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Study of a Class of Variational Inequalities

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Fatima & Amiour ✍

Study of a class of variational inequalities

Absract

In this thesis, we give a new proof of the existence of absolutely continuous solutions for a class of first-order state dependent maximal monotone differential inclusions. The existence result is obtained by using Schauder's fixed point theorem. In addition, a stability result is provided. Finally, using a suitable reduction of order technique, we give a new existence result for a general second order state dependent maximal monotone differential inclusion.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 34H05, 34K35, 28A25, 28C20, 35K90.
Key words: Differential inclusion, fixed point, maximal monotone operators, perturbations, pseudo-distance.

Étude d'une classe d'inégalités variationnelles

Résumé

Dans cette thèse, nous donnons une nouvelle preuve de l'existence de solutions absolument continues pour une classe d'inclusions différentielles de premier ordre gouvernée par des opérateurs maximaux monotones dépendant de l'état. Le résultat d'existence est obtenu en utilisant le théorème du point fixe de Schauder. En outre, un résultat de stabilité est fourni. Enfin, en utilisant une technique de réduction d'ordre appropriée, nous donnons un nouveau résultat d'existence pour des inclusions différentielles du deuxième ordre gouvernées par des opérateurs maximaux monotones dépendant de l'état.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 34H05, 34K35, 28A25, 28C20, 35K90.

Mots clés: Inclusion différentielle, point fixe, opérateur maximal monotone, perturbations, pseudo-distance.

دراسة فئة من المتباينات المتغيرة

ملخص

في هذه الأطروحة، نقدم برهان جديد على وجود حلول مستمرة مطلقا لفئة من التضمينات التفاضلية من الرتبة الأولى التي تعتمد عل المتغير. يتم الحصول على نتيجة الوجود باستخدام نظرية النقطة الثابتة لشودر. بالإضافة إلى ذلك، يتم توفير نتيجة الاستقرار. أخيراً، باستخدام تقنيات تخفيض الرتبة، نعطي نتيجة وجود جديدة لفئة من التضمينات التفاضلية من الرتبة الثانية والتي تتعلق بالمتغير.

الكلمات المفتاحية: تضمينات تفاضلية، النقطة الثابتة، عامل الرتابة القصوى، تشويش.

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Notations

Operations and Symbols

$i.e.$	Identically equivalent.
$a.e.$	Almost every.
$resp$	Respectively.
$:=$	Equal by definition.
$\equiv$	Identically equal.
$\langle \cdot, \cdot \rangle$	Inner product on a Hilbert space.
$\|\cdot\|$	Norm.
$\sup, \inf, \max, \min$	Supremum, Infimum, Maximum, Minimum Respectively.
$d_H(A, B)$	Pompieu-Hausdorff distance between sets.
$u_n \longrightarrow u$	u_n converges to u strongly.
$u_n \rightharpoonup u$	u_n converges to u weakly (in weak topology).
co	Convex hull of a set.
$\overline{co}$	Closed convex hull of a set.
$u.s.c$	Upper semicontinuous.
$l.s.c$	Lower semicontinuous.

Spaces

$\mathbb{R}$	Real line.
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{-\infty, +\infty\}$.
$\mathbb{R}^d$	d-dimensional Euclidean space.
E	Vector space.
E'	Dual vector space.
H	Hilbert space.
$\sigma(E, E')$	weakly topology.
$C([T_0, T]; H)$	Space of continuous functions over$[T_0; T]$.
$L^1([T_0, T]; H)$	H-valued Lebesgue integrable functions over$[T_0; T]$.
$L^q([T_0, T]; H)$	space of (classes of) measurable functions over$[T_0; T]$.
$L^\infty([T_0, T]; H)$	space of (classes of) measurable essentially bounded functions over$[T_0; T]$.
$W^{1,1}([T_0, T]; H)$	H-valued absolutely continuous functions.
$W^{k,p}([T_0, T]; H)$	$:= \left\{ u \in L^q([T_0, T]; H) : \|u^{(i)}\|_{L^q([T_0,T];H)} < \infty, \ \forall i \leq k \right\}$.

Sets

$\mathbb{B}_H$	Open unit ball of space H.
$\overline{\mathbb{B}}_H$	Closed unit ball of space H.
$\mathbb{B}$	Closed unit ball the space in question.
$\partial f(x_0)$	Clarke subdifferential of $f$at x_0.
$N_S(\cdot)$	Clarke normal cone to S at x
$epi(f)$	Epigraph of an extended real valued function f.
$dom(f)$	Effectif domain of an extended real valued function f.
$Rg(F)$	The range of a set-valued map F.
$Grph(F)$	Graph of a set-valued map F.
$D(F)$	Effectif domain of a set-valued map F.

Functions

$d_S(\cdot)$ *or* $d(\cdot, S)$	Distance function.
$\delta_S(\cdot)$	Indicator function of a set S.
$\sigma_S(\cdot)$ *or* $\sigma(\cdot, S)$	Support function of a set S.
$dis(A, B)$	Distance between two maximal monotone operators.

Mapping

$f : X \longrightarrow Y$	Single-valued mapping from X to Y.
$F : X \rightrightarrows Y$	Set-valued mapping from X to Y.

General introduction

The aim of this thesis is to give some contributions to the theory of differential inclusions involving maximal monotone operators from the point of the well-posedness (in the sense existence and uniquness of solution).

Let $T > 0$ and let H be a real separable Hilbert space. In this thesis, we study the following nonlinear evolution inclusion:

$$\begin{cases} \dot{x}(t) \in f(t, x(t)) - A_{t,x(t)}x(t) \quad \text{a.e. } t \in [0, T] \\ x(0) = x_0. \end{cases} \tag{1}$$

In this problem, the maximal monotone operator $A_{t,x} : D(A_{t,x}) \subset H \longrightarrow 2^H$ is of absolutely continuous variation in time and Lipschitz continuous in state, in the sense of the pseudo-distance introduced by A. A. Vladimirov and $f : [0, T] \times H \longrightarrow H$ is a Carathéodory mapping. In contrast to earlier works on the subject, to prove the existence of absolutely continuous solutions for (1), we do not use the discretization method. More precisely, in this thesis we show how a fixed point approach can lead to the general existence theorem in the infinite dimensional space to the problem (1). By using Schauder's fixed point theorem and the existence and uniqueness theorem for the problem (1) with time-dependent A (i.e. $A := A_t$), we give a new proof of the state-dependent maximal monotone evolution inclusion (1) in the infinite dimensional Hilbert setting. Our approach gives explicitly the upper bound of the velocity in terms of data. We also establish a stability result. Finally we show the existence of solutions of a general second-order state-dependent maximal monotone differential inclusion of the form

$$\begin{cases} f(t, x(t)) \in \ddot{x}(t) + A_{t,x(t)}\dot{x}(t) \quad \text{a.e. } t \in [0, T] \\ x(0) = x_0, \dot{x}(0) = \dot{x}_0 \in D(A(0, x_0)), \end{cases} \tag{2}$$

by using the well-posedness of the problem (1) and a suitable reduction of order technique. It is worth mentioning that in the particular case where A is time-dependent (i.e. $A := A_t$) with Lipschitz variation, problem (2) has been studied in [12]. To the best of our knowledge, there is no research paper for general second-order state-dependent maximal monotone differential inclusions, so our motivation is to fill this gap.

Vladimirov's paper [43] was one the first devoted to the study of nonlinear evolution inclusions with a time-dependent maximal monotone operator (that is, $A_{t,x} = A_t$). We note that in [43] $t \mapsto A_t$ was assumed to be regular. The most reasonable condition to ensure this regularity is that $\operatorname{int}(D(A_t)) \neq \emptyset$ for every $t \in [0, T]$ along with the continuity of the mapping $t \mapsto A_t$ with respect to the above-mentioned pseudo-distance. Later on, Kunze and Monteiro Marques [22] considered the case of $t \mapsto A_t$ being of bounded variation or absolutely continuous. In addition, they did not require that $\operatorname{int}(D(A_t)) \neq \emptyset$, because this condition restricts the possible applications of the theorems to PDEs or even to sweeping processes. Recently Le [25] studied a more general state-dependent maximal monotone differential inclusion. In [25] the well-posedness of problem (1) was proved by using an implicit discretization scheme and a kind of hypo-monotonicity assumption. Soon thereafter Slamnia et al. [35] used a more general forcing term (that is, the sum of a single-valued mapping $f(t, x)$ satisfying a Lipschitz condition and a scalarly upper semicontinuous set-valued map $F(t, x)$.

We mention that problems of the form (1) have an important number and variety of applications in PDEs (heat equations and obstacle problems), mechanics (rigid-body systems with impact, Coulomb friction), electricity (diodes and transistors). We refer to the works of Le [25, 24] and the references therein for such applications.

This thesis is divided in four chapters. In chapter 1, we recall some definitions and useful fundamental results of convex and variational analysis. The well-posdness of first order state dependent maximal monotone differential inclusions is analyzed throughly in chapter 2. Applications of the above chapter to the study of second order state dependent maximal monotone differential inclusions and the related stability results are presented in chapter 3. The lower semicontinuous set-valued perturbation of the first order state dependent maximal monotone differential inclusions end the thesis in chapter 4.

The results of chapters 2 and 3 are the subject of the pulication [3].

PRELIMINARIES

In this chapter we define, describe and introduce all basic results and concepts that are going to be used throughout of this thesis. Then we present some concepts of convex analysis as well as some theorems of compactness that will be used.

1.1) Convex sets and functions

For more details about this part see reference [42]

Definition 1.1.1. *Let E be a real vector space. A subset $S \subset E$ is called convex if and only if*

$$\forall\ a, b \in E,\ \forall \lambda \in [0, 1], \lambda a + (1 - \lambda) b \in S.$$

In other words C is convex if it contains all the line segment of these points.

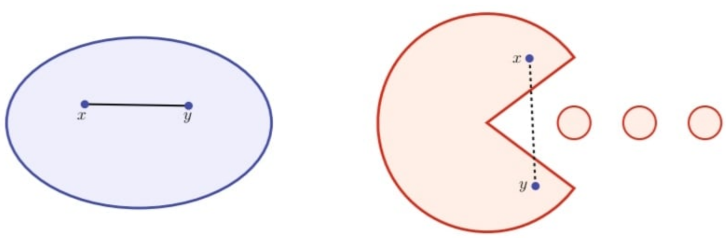

Figure 1.1: Convex and non convex set.

If the set is non convex, we can defined its convex hull as follows:

Definition 1.1.2. *Let E be a real vector space. The convex hull of subset $A \subset E$ is intersection* *of all convex sets containing A. Therfore, is the smallest convex that containing A. and we note* $co(A)$.

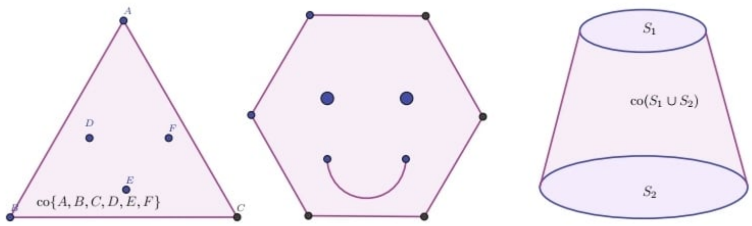

Figure 1.2: Convex hulls in $\mathbb{R}^d$.

In other words we have this characterization

$$co(A) = \left\{ \sum_{i=1}^{n} \beta_i x_i,\ \beta_i \geq 0,\ x_i \in A,\ \sum_{i=1}^{n} \beta_i = 1 \right\}. \tag{1.1}$$

The definition of closed convex hull is given by

Definition 1.1.3. *Let E be a real vector space. The closed convex hull of subset $A \subset E$ is intersection of all closed convex sets containing A. Therfore, is the smallest closed convex that containing A. and we note $\overline{co}(A)$.*

Now we are going to give some definitions and properties about convex functions in a real vector space E.

Definition 1.1.4. *Let f be a real function define from E to $\mathbb{R}$. The effective domain of f is the set*

$$dom(f) = \{x \in E,\ f(x) < \infty\}.$$

Definition 1.1.5. *The function f is said proper if $dom(f) \neq \emptyset$.*

Definition 1.1.6. *Let f be a real function define from E to $\overline{\mathbb{R}}$. The epigraph of f is the set*

$$epi(f) = \{(x,t) \in E \times \mathbb{R} : f(x) \leq t\}.$$

Definition 1.1.7. *The real function f define from E to $\overline{\mathbb{R}}$ is said convex if for all $\beta \in [0;1]$, we have*

$$f(\beta x + (1-\beta)y) \leq \beta f(x) + (1-\beta) f(y),\ \forall \beta \in [0,1], \forall x, y \in dom(f).$$

Definition 1.1.8. *Let X, Y be a metric spaces. The function $f : X \longrightarrow Y$ is a Lipshitz function if there exists a positive constant m such that*

$$d(f(x), f(y)) \leq md(x, y), \ \forall x, y \in X.$$

1.2) Normal cone and some special functions

In this part we are intrested in some convex sets that are more important like normal cone and some special functions indicator and support functions.

Definition 1.2.1. *Let E be a real vector space. we said that $S \subset E$ is cone if and only if $\lambda S \subset S$ for all $\lambda \geq 0$ i. e.;*

$$\lambda z \in S, \forall \lambda \geq 0, \forall z \in S. \tag{1.2}$$

If also we add the convexity of S then the set will be convex cone.

Definition 1.2.2. *In a Hibert space H. Given a convex subset S of H and let $x_0 \in S$. The normal cone is the set define by*

$$N_S(x_0) = \{\xi \in H : \langle \xi, x - x_0 \rangle \leq 0, \forall x \in S\}. \tag{1.3}$$

In other words the normal cone is the collection of all vecteurs that not forment an angle aigu with the vecteur $\vec{v} = (x - x_0)$ at the point x_0. If $x_0 \notin S$, we have

$$N_S(x_0) = \{\emptyset\}.$$

Let us give the definition of the support function

Definition 1.2.3. *the support function of $S \subset H$ that we denote by $\sigma_S(\cdot)$ is the function define in H by*

$$\begin{array}{rcl} \sigma(\cdot, S) : H & \longrightarrow & \overline{\mathbb{R}} \\ \xi & \longmapsto & \sigma(\xi, S) = \sup\limits_{x \in S} \langle \xi, x \rangle. \end{array}$$

It has the following properties

Proposition 1.2.1: [20]

$\boxed{1}$ $\sigma(\cdot, S)$ is convex even when S is not.

$\boxed{2}$ $\sigma(\cdot, S)$ is positively homogeneous of degree 1, i. e.;

$$\sigma(\alpha x, S) = \alpha\sigma(x, S) \quad \forall x \in H \ \forall \alpha > 0.$$

The next function is the indicator function

Definition 1.2.4. *Let S a nonempty convex set of H. We called indicator function of S that note by $\delta(\cdot, S)$ the function define by*

$$\begin{aligned} \delta(\cdot, S) : H \quad &\longrightarrow \quad \overline{\mathbb{R}} \\ x \quad &\longmapsto \quad \delta(x, S) = \begin{cases} 0 & \textit{if} \quad x \in S, \\ +\infty & \textit{if} \quad x \notin S. \end{cases} \end{aligned}$$

It satisfied this two properties

Proposition 1.2.2: [42]

$\boxed{1}$ $\delta(\cdot, S)$ is a convex set if and only if S is convex.

$\boxed{2}$ $\delta(\cdot, S)$ is a (l.s.c) function if and only if S is closed.

1.3) Subdifferential

In this section we are going to give the definitions of the subdifferential of convex functions and some properties. For more details see [1].

Definition 1.3.1. *Let H be a Hilbert space, $f : H \to \mathbb{R} \cup \{+\infty\}$ be convex proper. The subdifferential of f at the point $x_0 \in domf$ is the set-valued operator such that*

$$\partial f(x_0) = \{\xi \in H : \langle \xi, x - x_0 \rangle \leq f(x) - f(x_0), \ \forall x \in H\}. \tag{1.4}$$

If $x_0 \notin dom(f)$, the set $\partial f(x_0) \neq \emptyset$.
The elements of $\partial f(x_0)$ are called the subgradients of f at x_0.

The domain of ∂f is defined by

$$dom\,\partial f = \{x \in H : \partial f(x) \neq 0\}.$$

Proposition 1.3.1

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be proper convex function and let $x_0 \in dom f$. Then the following hold

$\boxed{1}$ $dom\,\partial f \subset dom f.$

$\boxed{2}$ If $x_0 \in dom\,\partial f$. Then f is lower semicontinuous at x_0.

$\boxed{3}$ $\partial(\lambda f) = \lambda \partial(f)$, for $\lambda \in \mathbb{R}_+$.

Proposition 1.3.2

Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper and convex function. For every $x_0 \in dom f$, the set $\partial f(x_0)$ is a closed convex set.

Example 1.3.1.

We consider the not differentiable function

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = |x| \,. \end{aligned}$$

If $x = 0$ then $\partial f(0) = [-1, 1]$. Indeed,

$$\begin{aligned} \partial f(0) &= \{y \in \mathbb{R} : f(x) \geq f(0) + \langle y, x\rangle \quad \forall x \in \mathbb{R}\} \\ &= \{y \in \mathbb{R} : |x| \geq x \cdot y \quad \forall x \in \mathbb{R}\} \\ &= \{y \in \mathbb{R} : x\,y \leq x,\ \forall\, x \in \mathbb{R}_+\} \cap \{y \in \mathbb{R} : x\,y \leq -x,\ \forall\, x < 0\} \\ &= \{y \in \mathbb{R} : y \leq 1\} \cap \{y \in \mathbb{R} : y \geq -1\} \\ &= [-1, 1]. \end{aligned}$$

Therefore, for $x_0 \in \mathbb{R}$ we obtain

$$\partial f(x_0) = \begin{cases} \{-1\} & \text{if } x < 0, \\ [-1,1] & \text{if } x = 0, \\ \{1\} & \text{if } x > 0. \end{cases} \tag{1.5}$$

The following figures give the subdifferential of some functions.

