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The $\mathcal{C}(t)\mathcal{PT}$-inner Product for Time-dependent non-Hermitian Quantum Systems

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Introduction

Quantum mechanics, the fundamental theory of the basic phenomena in the microscopic and macroscopic world, is an axiomatic theory because it is grounded on a few postulates:

I) The state of a particle is given by a vector $\psi(t)$ in a Hilbert space. The state is normalized: $\langle\psi(t)|\psi(t)\rangle = 1$. This is as opposed to the classical case where the position and momentum can be specified at any given time. Informally we can say that the wave function $\psi(x,t)$ contains all possible information about the particle.

II) There is a Hermitian operator corresponding to each observable property of the particle. Those corresponding to position x and momentum p satisfy $[x_i, p_j] = i\hbar\delta_{ij}$.

III) Measurement of the observable associated with the operator O will result in one of its eigenvalues o_i. Immediately after the measurement the particle will be in the corresponding eigenstate $|o_i\rangle$.

IV) The probability of obtaining the result o_i in the above measurement is $|\langle o_i|\psi\rangle|^2$. The state of the system will change from $|\psi\rangle$ to $|o_i\rangle$ as a result of the measurement.

V) The state vector $|\psi(t)\rangle$ obeys the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle\,,$$

where H is the Hamiltonian operator.

VI) The Hilbert space for a system of two or more particles is a product space.

These concepts of quantum theory have given birth to various exciting branches of physics such as quantum electrodynamics, quantum computation, quantum information theory, quantum optics, theory of quantum open systems etc. Despite of its huge success and great applicability in modern science, quantum theory has certain constraints. For example, the domain of a fully consistent quantum theory is usually restricted to self-adjoint operators (in the sense of Dirac), i.e. Hermitian quantum systems. A fully consistent quantum theory provides real values of energy and other observable quantities with a complete set of orthonormal eigenfunctions and upholds the unitary time evolution. Towards the end of the twentieth century, the domain of quantum theory (QT) has been extended to incorporate complex or non-self-adjoint systems [1–6].

In the year 1998, Carl M Bender and his collaborators have found a certain class of non-Hermitian (NH) Hamiltonians which holds an entire real discrete spectrum [7]. The reality of the spectrum is shown to be a direct consequence of unbroken symmetry under combined parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$) transformations [8]. Since then, such NH $\mathcal{PT}$-symmetric systems have acquired a great importance in quantum theory and have been studied rigorously [8–16].

The preliminary ideas of a $\mathcal{PT}$-symmetric NHQT are as follows. If a NH Hamiltonian $H \neq H^{\dagger}$ changes under parity ($\mathcal{P}$) and time reversal ($\mathcal{T}$) separately but remains invariant under the combined action of $\mathcal{PT}$ (i.e. H commutes with $\mathcal{PT}$), then all the energy eigenvalues E of the system are real when the eigenfunctions of H respect the $\mathcal{PT}$-symmetry. The operation of $\mathcal{P}$ in 1-D is simply a reflection in space whereas $\mathcal{T}$ executes time reversal transformation. These operations are defined in one dimension (1D) as

$$
\begin{aligned}
&\mathcal{P}\colon x \to -x \quad , \quad p \to -p \quad , \quad i \to i, \\
&\mathcal{T} : :x \to x \quad , \quad p \to -p \quad , \quad i \to -i, \quad \text{(as } \mathcal{T} \text{ is an anti-linear operator)}
\end{aligned} \tag{1}
$$

Since the combined $\mathcal{PT}$ is an anti-linear operator, $[H, \mathcal{PT}] = 0$ does not imply that H and $\mathcal{PT}$ have simultaneous eigenfunctions. There may be two possibilities (i) $H\psi = E\psi$, $\mathcal{PT}$ $\psi \neq a\psi$ and (ii) $H\psi = E\psi$, $\mathcal{PT}\psi = \pm a\psi$. In the latter case when the eigenfunctions ψ also respect the $\mathcal{PT}$-symmetry, then all the eigenvalues should be real [16] and the symmetry is unbroken. On the other hand, in the former case $\mathcal{PT}$-symmetry is breaking spontaneously and some parts of the spectrum (or the entire spectrum) may become complex. NH $\mathcal{PT}$-symmetric systems generally exhibit a phase transition [16] that separates the two parametric regions: (i) a region of unbroken $\mathcal{PT}$ symmetry in which the entire spectrum is real and eigenfunctions of the system respect $\mathcal{PT}$ symmetry and (ii) a region of broken $\mathcal{PT}$ symmetry in which the whole spectrum (or a part of it) is complex and eigenstates of the system are not the eigenstates of a $\mathcal{PT}$ operator.

However, even though the unbroken $\mathcal{PT}$ -symmetry of a Hamiltonian is sufficient to ensure the reality of the associated spectrum, one encounters both mathematical and physical constraints in developing a consistent QT with these $\mathcal{PT}$-symmetric Hamiltonians. The eigenfunctions in such $\mathcal{PT}$-symmetric NH theories may not form a complete orthonormal set and may have indefinite norms, which restricts the use of the probabilistic interpretation. Since the Hamiltonian is not Hermitian unlike the case of usual QT, such theories fail to have unitary time evolution. These problems prohibit one to have a consistent QT with $\mathcal{PT}$-symmetric NH systems. In usual QT, one uses the Hilbert space endowed with a Dirac inner product for self-adjoint (in the sense of Dirac) systems. Therefore, it is natural to introduce a modified Hilbert space, which is now endowed with a $\mathcal{PT}$ -inner product, for the $\mathcal{PT}$-symmetric nonself-adjoint theories. In such a Hilbert space, the time evolution becomes unitary as the Hamiltonian is self-$\mathcal{PT}$ -adjoint and the eigenfunctions form a complete set of orthonormal functions. But the norms of the eigenfunctions have alternate signs even in the new Hilbert space endowed with the $\mathcal{PT}$ -inner products. This again raises an obstacle in probabilistic interpretation despite the system being in an unbroken $\mathcal{PT}$ phase.

Later a new symmetry, inherent to all $\mathcal{PT}$-symmetric NH Hamiltonians, has been realized which is responsible for equal number of positive and negative norm states. This hidden symmetry (denoted as '$\mathcal{C}$') is described by a linear operator which represents the measurement of the signatures of the $\mathcal{PT}$ -norms of the eigenstates [16]. The notation '$\mathcal{C}$' has been assigned to this symmetry operator because of its analogous behavior and identical properties to the charge conjugation operator in quantum field theories. $\mathcal{C}$ commutes with both H and $\mathcal{PT}$ and fixes the problem of negative norms of the eigenfunctions when the inner products have been taken with respect to $\mathcal{CPT}$-adjoint. Thus by the notion of $\mathcal{CPT}$-invariance a fully consistent QT has been established with NH $\mathcal{PT}$-symmetric system which assures for a physical probabilistic description of the system with unitary time evolution and real energy spectra. Physical observables in parity-time reversal ($\mathcal{PT}$) symmetric non-Hermitian theories are also defined by considering the conservation of a new non-local flux [16]. These important realizations help the subject to grow in various directions [15, 16].

In addition, the complex QT study was written in rigorous mathematical language by the introduction of another bigger class of NH Hamiltonians, known as pseudo- Hermitian Hamiltonians [15, 16]. These NH Hamiltonians are not self-adjoint but satisfy the pseudo-hermiticity condition, i.e. $H = \eta^{-1}H^{\dagger}\eta$, where η is a linear Hermitian operator called the metric operator. The eigenvalues of pseudo-Hermitian Hamiltonians are either real or appear in complex conjugate pairs and the eigenfunctions satisfy bi-orthonormality relations in the conventional Hilbert space [15, 16]. Due to this reason, such Hamiltonians do not have a complete set of orthogonal eigenfunctions in the conventional Hilbert space and hence the probabilistic interpretation and unitarity of time evolution have not been satisfied by these pseudo-Hermitian Hamiltonians. Subsequently, such theories have been mapped to equivalent Hermitian theories [15, 16]. Due to these realizations, different pseudo-Hermitian systems have been studied extensively during the two past decades [15, 16].

On the other hand, the situation is less developed for non-Hermitian time-dependent quantum systems, and nowadays several research works are carried out for the elaboration of appropriate methods to solve such systems [17–25], like unitary and non-unitary transformations, the pseudo-invariant method, Dyson maps, point transformations and Darboux transformations. Moreover, when the non-Hermitian Hamiltonian is explicitly time-dependent, only few systems admit an exact analytical solution of the Schrödinger equation. In complex cases, approximation methods are used like perturbation theory, adiabatic approximation and numerical solutions.

In this thesis, we seek the exact analytical solutions of the Schrödinger equation for a class of explicitly time-dependent non-Hermitian quantum systems. In the first chapter, we introduce the basic concepts for non-Hermitian Hamiltonians, such as $\mathcal{PT}$-symmetry, $\mathcal{PT}$ and $\mathcal{CPT}$-inner products, and pseudo-hermiticity. In the second chapter, we present the Lewis-Riesenfeld invariant method for solving the Schrödinger equation for the explicitly time-dependent Hermitian and non-Hermitian Hamiltonians.

In the third chapter we present the main results of this thesis [26]. We use a unitary transformation $F(t)$ in order to transform the time-dependent non-Hermitian Hamiltonian

$H(t)$ to a time-independent $\mathcal{PT}$-symmetric one $\mathcal{H}_0^{\mathcal{PT}}$, and thus the analytical solution of the Schrödinger equation of the initial system can be easily deduced. Then, we define a new $\mathcal{C}(t)\mathcal{PT}$-inner product and show that the evolution preserves it, where $\mathcal{C}(t) = F^{+}(t)\mathcal{C}F(t)$. Moreover, we prove that the expectation value of a time-dependent non-Hermitian Hamiltonian $H(t)$ is real in the $\mathcal{C}(t)\mathcal{PT}$-normed states since the transformation $F(t)$ is unitary and commutes with the parity operator , i.e. $[\mathcal{P}, F(t)] = 0$.

At the end, this thesis is terminated by a conclusion.

Chapter 1

Non-Hermitian formalism of Quantum mechanics

Non-Hermitian formalism of quantum mechanics share many basic features with the standard Hermitian one. For example, the states of Hermitian and non-Hermitian quantum systems are defined in Hilbert spaces and all observables, except energy, are represented with Hermitian operators.

This chapter is devoted to give a short review of the properties and results concerning $\mathcal{PT}$ symmetry and pseudo-Hermitian operators that will be studied throughout this thesis. Also, in the chapter we will explain how non-Hermitian calculations are carried out or in what way the non-Hermitian systems are analogous to the standard Hermitian systems in quantum mechanics.

1.1 $\mathcal{PT}$-symmetric quantum mechanics

$\mathcal{PT}$-symmetric Hamiltonians have been introduced for the first time in 1998 by Bender and his collaborators [7–13]. It is a complex generalization of conventional quantum theory. The study of $\mathcal{PT}$-symmetric quantum systems started from a stimulating paper [7] on a class of

$\mathcal{PT}$-symmetric Hamiltonians

$$H = p^2 + mx^2 - (ix)^\epsilon, \qquad (\epsilon \in \mathbb{R}), \tag{1.1}$$

which are obviously not Dirac Hermitian operators, i.e. not invariant under combined matrix transposition and complex conjugation. They have a real and positive discrete spectrum and generate unitary time evolution for ($\epsilon \geq 2$), and thus define a consistent physical quantum theory. While the Hamiltonian (1.1) is not Dirac Hermitian, it is $\mathcal{PT}$-symmetric, thus the reality of the spectrum is a consequence of the $\mathcal{PT}$ symmetry.

1.1.1 Definitions and properties

A non-Hermitian Hamiltonian H is $\mathcal{PT}$-symmetric if it satisfies the following relation

$$H = H^{\mathcal{PT}} = (\mathcal{PT})H(\mathcal{PT}), \tag{1.2}$$

that is, invariant under both parity $\mathcal{P}$ (space reflection) and time reversal $\mathcal{T}$ transformations such that these two operators commute $[\mathcal{P}, \mathcal{T}] = 0$ and their square is equal to unit $(\mathcal{PT})^2 = 1$, $\mathcal{P}^2 = \mathcal{T}^2 = 1$ but $\mathcal{P} \neq \mathcal{T}$. However, the parity operator $\mathcal{P}$ is linear, whereas the time reversal operator $\mathcal{T}$ is antilinear. The operators $\mathcal{P}$ and $\mathcal{T}$ have the effect of transforming the operator position x, the momentum operator p and the imaginary number i as follows

$$\mathcal{P}\left\{x \to -x \quad , \quad p \to -p \quad , \quad i \to i\right\}, \tag{1.3}$$

$$\mathcal{T}\left\{x \to x \quad , \quad p \to -p \quad , \quad i \to -i\right\}, \tag{1.4}$$

In addition, if all eigenfunctions $|\varphi_n\rangle$ of the $\mathcal{PT}$-symmetric Hamiltonian H are eigenfunctions of the operator $\mathcal{PT}$, we say that the $\mathcal{PT}$-symmetry of H is unbroken

$$[H, \mathcal{PT}] = 0, \qquad \mathcal{PT}\,|\varphi_n\rangle = \pm\,|\varphi_n\rangle\,. \tag{1.5}$$

But, if there are eigenfunctions of the $\mathcal{PT}$-symmetric Hamiltonian H which are not eigenfunctions of the operator $\mathcal{PT}$, the $\mathcal{PT}$ symmetry of H is said to be broken.

In fact, to construct a quantum theory from $\mathcal{PT}$-symmetric Hamiltonians, we further require that the symmetry not be broken. It should be noted however that this condition is not trivial because there is no way to confirm if the symmetry of a $\mathcal{PT}$-symmetric Hamiltonian is broken or not. To this end, it is first necessary to determine the eigenfunctions of the $\mathcal{PT}$-symmetric Hamiltonian and the $\mathcal{PT}$ operator. With this additional condition, we can demonstrate the reality of eigenvalues of $\mathcal{PT}$-symmetric Hamiltonian, one requires that the operators H and $\mathcal{PT}$ admit φ_n as eigenfunctions with arbitrary eigenvalues such that

$$H\left|\varphi_n\right\rangle = E_n\left|\varphi_n\right\rangle, \tag{1.6}$$

$$\mathcal{PT}\left|\varphi_n\right\rangle = \lambda_n\left|\varphi_n\right\rangle,$$

where the eigenvalues E_n and λ_n are complex.
Using

$$(\mathcal{PT})^2 = 1, \tag{1.7}$$

leads to

$$|\lambda_n|^2 = 1. \tag{1.8}$$

The relation (1.5) allows to write

$$H\left|\varphi_n\right\rangle = \mathcal{PT}H\mathcal{PT}\left|\varphi_n\right\rangle = |\lambda_n|^2 E_n^*\left|\varphi_n\right\rangle = E_n\left|\varphi_n\right\rangle, \tag{1.9}$$

thus

$$E_n = E_n^*, \tag{1.10}$$

indeed, the eigenvalues E_n of the $\mathcal{PT}$-symmetric Hamiltonian H are real.

1.1.2 $\mathcal{PT}$ and $\mathcal{CPT}$ inner-products

In conventional quantum theory, the norm of an eigenvector of a Hermitian Hamiltonian in the Hilbert space must be positive. This arises from the fact that the norm represents the probability of presence, which must be defined positive. Furthermore, the inner product of any two eigenvectors in the Hilbert space must be conserved in time (unitarity and time independence) which is a fundamental property for quantum theory to be valid. However, in non-Hermitian quantum theory, to verify the orthogonality of the eigenvectors of a $\mathcal{PT}$-symmetric Hamiltonian we must specify a new inner product. By analogy with the inner product for Hermitian Hamiltonians, Bender [11] first introduced an inner product called "$\mathcal{PT}$-inner product" associated with $\mathcal{PT}$-symmetric Hamiltonians to have a coherent and unitary theory, defined by

$$(f,g)=\int_c dx\,[\mathcal{PT}f(x)]\,g(x), \tag{1.11}$$

where $\mathcal{PT}f(x)=f^*(-x)$ and c is a contour that is defined in the stokes sectors in which we impose the boundary conditions on the eigenvalue equation associated with the $\mathcal{PT}$-symmetric Hamiltonian. The advantage of this inner product is that the associated norm is conserved in time. The application of this definition to the eigenfunctions of H and $\mathcal{PT}$ implies

$$\langle\varphi_m\,|\varphi_n\rangle_{\mathcal{PT}}=\int_c dx\,[\mathcal{PT}\varphi_m(x)]\,\varphi_n(x)=\int_c dx\,[\varphi_m^*(-x)]\,\varphi_n(x)=(-1)^n\delta_{mn}. \tag{1.12}$$

The fermeture relation is written as a function of these eigenfunctions as

$$\sum_{n=0}^{\infty}(-1)^n\varphi_n(x)\varphi_n(y)=\delta(x-y). \tag{1.13}$$

In fact, from (1.12) we can establish the orthogonality, but unfortunately, the norm of a state is not necessarily positive. Thus, the inner product (1.12) is not acceptable for formulating a physical quantum theory.

