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Exponent generating functions of convolved generalized (p,q) numbers and symmetric functions

Presented by:

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Contents

Introduction

Many modern sciences depend in their research and theories on sequences, but they are used more in mathematics and physics sciences, similar to the Fibonacci sequence, which is considered one of the most famous sequences that has been studied by many researchers over several centuries, as it took great importance .From many researchers and scientists over several centuries, due to its abundant applications in several different fields and sciences (physics, biology, computer science, engineering, mathematics...). These numbers were generalized by several authors in different ways. Some authors preserved the regressive relationship and changed the initial conditions, while others tended to generalize this sequence by keeping the initial conditions (the first terms) and changing the regressive relationship slightly, you can refer to [27-38]

The generating functions have many applications in many branches of mathematics and mathematical physics. Although calculating the generating functions, for example the Tribonacci numbers, the Lucas Thribonacci and 2-orthogonal polynomials, is in simple and well-known ways, the calculation of the generating functions of the product of numbers with each other or with 2-orthogonal polynomials is much more difficult than the previous one, as it was not addressed for this type previously, most articles that we find in calculating the generating function of the product of numbers with polynomials of the second order, you can refer to [26, 41, 42, 44].

In 2021, N. Saba looking for generating function of the products of (p,q)-Fibonacci numbers, (p,q)-Pell and (p,q)-Jacobstall with 2-orthogonal Chebychev polynomial in their paper:

N. Saba, A. Boussayoud, S.Araci, M. Kerada, M.Acikgoz, Construction of a new class of symmetric function of binary products of (p,q)-numbers with 2-orthogonal Chebyshev poly-

nomials, Boletin de la SociedadMatematica Mexicana 27(1), 1-26 ,2021

In 2022, H. Merzouk looking for ,Ordinary Generating Functions of Binary Products of third-order Recurrence Relations and 2-Orthogonal Polynomials in their paper:

H. Merzouk, A. Boussayoud, A. Abderrezzak,Ordinary Generating Functions of Binary Products of third-order Recurrence Relations and 2-Orthogonal Polynomials, Math. Slovaca. 72, 11-34, 2022.

In 2022 H.Zerroug worked on Gaussian Padovan, Gaussian Pell-Padovan numbers and new generating functions with some numbers and polynomials, Journal of Information and Optimization Sciences in her paper

H. Zerroug, A. Boussayoud, B , Aloui, M. Kerada, Gaussian Padovan, Gaussian Pell-Padovan numbers and new generating functions with some numbers and polynomials, Journal of Information and Optimization Sciences, 43 , 16,2022

Recently, Gulec and Taskara (2012) defined and studied the (p,q)-Pell and (p,q)-Pell Lucas numbers. Accordingly, they showed some interesting properties of these numbers. For their part, Suvarnamani and Tatong (2015) investigated the (p,q)- Fibonacci numbers ,they studied and analyzed some results using the well-known Binet's formula. Furthermore, Suvarnamani (2016) derived some useful properties of the (p,q)- Lucas numbers and provided also in another paper (Suvarnamani, 2017) some novel identities for the (p,q)-Fibonacci numbers using the matrix methods.

On the other hand, N. Saba et.al (2021), defined and studied both the Symmetric and generating functions of generalized (p,q)-numbers. For $r \in \mathbb{R}$, convolved Fibonacci numbers can be defined by $\frac{1}{(1-t-t^2)^r} = \sum\limits_{n=0}^{\infty} F_{n+1}(r)t^n$, the work of Fengi Qi in his article (see[31]). Our work is divided into four chapters as follows:

In the first chapter, we present the preliminary tools and notions necessary for the understanding of the following chapters, we first give some reminders about formal series, recurrence relations of some numbers and polynomials and also some auxiliary results on functions and its applications to sequences defined by linear recurrences of order two, three and four. At the end we introduce the elementary and complete symmetric functions and their properties.

In the second chapter, we firstly define a new generalization of (p, q) numbers and then derive the appropriate Binet's formula and generating functions concerning (p, q)- Fibonacci numbers, (p, q) Pell numbers, (p, q) Mersenne numbers by using a new theorems, (I presented this work at the 5^{th}international conference in Istanbul, turkey)

In the third chapter, we introduce the new generating functions for the products of fourth-order symmetric functions in several variables and third-order making use the new symmetrizing endomorphism operators $\delta_{e_3e_4}\delta_{e_2e_3}\delta_{e_1e_2}$ on the formal series $\sum_{n=0}^{+\infty} S_n(E_4)a_1^n z^n$. so I generalize the work of [15].
For $e_4 = 0$ we get the results of [15]
For $a_3 = 0$ we get the results of [19]

Concerning the fourth chapter we give the generating function of the product of some numbers of the fourth and third order and polynomials , by using the theorems in the third chapter(it is an application of the third chapter).

Chapter 1

Preliminary Notions and Definitions

In this chapter we mention some definitions and basic theorems regarding formal series, linear relations of recurrences, generating functions and symmetric fonctions that we use it in our work, for more details, you can refer to [2-13]

1 Formal series

1.1 Definitions, operations

Let $\mathbb{K}$ be a commutative field $(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$.

Definition 1.1. *The elements of the set* $K[[x]] = \left\{ \sum_{n=0}^{\infty} a_n x^n, a_n \in A \right\}$ *are called formal series with coefficients in* $\mathbb{K}$. *for* $n \in \mathbb{N}$, x^n *called the monomial of degree* n *and* a_n *it's coefficient.*

Definition 1.2. *Let* $u = \sum_{n=0}^{\infty} a_n x^n$ *and* $v = \sum_{n=0}^{\infty} b_n x^n$ *two formal series. We can define the operations like that*

1.The sum

$$u + v = \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

2. The product

$$u \times v = \sum_{n=0}^{\infty} c_n x^n,$$

while $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

3. Multiplication by scalar

$$\lambda u = \sum_{n=0}^{\infty} \lambda a_n x^n.$$

4. Derivation

$$u' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

1.2 Inverse of formal serie

Definition 1.3. *We say that the series* $\sum_{n=0}^{\infty} b_n x^n$ *is the inverse of the series* $\sum_{n=0}^{\infty} a_n x^n$ *if*

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = 1.$$

Proposition 1.4. *A formal series* $\sum\limits_{n=0}^{\infty} a_n x^n$ *is inversible if and only if* $a_0 \neq 0$.

Proof.

Let $\sum_{n=0}^{\infty} b_n x^n$ be the inverse of the series $\sum_{n=0}^{\infty} a_n x^n$ such as

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) &= 1 \\
\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n &= 1 \\
a_0 b_0 + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n &= 1.
\end{aligned}$$

By identification; we find

$$a_0 b_0 = 1,$$

and

$$\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n = 0,$$

which gives the coefficient $a_0 \neq 0$.

Reciprocally, suppose that $a_0 \neq 0$, so the triangular system of equations

$$\left\{\begin{array}{c}
a_0 b_0 = 1 \\
a_1 b_0 + a_0 b_1 = 0 \\
. \\
. \\
. \\
a_0 b_n + a_{n-1} b_1 + ... + a_n b_0 = 0
\end{array}\right. , \forall n \in \mathbb{N}$$

have an unique solution.

This is complete the proof. □

Example 1.5.

1. *Series* $\sum_{n=0}^{\infty} z^n$ *is inversible and its inverse* $: 1 - z$.

2. *Series* $\sum_{n=0}^{\infty} (-1)^n z^n$ *is inversible and its inverse:* $1 + z$.

3. *Series* $\sum_{n=0}^{\infty} \frac{z^n}{n}$ *is inversible and its inverse:* $\sum\limits_{n=0}^{\infty} (-1)^n \frac{z^n}{n}$.

2 Homogeneous linear recurrence relations

Definition 1.6. *A recurrence relation is said homogeneous linear of order* k *with constants coefficient ; if it is of the form*

$$u_n + d_1 u_{n-1} + d_2 u_{n-2} + ... + d_k u_{n-k} = 0, \tag{1.1}$$

where $d_1,\ d_2,\ ...,d_k \in \mathbb{K}$ *and* $d_k \neq 0$.

Remark 1.7.

1. $u_n = 0$ *is a solution of the equation* (1.1) *is called trivial solution.*

2. $u_n = x^n$ *is a solution of the equation* (1.1) *with* $u_n \neq 0$*; verify*

$$x^n + d_1 x^{n-1} + d_2 x^{k-2} + ... + d_k x^{n-k} = 0,$$

so,

$$x^{n-k}(x^k + d_1 x^{k-1} + d_2 x^{k-2} + ... + d_k) = 0.$$

This last equation is the characteristic equation associated to the equation (1.1)

Definition 1.8. *The characteristic polynomial which corresponds is*

$$P(x) = x^k + d_1 x^{k-1} + d_2 x^{k-2} + ... + d_k.$$

Theorem 1.9. *[8] Let* $d_1, d_2, ..., d_k$ *a reals numbers and* d_k *non - zero . Suppose that the characteristic polynomial* P *admit* k *distinct roots* $x_1,\ x_2, ...,\ x_k$*, so* u_n *is the general solution of recurrence relation if and only if*

$$u_n = c_1 x_1^n + c_2 x_2^n + ... + c_k x_k^n,$$

with $c_i, i \in \overline{1,k} \in \mathbb{K}$.

Theorem 1.10. *[8] Let* $d_1, d_2, ..., d_k$ *reals numbers and* d_k *non-zero . Suppose that the characteristic polynomial* P *admit* r *root* $x_1, x_2, ..., x_r$ *of multiplicity* $m_1, m_2, ..., m_r$ *such that* $m_i \geq 1, i \in \overline{1.r}$ *and* $\sum\limits_{i=0}^{r} m_i = k$, *so* u_n *is a general solution of the recurrence relation if and only if*

$$u_n = \sum_{i=1}^{r} \left(c_{i1} + c_{i2}n + ... + c_{im_r} n^{m_{i-1}}\right) x_i^n,$$

with $c_{ij} \in \mathbb{K}, \forall i, j$ *such as* $1 \leq i \leq r$ *and* $0 \leq j \leq m_{i-1}$.

2.1 Recurrences relations of some numbers

Definition 1.11. *[41] The Fibonacci sequence generalized* $(U_n)_{n\in\mathbb{N}}$ *is defined by the following recurence ralation*

$$\begin{cases} U_n = aU_{n-1} + bU_{n-2} \\ U_0 = \alpha, U_1 = \beta \end{cases},$$

with a, $b \in \mathbb{Z}^*$ *and* α, $\beta \in \mathbb{C}$.

The following table gives the recurrence relation of some k numbers

the values of a, b, α, β	sequences	recurrence relations
$a = k, b = \alpha = 1, \beta = 1$	k-Fibonacci	$\begin{cases} F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \forall n \geq 2 \\ F_{k,0} = 0,\ F_{k,1} = 1 \end{cases}$
$a = \beta = k, b = 1, \alpha = 2$	k-Lucas	$\begin{cases} L_{k,n} = kL_{k,n-1} + L_{k,n-2}, \forall n \geq 2 \\ L_{k,0} = 2,\ L_{k,1} = k \end{cases}$
$a = 2, b = k, \alpha = 0, \beta = 1$	k-Pell	$\begin{cases} P_{k,n} = 2P_{k,n-1} + kP_{k,n-2},\ \forall n \geq 2 \\ P_{k,0} = 0,\ P_{k,1} = 1 \end{cases}$
$a = k, b = 2, \alpha = 0, \beta = 1$	k-Jacobstal	$\begin{cases} J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2},\ \forall n \geq 2 \\ J_{k,0} = 0,\ J_{k,1} = 1 \end{cases}.$
$a = 3k, b = -2, \alpha = 0, \beta = 1$	k-Mersenne	$\begin{cases} M_{k,n} = 3kM_{k,n-1} - 2M_{k,n-2},\ \forall n \geq 2 \\ M_{k,0} = 0,\ M_{k,1} = 1 \end{cases}.$

Table 1.1- Recurrence relation of some k numbers.

Remark 1.12. *If we put* $k = 1$ *in the previous table we get the reccurence relation of some numbers*

2.2 Recurrence relation of Some Gaussian numbers of the third order

Definition 1.13. *[30] The Tribonacci sequence generalized* $\{V_n\}_{n\in\mathbb{N}}$ *is defined by the following recurence ralation*

$$\begin{cases} V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}, n \geq 3 \\ V_0 = a, V_1 = b, V_2 = c \end{cases},$$

with $a,\ b,\ c \in \mathbb{C}$ *and* $r,\ s \in \mathbb{R}$ *and* $t \in \mathbb{R}^*$

Remark 1.14. *1. For* $r = 0, s = t = 1, a = 1, b = 1 + i, c = 1 + i$*, we obtain the sequence Gaussien Padovan numbers*

$$\begin{cases} GP_n = GP_{n-2} + GP_{n-3},\ ln \geq 3 \\ GP_0 = 1,\ GP_1 = 1 + i, GP_2 = 1 + i \end{cases}.$$

Remark 1.15. *2. For* $r = 0, s = t = 1, a = 1 - i, b = 1 + i, c = 1 + i$*, we obtain the sequence Gaussien Pell Padovan numbers*

$$\begin{cases} GR_n = GR_{n-2} + GR_{n-3},\ n \geq 3 \\ GR_0 = 1 - i,\ GR_1 = 1 + i, GR_2 = 1 + i \end{cases}.$$

2.3 Recurrence relations of certain orthogonal polynomials

Proposition 1.16. *[16] Let* $\{P_n\}_{n\geq 0}$ *be a monic polynomial sequence. Then* $\{P_n\}_{n\geq 0}$ *is orthogonal if and only if there exist two sequences of complex number* $\{\beta_n\}_{n\geq 0}$ *and* $\{\gamma_n\}_{n\geq 0}$*, such that* γ_n *non- zero,*$n \geq 1$ *and satisfies the three-term recurrence relation*

$$\begin{cases} P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), n \geq 0 \\ P_0(x) = 1, P_1(x) = x - \beta_0, \end{cases}.$$

Definition 1.17. *We define chebychev polynomials first kind* T_n *by*

$$T_n(\cos\theta) = \cos(n\theta),\ \ \theta \in [0, \pi]$$

if $x \in [-1, 1]$*, so we have*

$$T_n(x) = \cos(n(\arccos x)).$$

Proposition 1.18. *The Chebychev polynomial of the first kind are defined by the following recurrence relation*

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_n(x) = 2xT_{n+1}(x) - T_{n+2}(x), n \geq 0 \end{cases}.$$

Proof.

It suffices to prove that

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x).$$

For $\theta \in [0, \pi]$, $n \in \mathbb{N}$,

$$\begin{aligned} \cos(n\theta) + \cos((n+2)\theta) &= \cos((n+1)\theta)\cos\theta + \sin((n+1)\theta)\sin\theta \\ &\quad + \cos((n+1)\theta)\cos\theta - \sin((n+1)\theta)\sin\theta \\ &= 2\cos((n+1)\theta)\cos\theta, \end{aligned}$$

by using the definition 1.8, we have

$$\forall\theta \in [0, \pi] \qquad T_n(\cos\theta) + T_{n+2}(\cos\theta) = 2T_{n+1}(\cos\theta)\cos(\theta),$$

with $x = \cos\theta$, so

$$\forall x \in [-1, 1] \qquad T_n(x) + T_{n+2}(x) = 2xT_{n+1}(x).$$

□

Definition 1.19. *We define chebychev polynomials second kind U_n by*

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

if $x \in [-1, 1]$, so we have

$$U_n(x) = \frac{\sin(n+1)(\arccos(x))}{\sqrt{1-x^2}}.$$

Proposition 1.20. *The Chebychev polynomial of the second kind are defined by the following recurrence relation*

$$\begin{cases} U_0(x) = 1 \\ U_1(x) = 2x \\ U_n(x) = 2xU_{n+1}(x) - U_{n+2}(x), \quad \forall n \geq 0 \end{cases}.$$

Proof. (see [8]). □

Definition 1.21. *We define chebychev polynomials third kind* V_n *by*

$$V_n(x) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\left(\frac{1}{2}\theta\right)},\ \ \theta \in [0,\pi],$$

with $x = \cos\theta$ *and* $x \in [-1,1]$.

Proposition 1.22. *The Chebychev polynomial of the third kind are defined by the following recurrence relation*

$$\begin{cases} V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), n \geq 0 \\ V_0(x) = 1, V_1(x) = 2x - 1 \end{cases}.$$

Proof.
It suffices to prove that

$$V_{n-2}(x) = 2xV_{n-1}(x) - V_n(x)$$

,

for $\theta \in [0,\pi]$, $n \in \mathbb{N}$, we have

$$\begin{aligned} V_n(x) + V_{n+2}(x) &= \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\left(\frac{1}{2}\theta\right)} + \frac{\cos\left(n+\frac{5}{2}\right)\theta}{\cos\left(\frac{1}{2}\theta\right)} \\ &= \frac{\cos\left(n+\frac{3}{2}\right)\theta\cos\theta - \sin\left(n+\frac{3}{2}\right)\theta\sin\theta}{\cos\left(\frac{1}{2}\theta\right)} \\ &\quad + \frac{\cos\left(n+\frac{3}{2}\right)\theta\cos\theta + \sin\left(n+\frac{3}{2}\right)\theta\sin\theta}{\cos\left(\frac{1}{2}\theta\right)} \\ &= \frac{2\cos\left(n+\frac{3}{2}\right)\theta\cos\theta}{\cos\left(\frac{1}{2}\theta\right)} \\ &= 2xV_{n+1}(x). \end{aligned}$$

□

Definition 1.23. *We define the function* W_n *by*

$$W_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{1}{2}\theta\right)},\ \ \theta \in [0,\pi]$$

with $x = \cos\theta$ *and* $x \in [-1,1]$.

