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# Thesis

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**By**

## Boultif Yousra

**Theme**

# Slowly rotating Neutron stars in Teleparallel equivalent of general relativity

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**Before the jury:**

President : T.Boudjeda Prof. Jijel.Univ

Reporter : Kh. Nouicer Prof. Jijel.Univ

Examiner: N.Ferkous MCA. Jijel.Univ

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# Contents

# Chapter 1

# Introduction

After discovering that our universe was expanding faster and faster in 1998, our understanding of it has changed dramatically. This situation is entirely the opposite direction to what the researchers expected and who is responsible for it referred to as dark energy. It turns out that General Relativity (GR) can describe this by introducing a fluid with negative pressure and violation of energy conditions, effectively creating a repulsive force gravity. Due to the lack of theoretical motivation for this remaining the same, other methods have been developed to account for this late acceleration behavior of the universe. Take this approach believe that genetic resources can be expanded or modified and that these differences can be explained cosmological observations [1].

General Relativity is based on a special connection called Levi-Civita connection, it is symmetrical and twist-free. It is certainly not the most common or the most unique way to describe gravity. In this work we review another way of describing gravity called teleparallel gravity. The theory is based on Weitzenbock Compounds with no curvature with non-zero twist Tensor. This theory is equivalent to GR on the field equations, so it is known as the teleparallel equivalent of General Relativity (TEGR) [1].

Teleparallel gravity and its popular generalization $f(T)$-gravity can be formulated as completely invariant (under coordinate transformation and local Lorentz transformation) theory of gravity. There are some misunderstandings in the literature about teleparallel gravity and its generalizations, especially with regard to their local Lorentz invariance. Especially the center point of confusion seems to be related to inertial spin connections in parallel gravity in the literature. While inertial spin connections are common place, not something in special relativity inherent in teleparallel gravity, the role of inertial spin junctions in eliminating clutter inertia effects within a given frame of reference are emphasized here. Careful consideration of inertial spin connections leads to a completely invariant theory of teleparallel gravity and its generalizations. In fact, the nature of spin junctions distinguishes the relationship between so-called good and bad tetrads, clearly showing that in principle any tetrad can be used. The field equations and their generalizations for the completely invariant formulation of teleparallel gravity are given, and many examples with different frameworks and spin-connection assumptions are shown to illustrate the covariant approach. Various modified teleparallel gravity models are also briefly discussed [2].

Neutron stars are very dense and compact objects with a mass of about 1.5 $M_{\odot}$, but with radius 105 times smaller than the sun's radius. They are among the strangest objects in our universe and are a stellar laboratory for astrophysics and microphysics. In fact, their centers are so dense that atomic nuclei disappear into their cores. Their magnetic fields can be as high as 1015 G, and they are very compact and rapidly rotating objects. Their evolutionary dynamics requires an accurate description of their relativistic and microphysical properties. Therefore, they are of particular interest to astrophysicists and nuclear physicists [3].

This master thesis contains a description of neutrons stars in TEGR and it is structured as follows. The next section is devoted to a review of the general relativity theory. In chapter 3, we discuss the fundaments of the TEGR theory. Chapter 4 is then devoted to the theory of neutrons stars in GR and some of their properties. In chapter 5, we study the existence and the properties of neutrons stars in TEGR in the presence of a non minimally coupled field in an harmonic potential. A summary of our study in discussed in the conclusion.

# Chapter 2

# Introduction to general relativity

This chapter is essentially based on the book of L. Landau and E. Lifshitz [4].

## 2.1 Gravitational field in non-relativistic mechanics

Concurrently in the electromagnetic field, there are still gravitation fields in nature. Fields enjoy he following fundamental property : all bodies move in them , independently of their masses. Thus, the laws of free fall in the field of attraction of the earth are identical for bodies : whatever their masses, all acquire one and the same acceleration. It is the properties of motion in a non-inertial reference system are the same an inertial system gives in the presence of a gravitational field.

The motion of a particle in a gravitational field erected in tractive number mechanics by a lagrange function written in an inertial reference frame

$$L = \frac{mv^2}{2} - m\varphi \tag{2.1}$$

where $\varphi$ is some function of coordinates and time characterizing the field, called gravitation potential.

## 2.2 Gravitation field in relativistic mechanics

The fundamental property of gravitational fields which consists in the fact that all bodies move in them in the same way also remains a central concept in relativistic mechanics. In an inertial reference system referred to cartesian coordinates, the interval $ds$ is determined by the relation

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2. \tag{2.2}$$

when we pass to another reference inertial frame (i.e. under a the Lorentz transformation) we know that the expression of the interval $ds^2$ is not affected. But when we switch to a non-inertial reference system, the $ds^2$ is then no longer the sum of the squares of the differentials of the four coordinates. Thus, when we pass to an arbitrary coordinates system, for example, in uniform rotation

$$x = x' \cos \Omega t - y' \sin \Omega t, \quad y = x' \sin \Omega + y' \cos \Omega t, \quad z = z'$$

(where $\Omega$ is the angular velocity of rotation, directed along the z axis ) the interval takes the form

$$ds^2 = [c^2 - \Omega^2(x'^2 - y'^2)]dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y' dx' dt - 2\Omega x' dy' dt.$$

Whatever the law of transformation of time, this expression cannot be reduced to a sum of squares of the differentials of the four coordinates. Therefore, in a non-inertial frame the square of the interval is some general quadratic form of the differential of the coordinates, i.e.

$$ds^2 = g_{\alpha\beta} dx_\alpha dx_\beta \tag{2.3}$$

where the $g_{\alpha\beta}$, functions of the spatial coordinates $x_1, x_2, x_3$ and the temporal coordinates $x_0$, define the space-time metric ($g_{\alpha\beta} = g_{\beta\alpha}$), where $g_{\alpha\beta}$ and $g_{\beta\alpha}$ enter with the same factor $dx_\alpha dx_\beta$ . In an inertial reference frame, in cartesian spatial coordinates $x_{1,2,3} = x, y, z$ and temporal $x_0 = ct$, the $g_{\alpha\beta}$ are given by

$$g_{11} = g_{22} = g_{33} = 1, \quad g_{00} = -1, \quad g_{\alpha\beta} = 0, \quad \alpha \neq \beta. \tag{2.4}$$

## 2.3 Curvilinear coordinates

Consider a transformation from a coordinate system $x^0$, $x^1$, $x^2$, $x^3$ to another $x'^0$, $x'^1$, $x'^2$, $x'^3$:

$$x^i = f^i \left(x'^0, x'^1, x'^2, x'^3\right)$$

where the $f^i$ are certain functions. In a transformation of coordinates, the differentials are transformed according to the law

$$dx^i = \frac{\partial x^i}{\partial x'^k} dx'^k. \tag{2.5}$$

Any set of four quantities $A^i$ $(i = 0,\ 1, 2, 3)$ transforming in an arbitrary change of coordinates as their differential is called a contravariant four-vector. We thus have in coordinates transformation

$$A^i = \frac{\partial x^i}{\partial x'^k} A'^k. \tag{2.6}$$

Let $\varphi$ be a scalar. The four quantities $\frac{\partial \varphi}{\partial x^i}$ transform into a coordinate transformation according to the formulas

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial x'^k} \frac{\partial x'^k}{\partial x^i} \tag{2.7}$$

which is different from the last formula. Then any set of four quantities $A_i$ transforming under coordinate change as the derivative of a scalar is called a covariant four-vector. We thus have

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k. \tag{2.8}$$

As there exist two kinds of vector in curvilinear coordinates, we have in such a system three kinds of second order tensor. We call second-order contravariant tensor a set $A^{ik}$ of 16 quantities transforming like the products of the components of two contravariant vectors, that is to say according to the law

$$A^{ik} = \frac{\partial x^i}{\partial x'^l} \frac{\partial x^k}{\partial x'^m} A'^{lm}. \tag{2.9}$$

In the same way, a covariant tensor transforms according to the formulas

$$A_{ik} = \frac{\partial x'^l}{\partial x^i} \frac{\partial x'^m}{\partial x^k} A'_{lm}, \tag{2.10}$$

and a mixed tensor transforms as

$$A^i_k = \frac{\partial x^i}{\partial x'^l}\frac{\partial x'^m}{\partial x^k}A'^l_m. \tag{2.11}$$

We define in a completely analogous way the law of transformation of tensors of higher orders. For example, the tensor $A^m_{ikl}$ , covariant with respect to the lower 3 indices and contavariant with respect to one index transforms according to the formula

$$A^m_{ikl} = \frac{\partial x'^p}{\partial x^i}\frac{\partial x'^r}{\partial x^k}\frac{\partial x'^s}{\partial x^l}\frac{\partial x^m}{\partial x'^t}A'^t_{prs}. \tag{2.12}$$

### 2.3.1 Distance and time

We have already said that in general relativity the choice of the system of reference is not limited by anything, the three coordinates $x^1$, $x^2$, $x^3$ can be arbitrary quantities defining the position of bodies in space, and the time coordinate $x^0$ can be determined by a clock recording it's own time. Let's first determine the link between the real time, which we will denote below by $\tau$ and the $x^0$ coordinate. For this purpose, consider two infinitely close events taking place at one and the same point in space. Then, the interval $ds$ between these two events is nothing but $cd\tau$ where $d\tau$ is the (real) time interval between the two events. Setting $dx^1 = dx^2 = dx^3 = 0$ in the general expression $-ds^2 = g_{ik}dx^i dx^k$, we find

$$ds^2 = -c^2 d\tau^2 = g_{00}dx_0^2,$$

from which we get

$$d\tau = \frac{1}{c}\sqrt{-g_{00}}dx^0. \tag{2.13}$$

Again, for the time between two arbitrary events at one and the same point in space we obtain

$$\tau = \frac{1}{c}\int\sqrt{-g_{00}}dx^0. \tag{2.14}$$

This relation determines the real time (the proper time at a given point in space ) as a function of the coordinate $x^0$. We also note that the quantity $g_{00}$ is negative.

### 2.3.2 Covariant derivative

In cartesian coordinates the differentials $dA_i$ of a vector $A_i$ form a vector, and the partial derivatives $\frac{\partial A_i}{\partial x^i}$ of its components with respect to the coordinates a tensor. It is different in curvilinear coordinates : $dA_i$ is not longer a vector and $\frac{\partial A_i}{\partial x^k}$ a tensor. It is also easy to verify this directly, for this purpose, let us establish the formulas for transforming the differentials $dA_i$ into curvilinear coordinates. A covariant vector and it's differential transform according to the formulas

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_{k;}$$

and

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k d\frac{\partial x'^k}{\partial x^i} = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx'^l. \tag{2.15}$$

Therefore $dA_i$ does not transform like a vector, because of the second term. This is only in cases where the second derivatives cancel, $\frac{\partial^2 x' k}{\partial x^i \partial x^l} = 0$, that is, if the $x'^k$ are linear functions of the $x^k$(affine transformations). Then the transformation formulas have the form

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k, \tag{2.16}$$

and the $dA_i$'s transform as a vector.

When we compare two infinitely close vectors, we parallel transport one of them to the point where the other is. Let us consider an arbitrary contravariant vector, let $A^i$ be its components at the point of coordinates $x^i$ and $A^i + dA^i$ at the neighboring point $x^i + dx^i$ , and transport the vector $A^i$ in parallel to the infinitely neighboring point $x^i + dx^i$. Let $\delta A^i$ be its increase. Then the difference $DA^i$ between the two vectors, now found at the same point, is

$$DA^i = dA^i - \delta A^i \tag{2.17}$$

where is written as

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l \tag{2.18}$$

and the $\Gamma^i_{kl}$ are some function of the coordinates. In cartesian system all $\Gamma^i_{kl} = 0$ are zero. We can however choose a system of coordinates such that the $\Gamma^i_{kl}$ are canceled there in a point given in advance, the quantities $\Gamma^i_{kl}$ are called christoffel symbols. Subsequently, we will also have to use the quantities $\Gamma_{i,kl}$ , defined as follows

$$\Gamma_{i,kl} = g_{im}\Gamma^m_{kl}. \tag{2.19}$$

It is clear that conversely

$$\Gamma^i_{kl} = g^{im}\Gamma_{m,kl}. \tag{2.20}$$

Let $A_i$ and $B^i$ be covariant and contravariant vectors. From $\delta\left(A_iB^i\right) = 0$, we get

$$B^i\delta A_i = -A_i\delta B^i = \Gamma^i_{kl}B^kA_idx^l,$$

or, by changing the indices,

$$B^i\delta A_i = \Gamma^k_{il}A_kB^idx^l.$$

From where, the $B^i$ being arbitrary, we obtain

$$\delta A_i = \Gamma^k_{il}A_kdx^l. \tag{2.21}$$

Substituting (2.18) and $dA^i = \frac{\partial A^i}{\partial x^l}dx^l$ in (2.17), we find:

$$DA^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl}A^k\right)dx^l. \tag{2.22}$$

In an analogous way, we have for a covariant vector:

$$DA_i = \left(\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il}A_k\right)dx^l. \tag{2.23}$$

The expressions contained in the parentheses in (2.21) and (2.23) are tensors, since their products by the vector $dx^k$ yields a vector. It is obvious that they represent the tensors which in curvilinear coordinates play the role of the cartesian tensor $\frac{\partial A_i}{\partial x^k}$. These tensors are called the covariant derivatives of vectors $A^i_{;k}$ and $A_{i;k}$, respectively.

Therefore we write

$$DA^i = A^i_{;l}dx^l, \quad DA_i = A_{i;l}dx^l \tag{2.24}$$

where the covariant derivatives themselves being

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k, \tag{2.25}$$

$$A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k. \tag{2.26}$$

In cartesian coordinates the covariant derivatives coincide with the ordinary derivatives .

### 2.3.3 Relation between the christoffel symbols and the metric tensor

Let us first show that the covariant derivative of the mertic tensor $g_{\alpha\beta}$ is zero. In GR theory, this is called metric compatibility. Note for this purpose that we must have for the vector $DA_i$, as for any vector ($A_i = g_{ik}Ak$), the relation

$$DA_i = g_{ik} DA^k \tag{2.27}$$

so that

$$DA_i = D\left(g_{ik} A^k\right) = g_{ik} DA^k + A^k D g_{ik}. \tag{2.28}$$

Comparing with $DA_i = g_{ik} A^k$ , we get , given that the vector $A^i$ is arbitrary :

$$Dg_{ik} = 0. \tag{2.29}$$

It therefore follows that

$$g_{ik,l} = 0. \tag{2.30}$$

Therefore we must consider the $g_{ik}$ as a constant referred to in the covariant derivation . We can use the equality $g_{ik;l} = 0$ to express the christoffel symbols $\Gamma^i_{kl}$ by means of the tensor $g_{ik}$. Consequently, the derivatives of the $g_{ik}$ are expressed by means of the christoffel symbols. Let us write these derivatives, by circularly permuting the indices $i$, $k$, $l$ :

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,il} + \Gamma_{i,kl}, \tag{2.31}$$

$$\frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,kl} + \Gamma_{l,ik}, \tag{2.32}$$

$$-\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,ki} - \Gamma_{k,li}. \tag{2.33}$$

Taking the half-sum of these equalities, we find (remembering that $\Gamma_{ikl} = \Gamma_{ilk}$):

$$\Gamma_{ikl} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right). \tag{2.34}$$

From which we get for the $\Gamma^i_{kl} = g^{im}\Gamma_{m,kl}$:

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right). \tag{2.35}$$

Such are the formulas giving the desired expressions of the christoffel symbols as a function of the metric tensor .

Let us determine the differential $dg$ of the determinant $g$ formed with the components of the tensor $g_{ik}$. We can obtain $dg$ by taking the differential of each component of the tensor $g_{ik}$ and multiplying it by the corresponding minor. Moreover, the components of the tensor $g^{ik}$, the inverse of the tensor $g_{ik}$, are equal, as we know, to the quotients of the minors of the determinant formed with the quantities $g^{ik}$ by this determinant. The minors of the determinant $g$ are therefore $gg^{ik}$. Therefore,

$$dg = gg^{ik}dg_{ik} = -gg_{ik}dg^{ik} \tag{2.36}$$

where we have use the fact that $g_{ik}g^{ik} = \delta^i_i = 4 \Longrightarrow g^{ik}dg_{ik} = -g_{ik}dg^{ik}$. On the other hand we deduce that

$$\Gamma^i_{ki} = \frac{1}{2g}\frac{\partial g}{\partial x^k} = \frac{\partial \ln\sqrt{-g}}{\partial x^k}. \tag{2.37}$$

### 2.3.4 Rotations

A special case of stationary gravitational fields is the field which is generated by the transition to a reference system in uniform rotation. In order to determine the interval $ds$, let us pass from the stationary system to the uniformly rotating system $r'$, $\varphi'$, $z'$, $t$ (we are in cylindrical coordinates $r'$, $\varphi'$, $z'$ ) where the interval is written as

$$ds^2 = c^2dt^2 - dr'^2 - r'^2d\varphi'^2 - dz'^2. \tag{2.38}$$

Let $r$, $\varphi$, $z$ be the cylindrical coordinates in the rotating system. If the axis of rotation coincides with the axes $Z$ and $Z'$ , we have $r' = r$, $z' = z$, $\varphi' = \varphi + \Omega t$, where $\Omega$ is the angular velocity of the rotation. A simple calculation gives the expression for the interval in the rotation coordinate system:

$$ds^2 = \left(c^2 - \Omega^2 r^2\right) dt^2 - 2\Omega r^2 d\varphi dt - dz^2 - r^2 d\varphi^2 - dr^2. \tag{2.39}$$

## 2.4 The curvature tensor

If $x^i = x^i(s)$ are the parametric equation of a curve where $s$ is the infinitesimal arc measured from a given point, then the vector $u^i = \frac{dx^i}{ds}$ is the vector tangent to the curve. If the curve is a geodesic, we have $Du^i = 0$ along this curve, and this means that if we transport the vector $u^i$ in parallel from a point $x^i$ on the geodesic to another point $x^i + dx^i$ on the same geodesic , it coincides with the vector $u^i + du^i$ tangent to this line at point $x^i + dx^i$. Therefore, the parallel transport along a geodesic preserve the tangent vector to it. Moreover, the angle of two vectors is clearly invariant in their parallel transport. We can therefore affirm that during the transport of any vector along a geodesic, the projection of a vector on the tangent to a geodesic is invariant in the parallel transport along the geodesic. A fundamental fact is that, in a non-euclidean space, a vector parallel transported along a closed curve, it no longer coincides with the initial vector.

We express the general formula that determines the anisotropy of the vector during it's parallel transport along an infinitesimal closed contour $\Delta A_k$ in the form $\oint dA_k$, where the integral is taken over the given contour. By substituting for $dA_k$ its expression (2.18), we have

:

$$\Delta A_k = \oint \Gamma^i_{kl} A_i dx^l, \tag{2.40}$$

We can consider that the components of the vector $A_i$ at the points inside the infinitesimal contour are determined univocally by their values on the contour itself by virtue of the formulas $\delta A_i = \Gamma^n_{il} A_n dx^l$, that is , derivatives

$$\frac{\partial A_i}{\partial x^l} = \Gamma^n_{il} A_n. \tag{2.41}$$

Now applying Stokes theorem

$$\oint A_i dx_i = \int df_{ki} \frac{\partial A_i}{\partial x_k} = \frac{1}{2} \int df_{ik} \left( \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \right), \tag{2.42}$$

to the integral (2.40) and observing that the area bounded by the contour is an infinitesimal quantity $\Delta f^{im}$, we obtain:

$$\begin{aligned}
\Delta A_k &= \frac{1}{2} \left[ \frac{\partial \left( \Gamma^i_{km} A_i \right)}{\partial x^l} - \frac{\partial \left( \Gamma^i_{kl} A_i \right)}{\partial x^m} \right] \Delta f^{lm} \\
&= \frac{1}{2} \left[ \frac{\partial \Gamma^i_{km}}{\partial x^l} A_i - \frac{\partial \Gamma^i_{kl}}{\partial x^m} A_i + \Gamma^i_{km} \frac{\partial A_i}{\partial x^l} - \Gamma^i_{kl} \frac{\partial A_i}{\partial x^m} \right] \Delta f^{im}.
\end{aligned}$$

Substituting the derivatives deduced from (2.41) we obtain

$$\Delta A_k = \frac{1}{2} R^i_{klm} A_i \Delta f^{lm}, \tag{2.43}$$

where, $R^i_{klm}$ the fourth-order Riemann tensor given by

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}. \tag{2.44}$$

### 2.4.1 Properties of the curvature tensor

It follows immediately from expression (2.44) that the curvature tensor is antisymmetric with respect to the indices $l$ and $m$:

$$R^i_{klm} = -R^i_{kml}. \tag{2.45}$$

Moreover, we can easily verify that:

$$R^i_{klm} + R^i_{mkl} + R^i_{lmk} = 0. \tag{2.46}$$

We also use the covariant form given by

$$R_{iklm} = g_{in} R^n_{klm}. \tag{2.47}$$

After straightforward manipulations, we easily obtain the expression

$$R_{iklm} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l}\right) + g_{np}\left(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}\right). \tag{2.48}$$

We immediately deduce from this expression the following symmetry properties:

$$R_{iklm} = -R_{kilm}, \tag{2.49}$$

$$R_{iklm} = -R_{ikml}, \tag{2.50}$$

$$R_{iklm} = R_{lmik}. \tag{2.51}$$

It follows in particular that the components of $R_{iklm}$ such that $i = k$ or $l = m$ are zero. In the end, we have for $R_{iklm}$, as we dit it for $R^i_{klm}$, the identity (2.46) :

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0. \tag{2.52}$$

## 2.5 The action for the gravitational field

In order to find the equations that governs the evolution of the gravitational field, it is necessary to first determine the interaction of matter fields on the gravitational field. We then obtain the desired equation by varying the sum of the effects of the field and material particles. As usual this setup is build via the use of the action principle.