It is therefore necessary to construct a new inner product for a non-Hermitian Hamiltonian having an unbroken $\mathcal{PT}$-symmetry where the norm is positive. To this end, Bender et al. [11] noticed that a $\mathcal{PT}$-symmetric Hamiltonian with unbroken $\mathcal{PT}$-symmetry possesses a hidden symmetry that is generated by a new linear operator $\mathcal{C}$. We use the $\mathcal{C}$ notation because the properties of this operator are mathematically similar to those of the charge conjugation operator in quantum field theory. Moreover, the operator $\mathcal{C}$ can be represented in the coordinate-space representation as a sum of the normalized eigenfunctions of the $\mathcal{PT}$

symmetric Hamiltonian,

$$\mathcal{C}(x,y) = \sum_n \varphi_n(x)\varphi_n(y), \tag{1.14}$$

it is easy to verify that the square of $\mathcal{C}$ is equal to unity

$$\int dx \mathcal{C}(x,y)\mathcal{C}(y,z) = \delta(x-z), \tag{1.15}$$

consequently

$$\mathcal{C}^2 = 1. \tag{1.16}$$

One observes that the eigenvalues of the operator $\mathcal{C}$ are ± 1 and its action on the $\mathcal{PT}$ symmetric eigenfunctions is given by

$$\mathcal{C}\varphi_n(x) = (-1)^n \varphi_n(x). \tag{1.17}$$

In addition, $\mathcal{C}$ commutes with both the combination $\mathcal{PT}$ and the $\mathcal{PT}$-symmetric Hamiltonian H

$$[\mathcal{C}, \mathcal{PT}] = 0, \quad [\mathcal{C}, H] = 0, \quad [\mathcal{CPT}, H] = 0, \tag{1.18}$$

but not either $\mathcal{P}$ or $\mathcal{T}$ separately

$$[\mathcal{C}, \mathcal{P}] \neq 0 \quad , \quad [\mathcal{C}, \mathcal{T}] \neq 0. \tag{1.19}$$

Bender et al. have defined the $\mathcal{CPT}$ inner product of two $\mathcal{PT}$- symmetric eigenfunctions,

$$\langle \varphi_n, \varphi_m \rangle = \int_c dx \left[\mathcal{CPT}\varphi_n(x)\right] \varphi_m(x), \tag{1.20}$$

where

$$\mathcal{CPT}\varphi_n(x) = \int dx \mathcal{C}(x,y)\varphi_n^*(-y). \tag{1.21}$$

The $\mathcal{CPT}$ inner product is positive because the operator $\mathcal{C}$ contributes with (-1) when it acts on a state with negative $\mathcal{PT}$ norm, and the completeness condition reads

$$\langle \varphi_n, \varphi_m \rangle = \int_c dx \left[\mathcal{CPT}\varphi_n(x)\right] \varphi_m(x) = \delta_{mn}. \tag{1.22}$$

So this $\mathcal{CPT}$ inner product satisfies all the conditions for the quantum theory defined by H to be unitary [12].

1.1.3 Application: Time-independent complex forced harmonic oscillator governed by $\mathcal{PT}$-symmetric Hamiltonian

The time-independent oscillator in the presence of a complex linear potential can be represented by the following Hamiltonian [27]

$$\mathcal{H}_0 = \frac{p^2}{2m_0} + \frac{m_0\Omega^2}{2}q^2 + i\lambda_0 q, \tag{1.23}$$

where λ_0 is a real constant.

It is easy to verify that the Hamiltonian $\mathcal{H}_0$ is not Hermitian

$$\mathcal{H}_0^+ = \frac{p^2}{2m_0} + \frac{m_0\Omega^2}{2}q^2 - i\lambda_0 q \neq \mathcal{H}_0, \tag{1.24}$$

and satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t}\left|\chi(t)\right\rangle = \mathcal{H}_0\left|\chi(t)\right\rangle. \tag{1.25}$$

The Hamiltonian $\mathcal{H}_0$ is $\mathcal{CPT}$ symmetric if it satisfies the relation

$$\mathcal{H}_0 = \mathcal{CPT}\mathcal{H}_0\mathcal{CPT}, \tag{1.26}$$

where $\mathcal{C}$ is defined as [28]

$$\mathcal{C} = \exp\left[\frac{2\lambda_0\hat{p}}{m_0\Omega^2}\right]\mathcal{P}. \tag{1.27}$$

and satisfies the following properties:

i) $\mathcal{C}^2 = 1$, indeed

$$\exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\mathcal{P}\exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\mathcal{P} = \exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\exp\left(-\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right) = 1 \tag{1.28}$$

ii) $[\mathcal{C},\mathcal{PT}] = 0$, indeed

$$\begin{aligned}
[\mathcal{C},\mathcal{PT}] &= \exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\mathcal{T} - \mathcal{PT}\exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\mathcal{P} \\
&= \exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\mathcal{T} - \mathcal{T}\exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\mathcal{T}^2 \\
&= \exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\mathcal{T} - \exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\mathcal{T} = 0
\end{aligned} \tag{1.29}$$

iii) $[\mathcal{C}, \mathcal{H}_0] = 0$, indeed

$$\begin{aligned}\mathcal{C}\mathcal{H}_0\mathcal{C} &= \exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\mathcal{P}\mathcal{H}_0\exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\mathcal{P} \\ &= \exp\left(\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right)\left[\frac{p^2}{2m_0} + \frac{m_0\Omega^2 q^2}{2} - iq\right]\exp\left(-\frac{2\lambda_0}{m_0\Omega^2}\hat{p}\right) \\ &= \left[\frac{p^2}{2m_0} + \frac{m_0\Omega^2 q^2}{2} + i\lambda_0 q\right] = \mathcal{H}_0\end{aligned}$$

on the other hand, we do the following non unitary transformation on (1.25)

$$|\chi_n(q)\rangle = U\,|\varphi\rangle$$

we get

$$i\frac{\partial}{\partial t}U\,|\varphi\rangle = \mathcal{H}_0 U\,|\varphi\rangle \tag{1.30}$$

$$E_n\,|\varphi\rangle = U^{-1}\mathcal{H}_0 U\,|\varphi\rangle \tag{1.31}$$

we put

$$h = U^{-1}\mathcal{H}_0 U \tag{1.32}$$

where

$$U = \exp\left[-\frac{\lambda_0 P}{m_0\Omega^2}\right] \tag{1.33}$$

thus

$$h = \frac{p^2}{2m_0} + \frac{m_0\Omega^2}{2}q^2 + \frac{\lambda_0}{2m_0\omega^2} \tag{1.34}$$

where its eigenvalues

$$E_n = \hbar\Omega\left(n + \frac{1}{2}\right) + \frac{\lambda_0}{2m_0\Omega^2}, \tag{1.35}$$

are real, and its eigenfunctions are given by

$$\varphi_n(q) = \left[\frac{\sqrt{m_0\Omega}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2}\exp\left(-\frac{m_0\Omega}{2\hbar}q^2\right)H_n\left[\left(\frac{m_0\Omega}{\hbar}\right)^{1/2}q\right]. \tag{1.36}$$

we deduce the eigenfunctions associated with $\mathcal{H}_0$

$$|\chi_n(q)\rangle = U\,|\varphi_n\rangle \tag{1.37}$$

$$\chi_n(q) = U\left[\frac{\sqrt{m_0\Omega}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2}\exp\left(-\frac{m_0\Omega}{2\hbar}q^2\right)H_n\left[\left(\frac{m_0\Omega}{\hbar}\right)^{1/2}q\right], \tag{1.38}$$

thus, the time-dependent eigenfunctions are

$$\chi_n(t) = U \exp(-iE_n t) \left[\frac{\sqrt{m_0\Omega}}{n!2^n\sqrt{\pi\hbar}} \right]^{1/2} \exp\left(-\frac{m_0\Omega}{2\hbar}q^2\right) H_n\left[\left(\frac{m_0\Omega}{\hbar}\right)^{1/2} q \right] \tag{1.39}$$

$$= \exp(-\frac{\lambda_0}{m_0\omega^2}P) \exp(-iE_n t) \left[\frac{\sqrt{m_0\Omega}}{n!2^n\sqrt{\pi\hbar}} \right]^{1/2} \exp\left(-\frac{m_0\Omega}{2\hbar}q^2\right) H_n\left[\left(\frac{m_0\Omega}{\hbar}\right)^{1/2} q \right] \tag{1.40}$$

Finally, the $\mathcal{CPT}$-inner product is

$$\langle \chi_n(q) \mid \chi_n(q) \rangle_{\mathcal{CPT}} = \langle \chi_n(q) | \mathcal{CP} | \chi_n(q) \rangle = \langle \varphi_n | U\mathcal{CP}U | \varphi_n \rangle = \langle \varphi_n \mid \varphi_n \rangle = 1 \tag{1.41}$$

1.2 Pseudo-Hermitian Quantum Mechanics

The concept of pseudo-Hermiticity was introduced in the 1940s by Dirac and Pauli [1–4] and later discussed by Lee and Sudarshan [5, 6] who were trying to solve the problems that arise in quantization in electrodynamics and other quantum field theories, in which the negative norm states appear as a consequence of renormalization.

In 2000s, Mostafazadeh published several papers [29–32], in which he showed that $\mathcal{PT}$-symmetry is closely related to the concept of pseudo-Hermiticity and therefore pseudo-Hermitian quantum mechanics is a more general theory than $\mathcal{PT}$-symmetric quantum mechanics. Indeed, the pseudo-Hermitian quantum theory has many applications such as quantum computation [33], scattering theory [34], squeezed states [35], and so on.

1.2.1 Definitions and properties

From the viewpoint of pseudo-Hermitian quantum theory, the conventional quantum mechanics is a representation of pseudo-Hermitian quantum mechanics. Therefore, every pseudo-Hermitian Hamiltonian has an equivalent Hermitian Hamiltonian, and both Hermitian and pseudo-Hermitian Hamiltonians have the same energy spectrum, i.e. they are iso-spectral. The necessary and sufficient conditions for the reality of the spectrum of a pseudo-Hermitian

Hamiltonian H are the existence of a linear positive-definite, Hermitian and invertible operator η such that [29]

$$H^{+} = \eta H \eta^{-}, \tag{1.42}$$

the operator η (so) is called pseudo-metric operator and H^{+} is the adjoint Hamiltonian of H. In fact, the operator η is not unique such that for each Hamiltonian H there is an infinite set of such operators and the particular choice of η makes that H η-pseudo-Hermitian. In other words, the condition (1.42) reduces to ordinary Hermiticity when the operator η is equal to the identity 1. So pseudo-Hermiticity is already a generalization of Hermiticity. Also, the pseudo-Hermitian systems are shown to reduce to $\mathcal{PT}$-symmetric systems when $\eta = \mathcal{P}$.

The η-pseudo-Hermitian Hamiltonian H with discrete and non degenerate spectrum and its adjoint H^{+} verify the following eigenvalue equations

$$H\left|\psi_n\right\rangle = E_n \left|\psi_n\right\rangle, \tag{1.43}$$

$$H^{+}\left|\phi_n\right\rangle = E_n \left|\phi_n\right\rangle, \tag{1.44}$$

where E_n are the eigenvalues of H. The eigenvectors of H and those of H^{+} defined in (1.43) and (1.44), respectively, form a bi-orthonormal basis $\{\left|\psi_n\right\rangle, \left|\phi_n\right\rangle\}$. By definition [30, 36, 37]:

$$\left\langle\phi_m \middle| \psi_n\right\rangle = \delta_{mn} \ . \tag{1.45}$$

$$\sum_n \left|\psi_n\right\rangle\left\langle\phi_n\right| = \sum_n \left|\phi_n\right\rangle\left\langle\psi_n\right| = 1, \tag{1.46}$$

using the above fermeture relation, so that H and H^{+} admit a spectral representation of the form

$$H = \sum_n E_n \left|\psi_n\right\rangle\left\langle\phi_n\right| \quad , \quad H^{+} = \sum_n E_n \left|\phi_n\right\rangle\left\langle\psi_n\right|. \tag{1.47}$$

In the same representation, the pseudo-metric operator η and its inverse η^{-1} take the form

$$\eta = \sum_n \left|\phi_n\right\rangle\left\langle\phi_n\right| \quad , \quad \eta^{-1} = \sum_n \left|\psi_n\right\rangle\left\langle\psi_n\right|. \tag{1.48}$$

In order to determine the physical concept of the quantum system governed by the η-pseudo-Hermitian Hamiltonian H (1.42), we should research the Hermitian representation of the quantum system; i.e. examine the equivalent Hermitian Hamiltonian h of H. By definition, the equivalent Hermitian Hamiltonian h is given by

$$h = \rho H \rho^{-1}, \tag{1.49}$$

with ρ defined as a linear bounded, Hermitian and invertible operator. Also, it is easy to verify that

$$\eta = \rho^{+}\rho \quad , \quad \eta^{-1} = \rho^{-1}\left(\rho^{+}\right)^{-1}, \tag{1.50}$$

the operator ρ makes it possible to pass from the eigenvectors $|\varphi_n\rangle$ of h to the eigenvectors $|\psi_n\rangle$ of H as follows

$$|\varphi_n\rangle = \rho\,|\psi_n\rangle\,. \tag{1.51}$$

The eigenvectors $|\varphi_n\rangle$ form an orthonormal basis, i.e. preserve the ordinary inner product

$$\langle\varphi_m\,|\varphi_n\rangle = \delta_{mn}, \tag{1.52}$$

using the transformation (1.51) in (1.52), we get

$$\langle\psi_m|\,\rho^{+}\rho\,|\psi_n\rangle = \langle\psi_m|\,\eta\,|\psi_n\rangle = \langle\psi_m|\,\psi_n\rangle_\eta = \delta_{mn}, \tag{1.53}$$

the last relation defines the so called pseudo-inner product or η-inner product [20, 38, 39].

It is important to note that $|\varphi_n\rangle$ and $|\psi_n\rangle$ represent the same physical state in different Hilbert spaces because the Hermitian Hamiltonian h and the η-pseudo-Hermitian Hamiltonian H represent the same observable in different Hilbert spaces.

1.2.2 Application: Time-independent complex forced harmonic oscillator governed by pseudo-Hermitian Hamiltonian

Let us consider the time-independent complex forced harmonic oscillator governed by the following pseudo-Hermitian Hamiltonian

$$\mathcal{H}^{PH} = \frac{p^2}{2m_0} + \frac{m_0\Omega^2}{2}q^2 + i\lambda_0 q \tag{1.54}$$

that satisfies the eigenvalue equation

$$\mathcal{H}^{PH} |\chi(q)\rangle = E_n |\chi(q)\rangle \tag{1.55}$$

with λ_0 being a real constant. The Hamiltonian $\mathcal{H}^{PH}$ is pseudo-Hermitian with respect to the following metric operator

$$\eta = \exp\left(\frac{2\lambda_0}{m_0\Omega^2}P\right) \tag{1.56}$$

such that η is a Hermitian and invertible linear operator, and effectively

$$\left(\mathcal{H}^{PH}\right)^+ = \eta\mathcal{H}^{PH}\eta^{-1} \tag{1.57}$$

and therefore, the Hermitian equivalent is given by

$$h = \rho\mathcal{H}^{PH}\rho^{-1} \tag{1.58}$$

$$= \eta^{1/2}\mathcal{H}^{PH}\eta^{-1/2} \tag{1.59}$$

$$= \frac{p^2}{2m_0} + \frac{m_0\Omega^2}{2}q^2 + \frac{\lambda_0 q}{2m_0\Omega^2}, \tag{1.60}$$

then, it is easy to show that its eigenvalues

$$E_n = \Omega\hbar(n + \frac{1}{2}) + \frac{\lambda_0}{2m_0\Omega^2}, \tag{1.61}$$

are real, and its eigenfunctions are given by

$$\varphi(q) = \left[\frac{\sqrt{m_0\Omega}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2} \exp\left(-\frac{m_0\Omega}{2\hbar}q^2\right) H_n\left[\left(\frac{m_0\Omega}{\hbar}\right)^{1/2} q\right]. \tag{1.62}$$

Thus, the eigenfunctions associated with $\mathcal{H}^{PH}$

$$|\chi_n(q)\rangle = \rho^{-1}|\varphi\rangle = \eta^{-1/2}|\varphi\rangle \tag{1.63}$$

$$\chi_n(q) = \exp(-\frac{\lambda_0}{m_0\omega^2}P)\left[\frac{\sqrt{m_0\Omega}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2} \exp\left(-\frac{m_0\Omega}{2\hbar}q^2\right) H_n\left[\left(\frac{m_0\Omega}{\hbar}\right)^{1/2} q\right], \tag{1.64}$$

as well as

$$\chi_n(t) = \left[\frac{\sqrt{m_0\Omega}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2} \exp(-iE_n t)\exp(-\frac{\lambda_0}{m_0\omega^2}P)\exp\left(-\frac{m_0\Omega}{2\hbar}q^2\right) H_n\left[\left(\frac{m_0\Omega}{\hbar}\right)^{1/2} q\right]. \tag{1.65}$$

Finally, we calculate the pseudo-inner product

$$\langle\chi_m|\,\eta\,|\chi_n\rangle = \int dx\chi_n^*(x)\eta\chi_m(x) \tag{1.66}$$

$$= \langle\varphi_m|\,(\rho^{-1})^+\eta\rho^{-1}\,|\varphi_n\rangle \tag{1.67}$$

$$= \langle\varphi_m|\,(\rho^{-1})^+\rho^+\rho\rho^{-1}\,|\varphi_n\rangle \tag{1.68}$$

$$= \langle\varphi_m|\,(\rho\rho^{-1})^+\rho\rho^{-1}\,|\varphi_n\rangle \tag{1.69}$$

$$= \langle\varphi_m\,|\varphi_n\rangle = \delta_{mn}. \tag{1.70}$$

Chapter 2

Time-dependent non-Hermitian quantum systems

2.1 Introduction

The time-dependent Schrödinger equation (TDSE) is central and fundamental for the description of quantum systems. The basic principles for Hermitian quantum mechanics, i.e. when the quantum systems are described with a time-independent or even time-dependent Hermitian Hamiltonians, are very well understood and can be found in any standard quantum mechanics textbook or document where the frameworks are well-developed.