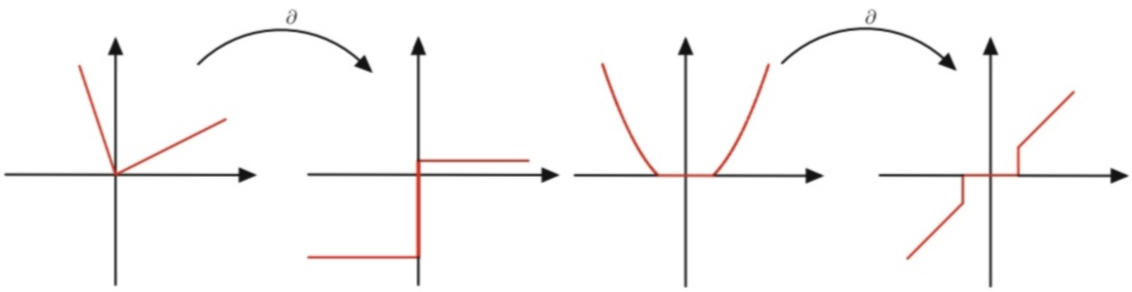

Figure 1.3: Some functions and their subdifferential.

1.4) Continuity of single-valued maps

In this part we will give the definitions of the semicontinuity (see [1, 15, 33, 38, 39]).

1.4.1 Lower and upper semicontinuous functions

Definition 1.4.1. *Let $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function, f is lower semicontinuous at x_0, if for each sequence $(x_k)_{k\in\mathbb{N}}$ in H with $x_k \longrightarrow x_0$ we get*

$$\liminf_{k\to+\infty} f(x_k) \geq f(x_0), \text{ as } k \longrightarrow +\infty.$$

Moreover, f is called lower semicontinuous if it's lower semicontinuous at every point of H.

We have the following characterization.

Definition 1.4.2. *Let E be a topological space, let $f : E \longrightarrow \overline{\mathbb{R}}$ be a function. We said that f is lower semicontinuous at x_0 in E if*

$$f(x_0) \leq \liminf_{x\to x_0} f(x).$$

Definition 1.4.3. *Let E be a topological space, let $f : E \longrightarrow \overline{\mathbb{R}}$ be a function. We said that f is upper semicontinuous at x_0 in E if*

$$f(x_0) \geq \limsup_{x \to x_0} f(x).$$

Definition 1.4.4. *We called that f is upper semicontinuous at x_0 if $-f$ is lower semicontinuous function at x_0. Moreover, recall that f is continuous at the point x_0 if it's lower semicontinuous and upper semicontinuous on x_0.*

Proposition 1.4.1: [1]

Let E be a topological space, let $f : E \longrightarrow \overline{\mathbb{R}}$ be a function. The following properties are equivalent

$\boxed{1}$ $epi(f)$ is closed in $E \times \mathbb{R}$.

$\boxed{2}$ f is lower semicontinuous at x_0.

Definition 1.4.5. *Let $(E_1, d_1), (E_2, d_2)$ be two metric space. We said that $f : E_1 \longrightarrow E_2$ is continuous on $x \in E_1$ if and only if*

$$\forall \varepsilon > 0,\, \exists\, \delta > 0, \forall x \in E_1 : d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \varepsilon. \tag{1.6}$$

Moreover, f is continuous if it is continuous at every point in E_1.

1.4.2 Absolutely continuous functions

Definition 1.4.6. *Let H be a Hilbert space. The function $f : [a; b] \to H$ is said to be absolutely continuous if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $]a_n; b_n[$ are pairwise disjoint subintervals of $[a; b]$*

$$\sum_{n \geq 0} (b_n - a_n) < \delta \Rightarrow \sum_{n \geq 0} \|f(a_n) - f(b_n)\| < \varepsilon.$$

Moreover, The function $f : [a, b] \longrightarrow H$ is absolutely continuous if and only if

$$f(b) - f(a) = \int_a^b f'(s) ds.$$

Any absolutely continuous function f is continuous.

1.5) The projection operator onto closed convex set

Recall the definition of the projection operator. For more details see [40]

Definition 1.5.1. *Let S be a nonempty closed convex subset of Hilbert space H , we recall that the distance between a point $x \in H$ and this subset is define by*

$$d(x, S) = \inf_{y \in S} ||y - x||.$$

If $x \in S$ then $d(x, S) = 0$.

The distance function $d(x, S)$ has this particular property.

Proposition 1.5.1: [4]

Let X be a normed space, $S \neq \emptyset$ be a closed subset of X. Then $d(x, S)$ is convex if and only if S is a convex set.

Definition 1.5.2. *Let $S \neq \emptyset$ be a subset of a Hilbert space H, $x \in H$ and z be a point of S. We said that z is the projection of x onto S if it satisfied the followig expression, i.e.;*

$$d(x, S) = ||x - z|| = \inf_{y \in S} ||x - y||. \tag{1.7}$$

We note the projection operator z by $proj_S(x)$.

Remark 1.5.1. *we have this two properties*

$\boxed{1}$ *If $x \in S$ then $Proj_S(x) = x$.*

$\boxed{2}$ *the projection operator $Proj(\cdot) : E \rightrightarrows E$ is a set-valued map.*

Theorem 1.5.2

Let H be a Hilbert space, $S \subset H$ be a nonempty closed convex set. Then all point $x \in H$ has a unique projection on S, noted $proj$.

The projection has this characterization.

Theorem 1.5.3

Let H be a Hilbert space, $S \subset H$ be a nonempty closed convex set. Then $proj \in H$ is a projection of y onto S if and only if

$$\langle x - proj, y - proj \rangle \leq 0, \ \forall \, x \in S.$$

Geometrically, This theorem states that the angle between the vectors $x - proj$ and $y - proj$ is nonacute, that is, right or obtuse, angle.

1.6) Monotone and maximal monotone operators

1.6.1 Definitions and properties

In this subsection we will define a monotone and maximal monotone operators which will be used in this thesis. We start by giving some definitions of the graph, the domain and the range of set-valued operator. For more details see[1, 10, 29].

Let H be a Hilbert space, with be a real Hilbert space withan inner space $\langle \cdot \rangle$ and a norm $\| \cdot \|$. A set-valued operator $A : H \longrightarrow H$ is an operator that associates to any $x \in H$ a subset $A(x) \subset H$. when $A(x)$ is a singleton $(x, y \in H)$, we write $Ax = y$ instead of $A(x) = y$.

Definition 1.6.1. *Let H be a Hilbert space and $A : H \rightrightarrows H$ be a set-valued operator*

$\boxed{1}$ *The domain of A that noted $D(A)$ is define by $D(A) = \{x \in H, A(x) \neq 0\}$.*

$\boxed{2}$ *The range of A that noted rg, is define by $rg(A) = \{\bigcup_{x \in H} A(x)\}$.*

$\boxed{3}$ *The graph of A that note $Grph(A) = \{(x, y) \in H \times H : y \in A(x)\}$.*

$\boxed{4}$ *The inverse of A, $\quad A^{-1}$ is define by $y \in A(x) \Leftrightarrow x \in A^{-1}(y)$.*

Definition 1.6.2. *We said that $A : H \rightrightarrows H$ is monotone if and only if, for every $(x, x^*) \in Grph(A)$, $(y, y^*) \in Grph(A)$*

$$\langle x^* - y^*, x - y \rangle \geq 0.$$

In other words

$$\forall x^* \in Ax, \forall y^* \in Ay: \ \langle x^* - y^*, x - y \rangle \geq 0.$$

Definition 1.6.3. *we said that A is hypomonotone if and only if, there exists a positive constant $k > 0$ such the following expression hold*

$$\langle x^* - y^*, x - y \rangle \geq -k \| \ x - y \|^2.$$

Definition 1.6.4. *Let H be a Hilbert space, we said that $A : H \rightrightarrows H$ is maximal monotone operator if it is monotone and its graph is maximal in the sense of inclusion,i.e., $Grph(A)$ is not contained in the graph of any other monotone operator.*

Example 1.6.1. *Let A, B two set-valued maps such that*

$$A(x) = \begin{cases} x - 1 & \text{if } x < 0, \\ \{-1, 0, 1\} & \text{if } x = 0, \\ x + 1 & \text{if } x > 0. \end{cases} \tag{1.8}$$

$$B(x) = \begin{cases} x - 1 & \text{if } x < 0, \\ [-1, 1] & \text{if } x = 0, \\ x + 1 & \text{if } x > 0. \end{cases} \tag{1.9}$$

A is monotone but not maximal monotone and B is maximal monotone.

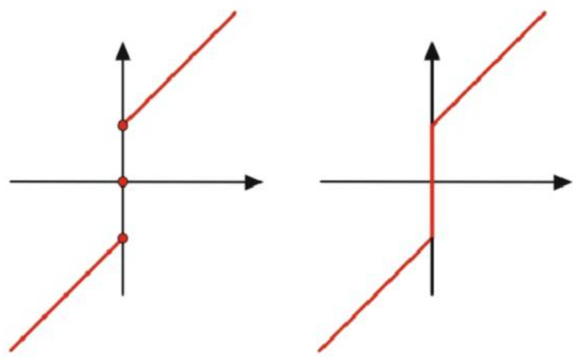

Figure 1.4: The left map is monotone but non maximal whereas the right is maximal monotone map.

The following theorem represent an important property of the subdifferential in convex analysis concerns maximal monotonicity.

Theorem 1.6.1: [29]

Let H be a Hilbert space. The subdifferential $\partial\phi(.)$ of a proper convex and lower semicontinuous function $\phi : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ is maximal monotone.

Remark 1.6.1. *Let $S \subset H$ be a nonempty closed and convex subset of H. Then the subdifferential of the indicator function coincides with the normal cone, i.e. $\partial\delta(x, S) = N_S(x)$.*

Remark 1.6.2. *we have this properties of monotone and maximal monotone operators.*

$\boxed{1}$ *If A is maximal monotone operator. Then the inverse operator A^{-1} is also maximal monotone.*

$\boxed{2}$ *If A is monotone operator. Then the inverse operator A^{-1} is monotone.*

$\boxed{3}$ *If A and B are monotone, then $A + B$ is monotone.*

This proposition gives an important characterization of maximal monotone operator.

Proposition 1.6.2: [10]

Let $A : D(A) \subset H \rightrightarrows H$ be a monotone operator, so the followig assertions are equivalent

$\boxed{1}$ A is maximal monotone operator.

$\boxed{2}$ For $\lambda > 0$ $I + \lambda A$ is surjective, i.e. $R(I + \lambda A) = H$.

Definition 1.6.5. *A^0 is the element of minimal norm of Ax wich defined by*

$$A^0x \in A(x) \quad \textit{and} \quad \|A^0x\| = \inf_{z\in A(x)} \|z\|.$$

Definition 1.6.6. *Let $A : D(A) \subset H \rightrightarrows H$. The principal section of A is all uni-valued operator $B \subset A$ with $D(A) = D(A')$ and for every $(x, y) \in \overline{D(A)} \times H$, the inequality*

$$\langle Bf - y, f - x\rangle \geq 0 \textit{ for each } f \in D(A)$$

imply that $y \in A(x)$.

The element of minimal norm satisfied the following property

Proposition 1.6.3: [10]

A^0 is a principal section of A.

Definition 1.6.7. *Let $A : D(A) \subset H \rightrightarrows H$ be a maximal monotone operator. Then the resolvent operator denoted by J_λ^A is a single-valued mapping define by*

$$\begin{array}{rcl} J_\lambda^A : H & \longrightarrow & H \\ x & \longmapsto & J_\lambda^A(x) = (I + \lambda A)^{-1}(x). \end{array}$$

The following definition is about Yosida approximation.

Definition 1.6.8. *Let $A : D(A) \subset H \rightrightarrows H$ be a maximal monotone operator and let $\lambda > 0$. Then the Yosida approximation is defined by*

$$\begin{array}{rcl} A_\lambda : H & \rightrightarrows & H \\ x & \rightarrow & A_\lambda(x) = \dfrac{1}{\lambda}(I - J_\lambda^A)(x). \end{array}$$

Lemma 1.6.4: [1]

Let A be a maximal monotone operator, Then $\overline{D(A)}$ is convex and

$$\lim_{\lambda \longrightarrow 0} J_\lambda^A x = prj_{\overline{D(A)}} x \ \forall x \in H.$$

1.6.2 Vladimirov distance

Now we are defined the psuedo distance between two maximal monotone operators that's defined by Vladimirov in [43].

Definition 1.6.9. *Let H be a Hilbert space, $A, B : H \rightrightarrows H$ are maximal monotone operators, the distance between A and B is defined by the forme*

$$dis(A, B) = \sup \left\{ \frac{\langle y - y', x' - x \rangle}{1 + \|y\| + \|y'\|} : x \in D(A), y \in Ax, x' \in D(B), y' \in Bx' \right\}.$$

Remark 1.6.3. *The distance dis is not metric because in the general case the triangle inequality is not hold.*

The properties of the pseudo-distance are summarized in the next lemma.

Lemma 1.6.5: [43]

Given $A, B : H \rightrightarrows H$ two maximal monotone operators then we have

$\boxed{1}$ $dis(A, B) = 0$ if and only if $A = B$.

$\boxed{2}$ $d(D(A), D(B)) \leq dis(A, B)$.

We end this subsection by the following important lemmas.

Lemma 1.6.6: [22]

Let $A_n (n \in \mathbb{N})$ and A be maximal monotone operators such that $dis(A_n, A) \to 0$. Suppose that $x_n \in D(A_n)$ with $x_n \to x$ and $y_n \in A_n(x_n)$ with $y_n \to y$ weakly for some $x, y \in H$. Then $x \in D(A)$ and $y \in A(x)$.

Lemma 1.6.7: [22]

Let A be a maximal monotone operator. If $x, y \in H$ such that

$$\langle A^0(z) - y, z - x \rangle \geq 0 \; \forall \; z \in D(A),$$

then $x \in D(A)$ and $y \in A(x)$.

Lemma 1.6.8: [22]

Let $A_n (n \in \mathbb{N})$ and A be maximal monotone operators such that $dis(A_n, A) \to 0$. Suppose that $\|A_n^0(x)\| \leq c(1 + \|x\|)$ for some constant $c > 0$, all $n \in \mathbb{N}, x \in D(A_n)$. Then for every $z \in D(A)$ there exists a sequence $(\zeta_n)_{n \in \mathbb{N}}$ such that

$$\zeta_n \in D(A_n), \quad \zeta_n \to z \quad \text{and} \quad A_n^0(\zeta_n) \to A^0(z). \tag{1.10}$$

1.7) Set-valued maps and selections

In this section we are going to give some definitions of set-valued maps, continuity and selections that's important in the study of the existence result in the last chapter, this result is taked from [5, 14, 18, 21, 32].

1.7.1 Continuity of set-valued maps

Definition 1.7.1. *Let X, Y be two sets. A set valued map F from X to Y is a map that associates with any $x \in X$ a subset $F(x)$ of Y, we can write $F : X \to \mathcal{P}(Y)$ or $F : X \rightrightarrows Y$*

$\boxed{1}$ *The domain of F is the subset $D(F) = \{x \in X, A(x) \neq 0\}$.*

$\boxed{2}$ *The range of F that note Rg, is define by $Rg(F) = \{\bigcup_{x \in X} F(x)\}$.*

$\boxed{3}$ *The graph of F is the subset $Grph(F) = \{(x, y) \in X \times Y : y \in F(x)\}$.*

Example 1.7.1. *Let $F : X \longrightarrow Y$ be a function, let us define the set-valued map*

$$F_+(x) = \begin{cases} F(x) + \mathbb{R}_+, & \text{if } F(x) < +\infty, \\ \emptyset & \text{if } F(x) = +\infty. \end{cases} \tag{1.11}$$

The domain of F_+ coincides with the set of the points x such that $F(x) < +\infty$, the graph of F_+ is the epigraph of F.

Definition 1.7.2. *We said that $F : X \rightrightarrows Y$ is upper semicontinuous (u.s.c.) at $x_0 \in X$ if for any open N containing $F(x_0)$ there exists a neighborhood M of x_0 such that $F(M) \in N$.*
If F is upper semicontinuous at every $x_0 \in X$, Then it's upper semicontinuous.

Proposition 1.7.1: [5]

The graph of an u.s.c. set-valued map with closed values from X to Y is closed.

Example 1.7.2. *Let G be the set-valued map defined by*

$$G(x) = \begin{cases} \{0\}, & \text{if } x \neq 0, \\ [-1, +1], & \text{if } x = 0. \end{cases} \tag{1.12}$$

G is u.s.c at 0 but not l.s.c at 0.

Definition 1.7.3. *We said that $F : X \rightrightarrows Y$ is lower semicontinuous (l.s.c.) at $x_0 \in X$ if for any open N of Y with $F(x_0) \cap N \neq \emptyset$, there exists a neighborhood M of x_0 such that $F(x) \cap M \neq \emptyset$, $\forall x \in M$.*
F is lower semicontinuous map, if it's lower semicontinuous at every $x_0 \in X$.

Definition 1.7.4. *The set-valued maps F is continuous if it's in both lower semicontinuous and upper semicontinuous.*

Theorem 1.7.2

Let X, Y be two metric spaces, the set-valued F is lower semicontinuous at x_0 if and only if for every sequence $x_n \subset X$ with $x_n \longrightarrow x_0$, for all $y_0 \in F(x_0)$ there exists $y_n \in F(x_n)$ such that $y_n \longrightarrow y_0$.

Example 1.7.3. *Let F be the set-valued map defined by*

$$F(x) = \begin{cases} [-1, +1], & \text{if } x \neq 0, \\ \{0\} & \text{if } x = 0. \end{cases} \tag{1.13}$$

F is l.s.c at 0 but not u.s.c at 0.

1.7.2 Hausdorff distance

In this subsection we will give the definition of the Hausdorff distance and some of their properties. We refer to [4] and [20].

