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Dedication

$\mathcal{TO}$

$\mathcal{M}$y parents,

my brothers,

to all those who have always believed in my success...

$\mathcal{W}$e dedicate this modest work.

$\preccurlyeq$ $\mathscr{B}$.**Abderrahim** $\succcurlyeq$

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المساهمات في الأنظمة الديناميكية غير العادية

ملخص

في هذه الأطروحة، ندرس، من ناحية، وجود الحل ووحدانيته للاحتواءات التكاملية-التفاضلية من نوع فولتيرا، ومن ناحية أخرى، وجود الحلول المثلى ومن ثم الحصول على الشروط المثلى اللازمة لفئة واسعة من المصغرات المحلية في مثل هذه المشاكل. يُخصص الموضوع الأول للاحتواءات مورو الشاملة المضطربة بمجموع تابع لكراتيودوري هيلبارتي ذو بعد غير منته. مجموعة القيود غير محدبة ومقدار تكاملي في فضاء وتتحرك على شكل استمرار مطلق. تم إعطاء تطبيقات في الدارات الكهربائية غير المنتظمة والمتراجحات المتغيرة. في الموضوع الآخر، نقدم شروط مثالية ضرورية يتم التعبير عنها بالكامل من حيث بيانات المشكلة ويتم توضيحها من خلال أمثلة غير بديهية تتضمن تطبيقات لنماذج التحكم المثلى للدوائر الكهربائية غير المنتظمة.

الكلمات المفتاحية: *عملية مورو الشاملة ، خوارزمية اللحاق بمورو ، معادلة فولتيرا التكاملية التفاضلية ، متراجحات جرونوال ، التحليل المتغير ، التحكم الأمثل ، التقريبات المنفصلة ،الشروط الضرورية المثالية ، أنظمة التكامل التفاضلي ،مشاكل الاتصال، مشاكل تطبيقات على الإلكترونيات.*

Contribution to nonsmooth dynamical systems

Absract

In this dissertation, we study, on the one hand, the existence and uniqueness of solution for integro-differential inclusions of the Volterra type and, on the other hand, the existence of optimal solutions and then obtain necessary optimality conditions for a broad class of local minimizers in such problems. The first topic is devoted to Moreau's sweeping processes perturbed by a sum of a Carathéodory mapping and an integral forcing term in infinite dimensional framework. The moving set is assumed to be prox-regular and moved in an absolutely variation way. Applications to the theory of complementarity problems, non regular electric circuits and evolution variational inequalities are given. In the other topic, we give necessary optimality conditions which are expressed entirely in terms of the problem data and are illustrated by nontrivial examples that include applications to optimal control models of non-regular electrical circuits.

Key words: Moreau's sweeping process, Moreau's catching-up algorithm, Volterra integro-differential equation, Gronwall's inequality, Variational analysis, Optimal control, Discrete approximations, Necessary optimality conditions, Differential complementarity systems, Contact problem, Applications to electronics.

Contribution aux systèmes dynamiques non- réguliers

Résumé

Cette thèse est consacrée, d'une part, à l'étude de l'existence et de l'unicité de solution pour des inclusions intégro-différentielles de type Volterra et, d'autre part, à l'existence de solutions optimales, puis obtenir les conditions nécessaires d'optimalité pour une large classe de minimiseur locaux dans de tels problèmes. Nous étudions dans la première partie des processus de rafle de Moreau perturbés par la somme d'une fonction de Carathéodory et un terme de force intégrale. L'ensemble mouvant est prox-régulier dans un espace de Hilbert réel quelconque et sa variation est contrôlé par une fonction absolument continue. Des applications à la théorie de la complémentarité, aux circuits électriques non réguliers et à celle des inéquations variationnelles sont présentées. Dans la seconde partie, on donne les conditions nécessaires d'optimalité qui sont exprimées entièrement en termes de données du problème et sont illustrées par des exemples non triviaux qui incluent des applications à des modèles de contrôle optimal de circuits électriques non réguliers.