Nevertheless, less developed is the situation regarding non-Hermitian quantum systems which can be investigated in various ways. So far, huge effort has gone into the development of a proper quantum framework for such systems. When it involves explicitly time-dependent Hamiltonians it is usually very difficult to solve. In fact, only few exact analytical solutions have been found using different methods. In some difficult cases, we resort to adopt approximative approaches to continue the physical study, and if it is impossible, we resort to the numerical solution.

Moreover, there are several methods for solving the Schrödinger equation for explicitly time-dependent Hermitian and non-Hermitan Hamiltonians. As an illustration, we have chosen to present the Lewis-Riesenfeld method in this chapter, and in the third chapter we will use the unitary transformation method.

Now, let us recall briefly various notions regarding the time-dependent non-Hermitian Hamiltonians.

2.2 Time-dependent pseudo-Hermitian Hamiltonians

The study of a class of time-dependent quantum systems with non-Hermitian Hamiltonians: pseudo-Hermitian Hamiltonians have been crucial so far but no consensus has been reached on a number of central questions. While the treatment for quantum systems governed by time-dependent non-Hermitian Hamiltonians with time-independent metric operators [19, 40] is vastly accepted, the generalization to time-dependent metric operators is still controversially discussed [20–22, 38, 39, 42–45]. The results in Refs. [12, 19–21, 39, 40, 46] reveal that the unity of time evolution can be guaranteed but the Hamiltonian remains unobservable in general. Moreover, the latest results which have been recently illustrated in Refs. [23, 48] argue that it is incompatible to preserve a unitary time evolution for time-dependent non-Hermitian Hamiltonians when the metric operator is explicitly time-dependent. It should be noted that the time-dependent Dyson equation and the time-dependent quasi-Hermiticity relation possess meaningful solutions [23].

We briefly recall the various viewpoints regarding this controversy. In Ref. [20], Mostafazadeh confirmed that with the help of a time-dependent metric operator, one cannot ensure the unitarity of the time evolution at the same time as the observability of the Hamiltonian. This viewpoint is adopted by Fring et al. [23, 48]. Instead, most of the authors resort to the use of non-unitary time evolution for time-dependent non-Hermitian Hamiltonians by insisting on a quasi-Hermiticity relation between a non-Hermitian and a Hermitian Hamiltonian [21, 22, 41–45].

On the other hand, there are two different viewpoints on the relation between the time-independence of the metric operator and the pseudo-Hermiticity of the time-dependent non-Hermitian Hamiltonian $H(t)$. One is that of Ali Mostafazadeh who says that: the independence of time of the metric operator is a necessary condition to ensure the pseudo-Hermiticity of the Hamiltonian $H(t)$, and the other is that of Miloslav Znojil who says that the independence of time is not a necessary condition to guarantee the pseudo-Hermiticity of $H(t)$.

2.3 Time-dependent Hamiltonian having $\mathcal{PT}$ symmetry

Given the importance of $\mathcal{PT}$-symmetric quantum mechanics (QM), it is necessary to seek a general theory to study the quantum problems described by time-dependent non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry. In several early studies of time-dependent $\mathcal{PT}$-symmetric QM, the usual Schrödinger equation was used without being modified [19, 20]. In these cases, a noticeable constraint is that although the $\mathcal{PT}$-symmetric Hamiltonian is time-dependent, the inner-product remains to be stationary, that for obtaining an unitary evolution. However, the time-dependent inner-product and the unitary condition may be made compatible with a time-dependent Schrödinger-like equation, thus this constraint can be lifted [43, 44, 47]. That is, to study the time-dependent $\mathcal{PT}$-symmetric QM with a time-dependent inner-product, one must go beyond the usual time-dependent Schrödinger equation. In Refs. [43, 47], the first attempt to create a Schrödinger-like equation for $\mathcal{PT}$-symmetric QM was performed by transforming a Hermitian quantum system to a non-Hermitian $\mathcal{PT}$-symmetric one using a known processing. In this sense, the evolution of a time-dependent $\mathcal{PT}$-symmetric quantum system is generated by a fully-known Hermitian one. On the other hand, there is an attempt in [44] to establish $\mathcal{PT}$-symmetric QM as a basic theory and treat the ordinary QM as a special case of $\mathcal{PT}$-symmetric QM. However, while creating a time-dependent Schrödinger-like equation giving unitary evolution for $\mathcal{PT}$-symmetric QM, there is ambiguity [44]. That is, for a time-dependent $\mathcal{PT}$- symmetric Hamiltonian with a time-dependent inner-product, there can be an endless number of time evolution operators providing a unitary evolution. This was expected because even in conventional QM, the unitarity condition is not adequate to determine the form of a time-dependent Schrödinger equation. In Ref. [45], the work was to get an equation of motion for $\mathcal{PT}$- symmetric QM that is as general as possible, an axiom has also been proposed to remove the aforementioned ambiguity.

In some recent studies of time-dependent $\mathcal{PT}$- symmetric QM, a new approach, which has been adopted, is an extension of the Lewis and Riesenfeld invariant approach [49, 50], developed in Refs. [24, 25, 51–53], to solve the time-dependent Schrödinger equation. The extension of the Lewis and Riesenfeld invariant or the so-called pseudo-invariant approach, has become an approved method for studying time-dependent quantum systems.

2.4 Invariant theories for time dependent non-Hermitian quantum systems

In conventional quantum mechanics, the invariant theory of Lewis and Riesenfeld is a very useful technique for determining the solutions of quantum systems described by explicitly time-dependent Hermitian Hamiltonians. The solution of the Schrödinger equation is expressed as a function of the eigenstates of the invariant multiplied by a phase [50], hence the problem comes down to finding the explicit form of the invariant operator and the phase associated with the evolution.

2.4.1 Conventional invariant theory

Let us consider a system whose Hamiltonian $h(t)$ is Hermitian and explicitly time dependent, the time-dependent Schrödinger equation is given by

$$i\hbar\partial_t \left|\Psi^h(t)\right\rangle = h(t)\left|\Psi^h(t)\right\rangle . \tag{2.1}$$

A Hermitian operator $I_h\ (t)$ is called an invariant for the system if it satisfies the Von-Neumann equation

$$\frac{dI_h(t)}{dt} = \frac{\partial I_h(t)}{\partial t} + \frac{1}{i\hbar}\left[I_h(t), h(t)\right] = 0. \tag{2.2}$$

We note that the action of the invariant $I_h(t)$ on a state vector $\left|\Psi^h(t)\right\rangle$ solution of the Schrödinger equation (2.1) associated with the Hermitian Hamiltonian $h(t)$, is also a solution of the following Schrödinger equation

$$i\hbar\partial_t \left(I_h(t) \left|\Psi^h(t)\right\rangle\right) = h(t) \left(I_h(t) \left|\Psi^h(t)\right\rangle\right). \tag{2.3}$$

The invariant operator $I_h(t)$ is assumed to admit a set of eigenstates $|\psi_{\lambda,\kappa}(t)\rangle$

$$I_h(t) |\psi_{\lambda,\kappa}(t)\rangle = \lambda |\psi_{\lambda,\kappa}(t)\rangle, \tag{2.4}$$

where λ corresponds to its eigenvalues and κ represents all the other necessary quantum numbers specifying the eigenstates of $I_h(t)$. These eigenfunctions are assumed to be orthonormal

$$\langle\psi_{\lambda',\kappa'}(t) |\psi_{\lambda,\kappa}(t)\rangle = \delta_{\lambda,\lambda'}\delta_{\kappa,\kappa'}. \tag{2.5}$$

Let us use the fact that $I_h(t)$ is a Hermitian invariant, then the eigenvalues λ are real and time independent. By differentiating Eq. (2.4) with respect to time, we obtain

$$(\partial_t I_h(t)) |\psi_{\lambda,\kappa}(t)\rangle + I_h(t)\partial_t |\psi_{\lambda,\kappa}(t)\rangle = (\partial_t\lambda) |\psi_{\lambda,\kappa}(t)\rangle + \lambda\partial_t |\psi_{\lambda,\kappa}(t)\rangle, \tag{2.6}$$

then, multiply the left-hand side of Eq. (2.6) by $\langle\psi_{\lambda',\kappa'}(t)|$, we will have

$$(\partial_t\lambda) = \langle\psi_{\lambda',\kappa'}(t)| \partial_t I_h(t) |\psi_{\lambda,\kappa}(t)\rangle, \tag{2.7}$$

The expectation value of Eq. (2.2) in the states is written as

$$i\hbar \langle\psi_{\lambda',\kappa'}(t)| \partial_t I_h(t) |\psi_{\lambda,\kappa}(t)\rangle + (\lambda' - \lambda) \langle\psi_{\lambda',\kappa'}(t)| h(t) |\psi_{\lambda,\kappa}(t)\rangle = 0. \tag{2.8}$$

which implies that for $\lambda' = \lambda$

$$\langle\psi_{\lambda',\kappa'}(t)| \partial_t I_h(t) |\psi_{\lambda,\kappa}(t)\rangle = 0, \tag{2.9}$$

from which we deduce that the eigenvalues of $I_h(t)$ are time-independent.

In order to study the connection between the eigenstates of $I_h(t)$ and the solutions of the Schrödinger equation $\left|\Psi^h(t)\right\rangle$ we first multiply the left-hand side of Eq. (2.6) by $\langle\psi_{\lambda',\kappa'}(t)|$,

$$(\lambda - \lambda') \langle\psi_{\lambda',\kappa'}(t)| \partial_t |\psi_{\lambda,\kappa}(t)\rangle = \langle\psi_{\lambda',\kappa'}(t)| \partial_t I_h(t) |\psi_{\lambda,\kappa}(t)\rangle \tag{2.10}$$

For $\lambda' \neq \lambda$, the above equation allows us to write Eq. (2.8) in the following form

$$i\hbar \langle\psi_{\lambda',\kappa'}(t)| \partial_t |\psi_{\lambda,\kappa}(t)\rangle = \langle\psi_{\lambda',\kappa'}(t)| h(t) |\psi_{\lambda,\kappa}(t)\rangle . \tag{2.11}$$

If Eq. (2.8) held for $\lambda' = \lambda$ as well as for $\lambda' \neq \lambda$, then we would immediately deduce that $|\psi_{\lambda,\kappa}(t)\rangle$ satisfies the Schrödinger equation, i.e., is a solution of the Schrödinger equation. This could be the case if we used the fact that the phases of the stationary states are not fixed. Indeed, we can therefore multiply $|\psi_{\lambda,\kappa}(t)\rangle$ by an arbitrarily time-dependent phase factor. That is, we can define a new set of eigenstates of $I_h(t)$ related to our initial set by a time-dependent gauge transformation

$$|\psi_{\lambda,\kappa}(t)\rangle_\alpha = \exp\left[i\alpha_{\lambda,\kappa}(t)\right] |\psi_{\lambda,\kappa}(t)\rangle , \tag{2.12}$$

where the $\alpha_{\lambda,\kappa}(t)$ are arbitrary real time-dependent functions. These $|\psi_{\lambda,\kappa}(t)\rangle_\alpha$ are also orthonormal eigenstates of $I_h(t)$ associated with the eigenvalues λ. If we choose the phases $\alpha_{\lambda,\kappa}(t)$ in order that Eq. (2.11) holds for $\lambda = \lambda'$ the objective will be achieved. It is just necessary to have the choice of the phases $\alpha_{\lambda,\kappa}(t)$ such as

$$\hbar\delta_{\kappa\kappa'} \frac{d\alpha_{\lambda,\kappa}(t)}{dt} = \langle\psi_{\lambda,\kappa'}(t)| \left(i\hbar\partial_t - h(t)\right) |\psi_{\lambda,\kappa}(t)\rangle . \tag{2.13}$$

This choice shows that Eq. (2.11) for $|\psi_{\lambda,\kappa}(t)\rangle_\alpha$ holds for $\lambda = \lambda'$ and the non-diagonal elements $\langle\psi_{\lambda',\kappa'}(t)\ (i\hbar\partial_t - h(t))\,|\psi_{\lambda,\kappa}(t)\rangle$ are null. Moreover, for $\kappa = \kappa'$, the phases $\alpha_{\lambda,\kappa}(t)$ are chosen to satisfy the equation

$$\hbar\frac{d\alpha_{\lambda,\kappa}(t)}{dt} = \langle\psi_{\lambda,\kappa}(t)| \left(i\hbar\partial_t - h(t)\right) |\psi_{\lambda,\kappa}(t)\rangle . \tag{2.14}$$

The solution of the Schrödinger equation (2.1) is written as a linear combination of the eigenstates

$$\left|\Psi^h(t)\right\rangle = \sum_{\lambda,\kappa} C_{\lambda,\kappa}(0) \exp\left[i\alpha_{\lambda,\kappa}(t)\right] |\psi_{\lambda,\kappa}(t)\rangle , \tag{2.15}$$

where the $C_{\lambda,\kappa}(0)$ are time-independent coefficients.

2.4.2 Pseudo-invariant approach

In this section we recall the results of the pseudo-invariant operator technique [24, 25, 53]. Given a time-dependent non-Hermitian Hamiltonian $H(t)$, the Schrödinger equation is given by

$$i\hbar\partial_t \left|\Psi(t)\right\rangle = H(t) \left|\Psi(t)\right\rangle . \tag{2.16}$$

It is possible to build a an explicitly time-dependent pseudo-invariant operator $I^{PH}(t)$ verifying the Von-Neumann equation

$$\frac{\partial I^{PH}(t)}{\partial t} = \frac{i}{\hbar}\left[I^{PH}\left(t\right), H(t)\right], \tag{2.17}$$

By introducing a time-dependent metric $\eta(t) = \rho^{+}(t)\rho(t)$ and in a completely analogous way to the time-independent case, the temporal relation of quasi-hermiticity for the invariant operator is

$$I^{PH+}\left(t\right) = \eta(t)I^{PH}\left(t\right)\eta^{-1}(t) \;\Leftrightarrow I^{h}(t) = \rho(t)I^{PH}(t)\rho^{-1}(t) = I^{h+}(t), \tag{2.18}$$

thus the metric operator connects $I^{PH}(t)$ to its Hermitian conjugate $I^{PH+}\left(t\right)$, and $I^{PH}\left(t\right)$ can also be related to the Hermitian invariant operator $I^{h}(t)$ via $\rho(t)$ transformation.

The virtue of such a conjugate pair $I^{h}(t)$ and I^{PH} is that they possess a similar spectrum because the invariants lie in the same similarity class. The reality of the spectrum is guaranteed, since the invariant $I^{h}(t)$ is Hermitian. It means that any self-adjoint invariant operator $I^{h}(t)$, i.e., an observable, in the Hermitian system possesses an observable counterpart I^{PH} in the non-Hermitian system and they are related to each other as

$$I^{PH} = \rho^{-1}(t)I^{h}(t)\rho(t), \tag{2.19}$$

in complete analogy to the time-independent scenario for any self-adjoint operator.

The corresponding eigenvalue equations are then

$$I_h(t)\left|\psi_n(t)\right\rangle = \lambda_n \left|\psi_n(t)\right\rangle , \tag{2.20}$$

and

$$I^{PH}(t)\left|\phi_n^H(t)\right\rangle=\lambda_n\left|\phi_n^H(t)\right\rangle, \tag{2.21}$$

where the eigenfunctions $|\psi_n(t)\rangle$ and $\left|\phi_n^H(t)\right\rangle$ are related as

$$|\psi_n(t)\rangle=\rho(t)\left|\phi_n^H(t)\right\rangle. \tag{2.22}$$

By introducing the time-dependent metric $\eta(t)$, the eigenfunctions $\left|\phi_n^H(t)\right\rangle$ associated with the pseudo-hermitian invariant $I^{PH}(t)$ satisfy the following pseudo inner product

$$\left\langle\phi_n^H(t)\right.\left|\phi_n^H(t)\right\rangle_{\eta(t)}=\left\langle\phi_n^H(t)\right|\eta(t)\left|\phi_n^H(t)\right\rangle=\delta_{mn}. \tag{2.23}$$

The eigenvalues λ_n are also independent of time and can be deduced in the following way, by differentiating the equation (2.21), it follows that

$$\frac{\partial I^{PH}}{\partial t}\left|\phi_n^H(t)\right\rangle+I^{PH}\frac{\partial\left|\phi_n^H(t)\right\rangle}{\partial t}=\frac{\partial\lambda_n}{\partial t}\left|\phi_n^H(t)\right\rangle+\lambda_n\frac{\partial\left|\phi_n^H(t)\right\rangle}{\partial t}, \tag{2.24}$$

by multiplying the left-hand side of Eq. (2.24) by $\left\langle\phi_n^H(t)\right|\eta(t)$ and using Eq.(2.17), we obtain

$$\frac{\partial\lambda_n}{\partial t}=\left\langle\phi_n^H(t)\right|\eta(t)\frac{\partial I^{PH}}{\partial t}\left|\phi_n^H(t)\right\rangle=0, \tag{2.25}$$

which means that the eigenvalues λ_n are constants.