The action $S_g$ , must be expressed as a scalar integral, just like the effect of an electromagnetic field

$$\int G\sqrt{-g}d\Omega \tag{2.53}$$

We will then assume that the gravitational field equation cannot contain (potential) derivatives of fields of order greater than two (as is the case with the electromagnetic field equation). To this end, since the field equations are obtained by varying the action, the expressions under the integral must not contain derivatives of $g_{il}$ of order greater than one; so $G$ only needs to contain the tensor $g_{ik}$ and the symbols $\Gamma^i_{kl}$. However, we cannot with the only quantities $g_{ik}$ and $\Gamma^i_{kl}$ form a scalar. We already see it from the fact that the $\Gamma^i_{kl}$can be canceled at a point by a suitable choice of coordinates. There is however a scalar $R$ — the scalar curvature of 4-space time — which, it is true, contains only the tensor $g_{ik}$ and its first derivatives and second derivatives of $g_{ik}$,but linearly. By application of Gauss's theorem to an integral of an expression containing no more second derivatives, we can put it in the form

$$S_g \sim \int R\sqrt{-g}d\Omega = \int G\sqrt{-g}d\Omega + \int \frac{\partial\sqrt{-g}\omega^i}{\partial x^i}d\Omega, \tag{2.54}$$

where, $G$ only contains $g_{ik}$ and its first derivatives, and where the second term is of the divergences of a certain quantity $\omega^i$. By virtu of Gauss's theorem, one can transform this second integral into an integral over the hypersurface bounding the 4-volume. When the action is varied, the variation of the second right-hand integral disappears, because, according to the principle of least action, the variation of the field is zero on the integration boundary. So we are let with the term

$$\delta S_g \sim \delta\int R\sqrt{-g}d\Omega = \delta\int G\sqrt{-g}d\Omega.$$

The quantity $G$ satisfies the condition stated above, because it contains only the $g_{ik}$ and their derivatives. We can therefore write :

$$\delta S_g \sim \delta\int G\sqrt{-g}d\Omega = \delta\int R\sqrt{-g}d\Omega, \tag{2.55}$$

The rule for calculating the energy-momentum tensor for any physical system is

$$S = \int \Lambda\left(q, \frac{\partial q}{\partial x_i}\right) dV dt \sim \int \Lambda d\Omega, \tag{2.56}$$

in 4-spaces, where $q$ is a set of generalized coordinates. In curvilinear coordinates this integral must be written in the form

$$S \sim \int \Lambda \sqrt{-g} d\Omega \tag{2.57}$$

where $\Lambda$ is some function of $q$ determining the state of the system and their derivatives with respect to coordinates and time ( in Galilean coordinates $g = 1$ and $S$ becomes $\int \Lambda dV dt$ ). The integration is made in the whole ( tree-dimensional ) space and between two given instant, that is to say in the infinite domain between two hypersurfaces .

We define energy-momentum tensor by formula

$$T_{ik} = \delta_{ik}\Lambda - \sum q^{(l)}_{,l} \frac{\partial \Lambda}{\partial q^{(l)}_{,k}}, \tag{2.58}$$

and is not symmetric in general.

To make it symmertic, we add to above expression a suitably chosen term of the form $\frac{\partial}{\partial x^l}\psi_{ikl}$ where $\psi_{ikl}$ is antisymmetric. We are now going to indicate another method of calculating the energy-momentum tensor, having the advantage of immediately providing an exact expression.

Let's move in (2.57) from the $x^i$ coordinates to the $x'^i = x^i + \xi^i$ coordinates , where the $\xi$'s are small displacements. In this transformation, the components $g^{ik}$ transform as:

$$\begin{aligned} g'^{ik}\left(x'^l\right) &= g^{lm}\left(x^l\right) \frac{\partial x'^i}{\partial x^l} \frac{\partial x'^k}{\partial x^m} = g^{lm}\left(\delta^i_l + \frac{\partial \xi^i}{\partial x^l}\right)\left(\delta^k_m + \frac{\partial \xi^k}{\partial x^m}\right) \\ &\approx g^{ik}\left(x^l\right) + g^{im}\frac{\partial \xi^k}{\partial x^m} + g^{kl}\frac{\partial \xi^i}{\partial x^l}. \end{aligned} \tag{2.59}$$

The tensor $g'^{ik}$ is here a function old $x'^l$, and the tensor $g^{ik}$ function of the old coordinates $x^l$. Expand the $g'^{ik}\left(x^l + \xi^l\right)$ in powers of $\xi^l$ and neglecting terms of second order in $\xi^l$, we obtain:

$$g'^{ik}\left(x^l\right) = g^{ik}\left(x^l\right) - \xi^l \frac{\partial g^{ik}}{\partial x^l} + g^{il}\frac{\partial \xi^k}{\partial x^l} + g^{kl}\frac{\partial \xi^i}{\partial x^l}. \tag{2.60}$$

$$g'^{ik} = g^{ik} + \delta g^{ik}, \quad \delta g^{ik} = \xi^{i;k} + \xi^{k;i}. \tag{2.61}$$

where we have used the fact $\xi^{i,k} + \xi^{k,i}$ is a tensor. Then:

$$g'_{ik} = g_{ik} + \delta g_{ik}, \quad \delta g_{ik} = -\xi_{i;k} - \xi_{k;i}. \tag{2.62}$$

The action $S$ is a scalar, and it is invariant under coordinate transformation. Moreover, the variation $\delta S$ of the action in a coordinate transformation can be written in the following form:

$$\delta S \sim \int \left\{ \frac{\partial\sqrt{-g}\Lambda}{\partial g^{ik}}\delta g^{ik} + \frac{\partial\sqrt{-g}\Lambda}{\partial\frac{\partial g^{ik}}{\partial x^l}}\delta\frac{g^{ik}}{\partial x^l} \right\} d\Omega$$
$$\sim \int \left\{ \frac{\partial\sqrt{-g}\Lambda}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g}\Lambda}{\partial\frac{\partial g^{ik}}{\partial x^l}} \right\} \delta g^{ik} d\Omega.$$

Let us set

$$\frac{1}{2}\sqrt{-g}T_{ik} = \frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g}\Lambda}{\partial\frac{\partial g^{ik}}{\partial x^l}} - \frac{\partial\sqrt{-g}\Lambda}{\partial g^{ik}}; \tag{2.63}$$

so $\delta S$ becomes

$$\delta S = -\frac{c}{2}\int T_{ik}\delta g^{ik}\sqrt{-g}d\Omega = \frac{c}{2}\int T^{ik}\delta g_{ik}\sqrt{-g}d\Omega \tag{2.64}$$

where $c$ is some constant. We have used the relation $g^{ik}\delta g_{ik} = -g_{ik}\delta g^{ik}$ so $T^{ik}\delta g_{ik} = -T_{ik}\delta g^{ik}$.

### 2.5.1 Gravitational field equations

The equations of the gravitational field are deduced from the principle of least action $\delta\left(S_m + S_g\right) = 0$, where $S_g$ and $S_m$ represent the action of the gravitational and mattersectors, respectively.

Let varying the action, $\delta S_g$. We have :

$$\delta\int R\sqrt{-g}d\Omega = \delta\int g^{ik}R_{ik}\sqrt{-g}d\Omega =$$
$$= \int \left(R_{ik}\sqrt{-g}\delta g^{ik} + R_{ik}g^{ik}\delta\sqrt{-g} + g^{ik}\sqrt{-g}\delta R_{ik}\right) d\Omega.$$

Using the relation

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2}\sqrt{-g}g_{ik}\delta g^{ik},$$

we obtain

$$\delta\int R\sqrt{-g}d\Omega = \int\left(R_{ik} - \frac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}d\Omega + \int g^{ik}\delta R_{ik}\sqrt{-g}d\Omega \tag{2.65}$$

We show that the second term is zero. Collecting eq.(2.64) and (2.65) we obtain the famous

Einstein equations of the gravitational field.

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R = \kappa T_{ik}. \tag{2.66}$$

# Chapter 3

# Introduction to Teleparallel Equivalent of General Relativity (TEGR)

This chapter is essentially based on the book of Ruben Aldrovandi and José Geraldo untilled "Teleparallel Gravity: an introduction" [5].

## 3.1 Linear frames and tetrads

Space-time is the common place where the four fundamental interaction known so far tack place. The theories describing the four interaction have all a strong geometrical flavor. We use the greek alphabet ($\mu, \nu, \rho, ... = 0, 1, 2, 3$) to denote indices related to space-time, and the first letters of the latin alphabet ($a, b, c, ... = 0, 1, 2, 3$) to denote indices related to tangent space. In Minkowski space-time the Lorentz metric has the following form:

$$\eta_{ab} = diag\,(+1, -1, -1, -1) \tag{3.1}$$

Space-time coordinates are represented by the set $\{x^\mu\}$ , while tangential space coordinates Denoted by $\{x^a\}$. Such coordinate systems determine their domain of definition, the local basis of the vector field formed by the gradient set

$$\{\partial_\mu\} = \left\{ \frac{\partial}{\partial x^\mu} \right\}, \{\partial_a\} = \left\{ \frac{\partial}{\partial x^a} \right\}, \tag{3.2}$$

as well as $\{dx^\mu\}$ and $\{dx^a\}$ for covector fields. The bases are dual in the sense that

$$dx^\mu (\partial_\nu) = \delta^\mu_\nu, \; dx^a (\partial_b) = \delta^a_b. \tag{3.3}$$

## 3.2 Trivial frames

General frames or tetrads, called also vierbeine (four-legs) will be denoted by

$$\{e_a\} \ , \ \{e^a\} .$$

Very particular cases are the mentioned "coordinate" bases

$$\{e_a\} = \{\partial_a\} , \quad \{e^a\} = \{dx^a\} , \tag{3.4}$$

its names cames from their relationship to the coordinate system

$$e^a (e_b) = \delta^a_b. \tag{3.5}$$

The rather special manifold where a vector field can be defined anywhere are called parallelizable like the euclidian space $E^n$

$$e_a = e_a{}^\mu \partial_\mu \quad e^a = e^a{}_\mu dx^\mu, \tag{3.6}$$

Conversely we have

$$\partial_\mu = e^a{}_u e_a \quad dx^\mu = e_a{}^\mu e^a. \tag{3.7}$$

On account of the orthogonality conditions (3.5), the frame components satisfy

$$e^a{}_\mu e_a{}^\nu = \delta^\nu_\mu \quad e^a{}_\mu e_b{}^\mu = \delta^a_b. \tag{3.8}$$

These frames and their bundles are components of space-time.

A general linear basis $\{e_a\}$ satisfies the commutation relation

$$[e_a, e_b] = f^c{}_{ab} e_c, \tag{3.9}$$

with $f^c{}_{ab}$ are the structure coefficients , or the anholonomy coefficients. We express the relationship with the Cartan structure equation as

$$de^c = -\frac{1}{2} f^c{}_{ab} e^a \wedge e^b c. \tag{3.10}$$

We have

$$\begin{aligned} f^c{}_{ab} &= e^c \left[ e^a \left( e_b \right) - e_b \left( e_a \right) \right] \\ &= e^c{}_\mu \left[ e_a \left( e_b{}^\mu \right) - e_b \left( e_a{}^\mu \right) \right] \\ &= e_a{}^\mu e_b{}^\nu \left( \partial_\nu e^c{}_\mu - \partial_\mu e^c{}_\nu \right), \end{aligned} \tag{3.11}$$

then, we substitute in the expression (3.10) to get

$$\begin{aligned} de^c &= -\frac{1}{2} e_a{}^\mu e_b{}^\nu \left( \partial_\nu e^c{}_\mu - \partial_\mu e^c{}_\nu \right) e^a \wedge e^b \\ &= -\frac{1}{2} e_a{}^\mu e_b{}^\nu \left( \partial_\nu e^c{}_\mu - \partial_\mu e^c{}_\nu \right) \left( e^a{}_\mu dx^\mu \right) \wedge \left( e^b{}_\nu dx^\nu \right) \\ &= \frac{1}{2} e_a{}^\mu e^a{}_\mu e_b{}^\nu e^b{}_\nu \left( \partial_\mu e^c{}_\nu - \partial_\nu e^c{}_\mu \right) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} \left( \partial_\mu e^c{}_\nu - \partial_\nu e^c{}_\mu \right) dx^\mu \wedge dx^\nu. \end{aligned} \tag{3.12}$$

When

$$f'^a{}_{cd} = 0 \Longrightarrow de' = 0,$$

the basis $\{e'^a\}$ is then said to be integrable or holonomic.

Consider now the metric of Minkowski space-time

$$\eta = \eta_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu \tag{3.13}$$

where $\{dx^\mu\}$ is an holonomic basis, and $\{x^\mu\}$ represents a set of cartesian coordinates, and

$$\eta_{\mu\nu} = diag\,(+1, -1, -1, -1)\,. \tag{3.14}$$

The $\eta_{\mu\nu}$ remain the same in any other coordinates.

The linear frame

$$e_a = e^u_a \partial_\mu$$

provides a relation between the tangent space-time $\eta_{ab}$ and the space-time metric as follows

$$\eta_{ab} = \eta_{\mu\nu} e^\mu_a e^\nu_b. \tag{3.15}$$

The inverse is expressed by

$$\eta_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \tag{3.16}$$

The $\{e_a\}$ always relate the tangent Minkowski space to a Minkowski space-time whether they are holonomic or not, inertial or not. These are the frames appearing in Spacial Relativity, and are called trivial frames or trivial tetrads.

## 3.3 Nontrivial frames

Nontrivial frames or tetrads will be denotes by

$$\{h_a\}\,, \quad \{h^a\}\,, \tag{3.17}$$

and they are defined like linear frames whose coefficient of anholonomy is related to both inertia and gravitation. To see the difference from trivial linear frames $e_a$, considering general pseudo-Riemannian space-time metric $g$ with components $g_{\mu\nu}$ in the dual holonomic basis $\{dx^\mu\}$

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = g_{\mu\nu} dx^\mu dx^\nu. \tag{3.18}$$

The tetrad field

$$h_a = h_a{}^\mu \partial_\mu, \quad h^a = h^a{}_\mu dx^\mu \tag{3.19}$$

is a linear basis that relates $g$ to the tangent-space metric

$$\eta = \eta_{ab} dx^a dx^b, \tag{3.20}$$

through the relation

$$\eta_{ab} = g_{\mu\nu} h_a^{\ \mu} h_b^{\ \nu}. \tag{3.21}$$

From

$$\eta_{ab} = diag\,(+1,-1,-1,-1) \tag{3.22}$$

the tetrad field is a linear frame and the members $h_a$ are orthogonal. Using the metric $g_{\mu\nu}$ the components of the dual basis tetrad fields $h_a = h^a_{\ \nu} dx^\nu$ satisfy

$$h^a_{\ \mu} h_a^{\ \nu} = \delta^\nu_\mu, \quad h^a_{\ \mu} h_b^{\ \mu} = \delta^a_b, \tag{3.23}$$

so that, we find $g_{\mu\nu}$ :

$$g_{\mu\nu} = \eta_{ab} h^a_{\ \mu} h^b_{\ \nu}. \tag{3.24}$$

Since the determinate is :

$$g = \det\left(g_{\mu\nu}\right), \tag{3.25}$$

we obtain

$$h = \det\left(h^a_{\ \mu}\right) = \sqrt{-g} \tag{3.26}$$

A tetrad basis $\{h_a\}$ satisfies the commutation relation

$$[h_a, h_b] = f^c_{\ \ ab} h_c \tag{3.27}$$

where, $f^c_{\ \ ab}$ are the structure coefficients (coefficients of anholonomy) of frame $\{h_a\}$

$$dh_c = -\frac{1}{2} f^c_{\ \ ab} h^a \wedge h^b. \tag{3.28}$$

We have

$$f^c{}_{ab} = h^c_\mu \left[h_a\left(h_b{}^\mu\right) - h_b\left(h_a{}^\mu\right)\right]$$
$$= h_a{}^\mu h_b{}^\nu \left(\partial_\nu h^c{}_\mu - \partial_\mu h^c{}_\nu\right). \tag{3.29}$$

We substitute in the previous sentence

$$dh_c = -\frac{1}{2} h_a{}^\mu h_b{}^\nu \left(\partial_\nu h^c{}_\mu - \partial_\mu h^c{}_\nu\right) h^a \wedge h^b$$
$$= -\frac{1}{2} h_a{}^\mu h_b{}^\nu \left(\partial_\nu h^c{}_\mu - \partial_\mu h^c{}_\nu\right) h^a{}_\mu h^b{}_\nu dx^\mu dx^\nu$$
$$= \frac{1}{2}\left(\partial_\mu h^c{}_\nu - \partial_\nu h^c{}_\mu\right) dx^\mu dx^\nu. \tag{3.30}$$

We have

$$h^a = dx^a.$$

when

$$f^c{}_{ab} = 0 \tag{3.31}$$

we find

$$dh^a = 0, \tag{3.32}$$

which means the absence of inertial effects in the presence of gravitation, then $f^c{}_{ab}$ represents both inertial effects and gravitation.

## 3.4 Lorentz connection

A lorentz connection (spin connection ) is 1-form assuming values in the lie algebra of the lorentz group

$$A_\mu = \frac{1}{2} A^{ab}{}_\mu S_{ab}, \tag{3.33}$$

where $S_{ab}$ represents Lorentz generators. These generators are antisymmetric in the algebra of indices, and $A^{ab}_\mu$ must be antisymmetric to be Lorentzian. This connection defines the Fock-Ivanenko covariant derivative

$$D_\mu = \partial_\mu - \frac{1}{2} A^{ab}{}_\mu S_{ab}. \tag{3.34}$$

The term in the right acts only on algebra or tangent-space indices for scalar as field is the generator

$$S_{ab} = 0 \tag{3.35}$$

where $S_{ab}$ is spinorial matrices for the Dirac spinor $\psi$

$$S_{ab} = \frac{1}{4}\left[\gamma_a, \gamma_b\right], \tag{3.36}$$

where $\gamma_a$ is the Dirac matrices .

