exp\left(-\beta H\left(\mathbf{X}, \mathbf{P}\right)\right), \tag{3.2}$$

where $(dX)$ and $(dP)$ stand as $dX_1 dX_2 \cdots dX_D$ and $dP_1 dP_2 \cdots dP_D$, respectively. $H$ is the classical Hamiltonian of a single-particle constituting the system, and $J$ is the Jacobian of the transformation, which links the variables $(X_i,\, P_i)$ and the canonically conjugated ones $(x_i,\ p_i)$ satisfying the Poisson brackets $\{x_i, p_j\} = \delta_{ij}, \{x_i, x_j\} = \{p_i, p_j\} = 0$.

The Jacobian can be read off from the deformed Poisson brackets, which are obtained from the quantum commutation relations as

$$\frac{1}{i\hbar}\left[\hat{A}, \hat{B}\right] \longrightarrow \{A, B\}\,. \tag{3.3}$$

The Jacobian of the transformation in $D$ dimensions is [51]

$$J = \frac{\partial\left(X_1, P_1, \ldots, X_D, P_D\right)}{\partial\left(x_1, p_1, \ldots, x_D, p_D\right)}. \tag{3.4}$$

In the one-dimensional case, the following expression holds [51]:

$$J = \frac{\partial\left(X, P\right)}{\partial\left(x, p\right)} = \{X, P\}\,. \tag{3.5}$$

In the three-dimensional case, it is concluded to [51]

$$\begin{aligned} J &= \frac{\partial\left(X_1, P_1, X_2, P_2, X_3, P_3\right)}{\partial\left(x_1, p_1, x_2, p_2, x_3, p_3\right)} \\ &= \{X_1, P_1\}\{X_2, P_2\}\{X_3, P_3\} \\ &- \{X_1, P_3\}\{P_1, P_2\}\{X_2, X_3\} - \{X_1, P_2\}\{X_2, P_1\}\{X_3, P_3\} \\ &- \{X_1, P_3\}\{X_2, P_2\}\{X_3, P_1\} - \{X_1, P_1\}\{X_2, P_3\}\{X_3, P_2\} \\ &+ \{X_1, X_2\}\{P_1, P_3\}\{X_3, P_2\} + \{X_1, P_3\}\{X_2, P_1\}\{X_3, P_2\} \\ &- \{X_1, X_2\}\{P_2, P_3\}\{X_3, P_1\} + \{X_1, P_2\}\{X_2, X_3\}\{P_1, P_3\} \\ &- \{X_1, X_3\}\{P_1, P_3\}\{X_2, P_2\} + \{X_1, X_3\}\{X_2, P_1\}\{P_2, P_3\} \\ &+ \{X_1, X_3\}\{P_1, P_2\}\{X_2, P_3\} - \{X_1, X_2\}\{P_1, P_2\}\{X_3, P_3\} \\ &- \{X_1, P_1\}\{X_2, X_3\}\{P_2, P_3\} + \{X_1, P_2\}\{X_2, P_3\}\{X_3, P_1\}. \end{aligned} \tag{3.6}$$

## 3.2 Semiclassical approach and minimal length GUP

For the deformed algebra (2.11), corresponding to the GUP with a minimal length (2.10), the Jacobian (3.6) reads [51]

$$J = \left(1 + \gamma \mathbf{P}^2\right)^2 \left(1 + \left(\gamma + \gamma'\right)\mathbf{P}^2\right). \tag{3.7}$$

Then, for a single-particle Hamiltonian

$$H = \frac{\mathbf{P}^2}{2m} + U\left(\mathbf{X}\right), \tag{3.8}$$

the partition function (3.2) takes the form [51]

$$Q_1 = \frac{1}{h^3}\int \left(dX\right)\exp\left(-\beta U\left(\mathbf{X}\right)\right)\int \left(dP\right)\frac{\exp\left(-\beta\frac{\mathbf{P}^2}{2m}\right)}{\left(1+\gamma\mathbf{P}^2\right)^2\left(1+\left(\gamma+\gamma'\right)\mathbf{P}^2\right)}. \tag{3.9}$$

For low temperature, i.e., $\frac{m\gamma}{\beta} << 1$ and $\frac{m\gamma'}{\beta} << 1$, the expression (3.9) gives the following formula for the partition function [51]:

$$Q_1 \approx q_1\left(1 - 3\left(3\gamma + \gamma'\right)\frac{m}{\beta} + o\left(T\right)\right), \tag{3.10}$$

where $q_1 = \frac{1}{h^3}\int\left(dX\right)\exp\left(-\beta U\left(\mathbf{X}\right)\right)\int\left(dP\right)\exp\left(-\beta\frac{\mathbf{P}^2}{2m}\right)$ is the partition function of one particle described by the same Hamiltonian for the undeformed case.

For large temperature, i.e., $\frac{m\gamma}{\beta} >> 1$, the expression (3.9) gives

$$Q_1 \approx q_1\left(\frac{2\pi m}{\gamma}\right)^{-3/2}\frac{\pi^2}{\sqrt{\gamma}\left(\sqrt{\gamma}+\sqrt{\gamma+\gamma'}\right)^2}\left(1 + O\left(\frac{1}{T}\right)\right). \tag{3.11}$$

## 3.3 Semiclassical approach and maximal length GUP

### 3.3.1 One-dimensional case

For the deformed algebra (2.62), corresponding to the GUP with a maximal length (2.59), the Jacobian (3.5) reads [72]

$$J = \frac{1}{1-\alpha X^2}. \tag{3.12}$$

Then, for a single-particle Hamiltonian

$$H = \frac{P^2}{2m} + U\left(X\right), \tag{3.13}$$

the partition function (3.2) takes the form [72]

$$Q_1 = \frac{1}{h}\int \left(1-\alpha X^2\right) \exp\left(-\beta U\left(X\right)\right) dX \int \exp\left(-\beta \frac{P^2}{2m}\right) dP. \tag{3.14}$$

### 3.3.2 Three-dimensional case

For the $3D$ maximal length deformed algebra (2.66)-(2.68), the Jacobian (3.6) reads [72]

$$J = \frac{1}{\left(1-\alpha \mathbf{X}^2\right)^3}. \tag{3.15}$$

For a single-particle Hamiltonian in $3D$

$$H = \frac{\mathbf{P}^2}{2m} + U\left(\mathbf{X}\right), \tag{3.16}$$

thus, the single-particle partition function (3.2) can be written in $3D$ as [72]

$$Q_1 = \frac{1}{h^3}\int \left(dX\right) \left(1-\alpha \mathbf{X}^2\right)^3 \exp\left(-\beta U\left(\mathbf{X}\right)\right) \int \left(dP\right) \exp\left(-\beta \frac{\mathbf{P}^2}{2m}\right), \tag{3.17}$$

where $\mathbf{X}^2 = \sum_{i=1}^{3} X_i X_i$ and $\mathbf{P}^2 = \sum_{i=1}^{3} P_i P_i$.

In what follows, the developed formalism, especially formulas (3.14) and (3.17) will be applied to study an ideal gas, an ensemble of $N$ harmonic oscillators and a relativistic gas in the framework of maximal-length deformed Heisenberg algebra.

Once the total partition function ($Q$) is known, the thermodynamics of the system flows in a straightforward manner. In the scope of the canonical ensemble defined through the parameters $[N, V, T]$[1], the internal energy, heat capacity at constant volume, free Helmholtz energy, entropy and chemical potential are given by

$$E = -\left(\frac{\partial \ln Q}{\partial \beta}\right)_{N,V}, \tag{3.18}$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V}, \tag{3.19}$$

$$A = -k_B T \ln Q, \tag{3.20}$$

$$S = -\left(\frac{\partial A}{\partial T}\right)_{N,V}, \tag{3.21}$$

$$M = \left(\frac{\partial A}{\partial N}\right)_{V,T}. \tag{3.22}$$

[1] In $1D$, the canonical ensemble is defined through the parameters $[N, L, T]$.

# Chapter 4

# Statistical applications of the minimal length GUP

## 4.1 Ideal gas

The statistical treatment of an ideal gas within the minimal length GUPs has been considered in several references [51–53, 69, 106], via both classical and quantum approaches. In this section, we summarize the contribution of Ref. [53] in order to make a comparison study with the results that will be obtained in the formalism of the GUP with a maximal length.

### 4.1.1 Partition function

In Ref. [53] the authors studied the thermodynamics of an ideal gas within the GUP (2.1), which incorporates a minimal length. In their treatment, the authors calculated the canonical partition function by using the quantum approach; in which the partition function of a single particle of the system is given by

$$Q_1 = \sum_n \exp\left(-\beta E_n\right), \tag{4.1}$$

where $\beta = \frac{1}{k_B T}$ is Boltzmann's factor with $k_B$ is Boltzmann's constant and $T$ is the temperature, and $E_n$ being the deformed spectrum of a particle in a box, which is given by [53]

$$E_n = \epsilon_n + 4m\gamma\epsilon_n^2, \tag{4.2}$$

where $\epsilon_n$ is the ordinary spectrum of a free particle in $3D$ box.

Based on the definition (4.1) and the deformed spectrum (4.2), it has been shown that the generalized partition function has the form

$$Q_1 = q_1 \left(1 - \gamma \frac{15m}{\beta}\right), \tag{4.3}$$

where $q_1$ is the ordinary partition function. For $N$ indistinguishable particles, the partition function is defined as

$$Q = \frac{Q_1^N}{N!}. \tag{4.4}$$

Inserting Eq. (4.3) in Eq. (4.4), the partition function of the system becomes

$$Q = \frac{q_1^N}{N!} \left(1 - \gamma \frac{15m}{\beta}\right)^N = \frac{1}{N!} \left(\frac{V}{\hbar^3}\right)^N \left(\frac{m}{2\pi\beta}\right)^{\frac{3N}{2}} \left(1 - \gamma \frac{15m}{\beta}\right)^N. \tag{4.5}$$

### 4.1.2 Modified thermodynamic properties

Knowing the generalized partition function of the system, Eq. (4.5), the modified thermodynamics may be determined in straightway manner. In Ref. [53], the authors derived the internal energy, $E$, up to first order of $\gamma$

$$E = -\left(\frac{\partial \ln Q}{\partial \beta}\right)_{N,V} \approx \frac{3}{2} N k_B T - \gamma 15 N m \left(k_B T\right)^2. \tag{4.6}$$

The heat capacity at constant volume, $C_V$, follows from

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V}, \tag{4.7}$$

therefore, up to $O(\gamma)$, one has

$$C_V \approx \frac{3}{2} N k_B - \gamma 30 N m k_B^2 T. \tag{4.8}$$

The first term is the ordinary heat capacity at constant volume, however, the second term is the GUP-induced correction $(-\gamma 30 N m k_B^2 T)$. Hence the GUP (2.1) contributes a negative correction to the heat capacity $C_V$.