In order to study the relation between the eigenstates of $I^{PH}(t)$ and $|\Psi(t)\rangle$ the solutions of the Schrödinger equation (2.16), we project the Eq.(2.24) on $\left\langle\phi_m^H(t)\right|\eta(t)$ and using Eq.(2.25), we get

$$i\hbar\left\langle\phi_m^H(t)\right|\eta(t)\frac{\partial}{\partial t}\left|\phi_n^H(t)\right\rangle=\left\langle\phi_m^H(t)\right|\eta(t)H(t)\left|\phi_n^H(t)\right\rangle, \quad (m\neq n). \tag{2.26}$$

For $m=n$ we can verify that $\left|\phi_n^H(t)\right\rangle$ is a solution of the Schrödinger equation. This can be the case if we use the fact that the phases of the stationary states are not fixed. Indeed, we can therefore multiply $\left|\phi_n^H(t)\right\rangle$ by a time-dependent phase factor, the new eigenstates $|\Psi(t)\rangle$ of I^{PH}(t) are

$$\left|\Phi_n^H(t)\right\rangle=e^{i\gamma_n(t)}\left|\phi_n^H(t)\right\rangle, \tag{2.27}$$

which must satisfy the Schrödinger equation (2.16). That is, $\left|\Phi_n^H(t)\right\rangle$ is a particular solution of the Schrödinger equation (2.16) where the phase $\gamma_n(t)$ is real and satisfies the following differential equation

$$\frac{d\gamma_n(t)}{dt} = \left\langle \phi_n^H(t)\right| \eta(t) \left[i\hbar\frac{\partial}{\partial t} - H(t)\right] \left|\phi_n^H(t)\right\rangle . \tag{2.28}$$

The general solution of the Schrödinger equation associated with a time-dependent non-hermitian Hamiltonian $H(t)$ is as follows

$$|\Psi(t)\rangle = \sum_n C_n e^{i\gamma_n(t)} \left|\phi_n^H(t)\right\rangle , \tag{2.29}$$

where the $C_n = \left\langle \phi_n^H(0)\right| \eta(0) \left|\Psi(0)\right\rangle$ are time-independent coefficients.

Chapter 3

Application: Harmonic Oscillator in imaginary linear potential

3.1 Introduction

In this chapter we present our results obtained in Ref. [26]. We use a unitary transformation to solve a time-dependent Schrödinger equation governed by a non-Hermitian Hamiltonian, we do so by mapping a time-dependent non-Hermitian Hamiltonian (TDNH) to an already-known $\mathcal{PT}$-symmetric time-independent one using a specific unitary transformation. Consequently, the solution of the time-dependent Schrödinger equation becomes easily deduced and the evolution preserves a new inner product. Moreover, the expectation value of the non-Hermitian Hamiltonian in the normed states is guaranteed to be real.

3.2 Unitary transformations for TDNH Hamiltonians

Let us consider a non-Hermitian time-dependent Hamiltonian $H(t)$ where the quantum time evolution of the system is governed by the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = H(t)\left|\psi(t)\right\rangle . \tag{3.1}$$

In order to study the evolution of the quantum system associated to the time-dependent

Hamiltonian $H(t)$, we seek that this Hamiltonian can be converted into a time-independent Hamiltonian by some time-dependent transformations. To this end, we initially perform a unitary transformation $F(t)$ on $|\psi(t)\rangle$

$$|\chi(t)\rangle = F(t)\,|\psi(t)\rangle\,, \tag{3.2}$$

by inserting (3.2) in Eq. (3.1),

$$\begin{aligned} i\frac{\partial}{\partial t}|\chi(t)\rangle &= i\frac{\partial F(t)}{\partial t}|\psi(t)\rangle + F(t)H(t)\,|\psi(t)\rangle\,, \\ &= \left(F(t)H(t)F^{+}(t) - iF(t)\frac{\partial F^{+}(t)}{\partial t}\right)|\chi(t)\rangle\,, \end{aligned} \tag{3.3}$$

we obtain the time-dependent Schrödinger equation for the state $|\chi(t)\rangle$

$$i\frac{\partial}{\partial t}|\chi(t)\rangle = \mathcal{H}\,|\chi(t)\rangle\,, \tag{3.4}$$

such that the new Hamiltonian

$$\mathcal{H} = F(t)H(t)F^{+}(t) - iF(t)\frac{\partial F^{+}(t)}{\partial t}, \tag{3.5}$$

is time-independent and $\mathcal{PT}$-symmetric, i.e.;

$$\mathcal{H} \equiv \mathcal{H}_0^{\mathcal{PT}}, \tag{3.6}$$

its eigenstates $|\chi(t)\rangle$ preserve the $\mathcal{CPT}$-inner product

$$\langle\chi(t)|\,\chi(t)\rangle_{\mathcal{CPT}} = \langle\chi(t)|\,\mathcal{CP}\,|\chi(t)\rangle\,, \tag{3.7}$$

and in this case the solution of the Schrödinger equation (3.4) can be written as

$$|\chi(t)\rangle = \exp(-iEt)\,|\chi\rangle\,. \tag{3.8}$$

where $|\chi\rangle$ is an eigenstate of $\mathcal{H}_0^{\mathcal{PT}}$.

3.3 The $\mathcal{C}(t)\mathcal{PT}$-inner product and expectation value

Knowing that our interest is the mean value of the non-Hermitian Hamiltonian $H(t)$, for this aim we calculate firstly the expectation value of the Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$

$$\left\langle \mathcal{H}_0^{\mathcal{PT}} \right\rangle_{\mathcal{CPT}} = \langle \chi(t)| \mathcal{CP}\mathcal{H}_0^{\mathcal{PT}} |\chi(t)\rangle = \langle \chi(t)| \mathcal{CP} \left[FH(t)F^{+} - iF\frac{\partial F^{+}}{\partial t} \right] |\chi(t)\rangle , \tag{3.9}$$

from which we deduce

$$\langle \chi(t)| \mathcal{CP} \left[FH(t)F^{+} \right] |\chi(t)\rangle = \left\langle \mathcal{H}_0^{\mathcal{PT}} \right\rangle_{\mathcal{CPT}} + \langle \chi(t)| \mathcal{CP} \left[iF\frac{\partial F^{+}}{\partial t} \right] |\chi(t)\rangle , \tag{3.10}$$

we note that the first term is nothing other than the expectation value of the Hamiltonian $H(t)$ with a new $\mathcal{C}(t)\mathcal{PT}$-inner product

$$\langle \chi(t)| \mathcal{CP} \left[FH(t)F^{+} \right] |\chi(t)\rangle = \langle \psi(t)| \mathcal{C}(t)\mathcal{P}H(t) |\psi(t)\rangle = \langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} \ , \tag{3.11}$$

where $[\mathcal{P}, F(t)] = 0$ and the new operator $\mathcal{C}(t)$ is defined as $\mathcal{C}(t) = F^{+}(t)\mathcal{C}F(t)$, which is similar to the operator $\mathcal{C}$ in the sense that it verifies the property $\mathcal{C}^2(t) = 1$ since $\mathcal{C}^2 = 1$.

Finally

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = \left\langle \mathcal{H}_0^{\mathcal{PT}} \right\rangle_{\mathcal{CPT}} + \langle \chi(t)| \mathcal{CP} \left[iF\frac{\partial F^{+}}{\partial t} \right] |\chi(t)\rangle . \tag{3.12}$$

Indeed, since $\mathcal{H}_0^{\mathcal{PT}}$ is $\mathcal{PT}$ -symmetric and F is unitary, the expectation value $\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}}$ is guaranteed to be real. This result is new for explicitly time-dependent non-Hermitian quantum systems.

3.4 Harmonic Oscillator in imaginary linear potential

Let us consider a class of one dimensional time-dependent harmonic oscillators with variable mass $m(t) = m_0\alpha(t)$ subjected to a driving linear complex time-dependent potential, in the form $i\lambda(t)x$, described by the following time-dependent non-Hermitian Hamiltonian

$$H(t) = \frac{p^2}{2m_0\alpha(t)} + \alpha(t)\frac{m_0\omega^2}{2}x^2 + i\lambda(t)x, \tag{3.13}$$

where $\alpha(t)$ and $\lambda(t)$ are real time-dependent functions which can be chosen to describe a specific quantum system, x and p are the canonical conjugates position and momentum operators ($[x, p] = i$). The function $\lambda(t)$ in the complex potential will be chosen later in order to obtain in Eq. (3.5) a time-independent $\mathcal{PT}$-symmetric Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$. The mass m_0 and the time-independent frequency ω are the characteristic parameters of the quantum oscillating system which can be precisely chosen according to the initial conditions for a controlled quantum system.

This exact solvable model of time dependent non-Hermitian quantum systems is well studied in the literature [18, 24, 52–56].

The corresponding Schrödinger equation is written as

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle\,. \tag{3.14}$$

The problem now is to solve the time-dependent Schrödinger equation (3.14). First, let us use a unitary transformation in order to eliminate the time-dependent parameter $\alpha(t)$ [26].

3.5 Time-dependent unitary transformations

A time-dependent transformation F is a general $SU(1,1)$ one-mode squeezing operator and constituted by the three quadratic canonical operators x^2, p^2 and $\{x, p\}$ which are three generators of $SU(1,1)$ Lie algebra. In the canonical representation (x, p), F is defined as follows [57, 58]

$$F = \exp\left[i\frac{C(t)}{2A(t)}x^2\right]\exp\left[-\frac{i}{2}\{x,p\}\ln A(t)\right]\exp\left[-i\frac{B(t)}{2A(t)}p^2\right], \quad A \neq 0, \tag{3.15}$$

where the time-dependent functions $A(t)$, $B(t)$ and $C(t)$ are arbitrary real functions in order to keep the transformation unitary. However, in some cases, these functions can be determined by solving a set of coupled partial differential equations as we will see later. Moreover, the operator F verifies the properties of a unit operator, thus

$$FF^{+} = F^{+}F = 1, \quad \text{and} \quad F^{-1} = F^{+}. \tag{3.16}$$

Using the above properties and the relation $[x, p] = i$, the canonical operators x and p are transformed under the action of F as follows

$$FxF^{+} = \quad D(t)\ x - B(t)p, \tag{3.17}$$

$$FpF^{+} = \quad -C(t)x + A(t)p, \tag{3.18}$$

and which can be written in the matrix form as

$$F\binom{x}{p}F^{+} = \left(\begin{array}{cc} D(t) & -B(t) \\ -C(t) & A(t) \end{array}\right)\binom{x}{p}, \tag{3.19}$$

where $D(t) = \frac{1+B(t)C(t)}{A(t)}$ and $A(t)D(t) - B(t)C(t) = 1$.

3.6 Solution of the time-dependent Schrödinger equation

The exact solution of the time-dependent Schrödinger equation (3.14) can be found by using the unitary transformations F defined in (3.15). By performing the unitary transformation F on $|\psi(t)\rangle$

$$|\psi(t)\rangle = F^{+}|\chi(t)\rangle, \tag{3.20}$$

and inserting (3.2) in Eq. (3.1), we easily obtain that

$$\begin{aligned} i\frac{\partial}{\partial t}|\chi(t)\rangle &= i\frac{\partial F}{\partial t}|\psi(t)\rangle + iF\frac{\partial}{\partial t}|\psi(t)\rangle \\ &= \left(i\frac{\partial F}{\partial t}F^{+} + FH(t)F^{+}\right)|\chi(t)\rangle \qquad (3.21) \\ &= \left(FH(t)F^{+} - iF\frac{\partial F^{+}}{\partial t}\right)|\chi(t)\rangle \equiv \mathcal{H}|\chi(t)\rangle, \qquad (3.22) \end{aligned}$$

where

$$\mathcal{H} = FH(t)F^{+} - iF\frac{\partial F^{+}}{\partial t}. \tag{3.23}$$

We note that the equation (3.4) implies that the unitary transformation F gives rise to a new system described by the time-dependent Hamiltonian $\mathcal{H}$.

Let us now calculate the expressions of the time-dependent functions $A(t)$, $B(t)$ and $C(t)$ in terms of $\alpha(t)$ and $\lambda(t)$. To this end, following the same steps of [58], we must calculate $\mathcal{H}$ (3.23) in terms of $A(t)$, $B(t), C(t)$ and $D(t)$ and compare it with the time-independent harmonic oscillator with variable mass m_0 subjected to a driving linear complex time-independent potential,

$$\mathcal{H} = \frac{p^2}{2m_0} + \frac{1}{2}m_0\Omega^2 x^2 + i\lambda_0 x, \tag{3.24}$$

and the global time-dependent frequency Ω will be determined later.

Using the Baker–Hausdorff formula and the first term in (3.4), we obtain

$$\frac{\partial F^+}{\partial t} = \frac{i}{2}F^+\left[(A(t)\dot{B}(t) - B(t)\dot{A}(t))p^2 + (C(t)\dot{D}(t) - D(t)\dot{C}(t))x^2 + (D(t)\dot{A}(t) - C(t)\dot{B}(t))\{x,p\}\right], \tag{3.25}$$

where the over-dots indicate the partial time derivative, and the second term in (3.4) becomes

$$FH(t)F^+ = \frac{1}{2m_0\alpha(t)}\ Fp^2F^+ + \alpha(t)\frac{m_0\omega^2}{2}Fx^2F^+ + i\lambda(t)FxF^+, \tag{3.26}$$

such that

$$Fp^2F^+ = A^2(t)p^2 - A(t)C(t)\{x,p\} + C^2(t)x^2, \tag{3.27}$$

$$Fx^2F^+ = D^2(t)x^2 - D(t)B(t)\{x,p\} + B^2(t)p^2. \tag{3.28}$$

By substituting (3.25), (3.26), (3.27) and (3.28) into (3.23), we find that

$$\begin{aligned}
\mathcal{H} &= FH(t)F^+ - iF\frac{\partial F^+}{\partial t} \\
&= \left[\frac{A^2(t)}{2m_0\alpha(t)} + \alpha(t)\frac{B^2(t)m_0\omega^2}{2} + \frac{1}{2}\left[A(t)\dot{B}(t) - B(t)\dot{A}(t)\right]\right]p^2 \\
&+ \left[\frac{C^2(t)}{2m_0\alpha(t)} + \alpha(t)\frac{D^2(t)m_0\omega^2}{2} + \frac{1}{2}\left[C(t)\dot{D}(t) - D(t)\dot{C}(t)\right]\right]x^2 \\
&- \left[\frac{A(t)C(t)}{2m_0\alpha(t)} + \alpha(t)\frac{B(t)D(t)m_0\omega^2}{2} - \frac{1}{2}\left[D(t)\dot{A}(t) - C(t)\dot{B}(t)\right]\right]\{x,p\} \\
&+ iD(t)\lambda(t)x - iB(t)\lambda(t)p,
\end{aligned} \tag{3.29}$$

comparing with (3.24), we get the following coupled differential equations

$$\left[\frac{A^2(t)}{2m_0\alpha(t)} + \alpha(t)\frac{B^2(t)m_0\omega^2}{2} + \frac{1}{2}\left[A(t)\dot{B}(t) - B(t)\dot{A}(t)\right]\right] = \frac{1}{2m_0}, \tag{3.30}$$

$$\left[\frac{C^2(t)}{2m_0\alpha(t)} + \alpha(t)\frac{D^2(t)m_0\omega^2}{2} + \frac{1}{2}\left[C(t)\dot{D}(t) - D(t)\dot{C}(t)\right]\right] = \frac{m_0\Omega^2}{2}, \tag{3.31}$$

$$\left[\frac{A(t)C(t)}{2m_0\alpha(t)} + \alpha(t)\frac{B(t)D(t)m_0\omega^2}{2} - \frac{1}{2}\left[D(t)\dot{A}(t) - C(t)\dot{B}(t)\right]\right] = 0, \tag{3.32}$$

$$D(t)\lambda(t) = 1, \tag{3.33}$$

$$-iB(t)\lambda(t) = 0. \tag{3.34}$$

From (3.34) we have $B(t) = 0$, and from (3.33) we get $D(t) = \frac{1}{\lambda(t)}$. So, substituting $B(t) = 0$ in (3.30) we obtain $A(t) = \sqrt{\alpha(t)}$, then from (3.32) we find $C(t) = m_0\dot{A}(t) = \frac{1}{2}m_0\dot{\alpha}(t)\alpha^{-\frac{1}{2}}(t)$. Moreover, since $A(t)D(t) - B(t)C(t) = 1$ we can note that $A(t) = \frac{1}{D(t)} = \sqrt{\alpha(t)}$, thus $D(t) = \frac{1}{\sqrt{\alpha(t)}}$ and therefore

$$A(t) = \sqrt{\alpha(t)},\ B(t) = 0,\ C(t) = \frac{1}{2}m_0\dot{\alpha}(t)\alpha^{-\frac{1}{2}}(t),\ D(t) = \frac{1}{\sqrt{\alpha(t)}} \text{ and } \lambda(t) = \sqrt{\alpha(t)}. \tag{3.35}$$

Using (3.35) and Eq.(3.31), we get

$$\Omega^2 = \left(\omega^2 + \frac{1}{4}\frac{\dot{\alpha}^2(t)}{\alpha^2(t)} - \frac{\ddot{\alpha}(t)}{2\alpha(t)}\right). \tag{3.36}$$

Substituting (3.35) again in (3.13) and (3.15), the Hamiltonian and its associated unitary transformation F become

$$H(t) = \frac{p^2}{2m_0\alpha(t)} + \alpha(t)\frac{m_0\omega^2(t)}{2}x^2 + ix\sqrt{\alpha(t)}, \tag{3.37}$$

$$F(t) = \exp\left[i\frac{m_0\dot{\alpha}(t)}{4\alpha(t)}x^2\right]\exp\left[-\frac{i}{2}\{x,p\}\ln\left(\sqrt{\alpha(t)}\right)\right]. \tag{3.38}$$

The task now is to make the time-dependent Hamiltonian (3.24) governing the evolution of $|\chi(t)\rangle$ time-independent. We set the global time-dependent frequency appearing in (3.36) equal to a real constant denoted by Ω_0^2 in order to have a time-independent Hamiltonian. As a result, we get an auxiliary equation of the form

$$\ddot{\alpha} - \frac{\dot{\alpha}^2}{2\alpha} + 2\alpha\left(\Omega_0^2 - \omega^2\right) = 0, \tag{3.39}$$

the resulting time-independent Hamiltonian

$$\mathcal{H}_0^{\mathcal{PT}} = \frac{p^2}{2m_0} + \frac{1}{2}m_0\Omega_0^2x^2 + i\lambda_0 x, \tag{3.40}$$

is $\mathcal{PT}$-symmetric.