The Fock-Ivanenko derivative of a scalar field is then

$$\begin{aligned} D_\mu \phi &= \left(\partial_\mu - \frac{i}{2} A^{ab}{}_\mu S_{ab}\right)\phi^c \\ &= \partial_\mu \phi^c - \frac{i}{2} A^{ab}{}_\mu S_{ab} \phi^c \\ &= \partial_\mu \phi^c - \frac{i}{2} A^{\eta\sigma}{}_\mu h^a{}_\eta h^b{}_\sigma S_{\gamma\rho} h_a{}^\gamma h_b{}^\rho h^c{}_d \phi^d \\ &= \partial_\mu \phi^c - \frac{i}{2} A^{\eta\sigma}{}_\mu S_{\gamma\rho} \delta^\gamma_\eta \delta^\rho_\sigma h^c{}_d \phi^c \\ &= \partial_\mu \phi^c - A^c{}_{d\mu} \phi^d. \end{aligned} \tag{3.37}$$

The tetrad relates tangent space internal tensors with space-time (external) tensors . If $\phi^\alpha$ is an internal or lorentz vectors

$$\phi^\rho = h_a{}^\rho \phi^a, \tag{3.38}$$

$$A^{ab}{}_\mu = h^a{}_\eta h^b{}_\sigma A^{\eta\sigma}{}_\mu, \tag{3.39}$$

$$S_{ab} = h_a{}^\gamma h_b{}^\rho S_{\gamma\rho}, \tag{3.40}$$

conversely, we write

$$\phi^a = h^a{}_\rho \phi^\rho, \tag{3.41}$$

$$A^{\eta\sigma}{}_\mu = h_a{}^\eta h_b{}^\sigma A^{ab}{}_\mu, \tag{3.42}$$

$$S_{\gamma\rho} = h^a{}_\gamma h^b{}_\rho S_{ab}. \tag{3.43}$$

On the other hand, due to its non-tensorial character, a connection will acquire a vacuum, or non-homogeneous term, under the same operation. For example, to each spin connection $A^a{}_{b\mu}$, there is a corresponding general linear connection $\Gamma^\rho{}_{\nu\mu}$ ,given by

$$
\begin{aligned}
\Gamma^\rho{}_{\mu\nu} &= h_a{}^\rho \partial_\mu h^a{}_\nu + h_a{}^\rho A^a{}_{b\mu} h^b{}_\nu \\
&= h_a{}^\rho \partial_\mu h^a{}_\nu + h_a{}^\rho A^a{}_{b\mu} h^b{}_a h^a{}_\nu \\
&= h_a{}^\rho \left(\partial_\mu + A^a{}_{b\mu} h^b{}_a\right) h^a{}_\nu \\
&= h_a{}^\rho \left(\partial_\mu - A^{ab}{}_\mu h^b{}_a\right) h^a{}_\nu \\
&= h_a{}^\rho D_\mu h^a{}_\nu, \qquad (3.44)
\end{aligned}
$$

where $D_\mu$ is the covariant derivative .

Consequently, the inverse relation is

$$
\begin{aligned}
A^a{}_{b\mu} &= h^a{}_\nu \partial_\mu h_b{}^\nu + h^a{}_\nu \Gamma^\nu{}_{\rho\mu} h_b{}^\rho \\
&= h^a{}_\nu \partial_\mu h_b{}^\nu + h^a{}_\nu \Gamma^\nu{}_{\rho\mu} h_\nu{}^\rho h_b{}^\nu \\
&= h^a{}_\nu \left(\partial_\mu + \Gamma^\nu{}_{\rho\mu} h_\nu{}^\rho\right) h_b{}^\rho \\
&= h^a{}_\nu \bigtriangledown_\mu h_b{}^\nu, \qquad (3.45)
\end{aligned}
$$

where

$$
\bigtriangledown_\mu = \partial_\mu + \Gamma^\nu{}_{\rho\mu} h^\rho{}_\nu, \qquad (3.46)
$$

is the standard covariant derivative endowed with the connection $\Gamma^\nu{}_{\rho\mu}$, which acts on external indices only for a space-time vector $\phi^\nu$,so we have

$$
\begin{aligned}
\bigtriangledown_\mu \phi^\nu &= \left(\partial_\mu + \Gamma^\nu{}_{\rho\mu} h^\rho{}_\nu\right) \phi^\nu \\
&= \partial_\mu \phi^\nu + \Gamma^\nu{}_{\rho\mu} h^\rho{}_\nu \phi^\nu \\
&= \partial_\mu \phi^\nu + \Gamma^\nu{}_{\rho\mu} \phi^\rho. \qquad (3.47)
\end{aligned}
$$

Now using

$$
\phi^\rho = h_a{}^\rho \phi^a, \qquad (3.48)
$$

$$
\phi^a = h^a{}_\rho \phi^\rho. \qquad (3.49)
$$

we obtain the relation between Fock-Ivanenko derivative and the covariat derivative

$$D_\mu \phi^d = h^d{}_\rho \bigtriangledown_\mu \phi^\rho. \tag{3.50}$$

## 3.5 Curvature and torsion

Curvature and torsion are tensorial properties of Lorentz connection, is evident if we note that are many different connections that can be defined in the case of general relativity where the zero-torsion spin connection is present. Universality of gravitation allows its curvature to be interprets together with the metric as part of space-time . So that we can talk about" space-time curvature". We take space-time simply to represent connection with different curvatures and torsions.

With the Lorentz connection $A^a{}_{b\mu}$, the curvature is a 2-form assuming values in the lie algebra of Lorentz group

$$R = \frac{1}{4} R^a{}_{b\nu\mu} S_a{}^b dx^\nu \wedge dx^\mu. \tag{3.51}$$

The torsion is also a 2-form given by

$$T = \frac{1}{2} T^a{}_{\nu\mu} P_a dx^\nu \wedge dx^\mu, \tag{3.52}$$

with $P_a = \partial_a$ is the translation generators.

The curvature and torsion components are

$$R^a{}_{b\nu\mu} = \partial_\nu A^a{}_{b\mu} - \partial_\mu A^a{}_{b\nu} + A^a{}_{e\nu} A^e{}_{b\mu} - A^a{}_{e\mu} A^e{}_{b\nu}, \tag{3.53}$$

$$T^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + A^a{}_{e\nu} h^e{}_\mu - A^a{}_{e\mu} h^e{}_\nu. \tag{3.54}$$

Though contraction with tetrads we can write tensors in space-time indexed forms

$$R^\rho{}_{\lambda\nu\mu} = \partial_\nu A^\rho{}_{\lambda\mu} - \partial_\mu A^\rho{}_{\lambda\nu} + A^\rho{}_{e\nu} A^e{}_{\lambda\mu} - A^\rho{}_{e\mu} A^e{}_{\lambda\nu}. \tag{3.55}$$

We calculate the first term :

$$\begin{aligned}
\partial_\nu A^\rho{}_{\lambda\mu} &= \partial_\nu \left( h_a{}^\rho h^b{}_\lambda A^a{}_{b\mu} \right) \\
&= \left(\partial_\nu h_a{}^\rho\right) h^b{}_\lambda A^a{}_{b\mu} + h_a{}^\rho \left(\partial_\nu h^b{}_\lambda\right) A^a{}_{b\mu} + h_a{}^\rho h^b{}_\lambda (\partial_\nu A^a{}_{b\mu}) \\
&= \left(\partial_\nu h_a{}^\rho\right) h^b{}_\lambda A^a{}_{b\mu} - \left(\partial_\nu h_a{}^\rho\right) h^b{}_\lambda A^a{}_{b\mu} + h_a{}^\rho h^b{}_\lambda \left(\partial_\nu A^a{}_{b\mu}\right) \\
&= h_a{}^\rho h^b{}_\lambda \partial_\nu A^a{}_{b\mu}.
\end{aligned} \tag{3.56}$$

The second term is :

$$\begin{aligned}
\partial_\mu A^\rho{}_{\lambda\nu} &= \partial_\mu \left( h_a{}^\rho h^b{}_\lambda A^a{}_{b\nu} \right) \\
&= \left(\partial_\mu h_a{}^\rho\right) h^b{}_\lambda A^a{}_{b\nu} + h_a{}^\rho \left(\partial_\mu h^b{}_\lambda\right) A^a{}_{b\nu} + h_a{}^\rho h^b{}_\lambda \left(\partial_\mu A^a{}_{b\nu}\right) \\
&= \left(\partial_\mu h_a{}^\rho\right) h^b{}_\lambda A^a{}_{b\nu} - \left(\partial_\mu h_a{}^\rho\right) h^b{}_\lambda A^a{}_{b\nu} + h_a{}^\rho h^b{}_\lambda \left(\partial_\mu A^a{}_{b\nu}\right) \\
&= h_a{}^\rho h^b{}_\lambda \partial_\mu A^a{}_{b\nu}.
\end{aligned} \tag{3.57}$$

We substitute (3.56) and (3.57) in (4.10) :

$$\begin{aligned}
R^\rho{}_{\lambda\nu\mu} &= h_a{}^\rho h^b{}_\lambda \partial_\nu A^a{}_{b\mu} - h_a{}^\rho h^b{}_\lambda \partial_\mu A^a{}_{b\nu} + h_a{}^\rho A^a{}_{e\nu} h^b{}_\lambda A^e{}_{b\mu} - h_a{}^\rho A^a{}_{e\mu} h^b{}_\lambda A^e{}_{b\nu} \\
&= h_a{}^\rho h^b{}_\lambda \left( \partial_\nu A^a{}_{b\mu} - \partial_\mu A^a{}_{b\nu} + A^a{}_{e\nu} A^e{}_{b\mu} - A^a{}_{e\mu} A^e{}_{b\nu} \right) \\
&= h_a{}^\rho h^b{}_\lambda R^a{}_{b\nu\mu},
\end{aligned} \tag{3.58}$$

$$\begin{aligned}
T^\rho{}_{\nu\mu} &= \partial_\nu h^\rho{}_\mu - \partial_\mu h^\rho{}_\nu + A^\rho{}_{e\nu} h^e{}_\mu - A^\rho{}_{e\mu} h^e{}_\nu \\
&= \partial_\nu h_a{}^\rho h^a{}_\mu - \partial_\mu h_a{}^\rho h^a{}_\nu + h_a{}^\rho h^e{}_\mu A^a{}_{e\nu} - h_a{}^\rho h^e{}_\nu A^a{}_{e\mu} \\
&= h_a{}^\rho \left( \partial_\nu h^a{}_\nu - \partial_\mu h^a{}_\nu + h^e{}_\mu A^a{}_{e\nu} - h^e{}_\nu A^a{}_{e\mu} \right) \\
&= h_a{}^\rho T^a{}_{\mu\nu}.
\end{aligned} \tag{3.59}$$

We use the relation

$$A^a{}_{b\mu} = h^a{}_\nu \partial_\mu h_b{}^\nu + h^a{}_\nu \Gamma^\nu{}_{\rho\mu} h_b{}^\rho \tag{3.60}$$

to find

$$R^{\rho}{}_{\lambda\nu\mu} = \partial_\nu \Gamma^{\rho}{}_{\lambda\mu} - \partial_\mu \Gamma^{\rho}{}_{\lambda\nu} + \Gamma^{\rho}{}_{\eta\nu}\Gamma^{\eta}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\eta\mu}\Gamma^{\eta}{}_{\lambda\nu}. \tag{3.61}$$

We have

$$\begin{aligned} R^{\rho}{}_{\lambda\nu\mu} &= h_a{}^{\rho} h^b{}_\lambda R^a{}_{b\nu\mu} \\ &= h_a{}^{\rho} h^b{}_\lambda \left[\partial_\nu A^a{}_{b\mu} - \partial_\mu A^a{}_{b\nu} + A^a{}_{e\nu} A^e{}_{b\mu} - A^a{}_{e\mu} A^e{}_{b\nu}\right]. \end{aligned} \tag{3.62}$$

We substitute (3.60) in (3.62) , and begin by a calculation of the first term:

$$\begin{aligned} term1 &= h_a{}^{\rho} h^b{}_\lambda \partial_\nu A^a{}_{b\mu} \\ &= h_a{}^{\rho} h^b{}_\lambda \partial_\nu \left(h^a{}_\gamma \partial_\mu h_b{}^\gamma + h^a{}_\gamma \Gamma^\gamma{}_{\sigma\mu} h_b{}^\sigma\right) \\ &= h_a{}^{\rho} h \left((\partial_\nu h^a{}_\gamma)\partial_\mu h_b{}^\gamma + h^a{}_\gamma(\partial_\nu\partial_\mu h_b{}^\gamma) + (\partial_\nu h^a{}_\gamma)\Gamma^\gamma{}_{\sigma\mu} h_b{}^\sigma + h^a{}_\gamma(\partial_\nu \Gamma^\gamma{}_{\sigma\mu}) h_b{}^\sigma + h^a{}_\gamma \Gamma^\gamma{}_{\sigma\mu}(\partial_\nu h_b{}^\sigma)\right) \\ &= h^b{}_\lambda \partial_\nu \partial_\mu h_b{}^\rho + \partial_\nu h^a{}_\lambda \partial_\mu h_a{}^\rho + h_a{}^\rho \partial_\nu h^a{}_\gamma \Gamma^\gamma{}_{\lambda\mu} + \partial_\nu \Gamma^\rho{}_{\sigma\mu} h_b{}^\sigma + h^b{}_\lambda \Gamma^\rho{}_{\sigma\mu} \partial_\nu h_b{}^\sigma. \end{aligned} \tag{3.63}$$

For the second term we get

$$\begin{aligned} term2 &= -h_a{}^{\rho} h^b{}_\lambda \partial_\mu A^a{}_{b\nu} \\ &= -h^b{}_\lambda \partial_\mu \partial_\nu h_b{}^\rho - \partial_\mu h^a{}_\lambda \partial_\nu h_a{}^\rho - h_a{}^\rho \partial_\mu h^a{}_\gamma \Gamma^\gamma{}_{\gamma\mu} - \partial_\mu \Gamma^\rho{}_{\sigma\nu} h_b{}^\sigma - h^b{}_\lambda \Gamma^\rho{}_{\sigma\nu} \partial_\mu h_b{}^\sigma. \end{aligned} \tag{3.64}$$

For the third term we have

$$\begin{aligned} term3 &= h_a{}^{\rho} h^b{}_\lambda A^a{}_{e\nu} A^e{}_{b\mu} \\ &= h_a{}^{\rho} h^b{}_\lambda \left[\left(h^a{}_\delta \partial_\nu h_e{}^\delta + h^a{}_\delta \Gamma^\delta{}_{\sigma\nu} h_e{}^\sigma\right)\left(h^e{}_\gamma \partial_\mu h_b{}^\gamma + h^e{}_\gamma \Gamma^\gamma{}_{\sigma\mu} h_b{}^\sigma\right)\right] \\ &= h_a{}^{\rho} h^b{}_\lambda \left(h^a{}_\delta \partial_\nu h_e{}^\delta h^e{}_\gamma \partial_\mu h_b{}^\gamma + h^a{}_\delta \partial_\nu h_e{}^\delta h^e{}_\gamma \Gamma^\gamma{}_{\sigma\mu} h_b{}^\sigma + h^a{}_\delta \Gamma^\delta{}_{\sigma\nu} h_e{}^\sigma h^e{}_\gamma \partial_\mu h_b{}^\gamma + h^a{}_\delta \Gamma^\delta{}_{\sigma\nu} h_e{}^\sigma h^e{}_\gamma \Gamma^\gamma{}_{\sigma\mu} h_b{}^\sigma\right) \\ &= -\partial_\nu h_e{}^\rho \partial_\mu h^e{}_\lambda + \partial_\nu h_e{}^\rho \Gamma^\gamma{}_{\lambda\mu} h^e{}_\gamma + h^b{}_\lambda \Gamma^\rho{}_{\sigma\nu} \partial_\mu h_b{}^\sigma + \Gamma^\rho{}_{\gamma\nu} \Gamma^\gamma{}_{\lambda\mu}, \end{aligned} \tag{3.65}$$

and finally the fourth term reads as

$$
\begin{aligned}
term4 &= -h_a{}^\rho h^b{}_\lambda A^a{}_{e\mu} A^e{}_{b\nu} \\
&= -h_a{}^\rho h^b{}_\lambda \left[ \left( h^a{}_\delta \partial_\mu h_e{}^\delta + h^a{}_\delta \Gamma^\delta{}_{\sigma\mu} h_e{}^\sigma \right) \left( h^e{}_\gamma \partial_\nu h_b{}^\gamma + h^e{}_\gamma \Gamma^\gamma{}_{\sigma\nu} h_b{}^\sigma \right) \right] \\
&= -h_a{}^\rho h^b{}_\lambda \left( h^a{}_\delta \partial_\mu h_e{}^\delta h^e{}_\gamma \partial_\nu h_b{}^\gamma + h^a{}_\delta \partial_\mu h_e{}^\delta h^e{}_\gamma \Gamma^\gamma{}_{\sigma\nu} h_b{}^\sigma + h^a{}_\delta \Gamma^\delta{}_{\sigma\mu} h_e^\sigma h^e{}_\gamma \partial_\nu h_b{}^\gamma + h^a{}_\delta \Gamma^\delta{}_{\sigma\mu} h_e^\sigma h^e{}_\gamma \Gamma^\gamma{}_{\sigma\nu} h_b{}^\sigma \right) \\
&= \partial_\mu h_e{}^\rho \partial_\nu h^e{}_\lambda - \partial_\mu h_e{}^\rho \Gamma^\gamma{}_{\lambda\mu} h^e{}_\gamma - h^b{}_\lambda \Gamma^\rho{}_{\sigma\mu} \partial_\nu h_b{}^\sigma - \Gamma^\rho{}_{\gamma\mu} \Gamma^\gamma{}_{\lambda\nu}.
\end{aligned} \tag{3.66}
$$

Considering the spin connection $A^a{}_{b\nu}$ is a (co)vector, we write

$$A^a{}_{bc} = A^a{}_{b\nu} h_c{}^\nu, \tag{3.67}$$

conversely, we write

$$A^a{}_{b\nu} = A^a{}_{bc} h^c{}_\nu. \tag{3.68}$$

In the anholonomic basis $\{h_a\}$, we write the curvature like

$$R^a{}_{bcd} = h_c \left( A^a{}_{bd} \right) - h_d \left( A^a{}_{bc} \right) + A^a{}_{ec} A^e{}_{bd} - A^a{}_{ed} A^e{}_{bc} - f^e{}_{cd} A^a{}_{be}, \tag{3.69}$$

and the torsion components is

$$T^a{}_{bc} = A^a{}_{cb} - A^a{}_{bc} - f^a{}_{bc}, \tag{3.70}$$

where ,

$$h_c = h_c{}^\mu \partial_\mu, \tag{3.71}$$

and

$$A^a{}_{bc} = \frac{1}{2} \left( f_b{}^a{}_c + T_b{}^a{}_c + f_c{}^a{}_b + T_c{}^a{}_b - f^a{}_{bc} - T^a{}_{bc} \right). \tag{3.72}$$

We can write

$$A^a{}_{bc} = \mathring{A}^a{}_{bc} + k^a{}_{bc}, \tag{3.73}$$

where

$$\mathring{A}^a{}_{bc} = \frac{1}{2} \left( f_b{}^a{}_c + f_c{}^a{}_b - f^a{}_{bc} \right), \tag{3.74}$$

where $\mathring{A}^a{}_{bc}$ is the usual expression of the general relativity spin connection in terms of the coefficient of anholonomy , and

$$K^{a}_{\ bc}=\frac{1}{2}\left(T_{b\ \ c}^{\ a}+T_{c\ \ b}^{\ a}-T^{a}_{\ bc}\right) \tag{3.75}$$

is the contortion tensor.

We have

$$K_{\mu}=\frac{1}{2}K^{a}_{\ b\mu}S_{a}^{\ \ b}, \tag{3.76}$$

and

$$\Gamma^{\rho}_{\ \mu\nu}=\overset{\circ}{\Gamma}{}^{\rho}_{\ \mu\nu}+K^{\rho}_{\ \mu\rho}, \tag{3.77}$$

where ,

$$\overset{\circ}{\Gamma}{}^{\rho}_{\ \mu\nu}=\frac{1}{2}g^{\sigma\rho}\left(\partial_{\mu}g_{\rho\nu}+\partial_{\nu}g_{\rho\mu}-\partial_{\rho}g_{\mu\nu}\right) \tag{3.78}$$

is the zero-torsion Christoffel connection, or levi-civita connection , and

$$K^{\rho}_{\ \mu\nu}=\frac{1}{2}\left(T_{\nu\ \ \mu}^{\ \rho}+T_{\mu\ \ \nu}^{\ \rho}-T^{\rho}_{\ \mu\nu}\right). \tag{3.79}$$

is space-time indexed contortion tensor .