Definition 1.7.5. *Let (X, d) be a metric space and let A, B be two nonempty closed sets of X. The excess of A over B is given by the form*

$$e(A, B) = \sup_{x \in A}\{d(x, B)\} = \sup_{x \in A}\Big(\inf_{z \in B} d(x, z)\Big).$$

Proposition 1.7.3

Let (X,d) be a metric space and let A,B,C be a nonempty closed sets of X. Then we have

$\boxed{1}$ $e(\emptyset,A)=0$ if $A\neq\emptyset$ and $e(B,\emptyset)=+\infty$.

$\boxed{2}$ $e(A,B)=0\Leftrightarrow A\subset\overline{B}$.

$\boxed{3}$ $e(A,C)\leq e(A,B)+e(B,C)$.

Definition 1.7.6. *Let (X,d) be a metric space and let A,B be a nonempty closed subsets of X. The Hausdorff distance $d_H(A,B)$ or we called Pompeiu-Hausdorff is defined to be*

$$d_H(A,B)=max\left\{e(A,B),e(B,A)\right\}.$$

We have the following properties

$\boxed{1}$ $d_H(A,B)=0\Leftrightarrow A=B$.

$\boxed{2}$ $d_H(A,B)=d_H(B,A)$.

$\boxed{3}$ $d_H(A,C)\leq d_H(A,B)+d_H(B,C)$.

1.7.3 Selections of set-valued maps

In this part we are going to give some definitions concerning the notion of decomposable function, measurable selection and continuous selection. For more details see ([14, 18]).

Definition 1.7.7. *Let $K\subset L^1_{\mathbb{R}^d}([0,T])$, we said that K is decomposable if for any $u,v\in K$ and $B\in G$ we have*

$$\mathbb{1}_B.u+\mathbb{1}_{I\setminus B}.v\in K, \tag{1.14}$$

where $\mathbb{1}_B$ is the characteristic function of B, and G is a measurable set.

Definition 1.7.8. *We said that $k \in D$ if there exists a measurable map $F : [0,T] \rightrightarrows \mathbb{R}^d$ with*

$$k = \left\{ u \in L^1_{\mathbb{R}^d}([0,T]) : u(t) \in F(t), a.e. \text{ in } [0,T] \right\}, \tag{1.15}$$

where the family of nonempty, closed and decomposable subset of $L^1_{\mathbb{R}^d}([0,T])$ is noted by D.

Definition 1.7.9. *Let $G : X \rightrightarrows Y$ be a set-valued map. We defined the selection of G as the function $g : X \to Y$, such that for every $x \in X$ we have*

$$g(x) \in G(x). \tag{1.16}$$

Definition 1.7.10. *Let (X, Ψ) be a measurable space, Y be a metrisable separable space. We said that $G : X \rightrightarrows Y$ is measurable if for all open $\Gamma \in Y$, we have*

$$G^{-1}(\Gamma) = \{t \in X,\ G(t) \cap \Gamma \neq \emptyset\} \in \Psi. \tag{1.17}$$

Or for every closed set $\upsilon \in Y$

$$G^{-1}(\upsilon) = \{t \in X,\ G(t) \cap \upsilon \neq \emptyset\} \in \Psi. \tag{1.18}$$

Definition 1.7.11. *(measurable selection). Let (X, Ψ) be a measurable space, Y be a Banach separable space, and $\Delta : X \rightrightarrows Y$ be a set-valued map. The set of measurable selections of Δ is defined by*

$$S_\Delta = \{f \text{ measurable} : f(t) \in \Delta(t)\}. \tag{1.19}$$

Definition 1.7.12. *(integrable selection). Let (X, Ψ) be a measurable space, Y be a Banach separable space, and $\Delta : X \rightrightarrows Y$ be a set-valued map. The set of integrable selections of Δ is defined by*

$$S^1_\Delta = \left\{ f \in L^1 : f(t) \in \Delta(t) \right\}. \tag{1.20}$$

1.8) Fixed point theorems

We refer to [31, 36, 44]

Definition 1.8.1. *Let $A : X \longrightarrow X$ be a univalued map, we said that $x \in X$ is a fixed point of A if $Ax = x$.*
Moreover, $x \in X$ is a fixed point of set-valued map $A : X \rightrightarrows X$ if $x \in A(x)$.

Let us give some important fixed point theorems Schauder's and Kakutani's theorems .

Theorem 1.8.1: [44]

Let K be a nonempty closed convex bounded subset of a Banach space X and $\Lambda : X \to X$ be a continuous relatively compact such that $\Lambda(K) \subset K$. Then Λ has a fixed point.

Theorem 1.8.2: [36]

Let S be a nonvoid, compact and convex subset of $\mathbb{R}^n$. Let $\phi : S \rightrightarrows S$ be a set-valued map on S with the following properties

$\boxed{1}$ ϕ has a closed graph.

$\boxed{2}$ $\phi(x)$ is non empty compact and convex set-valued map for all $x \in S$.

Then ϕ has a fixed point.

Remark 1.8.1. *If ϕ has a contiuous selection, then Kakutani's result would follow from the Brouwer fixed point Theorem.*

1.9) Some compactness results

Let H be a Hilbert space.

Definition 1.9.1. *Let a function $u : [T_0, T] \longrightarrow H$, a subinterval $I \subset [T_0, T]$, we define the variation of u on I by the following expression*

$$var(u, I) := \sup \left\{ \sum_{i=1}^{n} ||u(t_i) - u(t_{i-1})||, n \in \mathbb{N}, t_i \in I, t_0 < t_1 < ... < t_n \right\}.$$

We said that u has a bounded variation on the interval $[T_0, T]$ if

$$var(u, I) < +\infty.$$

Remark 1.9.1. *we have the following properties*

$\boxed{1}$ *The variation of a function u is null if and only if u is a constant.*

$\boxed{2}$ *Every function $u : [T_0, T] \longrightarrow H$ lipshitzien has a bounded variation.*

$\boxed{3}$ *All function $u : [T_0, T] \longrightarrow \mathbb{R}$ croissant has a bounded variation.*

Theorem 1.9.1: [28]

Let H be a Hilbert space, $(u_n)_{n\in\mathbb{N}}$ a sequence of functions $u_n : [0, T] \longrightarrow H$. Suppose that u_n is uniformly bounded in norm and in variation, i.e.,

$$||u_n|| \leq M_1, t \in [0, T], n \in \mathbb{N}, \qquad ||var(u_n, I)|| \leq M_2, n \in \mathbb{N},$$

for each $M_1, M_2 \geq 0$. Then, there exists a subsequence $(u_n)_k$ of (u_n) and a function $u : [0, T] \longrightarrow H$ such that

$$var(u) \leq M_2,$$

and

$$u_{n_k}(t) \rightharpoonup u(t) \text{ in } H, \forall\, t \in [0, T].$$

Now, we are intressed in the definition of L^p spaces and the weak topology (see [11, 15, 26]).

Definition 1.9.2. *Let $p \in \mathbb{R}$ such that $1 < p < \infty$, we define the L^p space by the form*

$$L^p = \left\{ f : \Omega \longrightarrow \mathbb{R}, f \text{is measurable and } |f|^p \in L^1(\Omega) \right\}. \tag{1.21}$$

With

$$\|f\|_{L^p} = \|f\|_p = \Big(\int_\Omega |f(x)|^p d\mu \Big)^{\frac{1}{p}}.$$

Definition 1.9.3. *For $p = \infty$ we define L^∞ as the follow*

$$L^\infty = \left\{ f : \Omega \longrightarrow \mathbb{R}, f \text{is measurable and } |f|^p \in L^1(\Omega) \right\}. \tag{1.22}$$

With

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{c,\ |f(x)| \leq c\ a.e. \text{on } \Omega\}.$$

Definition 1.9.4. *Given a Banach space E. The weak topology on E is the lowest topology on E making continuous all applications $\varphi \in E'$, we denote by $\sigma(E, E')$.*

Proposition 1.9.2: [11]

Let E be a Banach space. The weak topology $\sigma(E, E')$ is separate.

Definition 1.9.5. *Let $(x_n)_n$ be a sequence of E, we said that x_n coverges weakly to some x in E and we denote by $x_n \rightharpoonup x$ or we may write $x_n \rightharpoonup x$ on $\sigma(E, E')$ if and only if*

$$\langle f, x_n \rangle \longrightarrow \langle f, x \rangle, \text{ for all } \ f \in E'. \tag{1.23}$$

we give an useful following proposition

Proposition 1.9.3: [15]

Let x_n be a sequence of E. Then we have

$\boxed{1}$ $x_n \rightharpoonup x$ weakly for $\sigma(E, E')$ if and only if $f(x_n) \longrightarrow f(x)$, for all $f \in E'$.

$\boxed{2}$ If $x_n \longrightarrow x$ then x_n converge weakly to x for $\sigma(E, E')$.

$\boxed{3}$ If $(x_n)_{n\in\mathbb{N}}$ is a sequence in E converging weakly to x, then $(x_n)_{n\in\mathbb{N}}$ is bounded and

$$||x|| \leq \liminf_{n\to\infty} ||x_n||.$$

$\boxed{4}$ If $(x_n)_{n\in\mathbb{N}}$ is a sequence of E converging weakly to x and $(f_n)_{n\in\mathbb{N}}$ is a sequence in E' converging strongly to f, then

$$\lim_{n\longrightarrow\infty} \langle f_n, x_n \rangle \text{ exist and } \lim_{n\longrightarrow\infty} \langle f_n, x_n \rangle = \langle f, x \rangle.$$

Theorem 1.9.4: [11]

Let (f_n) be a sequence of functions in $L^1(\Omega)$ that satisfy the two properties

$\boxed{1}$ For almost every on Ω $f_n(x) \longrightarrow f(x)$.

$\boxed{2}$ There exists a function $g \in L^1(\Omega)$ such that for all n,

$$||f_n(x)|| \leq g(x) \text{ a.e. on } \Omega.$$

Then $f \in L^1(\Omega)$ and

$$||f_n(x) - f(x)|| \longrightarrow 0.$$

Lemma 1.9.5: [11, p. 61]

Let X be a Banach space, assume that $(x_n)_n \subset X$ converges weakly to x. Then there exists a sequence $(y_n)_n$ made up of the convex combination of the x_n that 's converges strongly to x.

Theorem 1.9.6: [11]

Let K be a compact metric space and let $\mathcal{H}$ be a bounded subset of $C(K)$. Assume that H is uniformly equicontinuous, that is,

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ such that } d(x_1, x_2) < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon \; \forall f \in \mathcal{H}. \tag{1.24}$$

Then the closure of $\mathcal{H}$ in $C(K)$ is compact.

Theorem 1.9.7: [19]

Let a set $K \subset C([T_0, T]; H)$ is relatively compact if and only if

$\boxed{1}$ for every $t \in [T_0, T]$, the set $K(t) := \{u(t) : u \in K\}$ is relatively compact in H.

$\boxed{2}$ K is uniformly equicontinuous, i.e.,for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, such that, if $t, s \in [T_0, T]$ and $|t - s| \leq \delta$, then

$$||u(t) - u(s)|| < \varepsilon, \forall u \in K.$$

We need the next result of Gronwall's inequality for bound the solution.

Theorem 1.9.8: [27]

Let $I = [T_0, T]$, α and u be continuous functions and β be a non-negative integrable function, all defined on I. Assume that α is non-increasing and that

$$u(t) \leq \alpha(t) + \int_{T_0}^{t} \beta(s)u(s)ds \qquad t \in [T_0, T]. \tag{1.25}$$

Then

$$u(t) \leq \alpha(t) \exp\left(\int_{T_0}^{t} \beta(s)ds\right) \quad t \in [T_0, T]. \tag{1.26}$$

Now we give the discret form of Gronwall's lemma as the following

Theorem 1.9.9: [27]

Let (α_i), (β_i), (γ_i) and (a_i) a sequence of real nembres nondecreasing such that

$$a_{i+1} \leq \alpha_i + \beta_i(a_0 + a_1 + .. + a_{i-1}) + (1 + \gamma_i)a_i, \tag{1.27}$$

therfore for $i \in \mathbb{N}_0$

$$a_j \leq \left(a_0 + \sum_{k=0}^{j-1} \alpha_k\right) \exp\left(\sum_{k=0}^{j-1} (k\beta_k + \gamma_k)\right). \tag{1.28}$$

The following lemma is the continuous form of Gronwall's lemma.

Lemma 1.9.10: [25]

Let $T > 0$ be given and $a(.), b(.) \in L^1([0,T], \mathbb{R})$ with $b(t) \geq 0$ for almost all $t \in [0;T]$. Let an absolutely continuous function $w : [0,T] \longrightarrow \mathbb{R}_+$ satisfy

$$(1-\alpha)w'(t) \leq a(t)w(t) + b(t)w^{\alpha}(t), a.e\ t \in [0,T]$$

where $0 \leq \alpha < 1$. Then for all $t \in [0,T]$, we have

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(0) \exp\Big(\int_0^t a(\tau)d\tau\Big) + \int_0^t \exp\Big(\int_s^t a(\tau)d\tau\Big)b(s)ds.$$

Lemma 1.9.11: [23]

Let $u : [0,T] \longrightarrow H$ be an absolutely continuous function. Then

- $\int_0^T \langle \dot{u}(t), u(t)\rangle = \frac{1}{2}||u(T)||^2 - \frac{1}{2}||u(0)||^2$.
- $\frac{1}{2}\left(\frac{d}{dt}||u(t)||^2\right) = \langle \dot{u}(t), u(t)\rangle = ||u(t)||^2$.

We finished this part by the Gronwall-like inequality.

Lemma 1.9.12: [13]

Let $\alpha, \beta, r : [0,T] \longrightarrow [0,\infty[$ such that

$$r(t) \leq \alpha(t) + \beta(t)\int_0^t r(s)ds,$$

then for all $t \in [0,T]$

$$r(t) \leq \alpha(t) + \beta(t)\int_0^t (\alpha(s)\exp(\int_s^t \beta(\tau)d\tau))ds.$$

FIRST-ORDER STATE DEPENDENT MAXIMAL MONOTONE DIFFERENTIAL INCLUSIONS

2.1) Introduction

Let $T > 0$ and let H be a real separable Hilbert space. In this chapter, we study the following first order maximal monotone differential inclusion:

$$\begin{cases} \dot{x}(t) \in f(t, x(t)) - A_{t,x(t)}x(t) \quad \text{a.e. } t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{2.1}$$

In (2.1), the maximal monotone operator $A_{t,x} : D(A_{t,x}) \subset H \longrightarrow 2^H$ is of absolutely continuous variation in time and Lipschitz continuous in state, in the sense of the pseudo-distance introduced by A. A. Vladimirov and $f : [0, T] \times H \longrightarrow H$ is a Carathéodory mapping. In contrast to earlier works on the subject, to prove the existence of absolutely continuous solutions for (2.1), we do not use the discretization method. More precisely, in this thesis we show how a fixed point approach can lead to the general existence theorem in the infinite dimensional space to the problem (2.1). By using Schauder's fixed point theorem and the existence and uniqueness theorem for the problem (2.1) with time-dependent A (i.e. $A := A_t$), we give a new proof of the state-dependent maximal monotone evolution inclusion (2.1) in the infinite dimensional Hilbert setting.

2.2) Standing Assumptions

In this subsection we are going to cite the hypotheses that will be used in the main theorem. The first assumption is according to the maximal monotone operator A, and the second one is about the single-valued perturation $f(\cdot,\cdot)$.

Assumption 2.2.1. *Let $T>0$ be given. For every $t\in[0,T]$ and $x\in H$, let $A_{t,x}: D(A_{t,x})\subset H\rightrightarrows H$ be a maximal monotone operator satisfying the following:*
(1.1) There exist a constant L with $L<1$ and $\zeta(\cdot)\in H^1([0,T],\mathbb{R}_+;dt)$ nondecreasing such that, for every $t,s\in[0,T]$ we have

$$dis(A_{t,x},A_{s,y})\leq|\zeta(t)-\zeta(s)|+L\|x-y\|,\ \forall\, t,s\in[0,T]. \tag{2.2}$$

(1.2) There exists a positive constant $c_0>0$ such that

$$\|A^0_{t,x}(y)\|\leq c_0(1+\|x\|+\|y\|),\ \forall\ t\in[0,T],x,y\in H. \tag{2.3}$$

(1.3) For any bounded subset $B\subset H$, the set $D(A_{I\times B})$ is relatively ball compact.

Assumption 2.2.2. *Let $f:[0,T]\times H\longrightarrow H$ be a function satisfying the following growth condition i.e. there exists $M>0$ such that for each element x of H we have*

$$\|f(t,x)\|\leq M(1+\|x\|),\ \forall\, t\in[0,T], \tag{2.4}$$

and that for every $r>0$ there exists a nonnegative real function $\alpha_r(\cdot)$ of $L^1([0,T],\mathbb{R},dt)$ such that

$$\|f(t,x)-f(t,y)\|\leq\alpha_r(t)\|x-y\|,\ \forall\ x,y\in B'(0,r), \tag{2.5}$$

where $B'(0,r)$ is the closed ball of radius r centered at 0.
We define $\mathcal{A}_1$ as follows:

$$\mathcal{A}_1=\{(t_0,x_0):x_0\in D(A_{t_0,x_0})\},$$

and for a fixed $x_0\in H$ we define the following constants

$$c=c_0(2+\|x_0\|),$$

$$\tilde{\kappa}_1 = \Big(\|x_0\| + (2 + M + 2c)(\zeta(T) + 1 + T)\Big) \cdot \exp\Big((M + 2c)(\zeta(T) + 1 + T)\Big)$$

and

$$\tilde{\kappa} := 2 + (M + 2c)(1 + \tilde{\kappa}_1),$$

where c_0 comes from (2.3).

2.3) Preparatory Lemmas

In this section we will give some important lemmas, that will be used in the proof of the main theorem.

Lemma 2.3.1: [11]

Let (I, Σ, μ) be a σ-finite, complete measure space, X be a Banach space and $f, f_n : I \to X$ be vector valued functions. Let $1 \leq p \leq +\infty$ and let $f_n \to f$ in $L^p(I, X)$. Then there is a subsequence $\{f_{k_n}\}_n$ which converges to f pointwise a.e.