Proposition 1.24. *The Chebychev polynomial of the fourth kind are defined by the following recurrence relation*

$$\begin{cases} W_n(x) = 2xW_{n+1}(x) - W_{n+2}(x) \\ W_0(x) = 1, W_1(x) = 2x + 1 \end{cases} \quad n \geq 2$$

Proof. (see [13]) □

We define the following polynomials refer to [16].

Fibonacci polynomials	$\begin{cases} F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ n \geq 2 \\ F_0(x) = 1,\ F_1(x) = x \end{cases}$
Lucas polynomials	$\begin{cases} L_n(x) = xL_{n-1}(x) + L_{n-2}(x), n \geq 2 \\ L_0(x) = 2,\ L_1(x) = x \end{cases}$
Pell polynomials	$\begin{cases} P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), n \geq 2 \\ P_0(x) = 0,\ P_1(x) = 1 \end{cases}$
Jacobstal polynomials	$\begin{cases} J_n(x) = J_{n-1}(x) + 2xJ_{n-2}(x),\ n \geq 2 \\ J_0(x) = 0,\ J_1(x) = 1 \end{cases}$

Definition 1.25. *The Fibonacci complex bivariate polynomials are defined by the following recurrence relation*

$$\begin{cases} F_n(x,y) = ixF_{n-1}(x,y) + F_{n-2}(x,y),\ n \geq 2 \\ F_0(x,y) = 0,\ F_1(x,y) = 1 \end{cases}.$$

Definition 1.26. *The Lucas complex bivariate polynomials are defined by the following recurrence relation*

$$\begin{cases} L_n(x,y) = ixL_{n-1}(x,y) + L_{n-2}(x,y), n \geq 2 \\ L_0(x,y) = 2,\ L_1(x,y) = ix \end{cases}.$$

3 Ordinary Generating Functions

Definition 1.27. *Let $(u_n)_{n\geq 0}$ the sequence of numbers , we call ordinary generating function of the sequence $(u_n)_{n\geq 0}$ the function*

$$G(z) = \sum_{n=0}^{\infty} u_n z^n.$$

Proposition 1.28. *Let $G(z)$the generating function of the sequence (a_n) and $H(z)$ the generating function of the sequence (b_n), so*

1 .$G(z)H(z)$ the generating function of the sequence $(a_nb_0 + a_{n-1}b_1 + ... + a_1b_{n-1} + a_0b_n)$.

2. $c_1G(z) + c_2H(z)$ the generating function of the sequence $(c_1a_n + c_2b_n)$.

3. $(1-z)\,G(z)$ *the generating function of the sequence* $(a_n - b_{n-1})$.

4. $nG(z)$ *the generating function of the sequence* (na_n).

5. $\frac{G(z)}{1-z}$ *the generating function of the sequence* $(a_0 + a_1 + ... + a_n)$.

Proof.
Its just verification. □

Proposition 1.29. *The generating function of Gaussian numbers of Padovan is given by*

$$G(z) = \frac{1 + (1+i)\,z + iz^2}{1 - z^2 - z^3}.$$

Proof.
By definition, we have

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} GP_n z^n \\ &= GP_0 + GP_1 + GP_2 + \sum_{n=3}^{\infty} GP_n z^n \\ &= GP_0 + GP_1 + GP_2 + \sum_{n=3}^{\infty} GP_{n-2} z^n + \sum_{n=3}^{\infty} GP_{n-3} z^n \\ &= \left(1 - z^2\right) GP_0 + GP_1 + GP_2 + z^2 \sum_{n=3}^{\infty} GP_n z^n + z^3 \sum_{n=3}^{\infty} GP_n z^n \\ &= \left(1 - z^2\right) GP_0 + GP_1 + GP_2 + z^2 G(z) + z^3 G(z), \end{aligned}$$

with $GP_0 = 1, GP_1 = GP_2 = 1 + i$, so

$$G(z) = \frac{1 + (1+i)\,z + iz^2}{1 - z^2 - z^3}.$$

□

Proposition 1.30. *The generating function of Gaussian numbers of Pell Padovan is given by*

$$G(z) = \frac{(1+i) + (1+i)\,z + (-1+3i)\,z^2}{1 - 2z^2 - z^3}.$$

Proof.
The proof is similar to that in the proposition 1.32. □

Theorem 1.31. *Let the sequence* $(w_n)_{n\geq 0}$ *definied by the following recurrence relation of the fourth order*

$$\begin{cases} w_n = r_1 w_{n-1} + r_2 w_{n-2} + r_3 w_{n-3} + r_4 w_{n-4}, \ n \geq 4 \\ w_0 = a, \quad w_1 = b, \quad w_2 = c, \quad w_3 = d \end{cases},$$

with $r_1,\ r_2,\ r_3, \in \mathbb{R},\ r_4 \in \mathbb{R}^*$ *and* $a,\ b,\ c,\ d \in \mathbb{C}$. *So the generating function of* $(w_n)_{n\geq 0}$ *is given by*

$$G(z) = \frac{a + (b - r_1 a) z + (c - r_1 b - r_2 a) z^2 + (d - r_1 c - r_2 b - r_3 a) z^4}{1 - r_1 z - r_2 z^2 - r_3 z^3 - r_4 z^4}.$$

Proof.
We have

$$\begin{aligned}
G(z) &= \sum_{n=0}^{\infty} w_n z^n \\
&= w_0 + w_1 z + w_2 z^2 + w_3 z^3 + \sum_{n=4}^{\infty} w_n z^n \\
&= w_0 + w_1 z + w_2 z^2 + w_3 z^3 + \sum_{n=4}^{\infty} \left(r_1 w_{n-1} + r_2 w_{n-2} + r_3 w_{n-3} + r_4 w_{n-4}\right) z^n \\
&= w_0 + w_1 z + w_2 z^2 + w_3 z^3 + r_1 \sum_{n=4}^{\infty} w_{n-1} z^n + r_2 \sum_{n=4}^{\infty} w_{n-2} z^n + r_3 \sum_{n=4}^{\infty} w_{n-3} z^n \\
&\quad + r_4 \sum_{n=4}^{\infty} w_{n-4} z^n \\
&= a + bz + cz^2 + dz^3 + z r_1 \sum_{n=4}^{\infty} w_{n-1} z^{n-1} + z^2 r_2 \sum_{n=4}^{\infty} w_{n-2} z^{n-2} + z^3 r_3 \sum_{n=4}^{\infty} w_{n-3} z^{n-3} \\
&\quad + z^4 r_4 \sum_{n=4}^{\infty} w_{n-4} z^{n-4}
\end{aligned}$$

$$\begin{aligned}
&= a + bz + cz^2 + dz^3 + z r_1 \left(\sum_{n=0}^{\infty} w_n z^n - a - bz - cz^2\right) + z^2 r_2 \left(\sum_{n=0}^{\infty} w_n z^n - a - bz\right) \\
&\quad + z^3 r_3 \left(\sum_{n=0}^{\infty} w_n z^n - a\right) + z^4 r_4 \sum_{n=0}^{\infty} w_n z^n \\
&= a + (b - r_1 a) z + (c - r_1 b - r_2 a) z^2 + (d - r_1 c - r_2 b - r_3 a) z^3 \\
&\quad + \left(r_1 z + r_2 z^2 + r_3 z^3 + r_4 z^4\right) \sum_{n=0}^{\infty} w_n z^n \\
&= a + (b - r_1 a) z + (c - r_1 b - r_2 a) z^2 + (d - r_1 c - r_2 b - r_3 a) z^3 \\
&\quad + \left(r_1 z + r_2 z^2 + r_3 z^3 + r_4 z^4\right) G(z)
\end{aligned}$$

so,

$$G(z)=\frac{a+(b-r_1a)z+(c-r_1b-r_2a)z^2+(d-r_1c-r_2b-r_3a)z^4}{1-r_1z-r_2z^2-r_3z^3-r_4z^4}.$$

□

3.1 Generating functions of some orthogonal polynomials

Theorem 1.32. *Let $(P_n)_{n\geq 0}$ the sequence of orthogonal polynomials defined by the following recurrence relation*

$$\begin{cases} P_n(x)=pxP_{n-1}(x)+qP_{n-2}(x), n\geq 2 \\ P_0(x)=\alpha, P_1(x)=\beta x+\delta \end{cases},$$

with p, q, α, $\beta\in K$.

the generating function of $(P_n)_{n\geq 0}$ is given by

$$G(z)=\frac{\alpha+(\beta x-p\alpha)z}{1-pxz-qz^2}.$$

Proof.
We have

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty}P_n(x)z^n \\ &= P_0(x)+P_1(x)z+\sum_{n=2}^{\infty}P_n(x)z^n \\ &= \alpha+\beta xz+\delta z+\sum_{n=2}^{\infty}(pxP_{n-1}(x)+qP_{n-2}(x))z^n \\ &= \alpha+\beta xz+\delta z+px\sum_{n=2}^{\infty}P_{n-1}(x)z^n+q\sum_{n=2}^{\infty}P_{n-2}(x)z^n \\ &= \alpha+\beta xz+\delta z+pxz\sum_{n=1}^{\infty}P_n(x)z^n+qz^2\sum_{n=0}^{\infty}P_n(x)z^n \\ &= \alpha+\beta xz+\delta z+pxz\left(\sum_{n=0}^{\infty}P_n(x)z^n-\alpha\right)+qz^2\sum_{n=0}^{\infty}P_n(x)z^n \\ &= \alpha+\beta xz+\delta z-p\alpha xz+pxzG(z)+qz^2G(z), \end{aligned}$$

and then

$$\left(1-pxz-qz^2\right)G(z)=\alpha+((\beta-p\alpha)x+\delta)z,$$

but

$$G(z) = \frac{\alpha + ((\beta - p\alpha)x + \delta)z}{1 - pxz - qz^2}.$$

□

According to the previous theorem, we deduce the generating function of some polynomials.

the values of $p, q, \alpha, \beta, \delta$	the coefficients of z^n	generating function
$p = 2, q = -1, \alpha = 1, \beta = 1, \delta = 0$	$T_n(x)$	$\frac{1-xz}{1-2xz+z^2}$
$p = 2, q = -1, \alpha = 1, \beta = 1, \delta = 0$	$U_n(x)$	$\frac{1}{1-2xz+z^2}$
$p = 2, q = -1, \alpha = 1, \beta = 2, \delta = -1$	$V_n(x)$	$\frac{1-z}{1-2xz+z^2}$
$p = 2, q = -1, \alpha = 1, \beta = 2, \delta = 1$	$W_n(x)$	$\frac{1+z}{1-2xz+z^2}$
$p = 1, q = 1, \alpha = 1, \beta = 1, \delta = 0$	$F_n(x)$	$\frac{1}{1-xz+z^2}$
$p = 1, q = 1, \alpha = 2, \beta = 1, \delta = 0$	$L_n(x)$	$\frac{2-xz}{1-xz-z^2}$
$p = 2, q = 1, \alpha = 0, \beta = 0, \delta = 1$	$P_n(x)$	$\frac{z}{1-2xz-z^2}$

Table 1.3-The generating function of some polynomials.

4 Symmetric Functions

In this section we set some definitions of elementntry and complete symmetric function, for more details, you can refer to [2, 10]

Definition 1.33. *A function $f(x_1, x_2, ..., x_n)$ with n variables is symmetric if for all permutation σ from the set of indices $\{1,\ 2, ...,\ n\}$ the following equality is verified*

$$f(x_1,\ x_2, ...,\ x_n) = f\left(x_{\sigma(1)},\ x_{\sigma(2)}, ...,\ x_{\sigma(2)}\right).$$

Which means, a function of several variables is symmetric if its values does not change when we swap variables

Definition 1.34. *We call the elementry symmetric function the function defined by*

$$e_k^{(n)} = e_k(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i_1+i_2+...+i_n=k} \lambda_1^{i_1}\lambda_2^{i_2}...\lambda_n^{i_n}, \quad 0 \leq k \leq n,$$

with $i_1, i_2, ..., i_n = 0$ or 1.

Proposition 1.35. *We have the following formulas*

1. $e_k^{(n+1)} = \lambda_{n+1} e_{k-1}^{(n)} + e_k^{(n)}$,

2. $e_k^{(n)} = \lambda_n e_{k-1}^{(n-1)} + \lambda_{n-1} e_k^{(n-2)} + ... + \lambda_{n-i} e_{k-1}^{(n-i-1)} + ... + \lambda_k e_k^{(k-1)}$.

Proof.
1. By definition of symmetric elementry function , we have

$$\begin{aligned}
\lambda_{n+1} e_{k-1}^{(n)} + e_k^{(n)} &= \lambda_{n+1} \left(\sum_{i_1+i_2+...+i_n=k-1} \lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n} \right) + \sum_{i_1+i_2+...+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n} \\
&= \sum_{i_1+i_2+...+i_n=k-1} \lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n} \lambda_{n+1} + \sum_{i_1+i_2+...+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n} \lambda_{n+1}^0 \\
&= \sum_{i_1+i_2+...+i_n+1=k} \lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n} \lambda_{n+1}^1 + \sum_{i_1+i_2+...+i_n+0=k} \lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n} \lambda_{n+1}^0 \\
&= \sum_{i_1+i_2+...+i_{n+1}=k} \lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n} \lambda_{n+1}^{i_{n+1}} \\
&= e_k^{(n+1)}.
\end{aligned}$$

2.By the formula 1, we write $e_k^{(n)}$ as following

$$\begin{aligned}
e_k^{(n)} &= \lambda_n e_{k-1}^{(n-1)} + e_k^{(n-1)} \\
&= \lambda_n e_{k-1}^{(n-1)} + \lambda_{n-1} e_{k-1}^{(n-2)} + e_k^{(n-2)} \\
&= \lambda_n e_{k-1}^{(n-1)} + \lambda_{n-1} e_{k-1}^{(n-2)} + \lambda_{n-2} e_{k-1}^{(n-3)} + e_k^{(n-3)} \\
&= \lambda_n e_{k-1}^{(n-1)} + \lambda_{n-1} e_{k-1}^{(n-2)} + \lambda_{n-2} e_{k-1}^{(n-3)} + e_k^{(n-3)} + ... + \lambda_{n-i} e_{k-1}^{(n-i-1)} + ... + \lambda_k e_{k-1}^{(k-1)}.
\end{aligned}$$

This completes the proof. □

Proposition 1.36. *[7] The elementry symmetric functions can also be defined as the coefficients of the formal series expansion*

$$E(z) = \sum_{k=0}^{\infty} e_k^{(n)} z^k = \prod_{i=1}^{n} (1 + \lambda_i z),$$

with $e_k(\lambda_1, \lambda_2, ..., \lambda_n) = 0$ *for* $k > n$.

4.1 Complete symmetric function

Definition 1.37. *We also define the complete symmetric functions of the roots as follows*

$$h_k^{(n)} = h_k(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i_1+i_2+...+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n},$$

with $i_1,\ i_2, ..., \ i_n \geq 0.$

Proposition 1.38. *[7] We have the following formulas*

1. $h_k^{(n+1)} = \lambda_{n+1} h_{k-1}^{(n+1)} + h_k^{(n)}$,

2. $h_k^{(n+1)} = \lambda_{n+1}^k + \lambda_{n+1}^{k-1} h_1^{(n)} + \lambda_{n+1}^{k-2} h_2^{(n)} + \lambda_{n+1}^{k-3} h_3^{(n)}. + ... + \lambda_{n+1} h_{k-1}^{(n)} + h_k^{(n)}.$

Proposition 1.39. *We can also define the k th complete functions as the coefficients of the expansion in formal series*

$$H\left(z\right) = \sum_{k=0}^{\infty} h_k z^k = \prod_{i \geq 1} \left(1 - \lambda_i z\right)^{-1}.$$

Proof.

We have

$$h_k^{(n)} = h_k\left(\lambda_1, \lambda_2, ..., \lambda_n\right) = \sum_{i_1+i_2+...+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} ... \lambda_n^{i_n}.$$

For $n = 2$, we have

$$\begin{aligned} \sum_{k=0}^{\infty} h_k^{(2)} z^k &= h_0^{(2)} + h_1^{(2)} z + h_2^{(2)} z^2 + ... \\ &= 1 + \left(\lambda_1 + \lambda_2\right) z + \left(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2\right) z^2 + ... \\ &= \left(1 + \lambda_1 z + \lambda_1^2 z^2 + ...\right) \left(1 + \lambda_2 z + \lambda_2^2 z^2 + ...\right) \\ &= \sum_{k=0}^{\infty} \left(\lambda_1 z\right)^k \times \sum_{k=0}^{\infty} \left(\lambda_2 z\right)^k \\ &= \frac{1}{\left(1 - \lambda_1 z\right)\left(1 - \lambda_2 z\right)} \\ &= \frac{1}{\prod_{i=1}^{2} \left(1 - \lambda_i z\right)}. \end{aligned}$$

Supposing that the property is true for n which means

$$\sum_{k=0}^{\infty} h_k^{(n)} z^k = \prod_{i \geq 1} \left(1 - \lambda_i z\right)^{-1},$$

and proving that it stays true for $n + 1$

$$\sum_{k=0}^{\infty} h_k^{(n+1)} z^k = \prod_{i \geq 1} \left(1 - \lambda_i z\right)^{-1},$$

in effect,

$$h_k^{(n+1)} = \lambda_{n+1} h_{k-1}^{(n+1)} + h_k^{(n)},$$

so,

$$\begin{aligned}
\sum_{k=0}^{\infty} h_k^{(n+1)} z^k &= \lambda_{n+1} \sum_{k=0}^{\infty} h_{k-1}^{(n+1)} z^k + \sum_{k=0}^{\infty} h_k^{(n)} z^k \\
&= \lambda_{n+1} \sum_{k=1}^{\infty} h_{k-1}^{(n+1)} z^k + \sum_{k=0}^{\infty} h_k^{(n)} z^k \\
&= \lambda_{n+1} z \sum_{k=0}^{\infty} h_{k-1}^{(n+1)} z^k + \sum_{k=0}^{\infty} h_k^{(n)} z^k \\
&= \lambda_{n+1} z \prod_{i=1}^{n+1} (1-\lambda_i z)^{-1} + \prod_{i=1}^{n} (1-\lambda_i z)^{-1} \\
&= \frac{\lambda_{n+1} z + (1-\lambda_{n+1} z)}{\prod_{n=0}^{n+1} (1-\lambda_i z)} \\
&= \frac{1}{\prod_{n=0}^{n+1} (1-\lambda_i z)}.
\end{aligned}$$

This completes the proof. □

Proposition 1.40. *Let $E(z)$ and $H(z)$ be elementry and complete symmetric functions ,respectively so we have*

$$E(-z) \times H(z) = 1.$$

Proof.