## 3.6 Local lorentz transformation

We consider a local lorentz transformation

$$x'^{a}=\Lambda^{a}_{\ \ b}\left(x\right)x^{b}, \tag{3.80}$$

under which the tetrad frames transforms as

$$h'^{a}=\Lambda^{a}_{\ \ b}\left(x\right)h^{b}, \tag{3.81}$$

$$h'_{a}=\Lambda_{a}^{\ \ b}\left(x\right)h_{b}. \tag{3.82}$$

Accordingly, the space-time metric becomes

$$g_{\mu\nu} = \eta_{cd} h'^{c}{}_{\mu} h'^{d}{}_{\nu}, \tag{3.83}$$

and

$$\eta_{ab} = \eta_{cd} \left( h'^{c}{}_{\mu} h_a{}^{\mu} \right) \left( h'^{a}{}_{\nu} h_b{}^{\nu} \right). \tag{3.84}$$

The matrix with entries

$$\Lambda^{a}{}_{b}\left(x\right) = h'^{a}{}_{\mu} h_b{}^{\mu} \tag{3.85}$$

with the transformation

$$h'^{a}{}_{\mu} = \Lambda^{a}{}_{b} h^{b}{}_{\mu}, \tag{3.86}$$

satisfies

$$\eta_{ab} = \eta_{cd} \left( h'^{c}{}_{\mu} h_a{}^{\mu} \right) \left( h'^{d}{}_{\nu} h_b{}^{\nu} \right). \tag{3.87}$$

From (3.87) we find

$$h'^{c}{}_{\mu} = \Lambda^{c}{}_{a} h^{a}{}_{\mu}, \tag{3.88}$$

$$h'^{d}{}_{\nu} = \Lambda^{d}{}_{b} h^{b}{}_{\nu}, \tag{3.89}$$

so, we substitute in (3.88)

$$\begin{aligned} \eta_{ab} &= \eta_{cd} \Lambda^{c}{}_{a} h^{a}{}_{\mu} h_a^{\mu} \Lambda^{d}{}_{b} h^{b}{}_{\nu} h_b{}^{\nu} \\ &= \eta_{cd} \Lambda^{c}{}_{a} \Lambda^{d}{}_{b}. \end{aligned} \tag{3.90}$$

Under a local lorentz transformation, $\Lambda^{a}{}_{b}$ the spin connection, undergoes the transformation

$$\Lambda'^{a}{}_{b\mu} = \Lambda^{a}{}_{c} \Lambda^{c}{}_{d\mu} \Lambda_b{}^{d} + \Lambda_c{}^{a} \partial_\mu \Lambda_b{}^{c}. \tag{3.91}$$

Let us calculate $T'^{a}_{\nu\mu}$ and $R'^{a}{}_{b\nu\mu}$ . In the same way $T^{a}_{\nu\mu}$ and $R^{a}{}_{b\nu\mu}$transform covariantly as

$$T^{a}_{\nu\mu} = \partial_\nu h^{a}{}_{\mu} - \partial_\mu h^{a}{}_{\nu} + A^{a}{}_{e\nu} h^{e}{}_{\mu} - A^{a}{}_{e\mu} h^{e}{}_{\nu}, \tag{3.92}$$

$$
\begin{aligned}
T'^{a}{}_{\nu\mu} &= \partial_\nu h'^{a}{}_{\mu} - \partial_\mu h'^{a}{}_{\nu} + A'^{a}{}_{e\nu} h'^{e}{}_{\mu} - A'^{a}{}_{e\mu} h'^{e}{}_{\nu} \\
&= \partial_\nu \left(\Lambda^{a}{}_{b} h^{b}{}_{\mu}\right) - \partial_\mu \left(\Lambda^{a}{}_{b} h^{b}{}_{\nu}\right) + \left(\Lambda^{a}{}_{c} A^{c}{}_{d\nu} \Lambda_{e}{}^{d} + \Lambda^{a}{}_{c} \partial_\nu \Lambda_{e}{}^{c}\right) \Lambda^{e}{}_{b} h^{b}{}_{\mu} + \left(\Lambda^{a}{}_{c} A^{c}{}_{d\mu} \Lambda_{e}{}^{d} + \Lambda^{a}{}_{c} \partial_\mu \Lambda_{e}{}^{c}\right) \Lambda^{e}{}_{b} h^{b}{}_{\nu} \\
&= h^{b}{}_{\mu} \partial_\nu \Lambda^{a}_{b} + \Lambda^{a}_{b} \partial_\nu h^{b}{}_{\mu} - h^{b}{}_{\nu} \partial_\mu \Lambda^{a}_{b} - \Lambda^{a}_{b} \partial_\mu h^{b}{}_{\nu} + \left(\Lambda^{a}_{b} A^{b}{}_{c\nu} h^{c}{}_{\mu} - \partial_\nu \Lambda^{a}_{b} h^{b}{}_{\mu}\right) - \left(\Lambda^{a}_{b} A^{b}{}_{c\mu} h^{c}{}_{\nu} - \partial_\mu \Lambda^{a}_{b} h^{b}{}_{\nu}\right) \\
&= \Lambda^{a}_{b} \partial_\nu h^{b}{}_{\mu} - \Lambda^{a}_{b} \partial_\mu h^{b}{}_{\nu} + \Lambda^{a}{}_{b} A^{b}{}_{c\nu} h^{c}{}_{\mu} - \Lambda^{a}{}_{b} A^{b}{}_{c\mu} h^{c}{}_{\nu} \\
&= \Lambda^{a}_{b} T^{b}{}_{\nu\mu}, \qquad (3.93)
\end{aligned}
$$

and

$$
R'^{a}_{b\nu\mu} = \partial_\nu A'^{a}_{b\mu} - \partial_\mu A'^{a}_{b\nu} + A'^{a}_{e\nu} A'^{e}_{b\mu} - A'^{a}_{e\mu} A'^{e}_{b\nu}. \qquad (3.94)
$$

We calculate each term separately :

$$
\partial_\nu A'^{a}_{b\mu} = \partial_\nu \Lambda^{a}{}_{c} A^{c}{}_{d\mu} \Lambda_{b}{}^{d} + \Lambda^{a}{}_{c} \partial_\nu A^{c}{}_{d\mu} \Lambda_{b}{}^{d} + \Lambda^{a}{}_{c} A^{c}{}_{d\mu} \partial_\nu \Lambda_{b}{}^{c} + \partial_\nu \Lambda^{a}{}_{c} \partial_\mu \Lambda_{b}{}^{c} + \Lambda^{a}{}_{c} \partial_\nu \partial_\mu \Lambda_{b}{}^{c}, \qquad (3.95)
$$

$$
\partial_\mu A'^{a}_{b\nu} = \partial_\mu \Lambda^{a}{}_{c} A^{a}{}_{d\nu} \Lambda_{b}{}^{d} + \Lambda^{a}{}_{c} \partial_\mu A^{c}{}_{d\nu} \Lambda_{b}{}^{d} + \Lambda^{a}{}_{c} A^{c}{}_{d\nu} \partial_\mu \Lambda_{b}{}^{d} + \Lambda^{a}{}_{c} \partial_\mu \partial_\nu \Lambda_{b}{}^{c}, \qquad (3.96)
$$

$$
A'^{a}_{e\nu} A'^{e}{}_{b\mu} = \Lambda^{a}{}_{c} \Lambda_{b}{}^{g} A^{c}{}_{f\nu} A^{f}{}_{g\mu} + \Lambda^{a}{}_{c} A^{c}{}_{f\nu} \partial_\mu \Lambda_{b}{}^{f} - \Lambda_{b}{}^{g} A^{f}{}_{g\mu} \partial_\nu \Lambda^{a}{}_{f} + \Lambda^{a}{}_{c} \Lambda_{f}{}^{e} \partial_\mu \Lambda_{b}{}^{f}, \qquad (3.97)
$$

$$
A'^{a}_{e\mu} A'^{e}_{b\nu} = \Lambda^{a}{}_{c} \Lambda_{b}{}^{g} A^{c}{}_{f\mu} A^{f}{}_{g\nu} + \Lambda^{a}{}_{c} A^{c}{}_{f\mu} \partial_\nu \Lambda_{b}{}^{f} - \Lambda_{b}{}^{g} A^{f}{}_{g\nu} \partial_\mu \Lambda^{a}{}_{f} + \Lambda^{a}{}_{c} \Lambda^{e}{}_{f} \partial_\nu \Lambda_{e}{}^{c} \partial_\mu \Lambda_{b}{}^{f}, \qquad (3.98)
$$

and we substitute eqs.(3.95-3.98) in (3.94) , we obtain

$$
\begin{aligned}
R'^{a}_{b\nu\mu} &= \Lambda^{a}{}_{c} \Lambda_{b}{}^{d} \left(\partial_\nu A^{c}{}_{d\mu} - \partial_\mu A^{c}{}_{d\nu} + A^{c}{}_{f\nu} A^{f}{}_{d\mu} - A^{c}{}_{f\mu} A^{f}{}_{d\nu}\right) \\
&= \Lambda^{a}{}_{c} \Lambda_{b}{}^{d} R^{c}{}_{d\nu\mu}. \qquad (3.99)
\end{aligned}
$$

## 3.7 Levi-civita symbol

The totally antisymmetric levi-civita symbol is :

$$
\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 \text{ if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 \text{ if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ \quad 0 \qquad\qquad \text{otherwise. for } \varepsilon^{0123}{=}1. \end{cases} \qquad (3.100)
$$

,

It satisfies

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\sigma} = -\begin{vmatrix} \delta^{\mu}_{\alpha} & \delta^{\mu}_{\beta} & \delta^{\mu}_{\gamma} \\ \delta^{\nu}_{\alpha} & \delta^{\nu}_{\beta} & \delta^{\nu}_{\gamma} \\ \delta^{\rho}_{\alpha} & \delta^{\rho}_{\beta} & \delta^{\rho}_{\gamma} \end{vmatrix} \tag{3.101}$$

,

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\right), \tag{3.102}$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\nu\rho\sigma} = -6\delta^{\mu}_{\alpha}, \tag{3.103}$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = -24, \tag{3.104}$$

we have

$$h = \det\left(h^{a}{}_{\mu}\right) = \sqrt{-g} \tag{3.105}$$

is a tensor density of weight -1 , $h^{-1}\varepsilon^{\mu\nu\rho\sigma}$ turn out to be an ordinary contravariant tensor. On the other hand

$$\varepsilon_{\alpha\beta\gamma\delta} = \varepsilon^{\mu\nu\rho\sigma}g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta} \tag{3.106}$$

it is a tensor density of weight +1, the quantity $h\varepsilon_{\mu\nu\rho\sigma}$ turns out to be an ordinary covariant tensor .

## 3.8 Purely inertial connection

To see how an inertial lorentz connection shows up, we denote by $a^{e}{}_{\mu}$ , a generic Minkowski space-time in general coordinate system

$$e'^{a}{}_{\mu} = \partial_{\mu}x'^{a}, \tag{3.107}$$

$x'^a$ is a space-time dependent lorentz vector , where

$$x'^a = x'^a\left(x\right). \tag{3.108}$$

The space-time metric is

$$\eta'_{\mu\nu} = e'^a{}_\mu e'^b{}_\nu \eta_{ab}. \tag{3.109}$$

In the specific case of cartesian coordinates the inertial frames assumes the form

$$e'^a{}_\mu = \delta^a_\mu \tag{3.110}$$

and the space-time metric is given by

$$\eta'_{\mu\nu} = diag\left(+1, -1, -1, -1\right) \tag{3.111}$$

Under a local lorentz transformation we have

$$x^a = \Lambda^a_b x'^b, \tag{3.112}$$

conversely , we find

$$x'^b = \Lambda^b{}_a x^a, \tag{3.113}$$

and

$$e'^a{}_\mu = \partial_\mu\left(x'^a\right). \tag{3.114}$$

The holonomic frame (3.107) is transformed accordingly

$$e^a{}_\mu = \Lambda^a_b e'^b{}_\mu, \tag{3.115}$$

as a simple computation shows, the explicit form is

$$e^a{}_\mu = \partial_\mu x^a + \dot{A}^a{}_{b\mu} x^b = D_\mu x^a, \tag{3.116}$$

where

$$\dot{A}^{a}{}_{b\mu} = \Lambda^{a}{}_{e}\partial_{\mu}\Lambda_{b}{}^{e}. \tag{3.117}$$

When $\dot{A}^{e}{}_{d\mu} \neq 0$, the expression (3.117) becomes

$$\dot{A}^{a}{}_{b\mu} = \Lambda^{a}{}_{e}\dot{A}'^{e}{}_{d\mu}\Lambda_{b}{}^{d} + \Lambda^{a}{}_{e}\partial_{\mu}\Lambda_{b}{}^{e}. \tag{3.118}$$

The coefficient of anholonomy is given by

$$f^{a}_{bc} = -\left(\dot{A}^{c}{}_{ab} - \dot{A}^{c}{}_{ba}\right), \tag{3.119}$$

where $\dot{A}^{a}{}_{bc}$ is

$$\dot{A}^{a}{}_{bc} = \dot{A}^{a}{}_{b\mu}e_{c}{}^{\mu}. \tag{3.120}$$

The inverse relation is

$$\dot{A}^{a}{}_{bc} = \frac{1}{2}\left(f_{b}{}^{c}{}_{a} + f_{c}{}^{a}{}_{b} - f^{a}{}_{bc}\right). \tag{3.121}$$

As a purely inertial connection $\dot{A}^{a}{}_{b\mu}$ has vanishing curvature and torsion :

$$\dot{R}^{a}{}_{b\nu\mu} = 0, \quad \dot{T}^{a}{}_{\nu\mu} = 0. \tag{3.122}$$

## 3.9 Particle Motion

In inertial frames $\left\{e'^{a}{}_{\mu}\right\}$, the equation of motion which describes free particles is

$$\frac{du'^{a}}{d\sigma} = 0, \tag{3.123}$$

with $u'^{a}$ the particle 4-velocity, and

$$d\sigma^{2} = \eta_{\mu\nu}dx^{\mu}dx^{\nu} \tag{3.124}$$

is the quadratic Minkowski invariant interval, with $\sigma$ the proper time .

In an anholonomic frame $\left\{e^{a}_{\ \mu}\right\}$ related to $e'^{a}_{\ \mu}$ , by the local lorentz transformation

$$e^{a}_{\ \mu} = \Lambda^{a}_{\ b}\,(x)\,e'^{b}_{\ \mu}, \tag{3.125}$$

the equation of motion assumes the following covariant form

$$\frac{du^{a}}{d\sigma} + \dot{A}^{a}_{\ b\mu} u^{b} u^{\mu} = 0, \tag{3.126}$$

where

$$u^{a} = \Lambda^{a}_{\ b}\,(x)\,u'^{b} \tag{3.127}$$

is the lorentz transformation of the 4-velocity, and his holonomic form is

$$u^{\mu} = u^{a} e_{a}^{\ \mu} \tag{3.128}$$

where is the space-time 4-velocity is

$$u^{\mu} = \frac{dx^{\mu}}{d\sigma} \tag{3.129}$$

where $e^{a}_{\ \mu}$ is related to inertial effects, and only the space-time metric still represent the Minkowski mertic

$$\eta_{\mu\nu} = e^{a}_{\ \mu} e^{b}_{\ \nu} \eta_{ab}. \tag{3.130}$$

Actually , the tetrad represents gravitational field can't be obtained through a Lorentz transformation from tetrad whose anholonomy represent inertial effects only .

In holonomic four-velocity , the equation of motion is given by

$$\frac{du^{\rho}}{d\sigma} + \dot{\gamma}^{\rho}_{\ \mu\nu} u^{\nu} u^{\mu} = 0, \tag{3.131}$$

where

$$\begin{aligned}
\dot{\gamma}^{\rho}_{\ \mu\nu} &= e_{c}^{\ \rho} \partial_{\mu} e^{c}_{\ \nu} + e_{c}^{\ \rho} \dot{A}^{c}_{\ b\mu} e^{b}_{\ \nu} = e_{c}^{\ \rho} \partial_{\mu} e^{c}_{\ \nu} + e_{c}^{\ \rho} \dot{A}^{c}_{\ b\mu} e^{b}_{\ c} e^{c}_{\ \nu} \\
&= e_{c}^{\ \rho} (\partial_{\mu} + \dot{A}^{c}_{\ b\mu} e^{b}_{\ c}) e^{c}_{\ \nu} = e_{c}^{\ \rho} \dot{D}_{\mu} e^{c}_{\ \nu},
\end{aligned} \tag{3.132}$$

where $\dot{\gamma}^{\rho}{}_{\mu\nu}$, symmetric in the lower 2 indexes, is the space-time indexed version of the inertial connection $\dot{A}^{a}{}_{b\mu}$ obtained through contraction with the trivial tetrad $e^{a}{}_{\mu}$.

The inverse relation is

$$\dot{A}^{a}{}_{b\mu} = e^{a}{}_{\rho}\partial_{\mu}e_{b}{}^{\rho} + e^{a}{}_{\rho}\dot{\gamma}^{\rho}{}_{\nu\mu}e_{b}{}^{\nu} = e^{a}{}_{\rho}\dot{\nabla}e_{b}{}^{\rho}. \tag{3.133}$$

In inertial frame $e'^{a}_{\mu}$ , where $\dot{A}^{a}{}_{b\mu} = 0$ , we get

$$\dot{\gamma}^{\rho}{}_{\nu\mu} = e'^{\rho}_{c}\partial_{\mu}e'^{c}{}_{\nu}. \tag{3.134}$$

In cartesian coordinates, we have

$$e'^{a}{}_{\mu} = \delta^{a}_{\mu} \tag{3.135}$$

and the connection $\dot{\gamma}'^{\rho}{}_{\nu\mu}$ vanishes , so the equation of motion reduces to

$$\frac{du'^{\rho}}{d\sigma} = 0. \tag{3.136}$$

with $u'^{\rho} = e'^{a}{}_{\mu}u'^{a}$.

From the above analysis, we have seen how inertial and coordinates effects can be brought into the equation of motion, and this formalism can be done for any classical relativistic theory.

## 3.10 Four-Acceleration and Parallel Transport

We consider the space-time metric

$$g_{\mu\nu} = \eta_{ab}h^{a}{}_{\mu}h^{b}{}_{\nu}, \tag{3.137}$$

$$ds^{2} = g_{\mu\nu}dx^{\mu}dx^{\nu}. \tag{3.138}$$

We have the components of the 4-velocity

$$u^{\mu} = \frac{dx^{\mu}}{ds}, \tag{3.139}$$

and the 4-acceleration is

$$a^{\rho} = u^{\nu}\nabla_{\nu}u^{\rho} = \frac{du^{\rho}}{ds} + \Gamma^{\rho}{}_{\mu\nu}u^{\mu}u^{\nu}. \tag{3.140}$$

Since $a^{\rho}$ is orthogonal to $u^{\rho}$, its disappearance means that $u^{\rho}$ remains parallel to itself along the curve. This leads to the concept of parallel transport: we say $u^{\rho}$ is parallel trasported along a curve if $a^{\rho} = 0$. Furthermore, since each vector field is locally tangent to the curve with the following conditions:

$$\boldsymbol{z}\nabla_{\mu}u^{\rho} = \nabla_{\boldsymbol{z}}u^{\rho} = 0. \tag{3.141}$$

This means that $u^{\rho}$ is parallel transported along the integral curve of $\boldsymbol{z}^{\rho}$ . The metric compatibility

$$\nabla^{\lambda}g_{\mu\nu} = \partial_{\lambda}g_{\mu\nu} - \Gamma^{\rho}{}_{\mu\lambda}g_{\rho\nu} - \Gamma^{\rho}{}_{\nu\lambda}g_{\mu\nu} \tag{3.142}$$

implies that

$$\boldsymbol{z}^{\rho}\nabla_{\rho}g_{\mu\nu} = 0 \tag{3.143}$$

for any vector field $\boldsymbol{z}^{\rho}$ . So, it is equivalent to say that the metric $g_{\mu\nu}$ is parallel transported every where on space-time. For Levi-civita connection

$$\overset{\circ}{\Gamma}{}^{\sigma}{}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_{\mu}g_{\rho\nu} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}\right), \tag{3.144}$$

we get also

$$\boldsymbol{z}^{\rho}\overset{\circ}{\nabla}_{\rho}g_{\mu\nu} = 0. \tag{3.145}$$

## 3.11 Inertial effects

Consider an observer attached to a particle moving along a curve $\gamma$. An observer is conceived as a timelike worldline[3, 4]. Notice that the 4 members of a tetrad are (pseudo-) orthogonal to each other . This means that one of them is timelike and the others are spacelike. As

$$\eta_{ab} = h_{a\nu}h_{b}{}^{\nu}, \tag{3.146}$$

and $\eta = diag\left(+1, -1, -1, -1\right)$ , we deduce that$h_0$ is timelike and $h_k\,(k = 1, 2, 3)$ is spacelike .