Lemma 2.3.2: ([16, Th. 1, p.101],[16, Corollary 13, p.76]

Let (I, Σ, μ) be a finite measure space and X be a reflexive Banach space and $\Omega \subset L^1(I, X)$. If the following three conditions hold:

i) Ω is bounded,

ii) Ω is uniformly integrable, i.e.

$$\int_J \|u\| \, d\mu \to 0 \quad \text{whenever} \quad \mu(J) \to 0,$$

where the convergence is uniform in $u(\cdot) \in \Omega$,

iii) for each $J \in \Sigma$ the set $\left\{\int_J u \, d\mu : u(\cdot) \in \Omega\right\}$ is relatively weakly compact,

then Ω is relatively weakly compact.

The next result known as Mazur's lemma that will be very useful throughout the thesis.

Lemma 2.3.3: [11, p. 61]

Let X be a Banach space, assume that $(x_n)_n \subset X$ converges weakly to x. Then there exists a sequence $(y_n)_n$ made up of the convex combination of the x_n that 's converges strongly to x.

2.4) Main results

We define $\mathcal{A}_1$ as follows:

$$\mathcal{A}_1 = \{(t_0, x_0) : x_0 \in D(A_{t_0,x_0})\},$$

and for a fixed $x_0 \in H$ we define the following constants:

$$c = c_0(2 + \|x_0\|),$$

$$\tilde{\kappa}_1 = \Big(\|x_0\| + (2 + M + 2c)(\zeta(T) + 1 + T)\Big) \cdot \exp\Big((M + 2c)(\zeta(T) + 1 + T)\Big),$$

and

$$\tilde{\kappa} := 2 + (M + 2c)(1 + \tilde{\kappa}_1), \tag{2.6}$$

where c_0 comes from (2.3).

The following theorem establishes the existence result of the non linear evolution problem (2.1).

Theorem 2.4.1

Suppose that assumption 2.2.1 and assumption 2.2.2 hold and that $L\tilde{\kappa} < 1$. Then for all $(t_0, x_0) \in \mathcal{A}_1$ the following differential inclusion

$$\begin{cases} \dot{x}(t) \in f(t, x(t)) - A_{t,x(t)}x(t) \quad \text{a.e. } t \in [t_0, T], \\ x(t_0) = x_0. \end{cases}$$

has at least one absolutely continuous solution $x(\cdot)$. Moreover $x(\cdot)$ satisfies the following estimate for almost every $t \in [t_0, T]$

$$\|\dot{x}(t)\| \leq \dot{\lambda}(t), \tag{2.7}$$

where $\lambda(\cdot)$ is the absolutely continuous solution of the following ordinary differential equation

$$\dot{\lambda}(t) := \frac{\kappa}{1 - L\kappa}(1 + \dot{\zeta}(t)), \quad \lambda(t_0) = 0,$$

with the constant

$$\kappa := 2 + (M + 2c)(1 + \kappa_1), \tag{2.8}$$

where

$$\kappa_1 = \Big(\|x_0\| + (2 + M + 2c)(\zeta(T) + L + T)\Big) \cdot \exp\Big((M + 2c)(\zeta(T) + L + T)\Big),$$

and $\zeta(\cdot)$ is given in Assumption 2.2.1.

Proof. Accordig to the definitions of κ and $\tilde{\kappa}$ in the relations (2.8), (2.6), we have $\kappa < \tilde{\kappa}$, we use also the hypothes $L\tilde{\kappa} < 1$ we obtain $L < \dfrac{1}{\kappa}$.

While $\lambda(\cdot)$ is absolutely continuous there exists some $\theta > 0$ such that the following holds

$$\int_t^{t+\theta} \dot{\lambda}(s)ds < 1 \quad \text{for every } t \in [0, T]. \tag{2.9}$$

We prove the existence theorem by using the Schauder's fixed point theorem, that 's based on two steps

Step 1.

Let $(t_0, x_0) \in \mathcal{A}_1$, $T_\theta := t_0 + \theta$, and for $I := [t_0, T_\theta]$, define the following set

$$\mathcal{K} = \left\{ u \in \mathcal{C}(I, H) : u(t) = x_0 + \int_{t_0}^{t} \dot{u}(s) ds \; \forall \, t \in I, \; \|\dot{u}(t)\| \leq \dot{\lambda}(t) \; a.e. \right\}.$$

It is easy to see that $\mathcal{K}$ is a bounded, nonempty subset in $\mathcal{C}(I, H)$.

We prove that $\mathcal{K}$ is convex. Let $u_1, u_2 \in \mathcal{C}(I, H)$, for $\alpha \in [0, 1]$ we have $\alpha u_1 + (1 - \alpha) u_2 \in \mathcal{K}$ it's clearly that $\alpha u_1 + (1 - \alpha) u_2 \in \mathcal{C}(I, H)$, as $u_1, u_2 \in \mathcal{K}$ we have

$$u_1(t) = x_0 + \int_{t_0}^{t} \dot{u_1}(s) ds,$$

$$u_2(t) = x_0 + \int_{t_0}^{t} \dot{u_2}(s) ds,$$

$$\begin{aligned} \alpha u_1(t) + (1 - \alpha) u_2(t) &= \alpha x_0 + \alpha \int_{t_0}^{t} \dot{u_1}(s) ds + (1 - \alpha) x_0 + (1 - \alpha) \int_{t_0}^{t} \dot{u_2}(s) ds, \\ &= x_0 + \int_{t_0}^{t} (\alpha \dot{u}_1(s) + (1 - \alpha) \dot{u}_2(s) ds. \end{aligned}$$

$$\begin{aligned} ||\alpha u_1(t) + (\dot{1} - \alpha) u_2(t)|| = ||\alpha \dot{u}_1(t) + (1 - \alpha) \dot{u}_2(t)|| &\leq \alpha ||\dot{u}_1(t)|| + (1 - \alpha) ||\dot{u}_2(t)|| \\ &\leq \alpha \dot{\lambda}(t) + (1 - \alpha) \dot{\lambda}(t) = \dot{\lambda}(t). \end{aligned}$$

Let us show that $\mathcal{K}$ is closed in $\mathcal{C}(I, H)$.

By Lemma 2.3.2 the set

$$\Omega := \{ y(\cdot) \in L^1(I, H) : \|y(t)\| \leq \dot{\lambda}(t) \}$$

is relatively weakly compact. Indeed, Ω is bounded by $\|\dot{\lambda}\|_{L^1(I, \mathbb{R})}$.

The set Ω is uniformly integrable, since for any $u \in \Omega$ and any measurable set $J \subset I$ the set

$$\int_J \|u(t)\| dt \leq \int_J \dot{\lambda}(t)\, dt,$$

and

for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every measurable set $J \subset I : \int_J dt < \delta,$ we have

$$\int_J \dot{\lambda}(t) dt < \varepsilon, \tag{2.10}$$

which has to hold, otherwise there will be some $\varepsilon_0 > 0$, a sequence of $\delta_i \to 0$ and measurable $J_i \subset I$ such that $\int_{J_i} dt \to 0$ but

$$\int_{J_i} \dot{\lambda}(t)\, dt \geq \varepsilon_0. \tag{2.11}$$

In such a case, notice that functions $g_i := \dot{\lambda} 1_{J_i}$ converge to zero pointwise a.e., and $g_i(\cdot) \leq \dot{\lambda} \in L^1(I, \mathbb{R})$. By Dominated Convergence Theorem $\int_J g_i(t) = \int_{J_i} \dot{\lambda}(t) dt \to 0$, which contradicts (2.11). Thus (2.10) holds and Ω is uniformly integrable.

Finally for each measurable $J \subset I$ and each $u(\cdot) \in \Omega$

$$\left\| \int_J u(t) dt \right\| \leq \int_J \|u(t)\| dt \leq \int_I \dot{\lambda}(t) dt = \|\dot{\lambda}\|_{L^1(I,\mathbb{R})},$$

hence the set $\left\{ \int_J u\, d\mu : u(\cdot) \in \Omega \right\}$ is bounded in Hilbert space H. This and the fact it is convex (due to convexity of Ω) imply that it is relatively weakly compact. We have shown that all the conditions of Lemma 2.3.2 hold for Ω, hence Ω is relatively weakly compact.

Moreover Ω is convex and strongly closed in $L^1(I, H)$, hence it is weakly closed in $L^1(I, H)$. Therefore Ω is weakly compact in $L^1(I, H)$. Let $(u_n)_n$ be a sequence of elements of $\mathcal{K}$ converging to $u(\cdot)$ in $C(I, H)$. Since $\|\dot{u}_n(t)\| \leq \dot{\lambda}(t)$ a.e. $t \in I$, by the Eberlein-Ŝmulian theorem, we can take a subsequence of $(\dot{u}_n(\cdot))_n$ (that we do not relabel) converging weakly in $L^1(I, H)$ to some mapping $w(\cdot) \in L^1(I, H)$.

Putting $v(t) := x_0 + \int_{t_0}^{t} w(s)\,ds$ we see that v is absolutely continuous with $\dot{v}(t) = w(t)$ a.e. and since Ω is weakly closed we have $\|\dot{v}(t)\| \leq \dot{\lambda}(t)$ a.e. in I. Then for any $\xi \in H$, we have for each $t \in I$

$$\lim_{n\to\infty} \int_{t_0}^{T_\theta} \left\langle 1_{[t_0,t]}(s)\xi, \dot{u}_n(s) \right\rangle \, ds = \int_{t_0}^{T_\theta} \left\langle 1_{[t_0,t]}(s)\xi, w(s) \right\rangle \, ds,$$

which entails for each $t \in I$ that $u_n(t) \longrightarrow v(t)$ weakly in H as $n \to \infty$.

Therefore $u = v$ and hence $u \in \mathcal{K}$, that is, $\mathcal{K}$ is closed in $C(I, H)$.

Step 2

Let $h \in \mathcal{K}$ be fixed and for each $t \in I$ define the mapping $B_h(t) : D(B_h(t)) \subset H \rightrightarrows H$ as follows

$$B_h(t) = A_{t,h(t)} \text{ that is } B_h(t)x = A_{t,h(t)}(x) \text{ for all } x \in H.$$

Further we define the element of minimal norm of $B_h(t)x$ by

$$B_h^0(t,x) \in B_h(t)x \quad \text{and} \quad \|B_h^0(t,x)\| = \inf_{z\in B_h(t)x} \|z\|.$$

It is clear that for each $t \in I$, $B_h(t)$ is maximal monotone and for $s, t \in I$ we have

$$\begin{aligned} \text{dis}(B_h(t), B_h(s)) &= \text{dis}(A_{t,h(t)}, A_{s,h(s)}) \\ &\leq |\zeta(t) - \zeta(s)| + L\|h(t) - h(s)\| \\ &\leq |\zeta(t) - \zeta(s)| + L\int_s^t \dot{\lambda}(\tau)d\tau. \end{aligned}$$

This ensures that

$$\text{dis}(B_h(t), B_h(s)) \leq |\beta(t) - \beta(s)|, \tag{2.12}$$

with

$$\beta(t) = \int_{t_0}^{t} \dot{\zeta}(s) + L\dot{\lambda}(s)\,ds.$$

On the other hand, from (2.3) we have, for all $t \in I$,

$$\begin{aligned} \|B_h^0(t,x)\| &= \|A^0_{t,h(t)}(x)\| \le c_0(1+\|x\|+\|h(t)\|) \\ &\le c_0\Big(1+\|x\|+\|x_0\|+\int_{t_0}^t \dot{\lambda}(s)ds\Big). \end{aligned}$$

It follows from this and (2.9) that

$$\|B_h^0(t,x)\| \le c(1+\|x\|), \tag{2.13}$$

with

$$c = c_0(2+\|x_0\|).$$

Therefore, by Assumption 2.2.2, inequalities (2.12) and (2.13) and Theorem 3.2 in [6] the following problem

$$\begin{cases} \dot{x}_h(t) \in f(t,x_h(t)) - B_h(t)x_h(t) \ \text{ a.e. } t \in I, \\ x_h(t_0) = x_0. \end{cases} \tag{P_h}$$

has a unique absolutely continuous solution $x_h(\cdot)$ satisfying

$$\|\dot{x}_h(t)\| \le K(1+\dot{\zeta}(t)+L\dot{\lambda}(t)), \tag{2.14}$$

for some constant

$$K := 2+(M+2c)(1+K_1),$$

with

$$K_1 := \Big(\|x_0\| + (2+M+2c)(\beta(T_\theta)+T_\theta)\Big)\cdot \exp\Big((M+2c)(\beta(T_\theta)+T_\theta)\Big).$$

Step 3: For every $h \in \mathcal{K},\ t \in I$, let define the function Ψ by the form

$$\begin{aligned} \Psi(h): I &\longrightarrow H \\ t &\longmapsto \Psi(h)(t) = x_0 + \textstyle\int_{t_0}^t \dot{x}_h(s)ds, \end{aligned}$$

such that $x_h(\cdot)$ is the unique solution of (P_h).

We shall prove that $\Psi(\mathcal{K}) \subset \mathcal{K}$ and that Ψ has a unique fixed point in $\mathcal{K}$ via Schauder's fixed point theorem 1.8.1.

Ψ is invariant by $\mathcal{K}$, that means $\Psi(\mathcal{K}) \subset \mathcal{K}$. Indeed

From (2.9), (2.14) and the fact that $\zeta(\cdot)$ is increasing we have $K < \kappa$. This assures us that

$$\|\dot{x}_h(t)\| \leq \kappa\Big(1 + \dot{\zeta}(t) + L\dot{\lambda}(t)\Big) = \dot{\lambda}(t). \tag{2.15}$$

We claim that $\Psi(\mathcal{K})$ is relatively compact in $\mathcal{C}(I, H)$. Indeed, let $h \in \mathcal{K}$ and $t,\ s \in I$

$$\begin{aligned} \|\Psi(h)(t) - \Psi(h)(s)\| &= \Big\| \int_s^t \dot{x}_h(\tau)d\tau \Big\| \\ &\leq \Big| \int_s^t \|\dot{x}_h(\tau)\| d\tau \Big| \\ &\leq |\lambda(t) - \lambda(s)|. \end{aligned}$$

Hence, $\Psi(\mathcal{K})$ is equicontinuous.

Moreover we have:

$$\|\Psi(h)(t)\| \leq \|x_0\| + \lambda(T), \quad \forall\, h \in \mathcal{K},\ t \in I, \tag{2.16}$$

otherwise stated

$$\|\Psi(h)\|_{C(I,H)} \leq \|x_0\| + \lambda(T), \forall\ h \in \mathcal{K}.$$

On the other hand, due to [7, Th. 2.1] and the fact that H is reflexive, we have for each $h \in \mathcal{K}$, $\Psi(h) = x_h$. Moreover, since $\dot{x}_h(t) \in f(t, x_h(t)) - B_h(t)x_h(t)$ a.e. $t \in I$, one has

$$\Psi(h)(t) \in D(A_{t,h(t)}) \cap B'(x_0, \lambda(T)).$$

Noting that $h(t) \in B'(x_0, \lambda(T))$ for all $h \in \mathcal{K}$ and $t \in I$, one has

$$\Psi(h)(t) \in D(A_{I \times B'(x_0,\lambda(T))}) \cap B'(x_0, \lambda(T)), \ \forall\, t \in I. \tag{2.17}$$

From this and (1.3) of Assumption 2.2.1, we deduce that for each $t \in I$, $\{\Psi(h)(t)\}_{h\in\mathcal{K}}$ is relatively compact. Therefore by Ascoli-Arzela theorem $\Psi(\mathcal{K})$ is relatively compact in $\mathcal{C}(I,H)$. Next, we prove the continuity of Ψ.

Let $(h_n)_{n\in\mathbb{N}} \subset \mathcal{K}$ be a sequence such that h_n converges to some h in $\mathcal{C}(I,H)$. For each $n \in \mathbb{N}$, let $x_{h_n}(\cdot)$ be the unique absolutely continuous solution of the problem (P_{h_n}). That is

$$\dot{x}_{h_n}(t) \in f(t, x_{h_n}(t)) - B_{h_n}(t)x_{h_n}(t) \text{ a.e. in } I \quad \text{and} \quad x_{h_n}(t_0) = x_0. \tag{2.18}$$

By (2.15) we have

$$\|\dot{x}_{h_n}(t)\| \leq \dot{\lambda}(t) \;\; \forall\, t \in I. \tag{2.19}$$

Moreover, due to [7, Th. 2.1] and the fact that H is reflexive, $\Psi(h_n) = x_{h_n}$.