By definition, we have

$$E(-z) = \sum_{k=0}^{\infty} e_k^{(n)} (-z)^k = \prod_{i=1}^{n} (1-\lambda_i z),$$

and

$$H(z) = \sum_{k=0}^{\infty} h_k z^k = \prod_{i \geq 1} (1-\lambda_i z)^{-1}.$$

so,

$$E(-z) \times H(z) = 1$$

□

4.2 Some properties of symmetric function

In this section, we set some properties of symmetric function, you can refer to [1, 8]

Definition 1.41. *[1] We give two alphabets A and B, we note $S_j(A-B)$ the coefficients rationnal series*

$$\frac{\prod_{b\in B}(1-bz)}{\prod_{a\in A}(1-az)}=\sum_{j=0}^{\infty}S_j(A-B)\,z^j.$$

Remark 1.42. *If $A=\emptyset$, so*

$$\prod_{b\in B}(1-bz)=\sum_{j=0}^{\infty}S_j(-B)\,z^i.$$

Proposition 1.43. *[1] Supposing that $A=\emptyset$ or $B=\emptyset$ we obtain*

$$\sum_{j=0}^{\infty}S_j(A-B)\,z^i=\sum_{j=0}^{\infty}S_j(A)\,z^i\cdot\sum_{j=0}^{\infty}S_j(-B)\,z^i.$$

Remark 1.44. *If $A=B$, so*

$$\sum_{j=0}^{\infty}S_j(A)\,z^i=\frac{1}{\sum_{j=0}^{\infty}S_j(-A)\,z^i}.$$

Lemma 1.45. *[1] Let $A=\{x\}$ et $B=\{b_1,b_2,\dots\}$ two alphabets, we have*

$$S_{j+k}(x-B)=x^kS_j(x-B),\forall k\in\mathbb{N}.$$

Proposition 1.46. *Let $A=\{x\}$ et $B=\{b_1,b_2,\dots\}$ two alphabets, we have*

$$\frac{\prod_{b\in B}(1-zb)}{(1-zx)}=1+...+z^{j-1}S_{j-1}(x-B)+z^j\frac{S_j(x-B)}{(1-zx)}.$$

Proof.
By the definition 1.40, we have

$$\begin{aligned}\frac{\prod_{b\in B}(1-zb)}{(1-zx)}&=\sum_{j=0}^{\infty}S_j(x-B)\,z^j\\&=1+...+S_{j-1}(x-B)\,z^{j-1}+S_j(x-B)\,z^j+S_{j+1}(x-B)\,z^{j+1}+...\\&=1+...+S_{j-1}(x-B)\,z^{j-1}+z^j\left(S_j(x-B)+S_{j+1}(x-B)\,z+...\right),\end{aligned}$$

by using the fact that $S_{j+k}(x-B) = x^k S_j(x-B), \forall k \in \mathbb{N}$, we obtain

$$\begin{aligned}
\frac{\prod_{b\in B}(1-bz)}{(1-xz)} &= 1+...+S_{j-1}(x-B)z^{j-1}+z^j(S_j(x-B)+xS_j(x-B)z+...) \\
&= 1+...+S_{j-1}(x-B)z^{j-1}+z^jS_j(x-B)\left(1+xz+x^2z^2+...\right) \\
&= 1+...+S_{j-1}(x-B)z^{j-1}+\frac{z^jS_j(x-B)}{1-xz}.
\end{aligned}$$

□

Proposition 1.47. *[9] Let $A = \{a_1, a_2\}$ an alphabet, so we have*

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])z^n = \frac{1}{1-(a_1-a_2)z-a_1a_2z^2}, \tag{1.2}$$

and

$$\sum_{n=0}^{\infty} S_{n-1}(a_1+[-a_2])z^n = \frac{z}{1-(a_1-a_2)z-a_1a_2z^2}. \tag{1.3}$$

Proposition 1.48. *Let $A = \{x\}$ and $B = \{b_1, b_2, \dots\}$ two alphabet,we have*

$$S_n(x-B) = x^nS_0(-B)+x^{n-1}S_1(-B)+...+S_n(-B).$$

Proof.
We have

$$\begin{aligned}
S_n(x-B) &= \sum_{k=0}^{n} S_{n-k}(x)S_k(-B) \\
&= \sum_{k=0}^{n} x^{n-k}S_k(-B) \\
&= x^nS_0(-B)+x^{n-1}S_1(-B)+...+S_n(-B).
\end{aligned}$$

This completes the proof. □

Remark 1.49. *Specifically, when $B = \{b, b, ..., b\} := nb$, we have $S_n(x-nb) = (x-b)^n$.*

Proposition 1.50. *If $B = \{1, 1, ..., 1\}$, the binomial coefficients is given by*

$$S_j(-n) = (-1)^j\binom{n}{j} \quad et \ S_j(n) = \binom{n+j-1}{j}$$

Proof.

We have $S_n(x-nb)=(x-nb)^n$, if $b=1$, so

$$\begin{aligned} S_n(x-n) &= (x-1)^n \\ &= \sum_{j=0}^{n}\binom{n}{j}(-1)^j x^{n-j}, \end{aligned}$$

so

$$S_j(-n)=(-1)^j\binom{n}{j}.$$

The same, we have

$$\begin{aligned} S_n(x+n) &= (x-1)^{-n} \\ &= \sum_{j=0}^{n}\binom{-n}{j}(-1)^j x^{n-j} \\ &= \sum_{j=0}^{n}\frac{(-n)!}{(-n-j)!j!}(-1)^j x^{n-j} \\ &= \sum_{j=0}^{n}\frac{(-n)(-n-1)\ldots(-n-j+1)}{j!}(-1)^j x^{n-j} \\ &= \sum_{j=0}^{n}\frac{n(n+1)\ldots(n+j-1)}{j!}x^{n-j}, \end{aligned}$$

so

$$S_j(n)=\binom{n+j-1}{j}.$$

This completes the proof . $\square$

Chapter 2

Some Properties of Convolved of Generalized (p,q)-Numbers

In this chapter, we will firstly define a new generalization of (p, q) numbers and then derive the appropriate Binet's formula and generating functions concerning (p, q)Fibonacci numbers, (p, q) Pell numbers, (p, q) Mersenne numbers.

1 Generalized (p,q) numbers

In this section we give the recurrence relation of generalized (p, q) numbers , also the appropriate Binet's formula.

Definition 2.1. *[36] For any positive real numbers p and q , the sequence of generalized (p, q) numbers $\{W_{p,q,n}\}_{n\in\mathbb{N}}$ is given by the following recurrence relation*

$$\begin{cases} W_{p,q,n} = apW_{p,q,n-1} + bqW_{p,q,n-2}, (n \geq 2), \\ W_{p,q,0} = \alpha, W_{p,q,1} = \beta p + \gamma, \end{cases} \quad a, b, \alpha, \beta, \gamma \in \mathbb{Z},$$

In the following table , we deduce some (p, q) numbers

the values of $a, b, \alpha, \beta, \gamma$	*sequences*	*recurrence relations*
$a = 1, b = 1, \alpha = 0, \beta = 0, \gamma = 1$	*(p,q)-Fibonacci*	$\begin{cases} F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2}, n \geq 2 \\ F_{p,q,0} = 0,\ F_{p,q,1} = 1 \end{cases}.$
$a = 2, b = 1, \alpha = 0, \beta = 0, \gamma = 1$	*(p,q)-Pell*	$\begin{cases} P_{p,q,n} = 2pP_{p,q,n-1} + qP_{k,n-2},\ n \geq 2 \\ P_{p,q,0} = 0,\ P_{p,q,1} = 1 \end{cases}.$
$a = 1, b = 2, \alpha = 0, \beta = 0, \gamma = 1$	*(p,q)-Jacobstal*	$\begin{cases} J_{p,q,n} = pJ_{p,q,n-1} + 2qJ_{p,q,n-2},\ n \geq 2 \\ J_{p,q,0} = 0,\ J_{p,q,1} = 1 \end{cases}.$
$a = 3, b = -2, \alpha = 0, \beta = 0, \gamma = 1$	*(p,q)-Mersenne*	$\begin{cases} M_{p,q,n} = 3pM_{p,q,n-1} - 2qM_{p,q,n-2},\ n \geq 2 \\ M_{p,q,0} = 0,\ M_{p,q,1} = 1 \end{cases}.$

Table 2.1-*Some (p, q) numbers.*

Theorem 2.2. *The Binet's formula for generalized (p, q) numbers is given by*

$$W_{p,q,n} = \frac{Aa_1^n - Ba_2^n}{a_1 - a_2}, \tag{2.1}$$

with $A = \beta p + \gamma - \alpha a_2$ and $B = \beta p + \gamma - \alpha a_1$, a_1, a_2 are characteristic roots

Proof.

According to the theory of difference equation, we have the following general term for generalized (p, q)- numbers is given by

$$W_{p,q,n} = C_1 a_1^n + C_2 a_2^n,$$

where C_1 and C_2 are the coefficients.
For $n = 0, 1$, we have:

$$\begin{cases} C_1 + C_2 = \alpha \\ C_1 a_1 + C_2 a_2 = \beta p + \gamma \end{cases}.$$

By these equalities

$$\begin{cases} C_1 = \frac{\beta p+\gamma-\alpha a_2}{a_1-a_2} = \frac{A}{a_1-a_2} \\ C_2 = \frac{\alpha a_1-(\beta p+\gamma)}{a_1-a_2} = -\frac{B}{a_1-a_2}. \end{cases}.$$

Therefore

$$W_{p,q,n} = \frac{Aa_1^n - Ba_2^n}{a_1 - a_2}.$$

This is completes the proof. □

By using the formula (2.1) we get the following table

general solution	(a_1, a_2)	γ	β	α	b	a	A	B
$F_{p,q,n} = \frac{a_1^n-a_2^n}{a_1-a_2}$	$a_{1,2} = \frac{p\pm\sqrt{p^2+4q}}{2}$	1	0	0	1	1	1	1
$P_{p,q,n} = \frac{a_1^n-a_2^n}{a_1-a_2}$	$a_{1,2} = p \pm \sqrt{p^2+q}$	1	0	0	1	2	1	1
$J_{p,q,n} = \frac{a_1^n-a_2^n}{a_1-a_2}$	$a_{1,2} = \frac{p\pm\sqrt{p^2+8q}}{2}$	1	0	0	2	1	1	1
$M_{p,q,n} = \frac{a_1^n-a_2^n}{a_1-a_2}$	$a_{1,2} = \frac{3p\pm\sqrt{9p^2-8q}}{2}$	1	0	0	2	1	1	1

Table 2.2*. Binet's formula of some (p, q) numbers*

2 Generating function of (p,q) generalized numbers

In this section we propose a theorem of the generating functions of (p, q) generalized numbers, after we deduce the generating function of some (p, q) numbers such as (p, q) Fibonacci numbers.

Theorem 2.3. *A generating function is derived for generalized (p, q) - numbers as follows :*

$$\sum_{n=0}^{+\infty} W_{p,q,n} z^n = \frac{\alpha + (p(\beta - \alpha a) + \gamma)z}{1 - apz - bqz^2}. \tag{2.2}$$

Proof.
Let

$$G(z) = \sum_{n=0}^{\infty} W_{p,q,n} z^n = W_{p,q,0} + W_{p,q,1} z + W_{p,q,2} z^2 + ... + W_{p,q,n} z^n + ...$$

the generating function of $(W_{p,q,n})_{n\geq 0}$.

on the other hand,

$$apzG(z) = \sum_{n=0}^{\infty} apW_{p,q,n}z^{n+1} = apW_{p,q,0}z + apW_{p,q,1}z^2 + apW_{p,q,2}z^3 + ... + apW_{p,q,n-1}z^n + ...$$

and

$$bqz^2G(z) = \sum_{n=0}^{\infty} bqW_{p,q,n}z^{n+2} = bqW_{p,q,0}z^2 + bqW_{p,q,1}z^3 + bqW_{p,q,2}z^4 + ... + bqW_{p,q,n-2}z^n + ...$$

we write, so

$$\left(1 - apz - bqz^2\right) G(z) = W_{p,q,0} + (W_{p,q,1} - apW_{p,q,0})\, z + (W_{p,q,2} - apW_{p,q,1} - bqW_{p,q,0})\, z^2 + \\ ... + (W_{p,q,n} - apW_{p,q,n-1} - bqW_{p,q,n-2})\, z^n + ...$$

Considering $W_{p,q,n} = \alpha, W_{p,q,n} = \beta p + \gamma, and\ W_{p,q,n} = apW_{p,q,n-1} + bqW_{p,q,n-2}z^n$, so we find

$$\left(1 - apz - bqz^2\right) G(z) = \alpha + ((\beta p + \gamma) - ap\alpha)\, z,$$

but

$$G(z) = \frac{\alpha + (p(\beta - \alpha a) + \gamma)z}{1 - apz - bqz^2}.$$

This is completes the proof. □

Theorem 2.4. *For $n \in \mathbb{N}$, the symmetric function of (p, q) generalized numbers as follows:*

$$W_{p,q,n} = \alpha S_n(a_1 + [-a_2]) + (p(\beta - \alpha a) + \gamma)S_{n-1}(a_1 + [-a_2]). \tag{2.3}$$

Proof. By putting $a_1 - a_2 = ap$ and $a_1a_2 = bq$ in (1.2) and (1.3) in chapter 1 we get :

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])z^n = \frac{1}{1 - apz - bqz^2}, \tag{2.4}$$

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + [-a_2])z^n = \frac{z}{1 - apz - bqz^2}, \tag{2.5}$$

multiplying (2.4) by α and summing (2.5) multiplying by $\alpha + (p(\beta - \alpha a) + \gamma)$, we get :

$$\alpha S_n(a_1 + [-a_2]) + (p(\beta - \alpha a) + \gamma)S_{n-1}(a_1 + [-a_2]) = \frac{\alpha + (p(\beta - \alpha a) + \gamma)z}{1 - apz - bqz^2}$$

This is complets the proof . □

From (2.2) and (2.4) we deduce the following table

(p, q) numbers	generating function	generating functions
(p, q) Fibonacci	$\frac{z}{1-pz-qz^2}$	$S_{n-1}(a_1 + [-a_2]), a_{1,2} = \frac{p\pm\sqrt{p^2+4q}}{2}$
$(p, q)Pell$	$\frac{z}{1-2pz-qz^2}$	$S_{n-1}(a_1 + [-a_2]), a_{1,2} = p \pm \sqrt{p^2 + q}$
$(p, q)Jacobstal$	$\frac{z}{1-pz-2qz^2}$	$S_{n-1}(a_1 + [-a_2]), a_{1,2} = \frac{p\pm\sqrt{p^2+8q}}{2}$
(p, q) Mersenne	$\frac{z}{1-3pz+2qz^2}$	$S_{n-1}(a_1 + [-a_2]), a_{1,2} = \frac{3p\pm\sqrt{9p^2-8q}}{2}$

Table 1.3 Generating function of some (p, q) numbers.

Proposition 2.5. *[36] The explicit formula of (p, q) Fibonacci is given by the following relation :*

$$F_{p,q,n} = \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-j-1}{j} (p)^{n-2j-1}(q)^j$$

Proposition 2.6. *[36] The explicit formula of (p, q) Pell is given by the following relation :*

$$P_{p,q,n} = \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-j-1}{j} (2p)^{n-2j-1}(q)^j$$

According to the previous theorem , we deduce the generating function of some numbers

values of p, q, α, β	Coefficients of z^n	generating functions
$p = k, q = \alpha = 1, \beta = 1$	$F_{k,n}$	$\frac{1}{1-kz-z^2}$
$p = \beta = k, q = 1, \alpha = 2$	$L_{k,n}$	$\frac{2-kz}{1-kz-z^2}$
$p = 2, q = k, \alpha = 0, \beta = 1$	$P_{k,n}$	$\frac{kz}{1-2z-kz^2}$
$p = k, q = 2, \alpha = 0, \beta = 1$	$J_{k,n}$	$\frac{z}{1-kz-2z^2}$
$p = 3k,\ q = -2, \alpha = 0, \beta = 1$	$M_{k,n}$	$\frac{z}{1-3kz+2z^2}$

Table 1.3 Generating function of some numbers.

Remark 2.7. *For $k = 1$ in the previous theorem we get the generating function of those numbers.*

3 Main theorems

In this section we are going to establish a new theorem so we can retrieve some convolved (p, q) numbers and by applying this theorem we deduce some results.