Then we choose $h_0$, as the observer velocity by identifying

$$u = h_0 = \frac{d}{ds} \tag{3.147}$$

with

$$u^\mu = h_0{}^\mu. \tag{3.148}$$

We take a general connection $\Gamma$ and examine the acceleration

$$a^a{}_{(f)} = h^a{}_\rho a^\rho{}_{(f)} = h^a{}_\rho \Gamma^\rho{}_{\mu\nu} h_0{}^\mu h_0{}^\nu + h^a{}_\rho h_0 \left(h_0{}^\rho\right). \tag{3.149}$$

Comparing with spin connection components

$$A^a{}_{bc} = h^a{}_\rho \nabla_{h_c} h_b{}^\rho = h^a{}_\rho \Gamma^\rho{}_{\mu\nu} h_b{}^\mu h_c{}^\nu + h^a{}_\rho h_c \left(h_b{}^\rho\right), \tag{3.150}$$

we deduce that

$$a^a{}_{(f)} = A^0{}_{00}. \tag{3.151}$$

The $A^a{}_{bc}$is antisymmetric in the first two indices, so $a^k{}_{(f)}$ is different from zero. The definition

$$A^a{}_{bc} = h^a{}_\rho \nabla_{hc} h_b{}^\rho \tag{3.152}$$

gives a new perception of the acceleration such that

$$a^k{}_{(f)} = \frac{du^k}{ds} + A^k{}_{bc} u^b u^c. \tag{3.153}$$

Now the torsion scalar is given by

$$T = S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}, \tag{3.154}$$

where

$$S_\rho{}^{\mu\nu} = \frac{1}{2}\left(K^{\mu\nu}{}_\rho + \delta^\mu_\rho T^{\theta\nu}{}_\theta - \delta^\nu_\rho T^{\theta\mu}{}_\theta\right), \tag{3.155}$$

and

$$K^{\mu\nu}{}_\rho = \frac{1}{2}\left(T^{\nu\mu}{}_\rho - T^{\mu\nu}{}_\rho + T_\rho{}^{\mu\nu}\right). \tag{3.156}$$

We substitute (3.155) and (3.156) in (3.154),

$$T = \frac{1}{2}\left[\frac{1}{2}\left(T^{\nu\mu}{}_{\rho} - T^{\mu\nu}{}_{\rho} + T_{\rho}{}^{\mu\nu}\right) + \delta^{\mu}_{\rho}T^{\theta\nu}{}_{\theta} - \delta^{\nu}_{\rho}T^{\theta\mu}{}_{\theta}\right]T^{\rho}{}_{\mu\nu}$$
$$= \frac{1}{4}T^{\mu\nu\rho}T_{\mu\nu\rho} + \frac{1}{2}T^{\mu\nu\rho}T_{\nu\mu\rho} - T^{\mu}T_{\mu}. \tag{3.157}$$

where $T^{\mu}$ is the vector of torsion.

## 3.12 Equations of the gravitational field in TEGR

Let us turn now to the derivation of the equations of motion of the gravitational field. We consider the action

$$S = c\int d^4xe\mathcal{L} = c\int d^4x\ eT + \int d^4x\ e\mathcal{L}_m, \tag{3.158}$$

where the second term is the matter action and $c$ is a constant.

The Variation of the gravitational action is given by

$$\delta S_G = c\int d^4x\delta\left(eT\right)$$
$$= c\int d^4x\left[\left(\delta e\right)T + e\left(\delta T\right)\right]. \tag{3.159}$$

Using this relations

$$\delta\left|e\right| = ee_a{}^{\mu}\delta e^a{}_{\mu}, \tag{3.160}$$

$$\delta e_a{}^{\mu} = -e_b{}^{\mu}e_a{}^{\nu}\delta e^b{}_{\nu}, \tag{3.161}$$

$$\delta g_{\mu\nu} = \eta_{ab}\left(e^a{}_{\mu}\delta e^b{}_{\nu} + e^a{}_{\nu}\delta e^b{}_{\mu}\right), \tag{3.162}$$

$$\delta g^{\mu\nu} = -fi\left(g^{\nu\rho}e_a{}^{\mu} + g^{\mu\rho}e_a{}^{\nu}\right)\delta e^a{}_{\rho}, \tag{3.163}$$

the first term in the eq.(3.159) becomes

$$T\left(\delta e\right) = eTe_a{}^{\mu}\delta e^a{}_{\mu}. \tag{3.164}$$

Then we substitute (3.157) in the second term in (3.159) to get

$$
\begin{aligned}
e\delta T &= e\delta\left(\frac{1}{4}T^{\mu\nu\rho}T_{\mu\nu\rho} + \frac{1}{2}T^{\mu\nu\rho}T_{\nu\mu\rho} - T^{\mu}T_{\mu}\right) \\
&= e\left[\frac{1}{4}\delta\left(T^{\mu\nu\rho}T_{\mu\nu\rho}\right) + \frac{1}{2}\delta\left(T^{\mu\nu\rho}T_{\nu\mu\rho}\right) - \delta\left(T^{\mu}T_{\mu}\right)\right]. \qquad (3.165)
\end{aligned}
$$

First, we calculate the third term in (3.165) because it's the easiest,

$$
\delta\left(T^{\mu}T_{\mu}\right) = \left(\delta T^{\mu}\right)T_{\mu} + T^{\mu}\left(\delta T_{\mu}\right). \qquad (3.166)
$$

Knowing that

$$
T_{\mu} = T^{\nu}_{\ \nu\mu}, \qquad (3.167)
$$

$$
T^{\mu} = g^{\mu\beta}T_{\beta}, \qquad (3.168)
$$

and substituting in (3.166), we obtain

$$
\begin{aligned}
\delta\left(T_{\mu}T^{\mu}\right) &= \left(\delta T^{\nu}_{\ \nu\mu}\right)T^{\mu} + T_{\mu}\delta\left(g^{\mu\beta}T_{\beta}\right) \\
&= \left(\delta T^{\nu}_{\ \nu\mu}\right)T^{\mu} + T_{\mu}\left[\left(\delta g^{\mu\beta}\right)T_{\beta} + g^{\mu\beta}\delta T_{\beta}\right] \\
&= \left(\delta T^{\nu}_{\ \nu\mu}\right)T^{\mu} + T_{\mu}T_{\beta}\delta g^{\mu\beta} + T_{\mu}g^{\mu\beta}\delta T_{\beta} \\
&= 2\left(\delta T^{\alpha}_{\ \alpha\mu}\right)T^{\mu} + T_{\mu}T_{\beta}\delta g^{\mu\beta} \\
&= 2T^{\mu}\left(-e_{b}^{\ \alpha}T^{\sigma}_{\ \mu\alpha}\delta e^{b}_{\ \sigma} + e_{a}^{\ \alpha}\left(\partial_{\alpha}\delta e^{a}_{\ \mu} - \partial_{\mu}\delta e^{a}_{\ \alpha}\right)\right) - \left(g^{\beta\rho}e_{a}^{\ \mu} + g^{\mu\rho}e_{a}^{\ \beta}\right)\delta e^{a}_{\ \rho}T_{\mu}T_{\beta} \\
&= -2\left(T^{\mu}T^{\sigma}_{\ \mu\alpha} + T_{\alpha}T^{\sigma}\right)e_{b}^{\ \alpha}\delta e^{b}_{\ \sigma} + 2\left(T^{\alpha}e_{a}^{\ \mu} - T^{\mu}e_{a}^{\ \alpha}\right)\partial_{\alpha}\delta e^{a}_{\ \mu}. \qquad (3.169)
\end{aligned}
$$

Secondly the first term in (3.165) gives

$$
\delta\left(T^{\mu\nu\rho}T_{\mu\nu\rho}\right) = \left(\delta T^{\mu\nu\rho}\right)T_{\mu\nu\rho} + T^{\mu\nu\rho}\left(\delta T_{\mu\nu\rho}\right). \qquad (3.170)
$$

The first term in (3.170) becomes

$$
\begin{aligned}
\left(\delta T^{\mu\nu\rho}\right) T_{\mu\nu\rho} &= \delta\left(g^{\nu\alpha} g^{\rho\beta} T^{\mu}{}_{\alpha\beta}\right) T_{\mu\nu\rho} \\
&= \left[\left(\delta g^{\nu\alpha}\right) g^{\rho\beta} T^{\mu}{}_{\alpha\beta} + g^{\nu\alpha}\left(\delta g^{\rho\beta}\right) T^{\mu}{}_{\alpha\beta} + g^{\nu\alpha} g^{\rho\beta}\left(\delta T^{\mu}{}_{\alpha\beta}\right)\right] T_{\mu\nu\rho} \\
&= \begin{matrix} \Big[-\left(g^{\alpha\lambda} e_a{}^{\nu} + g^{\nu\lambda} e_a{}^{\alpha}\right) \delta e^a{}_{\lambda} g^{\rho\beta} T^{\mu}{}_{\alpha\beta} - g^{\nu\alpha}\left(g^{\beta\lambda} e_a{}^{\rho} + g^{\rho\lambda} e_a{}^{\beta}\right) \delta e^a{}_{\lambda} T^{\mu}{}_{\alpha\beta} \\ + g^{\nu\alpha} g^{\rho\beta}\left(-e_b{}^{\mu} T^{\sigma}{}_{\alpha\beta} \delta e^b{}_{\sigma} + e_a{}^{\mu}\left(\partial_\alpha \delta e^a{}_{\beta} - \partial_\beta \delta e^a{}_{\alpha}\right)\right)\Big] T_{\mu\nu\rho} \end{matrix} \\
&= \left(-g^{\alpha\lambda} e_a{}^{\nu} g^{\rho\beta} T^{\mu}{}_{\alpha\beta} T_{\mu\nu\rho} - g^{\nu\lambda} e_a{}^{\alpha} g^{\rho\beta} T^{\mu}{}_{\alpha\beta} T_{\mu\nu\rho}\right) \delta e^a{}_{\lambda} \\
&+ \left(-g^{\nu\alpha} g^{\beta\lambda} e_a{}^{\rho} e_a{}^{\rho} T^{\mu}{}_{\alpha\beta} T_{\mu\nu\rho} - g^{\nu\alpha} g^{\rho\lambda} e_a{}^{\beta} T^{\mu}{}_{\alpha\beta} T_{\mu\nu\rho}\right) \delta e^a{}_{\lambda} \\
&- g^{\nu\alpha} g^{\rho\beta} e_b{}^{\mu} T^{\sigma}{}_{\alpha\beta} T_{\mu\nu\rho} \delta e^b{}_{\sigma} + g^{\nu\alpha} g^{\rho\beta} e_a{}^{\mu} T_{\mu\nu\rho}\left(\partial_\alpha \delta e^a{}_{\beta} - \partial_\beta \delta e^a{}_{\alpha}\right) \\
&= \left(-T_{\alpha\beta\sigma} T^{\alpha\beta\lambda} - T_{\alpha\beta\sigma} T^{\alpha\beta\lambda}\right) e_a{}^{\sigma} \delta e^a{}_{\lambda} + \left(-T^{\mu\rho\lambda} T_{\mu\rho\alpha} - T^{\mu\nu\lambda} T_{\mu\nu\alpha}\right) e_a{}^{\alpha} \delta e^a{}_{\lambda} \\
&- T^{\sigma\nu\rho} T_{\mu\nu\rho} e_b{}^{\mu} \delta e^b{}_{\sigma} + 2 T_a{}^{\alpha\beta} \partial_\alpha \delta e^a{}_{\beta} \\
&= \left(-T_{\alpha\beta\sigma} T^{\alpha\beta\lambda} - T_{\alpha\beta\sigma} T^{\alpha\beta\lambda}\right) e_a{}^{\sigma} \delta e^a{}_{\lambda} + \left(-T^{\mu\rho\lambda} T_{\mu\rho\sigma} - T^{\mu\nu\lambda} T_{\mu\nu\sigma}\right) e_a{}^{\sigma} \delta e^a{}_{\lambda} \\
&- T^{\alpha\nu\rho} T_{\sigma\mu\rho} e_b{}^{\sigma} \delta e^b{}_{\alpha} + 2 T_a{}^{\nu\rho} \partial_\nu \delta e^a{}_{\rho} \\
&= -4 T^{\mu\nu\lambda} T_{\mu\nu\sigma} e_a{}^{\sigma} \delta e^a{}_{\lambda} - T^{\alpha\rho\sigma} T_{\sigma\mu\rho} e_b{}^{\sigma} \delta e^b{}_{\sigma} + 2 T_a{}^{\nu\rho} \partial_\nu \delta e^a{}_{\rho},
\end{aligned}
\tag{3.171}
$$

while the second term in (3.170) gives

$$
\begin{aligned}
\left(\delta T_{\mu\nu\rho}\right) T^{\mu\nu\rho} &= \delta\left(g_{\mu\alpha} T^{\alpha}{}_{\nu\rho}\right) T^{\mu\nu\rho} \\
&= \left[\left(\delta g_{\mu\alpha}\right) T^{\alpha}{}_{\nu\rho} + g_{\mu\alpha}\left(\delta T^{\alpha}{}_{\nu\rho}\right)\right] T^{\mu\nu\rho} \\
&= \left[\eta_{ab}\left(e^a{}_{\mu} \delta e^b{}_{\alpha} + e^a{}_{\alpha} \delta e^b{}_{\mu}\right) T^{\alpha}{}_{\nu\rho} + g_{\mu\alpha}\left(-e_b{}^{\alpha} T^{\sigma}{}_{\nu\rho} \delta e^b{}_{\sigma}\right) + e_a{}^{\alpha}\left(\partial_\nu \delta e^a{}_{\rho} - \partial_\rho \delta e^a{}_{\nu}\right)\right] T^{\mu\nu\rho} \\
&= \eta_{ab}\left(e^a{}_{\mu} \delta e^b{}_{\alpha} T^{\alpha}{}_{\nu\rho} T^{\mu\nu\rho} + e^a{}_{\alpha} \delta e^b{}_{\mu} T^{\alpha}{}_{\nu\rho} T^{\mu\nu\rho}\right) - g_{\mu\alpha} e_b{}^{\alpha} T^{\sigma}{}_{\nu\rho} T^{\mu\nu\rho} \delta e^b{}_{\sigma} \\
&+ g_{\mu\alpha} e_a{}^{\alpha} T^{\mu\nu\rho} \partial_\nu \delta e^a{}_{\rho} - g_{\mu\alpha} e_a{}^{\alpha} T^{\mu\nu\rho} \partial_\rho \delta e^a{}_{\nu} \\
&= 2 T^{\alpha\nu\rho} T_{\sigma\nu\rho} e_b{}^{\sigma} \delta e^b{}_{\alpha} - T^{\alpha\nu\rho} T_{\sigma\nu\rho} e_b{}^{\sigma} \delta e^b{}_{\sigma} + 2 T_a{}^{\nu\rho} \partial_\nu \delta e^a{}_{\rho} \\
&= T^{\alpha\nu\rho} T_{\sigma\nu\rho} e_b{}^{\sigma} \delta e^b{}_{\alpha} + 2 T_a{}^{\nu\rho} \partial_\nu \delta e^a{}_{\rho}.
\end{aligned}
\tag{3.172}
$$

We substitute (3.171) and (3.172) in (3.170),

$$
\delta\left(T^{\mu\nu\rho} T_{\mu\nu\rho}\right) = -4 T^{\mu\nu\lambda} T_{\mu\nu\sigma} e_a{}^{\sigma} \delta e^a{}_{\lambda} + 4 T_{\alpha}{}^{\nu\rho} e_a{}^{\alpha} \partial_\nu \delta e^a{}_{\rho}. \tag{3.173}
$$

After that, we turn to the second term in (3.165)

$$\delta\left(T^{\mu\nu\rho}T_{\nu\mu\rho}\right) = \left(\delta T^{\mu\nu\rho}\right)T_{\nu\mu\rho} + T^{\mu\nu\rho}\left(\delta T_{\nu\mu\rho}\right). \tag{3.174}$$

The first term in (3.174) becomes

$$\begin{aligned}
\left(\delta T^{\mu\nu\rho}\right)T_{\nu\mu\nu} &= \delta\left(g^{\nu\alpha}g^{\rho\beta}T^{\mu}{}_{\alpha\beta}\right)T_{\nu\mu\rho} \\
&= \left[\left(\delta g^{\nu\alpha}\right)g^{\rho\beta}T^{\mu}{}_{\alpha\beta} + g^{\nu\alpha}\left(\delta g^{\rho\beta}\right)T^{\mu}{}_{\alpha\beta} + g^{\nu\alpha}g^{\rho\beta}\left(\delta T^{\mu}{}_{\alpha\beta}\right)\right]T_{\nu\mu\rho} \\
&= \Big[-\left(g^{\alpha\lambda}e_a{}^{\nu} + g^{\nu\lambda}e_a{}^{\alpha}\right)\delta e^a{}_{\lambda}g^{\rho\beta}T^{\mu}{}_{\alpha\beta} - g^{\nu\alpha}\left(g^{\beta\lambda}e_a{}^{\rho} + g^{\rho\lambda}e_a{}^{\beta}\right)\delta e^a{}_{\lambda}T^{\mu}{}_{\alpha\beta} \\
&+g^{\nu\alpha}g^{\rho\beta}\left(-e_a{}^{\mu}T^{\lambda}{}_{\alpha\beta}\delta e^a{}_{\lambda} + e_a{}^{\mu}\left(\partial_\alpha\delta e^a{}_{\beta} - \partial_\beta\delta e^a{}_{\alpha}\right)\right)\Big]T_{\nu\mu\rho} \\
&= -T_{\sigma\varepsilon\gamma}T^{\lambda\sigma\gamma}e_a{}^{\varepsilon}\delta e^a{}_{\lambda} - 2T^{\mu\nu\lambda}T_{\nu\mu\varepsilon}e_a{}^{\varepsilon}\delta e^a{}_{\lambda} + \left(T^{\alpha}{}_{\mu}{}^{\beta} - T^{\beta}{}_{\mu}{}^{\alpha}\right)e_a{}^{\mu}\partial_\alpha\delta e^a{}_{\beta} \\
&= T_{\sigma\gamma\varepsilon}T^{\lambda\sigma\gamma}e_a{}^{\varepsilon}\delta e^a{}_{\lambda} - 2T^{\mu\nu\lambda}T_{\nu\mu\varepsilon}e_a{}^{\varepsilon}\delta e^a{}_{\lambda} + \left(T^{\alpha}{}_{\mu}{}^{\beta} - T^{\beta}{}_{\mu}{}^{\alpha}\right)e_a{}^{\mu}\partial_\alpha\delta e^a{}_{\beta},
\end{aligned} \tag{3.175}$$