Further, since $\Psi(\mathcal{K})$ is relatively compact in $\mathcal{C}(I,H)$, for each $t \in I$ the set $\{x_{h_n}(t) : n \in \mathbb{N}\}$ is relatively compact in H. This and (2.19) ensures by [5, Th. 4] that there exists a subsequence, we denote it again by $(x_{h_n})_{n\in\mathbb{N}}$ converging to some function $x(\cdot)$ in the following sense

$$x_{h_n}(\cdot) \text{ converges uniformly to } x(\cdot) \text{ in } I, \tag{2.20}$$

and

$$\dot{x}_{h_n}(\cdot) \text{ converges weakly to } \dot{x}(\cdot) \text{ in } L^1(I,H). \tag{2.21}$$

So (2.20), (2.21) and (2.5) together give the following

$$-\dot{x}_{h_n}(\cdot) + f(\cdot, x_{h_n}(\cdot)) \quad \text{converges weakly to } -\dot{x}(\cdot) + f(\cdot, x(\cdot)) \text{ in } L^1(I,H). \tag{2.22}$$

Then by Mazur's lemma, one has for almost each $t \in I$,

$$-\dot{x}(t) + f(t, x(t)) \in \bigcap_n \overline{\mathrm{co}}\{-\dot{x}_{h_k}(t) + f(t, x_{h_k}(t)) : k \geq n\}. \tag{2.23}$$

Therefore (2.23) implies that for $z \in H$, one has

$$\langle -\dot{x}(t) + f(t, x(t)), z\rangle \leq \limsup \langle -\dot{x}_{h_n}(t) + f(t, x_{h_n}(t)), z\rangle. \tag{2.24}$$

Next, on one hand, since

$$\begin{aligned} \text{dis}(B_{h_n}(t), B_h(t)) &= \text{dis}(A_{t,h_n(t)}, A_{t,h(t)}) \\ &\leq L\|h_n(t) - h(t)\|, \end{aligned}$$

we have

$$\lim_{n\to\infty} \text{dis}(B_{h_n}(t), B_h(t)) = 0. \tag{2.25}$$

On the other hand, from (2.18) we have

$$-\dot{x}_{h_n}(t) + f(t, x_{h_n}(t)) \in B_{h_n}(t)x_{h_n}(t) \text{ a.e. } t \in I,$$

which means that there exists a negligible set $N_n \subset I$ such that

$$-\dot{x}_{h_n}(t) + f(t, x_{h_n}(t)) \in B_{h_n}(t)x_{h_n}(t)\ \forall\, t \in I \setminus N_n. \tag{2.26}$$

Moreover we have

$$B^0_{h_n}(t, \eta) \in B_{h_n}(t)(\eta) \quad \text{for all } \eta \in D(B_{h_n}(t)). \tag{2.27}$$

Since $B_{h_n}(t)$ is monotone, (2.26) and (2.27) entail that for all $t \in I \setminus N_n$ and $\eta \in D(B_{h_n}(t))$ one has

$$\langle -\dot{x}_{h_n}(t) + f(t, x_{h_n}(t)), \eta - x_{h_n}(t)\rangle \leq \langle B^0_{h_n}(t, \eta), \eta - x_{h_n}(t)\rangle. \tag{2.28}$$

Now, let $z \in D(B_h(t))$, by (2.13), (2.25) and Lemma 1.6.8 there exists a sequence (η_n) such that

$$\eta_n \in D(B_{h_n}(t)), \quad \eta_n \to z \quad \text{and} \quad B^0_{h_n}(t, \eta_n) \to B^0_h(t, z). \tag{2.29}$$

Therefore by (2.28), (2.15) and (2.4) we have, for $t \in I \setminus N_n$ and $z \in D(B_h(t))$

$$\begin{aligned}
\langle -\dot{x}_{h_n}(t) + f(t, x_{h_n}(t)), z - x(t)\rangle &= \langle -\dot{x}_{h_n}(t) + f(t, x_{h_n}(t)), \eta_n - x_{h_n}(t)\rangle \\
&+ \langle -\dot{x}_{h_n}(t) + f(t, x_{h_n}(t)), z - \eta_n + x_{h_n}(t) - x(t)\rangle \\
&\leq \langle B^0_{h_n}(t, \eta), \eta_n - x_{h_n}(t)\rangle \\
&+ \| - \dot{x}_{h_n}(t) + f(t, x_{h_n}(t))\| \left(\|x(t) - x_{h_n}(t)\| + \|z - \eta_n\|\right) \\
&\leq \langle B^0_{h_n}(t, \eta), \eta_n - x_{h_n}(t)\rangle + \Big(M(1 + \|x_{h_n}(t)\|) \\
&+ \dot{\lambda}(t)\Big) \times \Big(\|x(t) - x_{h_n}(t)\| + \|z - \eta_n\|\Big).
\end{aligned}$$

Taking the limsup and using (2.20) and (2.29) we obtain

$$\limsup_{n\to\infty} \langle -\dot{x}_{h_n}(t) + f(t, x_{h_n}(t)), z - x(t)\rangle \leq \langle B^0_h(t, \eta), z - x(t)\rangle \;\; \forall\, t \in I \setminus \bigcup_{n=1}^{\infty} N_n. \tag{2.30}$$

Hence, it follows from (2.24) and (2.30) that for all $z \in D(B_h(t))$ and for all $t \in I \setminus \bigcup_{n=1}^{\infty} N_n$

$$\langle -\dot{x}(t) + f(t, x(t)), z - x(t)\rangle \leq \langle B^0_h(t, \eta), z - x(t)\rangle. \tag{2.31}$$

Inequality (2.31) together with Lemma 1.6.7 give for all $t \in I \setminus \bigcup_{n=1}^{\infty} N_n$

$$x(t) \in D(B_h(t)) \quad \text{and} \quad -\dot{x}(t) + f(t, x(t)) \in B_h(t)x(t),$$

that is

$$\dot{x}(t) \in f(t, x(t)) - B_h(t)x(t) \;\text{ a.e. } t \in I.$$

Therefore $x(\cdot)$ is the unique solution x_h of (P_h) and so one has the equality

$$\Psi(h)(t) = x_0 + \int_{t_0}^{t} \dot{x}_h(s)ds.$$

By this we can write

$$\begin{aligned} \|\Psi(h_n)(t) - \Psi(h)(t)\| &= \|x_{h_n}(t) - x_h(t)\| \\ &\leq \|x_{h_n} - x_h\|_{\mathcal{C}(I,H)}, \end{aligned}$$

hence

$$\|\Psi(h_n) - \Psi(h)\|_{\mathcal{C}(I,H)} \leq \|x_{h_n} - x_h\|_{\mathcal{C}(I,H)}.$$

This and (2.20) justify that $\Psi(h_n)$ converges to $\Psi(h)$, which says that Ψ is continuous in $\mathcal{K}$. Therefore by Schauder's fixed point theorem there exists $h \in \mathcal{K}$ such that $h = \Psi(h)$, that is

$$h(t) = x_0 + \int_{t_0}^{t} \dot{x}_h(t),$$

where $x_h(\cdot)$ is the unique solution of (P_h), and this gives

$$\begin{cases} h(t) = x_h(t), \\ \dot{x}_h(t) \in f(t, x_h(t)) - A_{t,h(t)} x_h(t) \ \text{ a.e. } t \in I, \\ x_h(0) = x_0. \end{cases}$$

Therefore replacing h by x we have

$$\begin{cases} \dot{x}(t) \in f(t, x(t)) - A_{t,x(t)} x(t) \ \text{ a.e. } t \in I, \\ x(0) = x_0. \end{cases}$$

That is $x(\cdot)$ is a solution of the problem (2.1).

Step 4 solution of $(\mathcal{P})$

Let $n \in \mathbb{N}$ be such that $\theta_0 := \frac{T - t_0}{n} < \theta$. Consider the following subdivision of $[t_0, T]$, $t_i = t_0 + i\theta_0$, for $i = 0, 1, \cdots, n$. For each $i = 1, \cdots, n$ the above theorem provides a solution $x_i(\cdot)$

of the following problem

$$\begin{cases} \dot{x}_i(t) \in f(t, x_i(t)) - A_{t,x_i(t)} x_i(t) \ \text{ a.e. } t \in [t_i, t_{i+1}], \\ x_i(t_i) = x_{i-1}(t_i). \end{cases}$$

where we set $x_{-1}(t_0) = x_0$. Define $x(\cdot) : [t_0, T] \to H$ by

$$x(t) = x_i(t) \ \text{ if } \ t \in [t_i, t_{i+1}].$$

Then $x(\cdot)$ is a solution of $(\mathcal{P})$ in $[t_0, T]$ with $\|\dot{x}(t)\| \leq \dot{\lambda}(t)$. This completes the proof of the existence of solution.

■

To prove the uniqueness of solution for (2.1), we should add the following hypo-monotone like assumption on the operator A.

Theorem 2.4.2

Suppose that assumption 2.2.1 and assumption 2.2.2 hold and that $A : [0, T] \times H \longrightarrow H$ is hypo-monotone on bounded sets in the following sense; for given $r > 0$ there exists $\eta_r \geq 0$ such that for all $t \in [0, T], x_i \in r\mathbb{B}, x_i^* \in A_{t,x_i}(x_i), i = 1, 2$ we have

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\eta_r \|x_1 - x_2\|^2.$$

Then for each $(t_0, x_0) \in \mathcal{A}_1$, the problem (2.1) has a unique solution.

Proof. The existence is proved in the theorem 2.4.1.

Let $x_1(\cdot)$ and $x_2(\cdot)$ be two solutions of problem (2.1) with the same initial condition $x_1(0) = x_2(0) = x_0$. By the inequality (2.7) we have $x_i(t) \in B'(x_0, \lambda(T))$. Using the hypo-monotonicity of A and the Lipschitz continuity of $f(t, \cdot)$ on bounded sets there exist $\alpha(\cdot) \in L^1([0, T], \mathbb{R}_+, dt)$ and $\eta > 0$ such that for all $t \in [T_0, T], x_i \in k\mathbb{B}$ and $x_i^* \in A_{t,x_i}(x_i), i = 1, 2$

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\eta_r \|x_1 - x_2\|^2$$

and

$$\|f(t,x_1) - f(t,x_2)\| \leq \alpha_r \|x_1 - x_2\|.$$

It follows that

$$\begin{aligned}\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle &\leq \langle f(t,x_1(t)) - f(t,x_2(t)), x_1(t) - x_2(t)\rangle + \eta_r \|x_1(t) - x_2(t)\|^2 \\ &\leq (\alpha_r + \eta_r)\|x_1(t) - x_2(t)\|^2.\end{aligned}$$

This gives

$$\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 \leq 2(\alpha_r + \eta_r)\|x_1(t) - x_2(t)\|^2,$$

Use the continuous Gronwall's inequality in Lemma 1.9.10

$$||x_1(t) - x_2(t)||^2 \leq ||x_1(0) - x_2(0)||^2 \exp\Big(2(\alpha_r + \eta_r)(t)\Big), \forall t \in [0,T],$$

we conclude that $x_1 = x_2$, therfore we obtain the uniqueness of solutions .

■

CHAPTER THREE

STABILITY AND SECOND ORDER STATE DEPENDENT MAXIMAL MONOTONE DIFFERENTIAL INCLUSIONS

3.1) Stability results of first order state dependent maximal monotone differential inclusions

This section is devoted to the stability of the solution of the differential inclusion (2.1). For this purpose we introduce the following parametric differential inclusion

$$\begin{cases} \dot{x}(t) \in f(t, x(t), \xi) - A_{t,x(t),\xi}(x(t)) \quad \text{a.e. } t \in [0, T], \\ \\ x(0) = x_{\xi,0}. \end{cases} \tag{$\mathcal{P}_\xi$}$$

Note that the parameter appears in all the data of the evolution, including the set-valued maximal monotone operator, the single-valued mapping and the initial condition. The stability analysis of variational problems constitutes an important topic in applied mathematics, since it gives information on the robustness of the problem under data perturbation.

Consider the following assumptions:

Assumption $\mathcal{H}(A)$:

For each $t \in [0, T]$, $\xi > 0$ and $x \in H$, let $A_{t,x,\xi} : D(A_{t,x,\xi}) \subset H \rightrightarrows H$ be a maximal monotone operator satisfying the followings:

(1) There exist some constants $L_0 \geq 0$, $L_1 < 1$ and $\zeta(\cdot) \in H^1([0,T], \mathbb{R}_+; dt)$ such that

$$\text{dis}(A_{t,x,\xi}, A_{s,y,\xi'}) \leq |\zeta(t) - \zeta(s)| + L_0|\xi - \xi'| + L_1\|x - y\|, \; \forall\, t, s \in [0,T] \;\text{ and }\; \xi, \xi' > 0. \quad (3.1)$$

(2) There exists $c_0 > 0$ such that $\forall\, t \in [0,T], x, y \in H$ and $\xi > 0$ we have

$$\|A^0_{t,x,\xi}(y)\| \leq c_0(1 + \|x\| + \|y\|). \quad (3.2)$$

(3) There exist $\Gamma(\cdot)$ and $\delta(\cdot)$ in $L^2([0,T], \mathbb{R}_+, dt)$ such that for every $t \in [0,T], \xi > 0$ and $x \in H$

$$A_{t,x,\xi}(x) \subset (\Gamma(t)\|x\| + \delta(t))\mathbb{B}.$$

Assumption $\mathcal{H}(f)$:

(1) Let $f : [0,T] \times \mathbb{R}^+ \times H \longrightarrow H$ be a function satisfying the following growth condition:

$$\|f(t, x, \xi)\| \leq M(1 + \|x\|), \; \forall\, t \in [0,T], x \in H, \quad (3.3)$$

for some constant $M > 0$.

(2) For every $t \in [0,T]$ and $r > 0$, there exist a nonnegative real function $\alpha_r(\cdot)$ and $\beta_r(\cdot) \in L^2([0,T], \mathbb{R}, dt)$ such that:

$$\|f(t, x, \xi) - f(t, y, \xi')\| \leq \alpha_r(t)\|x - y\| + \beta_r(t)|\xi - \xi'|, \; \forall\, x, y \in B'(0,r) \;\text{ and }\; \xi, \xi' > 0. \quad (3.4)$$

Assumption $\mathcal{H}(0)$:

$\xi \mapsto x_{\xi,0}$ is continuous.

Definition 3.1.1. *Let $\{S_\xi\}_{\xi>0}$ be a family of set-valued mappings $S_\xi : H \rightrightarrows H$. We denote*

$$^{w}\limsup_{\substack{\xi \to 0^+ \\ x \to \bar{x}}} S_\xi(x) = \{z \in H : \exists\, \xi_k \to 0^+ \;\textit{ and }\; (x_k, z_k) \in gph S_{\xi_k} \;\textit{ such that } x_k \to x \;\textit{ and }\; z_k \rightharpoonup z\}.$$

We prove the following Lemma:

Lemma 3.1.1

Assume that the first hypothese in assumption $\mathcal{H}(A)$ is satisfied. Then we have

$$^{\omega}\limsup_{\substack{\xi\to 0^+\\ x\to\bar{x}}} A_{t,x,\xi}(x) \subset A_{t,\bar{x},0}(\bar{x}).$$

Proof. Let $z \in {}^{\omega}\limsup_{\substack{\xi\to 0^+\\ x\to\bar{x}}} A_{t,x,\xi}(x)$, there exist $\xi_k \to 0^+, x_k \to \overline{x}$ and $z_k \in A_{t,x_k,\xi_k}(x_k)$ such that $z_k \rightharpoonup z$.

According to the first hypothese of Assumption $\mathcal{H}(A)$ we obtain

$$\text{dis}(A_{t,x_k,\xi_k}, A_{t,\bar{x},0}) \leq |\zeta(t) - \zeta(t)| + L_0|\xi_k - 0| + L_1\|x_k - \overline{x}\|,$$

take the limit when $k \to \infty$ one has

$$\text{dis}(A_{t,x_k,\xi_k}, A_{t,\bar{x},0}) \to 0.$$

Applying Lemma 1.6.6 we have $z \in A_{t,\bar{x},0}(\bar{x})$.

■

Lemma 3.1.2: [30]

Let X be a Banach space and $1 \leq p < +\infty$. If $\{f_n(\cdot), f(\cdot)\}_n \subset L^p(I, X)$, $f_n(\cdot) \rightharpoonup f(\cdot)$ in $L^p(I, X)$ and $f_n(t) \in G(t)$ a.e. in I where for almost each $t \in I$, $G(t) \subset X$ is nonempty, weakly-compact and then

$$f(t) \in \bar{co}\ {}^{w}\limsup_{n\to\infty}\{f_n(t)\}_n \ \ a.e. \text{ in } \ I.$$

Theorem 3.1.3

Under the assumptions $\mathcal{H}(A)$ and $\mathcal{H}(f)$, the problem $(\mathcal{P}_\xi)$ has a unique absolutely continuous solution.

Proof. By introducing the operators $A^{\xi}_{t,x}(\cdot) := A_{t,x,\xi}(\cdot)$ and $f^{\xi}(\cdot,\cdot) := f(\cdot,\cdot,\xi)$, the problem $(\mathcal{P}_\xi)$

becomes

$$\begin{cases} \dot{x}(t) \in f^{\xi}(t,x(t)) - A^{\xi}_{t,x(t)}(x(t)) \quad \text{a.e. } t \in [0,T], \\ x(0) = x_{\xi,0}. \end{cases}$$

The existence and uniqueness of solution follow from Theorem 2.4.1.

■

Let us prove the stability of the solution in $W^{1,1}([0,T],H)$.

Theorem 3.1.4

Suppose that the assumptions $\mathcal{H}(A)(1)$, $\mathcal{H}(A)(2)$, $\mathcal{H}(f)$ and $\mathcal{H}(0)$ hold. Let $\xi_n \to 0$ and $x_n(\cdot)$ be a solution of $(\mathcal{P}_{\xi_n})$ such that $x_n(\cdot) \to x(\cdot)$ in $W^{1,1}([0,T],H)$. Then $x(\cdot)$ is a solution of $(\mathcal{P}_0)$.

Proof. Since $x_n(\cdot)$ is a solution of $(\mathcal{P}_{\xi_n})$ there exists $p_n(t) := f(t,x_n(t),\xi_n)$ such that

$$-\dot{x}_n(t) + p_n(t) \in A_{t,x_n(t),\xi_n}(x_n(t)). \tag{3.5}$$

Since $x_n(\cdot) \to x(\cdot)$ in $W^{1,1}([0,T],H)$, therefore up to a subsequence, $\dot{x}_n(t) \to \dot{x}(t)$ for each $t \in [0,T] \setminus N$ where N is a set of measure zero.

Fix $t \in [0,T] \setminus N$, by $\mathcal{H}(f)(1)$ we have

$$\begin{aligned} \|p_n(t)\| &\leq M(1 + \|x_n(t)\|) \\ &\leq M(2 + \|x(t)\|), \quad \forall\, n \geq n_0 \quad \text{for some } n_0 \in \mathbb{N}. \end{aligned} \tag{3.6}$$

In addition, as $x(\cdot) \in C([0,T],H)$ one has $C := \sup\{\|x(t)\| : t \in [0,T]\} < \infty$.