Definition 2.8. *[32] We consider the linear operator $\delta(A)$ of the Lagrange interpolation defined by*

$$\delta(A)(f) = \sum_{a\in A} \frac{f(a)}{R(a, A\ a)},$$

where $R(a,B) = R_{b\in B}(a-b)$ and $R(a,\emptyset) = 1$,

Accordingly, this operator sends a polynomial of degree k to a symmetric polynomial in A of degree $k-n$ with $card(A) = n+1$. In particular , it annihilates polynomials of degree $\leq n$, and $f(x) = x^n$ on constant 1.
Let $\delta(E_{n+1})$ be the Lagrange operator of an alphabet $E_{n+1} := \{e_1, e_2, \ldots, e_{n+1}\}$, we have

$$\delta(E_{n+1})(f) = \sum_{k=0}^{n} \frac{f(e_{k+1})}{R(e_{k+1}, E_{n+1}/e_{k+1})}.$$

Theorem 2.9. *Let a and b two real numbers so we have*

$$\frac{1}{(1-at)^r}\frac{1}{(1-bt)^r} = \sum_{n=0}^{\infty} \frac{(r)_n}{n!}\left((a+b)t - abt^2\right)^n,$$

with

$$(r)_n = r(r+1)(r+2)\ldots(r+n-1).$$

Proof.
Let E an alphabet, so newton's formula gives us:

$$\frac{1}{(1-at)^r}\frac{1}{(1-bt)^r} = \sum_{n=0}^{\infty} S_n(r-E_n)\delta_n\ldots\delta_1 \frac{1}{[(1-at)(1-bt)]^{e_1}}, \tag{2.6}$$

also Lagrange formula's gives us:

$$\delta_n\ldots\delta_1 \frac{1}{(1-at)(1-bt)^{e_1}} = \sum_{k=0}^{n} \frac{\frac{1}{(1-at)(1-bt)^{e_{k+1}}}}{R(e_{k+1}, E_k)R(e_{k+1}, E_{n+1}/E_{k+1})},$$

for $E = \{0, -1, -2\ldots\}$, we have

$$\begin{aligned} R(e_{k+1}, E_k) &= (e_{k+1}-e_1)(e_{k+1}-e_2)\ldots(e_{k+1}-e_k) \\ &= (-k-0)(-k-1)\ldots(-k+k+1)) \\ R(e_{k+1}, E_k) &= (-1)^k k!. \end{aligned}$$

And ,

$$\begin{aligned} R(e_{k+1}, E_{k+1}/E_{n+1}) &= (e_{k+1} - e_{k+2})(e_{k+1} - e_{k+3}) \dots (e_{k+1} - e_{n+1}) \\ &= (-k + k + 1)(-k + k + 2) \dots (-k + n) \\ &= (n - k)! \\ R(e_{k+1}, E_{k+1}/E_{n+1}) &= (n - k)!, \end{aligned}$$

so the formula of Lagrange is written by:

$$\begin{aligned} \delta_n \dots \delta_1 \frac{1}{(1-at)(1-bt)^{e_1}} &= \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} [(1-at)(1-bt)]^k \\ \delta_n \dots \delta_1 \frac{1}{(1-at)(1-bt)^{e_1}} &= \frac{1}{n!} [1 - (1-at)(1-bt)]^n \\ \delta_n \dots \delta_1 \frac{1}{(1-at)(1-bt)^{e_1}} &= \frac{1}{n!} [(a+b)t - abt^2]^n, \end{aligned}$$

for

$$\begin{aligned} E &= \{0, -1, -2 \dots\} \\ S_n(r - E_n) &= r(r+1) \dots (r + n - 1), \end{aligned}$$

so

$$S_n(r - E_n) = (r)_n. \tag{2.7}$$

Depending on (2.6),(2.7) give us:

$$\frac{1}{(1-at)^r(1-bt)^r} = \sum_{n=0}^{\infty} \frac{(r)_n}{n!} [(a+b)t - abt^2]^n.$$

This is completes the proof. □

The following theorem generalizes the work of [31].

Theorem 2.10. *For $n \geq 0$ and $r \in \mathbb{R}$, convolved (p,q) Fibonacci numbers $F_{p,q,n+1}(r)$ can be computed by*

$$F_{p,q,n+1}(r) = \frac{1}{n!} \left(\frac{p + \sqrt{p^2 + 4q}}{2} \right)^n \sum_{n=0}^{\infty} (-1)^k \binom{n}{k} \left(\frac{p - \sqrt{p^2 + 4q}}{p + \sqrt{p^2 + 4q}} \right)^k (r)_k (r)_{n-k}.$$

Proof. Considering the recurrence relation of (p,q) Fibonacci numbers with, $a = \frac{p - \sqrt{p^2+4q}}{2}$ and $b = \frac{p + \sqrt{p^2+4q}}{2}$

so

$$a+b=p,$$

$$ab=-q,$$

by using the previous theorem we get ,

$$\begin{aligned}
\frac{1}{(1-pt-qt^2)^r} &= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}\left(pt+qt^2\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}t^n\left(p+qt\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(p\right)^{n-k}\left(q\right)^k t^{n+k} \\
&= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}\sum_{m=n}^{2n}\binom{n}{n-m}\left(p\right)^{2n-m}\left(q\right)^{m-n} t^m. \qquad (2.8)
\end{aligned}$$

we get after the development the coefficients of t^n written:

$$\sum_{k=0}^{n}\frac{(r)_k}{k!}.$$

putting $F_{p,q,n}=\sum_{k=0}^{n}\binom{n}{k}\frac{(r)_k}{k!}p^{2k-n}q^{n-k}$.
So the formula(2.8) written by :

$$\frac{1}{(1-pt-qt^2)^r}=\sum_{n=0}^{\infty}F_{p,q,n+1}(r)t^n,$$

with

$$F_{p,q,n}(r)=\sum_{k=0}^{n}\binom{n}{k}\frac{(r)_k}{k!}p^{2k-n}q^{n-k}.$$

We are going to proof the second theorem of the article [31], the formula of newton gives us

$$\frac{1}{(1-ab)^r}=\sum_{n=0}^{\infty}S_n(r-E_n)\delta_n\ldots\delta_1\frac{1}{(1-ab)^{e_1}}.$$

For $E = \{0, -1, -2\ldots\}$, we have the formula,

$$\frac{1}{(1-at)^r} = \sum_{n=0}^{\infty} \frac{(r)_n}{n!} a^n t^n,$$

the same with

$$\frac{1}{(1-bt)^r} = \sum_{n=0}^{\infty} \frac{(r)_n}{n!} b^n t^n,$$

so

$$\frac{1}{(1-at)^r(1-bt)^r} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n=k}^{n} a^k b^{n-k} (r)_k (r)_{n-k}.$$

knowing that

$$\frac{1}{(1-at)^r(1-bt)^r} = \sum_{n=0}^{\infty} F_{p,q,n+1}(r) t^n,$$

so

$$\begin{aligned} F_{p,q,n+1}(r) &= \frac{1}{n!} \sum_{k=0}^{n} a^k b^{n-k} (r)_k (r)_{n-k} \\ &= \frac{1}{n!} b^n \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^k (r)_k (r)_{n-k}. \end{aligned}$$

For a$=\frac{p-\sqrt{p^2+4q}}{2}$ and $b = \frac{p+\sqrt{p^2+4q}}{2}$, we have

$$F_{p,q,n+1}(r) = \frac{1}{n!} \left(\frac{p+\sqrt{p^2+4q}}{2}\right)^n \sum_{n=0}^{\infty} (-1)^k \binom{n}{k} \left(\frac{p-\sqrt{p^2+4q}}{p+\sqrt{p^2+4q}}\right)^k (r)_k (r)_{n-k}.$$

This completes the proof. □

For $p = 1$ and $q = 1$ and $b = \frac{1}{a}$ in the theorem 2.10 we get the result of [31]

Corollary 2.11. *For $n \in \mathbb{N}$, and $r \in \mathbb{R}$ he form of convolved Fibonacci numbers, $F_{n+1}(r)$ is given by*

$$F_{n+1}(r) = \frac{1}{n!} \frac{1}{\left(\frac{\sqrt{5}-1}{2}\right)^n} \sum_{n=0}^{\infty} (-1)^k \binom{n}{k} \left(\frac{\sqrt{5}-1}{\sqrt{5}-1}\right)^k (r)_k (r)_{n-k}.$$

The following theorems are new results .

Theorem 2.12. *For $n \geq 0$ and $r \in \mathbb{R}$, convolved (p,q) Pell numbers $P_{p,q,n+1}(r)$ can be computed by:*

$$P_{p,q,n+1}(r) = \frac{1}{n!}\left(p - \sqrt{p^2+q}\right)^n \sum_{n=0}^{\infty}(-1)^k \binom{n}{k}\left(\frac{p-\sqrt{p^2+q}}{p+\sqrt{p^2+q}}\right)^k (r)_k(r)_{n-k}.$$

Proof. Considering the recurrence relation of (p,q)Pell numbers with, $a = p + \sqrt{p^2+q}$ and $b = p - \sqrt{p^2+q}$.
So

$$a + b = 2p,$$
$$ab = -q,$$

by using the previous theorem we get ,

$$\begin{aligned}
\frac{1}{(1-2pt-qt^2)^r} &= \sum_{n=0}^{\infty} \frac{(r)_n}{n!}\left(2pt + qt\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(r)_n}{n!} t^n \left(2p + qt\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(r)_n}{n!} \sum_{k=0}^{n} \binom{n}{k}\left(2p\right)^k (qt)^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{(r)_n}{n!} \sum_{m=n}^{2n} \binom{n}{n-m}\left(2p\right)^{n-m} (qt)^m.
\end{aligned}$$

So, we get

$$P_{p,q,n+1} = \frac{1}{n!} b^n \sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^k (r)_k(r)_{n-k},$$

by substituting the values of a and b we find

$$P_{p,q,n+1}(r) = \frac{1}{n!}\left(p - \sqrt{p^2+q}\right)^n \sum_{n=0}^{\infty}(-1)^k \binom{n}{k}\left(\frac{p-\sqrt{p^2+q}}{p+\sqrt{p^2+q}}\right)^k (r)_k(r)_{n-k}.$$

This completes the proof. □

For $p = 1$ and $q = 1$ in theorem 2.12 we have the following lemma ,

Lemma 2.13. *For $n \in \mathbb{N}$, and $r \in \mathbb{R}$,the form of convolved Pell numbers, $P_{n+1}(r)$ is given by*

$$P_{n+1}(r) = \frac{1}{n!}\left(1 - \sqrt{2}\right)^n \sum_{n=0}^{\infty}(-1)^k \binom{n}{k}\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)^k (r)_k(r)_{n-k}.$$

Theorem 2.14. *For $n \in \mathbb{N}$, and $r \in \mathbb{R}$ Jacobstal numbers $J_{p,q,n+1}(r)$ can be computed by:*

$$J_{p,q,n+1}(r) = \frac{1}{n!}\left(\frac{p+\sqrt{p^2+8q}}{2}\right)^n \sum_{n=0}^{\infty}(-1)^k \binom{n}{k}\left(\frac{p+\sqrt{p^2+8q}}{p-\sqrt{p^2+8q}}\right)^k (r)_k(r)_{n-k}.$$

Proof. Considering the recurrence relation of (p,q)Jacobstal numbers with, $a = \frac{p+\sqrt{p^2+8q}}{2}$ and $b = \frac{p-\sqrt{p^2+8q}}{2}$.
so

$$\begin{aligned} a+b &= p, \\ ab &= -2q, \end{aligned}$$

by using the previous theorem we get ,

$$\begin{aligned} \frac{1}{(1-pt-2qt^2)^r} &= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}\left(pt+2qt\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}t^n\left(p+2qt\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(p\right)^k(2qt)^{n-k} \\ &= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}\sum_{m=n}^{2n}\binom{n}{n-m}\left(p\right)^{n-m}(2qt)^m, \end{aligned}$$

so

$$J_{p,q,n+1} = \frac{1}{n!}b^n\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^k(r)_k(r)_{n-k}.$$

By substituting the values of a and b we find

$$J_{p,q,n+1}(r) = \frac{1}{n!}\left(\frac{p+\sqrt{p^2+8q}}{2}\right)^n \sum_{n=0}^{\infty}(-1)^k \binom{n}{k}\left(\frac{p+\sqrt{p^2+8q}}{p-\sqrt{p^2+8q}}\right)^k (r)_k(r)_{n-k}.$$

□

For $p=1$ and $q=1$ in theorem 2.14 we have the following lemma,

Lemma 2.15. *For $n \in \mathbb{N}$, and $r \in \mathbb{R}$ the form of convolved Jacobstal numbers $J_{n+1}(r)$ is given by*

$$J_{n+1}(r) = \frac{1}{n!}(2)^n\sum_{n=0}^{+\infty}(-1)^k\binom{n}{k}\left((-2)^k\right)(r)_k(r)_{n-k}.$$

Theorem 2.16. *For $n \in \mathbb{N}$, and $r \in \mathbb{R}$, convolved (p,q) Mersenne numbers, $M_{p,q,n+1}(r)$ can be computed by*

$$M_{p,q,n+1}(r) = \frac{1}{n!}\left(\frac{3p-\sqrt{9p^2-8q}}{2}\right)^n \sum_{n=0}^{\infty}(-1)^k\binom{n}{k}\left(\frac{3p+\sqrt{9p^2-8q}}{3p-\sqrt{9p^2-8q}}\right)^k (r)_k(r)_{n-k}.$$

Proof. Considering the recurrence relation of (p,q) Mersenne numbers with, $a = \frac{3p-\sqrt{9p^2-8q}}{2}$ and $b = \frac{3p+\sqrt{9p^2-8q)}}{2}$
so

$$a+b = \frac{3p}{2}$$
$$ab = 2q,$$

by using the previous theorem we get ,

$$\begin{aligned}\frac{1}{(1-3pt+2qt^2)^r} &= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}\left(\frac{3p}{q}t-2qt^2\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}t^n\left(\frac{3p}{2}t-2q\right)^n \\ &= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{3p}{q}\right)^k(-2qt)^{n-k} \\ &= \sum_{n=0}^{\infty}\frac{(r)_n}{n!}\sum_{m=n}^{2n}\binom{n}{n-m}\left(\frac{3p}{q}\right)^{n-m}(-2qt)^m,\end{aligned}$$

so

$$M_{p,q,n+1} = \frac{1}{n!}b^n\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^k(r)_k(r)_{n-k}.$$

By substituting the values of a and b we find

$$M_{p,q,n+1}(r) = \frac{1}{n!}\left(\frac{3p-\sqrt{9p^2-8q}}{2}\right)^n \sum_{n=0}^{\infty}(-1)^k\binom{n}{k}\left(\frac{3p+\sqrt{9p^2-8q}}{3p-\sqrt{9p^2-8q}}\right)^k (r)_k(r)_{n-k}$$

This completes the proof. □

For $p=1$ and $q=1$ in theorem 2.16 we have the following lemma ,

Corollary 2.17. *For $n \in \mathbb{N}$, and $r \in \mathbb{R}$ the form of convolved Mersenne numbers $M_{n+1}(r)$ is given by*

$$M_{n+1}(r) = \frac{1}{n!}\sum_{n=0}^{+\infty}(-1)^k\binom{n}{k}\left(2\right)^k(r)_k(r)_{n-k}.$$

Chapter 3

Some Theorems on Symmetric Functions

In this chapter , we introduce the new generating functions for the products of forth-order symmetric functions in several variables and third-order making use the new symmetrizing endomorphism operators $\delta_{e_3e_4}\delta_{e_2e_3}\delta_{e_1e_2}$ on the formal series $\sum_{n=0}^{+\infty} S_n(E_4)a_1^n z^n$.

1 Definitions and some properties

We need some preliminaries on symmetric functions and divided differences. For more details , you can refer to [2, 10]

Definition 3.1. *Let f be a function define in $\mathbb{R}^n$, we define the divided difference δ by*

$$\delta_{x_ix_{i+1}} = \frac{f(x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_{i+1}, x_i \ldots, x_n)}{x_i - x_{i+1}}.$$

Definition 3.2. *Let n be positive integer and $A_2 = \{a_1, a_2\}$ are set of given variables , then the n^{th} symmetric $S_n(a_1 + a_2)$ is defined by*

$$S_n(A_2) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2}, and\ S_n(E) = 0\ for\ n < 0.$$

with

$$\begin{aligned} S_0(A_2) &= S_0(a_1 + a_2) = 1, \\ S_1(A_2) &= S_1(a_1 + a_2) = a_1 + a_2, \\ S_2(A_2) &= S_2(a_1 + a_2) = a_1^2 + a_2^2 + a_1a_2, \\ &\vdots \end{aligned}$$

Definition 3.3. *The symmetrizing operator $\delta_{a_1a_2}^k$ is defined by*

$$\delta_{a_1a_2}^k f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}\ for\ all\ k \in \mathbb{N}_0. \tag{3.1}$$

Example 3.4. *If $f(a_1) = a_1$, the operator gives us :*

$$\delta_{a_1a_2}^k f(a_1) = \frac{a_1^{k+1} - a_2^{k+1}}{a_1 - a_2} = S_k(a_1 + a_2).$$

Definition 3.5. *[15] Let $A_3 = \{a_1, a_2, a_3\}$ an alphabet so we have :*

$$\delta_{a_2a_3}\delta_{a_1a_2} = \frac{a_1^2 f(a_1) + a_2a_3\delta_{a_2a_3}(f(a_2)) - a_1\delta_{a_2a_3}(a_2 f(a_2))}{(a_1 - a_2)(a_2 - a_3)}.$$

2 Motivation and Lemmas

In this section we define new symmetrizing operator and we base some assumptions and imperatives that help us in building some main results.