The Helmholtz free energy is given as

$$A = -k_B T \ln Q. \tag{4.9}$$

By substituting Eq. (4.5), the generalized Helmholtz free energy up to first order of $\gamma$ can be expressed as

$$A \approx Nk_B T \left( \ln \left[ \frac{N}{V} \left( \frac{2\pi\hbar^2}{mk_B T} \right)^{\frac{3}{2}} \right] - 1 \right) + \gamma 15 Nm \left(k_B T\right)^2. \tag{4.10}$$

The entropy of the system is given by

$$S = -\left( \frac{\partial A}{\partial T} \right)_{N,V}, \tag{4.11}$$

thus, up to first order in $\gamma$ , one can obtain

$$S \approx Nk_B \left( \ln \left[ \frac{V}{N} \left( \frac{mk_B T}{2\pi\hbar^2} \right)^{\frac{3}{2}} \right] + \frac{5}{2} \right) - \gamma 30 Nmk_B^2 T, \tag{4.12}$$

where the first term is the ordinary entropy ($\gamma = 0$), and the GUP induced correction is $\Delta S = -\gamma \left(30 Nmk_B^2 T\right) < 0$; therefore, in the framework of the minimal length GUP, the total entropy of the system decreases.

The chemical potential is given by

$$M = \left( \frac{\partial A}{\partial N} \right)_{V,T}, \tag{4.13}$$

which yields

$$M \approx k_B T \left( \ln \left[ \frac{N}{V} \left( \frac{2\pi\hbar^2}{mk_B T} \right)^{\frac{3}{2}} \right] \right) + \gamma 15 m \left(k_B T\right)^2. \tag{4.14}$$

Obviously, the minimal length GUP induces a positive correction to the chemical potential of the system $\Delta M = \gamma 15 m \left(k_B T\right)^2$.

The pressure of the system is defined within the canonical ensemble as

$$P = k_B T \left( \frac{\partial \ln Q}{\partial V} \right)_{N,T}, \tag{4.15}$$

which gives the familiar ideal gas law

$$PV = Nk_B T. \tag{4.16}$$

Therefore the minimal length presence, up to first order of $\gamma$, does not influence the pressure of the system. It is interesting to note that the equation of state preserves its form as in the undeformed case.

Another important thermodynamic characteristics is the density of states. An alternative definition of the canonical partition function $Q$ is given by [107]

$$Q=\int_0^{\infty} e^{-\beta E} G\left(E\right) dE, \tag{4.17}$$

where $G(E)$ denotes the density of states around the energy value $E$. This relation signifies that the partition function $Q$ is the Laplace transform of $G(E)$. Hence, one may write $G(E)$ as the inverse Laplace transform of $Q$ [107]

$$G\left(E\right)=\frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty} e^{\beta E} Q\left(\beta\right) d\beta, \quad \beta'>0. \tag{4.18}$$

The partition function is given by Eq. (4.5). The substitution in Eq. (4.18) results in

$$G\left(E\right)=\frac{1}{N!}\left(\frac{V}{\hbar^3}\right)^N\left(\frac{m}{2\pi}\right)^{\frac{3N}{2}}\frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty} e^{\beta E}\left(1-\gamma\frac{15Nm}{\beta}\right)d\beta. \tag{4.19}$$

This integral can be calculated by using the formula

$$I=\frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty}\frac{e^{\beta x}}{\beta^{n+1}}d\beta=\begin{cases}\frac{x^n}{n!}, & for \quad x\geq 0,\\ 0, & for \quad x\leq 0.\end{cases} \tag{4.20}$$

This leads to

$$G\left(E\right)=\frac{1}{N!}\left(\frac{V}{\hbar^3}\right)^N\left(\frac{m}{2\pi}\right)^{\frac{3N}{2}}\left(\frac{E^{\frac{3N}{2}-1}}{\left(\frac{3N}{2}-1\right)!}-\gamma\frac{15NmE^{\frac{3N}{2}}}{\frac{3N}{2}!}\right), \tag{4.21}$$

for $E\geq 0$, the first term in this equation is the ordinary density of states, the second term is the first order correction due to the minimal length GUP effect.

## 4.2 Harmonic oscillator

An ensemble of $N$ harmonic oscillators is an important statistical system. Within the minimal length GUPs several works addressed this system, see for instance, Refs. [51, 52].

By using the semiclassical approach, the author of Ref. [51] studied the $3D$ harmonic oscillators in the framework of the minimal-length deformed Heisenberg algebra given by Eq. (2.11). According to the developed method by the author (Eqs. (3.10) and (3.11)). For one-particle harmonic oscillator, with mass $m$ and frequency $\omega$, in low temperature regime $\left(\frac{m\gamma}{\beta} << 1 \text{ and } \frac{m\gamma'}{\beta} << 1\right)$, the partition function (3.10) reads

$$Q_1 \approx \left(\frac{1}{\beta\hbar\omega}\right)^3 \left(1 - 3\left(3\gamma + \gamma'\right)\frac{m}{\beta} + o\left(T\right)\right). \tag{4.22}$$

For an ensemble of $N$ distinguishable harmonic oscillators, the partition function is written as

$$Q = Q_1^N \approx \left(\frac{1}{\beta\hbar\omega}\right)^{3N} \left(1 - 3\left(3\gamma + \gamma'\right)\frac{m}{\beta} + o\left(T\right)\right)^N. \tag{4.23}$$

From this partition function, the thermodynamics of the system follows in usual way as given by Eqs. (3.18)-(3.22). We start with the internal energy $E$

$$E = -\left(\frac{\partial \ln Q}{\partial \beta}\right)_{N,V} \approx 3Nk_BT - 3\left(3\gamma + \gamma'\right)Nm\left(k_BT\right)^2 + o\left(T^2\right), \tag{4.24}$$

the first term is the ordinary internal energy, however, the second one is the leading induced correction due to the minimal length $(\Delta E) = -3Nm\left(3\gamma + \gamma'\right)\left(k_BT\right)^2$. Therefore the minimal length presence yields a decrease in the internal energy of the system.

The heat capacity at constant volume

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V} \approx 3Nk_B - 6\left(3\gamma + \gamma'\right)Nmk_B^2T + o\left(T\right). \tag{4.25}$$

The Helmholtz free energy follows from Eq. (4.23)

$$A = -k_BT\ln Q = -Nk_BT\left(3\ln\left[\frac{1}{\beta\hbar\omega}\right] + \ln\left[1 - 3\left(3\gamma + \gamma'\right)\frac{m}{\beta} + o\left(T\right)\right]\right). \tag{4.26}$$

From this relation, one can calculate the deformed entropy

$$S = -\left(\frac{\partial A}{\partial T}\right)_{N,V} \approx Nk_B\left(3\ln\left[\frac{1}{\beta\hbar\omega}\right] + 3 - 6\left(3\gamma + \gamma'\right)mk_BT + o\left(T\right)\right), \tag{4.27}$$

and the deformed chemical potential

$$M = \left(\frac{\partial A}{\partial N}\right)_{T,V} \approx -k_BT\left(3\ln\left[\frac{1}{\beta\hbar\omega}\right] + \ln\left[1 - 3\left(3\gamma + \gamma'\right)\frac{m}{\beta} + o\left(T\right)\right]\right). \tag{4.28}$$

For large temperatures, i.e., $\frac{m\gamma}{\beta} >> 1$, expression (3.11) gives

$$Q_1 \approx \left(\frac{1}{2\pi\beta m\hbar^2\omega^2}\right)^{3/2} \frac{\pi^2}{\sqrt{\gamma}\left(\sqrt{\gamma} + \sqrt{\gamma + \gamma'}\right)^2}\left(1 + O\left(\frac{1}{T}\right)\right), \tag{4.29}$$

For an ensemble of $N$ harmonic oscillators, the partition function is written as

$$Q \approx \left(\frac{1}{2\pi\beta m\hbar^2\omega^2}\right)^{3N/2} \left(\frac{\pi^2}{\sqrt{\gamma}\left(\sqrt{\gamma} + \sqrt{\gamma + \gamma'}\right)^2}\left(1 + O\left(\frac{1}{T}\right)\right)\right)^N. \tag{4.30}$$

On the internal energy

$$E = \frac{3}{2} N k_B T + O\left(1\right). \tag{4.31}$$

The heat capacity at high temperatures will therefore be

$$C_V = \frac{3}{2} N k_B + O\left(\tfrac{1}{T}\right). \tag{4.32}$$

Concerning Helmholtz free energy, the relation (4.30) makes it possible to write

$$A \approx -N k_B T \ln\left[\left(\tfrac{1}{2\pi\beta m\hbar^2\omega^2}\right)^{3/2} \tfrac{\pi^2}{\sqrt{\gamma}\left(\sqrt{\gamma}+\sqrt{\gamma+\gamma'}\right)^2}\left(1+O\left(\tfrac{1}{T}\right)\right)\right]. \tag{4.33}$$

From this equation, one can calculate directly the new entropy

$$S \approx -N k_B \left(\frac{3}{2} + \ln\left[\left(\tfrac{1}{2\pi\beta m\hbar^2\omega^2}\right)^{3/2} \tfrac{\pi^2}{\sqrt{\gamma}\left(\sqrt{\gamma}+\sqrt{\gamma+\gamma'}\right)^2}\right] + O\left(\tfrac{1}{T}\right)\right), \tag{4.34}$$

and the generalized chemical potential at high temperatures

$$M = -k_B T \ln\left[\left(\tfrac{1}{2\pi\beta m\hbar^2\omega^2}\right)^{3/2} \tfrac{\pi^2}{\sqrt{\gamma}\left(\sqrt{\gamma}+\sqrt{\gamma+\gamma'}\right)^2}\left(1+O\left(\tfrac{1}{T}\right)\right)\right]. \tag{4.35}$$

In summary, the obtained results turn out that the effect of the GUP with a minimal length would be important in the regime of high temperatures [51].

By using the quantum approach, the authors of Ref. [52] studied the harmonic oscillator in $1D$ within the minimal length GUP model given by Eq. (2.1). In their investigation, the authors employed the generalized one-dimensional spectrum for the harmonic oscillator up to $O\left(\gamma\right)$ [52]

$$E_n = \epsilon_n + \frac{1}{2} m\gamma\epsilon_n^2 + \frac{1}{2} m\gamma\hbar^2\omega^2. \tag{4.36}$$

where $\epsilon_n = \hbar\omega\left(n+\frac{1}{2}\right)$ is the ordinary spectrum.

By employing a developed approach (for further details see Ref. [52]), the authors evaluates the internal energy $E$ and the corresponding heat capacity $C$. Therefore, in the small $T$ limit [52],

$$E = k_B T - m\gamma\left(k_B T\right)^2, \tag{4.37}$$

$$C = k_B - 2m\gamma k_B^2 T. \tag{4.38}$$

From this equation, one finds that the heat capacity tends to zero at $T = \frac{1}{2\gamma m k_B}$.