This fact and (3.6) ensure that

$$\|p_n(t)\| \leq C_0, \ \forall\, n \in \mathbb{N}, \quad \text{for some constant} \quad C_0 > 0. \tag{3.7}$$

The last relation imply that the sequence $\{p_n(t)\}_{n\in\mathbb{N}}$ is bounded in H. Therfore there exists a subsequence $p_{k_n}(t) \rightharpoonup p(t)$ in H and thus

$$-\dot{x}_{k_n}(t) + p_{k_n}(t) \rightharpoonup -\dot{x}(t) + p(t) \text{ a.e. } t \in [0,T].$$

Therefore by Lemma 3.1.1 and (3.5) one has

$$-\dot{x}(t) + p(t) \in \bar{co}\ {}^{w}\limsup_{n\to\infty}\{-\dot{x}_{k_n}(t) + p_{k_n}(t)\} \subset A_{t,x(t),0}(x(t)) \text{ a.e. } t \in [0,T].$$

On the other hand, for each $t \in [0,T] \setminus N$, due to Mazur's lemma, for each $n \in \mathbb{N}$, there exists an integer $p(n) > n$ and some positive real numbers $\lambda_{k,n}$ for $k = n, \cdots, p(n)$ with $\sum_{k=n}^{p(n)} \lambda_{k,n} = 1$ such that the subsequence $\sum_{k=n}^{p(n)} \lambda_{k,n} p_{k_n}(t)$ is converges stongly to $p(t)$ in H, i.e.;

$$\sum_{k=n}^{p(n)} \lambda_{k,n} p_{k_n}(t) \to p(t) \qquad \text{in } H.$$

So, we have

$$p(t) \in \bigcap_n \bar{co}\{f(t, x_{k_n}(t), \xi_{k_n}) : k \geq n\} \ a.e.\ t \in [0,T]. \tag{3.8}$$

Moreover $x_{k_n}(\cdot) \to x(\cdot)$ in $L^1([0,T],H)$ and then from Lemma 2.3.1 we deduce a subsequence

$$x_{k_n}(t) \to x(t) \text{ a.e. } t \in [0,T]. \tag{3.9}$$

Therefore (3.8), (3.9) with (3.4) entail that for any $z \in H$, and for almost every $t \in [0,T]$.

$$\begin{aligned}\langle p(t), z\rangle &\leq \limsup_{n\to\infty} \langle f(t, x_{k_n}(t), \xi_{k_n}), z\rangle \\ &= \langle f(t, x(t), 0), z\rangle.\end{aligned}$$

From this, we obtain

$$\langle p(t) - f(t, x(t), 0), z\rangle = 0, \quad \forall\, z \in H,$$

which gives $p(t) = f(t, x(t), 0)$ a.e. $t \in [0,T]$.

By $\mathcal{H}(0)$ we have $x_{\xi_n,0} \to x(0)$. This completes the proof.

■

In the next theorem we study the stability of the solution in $C([0,T],H)$.

Theorem 3.1.5

Suppose that the assumptions $\mathcal{H}(A)$, $\mathcal{H}(f)$ and $\mathcal{H}(0)$ hold. Let $\xi_n \to 0$ and $x_n(\cdot)$ be a solution of $(\mathcal{P}_{\xi_n})$ such that $x_n(\cdot) \to x(\cdot)$ in $C([0,T],H)$. Then $x(\cdot)$ is a solution of $(\mathcal{P}_0)$.

Proof. Let $x_n(\cdot)$ be a solution of $(\mathcal{P}_{\xi_n})$ there exists $p_n(t) := f(t, x_n(t), \xi_n)$ such that

$$-\dot{x}_n(t) + p_n(t) \in A_{t,x_n(t),\xi_n}(x_n(t)) \text{ a.e. } t \in [0,T]. \tag{3.10}$$

By $\mathcal{H}(f)(1)$ and $\mathcal{H}(A)(3)$ we have for a.e. $t \in [0,T]$

$$\begin{aligned} \|\dot{x}_n(t)\| &\leq \Gamma(t)\|x_n(t)\| + \delta(t) + \|p_n(t)\| \\ &\leq (\Gamma(t) + M)\|x_n(t)\| + \delta(t) + M. \end{aligned} \tag{3.11}$$

Therefore by Gronwall Lemma there exists $M_0 > 0$ such that for every $n \in \mathbb{N}$ one has

$$\sup_{t\in[0,T]} \|x_n(t)\| \leq M_0.$$

From this and the relation (3.11), we have

$$\|\dot{x}_n(t)\| \leq (\Gamma(t) + M)M_0 + \delta(t) + M \qquad \text{a.e. } t \in [0,T]. \tag{3.12}$$

Further, for each $t \in [0,T]$

$$\|p_n(t)\| \leq M(1 + M_0). \tag{3.13}$$

From (3.12) and (3.13), we obtain the following inequality

$$\|\dot{x}_n(\cdot)\|_{L^2([0,T],H)} + \|p_n(\cdot)\|_{L^2([0,T],H)} \leq M_1 \quad \text{for some constant } M_1 > 0. \tag{3.14}$$

Therefore, there exists a sequence $(k_n)_n$ such that

$$\dot{x}_{k_n}(\cdot) \rightharpoonup y(\cdot), \quad p_{k_n}(\cdot) \rightharpoonup p(\cdot) \text{ in } L^2([0,T],H). \tag{3.15}$$

Using Mazur's Lemma, for each $n \in \mathbb{N}$, there exists an integer $\alpha(n) > n$ and some positive real numbers $s_{k,n}$ for $k = n, \cdots, \alpha(n)$ satisfying $\sum_{k=n}^{\alpha(n)} s_{k,n} = 1$ and such that the subsequence,

$$\sum_{k=n}^{\alpha(n)} s_{k,n}\dot{x}_{k_n}(\cdot) \longrightarrow y(\cdot) \quad \text{in} \quad L^2([0,T],H).$$

By Lemma 2.3.1 we have that up to a subsequence

$$\sum_{k=n}^{\alpha(n)} s_{k,n}\dot{x}_{k_n}(t) \longrightarrow y(t) \; a.e. \; t \in [0,T]. \tag{3.16}$$

Since $x_{k_n}(\cdot) \to x(\cdot)$ in $C([0,T],H)$, then by taking the integral in (3.16) we have $\dot{x}(\cdot) = y(\cdot)$. Combining this with (3.15), we have $\dot{x}_{k_n}(\cdot) \rightharpoonup \dot{x}(\cdot) \in L^2([0,T],H)$.

Consequently,

$$-\dot{x}_{k_n}(\cdot) + p_{k_n}(\cdot) \rightharpoonup -\dot{x}(\cdot) + p(\cdot) \text{ in } L^2([0,T],H). \tag{3.17}$$

It follows from (3.12) and (3.13) that for almost every $t \in [0,T]$,

$$\|-\dot{x}_{k_n}(t) + p_{k_n}(t)\| \leq \gamma(t) \quad \text{for some } \gamma(\cdot) \in L^2([0,T],\mathbb{R}). \tag{3.18}$$

From Lemma 3.1.2 and the relations (3.17) and (3.18), we have

$$-\dot{x}(t) + p(t) \in \overline{\text{co}}\ {}^{w}\limsup_{n\to\infty}\{-\dot{x}_{k_n}(t) + p_{k_n}(t)\}, \tag{3.19}$$

and Lemma 3.1.1 with the relation (3.10) give

$$\overline{\text{co}}\ {}^{w}\limsup_{n\to\infty}\{-\dot{x}_{k_n}(t) + p_{k_n}(t)\} \subset A_{t,x(t),0}(x(t)).$$

It ensues from this and (3.19) that

$$-\dot{x}(t) + p(t) \in A_{t,x(t),0}(x(t)) \text{ a.e. } t \in [0,T].$$

The same arguments as in the above Theorem 3.1.4 show that $p(t) = f(t, x(t), 0)$ a.e. $t \in [0,T]$.

By $\mathcal{H}(0)$ we have $x_{\xi_n,0} \to x(0)$. Therefore $x(\cdot)$ is a solution of $(\mathcal{P}_0)$.

■

3.2) Second order state dependent maximal monotone inclusions

3.2.1 Introduction

In this section we consider the second order state dependent maximal monotone inclusion

$$\begin{cases} f(t,u(t)) \in \ddot{u}(t) + A_{t,u(t)}\dot{u}(t) \quad \text{a.e. } t \in [0,T] \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0 \in D(A(0,u_0)), \end{cases} \tag{$\mathcal{S}_1$}$$

where A and f satisfy the conditions in Theorem 2.4.1 and Theorem 2.4.2.

The idea to study the second-order state-dependent maximal monotone inclusion $(\mathcal{S}_1)$ is motivated by the study of existence and stability results for the convex second-order sweeping process. Note that the following evolution problem associated with the second-order sweeping process by a closed convex Lipschitzian set-valued mapping $C : [0,T] \times H \longrightarrow H$:

$$\begin{cases} f(t,u(t)) \in \ddot{u}(t) + N_{C(t,u(t))}(\dot{u}(t)) \quad \text{a.e. } t \in [0,T] \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0 \in C(0,u_0), \end{cases} \tag{$\mathcal{SWP}$}$$

is a particular case of the problem $(\mathcal{S}_1)$, where $A_{t,u(t)} := N_{C(t,u(t))}$, the outward normal cone operator. Furthermore, we know (see [43]) that for all $t, s \in [0,T]$ and $u, v \in H$,

$dis(N_{C(t,u)}, N_{C(t,v)}) = \mathcal{H}(C(t,u), C(s,v))$; here $\mathcal{H}$ stands for the Hausdorff distance between closed subsets of H.

3.2.2 Existence theorem

To obtain the reduction of the second-order problem $(\mathcal{S}_1)$ to the first-order state-dependent maximal monotone differential inclusion, we shall use the next lemma.

Lemma 3.2.1

Suppose that assumption 2.2.1 and assumption 2.2.2 hold. Let $V := H \times H$ be equipped with the following norm

$$\|u\|_V = (\|x\|^2 + \|y\|^2)^{1/2}, \; \forall\, u = (x,y) \in V.$$

For $t \in [0,T]$ and $u = (x,y) \in V$, define the mapping $B_{t,u} : D(B_{t,u}) \subset V \rightrightarrows V$ in the following way:

$$D(B_{t,u}) = H \times D(A_{t,x}) \quad \text{and} \quad B_{t,u}(v) = \{0\} \times A_{t,x}(z), \; \forall\, v = (h,z) \in D(B_{t,u}),$$

and the mapping $F : [0,T] \times V \longrightarrow V$ as follows

$$F(t,u) = (y, f(t,x)), \; \forall\, t \in [0,T] \quad \text{and} \quad u = (x,y) \in V.$$

Then the following properties are satisfied:

$\mathcal{H}(B)(1)$: For all $t > 0$ and $u \in V$, $B_{t,u}$ is maximal monotone.

$\mathcal{H}(B)(2)$: For all $t, s \in [0,T]$ and $u, v \in V$, we have:

$$\text{dis}(B_{t,u}, B_{s,v}) \le |\zeta(t) - \zeta(s)| + L\|u - v\|_V. \tag{3.20}$$

$\mathcal{H}(B)(3)$: There exits $c_0 > 0$ such that for all $t \in [0,T], u \in V$ and $v \in D(B_{t,u})$ one has

Lemma: following of Lemma 3.2.1

$$\|B^0_{t,u}(v)\|_V \leq c_0(1 + \|u\|_V + \|v\|_V). \tag{3.21}$$

$\mathcal{H}(F)(1)$: There exists $M_0 > 0$ such that

$$\|F(t,u)\|_V \leq M_0(1 + \|u\|_V),\ \forall\, t \in [0,T], u \in V \tag{3.22}$$

$\mathcal{H}(F)(2)$: For every $t \in [0,T]$ and $R > 0$ there exists a nonnegative real function $\beta_R(\cdot) \in L^1([0,T], \mathbb{R}, dt)$ such that:

$$\|F(t,u) - F(t,v))\|_V \leq \beta_R(t)\|u - v\|_V,\ \forall\, u, v \in B'_V(0,R) \tag{3.23}$$

where $B'_V(0,R)$ is the closed ball in V of radius R centered at the origin.

Proof. Let $t \in [0,T]$, $u = (x,y) \in V$. Let $(0, w_i) \in B_{t,u}(x_i, y_i)$, for $i = 1,2$ with $(x_i, y_i) \in D(B_{t,u})$, for $i = 1,2$.

So by the definition of the operator $B_{t,u}$ we have $w_i \in A_{t,x}(y_i),\ i = 1,2$ where $x_i \in H,\ y_i \in D(A_{t,x}),\ i = 1,2$.

The fact that $A_{t,x}$ is monotone, implies

$$\langle (0,w_1) - (0,w_2), (x_1,y_1) - (x_2,y_2)\rangle = \langle (0, w_1 - w_2), (x_1 - x_2, y_1 - y_2)\rangle = \langle w_1 - w_2, y_1 - y_2\rangle \geq 0,$$

so $B_{t,u}$ is monotone.

This argument and while $A_{t,x}$ is also maximal monotone we can say that $B_{t,u}$ is maximal monotone.

Consequently B satisfies $\mathcal{H}(B)(1)$. Now let $t, s \in [0,T]$, $u = (x_1, y_1)$ and $v = (x_2, y_2) \in V$, we have for $y_1 \in D(A_{t,x_1}), w_1 \in A_{t,x_1}(y_1), y_2 \in D(A_{s,x_2}), w_2 \in A_{s,x_2}(y_2)$

$$\begin{aligned}
\text{dis}(B_{t,u}, B_{s,v}) &= \sup\left\{\frac{\langle (0,w_1)-(0,w_2),(x_2,y_2)-(x_1,y_1)\rangle}{1+\|(0,w_1)\|_V+\|(0,w_2)\|_V}\right\}, \\
&= \sup\left\{\frac{\langle w_1-w_2, y_2-y_1\rangle}{1+\|w_1\|+\|w_2\|}\right\}, \\
&= \text{dis}(A_{t,x_1}, A_{s,x_2}) \leq |\zeta(t)-\zeta(s)|+L\|x_1-x_2\|.
\end{aligned}$$

Or we have

$$\|x_1-x_2\| \leq (\|x_1-x_2\|^2+\|y_1-y_2\|^2)^{1/2}.$$

Therefore

$$\begin{aligned}
\text{dis}(B_{t,u}, B_{s,v}) &\leq |\zeta(t)-\zeta(s)|+L(\|x_1-x_2\|^2+\|y_1-y_2\|^2)^{1/2} \\
&= |\zeta(t)-\zeta(s)|+L\|u-v\|_V.
\end{aligned}$$

Comming back to verfy the growth condition. For $t \in [0,T], u=(x,y) \in V$ and $v=(h,z) \in D(B_{t,u})$, from (2.3), one has

$$\begin{aligned}
\|B^0_{t,u}(v)\|_V &= \|A^0_{t,x}(z)\| \\
&\leq c_0(1+\|x\|+\|z\|) \\
&\leq c_0(1+\|u\|_V+\|v\|_V).
\end{aligned}$$

Hence B satisfies both (3.20) and (3.21). Therfore B satisfies $\mathcal{H}(B)(2)$ and satisfies $\mathcal{H}(B)(3)$. Now let $u=(x,y) \in V$ and $t \in [0,T]$, from (2.4) we have

$$
\begin{aligned}
\|F(t,u)\|_V^2 &= \|(y, f(t,x))\|_V^2 \\
&\leq \|y\|^2 + M^2(1+\|x\|)^2 \\
&\leq (M+1)^2\left((1+\|x\|)^2 + \|y\|^2\right) \\
&\leq (M+1)^2\left(1+\|x\|+\|y\|\right)^2 \\
&\leq (M+1)^2\left(1+\sqrt{2}\|u\|_V\right)^2 \\
&\leq 2(M+1)^2\left(1+\|u\|_V\right)^2,
\end{aligned}
$$

that is,

$$\|F(t,u)\|_V \leq \sqrt{2}(M+1)\left(1+\|u\|_V\right).$$

This implies (3.22) with $M_0 = \sqrt{2}(M+1)$.

Let $t \in [0,T]$, $R > 0$ and $u = (u_1, u_2), v = (v_1, v_2) \in B'_V(0,R)$, from (2.5) we have

$$
\begin{aligned}
\|F(t,u) - F(t,v)\|_V^2 &= \|(u_2 - v_2, f(t,u_1) - f(t,v_1))\|_V^2 \\
&\leq \|u_2 - v_2\|^2 + \alpha_R^2(t)\|u_1 - v_1\|^2 \\
&\leq (1+\alpha_R(t))^2\left(\|u_1 - v_1\|^2 + \|u_2 - v_2\|^2\right) \\
&= (1+\alpha_R(t))^2\|u - v\|_V^2,
\end{aligned}
$$

which is equivalent to

$$\|F(t,u) - F(t,v)\|_V \leq (1+\alpha_R(t))\|u - v\|_V.$$

This gives (3.23) with $\beta_R(\cdot) = 1 + \alpha_R(t)$. The proof of the lemma is complete.

■

Let us now prove the following existence result for the second-order state-dependent maximal monotone differential inclusion $(\mathcal{S}_1)$.

Theorem 3.2.2

Suppose that assumption 2.2.1 and assumption 2.2.2 hold. Then for all $(t_0, u_0) \in \mathcal{A}_1$ the following differential inclusion

$$\begin{cases} f(t, u(t)) \in \ddot{u}(t) + A_{t,u(t)}\dot{u}(t) \quad \text{a.e. } t \in [0, T], \\ u(0) = u_0, \dot{u}(0) = \dot{u}_0 \in D(A(0, u_0)). \end{cases} \tag{$\mathcal{S}_1$}$$

has at least one solution $x(\cdot) \in W^{2,1}([0, T], H)$.