and the second term in (3.174) gives

$$\begin{aligned}
\left(\delta T_{\nu\mu\rho}\right)T^{\mu\nu\rho} &= \delta\left(g_{\nu\alpha}T^{\alpha}{}_{\mu\rho}\right)T^{\mu\nu\rho} \\
&= \left[\left(\delta g_{\nu\alpha}\right)T^{\alpha}{}_{\mu\rho} + g_{\nu\alpha}\left(\delta T^{\alpha}{}_{\mu\rho}\right)\right]T^{\mu\nu\rho} \\
&= \left[\eta_{ab}\left(e^a{}_{\nu}\delta e^b{}_{\alpha} + e^a{}_{\alpha}\delta e^b{}_{\nu}\right)T^{\alpha}{}_{\mu\rho} + g_{\nu\alpha}\left(-e_a{}^{\alpha}T^{\lambda}{}_{\mu\rho}\delta e^a{}_{\lambda} + e_a{}^{\alpha}\left(\partial_\mu\delta e^a{}_{\rho} - \partial_\rho\delta e^a{}_{\mu}\right)\right)\right]T^{\mu\nu\rho} \\
&= \eta_{ab}e^a{}_{\nu}T^{\alpha}{}_{\mu\rho}T^{\mu\nu\rho}\delta e^b{}_{\alpha} + \eta_{ab}e^a{}_{\alpha}T^{\alpha}{}_{\mu\rho}T^{\mu\nu\rho}\delta e^b{}_{\nu} - g_{\nu\alpha}e_a{}^{\alpha}T^{\lambda}{}_{\mu\rho}T^{\mu\nu\rho}\delta e^a{}_{\lambda} \\
&+ g_{\nu\alpha}e_a{}^{\alpha}\left(\partial_\mu\delta e^a{}_{\rho} - \partial_\rho\delta e^a{}_{\mu}\right)T^{\mu\nu\rho} \\
&= \eta_{ab}T^{\alpha}{}_{\mu\rho}T^{\mu a\rho}\delta e^b{}_{\alpha} + \eta_{ab}T^{a}{}_{\mu\rho}T^{\mu\nu\rho}\delta e^b{}_{\nu} - e_a{}^{\alpha}T^{\lambda}{}_{\mu\rho}T^{\mu}{}_{\alpha}{}^{\rho}\delta e^a{}_{\lambda} + g_{\nu\alpha}e_a{}^{\alpha}T^{\mu\nu\rho}\partial_\mu\delta e^a{}_{\rho} \\
&- g_{\nu\alpha}e_a{}^{\alpha}T^{\mu\nu\rho}\partial_\rho\delta e^a{}_{\mu} \\
&= T^{\alpha}{}_{\mu\rho}T^{\mu}{}_{b}{}^{\rho}\delta e^b{}_{\alpha} + T_{b\mu\rho}T^{\mu\nu\rho}\delta e^b{}_{\nu} - T^{\lambda}{}_{\mu\rho}T^{\mu}{}_{a}{}^{\rho}\delta e^a{}_{\lambda} + e_a{}^{\alpha}T^{\mu}{}_{\alpha}{}^{\rho}\partial_\mu\delta e^a{}_{\rho} - e_a{}^{\alpha}T^{\mu}{}_{\alpha}{}^{\rho}\partial_\rho\delta e^b{}_{\mu} \\
&= T^{\alpha}{}_{\mu\rho}T^{\mu}{}_{\gamma}{}^{\rho}e_b{}^{\gamma}\delta e^b{}_{\alpha} + T_{\gamma\mu\rho}T^{\mu\nu\rho}e_b{}^{\gamma}\delta e^b{}_{\nu} - T^{\lambda}{}_{\mu\rho}T^{\mu}{}_{\gamma}{}^{\rho}e_a{}^{\gamma}\delta e^a{}_{\lambda} + \left(T^{\mu}{}_{\alpha}{}^{\rho} - T^{\rho}{}_{\alpha}{}^{\mu}\right)\partial_\mu\delta e^b{}_{\rho} \\
&= T_{\gamma\mu\rho}T^{\mu\nu\rho}e_b{}^{\gamma}\delta e^b{}_{\nu} + \left(T^{\mu}{}_{\alpha}{}^{\rho} - T^{\rho}{}_{\alpha}{}^{\mu}\right)\partial_\mu\delta e^b{}_{\rho}.
\end{aligned} \tag{3.176}$$

Then , we substitute (3.176) and (3.175) in (3.174) to obtain

$$\delta\left(T^{\mu\nu\rho}T_{\nu\mu\rho}\right) = 2\left(T^{\beta\rho\mu} - T^{\rho\mu\beta}\right)T_{\rho\mu\nu}e_a{}^{\nu}\delta e^a{}_{\beta} + 2\left(T^{\alpha}{}_{\mu}{}^{\beta} - T^{\beta}{}_{\mu}{}^{\alpha}\right)e_a{}^{\mu}\partial_\alpha\delta e^a{}_{\beta} \tag{3.177}$$

Now we substitute the previous results (3.177) and (3.173) and (3.169)in (3.165), we obtain

$$
\begin{aligned}
e\delta T &= e\left[\frac{1}{4}\delta\left(T^{\mu\nu\rho}T_{\mu\nu\rho}\right)+\frac{1}{2}\delta\left(T^{\mu\nu\rho}T_{\nu\mu\rho}\right)-\delta\left(T_{\mu}T^{\mu}\right)\right]\\
&= e\left[-T^{\mu\nu\lambda}T_{\mu\nu\sigma}e_{a}{}^{\sigma}\delta e^{a}{}_{\lambda}+T_{\alpha}{}^{\nu\rho}e_{a}{}^{\alpha}\partial_{\nu}\delta e^{a}{}_{\rho}+\left(T^{\beta\rho\mu}-T^{\rho\mu\beta}\right)T_{\rho\mu\nu}e_{a}{}^{\nu}\delta e^{a}{}_{\beta}\right.\\
&\left.+\left(T^{\alpha}{}_{\mu}{}^{\beta}-T^{\beta}{}_{\mu}{}^{\alpha}\right)e_{a}{}^{\mu}\partial_{\alpha}\delta e^{a}{}_{\beta}-2\left(T^{\mu}T^{\sigma}{}_{\mu\alpha}+T^{\sigma}T_{\alpha}\right)e_{b}{}^{\alpha}\delta e^{b}{}_{\sigma}+2\left(T^{\mu}e_{a}{}^{\alpha}-T^{\alpha}e_{a}{}^{\mu}\right)\partial_{\alpha}\delta e^{a}{}_{\mu}\right].
\end{aligned}
$$

Collecting together the terms without derivations, we get

$$
\begin{aligned}
&-T^{\mu\nu\lambda}T_{\mu\nu\sigma}e_{a}{}^{\sigma}\delta e^{a}{}_{\lambda}+\left(T^{\beta\rho\mu}-T^{\rho\mu\beta}\right)T_{\rho\mu\nu}e_{a}{}^{\nu}\delta e^{a}{}_{\beta}-2\left(T^{\mu}T^{\sigma}_{\mu\alpha}+T^{\sigma}T_{\alpha}\right)e_{b}{}^{\alpha}\delta e^{b}{}_{\sigma}\\
&=T^{\mu\nu\lambda}T_{\mu\nu\sigma}e_{a}{}^{\sigma}\delta e^{a}{}_{\sigma}+T^{\beta\rho\mu}T_{\rho\mu\nu}e_{a}{}^{\nu}\delta e^{a}{}_{\beta}-T^{\rho\mu\beta}T_{\rho\mu\nu}e_{a}{}^{\nu}\delta e^{a}{}_{\beta}-2T^{\mu}T_{\mu\alpha}e_{b}{}^{\alpha}\delta e^{b}{}_{\sigma}-2T^{\sigma}T_{\alpha}e_{b}{}^{\alpha}\delta e^{b}{}_{\sigma}\\
&=\left(T^{\mu\nu\beta}T_{\mu\nu a}+T^{\beta\rho\mu}T_{\rho\mu a}-T^{\rho\mu\beta}T_{\rho\mu a}-2T^{\mu}T^{\beta}{}_{\mu a}+2T^{\beta}T_{a}\right)\delta e^{a}{}_{\beta}\\
&=\left(-T_{\lambda}{}^{\mu\beta}T^{\lambda}{}_{\mu a}+T^{\beta\mu}{}_{\lambda}T^{\lambda}{}_{\mu a}+T_{\lambda}{}^{\beta\mu}T^{\lambda}{}_{\mu a}-2T^{\alpha\mu}{}_{\alpha}T^{\mu\beta}{}_{\mu}+2T^{\alpha\beta}{}_{\alpha}T^{\mu}{}_{\mu a}\right)\delta e^{a}{}_{\beta}\\
&=4\left(T^{\lambda}{}_{\mu a}S_{\lambda}{}^{\beta\mu}\right)\delta e^{a}{}_{\beta}.
\end{aligned}
\tag{3.178}
$$

Then , applying integration by parts to the terms with derivations we obtain,

$$
\begin{aligned}
T_{\alpha}{}^{\nu\rho}e_{a}{}^{\alpha}\partial_{\nu}\delta e^{a}{}_{\rho} &= \int\partial_{\nu}\left(T_{\alpha}{}^{\nu\rho}e_{a}{}^{\alpha}\delta e^{a}{}_{\rho}\right)d^{4}x-\int\left(\partial_{\nu}T_{\alpha}{}^{\nu\rho}e_{a}{}^{\alpha}\right)\delta e^{a}{}_{\rho}d^{4}x\\
&= -\int\left(\partial_{\nu}T_{\alpha}{}^{\nu\rho}e_{a}{}^{\alpha}\right)\delta e^{a}{}_{\rho}d^{4}x\\
&= -\partial_{\nu}T_{a}{}^{\nu\rho}\delta e^{a}{}_{\rho},
\end{aligned}
\tag{3.179}
$$

$$
\begin{aligned}
\left(T^{\alpha}{}_{\mu}{}^{\beta}-T^{\beta}{}_{\mu}{}^{\alpha}\right)e_{a}{}^{\mu}\partial_{\alpha}\delta e^{a}{}_{\beta} &= \int\partial_{\alpha}\left[\left(T^{\alpha}{}_{\mu}{}^{\beta}-T^{\beta}{}_{\mu}{}^{\alpha}\right)e_{a}{}^{\mu}\delta e^{a}{}_{\beta}\right]d^{4}x\\
&-\int\partial_{\alpha}\left[\left(T^{\alpha}{}_{\mu}{}^{\beta}-T^{\beta}{}_{\mu}{}^{\alpha}\right)e_{a}{}^{\mu}\right]\delta e^{a}{}_{\beta}d^{4}x\\
&= -\left(\partial_{\alpha}T^{\alpha}{}_{a}{}^{\beta}-\partial_{\alpha}T^{\beta}{}_{a}{}^{\alpha}\right)\delta e^{a}{}_{\beta},
\end{aligned}
\tag{3.180}
$$

$$2\left(T^{\mu}e_{a}{}^{\alpha}-T^{\alpha}e_{a}{}^{\mu}\right)\partial_{\alpha}\delta e^{a}{}_{\mu}=\int\partial_{\alpha}\left[2\left(T^{\mu}e_{a}{}^{\alpha}-T^{\alpha}e_{a}{}^{\mu}\right)\delta e^{a}{}_{\mu}\right]d^{4}x$$
$$-2\int\partial_{\alpha}\left(T^{\mu}e_{a}{}^{\alpha}-T^{\alpha}e_{a}{}^{\mu}\right)\delta e^{a}{}_{\mu}d^{4}x$$
$$=-2\int\partial_{\alpha}\left(T^{\mu\alpha}{}_{\mu}e_{a}{}^{\mu}-T^{\alpha\mu}{}_{\alpha}e_{a}{}^{\alpha}\right)\delta e^{a}{}_{\mu}d^{4}x$$
$$=-2\partial_{\alpha}\left(T^{\mu\alpha}{}_{a}-T^{\alpha\mu}{}_{a}\right)\delta e^{a}{}_{\mu}. \tag{3.181}$$

Then , we add (3.179) , (3.180) and (3.181) , to get

$$\left(T^{\alpha}{}_{\mu}{}^{\beta}-T^{\beta}{}_{\mu}{}^{\alpha}\right)e_{a}{}^{\mu}\partial_{\alpha}\delta e^{a}{}_{\beta}-2\left(T^{\mu}T^{\sigma}{}_{\mu\alpha}+T^{\sigma}T_{\alpha}\right)e_{b}{}^{\alpha}\delta e^{b}{}_{\sigma}+2\left(T^{\mu}e_{a}{}^{\alpha}-T^{\alpha}e_{a}{}^{\mu}\right)\partial_{\alpha}\delta e^{a}{}_{\mu}=-4\partial_{\alpha}S_{a}{}^{\alpha\beta}. \tag{3.182}$$

Replacing (3.182) and (3.178) in (3.165), one finds

$$e\delta T=4\left[-\partial_{\alpha}\left(eS_{a}{}^{\alpha\beta}\right)+eT^{\lambda}{}_{\mu a}S_{\lambda}{}^{\beta\mu}\right]\delta e^{a}{}_{\beta}. \tag{3.183}$$

Finally , we find the variation of the action with respect to tetrad fields is

$$\delta S=c\int d^{4}x\left\{ee_{a}{}^{\beta}\delta e^{a}{}_{\beta}T+4\left[-\partial_{\alpha}\left(eS_{a}{}^{\alpha\beta}\right)+eT^{\lambda}{}_{\mu a}S_{\lambda}{}^{\beta\mu}\right]\delta e^{a}{}_{\beta}+\delta\left(e\mathcal{L}_{m}\right)\right\}=0. \tag{3.184}$$

from which, the equation of motion of the gravitational field in TEGR theory is obtained as

$$\frac{4}{e}\partial_{\alpha}\left(eS_{a}{}^{\alpha\beta}\right)-e_{a}{}^{\beta}T-T^{\lambda}{}_{\mu a}S_{\lambda}{}^{\beta\mu}=\frac{1}{c}T_{a}{}^{\beta}, \tag{3.185}$$

where the energy-momentum tensor is defined as

$$T_{a}{}^{\beta}=\frac{1}{e}\frac{\delta\left(e\mathcal{L}_{m}\right)}{\delta e^{a}{}_{\beta}}. \tag{3.186}$$

Let us now consider the action of a more general TEGR theory with non minimally coupled scalar field

$$S=\int d^{4}xe\left[\frac{\kappa}{2}\left(T+\alpha T\varphi^{2}\right)-X+V\left(\varphi\right)+\mathcal{L}_{m}\right], \tag{3.187}$$

where $V(\varphi)$ is the scalar potential and $X$ is the kinetic term

$$X = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi. \tag{3.188}$$

The variation with respect to the tetrad fields gives

$$\begin{aligned}
\delta_e X &= \delta \left( -\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi \right) \\
&= -\frac{1}{2} \left[ (\delta_e g^{\mu\nu}) \nabla_\mu \varphi \nabla_\nu \varphi + g^{\mu\nu} \delta_e (\nabla_\mu \varphi) \nabla_\nu \varphi + g^{\mu\nu} \nabla_\mu \varphi \delta_e (\nabla_\nu \varphi) \right] \\
&= -\frac{1}{2} - \left( g^{\nu\rho} e_a{}^{\mu} + g^{\mu\rho} e_a{}^{\nu} \right) \delta e^a{}_\rho \nabla_\mu \varphi \nabla_\nu \varphi \\
&= -\frac{1}{2} - \left( g^{\nu\rho} e_a{}^{\mu} \nabla_\mu \varphi \nabla_\nu \varphi + g^{\mu\rho} e_a{}^{\nu} \nabla_\mu \varphi \nabla_\nu \varphi \right) \delta e^a{}_\rho \\
&= \frac{2}{2} \left( \nabla_a \varphi \nabla^\rho \varphi \delta e^a{}_\rho \right) \\
&= \nabla_a \varphi \nabla^\rho \varphi \delta e^a{}_\rho,
\end{aligned} \tag{3.189}$$

$$\begin{aligned}
\delta_e \left( \frac{\alpha}{2} T \varphi^2 \right) &= \frac{\alpha}{2} \left[ (\delta_e T) \varphi^2 + T \left( \delta_e \varphi^2 \right) \right] \\
&= \frac{\alpha}{2} \varphi^2 \left( -\partial_\alpha S_a{}^{\alpha\beta} + T^\lambda{}_{\mu a} S_\lambda{}^{\beta\mu} \right) \delta e^a{}_\mu,
\end{aligned} \tag{3.190}$$

where we have used the fact that $\delta_e f(\varphi) = 0$.

The variation with respect the scalar field $\varphi$ gives

$$\delta_\varphi T = 0, \tag{3.191}$$

and

$$\begin{aligned}
\delta_\varphi X &= -\frac{1}{2} \delta_\varphi \left( g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi \right) \\
&= -\frac{1}{2} \left[ g^{\mu\nu} \left( \delta_\varphi \nabla_\mu \varphi \right) \nabla_\nu \varphi + g^{\mu\nu} \nabla_\nu \varphi \left( \delta_\varphi \nabla_\nu \varphi \right) \right] \\
&= -\frac{1}{2} \left[ \nabla_\mu \left( g^{\mu\nu} \nabla_\nu \varphi \delta \varphi \right) + \nabla_\nu \left( g^{\mu\nu} \nabla_\mu \varphi \delta \varphi \right) \right] \\
&= -\frac{1}{2} \left( \nabla_\mu \nabla^\mu \varphi \delta \varphi + \nabla_\nu \nabla^\nu \varphi \delta \varphi \right) \\
&= -\Box \varphi \delta \varphi,
\end{aligned} \tag{3.192}$$

and

$$\delta_{\varphi}\left(\alpha T\varphi^{2}\right)=\alpha\left[\left(\delta_{\varphi}T\right)\varphi^{2}+T\left(\delta_{\varphi}\varphi^{2}\right)\right]$$
$$=2\alpha T\varphi\delta\varphi, \tag{3.193}$$

$$\delta_{\varphi}V\left(\varphi\right)=\frac{\delta V\left(\varphi\right)}{\delta\varphi}\delta\varphi=V'\left(\varphi\right). \tag{3.194}$$

Substituting the above equations in the variation of eq.(3.187) , one finds the total variation of the action

$$\begin{aligned}\delta S=&\int d^{4}x\left\{ee_{a}{}^{\beta}F\left(T,X,\varphi\right)\delta e^{a}{}_{\beta}+2k\left[-\partial_{\alpha}\left(eS_{a}{}^{\alpha\beta}\right)+eT^{\lambda}{}_{\mu a}S_{\lambda}{}^{\beta\mu}\right]\delta e^{a}{}_{\beta}+\nabla_{a}\varphi\nabla^{\beta}\varphi\delta e^{a}{}_{\beta}\right.\\&\left.+2k\alpha\varphi^{2}\left[-\partial_{\alpha}\left(eS_{a}{}^{\alpha\beta}\right)+eT^{\lambda}{}_{\mu a}S_{\lambda}{}^{\beta\mu}\right]\delta e^{a}{}_{\beta}-\Box\varphi\delta\varphi+k\alpha T\varphi\delta\varphi+V'\left(\varphi\right)\delta\varphi\right\}\\=&\int d^{4}x\left\{\left[ee_{a}{}^{\beta}F\left(T,X,\varphi\right)+2k\left[-\partial_{\alpha}\left(eS_{a}{}^{\alpha\beta}\right)+eT^{\lambda}{}_{\mu a}S_{\lambda}{}^{\beta\mu}\right]\left(1+\alpha\varphi^{2}\right)+\nabla_{a}\varphi\nabla^{\beta}\varphi\right]\delta e^{a}{}_{\beta}\right.\\&\left.+\left[-\Box\varphi+k\alpha T\varphi+V'\left(\varphi\right)\right]\delta\varphi\right\},\end{aligned} \tag{3.195}$$

where the equation of motion of gravitational field the term proportional to $\delta e^{a}{}_{\beta}$

$$ee_{a}{}^{\beta}\left(T+\alpha T\varphi^{2}\right)+2\left[-\partial_{\alpha}\left(eS_{a}{}^{\alpha\beta}\right)+eT^{\lambda}{}_{\mu a}S_{\lambda}{}^{\beta\mu}\right]\left(1+\alpha\varphi^{2}\right)+\nabla_{a}\varphi\nabla^{\beta}\varphi=\kappa^{-1}T_{a}^{:\beta}, \tag{3.196}$$

and, the equation of motion of scalar field $\varphi$ is the one proportional to $\delta\varphi$.

$$\Box\varphi-V'\left(\varphi\right)=\kappa\alpha T\varphi. \tag{3.197}$$

# Chapter 4

# Neutron star

## 4.1 Introduction to Neutron Stars

The last decade has been the theater of an intense activity in astrophysical observation. The study of strong gravitational field regime in alternative theories was becoming more interesting [6, 7, 8, 9]. Neutron Stars (NSs) are one of the most compact objects for testing different theoretical models of gravity. In fact, the theory of General Relativity (GR) explains very well the observations at the weak gravitational background [10], but one can imagine that the theory of GR is a subject of modification around a NSs.