## 4.3 Relativistic gas: Semiclassical treatment

In this section, we present the contribution of Ref. [108] focusing on the thermodynamical quantities of a relativistic ideal gas. By using the Hamiltonian of the form [108, 109]

$$H = mc^2\left(\sqrt{1+\left(\frac{\mathbf{P}}{mc}\right)^2}-1\right). \tag{4.39}$$

In the context of the minimal-length deformed algebra (2.13), the modified partition function is given by [108]

$$Q_1 = \tfrac{4\pi V}{h^3}e^{\beta mc^2}\int_0^{\infty}\tfrac{P^2}{(1+\gamma P^2)^5}e^{-\beta mc^2\sqrt{1+\left(\frac{P}{mc}\right)^2}}dP, \tag{4.40}$$

the integration yields [108]

$$Q_1 = 4\pi V\left(\tfrac{mc}{h}\right)^3\left(K_2(u)-\gamma\tfrac{15}{2}\tfrac{K_3(u)}{u}\right)\frac{e^u}{u}. \tag{4.41}$$

here and hereafter $K_n(u)$ is the modified Bessel function of the second kind, of the integer order $n$, with $u=\beta mc^2$.

For $N$ indistinguishable particles, the partition function is defined as

$$Q = \frac{Q_1^N}{N!}. \tag{4.42}$$

Concerning the Helmholtz free energy, it follows from [108]

$$A = -k_BT\ln Q = -Nk_BT\left(\ln\left[4\pi V\left(\tfrac{mc}{h}\right)^3\tfrac{K_2(u)}{u}\right]+u+1\right)+\gamma\tfrac{15}{2}Nk_BT\tfrac{K_3(u)}{uK_2(u)}. \tag{4.43}$$

The first term denotes the ordinary Helmholtz free energy, however, the second one is the modification due to the minimal length GUP effects.

After calculating the Helmholtz free energy, the pressure is determined as

$$P = -\left(\frac{\partial A}{\partial V}\right)_{N,T} = \frac{Nk_BT}{V}. \tag{4.44}$$

Thus, the GUP incorporating a minimal length does not influence the equation of state of the relativistic ideal gas.

By using the modified Helmholtz free energy (4.43), the authors obtained the modified entropy ($S$), internal energy ($E$) and heat capacity ($C_V$):

$$S = -\left(\frac{\partial A}{\partial T}\right)_{N,V} = Nk_B\left(\ln\left[4\pi V\left(\frac{mc}{h}\right)^3\frac{K_2(u)}{u}\right] + u\frac{K_1(u)}{K_2(u)} + 4\right) + \Delta S, \tag{4.45}$$

the first term is the ordinary entropy of the relativistic ideal gas, and the second term represents the modification of the entropy due to the minimal length scale effect [108]

$$\Delta S = -\gamma\frac{15}{2}Nk_B\left(1 + \frac{3}{u}\frac{K_3(u)}{K_2(u)} - \frac{K_3(u)K_1(u)}{K_2^2(u)}\right). \tag{4.46}$$

The internal energy reads

$$E = A + TS = Nk_BT\left(3 + \frac{uK_1(u)}{K_2(u)} - u\right) + \Delta E, \tag{4.47}$$

the first term is the ordinary internal energy, and $\Delta E$ shows the modification [108]

$$\Delta E = -\gamma\tfrac{15}{2}Nk_BT\left(1 + \tfrac{2}{u}\tfrac{K_3(u)}{K_2(u)} - \tfrac{K_3(u)K_1(u)}{K_2^2(u)}\right). \tag{4.48}$$

It is noticeable that we can also acquire the nonrelativistic limit, when $u = \beta mc^2 >> 1$ and the ultrarelativistic limit, when $u = \beta mc^2 << 1$.

In the nonrelativistic limit, $u >> 1$, one gets [108]

$$E = \frac{3}{2}Nk_BT\left(1 - \gamma 10mk_BT\right), \tag{4.49}$$

which exactly conforms with the result of Ref. [53] for the ideal gas, see Eq. (4.6).

For the extreme relativistic gas, by using the limit $u << 1$, one obtains [108]

$$E = Nk_BT\left(1 - \gamma 40\left(\frac{k_BT}{c}\right)^2\right). \tag{4.50}$$

The heat capacity at constant volume holds from the internal energy

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V} = Nk_Bu\left(\tfrac{3}{u} + \tfrac{uK_0(u)}{K_2(u)} - \tfrac{K_1(u)}{K_2(u)}\left(1 + \tfrac{uK_1(u)}{K_2(u)}\right)\right) + \Delta C_V, \tag{4.51}$$

where the first term is the ordinary heat capacity and $\Delta C_V$ represents the minimal length GUP correction term [108]

$$\Delta C_V = -\gamma\tfrac{15}{2}Nk_Bu\left(\tfrac{3}{u} - \tfrac{K_1(u)}{K_2(u)} + \tfrac{6}{u^2}\tfrac{K_3(u)}{K_2(u)}\left(\tfrac{K_0(u)}{K_2(u)} + \frac{3}{u}\tfrac{K_1(u)}{K_2(u)}\right) + 2\tfrac{K_3(u)}{K_2(u)}\left(\tfrac{K_1(u)}{K_2(u)}\right)^2\right). \tag{4.52}$$

For the nonrelativistic case, $u >> 1$, one has [108]

$$C_V = \frac{3}{2}Nk_B\left(1 - \gamma 20mk_BT\right), \tag{4.53}$$

which matches the previous result of Ref. [53], see Eq. (4.8).

For the ultrarelativistic case, $u << 1$, Eq. (4.51) yields the heat capacity at constant volume for the extreme relativistic gas [108]

$$C_V = 3Nk_B \left(1 - \gamma 120 \left(\frac{k_B T}{c}\right)^2\right). \tag{4.54}$$

# Chapter 5

# Statistical applications of the maximal length GUP

## 5.1 Introduction

In this chapter, we present the main part of our work, which consists to study the effects of the new GUP of Ref. [19] incorporating a maximum length, in statistical physics. We consider the previous three statistical systems, namely, an ideal gas, an ensemble of $N$ harmonic oscillators and a relativistic gas.

## 5.2 Ideal gas

### 5.2.1 Partition function

#### Semiclassical treatment in 1D

Let us consider an ideal gas in $1D$ box of length $L$. Thus, the one-particle deformed Hamiltonian is $H = \frac{P^2}{2m}$. According to (3.14), the single-particle partition function is expressed by

$$Q_1 = \frac{1}{h}\int_0^L \left(1-\alpha X^2\right) dX \int_{-\infty}^{+\infty} \exp\left(-\beta\frac{P^2}{2m}\right) dP, \tag{5.1}$$

the integration yields

$$Q_1 = q_1\left(1-\frac{\alpha}{3}L^2\right), \tag{5.2}$$

where, $q_1 = \frac{L}{\hbar}\left(\frac{m}{2\pi\beta}\right)^{1/2}$ represents the one-particle partition function in the ordinary case. For $N$ indistinguishable particles constituting the system, the partition function is defined as

$$Q = \frac{Q_1^N}{N!}. \tag{5.3}$$

Inserting Eq. (5.2) into Eq. (5.3) gives

$$Q = \frac{q_1^N}{N!}\left(1-\frac{\alpha}{3}L^2\right)^N = q\left(1-\frac{\alpha}{3}L^2\right)^N, \tag{5.4}$$

where $q = \frac{q_1^N}{N!}$ represents the ordinary partition function of the system. Up to first order of $\alpha$, this generalized partition function can be written as

$$Q \approx q\left(1-\frac{\alpha}{3}NL^2\right). \tag{5.5}$$

### Semiclassical treatment in 3D

Now, let us consider an ideal gas in $3D$ box with a volume $V = L^3$. The $3D$ single-particle deformed Hamiltonian is $H = \frac{\mathbf{P}^2}{2m}$. Based on the definition (3.17), the partition function is given by [72]

$$Q_1 = \frac{1}{h^3}\int_0^L \left(1-\alpha\mathbf{X}^2\right)^3 (dX) \int_{-\infty}^{+\infty} \exp\left(-\beta\frac{\mathbf{P}^2}{2m}\right) (dP), \tag{5.6}$$

the integration over the six variables of the phase space results in [72]

$$Q_1 = q_1 F(\alpha, V), \tag{5.7}$$

where $q_1 = \frac{V}{\hbar^3}\left(\frac{m}{2\pi\beta}\right)^{3/2}$ is the ordinary partition function for a single particle, and $F$ denotes the term of deformation given by [72]

$$F(\alpha, V) = 1 - 3\alpha V^{2/3} + \tfrac{19}{5}\alpha^2 V^{4/3} - \tfrac{583}{315}\alpha^3 V^2. \tag{5.8}$$

The partition function of $N$ indistinguishable particles constituting the system, is then [72]

$$Q = \frac{q_1^N}{N!} \left[F\left(\alpha, V\right)\right]^N = q \left[F\left(\alpha, V\right)\right]^N . \tag{5.9}$$

where $q = \frac{q_1^N}{N!}$ represents the ordinary partition function of the system. Up to first order of $\alpha$, one gets

$$Q \approx q \left(1 - 3\alpha N V^{2/3}\right) . \tag{5.10}$$

Given that both partition functions (5.5) and (5.10) are similar, it is then expected to obtain the same modified thermodynamics in $1D$ and $3D$ cases, despite a numerical factor characterizing the dimension of space.

### Quantum treatment in 1D

In this paragraph, we show that the quantum approach allow us to establish the same canonical partition function of an ideal gas. It is well known that the quantum partition function of an ideal gas can be obtained from the energy spectrum of a particle in a box. Therefore, this problem will constitute our starting point.

Let us consider a non-relativistic quantum particle with a mass $m$ confined in an infinite square-well potential, defined as

$$U\left(x\right) = \begin{cases} 0, & 0 \leq 0 \leq L, \\ \infty, & otherwise. \end{cases} \tag{5.11}$$

Using the representation (2.64) for the position and momentum operators, inside the well the generalized time-independent Schrödinger equation takes the form [71]

$$\frac{d^2\psi}{dx^2} + \frac{2\alpha x}{1-\alpha x^2}\frac{d\psi}{dx} + k^2\left(1-\alpha x^2\right)^2\psi = 0, \tag{5.12}$$

with the usual notation $k^2 = \frac{2mE}{\hbar^2}$.

Let us make the following change of variable:

$$y = \left(1 - \frac{\alpha}{3}x^2\right)x, \tag{5.13}$$

which leads to the equation

$$\frac{d^2\psi}{dy^2} + k^2\psi = 0, \tag{5.14}$$

this equation has as general solution

$$\psi\left(y\right) = C_1 \sin\left(ky\right) + C_2 \cos\left(ky\right), \tag{5.15}$$

so, the general solution of Eq. (5.12) is

$$\psi\left(x\right) = C_1 \sin\left(\left(1 - \frac{\alpha}{3}x^2\right)kx\right) + C_2 \cos\left(\left(1 - \frac{\alpha}{3}x^2\right)kx\right), \tag{5.16}$$

where $C_1$ and $C_2$ are arbitrary constants.