Definition 3.6. *Given an alphabet $E_4 = \{e_1, e_2, e_3, e_4\}$, the new symmetrizing operator is defined by*

$$\delta_{e_3e_4}\delta_{e_2e_3}\delta_{e_1e_2}f(e_1) = \frac{\delta_{e_3e_4}(e_1^2 f(e_1)) + \delta_{e_3e_4}e_2e_3\delta_{e_2e_3}(f(e_2)) - \delta_{e_3e_4}(e_1\delta_{e_2e_3}(e_2 f(e_2)))}{(e_1 - e_2)(e_2 - e_3)}.$$

Proposition 3.7. *[15] Let an alphabet $A_3 = \{a_1, a_2, a_3\}$, then we have,*

$$S_n(A_3) = \frac{a_2 S_n(a_1 + a_2) - a_3 S_n(a_1 + a_3)}{a_2 - a_3}. \tag{3.2}$$

Proof. (see [15]). □

Proposition 3.8. *Let an alphabet $E_4 = \{e_1, e_2, e_3, e_4\}$, then we have,*

$$S_n(E_4) = \frac{e_3 S_n(e_1 + e_2 + e_3) - e_4 S_n(e_1 + e_2 + e_4)}{e_3 - e_4}. \tag{3.3}$$

Proof.
We have,

$$\sum_{n=0}^{+\infty} S_n(e_1 + e_2 + e_3)z^n = \frac{1}{(1 - e_1z)(1 - e_2z)(1 - e_3z)}. \tag{3.4}$$

Then

$$\sum_{n=0}^{+\infty} S_n(e_1 + e_2 + e_4)z^n = \frac{1}{(1 - e_1z)(1 - e_2z)(1 - e_4z)}. \tag{3.5}$$

Multiplying the equation (3.4) by e_3 and subtracting it from (3.5) multiplying by e_4, we have

$$\sum_{n=0}^{+\infty}[e_3 S_n(e_1 + e_2 + e_3) - e_4 S_n(e_1 + e_2 + e_4)]z^n = \frac{e_3 - e_4}{(1 - e_1z)(1 - e_2z)(1 - e_3z)(1 - e_4z)},$$

thus

$$\begin{aligned}\sum_{n=0}^{+\infty} \frac{e_3 S_n(e_1 + e_2 + e_3) - e_4 S_n(e_1 + e_2 + e_4)z^n}{e_3 - e_4} &= \frac{1}{(1 - e_1z)(1 - e_2z)(1 - e_3z)(1 - e_4z)} \\ &= \sum_{n=0}^{+\infty} S_n(e_1 + e_2 + e_3 + e_4)z^n.\end{aligned}$$

Therefore

$$\frac{e_3 S_n(e_1 + e_2 + e_3) - e_4 S_n(e_1 + e_2 + e_4)}{e_3 - e_4} = S_n(e_1 + e_2 + e_3 + e_4).$$

This completes the proof. □

Lemma 3.9. *Let $A_3 = \{a_1, a_2, a_3\}$ and $E_4 = \{e_1, e_2, e_3, e_4\}$ be two alphabets .Then action of operator $\delta_{e_3e_4}\delta_{e_2e_3}\delta_{e_1e_2}$ on the series $\sum_{n=0}^{+\infty} S_n(A_3)e_1^n z^n$is given by:*

$$\delta_{e_3e_4}\delta_{e_2e_3}\delta_{e_1e_2}\sum_{n=0}^{+\infty} S_n(A_3)e_1^n z^n = \sum_{n=0}^{+\infty} S_n(A_3)S_n(E_4)z^n.$$

Proof.
We have,

$$\begin{aligned}
\delta_{e_3e_4}\delta_{e_2e_3}\delta_{e_1e_2}\sum_{n=0}^{+\infty} S_n(A_3)e_1^n z^n &= \delta_{e_3e_4}\delta_{e_2e_3}\left(\frac{e_1\sum_{n=0}^{+\infty} S_n(A_3)e_1^n z^n - e_2\sum_{n=0}^{+\infty} S_n(A_3)e_2^n z^n}{e_1-e_2}\right)\\
&= \delta_{e_3e_4}\delta_{e_2e_3}\left(\sum_{n=0}^{+\infty} S_n(A_3)\frac{e_1^{n+1}-e_2^{n+1}}{(e_1-e_2)}z^n\right)\\
&= \delta_{e_3e_4}\delta_{e_2e_3}\left(\sum_{n=0}^{+\infty} S_n(A_3)S_n(e_1+e_2)z^n\right.\\
&= \delta_{e_3e_4}\left(\frac{e_2\sum_{n=0}^{+\infty} S_n(A_3)S_n(e_1+e_2)z^n - e_3\sum_{n=0}^{+\infty} S_n(A_3)S_n(e_1+e_3)z^n}{e_2-e_3}\right)\\
&= \delta_{e_3e_4}\sum_{n=0}^{+\infty} S_n(A_3)\frac{e_2S_n(e_1+e_2)z^n - e_3S_n(e_1+e_3)z^n}{(e_2-e_3)}\\
&= \delta_{e_3e_4}\sum_{n=0}^{+\infty} S_n(A_3)S_n(e_1+e_2+e_3)z^n\\
\delta_{e_3e_4}\delta_{e_2e_3}\delta_{e_1e_2}\sum_{n=0}^{+\infty} S_n(A_3)e_1^n z^n &= \sum_{n=0}^{\infty} S_n(A_3)S_n(E_4)z^n.
\end{aligned}$$

This completes the proof. □

Lemma 3.10. *Let $A = \{a_1, a_2\}$ and $E_4 = \{e_1, e_2, e_3, e_4\}$ be two alphabet. The action of the operator $\delta_{a_1a_2}$ on the series $\frac{1}{\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n}$ is given by:*

$$\delta_{a_1a_2}\frac{1}{\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n} = \frac{1 - a_1a_2S_2(-E_4)z^2 - a_1^2a_2S_3(-E_4)z^3 - a_1a_2^2S_3(-E_3)z^4}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)}. \tag{3.6}$$

Proof.
We have

$$\delta_{a_1a_2}\frac{1}{\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n} = \frac{1}{(a_1-a_2)}\left(\frac{a_1}{\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n} - \frac{a_2}{\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n}\right)$$

$$= \frac{a_1\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right) - a_2\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n\right)}{(a_1-a_2)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)}$$

$$= \frac{\sum\limits_{n=0}^{+\infty} S_n(-E_4)\frac{a_1a_2^n - a_2a_1^n z^n}{(a_1-a_2)}}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)}$$

$$= \frac{1 - a_1a_2\sum\limits_{n=0}^{+\infty} S_n(-E_4)S_{n-2}(a_1+a_2)z^n}{\left(\sum\limits_{n=2}^{+\infty} S_n(-E_4)a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)}$$

$$= \frac{1 - a_1a_2\left(S_2(-E_4)z^2 + S_3(-E_4)S_1(a_1+a_2)z^3 + S_4(-E_4)S_2(a_1+a_2)z^4\right)}{\left(\sum\limits_{n=2}^{+\infty} S_n(-E_4)a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)}$$

$$= \frac{1 - a_1a_2S_2(-E_3)z^2 - a_1^2a_2S_3(-E_4)z^3 - a_1a_2^2S_3(-E_4)z^4}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)}.$$

This is completes the proof. □

Lemma 3.11. *Given an alphabets $A = \{a_2, a_3\}$, and $E_4 = \{e_1, e_2, e_3, e_4\}$ we have*

$$\delta_{a_2a_3}\frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n} = \frac{1 - a_2a_3\left(S_2(-E_4)z^2 + S_3(-E_4)S_1(a_2+a_3)z^3\right)}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}.$$

Proof.

We have

$$\delta_{a_2a_3}\frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n} = \frac{1}{(a_2-a_3)}\left(\frac{a_2}{\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n} - \frac{a_3}{\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n}\right)$$

$$= \frac{a_2\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right) - a_3\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)}{(a_2-a_3)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}$$

$$= \frac{\sum\limits_{n=0}^{+\infty} S_n(-E_4)\frac{a_2a_3^n - a_3a_2^n z^n}{(a_2-a_3)}}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}$$

$$= \frac{1 - a_2a_3\sum\limits_{n=0}^{+\infty} S_n(-E_4)S_{n-2}(a_2+a_3)z^n}{\left(\sum\limits_{n=2}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}$$

$$= \frac{1 - a_2a_3\left(S_2(-E_4)z^2 + S_3(-E_4)S_1(a_2+a_3)z^3\right)}{\left(\sum\limits_{n=2}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}$$

$$= \frac{1 - a_2a_3S_2(-E_4)z^2 - a_2^2a_3S_3(-E_4)z^3 - a_2a_3^2S_3(-E_4)z^4}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}.$$

This completes the proof. □

Lemma 3.12. *Given an alphabet $A = \{a_2, a_3\}$, and $E_4 = \{e_1, e_2, e_3, e_4\}$ we have,*

$$\delta_{a_2a_3}\frac{a_2}{\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n} = \frac{(a_2+a_3) + a_2a_3S_1(-E_4)z - a_2^2a_3^2S_3(-E_4)z^3 - a_2^2a_3^2S_4(-E_4)(a_1+a_2)z^4}{\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}.$$

Proof.

We have,

$$\delta_{a_2a_3}\frac{a_2}{\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n} = \frac{1}{a_2-a_3}\left(\frac{a_2^2}{\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n} - \frac{a_3^2}{\sum\limits_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n}\right)$$

$$= \frac{\sum_{n=0}^{+\infty} S_n(-E_4)\frac{a_2^2a_3^n - a_3^2a_2^nz^n}{(a_2-a_3)}}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^nz^n\right)}$$

$$= \frac{(a_2+a_3) + a_2a_3S_1(-E_4)z - a_2^2a_3^2\sum_{n=0}^{+\infty} S_n(-E_4)S_{n-3}z^n}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^nz^n\right)}$$

$$= \frac{(a_2+a_3) + a_2a_3S_1(-E_4)z - a_2^2a_3^2S_3(-E_4)z^3 - a_2^2a_3^2S_4(-E_4)(a_1+a_2)z^4}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^nz^n\right)}.$$

This completes the proof . □

Lemma 3.13. *Given an alphabet $A = \{a_2, a_3\}$, and $E_4 = \{e_1, e_2, e_3, e_4\}$ We have,*

$$\delta_{a_2a_3}\left(\frac{a_2^2}{\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n}\right) = \frac{C_1(z)}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^nz^n\right)},$$

while :

$$C_1(z) = ((a_2+a_3)^2 - (a_2a_3)) + a_2a_3(a_2+a_3)S_1(-E_4)z + a_2^2a_3^2S_2(E_4)z^2 + a_3^3a_2^3S_4(-E_4)z^4.$$

Proof.
We have ,

$$\delta_{a_2a_3}\left(\frac{a_2^2}{\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n}\right) = \frac{1}{a_2-a_3}\left(\frac{a_2^3}{\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n} - \frac{a_3^3}{\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n}\right)$$

$$= \frac{\left(\sum_{n=0}^{+\infty} S_n(-E_4)\frac{a_2^3a_3^n - a_3^3a_2^n}{a_2-a_3} z^n\right)}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}$$

$$= \frac{((a_2+a_3)^2 - (a_2a_3)) + a_2a_3(a_2+a_3)S_1(-E_4)z + a_2^2a_3^2S_2(E_4)z^2}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}$$

$$+ \frac{+a_3^3a_2^3\sum_{n=4}^{+\infty} S_n(-E_4)S_{n-4}(a_2+a_3)z^n}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}$$

$$= \frac{((a_2+a_3)^2 - (a_2a_3)) + a_2a_3(a_2+a_3)S_1(-E_4)z + a_2^2a_3^2S_2(E_4)z^2 + a_3^3a_2^3S_4(-E_4)z^4}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}.$$

This completes the proof. □

Lemma 3.14. *Given an alphabet* $A = \{a_2, a_3\}$, *and* $E_4 = \{e_1, e_2, e_3, e_4\}$ *we have*

$$\delta_{a_2a_3}\left(\frac{a_2^3}{\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n}\right) = \frac{C_2(z)}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)},$$

while

$$C_2(z) = S_3(a_2+a_3) + a_2a_3S_2(a_2+a_3)S_1(-E_4)z + a_2^2a_3^2S_1(a_2+a_3)S_2(-E_4)z^2 + a_2^3a_3^3S_3(-E_4)z^3.$$

Proof.

We have ,

$$\delta_{a_2a_3}\left(\frac{a_2^3}{\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n}\right) = \frac{1}{a_2-a_3}\left(\frac{a_2^4}{\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n} - \frac{a_3^4}{\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n}\right)$$

$$= \frac{\left(\sum_{n=0}^{+\infty} S_n(-E_4)\frac{a_2^4a_3^n - a_3^4a_2^n}{a_2-a_3} z^n\right)}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}$$

$$= \frac{S_3(a_2+a_3) + a_2a_3S_2(a_2+a_3)S_1(-E_4)z + a_2^2a_3^2S_1(a_2+a_3)S_2(-E_4)z^2 + a_2^3a_3^3S_3(-E_4)z^3}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)}.$$

This completes the proof. □

Lemma 3.15. *We have,*

$$S_3(-A_3) - S_2(-A_3)S_1(-A_3) = a_1a_2^2 + a_1^2a_2 + a_1a_3^2 + a_1^2a_3 + a_2a_3^2 + a_2^2a_3.$$

Proof.
We have

$$\begin{cases} S_1(-A_3) = -(a_1 + a_2 + a_3) \\ S_2(-A_3) = a_1a_2 + a_1a_3 + a_2a_3 \\ S_3(-A_3) = -a_1a_2a_3 \end{cases}$$

and

$$(a_1a_2 + a_1a_3 + a_2a_3)(a_1 + a_2 + a_3) = 3a_1a_2a_3 + a_1a_2^2 + a_1^2a_2 + a_1a_3^2 + a_1^2a_3 + a_2a_3^2 + a_2^2a_3,$$

we get

$$S_3(-A_3) - S_2(-A_3)S_1(-A_3) = a_1a_2^2 + a_1^2a_2 + a_1a_3^2 + a_1^2a_3 + a_2a_3^2 + a_2^2a_3.$$

□

3 Main Results

In this section , we present three new theorems on the symmetric functions in several variables.

Theorem 3.16. *Let $A_3 = \{a_1, a_2, a_3\}$ and $E_4 = \{e_1, e_2, e_3, e_4\}$ two alphabets, we have:*

$$\sum_{n=0} S_n(E_4)S_n(A_3)z^n = \frac{\begin{matrix} 1 - S_2(-A_3)S_2(-E_4)z^2 \\ + \left[\begin{matrix} [S_2(-A_3)S_1(-A_3) - 3S_3(-A_3)]\, S_3(-E_4) \\ 2S_3(-A_3)S_2(-E_4)S_1(-E_4) \end{matrix} \right] z^3 \\ + \left[\begin{matrix} [S_2^2(-A_3) - S_2(-A_3)S_1^2(-A_3) + 2S_3(-A_3)S_1(-A_3)]\, S_4(-E_4) \\ -S_3(-A_3)S_1(-A_3)S_3(-E_4)S_1(-E_4) \end{matrix} \right] z^4 \\ +S_3(-A_3)\left[S_1^2(-A_3) - S_2(-A_3)\right] S_1(-E_4)S_4(-E_4)z^5 \\ -S_3^2(-A_3)\left[S_4(-E_4)S_2(-E_4) - S_3^2(-E_4)\right] z^6 \\ -S_3^2(-A_3)S_1(-A_3)S_4(-E_4)S_3(-E_4)z^7 \\ -S_3^2(-A_3)S_2(-A_3)S_4^2(-E_4)z^8 \end{matrix}}{D(a_1)D(a_2)D(a_3)}, \tag{3.7}$$

with

$$D(a_1)D(a_2)D(a_3) = \left[\sum_{k=0}^{4} S_k(-E_4)a_1^k z^k\right]\left[\sum_{k=0}^{4} S_k(-E_4)a_2^k z^k\right]\left[\sum_{k=0}^{4} S_k^4(-E_4)a_2^k z^k\right]$$

$$D(a_1)D(a_2)D(a_3) = \left[\begin{array}{c} [1 + S_1(-E_4)a_1 z + S_2(-E_4)a_1^2 z^2 + S_3(-E_4)a_1^3 z^3 + S_4(-E_4)a_1^4 z^4] \times \\ [1 + S_1(-E_4)a_2 z + S_2(-E_4)a_2^2 z^2 + S_3(-E_4)a_2^3 z^3 + S_4(-E_4)a_2^4 z^4] \times \\ [1 + S_1(-E_4)a_3 z + S_2(-E_4)a_3^2 z^2 + S_3(-E_4)a_3^3 z^3 + S_4(-E_4)a_3^4 z^4] \end{array}\right]$$

$$D(a_1)D(a_2)(D(a_3) = \begin{array}{c} 1+ \\ [a_1 + a_2 + a_3]\, S_1(-E_4)z+ \\ \left[\begin{array}{c} [a_1^2 + a_2^2 + a_3^2]\, S_2(-E_4)+ \\ [a_1a_2 + a_1a_3 + a_2a_3]\, S_1^2(-E_4) \end{array}\right] z^2+ \\ \left[\begin{array}{c} [a_1^3 + a_2^3 + a_3^3]\, S_3(-E_4)+ \\ \left[\begin{array}{c} a_1^2a_2 + a_1^2a_3+ \\ +a_2^2a_1 + a_2^2a_3 \\ a_3^2a_1 + a_3^2a_2 \end{array}\right] S_1(-E_4)S_2(-E_4) \\ a_1a_2a_3S_1^3(-E_4) \end{array}\right] z^3+ \\ \left[\begin{array}{c} [a_1^4 + a_2^4 + a_3^4]\, S_4(-E_4) \\ \left[\begin{array}{c} a_1^3a_2 + a_1^3a_3 \\ +a_2^3a_1 + a_2^3a_3 \\ +a_3^3a_1 + a_3^3a_2 \end{array}\right] S_1(-E_4)S_3(-E_4) \\ [a_1^2a_2a_3 + a_2^2a_1a_3 + a_3^2a_1a_2]\, S_1^2(-E_4)S_2(-E_4)+ \\ [a_1^2a_2^2]\, S_2^2(-E_4) \end{array}\right] z^4+ \\ \left[\begin{array}{c} [a_1^4a_2 + a_2^4a_1]\, S_1(-E_4)S_4(-E_4)+ \\ [a_1^2a_2^3 + a_1^3a_2^2]\, S_2(-E_4)S_3(-E_4) \end{array}\right] z^5+ \\ \left[\begin{array}{c} [a_1^2a_2^4 + a_1^2a_2^4]\, S_2(-E_4)S_4(-E_4)+ \\ [a_1^2a_2^4 + a_1^2a_3^4 + a_2^2a_1^4 + a_2^2a_3^4 + a_3^2a_1^4 + a_3^2a_2^4]\, S_2^3S_4(-E_4) \\ [a_1^3a_3^3 + a_2^3a_3^3]\, S_3^2(-E_4) \end{array}\right] z^6+ \end{array}$$