Note that, to write the above auxiliary equation (3.39) in a more familiar form, we make the change of variable $\alpha(t) = \frac{1}{\rho^2(t)}$. Then,

$$\ddot{\rho} + \left(\Omega_0^2 - \omega^2\right) \rho = 0. \tag{3.41}$$

which admits the following solutions:

- for $\Omega_0^2 > \omega^2$: $\rho(t) = A \exp\left(it\sqrt{\Omega_0^2-\omega^2}\right) + B \exp\left(-it\sqrt{\Omega_0^2-\omega^2}\right)$. For an appropriate choice of the constants: $A = B$, we obtain the expression of $\alpha(t)$ as $\alpha(t) = \frac{1}{A^2\cos^2\left(t\sqrt{\Omega_0^2-\omega^2}\right)}$.
- for $\Omega_0^2 < \omega^2$: $\rho(t) = A \exp\left(t\sqrt{\omega^2-\Omega_0^2}\right) + B \exp\left(-t\sqrt{\omega^2-\Omega_0^2}\right)$. For an appropriate choice of the constants: $A = B$, we obtain the expression of $\alpha(t)$ as $\alpha(t) = \frac{1}{A^2\cosh^2\left(t\sqrt{\omega^2-\Omega_0^2}\right)}$, and when $B = 0$ and $A \neq 0$ the expression of $\alpha(t)$ is $\alpha(t) = \frac{1}{A^2}\exp\left(-2t\sqrt{\omega^2-\Omega_0^2}\right)$ and the Hamiltonian $H(t)$ corresponds to the Caldirola-Kanai oscillator [59, 60].

Moreover, the eigenequation of the $\mathcal{PT}$ -symmetric Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$ has the form

$$\mathcal{H}_0^{\mathcal{PT}} \left|\chi_n(x)\right\rangle = E_n \left|\chi_n(x)\right\rangle, \tag{3.42}$$

the corresponding Schrödinger equation (3.4) has been solved in the second chapter, where

$$\chi_n(t) = \left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2} \exp(-iE_nt)\exp\left[-\frac{p}{m_0\Omega_0^2}\right]\exp\left(-\frac{m_0\Omega_0}{2\hbar}x^2\right) H_n\left[\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2} x\right], \tag{3.43}$$

and the eigenvalues

$$E_n = \hbar\Omega_0\left(n+\frac{1}{2}\right) + \frac{\lambda_0}{2m_0\Omega_0^2}, \tag{3.44}$$

are real and H_n is the Hermite polynomial of order n.

We can easily verify that the $\mathcal{CPT}$-inner product is conserved

$$\left\langle\chi_n(x,t)\right|\left.\chi_n(x,t)\right\rangle_{\mathcal{CPT}} = \left\langle\chi_n(x)\right|\mathcal{CP}\left|\chi_n(x)\right\rangle = \left\langle\varphi_n\right|U\mathcal{CP}U\left|\varphi_n\right\rangle = \left\langle\varphi_n(x)\right.\left|\varphi_n(x)\right\rangle = 1. \tag{3.45}$$

3.7 Analysis of the expectation value of the Hamiltonian

Now it is not difficult to calculate the expectation value $\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}}$ of the Hamiltonian (3.37)

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle \chi_n(x)| F\, F^{+}\mathcal{CP}FH(t)\, F^{+} |\chi_n(x)\rangle = \langle \chi_n(x)| \mathcal{CP}FH(t)F^{+} |\chi_n(x)\rangle \tag{3.46}$$

$$= \langle \chi_n(x)| \mathcal{CP}\left(\mathcal{H}_0\right) |\chi_n(x)\rangle + \langle \chi_n(x)| \mathcal{CP}\left(iF\frac{\partial F^{+}}{\partial t}\right) |\chi_n(x)\rangle \tag{3.47}$$

$$= E_n + \langle \chi_n(x)| \mathcal{CP}\left(-\frac{\dot{\alpha}(t)}{4\alpha(t)}\{x,p\} + \frac{m_0\ddot{\alpha}(t)}{4\alpha(t)}x^2\right) |\chi_n(x)\rangle \tag{3.48}$$

$$= E_n - \frac{\dot{\alpha}(t)}{4\alpha(t)} \langle \varphi_n(x)| U^{-1}\{x,p\}U |\varphi_n(x)\rangle + \left(\frac{m_0\ddot{\alpha}(t)}{4\alpha(t)}\right) \langle \chi_n(x)| \mathcal{CP}x^2 |\chi_n(x)\rangle, \tag{3.49}$$

thus

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = E_n - \frac{\dot{\alpha}(t)}{4\alpha(t)} \langle \varphi_n(x)| \{x,p\} |\varphi_n(x)\rangle + \frac{\dot{\alpha}(t)}{2\alpha(t)}\frac{i}{m_0\Omega_0^2} \langle \varphi_n(x)| p |\varphi_n(x)\rangle + \left(\frac{m_0\ddot{\alpha}(t)}{4}\right) \langle x^2\rangle_{\mathcal{CPT}}, \tag{3.50}$$

where $\langle x^2\rangle_{\mathcal{CPT}} = \langle \chi_n(x)| \mathcal{CP}x^2 |\chi_n(x)\rangle$.

By using the following relations

$$\langle \varphi_n(x)| x |\varphi_n(x)\rangle = \langle \varphi_n(x)| p |\varphi_n(x)\rangle = 0, \tag{3.51}$$

$$\langle \varphi_n(x)| x^2 |\varphi_n(x)\rangle = \frac{\hbar}{m_0\Omega_0}\left(n+\frac{1}{2}\right), \tag{3.52}$$

$$\langle \varphi_n(x)| p^2 |\varphi_n(x)\rangle = m_0\Omega_0\hbar\left(n+\frac{1}{2}\right), \tag{3.53}$$

$$\langle \varphi_n(x)| \{x,p\} |\varphi_n(x)\rangle = 0, \tag{3.54}$$

and

$$\langle x^2\rangle_{\mathcal{CPT}} = \frac{\hbar}{m_0\Omega_0}\left(n+\frac{1}{2}\right) - \left(\frac{\lambda_0}{m_0\Omega_0^2}\right)^2, \tag{3.55}$$

we get the expectation value of $H(t)$ as

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = E_n + \left(\frac{m_0\ddot{\alpha}(t)}{4\alpha(t)}\right) < x^2 >_{\mathcal{CPT}} = E_n + \frac{\ddot{\alpha}(t)}{4\alpha(t)}\left[\frac{\hbar}{\Omega_0}\left(n+\frac{1}{2}\right) - \frac{(\lambda_0)^2}{m_0\Omega_0^4}\right], \tag{3.56}$$

which is real for any positive real time-dependent function $\alpha(t)$. It is more simple than the result given in Eq. (28) in Ref. [18] with less constraints on the parameters of the quantum problem.

3.8 Uncertainty relation and probability density

Now, we calculate the expectation values $\langle x\rangle_{\mathcal{C}(t)\mathcal{PT}}$, $\langle x^2\rangle_{\mathcal{C}(t)\mathcal{PT}}$, $\langle p\rangle_{\mathcal{C}(t)\mathcal{PT}}$ and $\langle p^2\rangle_{\mathcal{C}(t)\mathcal{PT}}$ in the states $\psi_n(x,t)$ of $H(t)$ defined in Eq.(3.13). In the same way, using the $\mathcal{CPT}$-inner product (3.45) and after a straightforward calculation we obtain

$$\begin{aligned}
\langle x\rangle_{\mathcal{C}(t)\mathcal{PT}} &= \langle\psi_n(x,t)|\, F^{+}\mathcal{CP}Fx\,|\psi_n(x,t)\rangle && (3.57)\\
&= \langle\chi_n(x)|\, FF^{+}\mathcal{CP}FxF^{+}\,|\chi_n(x)\rangle && (3.58)\\
&= -\frac{i\lambda_0}{m_0\Omega_0^2}\frac{1}{\sqrt{\alpha(t)}}, && (3.59)
\end{aligned}$$

$$\begin{aligned}
\left\langle x^2\right\rangle_{\mathcal{C}(t)\mathcal{PT}} &= \langle\psi_n(x,t)|\, F^{+}\mathcal{CP}Fx^2\,|\psi_n(x,t)\rangle && (3.60)\\
&= \langle\varphi_n(x)|\left(x^2-\frac{\lambda_0}{(m_0^2\Omega_0)^2}-2\lambda_0x\frac{i}{m_0^2\Omega_0}\right)|\varphi_n(x)\rangle && (3.61)\\
&= \left(n+\frac{1}{2}\right)\frac{\hbar}{m_0\Omega_0\alpha(t)}-\frac{1}{\alpha(t)}\left(\frac{\lambda_0}{m_0\Omega_0^2}\right)^2, && (3.62)
\end{aligned}$$

$$\begin{aligned}
\langle p\rangle_{\mathcal{C}(t)\mathcal{PT}} &= \langle\psi_n(x,t)|\, F^{+}\mathcal{CP}Fp\,|\psi_n(x,t)\rangle && (3.63)\\
&= \langle\chi_n(x)|\, FF^{+}\mathcal{CP}FpF^{+}\,|\chi_n(x)\rangle && (3.64)\\
&= \frac{i}{2\Omega_0^2}\frac{\dot\alpha(t)}{\sqrt{\alpha(t)}}, && (3.65)
\end{aligned}$$

$$\left\langle p^2\right\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle\psi_n(x,t)|\, F^{+}\mathcal{CP}Fp^2\,|\psi_n(x,t)\rangle = \langle\chi_n(x)|\, FF^{+}\mathcal{CP}Fp^2F^{+}\,|\chi_n(x)\rangle\,, \tag{3.66}$$

$$\left\langle p^2\right\rangle_{\mathcal{C}(t)\mathcal{PT}} = \hbar\Omega_0\left(n+\frac{1}{2}\right)m_0\alpha(t)+\left(\frac{m_0\dot\alpha(t)}{2}\right)^2\left[\left(n+\frac{1}{2}\right)\frac{\lambda_0\hbar}{m_0\Omega_0\alpha(t)}-\frac{1}{\alpha(t)}\left(\frac{\lambda_0}{m_0\Omega_0^2}\right)^2\right]. \tag{3.67}$$

We also calculate the position and momentum uncertainties

$$\Delta x=\sqrt{\langle x^2\rangle_{\mathcal{C}(t)\mathcal{PT}}-\langle x\rangle^2_{\mathcal{C}(t)\mathcal{PT}}}=\left[\frac{\lambda_0\hbar}{m_0\Omega_0\alpha(t)}\left(n+\frac{1}{2}\right)\right]^{1/2}, \tag{3.68}$$

$$\Delta p=\sqrt{\langle p^2\rangle_{\mathcal{C}(t)\mathcal{PT}}-\langle p\rangle^2_{\mathcal{C}(t)\mathcal{PT}}}=\frac{1}{\Delta x}\left[\left(n+\frac{1}{2}\right)^2\hbar+\left(\frac{m_0\dot{\alpha}(t)}{2}\right)^2\Delta x^4\right]^{1/2}. \tag{3.69}$$

Thus, the uncertainty product is given by

$$\Delta x\Delta p=\left(n+\frac{1}{2}\right)\hbar\sqrt{1+\left(\frac{\lambda_0\dot{\alpha}(t)}{2\Omega_0\alpha(t)}\right)^2}, \tag{3.70}$$

it is easy to check that the uncertainty product (3.70) is always real and greater than or equal to $\frac{\hbar}{2}$ and, consequently, it is physically acceptable for any value of n. The uncertainty product takes the minimal value $\Delta x\Delta p=\frac{\hbar}{2}$ only for $n=0$ and $\alpha(t)=$constant, i.e., for time independent mass oscillators.

Finally, the probability density of the wavefunction $\psi_n(x,t)$ of $H(t)$ is in the form

$$\left|U^{-1}F\psi_n(x,t)\right|^2=\left|U^{-1}\chi_n(x,t)\right|^2=|\varphi(x)|^2=\varphi_n^*(x)\varphi_n(x), \tag{3.71}$$

thus

$$\left|U^{-1}F\psi_n(x,t)\right|^2=\left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]\exp\left(-\frac{m_0\Omega_0}{\hbar}x^2\right)\left(H_n\left[\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2}x\right]\right)^2, \tag{3.72}$$

it is the same as the probability density of the eigenstate $\chi_n(x,t)$ of the time-independent Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$ which is also equal to the probability density of the eigenstate $\varphi_n(x)$ (1.36) of the standard harmonic oscillator (1.34). Clearly, $\varphi_n(x)$ are elements from $L^2(R)$, and therefore the condition (3.72) yields that

$$\int|\varphi_n(x)|^2\,dx=\left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2}\exp\left(-\frac{m_0\Omega_0}{\hbar}x^2\right)\left(H_n\left[x\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2}\right]\right)^2dx=1 \tag{3.73}$$

under this observation, we deduce that the probability is finite.

3.9 Some examples

As an application of the above results, we study two examples of time-dependent mass $M(t) = m_0\alpha(t)$ where $\alpha(t)$ are the two solutions of the auxiliary equation (3.39).

3.9.1 Exemple 1: Trigonometrically growing mass

The first solution of the auxiliary equation (3.39) is a trigonometrically growing time-dependent mass:

- $M(t) = \alpha(t)m_0 = \frac{m_0}{A^2\cos^2\left(t\sqrt{\Omega_0^2-\omega^2}\right)}$ where $\Omega_0^2>\omega^2$

For this case, the Hamiltonian (3.37) is

$$H(t) = \frac{A^2\cos^2\left(t\sqrt{\Omega_0^2-\omega^2}\right)}{2m_0}p^2 + \frac{m_0\omega^2}{2A^2\cos^2\left(t\sqrt{\Omega_0^2-\omega^2}\right)}x^2 + \frac{i}{A\cos\left(t\sqrt{\Omega_0^2-\omega^2}\right)}x, \tag{3.74}$$

and the expression of the unitary operator (3.38) becomes

$$F(t) = \exp\left[i\frac{2m_0\sqrt{\Omega_0^2-\omega^2}\tan(t\sqrt{\Omega_0^2-\omega^2})\alpha(t)}{4\alpha(t)}x^2\right]\exp\left[-\frac{i}{2}\{x,p\}\ln\left(\sqrt{\frac{1}{A^2\cos^2\left(t\sqrt{\Omega_0^2-\omega^2}\right)}}\right)\right] \tag{3.75}$$

$$F(t) = \exp\left[i\frac{m_0\sqrt{\Omega_0^2-\omega^2}\tan(t\sqrt{\Omega_0^2-\omega^2})}{2}x^2\right]\exp\left[\frac{i}{2}\{x,p\}\ln\left(A\cos\left(t\sqrt{\Omega_0^2-\omega^2}\right)\right)\right]. \tag{3.76}$$

Then, the uncertainty product is

$$\Delta x\Delta p = \left(n+\frac{1}{2}\right)\hbar\sqrt{1+\left(\frac{\hbar\sqrt{\Omega_0^2-\omega^2}\tan(t\sqrt{\Omega_0^2-\omega^2})}{\Omega_0}\right)^2}, \tag{3.77}$$

which is always real and greater than or equal to $\frac{1}{2}$. Figure (3.1) represents the uncertainty product as a function of time for different values of n.