Using the boundary condition, $\psi\left(0\right) = 0$, the retained physical solution corresponds to $C_2 = 0$; and hence

$$\psi\left(x\right) = C_1 \sin\left(\left(1 - \frac{\alpha}{3}x^2\right)kx\right). \tag{5.17}$$

The energy spectrum follows from the boundary condition $\psi\left(L\right) = 0$, which leads to the quantization equation

$$\left(1 - \frac{\alpha}{3}L^2\right)kL = n\pi. \tag{5.18}$$

Thus, in the presence of a maximum length depending on $\alpha$, the generalized energy spectrum of a particle in one-dimensional box is written as [71]

$$E_n = \frac{\epsilon_n}{1 - \frac{\alpha}{3}L^2}, \left(n = 1, 2, 3, \ldots\right), \tag{5.19}$$

where $\epsilon_n = \frac{\pi^2\hbar^2}{2mL^2}n^2$ represents the ordinary spectrum.

To determine the normalization constant, $C_1$, one use the relation (2.65) defining the modified scalar product. One gets

$$1 = C_1 C_1^* \int_0^L \left(1 - \alpha x^2\right)\left[\sin\left(\left(1 - \frac{\alpha}{3}x^2\right)kx\right)\right]^2 dx = C_1 C_1^* \left(\frac{L}{2} - \alpha\frac{L^3}{6}\right). \tag{5.20}$$

Therefore, the normalized wave function reads [71]

$$\psi\left(x\right) = \sqrt{\frac{2}{\left(1 - \frac{\alpha}{3}L^2\right)L}} \sin\left(\frac{1 - \frac{\alpha}{3}x^2}{1 - \frac{\alpha}{3}L^2}\frac{n\pi}{L}x\right), \quad \left(n = 1, 2, 3, \ldots\right). \tag{5.21}$$

Since $\alpha$ is supposed to be a small parameter such as ($\alpha L^2 << 1$), one can expand Eq. (5.19) up to first order of $\alpha$; this yields [71]

$$E_n \approx \epsilon_n + \frac{\pi^2\hbar^2\alpha}{3m}n^2. \tag{5.22}$$

It follows that the introduction of a maximum length in the quantum treatment of a particle confined in a box leads to an increase of the energy levels, as well as the spacing between them. This effect is also observed in the minimal length scenario, GUP (2.1) [110]. However, the correction carried out by this GUP depends on $n^4$ (see Eq. (4.2) with $\epsilon_n = \frac{\pi^2\hbar^2}{2mL^2}n^2$) leading to a stronger confinement compared to that of the maximum length (the correction $\sim n^2$; see Eq. (5.22)), especially for large values of $n$.

Hereafter, the spectrum (5.19) will be used to study in detail an ideal gas in statistical physics.

The canonical partition function of a single particle of the system is given by

$$Q_1 = \sum_n \exp\left(-\beta E_n\right). \tag{5.23}$$

Using the generalized energy spectrum (5.19), the canonical partition function (5.23) reads

$$Q_1 = \sum_n \exp\left(-\beta\epsilon n^2\left(1-\frac{\alpha}{3}L^2\right)^{-2}\right), \tag{5.24}$$

with the notation $\epsilon = \frac{\pi^2\hbar^2}{2mL^2}$.

Because $L$ is not so small and $T$ is not too close to the absolute zero, the energy levels are extremely close together, one may then approximate the sum in Eq. (5.24) by an integral as follows [111]:

$$Q_1 = \int_0^\infty dn \exp\left(-\beta\epsilon n^2\left(1-\frac{\alpha}{3}L^2\right)^{-2}\right). \tag{5.25}$$

The integration yields [71]

$$Q_1 = q_1\left(1-\frac{\alpha}{3}L^2\right), \tag{5.26}$$

where $q_1 = \frac{L}{\hbar}\left(\frac{m}{2\pi\beta}\right)^{\frac{1}{2}}$ is the ordinary partition function for a single particle in one-dimensional box.

The partition function of an ideal gas (a system of non-interacting $N$ indistinguishable particles in a box) is defined as

$$Q = \frac{Q_1^N}{N!}. \tag{5.27}$$

Inserting (5.26) in (5.27), one obtains [71]

$$Q = \frac{q_1^N}{N!}\left(1 - \frac{\alpha}{3}L^2\right)^N = q\left(1 - \frac{\alpha}{3}L^2\right)^N, \tag{5.28}$$

where $q = \frac{q_1^N}{N!}$ represents the ordinary partition function of the system.

As in ordinary statistical physics, the classical partition function (5.4) is identical to the quantum one (5.28). Such conformity between the results of the classical and quantum treatments has also been concluded in the framework of minimal-length GUP [53]. In addition, due to the similarity between the $1D$ and $3D$ partition functions, one can probe the thermodynamic properties behavior under the maximal length assumption by considering the $1D$ partition function.

### 5.2.2 Thermodynamic properties

Now let us look into the thermodynamic properties of this system, including the internal energy ($E$), the Helmholtz free energy ($A$), the entropy ($S$) and the chemical potential ($M$).

For the internal energy, one has [71]

$$E = -\left(\frac{\partial \ln Q}{\partial \beta}\right)_{N,L} = \frac{1}{2}Nk_BT. \tag{5.29}$$

Equation (5.29) shows that the presence of a maximum length does not influence the internal energy of a canonical ideal gas, and hence the heat capacity at constant volume remains also unchanged: $C_V = \frac{1}{2}Nk_B$. This result is in contrast with the obtained one in the presence of a minimal length [51,52,70,106,112], where the GUPs with a minimal length led to the decreasing of these two quantities, see for instance Eqs. (4.6) and (4.8).

Concerning the Helmholtz free energy ($A$) with a maximum length, it has the expression [71]

$$A = -k_B T \ln Q = a - N k_B T \ln\left(1 - \frac{\alpha}{3} L^2\right), \tag{5.30}$$

here $a = -k_B T \ln q = N k_B T \left(\ln\left[\frac{N}{L}\left(\frac{2\pi\hbar^2}{m k_B T}\right)^{\frac{1}{2}}\right] - 1\right)$ is the ordinary Helmholtz free energy[1].

Now, one can deduce the deformed entropy from Eq. (5.30) as [71]

$$S = -\left(\frac{\partial A}{\partial T}\right)_{N,L} = s + N k_B \ln\left(1 - \frac{\alpha}{3} L^2\right), \tag{5.31}$$

where $s = -\left(\frac{\partial a}{\partial T}\right)_{N,L} = N k_B \left(\ln\left[\frac{L}{N}\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{\frac{1}{2}}\right] + \frac{3}{2}\right)$ is the ordinary entropy. In the limit $\alpha L^2 << 1$, one gets up to first order of $\alpha$

$$S \approx s - \alpha \frac{L^2}{3} N k_B. \tag{5.32}$$

One observes that the GUP with a maximal length carries out a negative correction $(\Delta S)_\alpha = -\alpha\left(\frac{1}{3} L^2 N k_B\right)$, which leads to a decreasing of the system entropy. This result is analogous to that obtained in the presence of a minimal length by using $3D$ representations of the GUP (2.1) [53, 106]. In $1D$, the approach of Ref. [53] leads to the following entropy correction: $(\Delta S)_\gamma = -\gamma\left(2 m N k_B^2 T\right)$.

In order to make a more reliable comparison between the two corrections, let us introduce the dimensionless parameters, $\widetilde{\alpha}$ and $\widetilde{\gamma}$, defined by $\widetilde{\alpha} = \alpha L^2$ and $\widetilde{\gamma} = \gamma m k_B T \sim \gamma \langle p^2 \rangle$, where $p$ represents the momentum. This allows us to write $(\Delta S)_\alpha = -\frac{1}{3}\widetilde{\alpha} N k_B$ and $(\Delta S)_\gamma = -2\widetilde{\gamma} N k_B$. Therefore, both GUPs (2.1) and (2.59) lead to identical corrections for $\widetilde{\alpha} = 6\widetilde{\gamma}$. In spite of this resemblance, the effect of the minimal length and that of the maximal length are fundamentally different. In fact, the correction of the minimal length on the entropy depends on the temperature and that of the maximal length does not depend on it. This can be explained from the structure of the two corresponding GUPs: the deformation term in GUP (2.1), is a function of the momentum (energy), and then its effect is expected to be sensitive

[1] The Stirling approximation $\ln N! \approx N \ln N - N$ for $N >> 1$ is used to establish this equation.

to $T$. However, the deformation term in GUP (2.59) depends on the position, and thus its effect is expected to be sensitive to the spatial size of the system ($L$). This claim is supported by the independence (dependence) of $\widetilde{\alpha}$ ($\widetilde{\gamma}$) on $T$.

From physical point of view, the minimal length is a microscopic scale, and then its corrections would depend on the parameters having a microscopic character, such as the temperature ($T$). On the other hand, the maximal length is a macroscopic scale, and hence its corrections would depend on the parameters having a macroscopic character, such as the spatial size of the system ($L$).

Finally, the generalized chemical potential, $M$, is given by [71]

$$M = \left(\frac{\partial A}{\partial N}\right)_{L,T} = \mu - k_B T \ln\left(1 - \frac{\alpha}{3}L^2\right), \tag{5.33}$$

where $\mu = \left(\frac{\partial a}{\partial N}\right)_{L,T} = k_B T \left(\ln\left[\frac{N}{L}\left(\frac{2\pi\hbar^2}{m k_B T}\right)^{\frac{1}{2}}\right]\right)$ stands for the chemical potential in the undeformed case. In the limit $\alpha L^2 << 1$, one gets up to first order of $\alpha$

$$M \approx \mu + \alpha\frac{L^2}{3}k_B T. \tag{5.34}$$

The chemical potential of the system increases in the presence of a maximal length. This result may be interpreted by an additional confinement induced by the maximal length assumption; since the chemical potential is the energy required for changing the number of particles of the system. Such a result, which is in consistency with the structure of the deformed spectrum (5.22) indicating the existence of a supplementary confinement under maximal length effect. The correction of this GUP, $(\Delta M)_\alpha = \frac{1}{3}\alpha L^2 k_B T$, is analogous to that of the GUP (2.1), obtained in Ref. [53], by using a $3D$ model. In $1D$, the leading correction of the minimal length is $(\Delta M)_\gamma = \gamma m \left(k_B T\right)^2$. In terms of the dimensionless parameters, $\widetilde{\alpha}$ and $\widetilde{\gamma}$, defined previously, these corrections take the forms $(\Delta M)_\alpha = \frac{1}{3}\widetilde{\alpha} k_B T$ and $(\Delta M)_\gamma = \widetilde{\gamma} k_B T$, which become identical when $\widetilde{\alpha} = 3\widetilde{\gamma}$. It is worth to mention that the $T$-dependence of the corrections to the chemical potential in both cases of GUPs comes from the definition $M = -T\left(\frac{\partial S}{\partial N}\right)_{E,L}$, which yields $\Delta M \equiv -T\left(\frac{\partial \Delta S}{\partial N}\right)_{E,L}$. Given that, $(\Delta S)_\alpha$ does not depend on $T$, and $(\Delta S)_\gamma$ linearly depends on $T$, then, the $(\Delta M)_\alpha$ correction is proportional to $T$ and $(\Delta M)_\gamma$ is proportional to $T^2$.