Mots clés: Processus de balayage de Moreau, Algorithme de rattrapage de Moreau, Equation intégro-différentielle de Volterra, Inégalité de Gronwall, Analyse variationnelle, Contrôle optimal, Approximations discrètes, Conditions d'optimalité nécessaires, Systèmes de complémentarité différentielle, Problème de contact, Applications à l'électronique.

PUBLICATIONS

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Notations

Operations and Symbols

$i.e.$	Identically equivalent.
$a.e.$	Almost every.
$s.t.$	Such that.
$resp$	Respectively.
$:=$	Equal by definition.
$\equiv$	Identically equal.
$\langle \cdot, \cdot \rangle$	Inner product on a Hilbert space.
$\|\cdot\|$	Norm of a Hilbert space.
$\lvert \cdot \rvert$	Euclidean norm.
$\sup, \inf, \max, \min$	Supremum, Infimum, Maximum, Minimum, respectively.
$u_n \longrightarrow u$	u_n converges to u strongly.
$u_n \rightharpoonup u$	u_n converges to u weakly (in weak topology).
$u_n \xrightarrow{S} u$	$u_n \longrightarrow u$ and $u_n \in S$ for all n.
$u_n \xrightarrow{f} u$	$u_n \longrightarrow u$ and $f(u_n) \longrightarrow f(u)$ for all n.
$u.s.c$	Upper semicontinuous.
$l.s.c$	Lower semicontinuous.
$k * I$	Convolution product between two functions k and I.
A^*	The transpose of a matrix A.

Sets

$\mathbb{B}$ or $\mathbb{B}_H$	Closed unit ball of space H.
$co(S)$	Convex hull of S.
$\overline{co}(S)$	Closed convex hull of S.
$\mathrm{bdr}\,(S)$	Boundary of S.
$\mathrm{int}\,(S)$	Interior of S.
$\mathrm{epi}\,(f)$	Epigraph of an extended real valued function f.
$\mathrm{dom}\,(f)$	Effectif domain of an extended real valued function f.
$rg(F)$	The range of a set-valued map F.
$\mathrm{gph}\,(F)$	Graph of a set-valued map F.
$N_S^P(x)$	Proximal normal cone to S at x.
$N_S^L(x)$	Limiting normal cone to S at x.
$\partial_P\, f(x)$	Proximal subdifferential of f at x.
$\partial_L\, f(x)$	Limiting subdifferential of f at x.
$\nabla\, f(x)$	Gradient vector of f at x.
$\nabla_2\, f(z,x)$ or $\nabla_x\, f(z,x)$	Gradient vector of f with respest to x for any z.
$\nabla^2\, f(x)$	Hessian matrix of f at x.
$\nabla_x^2\, f(z,x)$	Hessian matrix of f with respest to x for any z.

Spaces

$\mathbb{N}$	The set of positive integers.
$\mathbb{R}$	The real line.
$\mathbb{R}_+$	The set of nonnegative numbers.
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{-\infty, +\infty\}$.
$\mathbb{R}^d$	The d-dimensional Euclidean space.
$\mathbb{S}^d$	The space of second order symmetric tensors on $\mathbb{R}^d$.
Ω	An open, bounded, connected set in $\mathbb{R}^d$ with a Lipschitz boundary Γ.
Γ	The boundary of the domain; Ω.
$\bar{\Omega}$	The closure of Ω in $\mathbb{R}^d$, i.e. $\bar{\Omega} = \Omega \cup \Gamma$.
$\text{mes}\,(A)$	Lebesgue measure of the measurable subset $A \subset \Gamma$.
H, V	Hilbert spaces.
$u \perp v$	Two orthogonal elements in the space H, i.e. $\langle u, v\rangle = 0$.
$\mathcal{L}(V, X)$	The space of linear continuous operators from V to a normed space X.
$\mathcal{L}(V) \equiv \mathcal{L}(V, V)$.	
J	Any interval (resp. closed set) in $\mathbb{R}$ (resp. $\mathbb{R}^2$).
$\mathcal{C}(J; H)$	The space of continuous functions defined on J with values in H.
$L^1(J, H)$	The space of all mappings from J into H wich are Bochner integrable on J with respect to the Lebesgue measure.
$L^2(\Omega)$	The Lebesgue space of two integrable functions on Ω, with the usual modification if $p = \infty$.
$L^2(\Omega)^d$	The space of mapping $\boldsymbol{v} : \Omega \to \mathbb{R}^d$, with $v_i \in L^2(\Omega)^d$, $i = 1..., d$.