$$\begin{bmatrix} \begin{bmatrix} a_1^3a_2^4 + a_1^3a_3^4 + \\ +a_2^3a_1^4 + a_2^3a_3^4 \\ +a_3^3a_1^4 + a_3^3a_2^4 \end{bmatrix} S_3(-E_4)S_4(-E_4) \\ a_1^2a_2^2a_3^2\left[a_1 + a_2 + a3\right] S_2^2(-E_4)S_3(-E_4) \end{bmatrix} z^7 +$$

$$\begin{bmatrix} \left[a_1^4a_2^4 + a_1^4a_3^4 + a_2^4a_3^4\right] S_4^2(-E_4) + \\ \left[a_1^4a_2^2a_3^2 + a_2^4a_2^2a_3^2 + a_3^4a_1^2a_2^2\right] S_2^2(-E_4)S_4(-E_4) + \\ \left[a_1^2a_2^3a_3^3 + a_2^2a_1^3a_3^3 + a_3^2a_1^3a_3^3\right] S_2(-E_4)S_3^2(-E_4) \end{bmatrix} z^8 +$$

$$\begin{bmatrix} \left[a_1^3a_2^3a_3^3\right] S_3^3(-E_4) + \\ a_1a_2a_3\left[a_2^3a_3^3 + a_1^3a_2^3 + a_2^3a_3^3\right] S_1(-E_4)S_4^2(-E_4) + \\ a_1^2a_2^2a_3^2 \begin{bmatrix} a_2a_3^2 + a_3a_2^2 \\ +a_1a_3^2 + a_3^2a_1^2 \\ +a_1a_3^2 + a_3a_1^2 \end{bmatrix} S_2(-E_4)S_3(-E_4)S_4(-E_4) \end{bmatrix} z^9 + .$$

$$\begin{bmatrix} a_1^2a_2^2a_3^2\left[a_2^2a_3^2 + a_1^2a_3^2 + a_1^2a_2^2\right] S_2(-E_4)S_4^2(-E_4) + \\ a_1^3a_2^3a_3^3\left[a_1 + a_2 + a_3\right] S_3^2(-E_4)S_4(-E_4) \end{bmatrix} z^{10} +$$

$$a_1^3a_2^3a_3^3\left[a_2a_3 + a_1a_3 + a_1a_2\right] S_3(-E_4)S_4^2(-E_4)z^{11} +$$

$$a_1^4a_2^4a_3^4S_4^3(-E_4)z^{12}$$

Proof.

First step, the action of the operator $\delta_{a_1a_2}$ on $\sum\limits_{n=0}^{+\infty} S_n(E_4)a_1^nz^n = \frac{1}{D(a_1)}$ and by using the lemma (3.10) , we get

$$\sum_{n=0}^{+\infty} S_n(E_4)S_n(a_1 + a_2)z^n = \frac{1 - a_1a_2\sum\limits_{k=2}^{4} S_k(-E_4)S_{k-2}(a_1 + a_2)z^k}{D(a_1)D(a_2)}.$$

Considering $S_1(a_1 + a_2) = a_1 + a_2$ and $S_2(a_1 + a_2) = a_1^2 + a_1a_2 + a_2^2$, so the previous identity is written by:

$$\sum_{n=0} S_n(E_4)S_n(a_1+a_2)z^n = \frac{1}{D(a_1)} \begin{bmatrix} \frac{1}{D(a_2)} - \left[a_1S_2(-E_4)z^2 + S_3(-E_4)a_1^2z^3 + S_4(-E_4)a_1^3z^4\right] \frac{a_2}{D(a_2)} \\ -\left[a_1S_3(-E_4)z^3 + a_1^2S_4(-E_4)z^4\right] \frac{a_2^2}{D(a_2)} \\ -a_1S_4(-E_4)z^4 \frac{a_2^3}{D(a_2)} \end{bmatrix} \tag{3.8}$$

Second step, by the action of the operator $\delta_{a_2a_3}$ on (3.8) and by using the previous lemmas we get,

$$\sum_{n=0} S_n(E_4)S_n(a_1+a_2+a_3)z^n = \frac{1-a_2a_3\begin{bmatrix} S_2(-E_4)z^2 \\ +S_3(-E_4)S_1(a_2+a_3)z^3 \\ +S_4(-E_4)S_2(a_2+a_3)z^4 \\ a_1S_3(-E_4) \end{bmatrix}}{D(a_1)D(a_2)D(a_3)}$$

$$-\frac{\begin{matrix}[a_1S_2(-E_4)z^2+S_3(-E_4)a_1^2z^3+S_4(-E_4)a_1^3z^4]\times \\ \begin{bmatrix} S_1(a_2+a_3)+a_2a_3S_1(-E_4)z \\ -a_2^2a_3^2\begin{bmatrix} S_3(-E_4)z^3 \\ S_4(-E_4)S_1(a_2+a_3)z^4 \end{bmatrix} \end{bmatrix}\end{matrix}}{D(a_1)D(a_2)D(a_3)}$$

$$-\frac{\begin{matrix}[a_1S_3(-E_4)z^3+a_1^2S_4(-E_4)z^4]\times \\ \begin{bmatrix} S_2(a_2+a_3)+a_2a_3S_1(a_2+a_3)S_1(-E_4)z \\ +a_2^2a_3^2S_2(-E_4)z^2-a_2^3a_3^3S_4(-E_4)z^4 \end{bmatrix}\end{matrix}}{D(a_1)D(a_2)D(a_3)}$$

$$-\begin{bmatrix} \dfrac{\begin{matrix} a_1S_4(-E_4)z^4\times \\ \begin{bmatrix} S_3(a_2+a_3)+a_2a_3S_2(a_2+a_3)S_1(-E_4)z \\ +a_2^2a_3^2S_1(a_2+a_3)S_2(-E_4)z^2+a_2^3a_3^3S_3(-E_4)z^3 \end{bmatrix}\end{matrix}}{D(a_1)D(a_2)D(a_3)} \\ \\ \\ \end{bmatrix}$$

This completes the proof. □

Based on the Theorem 3.16 , we have the following proposition .

Proposition 3.17. *Let* $E_4 = \{e_1, e_2, e_3, e_4\}$ *ana* $A_3 = \{a_1, a_2, a_3\}$ *two alphabet , then we have ,*

$$\sum_{n=0} S_{n-1}(E_4)S_{n-1}(A_3)z^n = \frac{P_2(z)}{D(a_1)D(a_2)D(a_3)}, \tag{3.9}$$

with

$$\begin{aligned}
&P_2(z) = z - S_2(-A_3)S_2(-E_4)z^3 \\
&+ \left[\left[S_2(-A_3)S_1(-A_3) - 3S_3(-A_3)\right]S_3(-E_4) + 2S_3(-A_3)S_2(-E_4)S_1(-E_4)\right]z^4 \\
&+ \left[\begin{array}{l} \left[S_2^2(-A_3) - S_2(-A_3)S_1^2(-A_3) + 2S_3(-A_3)S_1(-A_3)\right]\times \\ S_4(-E_4) - S_3(-A_3)S_1(-A_3)S_3(-E_4)S_1(-E_4) \end{array}\right]z^5 \\
&+ S_3(-A_3)\left[S_1^2(-A_3) - S_2(-A_3)\right]S_1(-E_4)S_4(-E_4)z^6 \\
&- S_3^2(-A_3)\left[S_4(-E_4)S_2(-E_4) - S_3^2(-E_4)\right]z^7 \\
&- S_3^2(-A_3)S_1(-A_3)S_4(-E_4)S_3(-E_4)z^8 - S_3^2(-A_3)S_2(-A_3)S_4^2(-E_4)z^9.
\end{aligned}$$

Also ,based on the Theorem 3.16 , we have the following proposition .

Proposition 3.18. *Let $E_4 = \{e_1, e_2, e_3, e_4\}$ ana $A_3 = \{a_1, a_2, a_3\}$ two alphabet , then we have ,*

$$\sum_{n=0} S_{n-2}(E_4)S_{n-2}(A_3)z^n = \frac{P_3(z)}{D(a_1)D(a_2)D(a_3)}, \tag{3.10}$$

with

$$\begin{aligned}
&P_3(z) = z^2 - S_2(-A_3)S_2(-E_4)z^4 \\
&+ \left[\left[S_2(-A_3)S_1(-A_3) - 3S_3(-A_3)\right]S_3(-E_4) + 2S_3(-A_3)S_2(-E_4)S_1(-E_4)\right]z^5 \\
&+ \left[\begin{array}{l} \left[S_2^2(-A_3) - S_2(-A_3)S_1^2(-A_3) + 2S_3(-A_3)S_1(-A_3)\right]\times \\ S_4(-E_4) - S_3(-A_3)S_1(-A_3)S_3(-E_4)S_1(-E_4) \end{array}\right]z^6 \\
&+ S_3(-A_3)\left[S_1^2(-A_3) - S_2(-A_3)\right]S_1(-E_4)S_4(-E_4)z^7 \\
&- S_3^2(-A_3)\left[S_4(-E_4)S_2(-E_4) - S_3^2(-E_4)\right]z^8 \\
&- S_3^2(-A_3)S_1(-A_3)S_4(-E_4)S_3(-E_4)z^9 - S_3^2(-A_3)S_2(-A_3)S_4^2(-E_4)z^{10}.
\end{aligned}$$

Theorem 3.19. *Let $A_3 = \{a_1, a_2, a_3\}$ and $E_4 = \{e_1, e_2, e_3, e_4\}$ be two alphabets, then we have,*

$$\sum_{n=0} S_n(E_4)S_{n-1}(A_3)z^n = \frac{P_4(z)}{D(a_1)D(a_2)D(a_3)}, \tag{3.11}$$

with

$$\begin{aligned}
P_4(z) = [&-S_1(-E_4)z + S_2(-E_4)S_1(-A_3)z^2 - S_3(-E_4)(S_1(-A_3)^2 - S_2(-A_4))z^3 \\
&+ S_3(-A_3)^2(S_1(E_4)S_3(E_4) - S_2(E_4)^2)z^4 \\
&+ S_2(E_4)S_3(E_4)S_1(-A_3)S_3(-A_3)z^5 \\
&- S_3(-E_4)^2S_2(-A_3)S_3(-A_3)z^6 + S_4(-E_4)S_3^2(-E_4)2z^7 + S_4^2(-E_4)z^8].
\end{aligned}$$

Proof. we have (see[19])

$$\sum_{n=0}^{+\infty} S_n(E_4)S_{n-1}(a_1+a_2)z^n$$
$$= \frac{E_{a_1,a_2}(z)}{\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n \sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n},$$

while

$$E_{a_1,a_2}(z) = S_1(-E_4)z - (a_1+a_2)S_2(-E_4)z^2 + [(a_1+a_2)^2 + a_1a_2]S_3(-E_4)z^3$$
$$- (a_1-a_2)[(a_1-a_2)^2 + 2a_1a_2]S_4(-E_4)z^4.$$

It suffices to apply the operator $\delta_{a_2a_3}$ to the identity above , we obtain,

$$\delta_{a_2a_3} \sum_{n=0}^{+\infty} S_n(E_4)S_{n-1}(a_1+a_2)z^n$$
$$= a_2 \frac{E_{a_1,a_2}(z)}{(a_2-a_3)\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n \sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n}$$
$$-a_3 \frac{E_{a_1,a_3}(z)}{(a_2-a_3)\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n \sum_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n},$$

so
$\sum_{n=0}^{+\infty} S_n(E_4)S_{n-1}(a_1+a_2+a_3)z^n$

$$= [-(a_2-a_3)S_1(E_4)z - (a_2-a_3)(a_1+a_2+a_3)S_2(-E_4)z^2$$
$$- (a_2-a_3)S_3(E_4)((a_1+a_2+a_3)^2 - (a_1a_2+a_1a_3+a_2a_3))z^3$$
$$- (a_2-a_3)a_1a_2a_3(S_1(E_4)S_3(E_4) - S_2(E_4)^2)z^4$$
$$+ (a_2-a_3)a_1a_2a_3(a_1+a_2+a_3)S_2(E_4)S_3(E_4)z^5$$
$$+ (a_2-a_3)a_1a_2a_3(a_1a_2+a_1a_3+a_2a_3)S_3(E_4)^2z^6]$$
$$+ (a_2-a_3)[a_1^3(a_3^3-a_2^3) + a_1a_2a_3(a_3^2a_2+a_3^2-a_2^2(a_3+a_1))]S_4(-E_4)S_3(-E_4)z^7$$
$$+ [(a_2-a_3)(a_1^3(a_3^4-a_2^4) + a_1a_2a_3(a_3^3+a_3^3a_1-a_2^3a_1) - a_2^4a_3^3]S_4(-E_4)S_4(-E_4)z^8$$
$$\times \left[(a_2-a_3)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)\right]^{-1}.$$

$$\sum_{n=0}^{+\infty} S_n(E_4)S_{n-1}(A_3)$$

$$\begin{aligned}
&= [S_1(-E_4)z + S_2(-E_4)S_1(-A_3)z^2 - S_3(-E_4)(S_1(-A_3)^2 - S_2(-A_4))z^3 \\
&+ S_3(-A_3)^2(S_1(E_4)S_3(E_4) - S_2(E_4)^2)z^4 \\
&+ S_2(E_4)S_3(E_4)S_1(-A_3)S_3(-A_3)z^5 \\
&- S_3(-E_4)^2S_2(-A_3)S_3(-A_3)z^6 + S_4(-E_4)S_3^2(-E_4)2z^7 + S_4^2(-E_4)z^8] \\
&\times \left[\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_1^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^n z^n\right)\right]^{-1}.
\end{aligned}$$

Therefore

$$\sum_{n=0}^{+\infty} S_n(E_4)S_{n-1}(-A_3) = \frac{P_4(z)}{D(a_1)D(a_2)D(a_3)}$$

This completes the proof . □

Corollary 3.20. *From* (3.11) *we conclude the following relationships:*

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_{n-2}(-A_3) = \frac{P_5(z)}{D(a_1)D(a_2)D(a_3)} \tag{3.12}$$

with

$$\begin{aligned}
P_5(z) &= [S_1(-E_4)z^2 + S_2(-E_4)S_1(-A_3)z^3 - S_3(-E_4)(S_1(-A_3)^2 - S_2(-A_4))z^4 \\
&+ S_3(-A_3)^2(S_1(E_4)S_3(E_4) - S_2(E_4)^2)z^5 \\
&+ S_2(E_4)S_3(E_4)S_1(-A_3)S_3(-A_3)z^6 \\
&- S_3(-E_4)^2S_2(-A_3)S_3(-A_3)z^7 + S_4(-E_4)S_3^2(-E_4)2z^8 + S_4^2(-E_4)z^9].
\end{aligned}$$

Theorem 3.21. *Let* $A_3 = \{a_1, a_2, a_3\}$ *and* $E_4 = \{e_1, e_2, e_3, e_4\}$ *be two alphabets, Then we have,*

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_n(-A_3) = \frac{P_6(z)}{D(a_1)D(a_2)D(a_3)}, \tag{3.13}$$

with

$$\begin{aligned}
P_6(z) &= (2a_1 + a_2 + a_3)z + (a_1a_3 + 2a_3a_2 + a_1a_2)S_1(-E_4)z^2 \\
&+ [a_3^3(a_1 + a_2) - a_2^3(a_1 + a_3)]S_3(-E_4)z^3 \\
&+ S_3(-A_3)S_2(-E_4)S_1(-E_4)z^4 \\
&+ [a_3 + a_2]S_3(-A_3)S_3(-E_4)S_1(-E_4)z^5 \\
&+ [a_3^3 - a_2^3]S_3(-A_3)S_4(-E_4)S_1(-E_4)z^6 \\
&+ [a_3^3a_2 - a_2^3a_3)]S_3(-A_3)S_4^2(-E_4)z^8 \\
&+ S_3^2(-E_4)S_4(-E_4)S_2(-E_4)S_3(-A_3)z^7 \\
&+ [a_3^4(a_1^2a_3^2)(a_1 + a_3) - a_2^4(a_1^2a_2^2)(a_1 + a_2)]S_4^3(-E_4)z^9.
\end{aligned}$$

Proof. We have (see[19])

$$\sum_{n=0}^{+\infty} S_{n-1}(E_4)S_n(a_1 + a_2)z^n$$
$$= \frac{(a_1 + a_2)z + a_1a_2S_1(-E_4)z^2 - a_1^2a_2^2S_3(-E_4)z^4 - a_1^2a_2^2(a_1 + a_2)S_4(-E_4)z^5}{\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_1^nz^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n\right)}.$$