It follows that the corrections due to the maximum length become important at high temperature $T$ and for large space, where the system is confined ($L >> 1$).

In summary, although the corrections due to the maximum length and to the minimum length of some thermodynamic functions are similar, each GUP scenario introduces specific modified thermodynamics.

It is convenient to note that in $3D$ case, we have obtained the following generalized thermodynamic functions[2] [72]:

$$E = -\left(\frac{\partial \ln Q}{\partial \beta}\right)_{N,V} = \frac{3}{2} N k_B T, \tag{5.35}$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V} = \frac{3}{2} N k_B, \tag{5.36}$$

$$A = -k_B T \ln Q = a - N k_B T \ln F(\alpha, V), \tag{5.37}$$

the function $F$ encoding the corrections due to the maximal length is defined by Eq. (5.8).

$$S = -\left(\frac{\partial A}{\partial T}\right)_{N,V} = s + N k_B \ln F(\alpha, V), \tag{5.38}$$

up to $O(\alpha)$, from Eq. (5.8), one gets

$$S \approx s - 3\alpha V^{2/3} N k_B. \tag{5.39}$$

$$M = \left(\frac{\partial A}{\partial N}\right)_{V,T} = \mu - k_B T \ln F(\alpha, V), \tag{5.40}$$

up to $O(\alpha)$, from Eq. (5.8), one has

$$M \approx \mu + 3\alpha V^{2/3} k_B T. \tag{5.41}$$

The aforementioned functions (small letters) correspond to the undeformed case,

[2] The $3D$ partition function is given by Eq. (5.9).

and they are given by

$$a = -k_B T \ln q = N k_B T \left( \ln \left[ \frac{N}{V} \left( \frac{2\pi\hbar^2}{m k_B T} \right)^{\frac{3}{2}} \right] - 1 \right), \tag{5.42}$$

$$s = -\left( \frac{\partial a}{\partial T} \right)_{N,V} = N k_B \left( \ln \left[ \frac{V}{N} \left( \frac{m k_B T}{2\pi\hbar^2} \right)^{\frac{3}{2}} \right] + \frac{5}{2} \right), \tag{5.43}$$

$$\mu = \left( \frac{\partial a}{\partial N} \right)_{V,T} = k_B T \left( \ln \left[ \frac{N}{V} \left( \frac{2\pi\hbar^2}{m k_B T} \right)^{\frac{3}{2}} \right] \right). \tag{5.44}$$

As expected, the same modified thermodynamics in $1D$ and $3D$ cases are obtained, despite a numerical factor characterizing the dimension of space.

### 5.2.3 Maximum length and density of states

An alternative definition of the canonical partition function, $Q$, is given by Eq. (4.17), i.e., [107]

$$Q = \int_0^\infty e^{-\beta E} G(E)\, dE. \tag{5.45}$$

As mentioned in § 4.1.2, $G(E)$ denotes the density of states around the energy value $E$. Consequently, this relation signifies that the partition function $Q$ is the Laplace transform of $G(E)$. Thus, one may write $G(E)$ as the inverse Laplace transform of $Q$ [107]

$$G(E) = \frac{1}{2\pi i} \int_{\beta' - i\infty}^{\beta' + i\infty} e^{\beta E} Q(\beta)\, d\beta, \quad \beta' > 0. \tag{5.46}$$

Inserting the partition function (5.28) in (5.46) results in

$$G(E) = \frac{1}{N!} \left( \frac{L}{\hbar} \right)^N \left( \frac{m}{2\pi} \right)^{\frac{N}{2}} \left( 1 - \frac{\alpha}{3} L^2 \right)^N \int_{\beta' - i\infty}^{\beta' + i\infty} \frac{e^{\beta E}}{\beta^{\frac{N}{2}}} d\beta. \tag{5.47}$$

The calculation of this integral is quite possible for all positive $N$ (see Eq. (4.20)); one gets

$$\begin{aligned} G(E) &= \tfrac{1}{N!} \left( \tfrac{L}{\hbar} \right)^N \left( \tfrac{m}{2\pi} \right)^{\frac{N}{2}} \left( 1 - \tfrac{\alpha}{3} L^2 \right)^N \frac{E^{\frac{N}{2}-1}}{\left(\frac{N}{2}-1\right)!}, && for\ E \geq 0, \\ &= 0, && for\ E \leq 0. \end{aligned} \tag{5.48}$$

As a function of the ordinary density of states, $g(E)$,

$$g(E) = \frac{1}{N!}\left(\frac{L}{\hbar}\right)^N \left(\frac{m}{2\pi}\right)^{\frac{N}{2}} \frac{E^{\frac{N}{2}-1}}{\left(\frac{N}{2}-1\right)!}, \tag{5.49}$$

the generalized density of states (5.48) may be written as [71]

$$\begin{aligned} G(E) &= g(E)\left(1-\tfrac{\alpha}{3}L^2\right)^N, \qquad for\ E \geq 0, \\ &= 0, \qquad for\ E \leq 0. \end{aligned} \tag{5.50}$$

In the limit $\alpha L^2 << 1$, one gets up to first order of $\alpha$

$$G(E) = g(E)\left(1 - \frac{\alpha}{3}NL^2\right), \qquad for\ E \geq 0. \tag{5.51}$$

This result is analogous to that of the minimal length GUP (2.1), obtained by using $3D$ model in Ref. [53]. In $1D$, up to first order of the minimal length parameter, $\gamma$, the generalized density of states is $G(E) = g(E)(1 - 2\gamma m E)$, for $E \geq 0$. In terms of the dimensionless parameters, $\widetilde{\alpha}$ and $\widetilde{\gamma}$, defined previously, the induced corrections take the forms $(\delta G)_\alpha = -\frac{1}{3}\widetilde{\alpha}N$ and $(\delta G)_\gamma = -2\widetilde{\gamma}N$, which become identical when $\widetilde{\alpha} = 6\widetilde{\gamma}$.

The generalized number of microstates accessible to the system with energy lying between $E$ and $E + \delta E$ reads

$$\Omega(E) = \frac{1}{N!}\left(\frac{L}{\hbar}\right)^N \left(\frac{m}{2\pi}\right)^{\frac{N}{2}} \frac{E^{\frac{N}{2}-1}}{\left(\frac{N}{2}-1\right)!}\left(1 - \frac{\alpha}{3}L^2\right)^N \delta E, \qquad for\ E \geq 0. \tag{5.52}$$

By introducing the ordinary number of microstates accessible to the system with energy lying between $E$ and $E + \delta E$, which is noted $\omega(E)$

$$\omega(E) = \frac{1}{N!}\left(\frac{L}{\hbar}\right)^N \left(\frac{m}{2\pi}\right)^{\frac{N}{2}} \frac{E^{\frac{N}{2}-1}}{\left(\frac{N}{2}-1\right)!}\delta E, \tag{5.53}$$

one can write [71]

$$\Omega(E) = \omega(E)\left(1 - \frac{\alpha}{3}L^2\right)^N. \tag{5.54}$$

This expression allows us to study the thermodynamics of the system in the microcanonical ensemble, defined through the parameters $N$, $L$ and $E$, by utilizing of the deformed entropy [71]

$$S = k_B \ln \Omega = s + Nk_B \ln\left(1 - \frac{\alpha}{3}L^2\right), \tag{5.55}$$

where $s = k_B \ln \omega = Nk_B\left(\ln\left[\frac{L}{N}\left(\frac{mE}{\pi\hbar^2 N}\right)^{\frac{1}{2}}\right] + \frac{3}{2}\right)$ is the ordinary microcanonical entropy. The expression (5.55) is exactly the same given by Eq. (5.31). It is important to note that the decreasing of the entropy under this GUP effect may be explained by the fact that the maximum length reduces the number of microstates accessible to the system.

For the thermodynamic properties of the system, one gets the same expressions established in the previous section.

Concerning the internal energy, by using the formula $\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,L}$ with $N >> 1$, one obtains

$$E = \frac{1}{2} N k_B T. \tag{5.56}$$

The generalized chemical potential, defined by $M = -T\left(\frac{\partial S}{\partial N}\right)_{E,L}$, may be expressed by using the undeformed microcanonical chemical potential, $\mu = -T\left(\frac{\partial s}{\partial N}\right)_{E,L} = k_B T\left(\ln\left[\frac{N}{L}\left(\frac{N\pi\hbar^2}{mE}\right)^{\frac{1}{2}}\right]\right)$, as [71]

$$M = \mu - k_B T \ln\left(1 - \frac{\alpha}{3}L^2\right). \tag{5.57}$$

Finally, the generalized Helmholtz free energy is related to the internal energy and the entropy via the relationship $A = E - TS$. Therefore, it may be expressed in term of the undeformed microcanonical Helmholtz free energy, $a = E - Ts = N k_B T\left(\ln\left[\frac{N}{L}\left(\frac{\pi\hbar^2 N}{mE}\right)^{\frac{1}{2}}\right] - 1\right)$, as follows [71]

$$A = a - N k_B T \ln\left(1 - \frac{\alpha}{3}L^2\right). \tag{5.58}$$

As expected, similar to the ordinary case, both canonical and microcanonical descriptions yield identical results in the presence of a maximum length.

### 5.2.4 Maximum length and real gas behavior

Let us now write a formal equation of state, which includes a maximum length. Through the definition of the pressure in the canonical ensemble

$$P = k_B T\left(\frac{\partial \ln Q}{\partial V}\right)_{T,N}, \tag{5.59}$$

where $Q$ is given by Eq. (5.28) and $V \equiv L$. This leads to[3] [71]

$$PV = Nk_BT\left(1 - \frac{2}{3}\alpha V^2\left(1 - \frac{\alpha}{3}V^2\right)^{-1}\right). \tag{5.60}$$

By expanding $\left(1 - \frac{2}{3}V^2\right)^{-1}$ in power series of $\alpha$, one obtains

$$PV = Nk_BT\left(1 - \frac{2}{3}\alpha V^2 + \cdots\right), \tag{5.61}$$

which represents a modified equation of state including the maximum length effect.

This finding is new compared to that obtained previously in the presence of a minimum length [51, 52, 69, 70, 106, 112], where the equation of state preserves its ordinary form.