Finally, we deduce the probability density

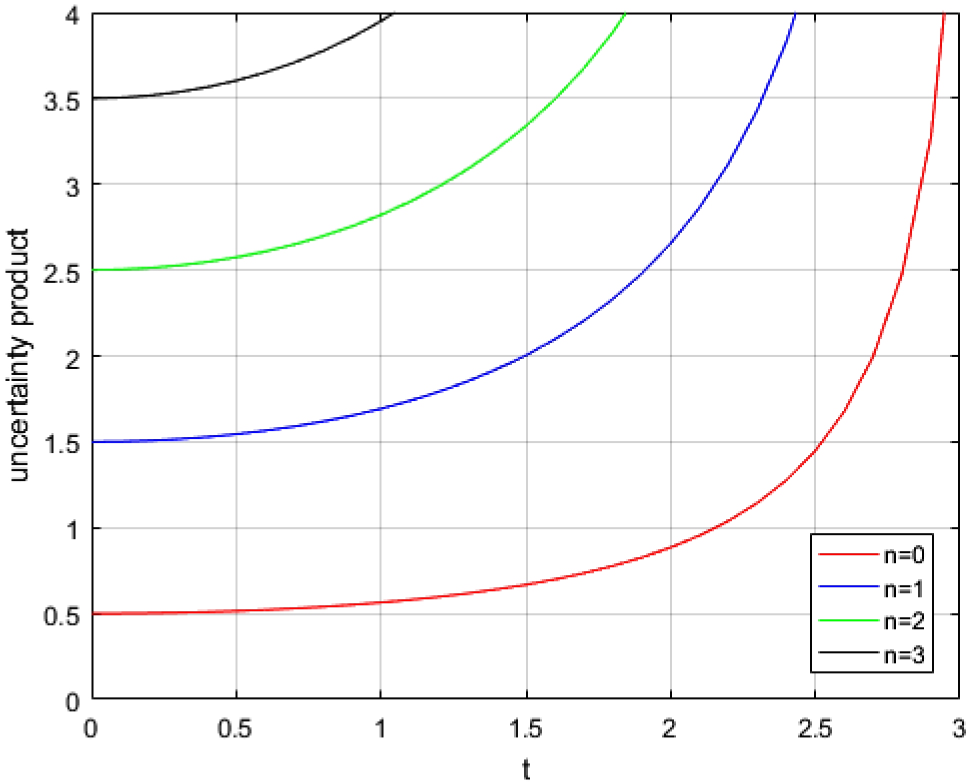

Figure 3.1: Uncertainty product as a function of time for different values of n with the following parameters: $(\Omega_0 = \hbar = 1)$. It is always real and greater than or equal to $\frac{1}{2}$

$$\left|U^{-1}F\psi_n(x,t)\right|^2 = \left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]\exp\left(-\frac{m_0\Omega_0}{\hbar}x^2\right)\left(H_n\left[\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2}x\right]\right)^2, \tag{3.78}$$

and figure (3.2) represents the probability density as a function of position for different values of n.

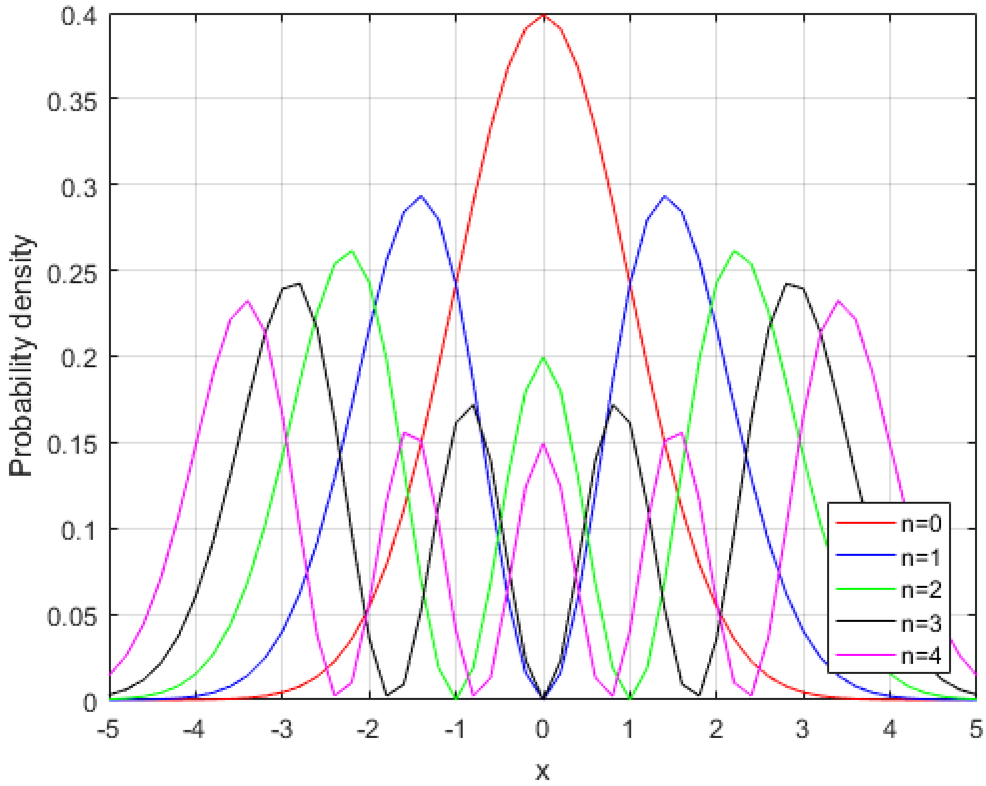

Figure 3.2: Probability density $|U^{-1}F\psi_n(x,t)|^2$ as a function of x for different values of n with the following parameters: $(m_0 = \Omega_0 = \hbar = 1)$. Its maximal value is at $x = 0$ and $n = 0$.

3.9.2 Example 2: Hyperbolically growing mass

The second solution of the auxiliary equation (3.39) is a hyperbolically growing time-dependent mass:

- $M(t) = \alpha(t) m_0 = \frac{m_0}{A^2 \cosh^2\left(t\sqrt{\omega^2 - \Omega_0^2}\right)}$ where $\Omega_0^2 < \omega^2$

For this case, the Hamiltonian (3.37) is

$$H(t) = \frac{A^2 \cosh^2\left(t\sqrt{\omega^2 - \Omega_0^2}\right)}{2m_0} p^2 + \frac{m_0 \omega^2}{2A^2 \cosh^2\left(t\sqrt{\omega^2 - \Omega_0^2}\right)} x^2 + \frac{i}{A\cosh\left(t\sqrt{\omega^2 - \Omega_0^2}\right)} x, \tag{3.79}$$

and the unitary operator (3.38) becomes

$$F(t) = \exp\left[-i\frac{m_0\sqrt{\omega^2 - \Omega_0^2}\tanh^2\left(t\sqrt{\omega^2 - \Omega_0^2}\right)\alpha(t)}{2\alpha(t)} x^2\right] \exp\left[-\frac{i}{2}\{x,p\}\ln\left(\sqrt{\frac{1}{A^2\cosh^2\left(t\sqrt{\omega^2 - \Omega_0^2}\right)}}\right)\right] \tag{3.80}$$

$$F(t) = \exp\left[-i\frac{m_0\sqrt{\omega^2 - \Omega_0^2}\tanh^2\left(t\sqrt{\omega^2 - \Omega_0^2}\right)}{2} x^2\right] \exp\left[\frac{i}{2}\{x,p\}\ln\left(A\cosh\left(t\sqrt{\omega^2 - \Omega_0^2}\right)\right)\right]. \tag{3.81}$$

Then, the uncertainty product is

$$\Delta x \Delta p = \left(n + \frac{1}{2}\right)\hbar\sqrt{1 + \left(-\frac{\hbar\sqrt{\omega^2 - \Omega_0^2}\tanh\left(t\sqrt{\omega^2 - \Omega_0^2}\right)}{\Omega_0}\right)^2}, \tag{3.82}$$

and figure (3.3) represents the uncertainty product as a function of time for different values of n.

Finally, we deduce the probability density

$$\left|U^{-1}F\psi_n(x,t)\right|^2 = \left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]\exp\left(-\frac{m_0\Omega_0}{\hbar}x^2\right)\left(H_n\left[\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2} x\right]\right)^2, \tag{3.83}$$

and figure (3.4) represents the probability density as a function of position for different values of n.

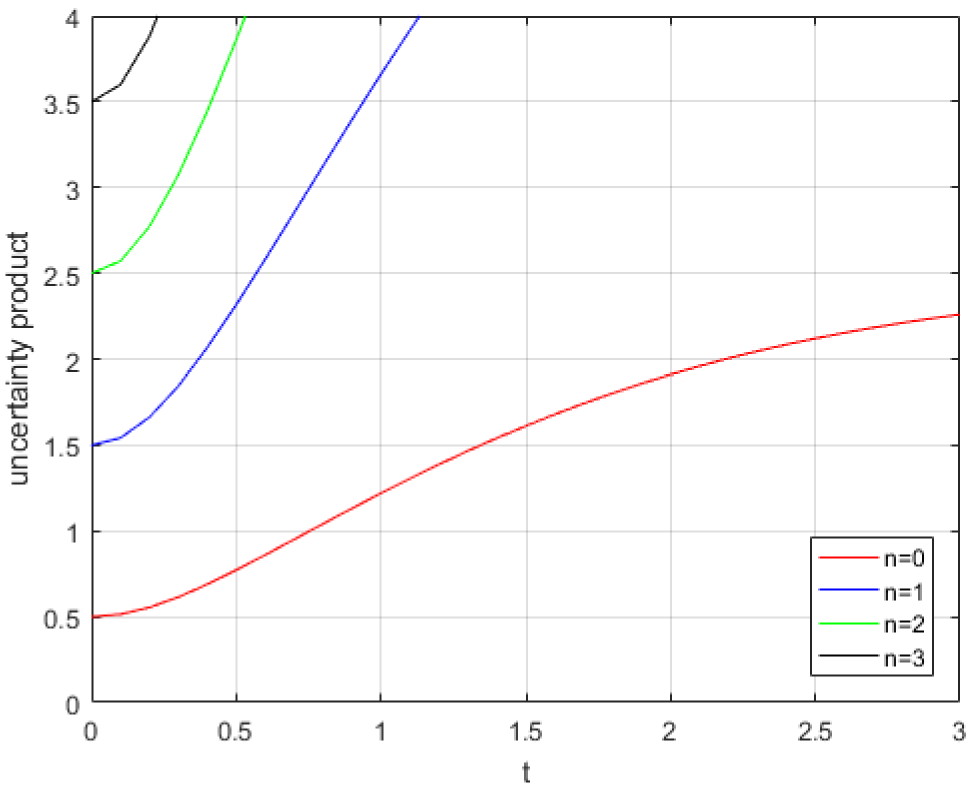

Figure 3.3: Uncertainty product as a function of time for different values of n with the following parameters: ($\Omega_0 = \hbar = 1$). It is always real and greater than or equal to $\frac{1}{2}$

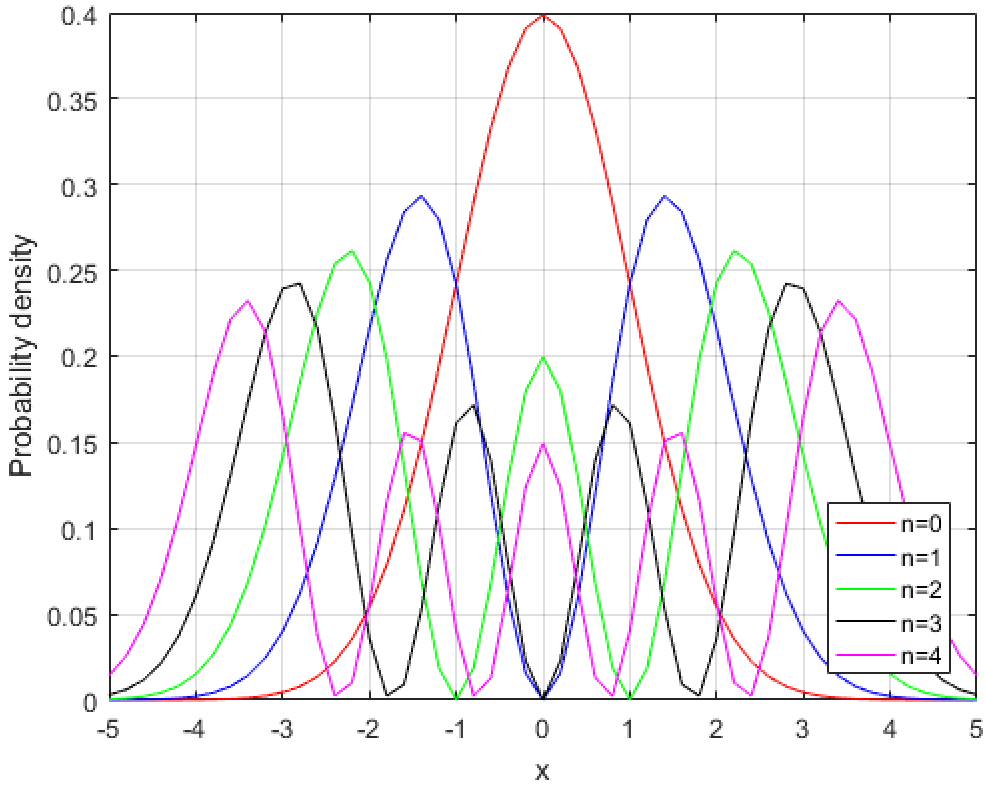

Figure 3.4: Probability density $|U^{-1}F\psi_n(x,t)|^2$ as a function of x for different values of n with the following parameters: $(m_0 = \Omega_0 = \hbar = 1)$. Its maximal value is at $x = 0$ and $n = 0$.

Conclusion

In this thesis, we have studied the analytical solutions of the Schrödinger equation for a class of explicit time dependent non-Hermitian quantum systems. The first chapter is concerned with the basic concepts used in quantum theory for non-hermitian Hamiltonians, such as $\mathcal{PT}$-symmetry, $\mathcal{PT}$ and $\mathcal{CPT}$-inner products, and pseudo-hermiticity. In the second chapter, we presented the Lewis-Riesenfeld invariant method for solving the Schrödinger equation for the explicitly time-dependent Hermitian and non-hermitian Hamiltonians.

In the last chapter, we have chosen a unitary transformation $F(t)$ that reduces the non-hermitian Hamiltonian $H(t)$ to a time-independent $\mathcal{PT}$-symmetric one $\mathcal{H}_0^{\mathcal{PT}}$, and thus the analytical solution of the Schrödinger equation of the initial system is easily obtained. Then, we defined a new $\mathcal{C}(t)\mathcal{PT}$-inner product and showed that the evolution preserves it, where $\mathcal{C}(t) = F^{+}(t)\mathcal{C}F(t)$. Moreover, we proved that the expectation value of a time-dependent non-Hermitian Hamiltonian $H(t)$ is real in the $\mathcal{C}(t)\mathcal{PT}$-normed states since the transformation $F(t)$ is unitary and $[\mathcal{P}, F(t)] = 0$, i.e.

$$\langle\chi(t)|\,\mathcal{CP}\left[FH(t)F^{+}\right]|\chi(t)\rangle = \left\langle\mathcal{H}_0^{\mathcal{PT}}\right\rangle_{\mathcal{CPT}} + \langle\chi(t)|\,\mathcal{CP}\left[iF\frac{\partial F^{+}}{\partial t}\right]|\chi(t)\rangle\,, \tag{3.84}$$

where $|\chi(t)\rangle$ is an eigenstate of $\mathcal{H}_0^{\mathcal{PT}}$.

As an illustration, we have studied a specific class of quantum time-dependent mass oscillators with a complex linear driving force. The expectation value of the Hamiltonian, the uncertainty relation and the probability density have also been calculated. The results of this chapter constitute the main results of this thesis.

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APPENDIX

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PAPER

A Real Expectation Value of the Time-dependent Non-Hermitian Hamiltonians*

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Keywords: Time-dependent Hamiltonians, non-Hermitian Hamiltonians, PT-symmetry, CPT-inner product, Time-dependent unitary transformation

Abstract

With the aim to solve the time-dependent Schrödinger equation associated to a time-dependent non-Hermitian Hamiltonian, we introduce a unitary transformation that maps the Hamiltonian to a time-independent $\mathcal{PT}$-symmetric one. Consequently, the solution of time-dependent Schrödinger equation becomes easily deduced and the evolution preserves the $\mathcal{C}(t)PT-$inner product, where $\mathcal{C}(t)$ is a obtained from the charge conjugation operator $\mathcal{C}$ through a time dependent unitary transformation. Moreover, the expectation value of the non-Hermitian Hamiltonian in the $\mathcal{C}(t)PT$ normed states is guaranteed to be real. As an illustration, we present a specific quantum system given by a quantum oscillator with time-dependent mass subjected to a driving linear complex time-dependent potential.

1. Introduction

It is commonly believed that the Hamiltonian must be Hermitian $H = H^{+}$ in order to ensure that the energy spectrum (the eigenvalues of the Hamiltonian) is real and that the time evolution of the theory is unitary (probability is conserved in time), where the symbol '$+$' denotes the usual Dirac hermitian conjugation; that is, transpose and complex conjugate. In 1998 this false impression has been challenged by Bender and Boettcher [1] who showed numerically that a few one-dimensional quantum potentials $V(x)$ may generate bound states $\psi(x)$ with real energies E even when the potentials themselves are not real. They show that because $\mathcal{PT}$-symmetry is an alternative condition to Hermiticity. The central idea of $\mathcal{PT}$-symmetric quantum theory is to replace the condition that the Hamiltonian of a quantum theory be Hermitian with the weaker condition: the invariance by space-time reflection. This allows one to construct and study many new Hamiltonians that would previously have been ignored.