Functions

$d_S(\cdot)$ *or* $d(\cdot, S)$	Distance function.
$\psi_S(\cdot)$ *or* $\psi(\cdot, S)$	Indicator function of a set S.
$\sigma_S(\cdot)$ *or* $\sigma(\cdot, S)$	Support function of a set S.
$\mathrm{Proj}_S(\cdot)$ *or* $\mathrm{Proj}(\cdot, S)$	Projection from H into S.

Mapping

$f : X \longrightarrow Y$	Single-valued mapping from X to Y.
$F : X \rightrightarrows Y$	Set-valued mapping from X to Y.

General Introduction

In the seventies, sweeping processes are introduced and deeply studied by J. J. Moreau through a series of papers, in particular [56, 57]. It is shown in [56] that such processes play an important role in mechanics, especially in elasto-plasticity, quasi-statics, dynamics. Roughly speaking, a point is swept by a moving closed convex set $C(t)$ in a Hilbert space H, which can be formulated in the form of differential inclusion as follows

$$(SP): \begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) \qquad a.e.\ t \in [T_0, T] \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $T_0, T \in \mathbb{R}$ with $0 \leq T_0 < T$ and $N_{C(t)}(\cdot)$ denotes here the normal cone of $C(t)$ in the sense of convex analysis. The need of consideration of systems with external forces (see, e.g., [16, 83] and [15] for more details) led to study the following perturbed variant

$$(PSP): \begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f(t, x(t)) \qquad a.e.\ t \in [T_0, T] \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $f : [T_0, T] \times H \to H$ is a Carathéodory mapping, i.e., $f(t, \cdot)$ is continuous and $f(\cdot, x)$ is Bochner measurable for $[T_0, T]$ endowed with the Borel σ-field $\mathcal{B}([T_0, T])$. By Bochner measurable mapping we mean here any limit of uniformly convergent sequence of simple mappings from $[T_0, T]$ into H with $[T_0, T]$ endowed with its Borel σ-field.

Actually, diverse approaches for existence of solutions of (SP) and (PSP) are available in the literature: Catching-up method (see, e.g., [57]), regularization procedure (see, e.g., [56, 63]), reduction to unconstrained differential inclusion (see, e.g., [77]). The method in [57] via the catching-up algorithm without the term $f(\cdot, \cdot)$ is based on a specific lemma of inequality involving the convexity of sets $C(t)$ along with the projection mapping $\operatorname{proj}_{C(t)}(\cdot)$ (see [57, Lemma 1.(2a)]), whereas in presence of the term $f(\cdot, \cdot)$ Gronwall-type inequalities are generally utilized for existence and uniqueness of solutions. In each method the corresponding lemma is applied to two suitable approximate solutions x_1, x_2 of (PSP), by means of the monotonicity of $N_{C(t)}(\cdot)$ (hypomonotonicity when $C(t)$ is prox-regular). Those features and the Lipschitz property of the forcing term $f(t, \cdot)$ with respect to the state variable are employed to obtain that the distance between $x_1(t)$ and $x_2(t)$ is nonincreasing with respect to time t. This reasoning allows in general the construction of a Cauchy net/sequence of approximate solutions, converging to a solution.

Several extensions of the sweeping process in diverse ways (well-posedness and optimal control) have been studied in the literature (see, e.g., [1], [2], [15], [30], [47], [77], [83] and references therein).