The corrective term in Eq. (5.61) is negative, so the pressure of the gas decreases under this GUP effect. This may be interpreted by the fact that the incorporation of a maximum length is equivalent to the introduction of additional attractive forces between the particles of the system, which in turn leads to a pressure decreasing.

Moreover, formula (5.61) might be viewed as a generalized equation of state taking into account the deviation from the ideal gas behavior, which is the aim of introducing real equations of state. For instance, one has in the literature the equation [113]

$$PV = ZNk_BT, \tag{5.62}$$

where the compressibility, $Z$, can be less than unity for real gases.

One can also compare Eq. (5.61) with the well known virial expansion [111]

$$PV = Nk_BT\left(1 + \frac{N}{V}B\left(T\right) + \cdots\right), \tag{5.63}$$

where the virial coefficient $B(T)$ may have negative values. Therefore, the real behavior appears naturally by introducing a maximum length in the quantum treatment of an ideal gas. This would represent an advantage of the GUP (2.59) formalism.

It is worthwhile to mention that the present particle horizon of the Universe

[3] The same modified equation of state occurs form the definition of the pressure in the microcanonical ensemble, $P = T\left(\frac{\partial S}{\partial V}\right)_{E,N}$ with $S$ is given by Eq. (5.55).

leads to a large maximum length $l_{\max} \simeq 10^{26}$ m [19]. Recall that $l_{\max} = \frac{1}{\sqrt{\alpha}}$. For a typical system, $V = 1$ m$^3$, Eq. (5.61) deviates from the ideal equation of state as $\frac{PV}{Nk_BT} = 1 - \delta$, where $\delta \equiv \frac{2}{3}\alpha V^2 \sim 10^{-52}$, which is too small to be measurable in current experiments. However, for huge volumes or in the early Universe, with small cosmological particle horizon, the deviation could be more significant, $\delta \sim 1$, and then the maximum length effect might be brought out.

Otherwise, the parameter $\alpha_0$, which is related to the maximal length by $\alpha = l_{\max}^{-2} = \alpha_0 \left(\frac{H_0}{c}\right)^2$, with $H_0$ is the Hubble constant and $c$ is the speed of light, may be viewed as scale-dependent. Therefore, $\alpha_0$ may take large values so that the deviation $\delta$ would be in the range of experimentally accessible errors. One can then estimate lower bounds of $\alpha_0$; this idea might be deeply explored in follow-up works.

It is worth to mention that the $3D$ treatment (see § 5.2.2) provides a similar equation of state, beside a numerical factor characterizing the considered spatial dimension [72]. From Eq. (5.37) and the definition of the pressure

$$P = -\left(\frac{\partial A}{\partial V}\right)_{N,T}, \tag{5.64}$$

up to first order in the deformation parameter, $\alpha$, one has [72]

$$PV = Nk_BT\left(1 - 2\alpha V^{2/3}\right). \tag{5.65}$$

### 5.2.5 Mayer's relation

For an ideal gas, the well known Mayer's relation reads

$$c_P - c_V = Nk_B, \tag{5.66}$$

where, $c_P = \frac{5}{2}NK_B$ and $c_V = \frac{3}{2}NK_B$ are the heat capacity at constant pressure and the one at constant volume in the undeformed case, respectively.

Let us examine this formula in the presence of a maximal length. From Eqs.

(5.35) and (5.65), up to $O(\alpha)$, one finds

$$C_P = \left(\frac{\partial(E+PV)}{\partial T}\right)_{N,P} = \frac{5}{2}Nk_B\left(1-\frac{20}{15}\alpha V^{\frac{2}{3}}\right), \tag{5.67}$$

thus

$$C_P - C_V = Nk_B\left(1-\frac{10}{3}\alpha V^{\frac{2}{3}}\right). \tag{5.68}$$

This might be viewed as a generalized Mayer's relation including the maximal length effect. Formula (5.68) is consistent with the nonideality of gases, which manifests by the deviation of the ratio $(C_P - C_V)/N$ from its ideal value of $k_B$ [114].

To end this section, it is significant to estimate, even roughly, a lower bound for the maximal length by considering its correction on the heat capacity at constant pressure. For this purpose, we fellow the method of Ref. [54]. From Eq. (5.67), one gets

$$\frac{\Delta C_P}{c_P} = \frac{C_P - c_P}{c_P} = -\frac{20}{15}\alpha V^{\frac{2}{3}} \approx -\alpha_0 10^{-52} V^{\frac{2}{3}}. \tag{5.69}$$

Recall that the dimensionless deformation parameter, $\alpha_0$, is related to the maximal length by $\alpha = l_{\max}^{-2} = \alpha_0\left(\frac{H_0}{c}\right)^2$, where $H_0$ is the Hubble constant and $c$ is the speed of light.

Nowadays, accuracy on the experimental values of heat capacities is about $10^{-7}$ [54, 115]. For a monatomic gas confined in a volume of 1 m$^3$, within this precision and by using Eq. (5.69), the following upper bound can be set for the parameter $\alpha_0$:

$$\alpha_0 < 10^{45}. \tag{5.70}$$

This upper bound is comparable to the minimal length dimensionless parameter, estimated in Ref. [54].

The lower bound for the maximal length that arises from (5.70) is then

$$l_{\max} > 10^{7/2}\ \text{m}. \tag{5.71}$$

It follows that the maximal length is size-dependent and hence, it would not be always close to the maximum measurable length in the Universe ($10^{26}$ m [19]), but it might be associated to a macroscopic scale of the system under study.

## 5.3 Harmonic oscillator: Semiclassical treatment

In this section, we consider in the formalism of the maximum-length GUP a system consisting of $N$ harmonic oscillators, assumed free and independent. We will employ the semiclassical treatment by following the approach of Ref. [51].

### 5.3.1 One-dimensional case

The $1D$ Hamiltonian of a harmonic oscillator of mass $m$ and frequency $\omega$ is

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2. \tag{5.72}$$

Based on the definition (3.14), the partition function for one oscillator reads

$$Q_1 = \frac{1}{h} \int_{-l_{\max}}^{+l_{\max}} \left(1 - \alpha X^2\right) \exp\left(-\frac{\beta m\omega^2}{2} X^2\right) dX \int_{-\infty}^{+\infty} \exp\left(-\frac{\beta}{2m} P^2\right) dP, \tag{5.73}$$

the integration yields

$$Q_1 = q_1 f\left(T, \alpha\right), \tag{5.74}$$

where $q_1 = \frac{1}{\beta\hbar\omega} = \frac{k_B T}{\hbar\omega}$ is the ordinary partition function, and $f$ representing the term of deformation due to the presence of a maximal length is given by [72]

$$f\left(T, \alpha\right) \equiv f\left(\beta, \alpha\right) = \left(1 - \frac{\alpha}{\beta m\omega^2}\right) \operatorname{erf}\left(\sqrt{\frac{\beta m\omega^2}{2\alpha}}\right) + \frac{1}{\sqrt{\pi}} \sqrt{\frac{2\alpha}{\beta m\omega^2}} \exp\left(-\frac{\beta m\omega^2}{2\alpha}\right). \tag{5.75}$$

Up to first order of $\alpha$, one has

$$f\left(T, \alpha\right) \approx 1 - \frac{\alpha}{\beta m\omega^2} = 1 - \alpha \frac{k_B T}{m\omega^2}. \tag{5.76}$$

The partition function of $N$ distinguishable particles constituting the system of $N$ harmonic oscillators, is defined as

$$Q = Q_1^N. \tag{5.77}$$

Therefore

$$Q = q\left(f\left(T,\alpha\right)\right)^N, \tag{5.78}$$

where $q = \left(\frac{1}{\beta\hbar\omega}\right)^N$ is the ordinary partition function of the system.

### 5.3.2 Three-dimensional case

Now, let us show that the partition function of $3D$ harmonic oscillators can also be computed in the same manner. To this end, by considering the $3D$ Hamiltonian

$$H = \frac{\mathbf{P}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{X}^2, \tag{5.79}$$

and using the definition (3.17), the partition function for one oscillator reads

$$Q_1 = \frac{1}{h^3}\int_{-1/\sqrt{\alpha}}^{1/\sqrt{\alpha}}\left(1-\alpha\mathbf{X}^2\right)^3\exp\left(-\frac{\beta m\omega^2}{2}\mathbf{X}^2\right)(dX)\int_{-\infty}^{+\infty}\exp\left(-\beta\frac{\mathbf{P}^2}{2m}\right)(dP), \tag{5.80}$$

the integration yields

$$Q_1 = q_1 F\left(T,\alpha\right). \tag{5.81}$$

where $q_1 = \frac{1}{(\beta\hbar\omega)^3}$ is the ordinary partition function, and $F$ representing the term of deformation given by [72]

$$\begin{aligned} F\left(T,\alpha\right) \equiv F\left(\eta,\alpha\right) = &\left(1-\frac{9}{2}\frac{\alpha}{\eta}+\frac{45}{4}\left(\frac{\alpha}{\eta}\right)^2-\frac{105}{8}\left(\frac{\alpha}{\eta}\right)^3\right)\operatorname{erf}\left[\sqrt{\frac{\eta}{\alpha}}\right]^3 \\ &+\frac{3}{\sqrt{\pi}}\left(\sqrt{\frac{\alpha}{\eta}}-5\left(\frac{\alpha}{\eta}\right)^{3/2}+\frac{57}{4}\left(\frac{\alpha}{\eta}\right)^{5/2}\right)e^{-\frac{\eta}{\alpha}}\operatorname{erf}\left[\sqrt{\frac{\eta}{\alpha}}\right]^2 \\ &-\frac{36}{\pi}\left(\frac{\alpha}{\eta}\right)^2 e^{-2\frac{\eta}{\alpha}}\operatorname{erf}\left[\sqrt{\frac{\eta}{\alpha}}\right]+\frac{6}{\pi^{3/2}}e^{-3\frac{\eta}{\alpha}}\left(\frac{\alpha}{\eta}\right)^{3/2}, \end{aligned} \tag{5.82}$$

where $\eta = \frac{\beta m\omega^2}{2} = \frac{m\omega^2}{2k_BT}$.

The partition function of the $N$ distinguishable harmonic oscillators, is defined as

$$Q = Q_1^N. \tag{5.83}$$

Therefore

$$Q = q\left[F\left(T,\alpha\right)\right]^N. \tag{5.84}$$

where $q = q_1^N = \left(\frac{1}{\beta\hbar\omega}\right)^{3N}$ represents the ordinary partition function of the system in $3D$.

### 5.3.3 Thermodynamic functions

As illustrated in § 5.2.2, the effect of the maximal length does not depend on the spatial dimension. Thus, for the sake of simplicity, we consider in what follows the $1D$ partition function (5.78) to probe the thermodynamic properties of the system.