Neutron stars (NS) are compact stars that contain very high-density matter. NS has typical mass of order $1.4M_{\odot}$, with a radius of about 10 km [11]. The NS has similar mass compared to our sun but their radius is 105 times smaller than the radius of the sun. Therefore, NS are extremely dense objects. The average mass density can be estimated by

$$\bar{\rho} = \frac{3M}{4\pi R^3} \approx 7.10^{14} \frac{g}{cm^3} \approx (2-3)\,\rho_0, \tag{4.1}$$

where $\rho_0 = 2,8 \times 10^{14} g/cm^3$ is the normal nuclear mass density.

The new astrophysical imprints beyond GR theory can be explored via Gravitational Waves (GWs) signal, from the collision between binary neutron stars (NSs), binary black holes (BHs) or binary BH-NS, which carry information about the properties of NSs [12, 13]. The golden era of gravitational-wave astronomy was recently launched by the first observation of GWs signal coming from a binary black hole merger at LIGO and Virgo observatories [14, 15]. After two years, the detection of gravitational waves from a binary neutron star merge GW170817 [16] together with gamma-ray burst GRB 170817A [17, 18, 19] has considerably advanced our understanding in alternative theory of gravity [20, 21, 22, 23, 24]. On August 14, 2019, the LIGO/Virgo Collaboration (LVC) announced the detection of GWs sourced by the collision of a black hole with mass $M \approx 23.2^{+1.1}_{-1.0} M_{\odot}$ and a compact object with mass $M \approx 2.59^{+0.08}_{-0.09} M_{\odot}$ [25], where $M_{\odot}$ is the solar mass. It is difficult to identify the nature of this object, whether it is the most massive NS or the least massive BH, because neither electromagnetic counterpart nor measurable tidal deformation signature was imprinted in GW19081-event.

## 4.2 Birth of a neutron star

There are two types of supernovae, which can be classified as two different phenomena. Names: Thermonuclear Supernovae and Core Collapse (or Gravitational) Supernovae. The term supernova first appeared in a 1934 article by Baade and Zwicky. The first distinction novæ is due to a sudden thermonuclear reaction The accumulation of matter from red giants to white dwarfs in binary star systems, from such- called a supernova. White dwarfs are composed of degenerate electronic matter and are the last stage of the evolution of stars with masses less than 8 solar masses (about 97% stars in our galaxy) after the stellar envelope expands into space, forming a Planetary Nebula. A given white dwarf may or may not be nova multiple times. All material accumulated since the last eruption must be expelled or burned. Supernovae are classified according to the absorption lines in their spectrum shortly after the explosion. When hydrogen lines are present, supernovae are type I, Otherwise type II. Type I supernovae are further subdivided into:

- Type Ia, when silicone II lines are visible,
- Type Ib, when helium I line is visible and no silicon II line,
- Type Ic, when neither silicon II nor helium I lines are visible.

No Type Ib, Ic or II supernovae have been observed in elliptical galaxies where no new stars are formed and all massive stars have exploded. Therefore, it can be concluded that in contrast to Types Ib, Ic and II, a Type Ia supernova is not the result of the collapse of a massive star into a compact body, but the result of the collapse of a white dwarf, as its internal degenerate pressure can no longer sustain the pressure from the companion auxiliary accumulation of substances. When the white dwarf collapses, the temperature rises, Carbon fusion begins throughout the white dwarf, immediately leading to the observation of a thermonuclear supernova. Differences between other species come from ancestors. In Type Ib and Type Ic supernovae, the precursors have lost their hydrogen shells (eg Wolf-Rayet stars).

## 4.3 Structure of Neutron Stars

An NS is usually divided into five layers. Atmosphere, shell, inner core, outer core and inner core. The core of a star makes up most of the stellar mass [11]. The thickness of the atmosphere can vary from about 10 centimeters to a few millimeters. It consists of hot plasma. Radiation from stars is produced in the atmosphere. outside of Looking at the radiation provides insight into the radius and mass of a star. If the NS is very cold, a solid or liquid surface may appear. The crust is about 100 meters thick and has a density of about $4.10^{11} g/cm^3$. It is mainly composed of iron ions and electrons. The Fermi energy of an electron increases with increasing pressure. Beta trapping enriches the nucleus with neutrons. mostly shell celebration. A few kilometers thick inner crust reaches a density of $0,5\rho_0$. inner shell by Free neutrons, electrons, and neutron-rich nuclei.

At this layer, the pressure is high enough to be free Neutrons are stable. The higher the pressure, the higher the Fermi energy of the electron. That Neutron-rich nuclei start releasing neutrons at a density of about $4,3.10^{11} g/cm^3$[26] In the outer core, which reaches a density of around $2\rho_0$ and extends for several kilometers, the matter consists mainly of free neutrons. Some protons, electrons and muons will also be encountered. The composition of the one plasma is strongly degraded [11]. Protons and neutrons form a Strongly interacting superfluid Fermi liquids. Regions with a density greater than $2\rho_0$ are the kernels. stellar density near the center Several $\rho_0$ are possible. The inner core is several kilometers thick. This layer in particular is not well understood because it is so dense that they have not yet been replicated on Earth. Several different exotic states of matter have been predicted for this layer. Matter is highly interactive on basis of

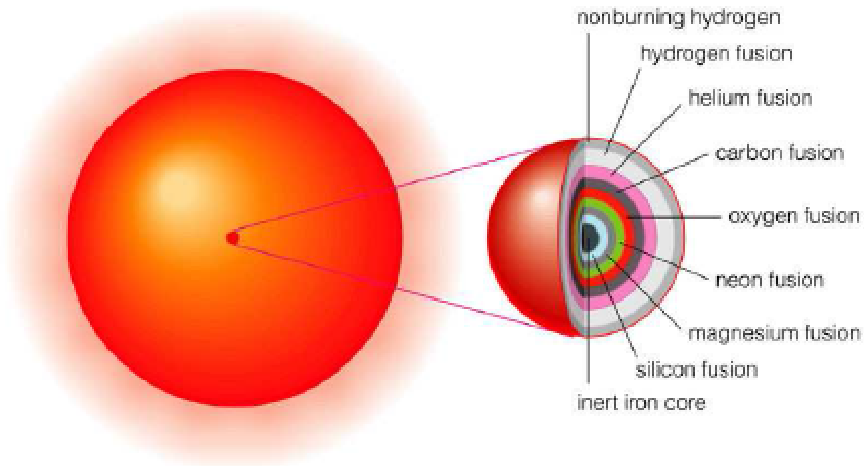


Figure 4.3: Structure of a neutron star

## 4.4 Equations for the stellar structure

### 4.4.1 The Schwarzschild Solution

Schwarzschild's solution is the first solution of Einstein's equations of motion

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}. \tag{4.2}$$

It is a solution in the vacuum ($T_{\alpha\beta} = 0$) static $\left(g_{\alpha\beta,x^0} = 0\right)$ and with spherical symmetry

$$ds^2 = -B(r)dt^2 + A\,(r)\,dr^2 + r^2\left(d\theta^2 + \sin^2 d\varphi^2\right). \tag{4.3}$$

The functions $A(r)$ and $B(r)$ must be infinitely far from any gravitational source ($r \longrightarrow \infty$), converges to unity to satisfy the asymptotic flatness. A substantial ansatz is

$$A(r) = \exp[2\lambda(r)], \quad B(r) = \exp[2\nu(r)]. \tag{4.4}$$

Then in vacuum

$$\begin{aligned} G_{\alpha\beta} = 0 \Longrightarrow G = 0 \Longrightarrow R - 2R = -R = 0 \\ \Longrightarrow R_{\alpha\beta} = 0 \end{aligned} \tag{4.5}$$

Then, Einstein's equations in vacuum become

$$R_{\alpha\beta} = 0. \tag{4.6}$$

A lengthily calculation leads to the components of Ricci tensor

$$R_{00} = \left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{2\nu'}{r}\right) e^{2\nu - 2\lambda} = 0, \tag{4.7}$$

$$R_{11} = -\nu'' + \nu'\lambda' + \frac{2\lambda'}{r} - \nu'^2 = 0, \tag{4.8}$$

$$R_{22} = 1 + \left(-1 - r\nu' + r\lambda'\right) e^{-2\lambda} = 0, \tag{4.9}$$

$$R_{33} = R_{22} \sin^2\theta. \tag{4.10}$$

We multiply (4.7) by $e^{-2\nu+2\lambda}$ and add to . This leads to

$$\frac{d}{dr}(\nu + \lambda) = 0 \Longrightarrow \nu(r) + \lambda(r) = c \tag{4.11}$$

where $c$ is a constant. We impose asymptotic flatness such that $g_{\alpha\beta} \to \eta_{\alpha\beta}$ (when $r \to \infty$)

$$\lim_{r\to\infty} \nu = 0, \quad \lim_{r\to\infty} \lambda = 0$$

Then we obtain $c = 0$ and

$$\nu(r) + \lambda(r) = 0$$
$$\lambda(r) = -\nu(r). \tag{4.12}$$

We substitute in (4.9) to get

$$1 - \frac{d}{dt}\left(re^{2\nu}\right) = 0,$$
$$re^{2\nu} = r + c,$$

where $c$ is an other constant. It follows that

$$e^{2\nu} = 1 + \frac{c}{r} = 1 - \frac{2m}{r}, \tag{4.13}$$

$$e^{2\lambda} = \frac{1}{1 - \frac{2m}{r}}, \tag{4.14}$$

where we have set $c = 2m$. In the weak field limit (the Newtonian approximation) we have $g_{tt} = -\left(1 - \frac{2MG}{rc^2}\right)$, from which we fix the constant $m$ by $m = \frac{MG}{c^2}$, where $M$ is the mass of the central object which creates the gravitational field.

Finally, the schwarzschild's solution( in the schwarzschild's coordinate) is given by

$$dS^2 = -\left(1 - \frac{2m}{r}\right)c^2dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2d\Omega^2, \tag{4.15}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2 d\varphi^2.$$

### 4.4.2 Tolman-Oppenheimer-Volkoff Equations

In this section we derive the equations governing the structure of a static, spherically symmetric, relativistic star considered as a perfect fluid. Then its matter content is described by the energy-momentum tensor of a perfect fluid. Inside the star, where there is no vacuum, the curvature scalar is no longer zero as it was established for empty space. Let us postulate that the interior of the star is described by a static and spherically symmetric solution as Eq. (4.3)

$$dS^2 = -e^{2\nu(r)}c^2dt^2 + e^{2\lambda(r)}dr^2 + r^2d\Omega^2, \tag{4.16}$$

from which we obtain

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\nu} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin\theta \end{pmatrix} \tag{4.17}$$

and

$$\begin{aligned}
&\Gamma^1_{00} = \nu' e^{2(\nu-\lambda)}, \qquad && \Gamma^0_{10} = \nu', \qquad && \Gamma^1_{11} = \nu\lambda', \\
&\Gamma^1_{22} = -re^{-\lambda}, && \Gamma^1_{33} = -re^{-2\lambda}, && \\
&\Gamma^2_{12} = \Gamma^3_{13}\frac{1}{r}, && \Gamma^2_{33} = \sin^2\theta\cos\theta, && \\
&\Gamma^3_{23} = \cot\theta. && &&
\end{aligned}$$

we have already calculated the Ricci tensor, and the Ricci scalar reads

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ &= -e^{-2\nu}R_{00} + e^{-2\lambda}R_{11} + \frac{2}{r^2}R_{22}. \end{aligned} \tag{4.18}$$

**Energy-momentum tensor of a perfect fluid**

The energy-momentum tensor of a perfect fluid (in units $c$=1) is given by

$$T^\mu_\nu = (\rho + p)\, u^\mu u^\nu + g^{\mu\nu}p. \tag{4.19}$$

The geometrical part is expressed by Einstein's equations of motion

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{1}{2}\delta^\mu_\nu R, \tag{4.20}$$

where

$$G_t^t = e^{-2\lambda}\left(\frac{1}{r^2} - 2\frac{\lambda'}{r}\right) - \frac{1}{r^2} = -\frac{d}{dr}\left[\frac{r\left(-1 - e^{-2\lambda(r)}\right)}{r^2}\right] = -8\pi\rho, \tag{4.21}$$

$$G_r^r = e^{-2\nu}\left(\frac{2\nu'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi p, \tag{4.22}$$

$$G_\theta^\theta = e^{-2\lambda}\left(\nu'' - \nu'\lambda' + \left(\nu'\right)^2 + \frac{\nu' - \lambda'}{r}\right) = 8\pi p, \tag{4.23}$$

$$G_\varphi^\varphi = G_\theta^\theta = 8\pi p. \tag{4.24}$$

In the rest frame of the star the 4-velocity is given by

$$u^\nu = \left(u^t, 0, 0, 0\right).$$

Using the normalization

$$g_{\mu\nu} u^\mu u^\nu = -1,$$

we obtain

$$\left(u^t\right)^2 g_{tt} = -1 \Longrightarrow u^t = \frac{1}{\sqrt{-g_{tt}}}, \tag{4.25}$$

and the energy density and pressure are

$$T_t^t = -\rho \quad T_i^i = 0.$$

From the Einstein equations we form the quantity

$$r^2 G_t^t = \frac{d}{dt}\left[r\left(1 - e^{-2\lambda}\right)\right] = kr^2 T_0^0 = 8\pi r^2 \rho, \tag{4.26}$$

which is integrated to give

$$e^{-2\lambda} = 1 - \frac{8\pi}{r}\int_0^r \rho\left(r\right) r^2 dr. \tag{4.27}$$

Now defining a quantity

$$m\left(r\right)=4\pi\int_{0}^{r}\rho\left(r\right)r^{2}dr, \tag{4.28}$$

we get

$$e^{-2\lambda}=1-\frac{2m(r)}{r}. \tag{4.29}$$

Adding eq(4.21) and eq(4.22) , we have

$$8\pi\left(\rho+p\right)=\frac{2}{r}e^{-2\lambda}\left(\lambda'+\nu'\right). \tag{4.30}$$

The function $\lambda$ can be eliminated with help the eq(4.29),

$$-2\lambda e^{-2\lambda}=\frac{2}{r}\left(\frac{m}{r}-m\right). \tag{4.31}$$

Substituting in equation (4.30) we obtain

$$8\pi p=\frac{-2m}{r^{3}}+2\left(1-\frac{2m}{r}\right)\frac{r'}{r},$$

$$\nu'=\frac{4\pi r^{3}p+m}{r^{2}\left(1-\frac{2m}{r}\right)}. \tag{4.32}$$

From Eq.(4.21) we have

$$2r\lambda'=\left(1-2\pi r^{2}\rho\right)e^{2\lambda}-1, \tag{4.33}$$

and Eq. (4.22) gives

$$2r\nu'=\left(1+2\pi r^{2}p\right)e^{2\lambda}-1. \tag{4.34}$$

The derivation of Eq.(4.34)with respect to $r$

$$2\nu'+2r\nu''=\left(2\lambda'+16\pi pr+16\pi pr\lambda'+8\pi r^{2}p'\right)e^{2\lambda},$$

and multiplying the above equation by $r$ and after some manipulations we obtain

$$2r^{2}\nu''=1+\left(16\pi r^{2}p+8\pi r^{3}p'\right)e^{2\lambda}-\left(1+8\pi r^{2}p\right)\left(1-8\pi r^{2}\rho\right)e^{4\lambda}, \tag{4.35}$$

$$2r^2\nu'^2 = \frac{1}{2}\left(1+8\pi r^2 p\right)^2 e^{4\lambda} - \left(1+8\pi r^2 p\right)e^{2\lambda} + \frac{1}{2}. \tag{4.36}$$

From Eqs.(4.29) , (4.35) , (4.36) we have $\lambda', \nu', \nu''$, therefore substituting in Eq.(4.23) ,and solving for $p'$ we obtain

$$\frac{dp}{dr} = -\frac{p+\rho\left[m(r)+4\pi r^3 p(r)\right]}{r\left[r-2m(r)\right]}, \tag{4.37}$$

and

$$\frac{d\nu}{dr} = \frac{4\pi r^3 p(r) + m(r)}{r^2\left(1-\frac{2m}{r}\right)}, \tag{4.38}$$

and

$$e^{-2\lambda} = 1 - \frac{2m}{r}. \tag{4.39}$$

This set of equations along with the mass equation, (4.28) , is known as Tolman-Oppenheimer-Volkoff Equations. The mass of the star is then obtained from

$$m'(r) = 4\pi r^2 \rho(r). \tag{4.40}$$

To complete the set of equations we need the equation of state (EoS), $p = f(\rho)$ , relating the pressure of the star to its energy density. The solution of this set of equations describe the complete structure of the star for a given central pressure $p_c = p(r = 0)$.

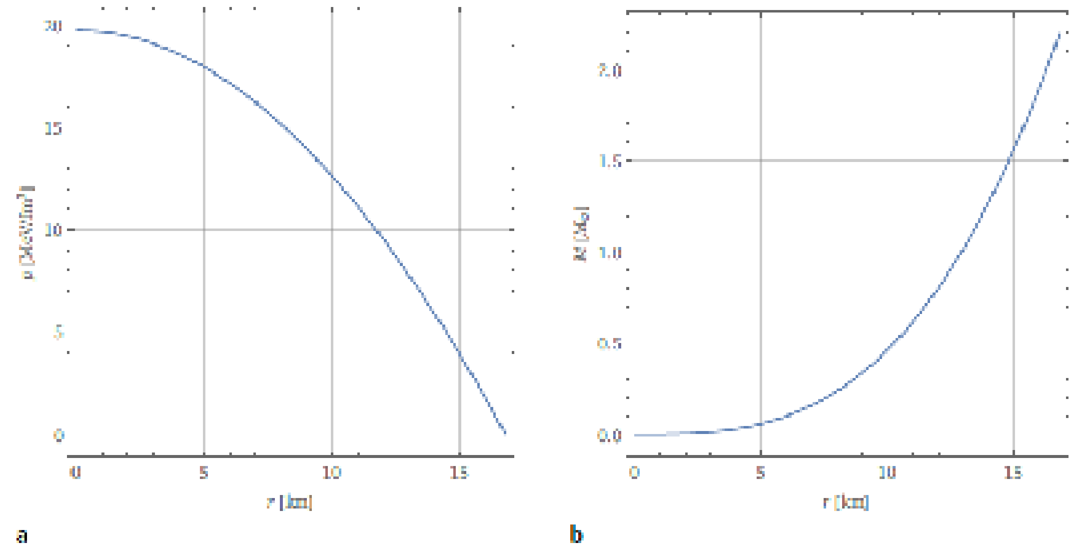


Figure 4.4.2: Pressure and mass of a neutron star vs the radius.

### 4.4.3 Interior Schwarzschild Solution

In addition to the solution in vacuum, K. Schwarzschild also published an analytical solution for the interior of an incompressible liquid sphere, known as the *Interior Schwarschild Solution* (ISS). We assume that the star energy density is constant $\rho = \rho_c = const$. Then, the mass of the liquid sphere is

$$M_* = \frac{4\pi}{3} R_*^3 \rho_c \Longrightarrow \rho_c = \frac{3M_*}{4\pi R_*^3}, \tag{4.41}$$

where $R_*$ is the star radius.

Now we can obtain analytically the mass (4.40) at a coordinate $r$ and one finds,

$$M(r) = \frac{4\pi}{3} r^3 \rho_c = \frac{M_* r^3}{R_*^3}. \tag{4.42}$$

The eqs.(4.41) and (4.42) can be inserted into the TOV Equation(4.37),

$$
\begin{aligned}
p' &= -\frac{\left[p\left(r\right)+\frac{3M_*}{4\pi R_*^3}\right]\left[\frac{M_* r^3}{R_*^3}+4\pi r^3 p\left(r\right)\right]}{r\left[r-\frac{2M_* r^3}{R_*^3}\right]} \\
&= -\frac{r\left[4\pi R_*^3 p\left(r\right)+3M_*\right]\left[M_*+4\pi R_*^3 p\left(r\right)\right]}{4\pi R_*^3\left[R_*^3-2M_* r^2\right]}.
\end{aligned} \tag{4.43}
$$

When reaching the center of star, the pressure should of course be zero, which leads to the initial condition $p\left(R_* = 0\right)$. The solution of (4.43) is given by

$$
p\left(r\right) = -\frac{3M_*\left(\sqrt{r^2\left(R_*-2M_*\right)}-\sqrt{R_*^3-2M_* r^2}\right)}{4\pi R_*^3\left(3\sqrt{R_*^3\left(R_*-2M_*\right)}-\sqrt{R_*-2M_* r^2}\right)}. \tag{4.44}
$$

In Fig. 4.4.3, the pressure and mass are plotted over the radius for a star with mass $M_* \approx 2.193 M_\odot$ and radius $R_* \approx 16.787$ km. The resulting central pressure is $p(r=0) = p_c = 19.7588$ Mev/fm$^3$.