Proof. It is easy to see that u is a solution of $(\mathcal{S}_1)$ if and only if $X = (u, \dot{u})$ is a solution of the following differential inclusion

$$\begin{cases} \dot{X}(t) \in -B_{(t,X(t))}(X(t)) + F(t, X(t)). \\ X(0) = X_0 = (u_0, \dot{u}_0). \end{cases} \tag{$\mathcal{P}$}$$

Consequently, from (3.20), (3.21), (3.22) and (3.23) of Lemma 3.2.1 and using the result of Theorem 2.4.1, the problem $(\mathcal{P})$ has a solution in $W^{1,1}([0, T], H \times H)$.

■

3.2.3 Stability results

This subsection is devoted to the study of the stability of the solution of the above second-order differential inclusion $(\mathcal{S}_1)$. In what follows we consider the second-order differential inclusion with parameter ξ,

$$\begin{cases} f(t, u(t), \xi) \in \ddot{u}(t) + A_{t,u(t),\xi}(\dot{u}(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) = u_{\xi,0}, \dot{u}(0) = \dot{u}_{\xi,0} \in D(A(0, u_{\xi,0}). \end{cases} \tag{$\mathcal{S}_\xi$}$$

The existence of solutions of $(\mathcal{S}_\xi)$ is obtained in the following theorem.

Theorem 3.2.3

Under the assumptions $\mathcal{H}(A)$ and $\mathcal{H}(f)$, the problem $(\mathcal{S}_\xi)$ has at least one solution in $W^{2,1}([0,T],H)$.

Proof. By introducing the operators $A^{\xi}_{t,x}(\cdot) := A_{t,x,\xi}(\cdot)$ and $f^{\xi}(\cdot,\cdot) := f(\cdot,\cdot,\xi)$ the problem $(\mathcal{S}_\xi)$ becomes

$$\begin{cases} f^{\xi}(t,u(t)) \in \ddot{u}(t) + A^{\xi}_{t,u(t)}(\dot{u}(t)) \quad \text{a.e. } t \in [0,T] \\ u(0) = u_{\xi,0}, \dot{u}(0) = \dot{u}_{\xi,0} \in D(A(0,u_{\xi,0}) \end{cases}$$

Moreover $A^{\xi}_{t,x}(\cdot)$ and $f^{\xi}(\cdot,\cdot)$ satisfy the assumptions 1 and 2. Therefore the existence of solution follow from Theorem 3.2.2.

■

Our next goal is to prove the stability of the solution in $W^{2,1}([0,T],H)$.

Theorem 3.2.4

Suppose that assumptions $\mathcal{H}(A)(1)$ and $\mathcal{H}(A)(2)$, $\mathcal{H}(f)$ and $\mathcal{H}(0)$ hold. Let $\xi_n \to 0$ and $u_n(\cdot)$ be a solution of $(\mathcal{S}_{\xi_n})$ such that $u_n(\cdot) \to u(\cdot)$ in $W^{2,1}([0,T],H)$. Then $u(\cdot)$ is a solution of $(\mathcal{S}_0)$.

Proof. By putting the following change of variables $X_n = (u_n, \dot{u}_n)$ and $X = (u, \dot{u})$, we have: Since $u_n(\cdot) \to u(\cdot)$ in $W^{2,1}([0,T],H)$, then $X_n(\cdot) \to X(\cdot)$ in $W^{1,1}([0,T], H \times H)$. Moreover, for every $t > 0$, $\xi > 0$ and $X = (x,y) \in H \times H$, define

$$B_{t,X,\xi}(h,z) = \{0\} \times A_{t,x,\xi}(z) \quad \text{and} \quad F(t,X,\xi) = (y, f(t,x,\xi)).$$

Then X_n is a solution of the following problem

$$\begin{cases} \dot{X}_n(t) \in -B_{t,X_n(t),\xi_n}(X_n(t)) + F(t,X_n(t),\xi_n). \\ X_n(0) = (u_{\xi_n,0}, \dot{u}_{\xi_n,0}), \end{cases} \tag{$\mathcal{P}_{\xi_n}$}$$

where $B_{(\cdot,\cdot,\xi)}$ and $F(\cdot,\cdot,\xi)$ satisfy the assumptions $\mathcal{H}(A)(1)$, $\mathcal{H}(A)(2)$, $\mathcal{H}(f)$ and $\mathcal{H}(0)$. Therefore by the Theorem 3.1.4, X_n converges to the solution $X = (u, \dot{u})$ of $(\mathcal{P}_0)$. But $(\mathcal{P}_0)$ is equivalent to $(\mathcal{S}_0)$, hence u_n converges to u, a solution of $(\mathcal{S}_0)$.

■

Following the same arguments with the result of Theorem 3.1.5, we have the following stability result in $C^1([0,T],H)$.

Theorem 3.2.5

Suppose that assumptions $\mathcal{H}(A)$, $\mathcal{H}(f)$ and $\mathcal{H}(0)$ hold. Let $\xi_n \to 0$ and $u_n(\cdot)$ be a solution of $(\mathcal{S}_{\xi_n})$ such that $u_n(\cdot) \to u(\cdot)$ in $C^1([0,T],H)$. Then $u(\cdot)$ is a solution of $(\mathcal{S}_0)$.

CHAPTER FOUR

LOWER SEMICONTINUOUS SET-VALUED PERTURBATIONS

4.1) Introduction

The present chapter is essentially for a continuation of the work in [17] dealing with lower semicontinuous perturbations of sweeping process. Namely, we are intressted in the existence of solutions of the perturbed problem

$$\begin{cases} \dot{u} \in G(t, u(t)) - A(t)u(t) \ a.e. \ t \in [0, T], \\ u(0) = u_0 \in D(A(0)), \end{cases} \tag{4.1}$$

where $G(.,.)$ is lower semi-continuous set-valued mapping. The case where the perturbation $G(.,.)$ is convex-valued and upper semicontinuous has been studied in [6]. Here, we investigate in the following sections, under the same assumptions about the operator $A(t)$, the case where $G(t,.)$ is lower semicontinuous and takes non convex values.

4.2) Standing Assumptions

Assumption 4.2.1. *Let $T > 0$. For all $t \in [0, T]$, Let $A(t) : D(A(t)) \subset \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a maximal monotone operator satisfying:*
(1.1) There exists a positive croissant function $\xi(.) \in W^{1,1}([0, T], \mathbb{R}; dt)$ with $\xi(0) = 0, \ \xi(T) < \infty$

such that for all $t, s \in [0,T]$

$$dis(A(t), A(s)) \leq |\xi(t) - \xi(s)|. \tag{4.2}$$

(1.2) For all $t \in I, x \in D(A(t))$ *there exists a nonnegative constant* c *such that*

$$||A^0(t,x)|| \leq c(1 + ||x||). \tag{4.3}$$

(1.3) For any $t \in [0,T]$*, for any* $x \in D(A(t)),\ A(t)x$ *is cone-valued.*

4.3) Auxiliary lemmas

In this section we give some usefull results in this chapter.

Theorem 4.3.1: [18]

A l.s.c set-valued map $G : S \rightrightarrows L^1_{\mathbb{R}^d}([0,T])$ with non empty closed decomposable values has a continuous selection, i.e., there exists a continuous map $g : S \longrightarrow L^1_{\mathbb{R}^d}([0,T])$ such that

$$g(s) \in G(s), \forall s \in S. \tag{4.4}$$

Proposition 4.3.2: [14]

Let (X, Ψ) be a measurable space, Y be a separable metrisable complete space, and $G : X \rightrightarrows Y$ be a closed measurable set-valued map. Then G has a measurable selection.

Theorem 4.3.3: [14]

Let (X, Ψ) be a measurable space, Y be a Banach separable space, and $\Delta : X \rightrightarrows Y$ be a integrable, bounded, convex and weakly compact set-valued map. Then the set of integrable selections S^1_Δ is $\sigma(L^1_{\mathbb{R}^d}([0,T]), L^\infty_{\mathbb{R}^d}([0,T])$-compact.

4.4) Main result

Recall that the fact that $D(A(t))$ is closed convex ensures that for each non negative number $\delta < \rho$, each point of $D(A(t)) + \delta\overline{\mathbb{B}_E}$ has the nearest point in $D(A(t))$ (see [34]).

We need the two following results, we refer to [6].

Proposition 4.4.1

Assume Assumption 4.2.1. Then for all mapping $h \in L^1_{\mathbb{R}^d}([0,T])$, the differential inclusion

$$\begin{cases} -\dot{u}(t) \in A(t)u(t) + h(t) \ a.e \ \ t \in [0,T], \\ u(0) = u_0 \in D(A(0)), \end{cases} \tag{4.5}$$

has a unique absolutely continuous solution $u(.) \in W^{1,1}([0,T],\mathbb{R}^d)$ such that

$$||\dot{u}(t) + h(t)|| \leq K'(1 + |\dot{\xi}(t)|) + (K'+1)||h(t)||, \text{ for almost } t \in [0,T],$$

where, for all almost $t \in [0,T]$ we have

$$||\dot{u}(t)|| \leq K'(1 + \dot{\xi}(t)) + (K'+1)||h(t)||, \tag{4.6}$$

such that

$$K' := 2\Big(1 + (1 + \|h(t)\|_{L^1})(1 + K_1')\Big),$$

and $K_1' := (\|u_0\| + (2(1 + c(1 + \|h(t)\|_{L^1}))(\xi(T) + T + \|h(t)\|_{L^1}))$

$$\cdot \exp(2c(1 + \|h(t)\|_{L^1}))(\xi(T) + T + \|h(t)\|_{L^1}).$$

Lemma 4.4.2

Assume Assumption 4.2.1 is satisfying, suppose also that $D(A(t))$ is ball-compact, let m be a non-negative Lebesgue-integrable function defined on $[0, T]$ and let

$$\mathcal{H} = \left\{h \in L^1_{\mathbb{R}^d}([0,T]) : ||h(t)|| \leq m(t)\, a.e\right\}.$$

Then, the set $\{u_h, h \in S^1_\Delta\}$ of absolutely continuous solutions to the evolution inclusions

$$\begin{cases} -\dot{u}_h(t) \in A(t)u_h(t) + h(t) \;\; a.e \;\; t \in [0,T], \\ u(0) = u_0 \in D(A(0)). \end{cases} \tag{P-h}$$

is compact in $C_{\mathbb{R}^d}([0,T])$.

Lemma 4.4.3

Assume Assumption 4.2.1, let m be a non-negative Lebesgue-integrable function defined on $[0, T]$ and let

$$\mathcal{H} = \left\{h \in L^1_{\mathbb{R}^d}([0,T]) : ||h(t)|| \leq m(t)\, a.e\right\}.$$

Then, the mapping $h \mapsto u_h$ is continuous on $\mathcal{H}$. Such that the set $\{u_h, h \in S^1_\Delta\}$ is of absolutely continuous solutions to the evolution inclusions (P-h).

Proof. The following proof is inspired from [9]. Let defined the mapping Λ by the forme

$$\begin{array}{rcl} \Lambda(\cdot) : L^1_{\mathbb{R}^d}([0,T]) & \longrightarrow & L^\infty_{\mathbb{R}^d}([0,T]) \\ h & \longmapsto & \Lambda(h) = u_h \end{array}$$

we are going to prove that Λ is continuous on $\mathcal{H}$.

Let $(h_n)_n$ be a sequence on $\mathcal{H}$, where $h_n \rightharpoonup h$ in $L^1_{\mathbb{R}^d}([0,T])$ and we will prove that $\Lambda(h_n) \longrightarrow \Lambda(h)$ in $L^\infty_{\mathbb{R}^d}([0,T])$ that means we will prove that $u_{h_n} \longrightarrow u_h$.

Given $u_{h_n} \in \mathcal{H}$, such that for every $n \in \mathbb{N}, u_{h_n}$ is the unique solution of this problem

$$\begin{cases} -\dot{u_{h_n}}(t) \in A(t)u_{h_n}(t) + h_n(t) \ a.e \ \ t \in [0,T], \\ u_{h_n}(0) = u_0 \in D(A(0)). \end{cases} \tag{4.7}$$

While $u_{h_n} \in \mathcal{H}$, the following estimate is satisfying

$$||\dot{u}_{h_n}(t)|| \leq K'(1+\dot{\xi}) + (K'+1)||h_n(t)||. \tag{4.8}$$

Use the fact that $h_n \in \mathcal{H}$, we obtain

$$||\dot{u}_{h_n}(t)|| \leq K'(1+\dot{\xi}) + (K'+1)m(t). \tag{4.9}$$

therfore $\dot{u}_{h_n} \in K$, moreovere by extracting a subsequence, we may suppose that $\dot{u}_{h_n}$ weakly converges in $L^1_{\mathbb{R}^d}([0,T])$ to $z \in K$ and also we have u_{h_n} converges uniformely to a continuous function $u \in C_{\mathbb{R}^d}([0,T])$ with $u(t) = u_0 + \int_0^t z(\tau)d\tau$, so we have $\dot{u} = z$.
As u_{h_n} is absolutely continuous, we deduce that

$$u_{h_n}(t) = u_0 + \int_0^t \dot{u}_{h_n}(\tau)d\tau. \tag{4.10}$$

so,

$$\begin{aligned} \lim_{n\longrightarrow\infty} u_{h_n}(t) &= u_0 + \lim_{n\longrightarrow\infty}\left(\int_0^t \dot{u}_{h_n}(\tau)d\tau\right) \\ &= u_0 + \int_0^t \left(\lim_{n\longrightarrow\infty} \dot{u}_{h_n}(\tau)\right) d\tau \\ &= u_0 + \int_0^t z(\tau)d\tau = u(t). \end{aligned}$$

while $-\dot{u}_{h_n} \rightharpoonup -\dot{u}$ in $L^1_{\mathbb{R}^d}([0,T])$, so $-\dot{u}_{h_n}$ converge komlos to $-\dot{u}$,i.e. there exists a negligeable N such that $\lim_n \dfrac{1}{n}\sum(\dot{h}_j) = \dot{u}$.

Let $\mu \in D(A(t))$. We have

$$\begin{aligned}\langle \dot{u}_{h_n}(t) + h_n(t), u(t) - \mu \rangle &\leq \langle \dot{u}_{h_n}(t) + h_n(t), u_{h_n}(t) - \mu \rangle + \langle \dot{u}_{h_n}(t) + h_n(t), u(t) - u_{h_n}(t) \rangle \\ &\leq \langle A^0(t, \mu), \mu - u_{h_n}(t) \rangle + (\xi(t) + M(t)||u(t) - u_{h_n}(t)||,\end{aligned}$$

that imply

$$\frac{1}{n}\sum_{j=1}^{n}\langle \dot{u}_{h_j}(t) + h_j(t), u(t) - \mu \rangle \leq \frac{1}{n}\sum_{j=1}^{n}\langle A^0(t, \mu), \mu - u_{h_j}(t) \rangle + (\xi(t) + M(t)||u(t) - u_{h_n}(t)||.$$

Taking limits as $n \longrightarrow \infty$,

$$\langle \dot{u}(t) + h(t), u(t) - \mu \rangle \leq \langle A^0(t, \mu), \mu - u(t) \rangle \ a.e.$$

The fact that $A^0(t, \mu)$ is a principal selection. Thus, $\dot{u}(t) + h(t) \in A(t)u(t)\, a.e.$ and $u(t) \in D(A(t))$. Therefore u is the unique solution u_h of (P-h) and so one has the following equality

$$\Lambda(h)(t) = u_0 + \int_0^t \dot{u}_h(t),$$

so

$$\|\Lambda(h_n)(t) - \Lambda(h)(t)\| = \|u_{h_n}(t) - u_h(t)\| \leq \|u_{h_n} - u_h\|_{C_{\mathbb{R}^d}([0,T])},$$

therfore

$$\|\Lambda(h_n) - \Lambda(h)\| \leq \|u_{h_n} - u_h\|_{C_{\mathbb{R}^d}([0,T])}.$$

Consequentely Λ is continuous.

■

We need this lemma in the proof of the principal theorem.

Lemma 4.4.4

Assume that Assumption 4.2.1 satisfied, $D(A(t))$ is closed and ball-compact and let $G : [0,T] \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a compact-valued multifunction such that

(i) For every absolutely continuous function $u : [0,T] \longrightarrow \mathbb{R}^d$ the multifunction $t \mapsto G(t,u(t))$ is Lebesgue measurable on $[0,T]$.

(ii) There exist two functions p and q in $L^1_{\mathbb{R}_+}(0,T)$ such that for every $(t,x) \in [0,T] \times \mathbb{R}^d$, we have

$$G(t,x) \subset (p(t) + q(t)||x||)\overline{\mathbb{B}}_{\mathbb{R}^d}.$$

If $u(.)$ is an absolutely continuous solution to the differential inclusion

$$\begin{cases} \dot{u} \in G(t,u(t)) - A(t)u(t) \ a.e. \ t \in [0,T], \\ u(0) = u_0 \in D(A(0)). \end{cases} \tag{4.11}$$

Then, for almost all $t \in [0,T]$ we have this estimation

$$||\dot{u}(t)|| \leq \alpha(t) + \beta(t),$$

where

$$\alpha(t) = K'(1 + \dot{\xi}(t)) + (K'+1)p(t) + (K'+1)q(t)||u_0||,$$

and

$$\beta(t) = (K'+1)q(t) \int_0^t \left(\alpha(s) \exp\left((K'+1) \int_s^t q(\tau) d\tau \right) \right) ds.$$

Proof. assume that $u(.)$ is one of the absolutely continuous solution of (4.11). According to the hypothese (i), and usual techniques for measurable set-valued mappings, there exists a

Lebesgue-measurable mapping $\varphi : [0,T] \longrightarrow \mathbb{R}^d$ such that

$$\varphi(t) \in G(t, u(t)), \tag{4.12}$$

and, we have

$$-\dot{u}(t) - \varphi(t) \in A(t)u(t), \tag{4.13}$$

applying the hypothesis (ii) we find

$$||\varphi(t)|| \leq p(t) + q(t)||u(t)||, \ \forall \ t \in [0,T], \tag{4.14}$$

while q is integrable and u is bounded on $[0,T]$, the mapping φ is Lebesgue-integrable on $[0,T]$.