Let us start with the internal energy $E$, and the heat capacity at constant volume $C_V$. The internal energy

$$E = -\left(\frac{\partial \ln Q}{\partial \beta}\right)_{N,L} = \varepsilon - \frac{N}{f}\frac{\partial f}{\partial \beta}. \tag{5.85}$$

where $\varepsilon = Nk_BT$ is the ordinary internal energy. Hence, the hat capacity at constant volume reads [72]

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,L} = c_V - N\frac{\partial}{\partial T}\left(\frac{1}{f}\frac{\partial f}{\partial \beta}\right), \tag{5.86}$$

where $c_V = \left(\frac{\partial \varepsilon}{\partial T}\right)_{N,L} = Nk_B$ is the ordinary heat capacity at constant volume.

From both Eqs. (5.85) and (5.86), one can get in the first order of the deformation parameter $\alpha$, the following expressions [72]:

$$E \approx Nk_BT - \frac{\alpha N}{m\omega^2}\left(k_BT\right)^2, \tag{5.87}$$

$$C_V \approx Nk_B - \frac{2\alpha N}{m\omega^2}k_B^2 T. \tag{5.88}$$

In contrast with the ideal gas, where the maximal length GUP does not influence the internal energy and heat capacity at constant volume, in the case of harmonic oscillators these quantities are affected. Moreover, Eqs. (5.87) and (5.88) indicate that, for a given $m$ and $\omega$, the effect of the maximal length grows at high temperatures, and at $T = \frac{m\omega^2}{2\alpha k_B}$ the heat capacity at constant volume goes to zero. These features are similar to those of the minimal length; see Eqs. (4.37) and (4.38).

The new Helmholtz free energy

$$A = -k_B T \ln Q = a - N k_B T \ln f(T, \alpha), \tag{5.89}$$

here $a = N k_B T \ln\left(\frac{\hbar\omega}{k_B T}\right)$ is the ordinary Helmholtz free energy. From Eq. (5.76), one has in the first order of $\alpha$ the following generalized Helmholtz free energy [72]:

$$A \approx N k_B T \ln\left(\frac{\hbar\omega}{k_B T}\right) + \alpha \frac{N (k_B T)^2}{m\omega^2}. \tag{5.90}$$

From this equation, the entropy $S$, and the chemical potential $M$ follow straightforwardly [72]

$$S = -\left(\frac{\partial A}{\partial T}\right)_{N,V} \approx N k_B \left(1 + \ln\left(\frac{k_B T}{\hbar\omega}\right)\right) - 2\alpha \frac{N k_B^2 T}{m\omega^2}, \tag{5.91}$$

and

$$M = \left(\frac{\partial A}{\partial N}\right)_{V,T} \approx k_B T \ln\left(\frac{\hbar\omega}{k_B T}\right) + \alpha \frac{(k_B T)^2}{m\omega^2}, \tag{5.92}$$

In Eqs. (5.91) and (5.92), the first terms represent the ordinary entropy, $s = -\left(\frac{\partial a}{\partial T}\right)_{N,V} = N k_B \left(1 + \ln\left(\frac{k_B T}{\hbar\omega}\right)\right)$, and ordinary chemical potential, $\mu = \left(\frac{\partial a}{\partial N}\right)_{V,T} = k_B T \ln\left(\frac{\hbar\omega}{k_B T}\right)$, of the system, however, the second terms are the corrections induced by the maximal length presence. Therefore, one observes that the maximal length GUP induces a negative correction to the entropy, and a positive correction to the chemical potential of the system, similarly to the ideal gas (see Eqs. (5.32) and (5.34)).

## 5.4 Relativistic gas: Semiclassical treatment

### 5.4.1 Partition function

In this section, let us consider $N$ noninteracting monatomic particles which are confined in $3D$ box of volume $V = L^3$, and with the relativistic Hamiltonian (4.39); $H = mc^2\left(\sqrt{1 + \left(\frac{\mathbf{P}}{mc}\right)^2} - 1\right)$. Based on the definition (3.2), the partition function of the relativistic ideal gas is given by [94]

$$Q_1 = \frac{1}{h^3}\int_0^L \left(1-\alpha\mathbf{X}^2\right)^3 (dX) \int_{-\infty}^{+\infty} \exp\left(-\beta mc^2\left(\sqrt{1+\left(\tfrac{\mathbf{P}}{mc}\right)^2}-1\right)\right)(dP)\,, \tag{5.93}$$

the integration over the six coordinates of the phase space yields [94]

$$Q_1 = q_1 F\left(\alpha, V\right), \tag{5.94}$$

where $q_1$ represents the ordinary partition function for a single relativistic particle, given by [109]

$$q_1 = 4\pi V\left(\frac{mc}{h}\right)^3 e^u \frac{K_2\left(u\right)}{u}, \tag{5.95}$$

as noted above in § 4.3, here and hereafter $K_n\left(u\right)$ is the modified Bessel function of the second kind, of the integer order $n$, with $u=\beta mc^2$, however, $F$ is the deformation term reflecting the presence of the maximal length given by Eq. (5.8).

The partition function of $N$ indistinguishable particles constituting the system is

$$Q = \frac{q_1^N}{N!} = q\left[F\left(\alpha, V\right)\right]^N. \tag{5.96}$$

where $q=\frac{q_1^N}{N!}$ represents the ordinary partition function of the system.

### 5.4.2 Thermodynamic properties

Let us start with the internal energy [94]

$$E = -\left(\frac{\partial \ln Q}{\partial\beta}\right)_{N,V} = Nmc^2\left(\tfrac{K_1(u)}{K_2(u)}+\tfrac{3}{u}-1\right), \tag{5.97}$$

and hence the heat capacity at constant volume

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V} = \tfrac{Nmc^2}{T}\left(u+\tfrac{3}{u}-\tfrac{K_1(u)}{K_2(u)}\left(3+u\tfrac{K_1(u)}{K_2(u)}\right)\right). \tag{5.98}$$

One observes that the maximal length does not influence the internal energy and the heat capacity of a relativistic ideal gas, exactly as in the nonrelativistic regime [72]. However, the minimal length GUP of Ref. [5] modifies these both quantities for this system; see Eqs. (4.47) and (4.51).

The deformed Helmholtz free energy

$$A = -k_B T \ln Q = a - N k_B T \ln F(\alpha, V), \tag{5.99}$$

here $a = -k_B T \ln q = -N k_B T \left(1 + \ln\left[4\pi \frac{V}{N} \left(\frac{mc}{h}\right)^3 \frac{K_2(u)}{u}\right]\right) - Nmc^2$ is the ordinary Helmholtz free energy.

From Eq. (5.99), the generalized entropy, $S$, is given by

$$S = -\left(\frac{\partial A}{\partial T}\right)_{N,V} = s + N k_B \ln F(\alpha, V), \tag{5.100}$$

where $s = -\left(\frac{\partial a}{\partial T}\right)_{N,V} = N k_B \left(4 + u\frac{K_1(u)}{K_2(u)} + \ln\left[4\pi \frac{V}{N} \left(\frac{mc}{h}\right)^3 \frac{K_2(u)}{u}\right]\right)$ is the ordinary entropy. From Eq. (5.8), up to $O(\alpha)$, one obtains

$$S \approx s - 3\alpha V^{2/3} N k_B, \tag{5.101}$$

The correction introduced by the maximal length is identical to the one obtained in the nonrelativistic regime [72]; see Eq. (5.39). Furthermore, Eq. (5.101) indicates that the entropy decreases also for the relativistic ideal gas within maximal-length GUP formalism.

The generalized chemical potential, $M$, is given by

$$M = \left(\frac{\partial A}{\partial N}\right)_{V,T} = \mu - k_B T \ln F(\alpha, V), \tag{5.102}$$

where $\mu = \left(\frac{\partial a}{\partial N}\right)_{V,T} = -k_B T \ln\left[4\pi \frac{V}{N} \left(\frac{mc}{h}\right)^3 \frac{K_2(u)}{u}\right] - mc^2$ is the ordinary chemical potential. From Eq. (5.8), up to first order of $\alpha$, one gets

$$M \approx \mu + 3\alpha V^{2/3} k_B T, \tag{5.103}$$

up to $O(\alpha)$, the maximal length GUP induces a positive correction to the chemical potential of the system: $\Delta M = 3\alpha V^{2/3} k_B T$. As mentioned in § 5.2.2, this finding may be interpreted by an additional confinement induced by the maximal length assumption.

In the canonical ensemble, the pressure $P$ reads

$$P = k_B T \left(\frac{\partial \ln Q}{\partial V}\right)_{N,T} = \frac{N k_B T}{V}\left(1 + \frac{V}{F}\frac{\partial F}{\partial V}\right), \tag{5.104}$$

where $\frac{V}{F}\frac{\partial F}{\partial V} = \left(-2\alpha V^{2/3} + \frac{76}{15}\alpha^2 V^{4/3} - \frac{1166}{315}\alpha^3 V^2\right)\left(1 - 3\alpha V^{2/3} + \frac{19}{5}\alpha^2 V^{4/3} - \frac{583}{315}\alpha^3 V^2\right)^{-1}$, up to $O(\alpha)$, on has the following equation of state:

$$PV = Nk_BT\left(1 - 2\alpha V^{2/3}\right). \tag{5.105}$$

The maximal length modifies then the equation of state of relativistic ideal gases. This result has not been observed previously when studying the effects of the minimal length in statistical mechanics; the equation of state of relativistic ideal gases preserves its ordinary form even in the presence of a minimal length [108], see Eq. (4.44). Moreover, as in the ordinary case, where the ideal gas equation of state is also valid for the relativistic ideal gas [109], Eq. (5.105) is identical to the one obtained for the nonrelativistic ideal gas in the presence of a maximal length [72]; see Eq. (5.65).

### 5.4.3 Ultrarelativistic and nonrelativistic limits

Now it is convenient to discuss the ultrarelativistic limit, $u = \beta mc^2 << 1$, and the nonrelativistic one, $u = \beta mc^2 >> 1$.

We begin by noting that the maximal length corrections do not depend on the parameter $u$, as showed by Eqs. (5.99), (5.100), (5.102) and (5.105). Therefore, taking one of the aforementioned limits comes back to consider only the limit in the ordinary part of each thermodynamic function, and the maximal length correction remains unchanged regardless the considered regime.