Chapter 2: Nonconvex Integro-Differential Sweeping Process with Applications

In this chapter, aims to study the following new variant of the sweeping process

$$(P_{f_1,f_2}): \begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))ds, \quad \text{a.e. } t \in [T_0, T] \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $N_{C(t)}(\cdot)$ denotes here a suitable normal cone to the subset $C(t)$ of the Hilbert space H. Assumptions will be considered below to ensure the desirable integrability in t or (s, t) of $f(t, s, x(s))$. We called the differential inclusion (P_{f_1,f_2}) an *integro-differential sweeping process* because the integral of the state and the velocity are defined in the dynamical system. One can interpret (P_{f_1,f_2}) as follows: as long as $x(t)$ is in the interior of the set $C(t)$, we get $N_{C(t)}(x(t)) = 0$ and (P_{f_1,f_2}) reduces to a Volterra integro-differential equation

$$(E_{f_1,f_2}): \begin{cases} -\dot{x}(t) = f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))ds \quad \text{a.e. } t \in [T_0, T] \\ x(T_0) = x_0, \end{cases}$$

(for at least a small period of time) to satisfy the constraint $x(t) \in C(t)$, until $x(t)$ hits the boundary of the set $C(t)$. At this moment, if the vector field $-\big(f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))ds\big)$ is pointed outside of the set $C(t)$, then any component of this vector field in the direction normal to $C(t)$ at $x(t)$ must be annihilated to maintain the motion of x within the constraint set. So, the system (P_{f_1,f_2}) can be considered as a Volterra integro-differential equation (E_{f_1,f_2}) under control term $u(t) \in N_{C(t)}(\cdot)$ which guarantees that the trajectory $x(t)$ always belongs to the desired set $C(t)$ for all $t \in [T_0, T]$.

The well-posedness of the classical perturbed sweeping process (PSP), i.e.,$P_{f_1,0}$ ($f_2 \equiv 0$), has been studied by many authors with different assumptions on data, see, e.g., [39, 40, 64] and references therein. Sweeping process involving integral perturbation, i.e., $P_{0,f_{0,2}}$ ($f_1 \equiv 0$ and the particular mapping $f_{0,2}(s,x)$) was considered earlier by Brenier, Gangbo and Savare [17] and recently by Colombo and Kozaily [34]. In the latter paper [34] the authors proved the existence and uniqueness of solution with the particular integral $\int_{T_0}^{t} f_{0,2}(s,x(s))ds$, i.e., for the following problem

$$(P_{0,f_{0,2}}):\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + \displaystyle\int_{T_0}^{t} f_{0,2}(s,x(s))ds, \quad \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

It is also worth mentioning that Colombo and Kozaily say in their paper [34]: "*of course, existence and uniqueness to* $(P_{0,f_{0,2}})$ *is not surprising*". We point out that with the above integral $\int_{T_0}^{t} f_{0,2}(s,x(s))ds$, the integro-differential sweeping process $(P_{f_1,f_{0,2}})$ is equivalent to

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t,x(t)) + y(t),\ \dot{y}(t) = f_{0,2}(t,x(t)),\ \ x(T_0) = x_0,\ y(T_0) = 0,$$

and so

$$\overbrace{\begin{pmatrix} -\dot{x}(t) \\ -\dot{y}(t) \end{pmatrix}}^{-\dot{X}(t)} \in N_{C(t)\times H}\overbrace{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}}^{X(t)} + \overbrace{\begin{pmatrix} f_1(t,x(t)) + y(t) \\ -f_{0,2}(t,x(t)) \end{pmatrix}}^{f(t,X(t))},$$

which is a special case of the classical perturbed sweeping process (PSP), see, e.g., [57, 39, 64] for the situation of unbounded moving sets. Otherwise stated, $(P_{0,f_{0,2}})$ is reduced to the now classical perturbed sweeping process (PSP).