These two important discrete symmetry operators are parity $\mathcal{P}$ and time reversal $\mathcal{T}$. The operators $\mathcal{P}$ and $\mathcal{T}$ are defined by their effects on the dynamical variables x and p. The operator $\mathcal{P}$ is linear and has the effect of changing the sign of the momentum operator p and the position operator $x: p \to -p$ and $x \to -x$. The operator $\mathcal{T}$ is antilinear and has the effect $p \to -p$, $x \to x$, and $i \to -i$. It is crucial, of course, that when replacing the condition of Hermiticity by $\mathcal{PT}$-symmetry, we preserve the key physical properties that a quantum theory must have. We see that if the $\mathcal{PT}$- symmetry of the Hamiltonian is not broken, then the Hamiltonian exhibits all of the features of a quantum theory described by a Hermitian Hamiltonian.

In order to have a coherent and unitary theory, Bender *et al* [2] have defined the $\mathcal{PT}$ inner-product associated to $\mathcal{PT}$-symmetric Hamiltonians as follows

* This paper is dedicated to the memory of our friends and colleagues Lahcene Alouani and Mebrouk Gherda died due to covid 19.

$$\langle f, g\rangle_{\mathcal{PT}} = \int_C dx[\mathcal{PT}f(x)]g(x), \tag{1}$$

where $\mathcal{PT}f(x) = f^*(-x)$. The advantage of this inner product is that the associated norm (f, f), which independent of the global phase of $f(x)$, is conserved in time. The application of this definition to the eigenfunctions of H and $\mathcal{PT}$ implies

$$\langle \psi_m, \psi_n\rangle_{\mathcal{PT}} = (-1)^n \delta_{mn}, \tag{2}$$

The situation here (that half of the the eigenfunctions of H and $\mathcal{PT}$ have positive norm and the other half have negative norm) is analogous to the problem that Dirac encountered in formulating the spinor wave equation in relativistic quantum theory. Following Dirac, Bender *et al* [2] constructed a linear operator denoted by $\mathcal{C}$ and represented in position space as a sum over the energy eigenstates of the Hamiltonian. The operator $\mathcal{C}$ is the observable that represents the measurement of the signature of the $\mathcal{PT}$ norm of a state. The properties of the new operator $\mathcal{C}$ resemble those of the charge conjugation operator in quantum field theory. Specifically, if the energy eigenstates satisfy (2), then we have $\mathcal{C}\psi_n = (-1)^n\psi_n$. $\mathcal{C}$, called the charge conjugation symmetry with eigenvalues ±1, $\mathcal{C}^2 = 1$, such that $\mathcal{C}$ commutes with the operator $\mathcal{PT}$ but not with the operators $\mathcal{P}$ and $\mathcal{T}$ separately, is the operator observable that represents the measurement of the signature of the $\mathcal{PT}$ norm of a state which determines its parity type. We can regard $\mathcal{C}$ as representing the operator that determines the $\mathcal{C}$ charge of the state. Quantum states having opposite $\mathcal{C}$ charge possess opposite parity type.

The introduction of operator $\mathcal{C}$ permits to formulate a positive $\mathcal{CPT}$ inner-product

$$\langle f, g\rangle_{\mathcal{CPT}} = \int_C dx[\mathcal{CPT}f(x)]g(x), \tag{3}$$

thus equation (2) becomes

$$\langle \psi_m, \psi_n\rangle_{\mathcal{CPT}} = \delta_{mn}. \tag{4}$$

The non-Hermitian $\mathcal{PT}$-symmetric models have been successfully used for describing several physical systems such the plasmons in nanoparticle systems [3], the problems related to the quantum information theory [4], nonclassical light [5] and the stability of hydrogen molecules [6].

The generalization to time-dependent non-Hermitian case have been studied in [7–29]. Note that the authors of [30] emphasize that in nonrelativistic quantum mechanics and in relativistic quantum field theory, the time coordinate t is a parameter and thus the time-reversal operator $\mathcal{T}$ does not actually reverse the sign of t. Some authors adopt the fact that the operator $\mathcal{T}$ changes also the sign of time $t \to -t$ [7, 31–40], this case could lead sometimes to incorrect results.

In this work, we adopt the following strategy: we introduce a unitary transformation $F(t)$ which commutes with the parity $\mathcal{P}$ and maps the solution $|\psi(t)\rangle$ of the time-dependent Schrödinger equation involving a non-Hermitian Hamiltonian $H(t)$ to the solution $|\chi(t)\rangle$ involving a non Hermitian Hamiltonian $\mathcal{H}$ required to be time-independent and $\mathcal{PT}$-symmetric. After performing this transformation, the problem becomes exactly solvable and the evolution preserve the $\mathcal{CPT}$-scalar product $\langle\chi(t)|\ \chi(t)\rangle_{\mathcal{CPT}} = \langle\chi(t)|\ \mathcal{CP}|\chi(t)\rangle$. The other essential ingredient of this theory is the construction of a positive-definite inner product with respect to $H(t)$ being non self-adjoint, so that its time-evolution operator is unitary and we obtain a consistent probabilistic interpretation so that the Hamiltonian under study exhibits real mean values. The most important step towards finding this positive-definite inner product is thus to find a new operator, which we call $\mathcal{C}(t) = F^+(t)\mathcal{C}F(t)$ such that, we obtain a conserved norm for our original system described by the solution $|\psi(t)\rangle$ that is $\langle\psi(t)|\psi(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle\chi(t)|\chi(t)\rangle_{\mathcal{CPT}}$, and the mean value of the time-dependent non-Hermitian Hamiltonian $H(t)$ is real in the new $\mathcal{C}(t)PT-$inner product. This is the main result of this paper.

For this we introduce, in section 2, a formalism based on the time-dependent unitary transformations is given in order to prove that the expectation value of the time-dependent non-Hermitian Hamiltonian $H(t)$ is real in the new $\mathcal{C}(t)PT-$inner product. In section 3, we illustrate our formalism introduced in the previous section by treating a non-Hermitian time-dependent quantum oscillator with time-dependent mass in linear complex time-dependent potential. On the hand, in the Hermitian case the time-dependent quantum harmonic have been extensively studied in the literature in different ways [41–48]. Finally, section 4 concludes our work.

2. Mean value of non-Hermitian time-dependent Hamiltonian

Let us consider a non-hermitian time-dependent Hamiltonian $H(t)$ where the quantum time evolution of the system is governed by the time-dependent Schrödinger equation (for simplicity we take $\hbar = 1$)

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{5}$$

In order to study the evolution of the quantum systems associated to the time-dependent Hamiltonian $H(t)$, we seek that this Hamiltonian can be converted into a time-independent Hamiltonian by some time-dependent transformations. To this end, we initially perform a unitary transformation $F(t)$ on $|\psi(t)\rangle$

$$|\chi(t)\rangle = F(t)|\psi(t)\rangle, \tag{6}$$

by inserting (6) in equation (5), we obtain the time dependent Schrödinger equation for the state $|\chi(t)\rangle$

$$i\frac{\partial}{\partial t}|\chi(t)\rangle = \mathcal{H}|\chi(t)\rangle, \tag{7}$$

such that the new Hamiltonian

$$\mathcal{H} = F(t)H(t)F^{+}(t) - iF(t)\frac{\partial F^{+}(t)}{\partial t}, \tag{8}$$

is time-independent and $\mathcal{PT}$-symmetric, i.e.;

$$\mathcal{H} \equiv \mathcal{H}_0^{\mathcal{PT}}, \tag{9}$$

its eigenstates $|\chi(t)\rangle$ preserve the $\mathcal{CPT}$-inner product

$$\langle\chi(t)|\ \chi(t)\rangle_{\mathcal{CPT}} = \langle\chi(t)|\ \mathcal{CP}|\chi(t)\rangle, \tag{10}$$

and in this case the solution of the Schrödinger equation (7) can be written as

$$|\chi(t)\rangle = \exp(-iEt)|\chi\rangle. \tag{11}$$

where $|\chi\rangle$ is an eigenstate of $\mathcal{H}_0^{\mathcal{PT}}$.

Knowing that our interest is the mean value of the non-Hermitian Hamiltonian $H(t)$, for this aim we calculate firstly the expectation value of the Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$

$$\langle\mathcal{H}_0^{\mathcal{PT}}\rangle_{\mathcal{CPT}} = \langle\chi(t)|\ \mathcal{CP}\mathcal{H}_0^{\mathcal{PT}}|\chi(t)\rangle = \langle\chi(t)|\ \mathcal{CP}\left[FH(t)F^{+} - iF\frac{\partial F^{+}}{\partial t}\right]|\chi(t)\rangle, \tag{12}$$

from which we deduce that is

$$\langle\chi(t)|\ \mathcal{CP}[FH(t)F^{+}]|\chi(t)\rangle = \langle\mathcal{H}_0^{\mathcal{PT}}\rangle_{\mathcal{CPT}} + \langle\chi(t)|\ \mathcal{CP}\left[iF\frac{\partial F^{+}}{\partial t}\right]|\chi(t)\rangle, \tag{13}$$

we note that the first term is nothing other than the expectation value of the Hamiltonian $H(t)$ with a new $\mathcal{C}(t)\mathcal{PT}$-inner product

$$\langle\chi(t)|\ \mathcal{CP}[FH(t)F^{+}]|\chi(t)\rangle = \langle\psi(t)|\ \mathcal{C}(t)\mathcal{P}H(t)|\psi(t)\rangle = \langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}}, \tag{14}$$

where $[\mathcal{P},\ F(t)] = 0$ and the new operator $C(t)$ is defined as $C(t) = F^{+}(t)CF(t)$, which is similar to the operator C in the sense that verifies the property $\mathcal{C}^2(t) = 1$ since $\mathcal{C}^2 = 1$.

Finally,

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle\mathcal{H}_0^{\mathcal{PT}}\rangle_{\mathcal{CPT}} + \langle\chi(t)|\ \mathcal{CP}\left[iF\frac{\partial F^{+}}{\partial t}\right]|\chi(t)\rangle. \tag{15}$$

Indeed, since $\mathcal{H}_0^{\mathcal{PT}}$ is $\mathcal{PT}$ symmetric and F is unitary, the expectation value $\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}}$ is guaranteed to be real. To our knowledge, this general result is new for explicitly time-dependent non-Hermitian systems.

3. Application: non-Hermitian time-dependent mass forced oscillators

Let us consider a class of one dimensional time-dependent harmonic oscillators with variable mass $m(t) = m_0\alpha(t)$ subjected to a driving linear complex time-dependent potential, in the form $i\lambda(t)x$, described by the following non-Hermitian Hamiltonian

$$H(t) = \frac{p^2}{2m_0\alpha(t)} + \alpha(t)\frac{m_0\omega^2(t)}{2}x^2 + ix\sqrt{\alpha(t)}, \tag{16}$$

where $\alpha(t)$ is a positive real time-dependent function, x and p are the canonical conjugates position and momentum operators satisfying $[x,\ p] = i$. The function $\lambda(t)$ in the complex potential has been choosen $\lambda(t) = \sqrt{\alpha(t)}$ in order to obtain in equation (8) a time-independent $\mathcal{PT}$-symmetric Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$. Without loss of generalities, we choose $\omega(t) = \omega$ as a constant. The mass m_0 and the frequency ω are the characteristic parameters of the quantum system.

We show that the exact solution of the time-dependent Schrödinger equation (5) can be found by introducing two consecutive unitary transformations. In order to solve the Schrödinger with the Hamiltonian specified by (16), we first try to eliminate the time-dependent parameter $\alpha(t)$.This can be achieved by the transformation

$$F_1(t) = \exp\left[-\frac{i}{2}\{x, p\}\ln(\sqrt{\alpha(t)})\right] \tag{17}$$

The unitary operator $F_1(t)$ has the properties

$$F_1 x F_1^+ = \frac{x}{\sqrt{\alpha(t)}}, \quad F_1 p F_1^+ = p\sqrt{\alpha(t)}, \tag{18}$$

In a representation x, the wave function is given by

$$\langle x|\ F_1|\phi\rangle = \alpha^{-\frac{1}{2}}\phi(x\alpha^{-\frac{1}{2}}). \tag{19}$$

Suppose that

$$|\phi(t)\rangle = F_1(t)|\psi(t)\rangle, \tag{20}$$

Substituting (20) into (5) ruled by the Hamiltonian (16), we find the equation of motion for $|\phi(t)\rangle$

$$i\frac{\partial}{\partial t}|\phi(t)\rangle = H_1(t)|\phi(t)\rangle, \tag{21}$$

where the Hamiltonian

$$H_1(t) = F_1(t)H(t)F_1^+(t) - iF_1(t)\frac{\partial F_1^+(t)}{\partial t} \tag{22}$$

$$= \frac{p^2}{2m_0} + \frac{m_0\omega^2}{2}x^2 + ix + \frac{1}{4}\frac{\dot{\alpha}(t)}{\alpha(t)}(xp + px) \tag{23}$$

look like the time-independent harmonic oscillators with variable mass m_0 subjected to a driving linear complex time-independent potential plus a time dependent $(xp + px)$ terms. In order to obtain the usual time-dependent harmonic oscillator with a perturbative linear potential, we remove the cross term in (23) via the transformation

$$F_2(t) = \exp\left[i\frac{m_0\dot{\alpha}(t)}{4\alpha(t)}x^2\right], \tag{24}$$

where its properties are

$$F_2 x F_2^+ = x, \quad F_2 p F_2^+ = -\frac{m_0\dot{\alpha}(t)}{2\alpha(t)}x, \tag{25}$$

Thus, the following unitary transformation $F(t) = F_2(t)F_1(t)$

$$F(t) = \exp\left[i\frac{m_0\dot{\alpha}(t)}{4\alpha(t)}x^2\right]\exp\left[-\frac{i}{2}\{x, p\}\ln(\sqrt{\alpha(t)})\right], \tag{26}$$

transforms the canonical operators x and p and their squares x^2 and p^2 as follows

$$FxF^+ = \frac{x}{\sqrt{\alpha(t)}}, \quad FpF^+ = p\sqrt{\alpha(t)} - \frac{m_0\dot{\alpha}(t)}{2\sqrt{\alpha(t)}}x,$$

$$Fp^2F^+ = \alpha(t)x^2 - \frac{1}{2}m_0\dot{\alpha}(t)\{x, p\} + \frac{m_0^2\dot{\alpha}^2(t)}{4\alpha(t)}x^2, \quad Fx^2F^+ = \frac{x^2}{\alpha(t)}, \tag{27}$$

therefore, the transformed Hamiltonian (8) reads

$$\mathcal{H} = \frac{p^2}{2m_0} + \frac{1}{2}m_0\Omega^2x^2 + ix. \tag{28}$$

where

$$\Omega^2 = \left(\omega^2 + \frac{1}{4}\frac{\dot{\alpha}^2(t)}{\alpha^2(t)} - \frac{\ddot{\alpha}(t)}{2\alpha(t)}\right) \tag{29}$$

The central idea in this procedure is to require that the Hamiltonian (28) governing the evolution of $|\chi(t)\rangle$ is time-independent. This is achieved by setting the global time-dependent frequency appearing in (28) equal to a real constant denoted by Ω_0^2 so that its time-derivatve leads to an auxiliary equation of the form

$$\ddot{\alpha} - \frac{\dot{\alpha}^2}{2\alpha} + 2\alpha(\Omega_0^2 - \omega^2) = 0, \tag{30}$$

the resulting time independent non-Hermitian Hamiltonian

$$\mathcal{H}_0^{\mathcal{PT}} = \frac{p^2}{2m_0} + \frac{1}{2}m_0\Omega_0^2 x^2 + ix, \tag{31}$$

is $\mathcal{PT}$-symmetric.

Note that when taking $\alpha(t) = \frac{1}{\rho^2(t)}$, the above auxiliary equation (30) is transformed to the following new auxiliary equation

$$\ddot{\rho} + (\Omega_0^2 - \omega^2)\rho = 0. \tag{32}$$

which admits the following solutions:

- for $\Omega_0^2 > \omega^2$: $\rho(t) = A\exp(it\sqrt{\Omega_0^2 - \omega^2}) + B\exp(-it\sqrt{\Omega_0^2 - \omega^2})$. For an appropriate choice of the constants: $A = B$, we obtain the expression of $\alpha(t)$ as $\alpha(t) = \frac{1}{A^2\cos^2(t\sqrt{\Omega_0^2 - \omega^2})}$.
- for $\Omega_0^2 < \omega^2$: $\rho(t) = A\exp(t\sqrt{\omega^2 - \Omega_0^2}) + B\exp(-t\sqrt{\omega^2 - \Omega_0^2})$. For an appropriate choice of the constants: $A = B$, we obtain the expression of $\alpha(t)$ as $\alpha(t) = \frac{1}{A^2\cosh^2(t\sqrt{\omega^2 - \Omega_0^2})}$, and when $B = 0$ and $A \neq 0$ the expression of $\alpha(t)$ is $\alpha(t) = \frac{1}{A^2}\exp(-2t\sqrt{\omega^2 - \Omega_0^2})$ and the Hamiltonian $H(t)$ corresponds to the Caldirola-Kanai oscillator [41, 42].