It suffices to apply the operator $\delta_{a_2a_3}$ to the identity above , we obtain

$$\delta_{a_2a_3}\sum_{n=0}^{+\infty} S_n(E_4)S_{n-1}(a_1 + a_2)z^n$$
$$= a_2\frac{(a_1 + a_2)z + a_1a_2S_1(-E_4)z^2 - a_1^2a_2^2S_3(-E_4)z^4 - a_1^2a_2^2(a_1 + a_2)S_4(-E_4)z^5}{\left((a_2 - a_3)\sum_{n=0}^{+\infty} S_n(-E_4)a_1^nz^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_2^nz^n\right)}$$
$$-a_3\frac{(a_1 + a_3)z + a_1a_3S_1(-E_4)z^2 - a_1^2a_3^2S_3(-E_4)z^4 - a_1^2a_3^2(a_1 + a_3)S_4(-E_4)z^5}{\left((a_2 - a_3)\sum_{n=0}^{+\infty} S_n(-E_4)a_1^nz^n\right)\left(\sum_{n=0}^{+\infty} S_n(-E_4)a_3^nz^n\right)},$$

$$\begin{aligned}
P_6(z) &= (2a_1 + a_2 + a_3)z + (a_1a_3 + 2a_3a_2 + a_1a_2)S_1(-E_4)z^2 \\
&+ [a_3^3(a_1 + a_2) - a_2^3(a_1 + a_3)]S_3(-E_4)z^3 \\
&+ S_3(-A_3)S_2(-E_4)S_1(-E_4)z^4 \\
&+ [a_3 + a_2]S_3(-A_3)S_3(-E_4)S_1(-E_4)z^5 \\
&+ [a_3^3 - a_2^3]S_3(-A_3)S_4(-E_4)S_1(-E_4)z^6 \\
&+ [a_3^3a_2 - a_2^3a_3)]S_3(-A_3)S_4^2(-E_4)z^8 \\
&+ S_3^2(-E_4)S_4(-E_4)S_2(-E_4)S_3(-A_3)z^7 \\
&+ [a_3^4(a_1^2a_3^2)(a_1 + a_3) - a_2^4(a_1^2a_2^2)(a_1 + a_2)]S_4^3(-E_4)z^9.
\end{aligned}$$

□

Corollary 3.22. *Based to the relation* (3.13) *we obtain*

$$\sum_{n=0}^{+\infty} S_{n-2}(E_4)S_{n-1}(-A_3) = \frac{P_7(z)}{D(a_1)D(a_2)D(a_3)}, \tag{3.14}$$

with

$$\begin{aligned} P_7(z) = {} & (2a_1 + a_2 + a_3)z^2 + (a_1a_3 + 2a_3a_2 + a_1a_2)S_1(-E_4)z^3 \\ & + [a_3^3(a_1 + a_2) - a_2^3(a_1 + a_3)]S_3(-E_4)z^4 \\ & + S_3(-A_3)S_2(-E_4)S_1(-E_4)z^5 \\ & + [a_3 + a_2]S_3(-A_3)S_3(-E_4)S_1(-E_4)z^6 \\ & + [a_3^3 - a_2^3]S_3(-A_3)S_4(-E_4)S_1(-E_4)z^7 \\ & + [a_3^3a_2 - a_2^3a_3)]S_3(-A_3)S_4^2(-E_4)z^8 \\ & + S_3^2(-E_4)S_4(-E_4)S_2(-E_4)S_3(-A_3)z^9 \\ & + [a_3^4(a_1^2a_3^2)(a_1 + a_3) - a_2^4(a_1^2a_2^2)(a_1 + a_2)]S_4^3(-E_4)z^{10}. \end{aligned}$$

Chapter 4

Ordinary Generating Functions of Binary Products of Fourth-Order Reccurrence Relations And Third- order

In this chapter we are going to give the generating function of the product of some numbers of the fourth and third order and polynomials , by using the theorems in the previous chapter

1 Definitions and Preliminaries

In this section , we give some definitions of reccurence relations and q- analog that we need for application. For more details you can refer to [15, 19, 25].

Definition 4.1. *The reccurence relation of Tetranacci numbers* $\{T_n^{(4)}\}_{n\in\mathbb{N}}$ *is given by*

$$\begin{cases} T_n^{(4)} = T_{n-1}^{(4)} + T_{n-2}^{(4)} + T_{n-3}^{(4)} + T_{n-4}^{(4)}, n \geq 4, \\ T_0^{(4)} = 0, T_1^{(4)} = 1, T_2^{(4)} = 1, T_3^{(4)} = 2. \end{cases}$$

Definition 4.2. *The reccurence relation of Jacobstal of the third order numbers* $\{J_n^{(3)}\}_{n\in\mathbb{N}}$ *is given by*

$$\begin{cases} J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}, n \geq 3, \\ J_0^{(3)} = 0, J_1^{(3)} = 1, J_2^{(3)} = 1. \end{cases}$$

Definition 4.3. *The reccurence relation of Narayana numbers* $\{N_n\}_{n\in\mathbb{N}}$ *is given by*

$$\begin{cases} N_n = N_{n-1} + N_{n-3}, n \geq 3 \\ N_0 = 0, N_1 = 1, N_2 = 1. \end{cases}$$

Definition 4.4. *The reccurence relation of Padovan Pira numbers* $\{S_n\}_{n\in\mathbb{N}}$ *is given by :*

$$\begin{cases} S_n = S_{n-2} + S_{n-3}, n \geq 3, \\ S_0 = 0, S_1 = 0, S_2 = 1. \end{cases}$$

Definition 4.5. *The reccurence cheybechev 2- orthogonal polynomials* $\{\hat{T}_n\}_{n\in\mathbb{N}}$ *is given by*

$$\begin{cases} \hat{T}_{n+1}(x) = x\hat{T}_n(x) - \alpha\hat{T}_{n-1}(x) - \beta\hat{T}_{n-2}(x)\ , \alpha, \beta \in \mathbb{R}, n \geq 3, \\ \hat{T}_0(x) = 1, \hat{T}_1(x) = x, \hat{T}_2(x) = x^2 - \alpha. \end{cases}$$

Definition 4.6. *The reccurence Tricobstal polynomials* $\{J_n^{(3)}\}_{n\in\mathbb{N}}$ *is given by*

$$\begin{cases} J_n^{(3)}(x) = J_{n-1}^{(3)}(x) + xJ_{n-2}^{(3)}(x) + x^2 J_{n-3}^{(3)}, n \geq 3, \\ J_1^{(3)}(x) = 1, J_2^{(3)}(x) = 1, J_3^{(3)}(x) = x + 1. \end{cases}$$

Theorem 4.7. *Let* $E_4 = \{1, 2, 3, 4\}$ *an alphabet ,we have*

$$S_n(E_4) = \frac{4^{n+3} - 1}{6} + \frac{2^{n+3} - 3^{n+3}}{2}.$$

Proof.
we have

$$\delta_{e_n e_{n+1}} \delta_{e_{n-1} e_n} \dots \delta_{e_1 e_2} f(e_1) = \sum_{k=0}^{n} \frac{f(e_{i+1})}{R(e_{k+1}, E_k) R(e_{k+1}, E_{n+1}/E_{k+1})},$$

when $E_4 = \{e_1, e_2, e_3, e_4\}$ and $f(e_1) = e_1^{n+3}$, the formula of lagrange becomes,
we have

$$\delta_{e_3 e_4} \delta_{e_2 e_3} \delta_{e_1 e_2} f(e_1) = \sum_{k=0}^{3} \frac{e_{k+1}^{n+3}}{R(e_{k+1}, E_k) R(e_{k+1}, E_{n+1}/E_{k+1})}$$

$$S_n(E) = \sum_{k=0}^{3} \frac{e_{k+1}^{n+3}}{R(e_{k+1}, E_k) R(e_{k+1}, E_{n+1}/E_{k+1})}.$$

So

$$S_n(E) = \frac{e_1^{n+3}}{(e_1 - e_2)(e_1 - e_3)(e_1 - e_4)} + \frac{e_2^{n+3}}{(e_2 - e_1)(e_2 - e_3)(e_2 - e_4)} + \frac{e_3^{n+3}}{(e_3 - e_1)(e_3 - e_2)(e_3 - e_4)} + \frac{e_4^{n+3}}{(e_4 - e_1)(e_4 - e_2)(e_4 - e_3)}.$$

For $E_4 = \{1, 2, 3, 4\}$ we have ,

$$S_n(E_4) = \frac{4^{n+3} - 1}{6} + \frac{2^{n+3} - 3^{n+3}}{2}.$$

This completes the proof. □

Definition 4.8. *For* $n \in \mathbb{N}$ *, we define* $q-analog$ *by*

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}$$

Definition 4.9. *We define Stiriling numbers from the first kind* $s(n, k)$ *is the coefficients of polynomial* $(x)_i$ *defined by*

$$(x)_k = x(x - 1)(x - 2) \dots (x - n + 1) = \sum_{k=0}^{n} s(n, k) x^k.$$

Definition 4.10. *The Gaussien binomial coefficients are defined by :*

$$\binom{n}{k}_q = \frac{(1 - q^n)(1 - q^{n-1}) \dots (1 - q^{n-k+1})}{(1 - q)(1 - q^2)(1 - q^k)}.$$

Theorem 4.11. *Let $A_3 = \{1, q, q^2\}$ an alphabet, so we have*

$$S_n(A_3) = \begin{bmatrix} n+2 \\ n \end{bmatrix}_q .$$

Proof.
We have $A = \{1, q, q^2, \ldots, q^{n-1}\} := [n]$ Guass polynomials given by the following relation [1]

$$S_j([n]) = \begin{bmatrix} n+j-1 \\ j \end{bmatrix}_q .$$

For $n = 3$ and $j = n$ we find

$$S_n([3]) = \begin{bmatrix} n+2 \\ n \end{bmatrix}_q .$$

This completes the proof. □

2 q- calculs and some propreties

Definition 4.12. *For $w \in \mathbb{C}$, we define q binomial for w by:*

$$(wq)_n = \begin{cases} 1 & for\ n = 0 \\ (1-w)(1-wq)\ldots(1-wq^{n-1}) & for\ n \geq 1 \end{cases} .$$

Remark 4.13. *We have ,*

$$(wq)_\infty = \lim_{n\longrightarrow\infty}(wq)_n = \prod(1 - wq^n).$$

Remark 4.14.

$$(qq)_n = \begin{cases} 1 & for\ n = 0 \\ (1-q)(1-q^2)\ldots(1-q^n) & for\ n \geq 1 \end{cases} .$$

and

$$(qq)_\infty = \lim_{n\longrightarrow\infty}(qq)_n = \prod(1 - q^{n+1}).$$

Now we are going to retrieve some relations of [42]
Taking A=$\{1, q, q^2, \dots, q^n\}$ we get

$$\sum_{n=0}^{\infty} S_n(A)z^n = \frac{1}{(1-z)(1-qz)\dots(1-q^n z)} = \frac{1}{(zq)_\infty}.$$

From the theorem of q binomial we get,

$$S_n(A) = \frac{1}{(qq)_n}$$
$$S_0(A) = 1$$
$$S_1(A) = 1 + q + q^2 \dots + q^n = \frac{1}{1-q} = \frac{1}{(qq)_1}$$
$$S_2(A) = \frac{1}{(qq)_2} = \frac{1}{(1-q)(1-q^2)}.$$

Taking $A = xA$ and $B = yB$ in definition 1.46 we get ,

$$\sum_{n=0}^{+\infty} S_n(A-E)z^n = \frac{\prod_{y\in B}(1-yq^n z)}{\prod_{x\in A}(1-xq^n z)} = \frac{(y^t q)_\infty}{(x^t q)_\infty},$$

so

$$S_n(A-E) = S_n(xA) + S_n(-yB)$$
$$S_1(A-E) = \frac{x}{1-q} - \frac{y}{1-q} = \frac{x-y}{1-q} = \frac{x-y}{(qq)_1},$$

we deduce that

$$S_n(xA - yE) = \frac{f(x,y)}{(qq)_n}.$$

By replacing e_1 by $e_1 x$ and e_2 by $e_2 x$ in definition 3.1 we get,

$$\delta^k_{e_1 e_2} = \frac{e_1^k f(e_1 x) - e_2^k f(e_2 x)}{e_1 x - e_2 x}. \tag{4.1}$$

For $A = \{[0], [1], [2], \dots\}$,
while

$$[0] = 1, [1] = 1, [2] = 1 + q, [3] = 1 + q + q^2 \dots, [n] = \frac{1-q^n}{1-q}.$$

And q binomial is given by

$$[n]! = [n]_q[n-1]_q \dots [2]_q[1]_q = \frac{(qq)_n}{(1-q)^n},$$

so

$$[n]! = \prod_{k=1}^{n} \frac{(1-q^k)}{1-q},$$

while

$$[0]! = 1.$$

We have

$$S_j(-[n]) = (-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix},$$

and

$$S_j([n]) = \begin{bmatrix} n+j-1 \\ j \end{bmatrix},$$

while

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{(qq)_n}{(qq)_j(qq)_{n-j}}.$$

When $q \to 1$, we have the following lemma

Lemma 4.15.

1) $\lim\{D_q\{f(x)\}\} = f'(x)$.

2) $\lim\{D_q^{-1}\{f(x)\}\} = f'(x)$.

3) $f(x)D_q\{g(x)\} + g(x)D_q\{f(x)\} = D_q\{f(x)g(x)\}$.

4) $f(x)D_q^{-1}\{g(x)\} + g(q^{-1}x)D_q^{-1}\{f(x)\} = D_q^{-1}\{f(x)g(x)\}$.

Proof.

For $e_1 = 1$ and $e_2 = q$, for $k = 1$ in (4.1) we get

$$\delta_{1q} = \frac{f(x) - f(xq)}{(1-q)x} = D_q\{f(x)\}.$$

$$\delta_{\frac{1}{q}1} = \frac{f(q^{-1}x) - f(x)}{(q^{-1}-1)x} = D_q^{-1}\{f(x)\}.$$

Also

$$\begin{aligned}\delta_{1q} &= \frac{f(x)g(x)-f(xq)g(xq)}{(1-q)x} = \frac{f(x)g(x)-f(xq)g(xq)+f(x)g(xq)-f(x)g(xq)}{(1-q)x}\\ &= f(x)\frac{[g(x)-g(xq)]}{(1-q)x} + g(xq)\frac{[f(x)-f(xq)]}{(1-q)x}\\ &= f(x)D_q\{g(x)\} + g(qx)D_q\{f(x)\}.\end{aligned}$$

And

$$\begin{aligned}\delta_{\frac{1}{q}1} &= \frac{f(q^{-1}x)g(q^{-1}x)-f(x)g(x)}{(q^{-1}-1)x} = \frac{f(q^{-1}x)g(q^{-1}x)-f(x)g(x)+f(x)g(q^{-1}x)-f(x)g(q^{-1}x)}{(q^{-1}-1)x}\\ &= f(x)\frac{g(q^{-1}x)-g(x)}{(q^{-1}-1)x} + g(q^{-1}x)\frac{f(q^{-1}x)-f(x)}{(q^{-1}-1)x}\\ &= f(x)D_q^{-1}\{g(x)\} + g(q^{-1}x)D_q^{-1}\{f(x)\}.\end{aligned}$$

This completes the proof . □

3 Generating functions of product of some known numbers

In this part, we now derive the new generating functions of square of some known numbers.This case consists of four relaterd parts. **Firstly**, the substitutions: $\begin{cases} S_1(-A_3) = -1 \\ S_2(-A_3) = -1 \\ S_3(-A_3) = -1 \end{cases}$

and $\begin{cases} S_1(-E_4) = -1 \\ S_2(-E_4) = -1 \\ S_3(-E_4) = -1 \\ S_4(-E_4) = -1 \end{cases}$ by using (3.13) we get the following proposition.

Proposition 4.16. *For $n \in \mathbb{N}$ the new generating function of Tetranacci and Tribonacci numbers is given by :*

$$\sum_{n=0} T_n^{(4)} T_n z^n = \frac{z+z^2-z^4-z^5-z^6+z^7+z^8+z^9}{1-z-z^4+2z^5+z^6-2z^8-2z^9-2z^{10}-z^{11}-z^{12}}.$$

Corollary 4.17. *The following identity holds true:*

$$T_n^{(4)} T_n = S_{n-1}(E_4) S_n(A_3).$$

Secondly, let us now consider the following conditions in (3.9) $\begin{cases} S_1(-A_3) = -1 \\ S_2(-A_3) = -1 \\ S_3(-A_3) = -2 \end{cases}$ and $\begin{cases} S_1(-E_4) = -1 \\ S_2(-E_4) = -1 \\ S_3(-E_4) = -1 \\ S_4(-E_4) = -1 \end{cases}$, we get the following proposition .

Proposition 4.18. *For $n \in \mathbb{N}$ the new generating function of Tetranacci and Jacobstal third kind numbers is given by*

$$\sum_{n=0} T_n^{(4)} J_n^{(3)} z^n = \frac{z^2 + z^3 + 2z^4 + 2z^6 - 2z^7 + z^8 + z^9}{1 + z - 4z^2 - 11z^3 - 7z^4 + 2z^5 + z^6 + 3z^8 - 2z^9 + 4z^{10} - 2z^{11} - 8z^{12}}.$$

Corollary 4.19. *The following identity holds true*

$$T_n^{(4)} J_n^{(3)} = S_{n-1}(E_4) S_{n-1}(A_3).$$

Thirdly , we consider the following conditions in(3.9) $\begin{cases} S_1(-A_3) = -1 \\ S_2(-A_3) = 0 \\ S_3(-A_3) = -1 \end{cases}$ and $\begin{cases} S_1(-E_4) = -1 \\ S_2(-E_4) = -1 \\ S_3(-E_4) = -1 \\ S_4(-E_4) = -1 \end{cases}$, we get the following proposition.