The mass-radius-curves for different polytropic EoS are shown in Fig.4.4.3. The star with maximum mass for each EoS is marked by a cross. The stars right of the maximum mass star are stable, the ones on the left are unstable. The maximum mass for an EoS decreases with the polytropic index, i.e. stiffer EoS have higher maximum masses, than softer EoS.

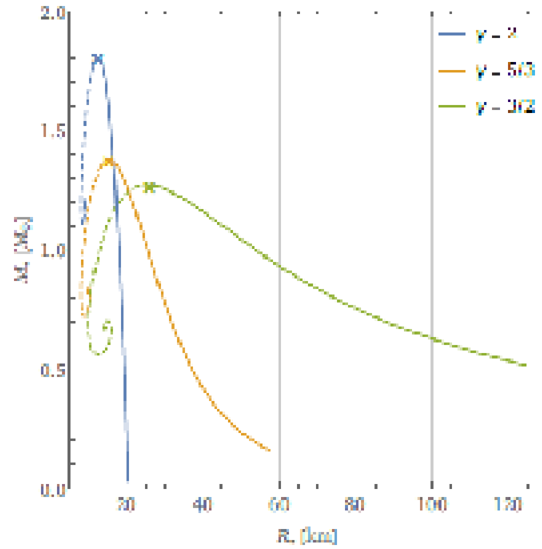


Figure 4.4.3: The M-R diagram vs the star radius for different valeus of the polytropic index in GR.

# Chapter 5

# Neutron stars in TEGR

## 5.1 Choice of the tetrad

In this chapter we exploit the framework of the TEGR theory developed in the last chapter to study the existence of NSs in TEGR theory with non minimal coupling to a scalar field and in the presence of a potential energy. This is just an approximate model of realistic rotating NSs.

We begin by introducing the non diagonal tetrad given by

$$\overline{e}^a_\mu = \begin{pmatrix} \sqrt{f(r)} & 0 & 0 & 0 \\ 0 & \sqrt{h(r)}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sqrt{h(r)}\sin\theta\cos\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \sqrt{h(r)}\cos\theta & -r\sin\theta & 0 \end{pmatrix}. \tag{5.1}$$

This off diagonal tetrad can be obtained from a diagonal tetrad by the following local Lorentz transformation, $e_{nd} = \Lambda e_d$,where

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & \sin\theta\sin\phi \\ 0 & \sin\theta\cos\phi & \cos\theta\sin\phi & \sin\theta\cos\phi \\ 0 & \cos\theta & \sin\theta & 0 \end{pmatrix}. \tag{5.2}$$

Now inserting the tetrad components in the relation of the scalar of torsion we obtain

$$T=\frac{2\left(\sqrt{h}-1\right)}{r^2h}\frac{f'}{f}-\frac{2\left(\sqrt{h}-1\right)^2}{r^2h}. \tag{5.3}$$

In the following we consider the action

$$S=\int d^4xe\left(\frac{\kappa}{2}T-X+\frac{1}{2}\mathcal{D}T\varphi^2+V(\varphi)\right), \tag{5.4}$$

where $X=-\varphi^{,\mu}\varphi_{,\mu}/2$ is the kinetic term, $\mathcal{D}$ is the non minimal coupling constant and $V$ is a scalar potential. Using EEs given by (3.196) we obtain the following set of equations

$$\begin{aligned}\rho_m=\frac{\kappa}{r^2}-\frac{\kappa}{r^2h}-V+\frac{\kappa h'}{rh^2}+\frac{\mathcal{D}}{r}\left(\frac{1}{r}-\frac{1}{rh}+\frac{h'}{h^2}\right)\varphi^2+\frac{4\mathcal{D}}{r}\left(\frac{1}{r\sqrt{h}}-\frac{1}{h}\right)\varphi\varphi'-\frac{\varphi'^2}{2h},\\ \frac{P_m}{c^2}=-\frac{\kappa}{r^2}+\frac{\kappa}{r^2h}+V+\frac{\kappa f'}{rfh}-\frac{\mathcal{D}}{r}\left(\frac{1}{r}-\frac{1}{rh}-\frac{f'}{fh}\right)\varphi^2-\frac{\varphi'^2}{2h}.\end{aligned} \tag{5.5}$$

The equation of energy conservation $\nabla_\alpha T^{\alpha\beta}=0$ gives

$$P'_m+\frac{\left(P_m+c\rho_m^2\right)}{2}\frac{f'}{f}=0. \tag{5.6}$$

The equation of motion of the scalar field is now obtained from (3.197)

$$\varphi''+\frac{1}{2}\left(\frac{4}{r}-\frac{f'}{f}-\frac{h'}{h}\right)\varphi'-\frac{2\mathcal{D}\left(\sqrt{h}-1\right)\left(f\left(1-\sqrt{h}\right)+rf'\right)}{r^2f}\varphi-hV_{,\varphi}=0. \tag{5.7}$$

At this stage we solve eqs.(5.5) for $f',h'$ in terms of $h,f$, we get

$$f'\left(r\right)=\frac{2f(r)h(r)(c^2\kappa-c^2r^2V(\varphi(r))+c^2\mathcal{D}\varphi(r^2)+r^2p_m+c^2(-2\kappa-2\mathcal{D}\varphi\left(r\right)^2+r^2\varphi\left(r\right)^2))}{2c^2r\left(\kappa+\mathcal{D}\varphi\left(r\right)^2\right)}, \tag{5.8}$$

$$\begin{aligned}h'\left(r\right)=&\frac{1}{2r\left(\kappa+\mathcal{D}\varphi\left(r\right)^2\right)}h\left(r\right)(2\kappa+2\mathcal{D}\varphi\left(r\right)^2-2h\left(r\right)(\kappa-r^2V\left(\varphi\left(r\right)\right)+\mathcal{D}\varphi\left(r\right)^2-r^2\rho_m\left(r\right))\\ &+8r\mathcal{D}\varphi\left(r\right)\varphi'\left(r\right)-8r\mathcal{D}\sqrt{h\left(r\right)}\varphi\left(r\right)\varphi'\left(r\right)+r^2\varphi'r\left(r\right)^2),\end{aligned} \tag{5.9}$$

$$P_m'(r) = -\frac{1}{4c^2 r\left(\kappa + \mathcal{D}\varphi(r)^2\right)}$$

$$P_m(r) + c^2\rho_m(r)\left(2h(r)\left(c^2\kappa - c^2r^2V(\varphi(r)) + c^2\mathcal{D}\varphi\left(r^2\right) + r^2P_m(r)\right) + \right. \tag{5.10}$$

$$\left. c^2\left(-2\kappa - 2\mathcal{D}\varphi(r)^2 + r^2\varphi^2(r)^2\right)\right), \tag{5.11}$$

$$\varphi''(r) = \frac{1}{2c^2r^2\left(\kappa + \mathcal{D}\varphi(r)^2\right)}$$

$$\begin{aligned}
&\Big(4\mathcal{D}h(r)^{\frac{3}{2}}\varphi(r)\left(c^2\kappa - c^2r^2V(\varphi(r)) + c^2\mathcal{D}\varphi(r)^2 + r^2P_m(r)\right)\\
&+ 2c^2\mathcal{D}\sqrt{h(r)}\varphi(r)\left(2\kappa + 2\mathcal{D}\varphi(r)^2 - r^2\varphi'(r)\right) + \\
&h(r)\left(-8c^2\mathcal{D}^2\varphi(r)^3 - 4\mathcal{D}\varphi(r)\left(2c^2\kappa - c^2r^2V(\varphi(r)) + r^2P_m(r)\right) + 2c^2r\mathcal{D}\varphi(r)^2\right.\\
&\left(rV'(\varphi(r)) - \varphi'(r)\right) + \left(r(2c^2rV'(\varphi(r)) + \left(2c^2r^2V(\varphi(r)) - r^2P_m(r) + \right.\right.\\
&\left.\left.\left. c^2\left(-2\kappa + r^2P_m(r)\right)\varphi'(r)\right)\right)\right)\Big).
\end{aligned} \tag{5.12}$$

Instead of the metric $h$, it is convenient to introduce the mass $\mathcal{M}(r)$ defined by

$$h(r) = 1 - \frac{2G\mathcal{M}}{rc^2}. \tag{5.13}$$

Deriving this equation with respect to $r$ and substituting for $h'$ we obtain

$$M' = \frac{2\pi r\kappa\left(2rh(r)\left(V(\varphi(r)) + \rho_m(r)\right) - 8\mathcal{D}\sqrt{h(r)}\varphi(r)\varphi'(r) + \varphi'(r)\left(8\mathcal{D}\varphi(r) + r\varphi'(r)\right)\right)}{h(r)\left(\kappa + \mathcal{D}\varphi(r)^2\right)}. \tag{5.14}$$

The solution of the set of equations given by (5.8-5.12) must be subjected to regularity conditions and initial conditions. The regularity conditions impose that

$$f'(r=0) = 0, \quad h'(r=0) = 0, \quad \varphi'(r=0) = 0, \quad P_m'(r=0) = 0. \tag{5.15}$$

while the initial conditions are fixed at the center of the NS and are obtained by expanding any quantity as

$$X\left(r\right)=X_c+\sum_{j=2}^{\infty}X_j r^j, \tag{5.16}$$

where $X_0 = X(r = 0)$. For $X = f$ and $X = h$ we set $f_c = 1$ and $h_c = 1$, respectively. In the presence of the potential we use the expansion

$$V\left(\varphi\right)=V\left(\varphi_c\right)+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^nV}{d\varphi^n}\bigg|_{\varphi=\varphi}\left(\varphi-\varphi_c\right)^n. \tag{5.17}$$

Before closing this section, we define the ADM mass $M$ of the star, as

$$M=\lim_{r\longrightarrow\infty}\mathcal{M}(r)=\frac{rc^2}{2G}(1-h)\bigg|_{r\longrightarrow\infty}. \tag{5.18}$$

The star radius $r_s$ is determined by the condition

$$P_m\left(r_s\right)=0. \tag{5.19}$$

## 5.2 NS in an harmonic potential

In the following, we choose the potential as:

$$V\left(\varphi\right)=\frac{1}{2}m^2\varphi^2 \tag{5.20}$$

where $m$ is the mass of the scalar field.

Let us make the following substitution $f\longrightarrow f^2, h\longrightarrow h^2$ and introduce the typical parameters of a NS:

$$r_0=\sqrt{\frac{8\pi\kappa}{\rho_0}}=89.664:\text{Km},\quad \rho_0=m_n n_0=1.6749\times10^{14}:\text{g:cm}^{-3} \tag{5.21}$$

where $m_n = 1.6749 \times 10^{24} g$ is the neutron mass and $n_0 = 0.1(fm)^{-3}$is the typical number density of NSs. Let us now define the following convenient variables

$$P_m \longrightarrow \rho_0 P_m, \quad \rho \longrightarrow \rho_0 \rho, \quad m \longrightarrow m\sqrt{\frac{\rho_0}{8\pi\kappa}}, \quad r \longrightarrow r_0 e^s. \tag{5.22}$$

In terms of the new time $s$ the Einstein equations of motion $(t,t)$ and $(r,r)$ give

$$\begin{aligned} f'(s) = & \frac{1}{4\left(1+\mathcal{D}\varphi(s)^2\right)} \\ & \left(f(s)\left(-2-2\mathcal{D}\varphi(s)^2+h(s)^2\right.\right. \\ & \left.\left.\left(2+16e^{2s}\pi P_m(s)+\left(-e^{2s}m^2+2\mathcal{D}\right)\varphi(s)^2\right)+\varphi'(s)^2\right), \end{aligned} \tag{5.23}$$

$$\begin{aligned} h'(r) = & \frac{1}{4\left(1+\mathcal{D}\varphi(s)^2\right)} \\ & h(s)\left(2+2\mathcal{D}\varphi(s)^2+h(s)^2\left(-2+16e^{2s}\pi\rho(s)+\left(e^{2s}m^2-2\mathcal{D}\right)\varphi(s)^2\right)+\right. \\ & \left. 8\mathcal{D}\varphi(s)\varphi'(s)-8\mathcal{D}h(s)\varphi(s)\varphi'(s)+\varphi'(s)^2\right), \end{aligned} \tag{5.24}$$

and the equation of motion of the scalar field becomes

$$\begin{aligned} \varphi''(s) = & \frac{1}{2\left(1+\mathcal{D}\varphi(s)^2\right)} \\ & \left(2\mathcal{D}h(s)^3\varphi(s)\left(2+16e^{2s}\pi P_m(s)+\left(-e^{2s}m^2+2\mathcal{D}\right)\varphi(s)^2\right)\right. \\ & 2\mathcal{D}\varphi(s)\varphi'(s)^2+h(s)^2\left(2\left(e^{2s}m^2-4\mathcal{D}-16e^{2s}\pi\mathcal{D}P_m(s)\right)\varphi(s)+\right. \\ & 4\left(e^{2s}m^2-2\mathcal{D}\right)\mathcal{D}\varphi(s)^3-2\left(1+4e^{2s}\pi P_m(s)-4e^{2s}\pi\rho(s)\right)\varphi'(s)+ \\ & \left.\left.\left(e^{2s}m^2-2\mathcal{D}\right)\varphi'(s)\right)+2\mathcal{D}h(s)\varphi(s)\,2+2\mathcal{D}\varphi(s)^2-\varphi'(s)^2\right), \end{aligned} \tag{5.25}$$

while the energy-momentum conservation and the mass variation equations lead to

$$\rho'(s) = -\frac{1}{4\left(1+\mathcal{D}\varphi\left(s\right)^{2}\right)P_{m}'\left(\rho(s)\right)}$$
$$\left(\left(P_{m}\left(s\right)+\rho\left(s\right)\right)\right.$$
$$\left.\left(-2-2\mathcal{D}\varphi\left(s\right)^{2}+h\left(s\right)^{2}\left(2+16e^{2s}\pi P_{m}\left(s\right)+\left(e^{2s}m^{2}-2\mathcal{D}\right)\varphi\left(s\right)^{2}\right)+\varphi'\left(s\right)^{2}\right)\right). \quad (5.26)$$

$$m'\left(s\right) = \frac{1}{16h\left(s\right)^{2}\pi\left(1+\mathcal{D}\varphi\left(s\right)^{2}\right)}$$
$$\left(3\theta^{3}\left(e^{2s}h\left(s\right)^{2}16e^{2s}\pi P_{m}\left(s\right)+m^{2}\varphi\left(s\right)^{2}\right)-\right.$$
$$\left.8\mathcal{D}h\left(s\right)\varphi\left(s\right)\varphi'\left(s\right)+\varphi'\left(s\right)\left(8\mathcal{D}\varphi\left(s\right)+\varphi'\left(s\right)\right)\right), \quad (5.27)$$

where $P_{m}'\left(\rho(s)\right) = \frac{dP_{m}(s)}{d\rho(s)}$.

These equations are completed with the following initial conditions

$$f\left(s_{0}\right) = 1+\frac{e^{2s_{0}}\left(24P_{m,c}\pi+8\pi\rho_{c}-m^{2}\varphi_{c}^{2}\right)}{6(1+\mathcal{D}\varphi_{c}^{2})}, \quad (5.28)$$

$$h\left(s_{0}\right) = 1+\frac{e^{2s}\left(16\pi\rho_{c}+m^{2}\varphi_{c}^{2}\right)}{6(1+\mathcal{D}\varphi_{c}^{2})}, \quad (5.29)$$

$$P\left(s_{0}\right) = P_{m,c}+\frac{e^{2s_{0}}\left(P_{m,c}+\rho_{c}\right)\left(8\pi\left(3P_{m,c}+\rho_{c}\right)-m^{2}\varphi_{c}^{2}\right)}{12\left(1+\mathcal{D}\varphi_{c}^{2}\right)}, \quad (5.30)$$

$$\varphi\left(s_{0}\right) = \varphi_{c}+\frac{1}{6}e^{2s_{0}}m^{2}\varphi_{c}, \quad (5.31)$$

$$m\left(s_{0}\right) = m_{c}+\frac{3e^{3s_{0}}\left(16\pi\rho_{c}+m^{2}\varphi_{c}^{2}\right)}{32\pi\left(1+\mathcal{D}\varphi_{c}^{2}\right)}. \quad (5.32)$$

In our numerical solution we have used the following set of EOS known as FPS, SSLy, BSK19, BSK20 and BSK21 given by

$$\xi = \log_{10}\left(\rho/\mathrm{gcm}\ ^{3}\right), \quad \zeta = \log_{10}\left(P_m/\mathrm{dyncm}\ ^{2}\right)$$

where for the five EOS we can write

$$\begin{aligned} \varsigma(\xi) \quad = \quad & \frac{a_1 + a_2\xi + a_3\xi^3}{1 + a_4\xi} f_0\left(a_5\left(\xi - a_6\right)\right) + \left(a_7 + a_8\xi\right) f_0\left(a_9\left(a_{10} - \xi\right)\right) + \\ & \left(a_{11} + a_{12}\xi\right) f_0\left(a_{13}\left(a_{14} - \xi\right)\right) + \left(a_{15} + a_{16}\xi\right) f_0\left(a_{17}\left(a_{18} - \xi\right)\right), \end{aligned} \tag{5.33}$$

where

$$f_0(x) = \frac{1}{1 + e^x}, \tag{5.34}$$

and the numerical coefficients $a_i$ can be found in [27].

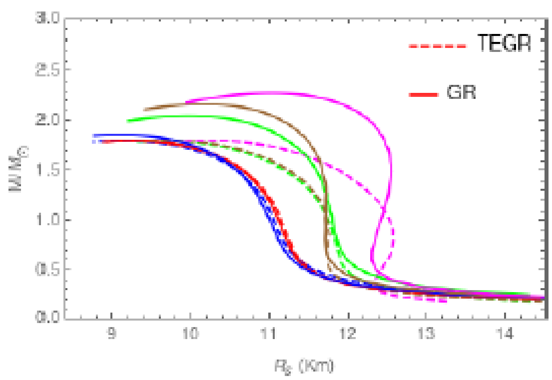


Figure 5.2: The M-R diagram in TEGR for m=$10^{-10}$, $\mathcal{D} = 0.1$ and $\varphi_c = 0.01$.

In Fig. 5.2 we show the $M - R$ diagram for the five EoS listed above for the TEGR and RG theory. We observe that we have NSs with mass of order 1.7 mass solar, which is compatible with all the measurements.

# Chapter 6

# Conclusion

This thesis deals with a study of the fundamental theory of gravity on torsional manifolds, but with vanishing curvature tensors. The theory is called Teleparallel Gravity (TEGR). Their field equations are equivalent to the General Relativity (GR). The lack of research in this field is mainly due to historical reasons since Einstein developed GR before TEGR. Both theories have the same equations, and can be regarded as the basic theory of gravitation, but their physical interpretations are very different. GR understands gravity as deformation of the space-time while the TEGR return to the view that gravity is one force mediated by the torsion tensor. Then even if they are similar on the field equations, their physical interpretations are very different. However, generalized TEGR theories are completely different from GR and even from generalizations of GR.

All Teleparallel theories of gravity are based on the scalar torsion $T$ which contains only first derivatives of tetrads. On the other hand, GR is based on the Ricci scalar R which depends on second derivatives of the metric tensor. Therefore, some Teleparallel theories are mathematically simpler than other modified theories coming from GR. In fact modified GR theories need some mechanisms to cancels terms of order higher than 2.

The main topi of this thesis is to study the existence of Neutron stars in TEGR theory non minimally coupled to a scalar field with a scalar potential. The results obtained are new and accurate. Indeed, for particular values of the theory parameters, we have obtained NSs with masses less than the ones in GR theory, but which remain consistent with the astrophysical observations.

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