We define $\psi(\cdot)$ by the form

$$\psi(t) = \int_0^t \varphi(s)ds, \ \forall t \in [0,T],$$

therfore ψ is absolutely continuous and we get

$$\dot{\psi} = \varphi(t), \ a.e. \ t \in [0,T]. \tag{4.15}$$

From the inclusion (4.13), the hypomonotonicity of the operator A we have for all $x \in D(A(t))$

$$\langle -\dot{u}(t) - \varphi(t), x - u(t) \rangle \leq \langle A^0(t, x(t)), x - u(t) \rangle. \tag{4.16}$$

Let $t \in [0,T]$, $s < t$, from (1.1), (4.2), the second property of maximal monotone in

Lemma 1.6.5 and since $u(s) \in D(A(s))$, we have

$$\begin{aligned} d\left(u(s)+\psi(s)-\psi(t), D(A(t)\right) &\leq ||\psi(s)-\psi(t)|| + d(u(s), D(A(t)) \\ &\leq ||\psi(s)-\psi(t)|| + d(D(A(s), D(A(t)) \\ &\leq ||\psi(s)-\psi(t)|| + d_H(D(A(s), D(A(t)) \\ &\leq ||\psi(s)-\psi(t)|| + dis(A(s), A(t)) \\ &\leq ||\psi(s)-\psi(t)|| + ||\xi(s)-\xi(t)||. \end{aligned}$$

While $D(A(t))$ is closed convex so by definition 1.1 in [34], for each $y(s) \in D(A(s))$ we get

$$||u(s)+\psi(s)-\psi(t)-y(s)|| \leq ||\psi(s)-\psi(t)|| + ||\xi(s)-\xi(t)||. \tag{4.17}$$

According to (4.16), (4.17) we obtain
$\langle -\dot{u}(t)-\varphi(t), u(s)+\psi(s)-\psi(t)-y(s)+y(s)-u(t)\rangle \leq ||\dot{u}(t)+\varphi(t)||(||\psi(s)-\psi(t)|| + |\xi(s| - \xi(t)| + ||A^0(t,u(t))||||y(s)-u(t)||)$.
Or as $K' \in]0,\infty[$ then

$$|\xi(s| - \xi(t)| \leq K'(1+|\xi(s)-\xi(t)|),$$

and

$$||\psi(s)-\psi(t)|| \leq K'||\psi(s)-\psi(t)||,$$

so

$$\begin{aligned} \langle -\dot{u}(t)-\varphi(t), u(s)+\psi(s)-\psi(t)-y(s)+y(s)-u(t)\rangle \leq\ & ||\dot{u}(t)+\varphi(t)||(K'||\psi(s)-\psi(t)|| \\ & + K'(1+|\xi(s)-\xi(t)|) \\ & + ||A^0(t,u(t))||||y(s)-u(t)||), \end{aligned}$$

if we use the growth condition of A in Assumpption 4.2.1, one get

$$\langle -\dot{u}(t) - \varphi(t), u(s) + \psi(s) - \psi(t) - y(s) + y(s) - u(t) \rangle \leq ||\dot{u}(t) + \varphi(t)||(K'||\psi(s) - \psi(t)||$$

$$+K'(1 + |\xi(s) - \xi(t)|) + (1 + ||y(s)||)||y(s) - u(t)||.$$

As $t - s > 0$

$$\langle \dot{u}(t) + \varphi(t), \frac{u(t) - u(s)}{t - s} + \frac{\psi(t) - \psi(s)}{t - s} \rangle \leq ||\dot{u}(t) + \varphi(t)|| \left(K' \| \frac{\psi(s) - \psi(t)}{t - s} \| + K'(1 + |\frac{\xi(s) - \xi(t)}{t - s}|) \right)$$

$$+ \zeta(s),$$

such that

$$\lim_n \zeta(s) = 0 \text{ for } y(s) \longrightarrow u(t),$$

where

$$\zeta(s) = (1 + ||y(s)||) \, ||y(s) - u(t)||,$$

the last inequalities imply that

$$\langle \dot{u}(t) + \varphi(t), \dot{u}(t) + \varphi(t) \rangle \leq ||\dot{u}(t) + \varphi(t)|| \left(K'||\dot{\psi}(t)|| + K' \left(1 + \dot{\xi}(t) \right) \right),$$

so

$$||\dot{u}(t) + \varphi(t)||^2 \leq ||\dot{u}(t) + \varphi(t)|| \left(K'||\varphi(t)|| + K' \left(1 + \dot{\xi}(t) \right) \right),$$

that's

$$||\dot{u}(t) + \varphi(t)|| \leq K'||\varphi(t)|| + K'(1 + \dot{\xi}(t)),$$

hence

$$||\dot{u}(t)|| \leq (K' + 1)||\varphi(t)|| + K'(1 + \dot{\xi}(t)),$$

get ii) we have

$$\begin{aligned}||\dot{u}(t)|| &\leq K'(1+\dot{\xi}(t)) + (K'+1)p(t) + (K'+1)q(t)||u(t)|| \\ &\leq K'(1+\dot{\xi}(t)) + (K'+1)p(t) + (K'+1)q(t)||u_0 + \int_0^t \dot{u}(s)ds|| \\ &\leq K'(1+\dot{\xi}(t)) + (K'+1)p(t) + (K'+1)q(t)||u_0|| + (K'+1)q(t)\int_0^t ||\dot{u}(s)||ds,\end{aligned}$$

the Gronwall-like inequality in lemma 1.9.12 imply that

$$||\dot{u}(t)|| \leq \alpha(t) + (K'+1)q(t)\int_0^t \left(\alpha(s)\exp\left((K'+1)\int_s^t q(\tau)d\tau\right)\right)ds,$$

for

$$\alpha(t) = K'(1+\dot{\xi}(t)) + (K'+1)p(t) + (K'+1)q(t)||u_0||,$$

and

$$\beta(t) = (K'+1)q(t)\int_0^t \left(\alpha(s)\exp\left((K'+1)\int_s^t q(\tau)d\tau\right)\right)ds.$$

■

Now coming back to prove the existence of absolutely continuous solution for the diffirential inclusion

$$\begin{cases} \dot{u} \in G(t,u(t)) - A(t)u(t) \ a.e. \ t \in [0,T], \\ u(0) = u_0 \in D(A(0)). \end{cases} \tag{4.18}$$

such that the perturbation $G(t,.)$ is lower semicontinuous on $[0,T]$.

Now, we proceed to prove our main result in this chapter.

Theorem 4.4.5

Assume that Assumption 4.2.1 is satisfied and let $G : [0,T] \times \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ be a compact-valued multifunction such that

(i) The multifunction $G(t,.)$ is lower semicontinuous on $\mathbb{R}^d$, for all $t \in [0,T]$,

(ii) for every $u : [0,T] \to \mathbb{R}^d$ the multifunction $t \longmapsto G(t,u(t))$ is Lebesgue-measurable on $[0,T]$,

(iii) there exist two functions p and q in $L^1_{\mathbb{R}_+}([0,T])$ such that for every $(t,x) \in [0,T] \times \mathbb{R}^d$

$$G(t,x) \subset (p(t) + q(t)||x||)\,\overline{\mathbb{B}}_{\mathbb{R}^d}.$$

Then, for every $u_0 \in D(A(0))$, there is an absolutely continuous solution $u : [0,T] \longrightarrow \mathbb{R}^d$ for the problem 4.18 and for any solution $u(.)$ we have this estimate

$$||\dot{u}(t)|| \leq \alpha(t) + \beta(t),\ a.e\ t \in [0,T],$$

where $\alpha(t), \beta(t)$ are given in Lemma 4.4.4

Proof. the proof of this theorem is divided on two steps, in the first step we are going to proof the existence of an absolutely continuous solution and in the second step we are interested in the generalized case.

Step 1:

For each $(t,x) \in [0,T] \times \overline{\mathbb{B}}_{\mathbb{R}^d}$, we suppose that

$$G(t,x) \subset m(t)\overline{\mathbb{B}}_{\mathbb{R}^d}, \tag{4.19}$$

such that m is a nonnegative Lebesgue-integrable function defined on $[0,T]$. Putting

$$\Delta(t) = m(t)\overline{\mathbb{B}}_{\mathbb{R}^d}, \forall\, t \in [0,T].$$

Then Δ is a convex compact valued integrable bounded multifunction and the set S^1_Δ of all integrable selections of Δ is nonempty and $\sigma(L^1_{\mathbb{R}^d}([0,T]), L^\infty_{\mathbb{R}^d}([0,T])$-compact.

We consider for every $f \in S^1_\Delta$, u_f be the unique absolutely continuous solution to

$$\begin{cases} \dot{u} \in f(t) - A(t)u(t) \ a.e \ \ t \in [0,T], \\ u(0) = u_0 \in D(A(0)). \end{cases}$$

the existence of this solution is proved in the Proposition 4.4.1.

For each $f \in S^1_\Delta$, we have

$$G(t, u_f(t)) \subset \Delta(t), \ \ \forall t \in [0,T]. \tag{4.20}$$

By Proposition 4.4.1, the set $\mathcal{X} = \{u_f, f \in S^1_\Delta\}$ is compact in $\mathbb{C}_{\mathbb{R}^d}([0,T])$. For each $f \in S^1_\Delta$, let define

$$\Phi(u) = \left\{ f \in L^1_{\mathbb{R}^d}([0,T]) : f(t) \in G(t, u(t)) \ a.e \ \ t \in [0,T] \right\}. \tag{4.21}$$

According to the hypothese ii) we deduce that the multifunction $\Phi(.)$ is nonempty, closed and decomposable in the integrable space $L^1_{\mathbb{R}^d}([0,T])$.

While the multifunction is decomposable, we obtain for $u \in \mathcal{X}$, for every measurable set A, and for two Lebesgue integrable functions $f, g \in \Phi(u)$

$$\mathbb{1}_A.f + \mathbb{1}_{A^c}.g \in \Phi(u). \tag{4.22}$$

It remains to prove that $\Phi(.) : \mathcal{X} \rightrightarrows L^1_{\mathbb{R}^d}([0,T])$ is lower semicontinuous for aplying the continuous selection theorem 4.3.1. Consider a closed subset of $L^1_{\mathbb{R}^d}([0,T])$ wich denote F. We take (u_{f_n}) is a sequence in $\mathcal{X}$ such that

$$\Phi(u_{f_n}) \subset F, \ \forall n$$

where $u_{f_n} \longrightarrow u_f$ in the compact subset $\mathcal{X}$ of $C_{\mathbb{R}^d}([0,T])$, we are going to prove that

$$\Phi(u_{f_n}) \longrightarrow \Phi(u_f) \in F.$$

Let a function $g \in \Phi(u_f)$, we define $R_n(t) : [0,T] \rightrightarrows \mathbb{R}^d$ for all $n \in \mathbb{N}$ by the form

$$R_n(t) := \{y \in G(t,(u_{f_n}(t)) : ||y - g(t)|| \leq d\,(g(t), G(t,(u_{f_n}(t))))\}\,,\ \forall t \in [0,T]. \tag{4.23}$$

Due to the hypothese ii) that $R_n(t)$ is a nonempty, closed valued measurable multifunction, by Proposition 4.3.2 the multifunction $R_n(t)$ has a measurable selection.

Let $g_n : [0,T] \longrightarrow \mathbb{R}^d$, with $g_n(t) \in G(t,(u_{f_n}(t))$, while also $g \in \Phi(u_f)$ so $g \in G(t,u_f(t))$ we have

$$||g_n(t) - g(t)|| \leq d(G(t,(u_f(t)), G(t,(u_{f_n}(t))) \leq e(G(t,(u_f(t)), G(t,(u_{f_n}(t)))$$

$e(A,B)$ denotes the Hausdorff ecart of the compact sets A and B.

Since $u_{f_n} \longrightarrow u_f$, and use the fact that $G(t,.)$ is lower semicontinuous for all $t \in ([0,T]$ by (i), we obtain

$$\lim_{n\longrightarrow\infty} G(t,u_{f_n}(t)) = G(t,u_f(t)),\ \forall t \in [0,T].$$

we deduce that for all $t \in [0,T]$

$$\lim_{n\longrightarrow\infty} ||g_n(t) - g(t)|| = 0,\ \ .$$

applying the Lebesgue dominated convergence theorem 1.9.4, we conclude that

$$\lim_n g_n(t) = g(t) \text{ in } L^1_{\mathbb{R}^d}([0,T]). \tag{4.24}$$

Since $g \in \Phi(u_f) \subset G(t,u_f(t)$ and $g_n(t) \in \Phi(u_{f_n}) \subset F$ and we know that F is closed, then $g \in F$, hence Φ is lower semicontinuous in $\mathcal{X}$.

All the hypotheses of continuous selection theroem 4.3.1 are satisfying, Therefore Φ admits a continuous selection $S : \mathcal{X} \longrightarrow L^1_{\mathbb{R}^d}([0,T])$, it means that for every $u \in \mathcal{X}$

$$S(u)(t) \in G(t,u(t))\ a.e\ \ t \in [0,T]. \tag{4.25}$$

While $f \longmapsto u_{f_n}$ is continuous on S^1_Δ which is $\sigma(L^1, L^\infty)$ compact metrisable set, according to

Proposition 4.4.1 the following mapping $\psi(\cdot) : S^1_\Delta \rightrightarrows S^1_\Delta$ such that $\psi(f) = S(u_f)$ is $\sigma(L^1, L^\infty)$ continuous.

Therfore the fact that S^1_Δ is weakly compact, ψ is continuous, and acoording to Kakutani-Ky Fan fixed point theorem we obtain ψ has a fixed point $f \in S^1_\Delta$. Then u_f is an absolutely continuous solution of this inclusion

$$\begin{cases} -\dot{u}(t) \in A(t)u(t) + f(t) \ a.e \ \ t \in [0,T], \\ u(0) = u_0 \in D(A(0)), \end{cases}$$

with

$$f(t) = S(u_f)(t) \in G(t, u_f(t)) \ a.e \ \ t \in [0,T].$$

Step2.

In this step we are going to generalize the above case.

Suppose G satisfies (i) ,(iii). For $\alpha(t)$ and $\beta(t)$ given by Lemma 4.4.4, put

$$\gamma := ||u_0|| + \int_0^t (\alpha(s) + \beta(s))ds.$$

Let us consider the mapping $\Pi : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ with

$$\Pi(t,x) = \begin{cases} x, Si||x|| \leq \gamma(t) \\ \gamma(t)x/||x||, Si||x|| > \gamma(t) \end{cases}$$

and put

$$G_0(t,x) = G(t, \Pi(t,x)).$$

Then $G_0(t,x) = G(t, \Pi(t,x))$ inherits the the lower semicontinuous property (i) from $G(t,.)$ and measurable property (ii) from $G(.,x)$ and for $m(t) = p(t) + q(t)\gamma(t)$, one has for all $x \in \mathbb{R}^d$

$$G_0(t,x) \subset m(t)\overline{\mathbb{B}}_{\mathbb{R}^d}.$$

Apply the result of the step 1 to get that $u(.)$ is an absolutely continuous solution to

$$\begin{cases} \dot{u}(t) \in G_0(t, u(t)) - A(t)u(t) \ a.e. \ t \in [0, T], \\ u(0) = u_0 \in D(A(0)). \end{cases} \tag{4.26}$$

It easy to see that $u(.)$ is a solution to

$$\begin{cases} \dot{u}(t) \in G(t, u(t)) - A(t)u(t) \ a.e. \ t \in [0, T], \\ u(0) = u_0 \in D(A(0)), \end{cases}$$

if and only if $u(.)$ is a solution to

$$\begin{cases} \dot{u}(t) \in G_0(t, u(t)) - A(t)u(t) \ a.e. \ t \in [0, T], \\ u(0) = u_0 \in D(A(0)). \end{cases} \tag{4.27}$$

$\blacksquare$

Conclusion

Although the main focus of this thesis has been the maximal monotone differential inclusions, the developed methods have allowed us to address several differential inclusions involving normal cones.

In the future, we would like to continue our research on the following issues:

- The situation where the maximal monotone operators $A(t)$ move in a BV way, i.e., with a bounded variation, would also have a great interest.

- A challenging issue remains on deriving necessary optimality conditions for local solutions to absolutely continuous-time maximal monotone control problems of this class by passing to the limit from those obtained for their finite-difference counterparts. Besides their own theoretical interest, explicit necessary optimality conditions for absolutely continuous-time maximal monotone systems may be convenient for calculating optimal solutions. We pursue these goals in both theory and applications, particularly to non regular circuit model in a more general setting in our on going research.

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Abstract

In this thesis, we give a new proof of the existence of absolutely continuous solutions for a class of first order state dependent maximal monotone differential inclusions. The existence result is obtained by using Schauder's fixed point theorem. In addition, a stability result is provided. Finally, using a suitable reduction of order technique, we give a new existence result for a general second-order state-dependent maximal monotone differential inclusion.

Résumé

Dans cette thèse, nous donnons une nouvelle preuve de l'existence de solutions absolument continues pour une classe d'inclusions différentielles de premier ordre gouvernée par des opérateurs maximaux monotones dépendant de l'état. Le résultat d'existence est obtenu en utilisant le théorème du point fixe de Schauder. En outre, un résultat de stabilité est fourni. Enfin, en utilisant une technique de réduction d'ordre appropriée, nous donnons un nouveau résultat d'existence pour des inclusions différentielles du deuxième ordre gouvernées par des opérateurs maximaux monotones dépendant de l'état.

ملخص

في هذه الأطروحة، نقدم برهان جديد على وجود حلول مستمرة مطلقا لفئة من التضمينات التفاضلية من الرتبة الأولى التي تعتمد عل المتغير. يتم الحصول على نتيجة الوجود باستخدام نظرية النقطة الثابتة لشودر. بالإضافة إلى ذلك، يتم توفير نتيجة الاستقرار. أخيراً، باستخدام تقنيات تخفيض الرتبة، نعطي نتيجة وجود جديدة لفئة من التضمينات التفاضلية من الرتبة الثانية والتي تتعلق بالمتغير.