3.1. Analysis of the expectation value of the Hamiltonian

The eigenequation of the $\mathcal{PT}$-symmetric Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$ has the form

$$\mathcal{H}_0^{\mathcal{PT}}|\chi_n(x)\rangle = E_n|\chi_n(x)\rangle, \tag{33}$$

and the solution of the corresponding Schrödinger equation (7) can be written as

$$|\chi_n(x,\ t)\rangle = \exp(-iE_n t)|\chi_n(x)\rangle. \tag{34}$$

Let us introduce a non unitary transformation of the form

$$U = \exp\left[-\frac{p}{m_0\Omega_0^2}\right], \tag{35}$$

such that

$$|\chi_n(x)\rangle = U|\varphi_n(x)\rangle. \tag{36}$$

The action of U maps the $\mathcal{PT}$-symmetric Hamiltonian $H_0^{\mathcal{PT}}$ to a Hermitian one as

$$h = U^{-1}\mathcal{H}_0^{\mathcal{PT}}U = \frac{p^2}{2m_0} + \frac{m_0\Omega_0^2}{2}x^2 - \frac{1}{2m_0\Omega_0^2}, \tag{37}$$

where the eigenfunctions $|\varphi_n(x)\rangle$ of the Hermitian Hamiltonian h are

$$|\varphi_n(x)\rangle = \left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2}\exp\left(-\frac{m_0\Omega_0}{2\hbar}x^2\right)H_n\left[x\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2}\right]. \tag{38}$$

Then, the solutions $|\chi_n(x,\ t)\rangle$ are obtained as

$$\begin{aligned} |\chi_n(x,\ t)\rangle &= \exp(-iE_n t)U|\varphi_n(x)\rangle,\\ |\chi_n(x,\ t)\rangle &= \left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2}\exp(-iE_n t)\exp\left[-\frac{p}{m_0\Omega_0^2}\right]\exp\left(-\frac{m_0\Omega_0}{2\hbar}x^2\right)H_n\left[\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2}x\right], \end{aligned} \tag{39}$$

where the eigenvalues

$$E_n = \hbar\Omega_0\left(n + \frac{1}{2}\right) - \frac{1}{2m_0\Omega_0^2}, \tag{40}$$

are real and H_n is the Hermite polynomial of order n.

A more general way to represent the $\mathcal{C}$ operator is to express it generically in terms of the fundamental dynamical operators x and p: $\mathcal{C} = e^{Q(x,p)}$. The exact formula of $\mathcal{C}$ associated to the theory described by the Hamiltonian (31) is given as a function of the parity operator $\mathcal{P}$ as

$$\mathcal{C} = \exp\left[\frac{2}{m_0\Omega_0^2}p\right]\mathcal{P}, \tag{41}$$

such that the operator $\mathcal{C}$ commute with $\mathcal{PT}$ and $\mathcal{H}_0^{\mathcal{PT}}$, i.e., $[\mathcal{C}, \mathcal{PT}] = [\mathcal{C}, \mathcal{H}_0^{\mathcal{PT}}] = 0$.

We can easily verify that the $\mathcal{CPT}$-inner product is conserved

$$\langle \chi_n(x, t)\, |\chi_n(x, t)\rangle_{\mathcal{CPT}} = \langle \chi_n(x)|\, \mathcal{CP}|\chi_n(x)\rangle = \langle \varphi_n|\, U\mathcal{CP}U|\varphi_n\rangle = \langle \varphi_n(x)|\varphi_n(x)\rangle = 1. \tag{42}$$

Now it is not difficult to calculate the expectation value of the Hamiltonian $\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}}$ defined previously

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = E_n - \frac{\dot{\alpha}(t)}{4\alpha(t)}\langle \varphi_n(x)|\, U^{-1}\{x, p\}U|\varphi_n(x)\rangle + \left(\frac{m_0\ddot{\alpha}(t)}{4\alpha(t)}\right)\langle \chi_n(x)|\, \mathcal{CP}x^2|\chi_n(x)\rangle, \tag{43}$$

thus

$$\begin{aligned}\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = E_n - \frac{\dot{\alpha}(t)}{4\alpha(t)}\langle \varphi_n(x)|\{x, p\}|\varphi_n(x)\rangle \\ + \frac{\dot{\alpha}(t)}{2\alpha(t)}\frac{i}{m_0\Omega_0^2}\langle \varphi_n(x)|\, p|\varphi_n(x)\rangle + \left(\frac{m_0\ddot{\alpha}(t)}{4}\right)\langle x^2\rangle_{\mathcal{CPT}},\end{aligned} \tag{44}$$

where $\langle x^2\rangle_{\mathcal{CPT}} = \langle \chi_n(x)|\, \mathcal{CP}x^2|\chi_n(x)\rangle$. By using the following relations

$$\langle \varphi_n(x)|\, x|\varphi_n(x)\rangle = \langle \varphi_n(x)|\, p|\varphi_n(x)\rangle = 0, \tag{45}$$

$$\langle \varphi_n(x)|\, x^2|\varphi_n(x)\rangle = \frac{\hbar}{m_0\Omega_0}\left(n + \frac{1}{2}\right), \tag{46}$$

$$\langle \varphi_n(x)|\, p^2|\varphi_n(x)\rangle = m_0\Omega_0\hbar\left(n + \frac{1}{2}\right), \tag{47}$$

$$\langle \varphi_n(x)|\{x, p\}|\varphi_n(x)\rangle = 0, \tag{48}$$

and

$$\langle x^2\rangle_{\mathcal{CPT}} = \frac{\hbar}{m_0\Omega}\left(n + \frac{1}{2}\right) - \frac{1}{(m_0\Omega_0^2)^2}, \tag{49}$$

we get the expectation value of $H(t)$ as

$$\langle H(t)\rangle_{\mathcal{C}(t)\mathcal{PT}} = E_n + \left(\frac{m_0\ddot{\alpha}(t)}{4\alpha(t)}\right)\langle x^2\rangle_{\mathcal{CPT}} = E_n + \frac{\ddot{\alpha}(t)}{4\alpha(t)}\left[\frac{\hbar}{\Omega_0}\left(n + \frac{1}{2}\right) - \frac{1}{m_0\Omega_0^4}\right], \tag{50}$$

which is real for any positive real time-dependent function $\alpha(t)$ and more simple than the result given in equation (28) in [7] with less constraints on the parameters of the problem.

3.2. Uncertainty relation and probability density

Now, we calculate the expectation values $\langle x\rangle_{\mathcal{C}(t)\mathcal{PT}}$, $\langle x^2\rangle_{\mathcal{C}(t)\mathcal{PT}}$, $\langle p\rangle_{\mathcal{C}(t)\mathcal{PT}}$ and $\langle p^2\rangle_{\mathcal{C}(t)\mathcal{PT}}$ in the states $\psi_n(x, t)$ of $H(t)$ defined in equation (16). In the same way, using the $\mathcal{CPT}$-inner product (42) and after straightforward calculation we obtain that

$$\langle x\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle \psi_n(x, t)|\, F^{+}\mathcal{CP}Fx|\psi_n(x, t)\rangle = -\frac{i}{m_0\Omega_0^2}\frac{1}{\sqrt{\alpha(t)}}, \tag{51}$$

$$\langle x^2\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle \psi_n(x, t)|\, F^{+}\mathcal{CP}Fx^2|\psi_n(x, t)\rangle = \left(n + \frac{1}{2}\right)\frac{\hbar}{m_0\Omega_0\alpha(t)} - \frac{1}{\alpha(t)(m_0\Omega_0^2)^2}, \tag{52}$$

$$\langle p\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle \psi_n(x, t)|\, F^{+}\mathcal{CP}Fp|\psi_n(x, t)\rangle = \frac{i}{2\Omega_0^2}\frac{\dot{\alpha}(t)}{\sqrt{\alpha(t)}}, \tag{53}$$

$$\begin{aligned}&\langle p^2\rangle_{\mathcal{C}(t)\mathcal{PT}} = \langle \psi_n(x, t)|\, F^{+}\mathcal{CP}Fp^2|\psi_n(x, t)\rangle,\\ &\langle p^2\rangle_{\mathcal{C}(t)\mathcal{PT}} = \hbar\Omega_0\left(n + \frac{1}{2}\right)m_0\alpha(t) + \left(\frac{m_0\dot{\alpha}(t)}{2}\right)^2\left[\left(n + \frac{1}{2}\right)\frac{\hbar}{m_0\Omega_0\alpha(t)} - \frac{1}{\alpha(t)(m_0\Omega_0^2)^2}\right].\end{aligned} \tag{54}$$

We calculate also the position and momentum uncertainties

$$\Delta x = \sqrt{\langle x^2\rangle_{\mathcal{C}(t)\mathcal{PT}} - \langle x\rangle_{\mathcal{C}(t)\mathcal{PT}}^2} = \left[\frac{\hbar}{m_0\Omega_0\alpha(t)}\left(n + \frac{1}{2}\right)\right]^{1/2}, \tag{55}$$

$$\Delta p = \sqrt{\langle p^2\rangle_{\mathcal{C}(t)\mathcal{PT}} - \langle p\rangle^2_{\mathcal{C}(t)\mathcal{PT}}} = \frac{1}{\Delta x}\left[\left(n+\frac{1}{2}\right)^2 + \left(\frac{m_0\dot{\alpha}(t)}{2}\right)^2\Delta x^4\right]^{1/2}. \tag{56}$$

Thus, the uncertainty product is given by

$$\Delta x\Delta p = \left(n+\frac{1}{2}\right)\sqrt{1+\left(\frac{\hbar\dot{\alpha}(t)}{2\Omega_0\alpha(t)}\right)^2}, \tag{57}$$

it is easy to check that the uncertainty product (57) is always real and greater than or equal to $\frac{1}{2}$ and, consequently, it is physically acceptable for any value of n. The uncertainty product takes the minimal value $\Delta x\Delta p = \frac{1}{2}$ only for $n = 0$ and $\alpha(t)$= constant, i.e., for time independent mass oscillators.

Finally, the probability density of the wavefunction $\psi_n(x, t)$ of $H(t)$ is in the form

$$|U^{-1}F\psi_n(x, t)|^2 = |U^{-1}\chi_n(x, t)|^2 = |\varphi(x)|^2 = \varphi_n^*(x)\varphi_n(x), \tag{58}$$

thus

$$|U^{-1}F\psi_n(x, t)|^2 = \left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]\exp\left(-\frac{m_0\Omega_0}{\hbar}x^2\right)\left(H_n\left[\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2}x\right]\right)^2, \tag{59}$$

is the same as the probability density of the eigenstate $\chi_n(x, t)$ of time independent $\mathcal{H}_0^{\mathcal{PT}}$ which is also equal to the probability density of the eigenstate $\varphi_n(x)$ (38) of the standard harmonic oscillator (37). Clearly, $\varphi_n(x)$ are elements from $L^2(R)$, and therefore the condition (59) yields that

$$\int|\varphi_n(x)|^2dx = \left[\frac{\sqrt{m_0\Omega_0}}{n!2^n\sqrt{\pi\hbar}}\right]^{1/2}\int\exp\left(-\frac{m_0\Omega_0}{\hbar}x^2\right)\left(H_n\left[x\left(\frac{m_0\Omega_0}{\hbar}\right)^{1/2}\right]\right)^2dx = 1 \tag{60}$$

under this observation, we deduce that the probability is finite.

4. Conclusion

The essential ingredient of quantum mechanical non Hermitian theory is the construction of a positive-definite inner product, so that its probability is conserved in time. The operator $\mathcal{C}(t) = F^{+}(t)\mathcal{C}F(t)$ confer to the norm its conservation. The main result of this paper is that the mean value of a time-dependent non-Hermitian Hamiltonian $H(t)$ is real in the new $\mathcal{C}(t)PT-$inner product. For this, we introduced a unitary transformation $F(t)$ that reduces the study of time-dependent non-Hermitian Hamiltonian $H(t)$ to the study of time-independent $\mathcal{PT}$-symmetric Hamiltonian $\mathcal{H}_0^{\mathcal{PT}}$, and derived the analytical solution of the Schrödinger equation of the initial system. Then, we defined a new $\mathcal{C}(t)PT$-inner product and showed that the evolution preserves it. Furthermore, we proved that the expectation value of the time-dependent non-Hermitian Hamiltonian $H(t)$ is real in the $\mathcal{C}(t)PT$ normed states since the transformation $F(t)$ is unitary and $[\mathcal{P}, F(t)] = 0$. As an illustration, we have investigated a class of quantum time-dependent mass oscillators with a complex linear driving force.
The expectation value of the Hamiltonian, the uncertainty relation and probability density have been also calculated.

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Data availability statement

No new data were created or analysed in this study.

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Abstract :

In this thesis, we have studied the analytical solutions of the Schrödinger equation for a class of explicit time dependent non-Hermitian systems. In the first chapter we introduced the basic concepts used in quantum theory for non-Hermitian systems, such *PT*-symmetry, *PT* and *CPT*-inner-products and pseudo-Hermiticity. The second chapter is dedicated to the Lewis-Riesenfeld invariant method for solving the Schrödinger equation for explicitly time dependent Hermitian and non-Hermitian Hamiltonians. In the last chapter, we have used a unitary transformation *F(t)* that reduces the non-Hermitian Hamiltonian *H(t)* to a time-independent PT-symmetric one, and thus the analytical solution of the Schrödinger equation of the initial system is easily obtained. Then, we defined a new C(t)PT-inner product and showed that the evolution preserves it, where $C(t)=F^{+}(t)CF(t)$. Moreover, we proved that the expectation value of the time-dependent non-Hermitian Hamiltonian H(t) is real in the C(t)PT-normed states since the transformation F(t) is unitary and *[P,F(t)]=0*. As an illustration, we have study a class of quantum time-dependent mass oscillators with a complex linear driving force. The expectation value of the Hamiltonian, the uncertainty relation and the probability density have also been calculated. The results of this chapter constitute the main results of this thesis.

Résumé :

Dans cette thèse, nous avons étudié les solutions analytiques de l'équation de Schrödinger d'une classe de systèmes non-hermiticiens dépendant explicitement du temps. Dans le premier chapitre, on a introduit les concepts de base utilisés en théorie quantique des systèmes non hermitiens, tels que la *PT*-symétrie, les produits scalaires *PT* et *CPT* et la pseudo-herméticité. Le deuxième chapitre est dédié à la méthode des invariants de Lewis-Riesenfeld pour la résolution de l'équation de Schrödinger pour les Hamiltoniens Hermitiens et non-Hermitiens dépendant explicitement du temps. Dans le dernier chapitre, nous avons utilisé une transformation unitaire *F(t)* qui transforme l'hamiltonien non hermitien *H(t)* en un hamiltonien *PT*-symétrique indépendant du temps, et donc la solution analytique de l'équation de Schrödinger du système initial s'obtient facilement. Ensuite, nous avons défini un nouveau produit scalaire *C(t)PT* et montré que son évolution est conservée au cours du temps, où $C(t)=F^{+}(t)CF(t)$. De plus, nous avons prouvé que la valeur moyenne de l'hamiltonien non-hermiticien dépendant du temps H(t) est réelle dans les états normés *C(t)PT* puisque la transformation *F(t)* est unitaire et *[P,F(t)]=0*. A titre d'illustration, nous avons étudié une classe d'oscillateurs de masse dépendante du temps en présence d'un potentiel complexe et linéaire. La valeur moyenne de l'hamiltonien, la relation d'incertitude et la densité de probabilité ont également été calculées. Les résultats de ce chapitre constituent les principaux résultats de cette thèse.

ملخص:

في هذه الأطروحة، درسنا حلول معادلة شرودنجر للجمل غير الهرميتية المتعلقة صراحة بالزمن. في الفصل الأول قدمنا المفاهيم الأساسية للنظرية الكمية للجمل غيرالهرميتية، مثل التناظرPT، الجداءات السلمية PT وCPT وكذلك الهرميتية الزائفة. في الفصل الثاني، قدمنا بإيجاز طريقة لويس و ريزنفيلد لحل معادلة شرودنجر المتعلقة صراحة بالزمن للجمل الهرميتية وغير الهرميتية. وفي الفصل الأخير، استخدمنا تحويل وحدوي $\mathrm{F(t)}$ للإنتقال من هاملتونيان غير هرميتي متعلق على الزمن H(t) إلى هاملتونيان مستقل عن الزمن وصامد بالنسبة للتناظرPT، وبالتالي يمكن الحصول على حل معادلة شرودنجر بسهولة. بعد ذلك، عرفنا جداء سلمي جديد-C(t)PT وبينا أن تطوره مع الزمن محفوظ، حيث $C\ (t)\ =\ F^{+}\ (t)\ CF(t)$. علاوة على ذلك، أثبتنا أن القيمة المتوسطة للهاملتونيان الغير هرميتي المتعلق بالزمنH(t) هي قيمة حقيقية في الحالات المقننة بالجداء السلمي-$\mathrm{C\ (t)\ PT}$ لأن التحويل$\mathrm{F(t)}$ وحدوي و $\mathrm{[F\ (t)،P]} = 0$. وكمثال تطبيقي للنتائج المحصل عليها، درسنا فئة من الهزازات التوافقية ذات كتلة متعلقة بالزمن مع وجود كمون خطي مركب. حيث تم حساب القيمة المتوسطة للهاملتونيان، علاقة الارتياب وكثافة الاحتمال. تمثل نتائج هذا الفصل النتائج الرئيسية لهذه الأطروحة.