For the sake of illustration, we consider some thermodynamic functions of an extreme relativistic ideal gas, which can be directly deduced from the previous results by simply taking the limit[4] $u = \beta mc^2 << 1$; one obtains the following quantities [94]:

$$E = 3Nk_BT, \tag{5.106}$$

$$C_V = 3Nk_B, \tag{5.107}$$

[4] By using the following approximations for $u << 1$: $K_1(u)/K_2(u) \approx u/2, K_2(u) \approx 2/u^2, K_2(u)/u \approx 2/u^3$ and $u\left(K_1(u)/K_2(u)\right) \approx u^2/2 \approx 0$.

$$A = -Nk_BT\left(\ln\frac{V}{N} + 3\ln\left[\frac{2\pi^{1/3}k_BT}{hc}\right] + 1\right) - Nk_BT\ln F(\alpha, V), \tag{5.108}$$

$$S = Nk_B\left(\ln\frac{V}{N} + 3\ln\left[\frac{2\pi^{1/3}k_BT}{hc}\right] + 4\right) + Nk_B\ln F(\alpha, V), \tag{5.109}$$

$$M = -k_BT\left(\ln\frac{V}{N} + 3\ln\left[\frac{2\pi^{1/3}k_BT}{hc}\right]\right) - k_BT\ln F(\alpha, V). \tag{5.110}$$

Similarly to the relativistic gas, Eqs. (5.106) and (5.107) show that, the maximal length does not influence the internal energy and heat capacity at constant volume of an extreme relativistic gas.

It is worth to mention that the modified equation of state of relativistic ideal gases holds also in the extreme relativistic regime; due to the structure of Eq. (5.105). We note that in the context of GUP model of Refs. [11, 12], the equation of state of an extreme relativistic gas preserves its ordinary form, i.e., the minimal length does not influence the equation of state of an extreme relativistic gas [69].

It is important to highlight that the generalized partition function for the extreme relativistic gas can be established by using Eq. (3.2), with the corresponding Hamiltonian $H = \mathbf{P}c$; then, the modified thermodynamic functions may be extracted in usual way. The obtained results are exactly identical to the ones obtained here by using the ultrarelativistic limit.

Finally, it has been checked that in the nonrelativistic limit[5] $u = \beta mc^2 >> 1$, all thermodynamic functions of the relativistic ideal gas reduce to those of the ideal gas obtained in Ref. [72]; see Eqs. (5.35)-(5.41).

We end this section by commenting on the fact that the maximal length corrections are independent of the considered regime, unlike the minimal length corrections, which are sensitive to the considered regime [53, 69, 108]. This is a consequence of the dependence (independence) of the minimal length (maximal length) contribu-

[5] By using the following approximations for $u >> 1$:
$K_1(u)/K_2(u) \approx 1 - 3/2u + 15/8u^2$, $K_2(u) \approx \sqrt{\pi/2}u^{-1/2}e^{-u}$, $K_2(u)/u \approx \sqrt{\pi/2}u^{-3/2}e^{-u}$ and $u(K_1(u)/K_2(u)) \approx u - 3/2$.

tions in the thermodynamic functions with the parameter $u = \beta mc^2$, from which a limit can be defined. Such a result may be explained from the structure of the corresponding GUPs: the deformation factor in the minimal length GUPs is a function of the momentum (energy), and then its effect is expected to be sensitive to $u = \beta mc^2 = mc^2/k_B T$. However, the deformation factor in GUP (2.59) depends on the position, and thus its effect would be sensitive to the spatial size of the system ($V$). This claim is consistent with our analysis in § 5.2.2. Otherwise, this outcome may be also viewed as a consequence of the fact that the maximal length is a macroscopic scale, corresponding to the low-energy regime where relativistic effects are hidden; unlike the minimal length, which is a microscopic scale, correlated to the high-energy regime where relativistic effects manifest [94].

# Conclusion

In this thesis, we studied the formalism of quantum mechanics (QM) based on a Generalized Uncertainty Principle (GUP), where different forms of GUPs have been investigated. As applications in this framework, we considered some fundamental statistical systems.

In the first part of the thesis, two chapters have been devoted to address theoretical developments of this new version of QM. The first chapter deals with several proposals that suggest the existence of limit values of measurable quantities, as well as the corresponding GUPs that include the concerned quantities. Especially, we discussed the minimum length in connection with quantized space-time hypothesis, string theory and black hole physics. The maximum momentum, suggested in Doubly Special Relativity (DSR), and the maximum length corresponding to the cosmological particle horizon, have also been discussed. We showed in particular, how to implement these optimum quantities in QM by generalizing the HUP to the so-called GUP. The second chapter represents a literature survey on different theoretical implications of the GUP in the mathematical bases of quantum theory: modification of the commutation relations between position and momentum operators, representation of operators in position and momentum spaces, modification of the scalar product,...etc. Firstly, we presented the GUP with minimal length, well investigated by Kempf *et al.* [5], and then, we brought out that such a GUP can be established from a modified de Broglie relation by following the approach of Hossenfelder *et al.* [6]. The GUP of Ali *et al.* [8–10], incorporating a minimal length and a maximal momentum, has been also studied. In addition, we discussed the higher-order GUP proposed by Pedram, which predicts a minimal length and a maximal momentum, as required by DSR theories [11, 12]. A special attention has been given to the GUP with a maximal length, presented recently in Ref. [19]. The one-dimensional case of the formalism has been presented; furthermore, we generalized the equations of this GUP to arbitrary dimensions [72], which allowed us to investigate some physical problems. Finally, within the minimal length GUP model of Kempf *et al.* [5], we established the corresponding Modified Dispersion Relation (MDR) and its consequences, namely, the varying speed of light and the generalized de Broglie wavelength.

In the second part, applications in statistical physics have been performed in three chapters. In the third chapter, the statistical physics and the GUP scenario has been discussed. In the fourth chapter, a summary of some studies focusing on the statistical applications of the minimal length GUP is given. In fact, three statistical systems have been considered, namely, an ideal gas, a system of $N$ harmonic oscillators and a relativistic gas. The last chapter is dedicated to the maximal length GUP applications for the concerned statistical systems. Firstly, the statistical description of an ideal gas within the maximal length GUP formalism has been investigated in detail. The generalized Schrödinger equation has first been established and analytically solved for a particle in $1D$ infinite square-well potential, and then the modified energy spectrum has been obtained. By using a $1D$ model for an ideal gas, we derived the generalized canonical partition function, which allowed to extract the corresponding thermodynamic properties of the system. We also studied the system in the framework of the microcanonical ensemble through the generalized density of states. As expected, the complete consistence between both statistical descriptions is preserved in the presence of a maximum length. Besides, our results have been compared with those of the minimal length GUP, available in the literature. This showed that the two GUPs bring out the same effects on some thermodynamic properties, however, this new deformation of QM leads fundamentally to a novel deformed thermostatistics. Indeed, a modified equation of state naturally appears in this deformed formalism; this might be associated with the real behavior of gases. In addition, a generalized Mayer's relation incorporating a maximal length has been obtained. Using experimental data, a lower bound of the maximal length is estimated of about $10^{7/2}$ m, which is so far from the limit of maximum measurable length scale in the Universe ($10^{26}$ m [19]). This result led us to guess that the maximal length would depend on the size of the system under study, and hence, it might be viewed as a scale associated to the concerned system.

Then, by employing the semiclassical approach, the thermodynamics of an ensemble of harmonic oscillators has been probed in the framework of this new version of QM. The effects of the maximal length have been examined by computing its corrections on several thermodynamic functions. The results showed that the maximal length effect depends on the studied system. Furthermore, a comparison with the minimal-length thermodynamics of the harmonic oscillators, indicated the existence of a qualitative similarity between the maximal and minimal length effects. This fact is somehow, in contrast with what has been observed in the case of an ideal gas, where the minimal length and maximal length formalisms lead to different thermostatistics. It would be of interest to examine this issue by studying a system of quantum harmonic oscillators in the presence of a maximal length. Actually, this task cannot be realized since the energy spectrum of the harmonic oscillator is not available in the literature, and deserves to be addressed.

Finally, the thermodynamics of a relativistic gas has been investigated using the semiclassical approach. In fact, the relativistic gas exhibits similar deformation terms as those of the ideal gas, including, the real-like equation of state.

To fence our manuscript, let us mention some envisaged perspectives, other investigations may be performed in connection to the topic of this thesis, for instance, considering systems of fermions and bosons, which are very important in statistical physics to examine the effect of such a deformation in different areas of physics. Furthermore, the study of black holes thermodynamics with the maximal length assumption is of special interest and it is worth to be addressed.

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**بعض جوانب مبدأ الارتياب المعمم: بحث نظري وتطبيقات**

**ملخص**

في هذه الأطروحة ندرس نتائج بعض التعديلات على مبدأ عدم اليقين لهيزنبرغ (HUP)، المفترضة في نظرية الكم من أجل مراعاة وجود قيم حدية للأطوال وكميات الحركة. في هذا السياق، يتم استبدال HUP بما يسمى مبدأ عدم اليقين المعمم (GUP). أولاً، بناءً على الاعتبارات الفيزيائية الأساسية، يتم تقديم الفرضيات المختلفة التي تشير إلى وجود حدود عليا ودنيا لكميات معينة قابلة للقياس بالإضافة إلى GUPs الناتجة. نهتم بشكل خاص بثلاثة GUPs: واحد يؤدي إلى وجود حد أدنى للطول (مقترح في سياقات مختلفة: مثل نظرية الزمكان المكمم، نظرية الأوتار وفيزياء الثقب الأسود)، GUP الذي يشتمل على زخم أقصى (الناشئ في النسبية الخاصة المضاعفة) و GUP بأقصى طول (متوقع في الكسمولوجيا). بناءًا على هذه GUPs، تمت دراسة شكليات ميكانيكا الكم المشوهة الموافقة بالتفصيل في الفصل الثاني. بالخصوص، علاقات التبديل المشوهة بين مؤثرات الموضع و الزخم، التمثيلات المقابلة في فضاء هيلبرت وتعريف الجداء السلمي. علاوة على ذالك، تمت دراسة بعض التطورات النظرية التي تم تناولها مع الاعتماد على أهم الأعمال المنجزة في المراجع المتوفرة.

قمنا أيضًا بدراسة بعض التطبيقات في الفيزياء الإحصائية من خلال التركيز على GUP الأخير بأقصى طول. تمت دراسة ثلاثة أنظمة: غاز مثالي، مجموعة من المهتزات التوافقية وغاز نسبي. في هذا السياق، درسنا الخصائص الديناميكية الحرارية لهذه الأنظمة في الإطار القانوني باستخدام المقاربتين الكمية والشبه الكلاسيكية. تُظهر المقارنة مع النتائج التي تم الحصول عليها في سياق GUPs ذات الطول الأدنى أن الحد الأقصى للطول يمكن أن يؤدي إلى تأثيرات جديدة، والتي تصبح مهمة في درجات الحرارة العالية والأحجام الكبيرة. على وجه الخصوص، تبرز معادلة حالة معدلة للغازات المثالية في إطار هذه الشكلية الجديدة. من خلال تحليل بعض البيانات التجريبية، وجدنا بأن الطول الأقصى يمكن اعتباره مقياسًا عيانيا مرتبط بالنظام قيد الدراسة.

**الكلمات المفتاحية**

مبدأ عدم اليقين المعمم، علاقة التشتت المعدلة، الطول الأقصى، الطول الأدنى، الغاز المثالي، المهتز التوافقي، الغاز النسبي، دالة التقسيم، معادلة الحالة.