Proposition 4.20. *For $n \in \mathbb{N}$ the new generating function of Tetranacci and Narayana numbers is given by*

$$\sum_{n=0} T_n^{(4)} N_n z^n = \frac{z^2 + z^3 + z^4 + z^5 + z^6 + z^8 + z^9}{1 + z - z^2 - 2z^3 - 4z^4 + 2z^5 + z^6 - z^8 - 2z^9 - z^{10} - z^{12}}.$$

Corollary 4.21. *The following identity holds true*

$$T_n^{(4)} N_n = S_{n-1}(E_4) S_{n-1}(A_3).$$

Fourthly, let us now consider the following conditions in (3.12) $\begin{cases} S_1(-A_3) = 0 \\ S_2(-A_3) = -1 \\ S_3(-A_3) = -1 \end{cases}$ and $\begin{cases} S_1(-E_4) = -1 \\ S_2(-E_4) = -1 \\ S_3(-E_4) = -1 \\ S_4(-E_4) = -1 \end{cases}$, we have the following proposition.

Proposition 4.22. *For $n \in \mathbb{N}$ the new generating function of Tetranacci and Padovan Pira numbers is given by*

$$\sum_{n=0} T_n^{(4)} P_n z^n = \frac{z^2 + z^4 - z^5 - z^7 - z^8 + z^9}{1 - 2z^2 - 3z^3 - 3z^4 + 2z^5 + z^6 + z^8 - 2z^9 - z^{10} - z^{11} - z^{12}}.$$

Corollary 4.23. *The following identity holds true:*

$$T_n^{(4)} P_n = S_{n-1}(E_4) S_{n-2}(A_3).$$

4 Generating function of product of Tetranacci numbers and some polynomials

In this section ,we are going to give the generating function of the product of Tetranacci numbers (fourth order) and some polynomials of third order.

Firstly, by using (3.13), we consider the following conditions : $\begin{cases} S_1(-A_3) = x^2 \\ S_2(-A_3) = -x \\ S_3(-A_3) = -1 \end{cases}$ and

$\begin{cases} S_1(-E_4) = -1 \\ S_2(-E_4) = -1 \\ S_3(-E_4) = -1 \\ S_4(-E_4) = -1 \end{cases}$,we have the following proposition

Proposition 4.24. *For $n \in \mathbb{N}$ the new generating function of Tetranacci and Tribonacci polynomials is given by*

$$\sum_{n=0} T_n^{(4)} T_n(x) z^n = \frac{Q_1(z)}{D_1(z)},$$

while

$$D_1(z) = 1 + x^2 z + (-x^5 + x^4 - 2x)z^2 + (x^4 - 2x^3)z^3 + (-x^8 + x^6 - x^5 - 4x^3 - 5x^2 - 1)z^4 \\ - x^3 z^6 + x^3 z^7 + (-4x^7 - x^6 - 2x^4 + x^2 - 3x)z^8 + (-x^3 + 3)z^9 - xz^{11} - z^{12}$$

and

$$Q_1(z) = z + xz^3 - +(-x^3 - 5)z^4 + r(x^2 + x^5 + 2x^2)(1 - x^2)z^5 \\ - (x^4 + x)z^6 + (1 - x)z^7 - z^8 + xz^9$$

Secondly, we consider the following conditions in (3.9) $\begin{cases} S_1(-A_3) = -1 \\ S_2(-A_3) = -x \\ S_3(-A_3) = -x^2 \end{cases}$ and

$\begin{cases} S_1(-E_4) = -1 \\ S_2(-E_4) = -1 \\ S_3(-E_4) = -1 \\ S_4(-E_4) = -1 \end{cases}$,we have the following proposition.

Proposition 4.25. *For $n \in \mathbb{N}$ the new generating function of Tetranacci and Tricobstal polynomials is given by*

$$\sum_{n=0} T_n^{(4)} J_n^{(3)}(x) z^n = \frac{Q_2(z)}{D_2(z)},$$

while

$$D_2(z) = 1 + z + (1-3x)z^2 + (-3x^2 - 4x + 1)z^3 + (-5x^2 + 3x - 1)z^4$$
$$- x^4 z^6 + x^4 z^7 + (5x^5 + 6x^4 - 2x)z^8 + (-3x^7 - x^6 - x^5 - x^2)z^9 + (x^8 + x^6)z^1 0 - x^3 z^{11} - x^8 z^{12},$$

and

$$Q_2(z) = z - xz^3 + (-x - 5x^2)z^4 + (3x^2 + x)(1 + x^2)z^5 - (x^3 - x^2)z^6 + (1 - x^4)z^7 - x^4 z^8 + x^5 z^9.$$

Thirdly , we consider the following conditions: $\begin{cases} S_1(-A_3) = x \\ S_2(-A_3) = \alpha \\ S_3(-A_3) = -\beta \end{cases}$ and $\begin{cases} S_1(-E_4) = -1 \\ S_2(-E_4) = -1 \\ S_3(-E_4) = -1 \\ S_4(-E_4) = -1 \end{cases}$,

we have the following proposition.

Proposition 4.26. *For $n \in \mathbb{N}$ the new generating function of Tetranacci and 2-orthogonal cheybechev polynomials of first second is given by*

$$\sum_{n=0} T_n^{(4)} \hat{T}_n(x) z^n = \frac{Q_3(z)}{D_3(z)},$$

while

$$D_3(z) = 1 - xz + (-x^2 + 3\alpha)z^2 + (x^2 - 4\alpha x - 3\beta)z^3 + (-x^4 + 5\alpha x + 6\beta x + \beta - 4\alpha^2)z^4$$
$$- \beta^2\alpha^2 z^6 - \beta^2 x z^7 + (-2\beta^3 x^2 + (\alpha^4 + 4\beta^2\alpha + 2\beta\alpha^2 - \beta^2)x + \beta^2\alpha)z^8 + (-\alpha\beta^2 x - 2\beta^3 + \beta)z^9$$
$$+ (-\beta^3 x - \alpha^2\beta^2)z^{10} + \beta^3\alpha z^{11} + \beta^4 z^{12},$$

and

$$Q_3(z) = xz - \alpha z^2 - \beta z^4 + \beta x z^5 + \beta x z^6 + \beta z^8 + \beta z^7 + z^9.$$

5 Application

5.1 The case : $E_4 = \{1, 2, 3, 4\}$ and $A_3 = \{1, 2, 3\}$

Let us consider the following conditions in theorem 3.7 $\begin{cases} S_1(-A_3) = -6 \\ S_2(-A_3) = 11 \\ S_3(-A_3) = -6 \end{cases}$ and $\begin{cases} S_1(-E_4) = -10 \\ S_2(-E_4) = 35 \\ S_3(-E_4) = -38 \\ S_4(-E_4) = 24 \end{cases}$,

we have the new results .

Proposition 4.27. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind is given by the following relation*

$$\sum_{n=0}^{\infty} \left(\frac{4^{n+3} - 1}{6} + \frac{2^{n+3} - 3^{n+3}}{2} \right) \left(\frac{3^{n+2} + 1}{2} - 2^{n+2} \right) z^n = \frac{Q_4(z)}{D_4(z)},$$

while

$$Q_4(z) = 1 - 385z^2 + 3048z^3 + 8808z^4 + 36000z^5 - 73804z^6 + 196992z^7 - 228096z^8,$$

and

$$D_4(z) = 1-60z+1590z^2-21432z^3-8468z^4+202850z^5+66572z^6-8949456z^7+114049548z^8 + 18254592z^9 + 80476416z^{10} + 5702400z^{11} + 17915904z^{12}.$$

Proposition 4.28. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind is given by the following relation*

$$\sum_{n=0}^{\infty} \left(\frac{4^{n+2} - 1}{6} + \frac{2^{n+2} - 3^{n+2}}{2} \right) \left(\frac{3^{n+1} + 1}{2} - 2^{n+1} \right) z^n = \frac{Q_5(z)}{D_4(z)},$$

while

$$Q_5(z) = z - 385z^3 + 3048z^4 + 8808z^5 + 36000z^6 - 73804z^7 + 196992z^8 - 228096z^9.$$

Proposition 4.29. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind is given by the following relation*

$$\sum_{n=0}^{\infty} \left(\frac{4^{n+3} - 1}{6} + \frac{2^{n+3} - 3^{n+3}}{2} \right) \left(\frac{3^{n+1} + 1}{2} - 2^{n+1} \right) z^n = \frac{Q_6(z)}{D_4(z)},$$

while

$$Q_6(z) = -10z - 210z^2 - 600z^3 - 7056z^4 + 47880z^5 + 1368z^6 + 34656z^7 + 576z^8.$$

Proposition 4.30. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind is given by the following relation*

$$\sum_{n=0}^{\infty}\left(\frac{4^{n+2}-1}{6}+\frac{2^{n+2}-3^{n+2}}{2}\right)\left(\frac{3^{n}+1}{2}-2^{n}\right)z^{n}=\frac{Q_6(z)}{D_4(z)},$$

while

$$Q_6(z) = -10z^2 - 210z^3 - 600z^4 - 7056z^5 + 47880z^6 + 1368z^7 + 34656z^8 + 576z^9.$$

Proposition 4.31. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind is given by the following relation*

$$\sum_{n=0}^{\infty}\left(\frac{4^{n+2}-1}{6}+\frac{2^{n+2}-3^{n+2}}{2}\right)\left(\frac{3^{n+2}+1}{2}-2^{n+2}\right)z^{n}=\frac{Q_4(z)}{D_4(z)},$$

while

$$Q_7(z) = 7z + -170z^2 - 1862z^3 + 2100z^4 - 11400z^5 + 27360z^6 + 7277760z^7 + 20736z^8.$$

Proposition 4.32. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind is given by the following relation*

$$\sum_{n=0}^{\infty}\left(\frac{4^{n+1}-1}{6}+\frac{2^{n+1}-3^{n+1}}{2}\right)\left(\frac{3^{n+1}+1}{2}-2^{n+1}\right)z^{n}=\frac{Q_8(z)}{D_4(z)},$$

while

$$Q_8(z) = 7z^2 + -170z^3 - 1862z^4 + 2100z^5 - 11400z^6 + 27360z^7 + 7277760z^8 + 20736z^9.$$

5.2 The case :$E_4 = \{1, 2, 3, 4\}$ and $A_3 = \{1, q, q^2\}$

Let consider the following conditions $\begin{cases} S_1(-A_3) = -[3] \\ S_2(-A_3) = q[3] \\ S_3(-A_3) = -q^3 \end{cases}$ and $\begin{cases} S_1(-E_4) = -10 \\ S_2(-E_4) = 35 \\ S_3(-E_4) = -38 \\ S_4(-E_4) = 24 \end{cases}$, we have the following new results.

Proposition 4.33. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind and $q-$ analog is given by the following relation*

$$\sum_{n=0}^{\infty}\left(\frac{4^{n+3}-1}{6}+\frac{2^{n+3}-3^{n+3}}{2}\right)\begin{bmatrix} n+2 \\ n \end{bmatrix}_q z^{n}=\frac{Q_9(z)}{D_9(z)},$$

while

$$Q_9(z) = 1-35q[3]z^2+(-814q^3+38q[3]^2)z^3+(q^2[3]^2+q[3]^3-332q^3[3])z^4+240q^3([3]^2-q[3])z^5 + 604q^6z^6 + 1292[3]q^6z^7 + 20160q^6z^8,$$

and,

$$D_9(z) = 1 + 10[3]z + ([3]^2 - 70q[3])z^2 + (-1050q^3 + 38[3]^2 - 350[3]^2 + 114q[3])z^3 + (-1570q^4 - 1330q^3 - 240q)z^5 + (1680q^4 + 105q^9 + 1444q^3)z^6 + (-912(q^4+q^8+q^3+q^{12}+q^6+q^{14})+46550[2]^3q^4)z^7+(576(q^4+q^8+q^{12})+29400(q^6+2q^8))z^8 + (13824q^8 + 5760q^3(2q^9 + q^3))z^9 + (20160q^6(q^6 + 2q^2) + 34656q^9[3])z^{10} + 3800q^9(q^4 + q^2 + q)z^{11} + 13824q^{12}z^{12}.$$

Proposition 4.34. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind and $q-$analog is given by the following relation*

$$\sum_{n=0}^{\infty}\left(\frac{4^{n+2}-1}{6}+\frac{2^{n+2}-3^{n+2}}{2}\right)\begin{bmatrix} n+1 \\ n \end{bmatrix}_q z^n = \frac{Q_{10}(z)}{D_9(z)},$$

while

$$Q_{10}(z) = 1-35q[3]z^3+(-814q^3+38q[3]^2)z^4+(q^2[3]^2+q[3]^3-332q^3[3])z^5+240q^3([3]^2-q[3])z^6+ 604q^6z^7 + 1292[3]q^6z^8 + 20160q^6z^9.$$

Proposition 4.35. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind and $q-$analog is given by the following relation*

$$\sum_{n=0}^{\infty}\left(\frac{4^{n+1}-1}{6}+\frac{2^{n+1}-3^{n+1}}{2}\right)z^n = \frac{Q_{11}(z)}{D_9(z)},$$

while

$$Q_{11}(z) = 1-35q[3]z^4+(-814q^3+38q[3]^2)z^5+(q^2[3]^2+q[3]^3-332q^3[3])z^6+240q^3([3]^2-q[3])z^7+ 604q^6z^8 + 1292[3]q^6z^9 + 20160q^6z^{10}.$$

Proposition 4.36. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind and $q-$analog is given by the following relation*

$$\sum_{n=0}^{\infty}\left(\frac{4^{n+3}-1}{6}+\frac{2^{n+3}-3^{n+3}}{2}\right)\begin{bmatrix} n+1 \\ n \end{bmatrix}_q z^n = \frac{Q_{12}(z)}{D_9(z)},$$

while

$$Q_{12}(z) = -10z+35[3]z^2-24([3]^2-q[3])z^3-1465q^6z^4+1330[3]q^3z^5+1444q^4[3]z^6+7296qz^7+576z^8.$$

Proposition 4.37. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind and $q-$ analog is given by the following relation*

$$\sum_{n=0}^{\infty}\left(\frac{4^{n+3}-1}{6}+\frac{2^{n+3}-3^{n+3}}{2}\right)z^n=\frac{Q_{12}(z)}{D_9(z)},$$

while

$$Q_{12}(z)=-10z^2+35[3]z^3-24([3]^2-q[3])z^4-1465q^6z^5+1330[3]q^3z^6+1444q^4[3]z^7+7296qz^8+576z^9.$$

5.3 The case: $E_4=\{1,2,3,4\}$ and $A_3=\{1,1,1\}$

Let consider the following conditions in (3.7) $\left\{\begin{array}{c} S_1(-A_3)=-3 \\ S_2(-A_3)=3 \\ S_3(-A_3)=-1 \end{array}\right.$ and $\left\{\begin{array}{c} S_1(-E_4)=-10 \\ S_2(-E_4)=35 \\ S_3(-E_4)=-38 \\ S_4(-E_4)=24 \end{array}\right.$,

then we have new results .

Proposition 4.38. *For all $n \in \mathbb{N}_0$, the generating function of the product of Stirling numbers from first kind and binomial is given by the following relation*

$$\sum_{n=0}^{\infty}\left(\frac{4^{n+3}-1}{6}+\frac{2^{n+3}-3^{n+3}}{2}\right)\binom{2n+1}{n}=\frac{Q_{13}(z)}{D_{13}(z)},$$

while

$$Q_{13}(z) = 1-105z^2+928z^3-1428z^4+1440z^5+604z^6-2736z^7-1728z^8,$$

and,

$$D_{13}(z)=1-30z+405z^2-1978z^3-8063z^4+3140z^5+4349z^6-145122z^7+241548z^8-263672z^9 - 3890304z^{10}-65664z^{11}+13824z^{12}.$$

Proposition 4.39. *For all $n \geq 1$, the generating function of the product of Stirling numbers from first kind and binomial is given by the following relation*

$$\sum_{n=0}^{\infty}\left(\frac{4^{n+2}-1}{6}+\frac{2^{n+2}-3^{n+2}}{2}\right)\binom{2n}{n-1}=\frac{Q_{14}(z)}{D_{13}(z)},$$

while

$$Q_{14}(z) = z-105z^3+928z^4-1428z^5+1440z^6+604z^7-2736z^8-1728z^9.$$

Conclusion

In conclusion , we were able through this thesis to find generating functions with an exponent (convolved), and generating functions of numbers of the fourth order with numbers of the third order, depending on the basic theorem in the third chapter, based on the results obtained in this thesis, we can get several new normal and exponential generating functions.
Here are some possible suggestion :
1. Taking the alphabets $A = \{a_1, a_2, a_3, a_4\}$ and $E = \{e_1, e_2, e_3, e_4\}$ we suggest new theorem, so we can get the generating functions of the product of numbers and polynomials of fourth order.
2. Taking the alphabets $E = \{e_1, e_2, e_3, e_4, \ldots, e_{d+1}\}$ we suggest new theorem, so we can get d orthogonal polynomials.
3. Taking the alphabets $A = \{a_1, a_2, a_3\}$, $E = \{e_1, e_2\}$ and $B = \{b_1, b_2\}$ we suggest new theorem, so we can get new results.
4. Taking the alphabets $A = \{a_1, a_2, a_3\}$, $E = \{e_1, e_2, e_3\}$ and $B = \{b_1, b_2\}$ we suggest new theorem, so we can get new results.
5. Taking the alphabets $A = \{a_1, a_2, a_3\}$, $E = \{e_1, e_2, e_3\}$ and $B = \{b_1, b_2, b_3\}$ we suggest new theorem, so we can get new results.
Also concerning the second chapter we suggest the form of convolved polynomials such as Mersenne polynomials, Pell polynomials,...

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