In [34] the motivation of the authors for studying $(P_{0,f_{0,2}})$ was designing a smoother method of penalization, the motivation of which comes from applications to deriving necessary optimality conditions for optimal control problems with sweeping processes. Notice that the more general differential inclusion

$$(P_{\varphi}):\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t,x(t)) + \varphi(t)\displaystyle\int_{T_0}^{t} f_{0,2}(s,x(s))ds, \quad \text{a.e. } t \in [T_0,T] \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

with $\varphi : [T_0,T] \to \mathbb{R}$, can be reduced as above to (PSP) via

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t,x(t)) + \varphi(t)\,y(t),\ \dot{y}(t) = f_{0,2}(t,x(t)),\ \ x(T_0) = x_0,\ y(T_0) = 0,$$

that is,

$$\overbrace{\begin{pmatrix} -\dot{x}(t) \\ -\dot{y}(t) \end{pmatrix}}^{-\dot{X}(t)} \in N_{C(t)\times H}\overbrace{\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}}^{X(t)} + \overbrace{\begin{pmatrix} f_1(t,x(t)) + \varphi(t)\,y(t) \\ -f_{0,2}(t,x(t)) \end{pmatrix}}^{f(t,X(t))}.$$

Now, if the integral involving f_2 depends on the two *time*-variables, the reduction of (P_{f_1,f_2}) there to the perturbed sweeping process (PSP) cannot be applied.

To the best of our knowledge, for the problem under consideration in the case of the function f_2 depending on two *time*-variables, that is, in the case of a general integro-differential sweeping process of Volterra type (P_{f_1,f_2}), a well-posedness result, including the existence, uniqueness, and stability of the solution, has not been obtained up to now.

In this work,, we obtain results on the existence and uniqueness of a solution to the Volterra sweeping process (P_{f_1,f_2}) in a Hilbert space. This is done with the help of a new Gronwall-like inequality (see Section 2.1) and of a new scheme corresponding to the existence of absolutely continuous solutions for the quasi-autonomous sweeping processes

$$\begin{cases} -\dot{x}_n(t) \in N_{C(t)}(x_n(t)) + f_1(t, x_n(t_k)) + \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} f_2(t, s, x_n(t_j))\, ds \\ + \int_{t_k}^{t} f_2(t, s, x_n(t_k))\, ds \quad \text{a.e. } t \in [t_k, t_{k+1}] \\ x_n(T_0) = x_0 \in C(T_0), \end{cases}$$

where $T_0 = t_0 < t_1 < ... < t_n = T$ is a discretization of the interval $[T_0, T]$. Such discretization methods via suitable catching-up algorithms are desirable approaches, especially for nummerical simulations[1]. They are used numerically for integro-differential sweeping process in [10, 11]. We must also say that our approach only assumes for (P_{f_1,f_2}) the growth condition $\|f_2(t, s, x)\| \leq \beta(t, s)(1 + \|x\|)$, while for the particular inclusion $(P_{0,f_{0,2}})$ the authors of [34] require for some real $M > 0$ the boundedness condition $\|f_{0,2}(s, x)\| \leq M$ for all $s \in [T_0, T]$ and all x in an open set $\Omega \supset \bigcup_{t\in[T_0,T]} C(t)$.

The outline of the chapter is as follows. In Section 2.1, we prove a new Gronwall-like inequality (differential inequality). Then, in Section 2.2, we present our main existence, uniqueness, and stability result. In Section 2.3 we use those results in the study of nonlinear integro-differential complementarity systems. This is realized by transforming such systems into integro-differential sweeping processes of the form (2.2) where the moving set $C(t)$ is described by a finite number of inequalities. We also provide sufficient verifiable conditions ensuring the absolute continuity of the moving set. In Section 2.4, we give a second application of our results to non-regular electrical circuits containing time-varying capacitors and nonsmooth electronic device like diodes. A circuit with tranmission line, diode and inductor is also presented. Section 2.5 is concerned with an application to frictionless contact problems in mechanics.

[1] The books, e.g., [8, 12] show how sweeping processes provide efficient tools and catching-up discretizations which can help engineers for the simulation of complicated electrical circuits.

Chapter 3: On the Discretization of Truncated Integro-Differential Sweeping Process and Optimal Control

The purpose of this chapter is twofold: first, to show the solvability (in the sense existence and uniqueness) of general integro-differential sweeping process of Volterra type (P_{f_1,f_2}) by using an appropriate catching-up algorithm (full discretization), and second, to show the existence of optimal solutions to an optimal control problem involving the Volterra integro-differential sweeping process.

The Moreau catching-up algorithm is a quite old approach to deal with sweeping processes. From a numerical point of view, the time-integration (also known as time-stepping) schemes have been applied to find an approximation of the solution to the sweeping process. The so-called catching-up algorithm was introduced by Moreau [61, 57, 62] to prove the existence of a solution to sweeping process. It consists in building discretized solutions in dividing the time interval into sub-intervals where the moving set does not vary too much. Then by compactness arguments or Cauchy property, one can construct a limit mapping (when the length of subintervals tends to zero) which satisfies the desired differential inclusion. The catching-up algorithm has never been used, even in the convex case, to study the Volterra integro-differential sweeping process.

The chapter is organized as follows. Section 3.1, by extending the catching-up scheme of Moreau [57, 62] to integro-differential sweeping processes, we prove solvability of absolutely continuous integro-differential sweeping processes with hypomonotone dependence on the state of the external dynamic perturbations, and Lipschitz dependence on the state of the integral parts of the sweeping dynamics. Next, in Section 3.2, we apply our results to a model appearing in non-regular electrical circuits with nonlinear resistors which generate the non Lipschitz parts of the sweeping dynamics and time-varying capacitors which generate the Lipschitz parts of the sweeping dynamics. Section 3.3 contains some numerical simulations and presents a realistic example showing that the obtained algorithm allows us to compute solutions. Finally, in Section 3.4, an optimal control problem governed by Volterra integro-differential sweeping process is introduced, and a solvability result for the optimal control problem is established.

Chapter 4: Optimal Control of Nonconvex Integro-Differential Sweeping Processes

This chapter continues a series of recent publications devoted to the rather new optimal control theory for discontinuous differential inclusions governed by *controlled sweeping processes*.

One of the most remarkable features of Moreau's sweeping process and its extensions is that the Cauchy problem for them has a *unique* solution. This excludes any optimization of the discontinuous sweeping dynamics of type (SP). A new view on sweeping processes was offered in [25], were the authors suggested to parameterize the moving sets in (SP) by control functions $C(t) = C(u(t))$ that allowed them to formulate an *optimal control* problem and derive first necessary optimality conditions in sweeping control theory. Since that, optimal control theory for various types of controlled sweeping processes governed by ordinary differential inclusions of type (SP) has been developed in many publications with deriving necessary optimality conditions and applications; see, e.g., [5, 7, 18, 20, 21, 22, 26, 27, 68, 81, 53, 54, 85] and the references therein.

In contrast to the previous publications, in this chapter we consider controlled sweeping processes with the dynamics governed by *integro-differential inclusions* of the Volterra type:

$$\begin{cases} -\dot{x}(t) \in N_{C(u(t))}(x(t)) + f_1(a(t), x(t)) + \displaystyle\int_0^t f_2\big(b(s), x(s)\big)\mathrm{d}s, \quad \text{a.e. } [0,T], \\ \big(u(\cdot), a(\cdot), b(\cdot)\big) \in W^{1,2}([0,T], \mathbb{R}^n) \times L^2([0,T], \mathbb{R}^{m+d}), \\ (a(t), b(t)) \in A \times B \subset \mathbb{R}^m \times \mathbb{R}^d \ \text{ a.e. } \ t \in \ [0,T], \\ x(0) = x_0 \in C(0), \end{cases} \tag{1}$$

on the fixed time interval $[0,T]$, where the triple $(u(\cdot), a(\cdot), b(\cdot))$ signifies *feasible controls* acting in the moving sets, additive perturbations, and the integral part of the sweeping dynamics, respectively. The controlled moving sets are given in the form

$$C(t) := C(u(t)) = C + u(t), \ C := \big\{x \in \mathbb{R}^n \ \big|\ g_i(x) \geq 0, \quad i = 1, \ldots, s\big\}. \tag{2}$$

Since the sets $C(t)$ are generally *nonconvex*, the normal cone in (1) is understood in the generalized sense defined in Section 4.1, which reduces to the one in (SP) for the case of convexity.

Given further a terminal cost function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ and a running cost $l_0 : [0,T] \times \mathbb{R}^{4n+m+d} \to \mathbb{R}$, the *sweeping optimal control problem* (P) consists of minimizing the

Bolza-type functional

$$J_0[x,u,a,b] := \varphi(x(T)) + \int_0^T l_0\big(t, x(t), u(t), a(t), b(t), \dot{x}(t), \dot{u}(t)\big)\, dt,$$

on the set of feasible control $(u(\cdot), a(\cdot), b(\cdot))$ where $(a(\cdot), b(\cdot)) \in L^2([0,T]; \mathbb{R}^{m+d})$ are measurable functions and the corresponding trajectories $x(\cdot)$ of (1) from the space $W^{1,2}([0.T], \mathbb{R}^n)$. Such quadruples $(x(\cdot), u(\cdot), a(\cdot), b(\cdot))$ are called *feasible solutions* to (P). The existence results for feasible and optimal solutions to (P) are given in Section 4.2. The required assumptions on the functions φ, l_0, g_i and the mappings $f_1 : \mathbb{R}^{m+n} \to \mathbb{R}^n$ and $f_2 : \mathbb{R}^{d+n} \to \mathbb{R}^n$ will be formulated in Section 4.1.

Note that controls $u(\cdot)$ from one side and $(a(\cdot), b(\cdot))$ from the other have different functional natures. This is the most natural and essential for our model and developed approach to derive necessary optimality conditions for (P). Various attempts to unify control actions by moving one group of controls to the other, as well as reducing integral perturbations to the differential sweeping dynamics with the subsequent application of known results, lead us to extra regularity assumptions and would not allow us to obtain new optimality conditions specific for the controlled integro-differential sweeping process under consideration, see below.

As follows from the sweeping inclusion (1) and the structure of the controlled moving sets (2), problem (P) automatically involves the pointwise *mixed state-control constraints*

$$g_i(x(t) - u(t)) \geq 0 \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad i = 1, \ldots, s,$$

which have been recognized among the most challenging issues even in standard optimal control theory for systems governed by smooth controlled ordinary differential equations.

The *uncontrolled* integro-differential sweeping process (1) was first introduced in [17] for the case where $f_1 \equiv 0$, $f_2 \equiv f_2(x)$, and where $C(t) := \mathbb{E}^n$ is the nonnegative orthant of $\mathbb{R}^n$. The motivation in [17] came from the study a one-dimensional flow of particles subject to a force field generated by the fluid itself. Then (again uncontrolled) integro-differential sweeping process (1) with $f_1 \equiv 0$ and $f_2 \equiv f_2(x) \in \gamma\mathbb{B}$ was considered in [34], where the authors established the existence and uniqueness of solutions by developing a new penalization approach. Quite recently, the existence and uniqueness issues for a generalized uncontrolled version of (1) have been revisited in [9] with the extension of the results in [34] to a more general framework of (1) by using a new Gronwall-like differential inequality within a developed semi-discretization method. Furthermore, paper [9] contains results on the continuous dependence of solutions of (1) on the initial data with some applications to sweeping dynamical models arising in electronics.

To the best of our knowledge, optimal control problems for integro-differential sweeping processes have been *never formulated* and studied in the literature even in the case of convex and uncontrolled moving sets. The closest prototype of (P) in the case of sweeping differential inclusions with no integral part was considered in [22], where the authors established the existence of optimal solutions and obtained necessary optimality conditions for local minimizers. Furthermore, in [23] they applied the necessary optimality conditions from [22] to optimal control of the planar version of the crowd motion model in traffic equilibria. The approach to deriving necessary optimality conditions in [22] was based on the method of discrete approximations developed in [51, 49] for optimal control of Lipschitzian differential inclusions and then extended in [20, 21, 22, 26, 27, 81, 53, 54] to various control systems governed by discontinuous differential inclusions of the sweeping type.

Here we conduct a detailed study of the formulated optimal control problem (P) for integro-differential sweeping processes with proving the *existence of optimal solutions* and deriving comprehensive *necessary optimality conditions* for a broad family of local minimizers of (P), where the obtained conditions are expressed entirely in terms of the problem data. On one side, the achieved results extend those from [22] to the new class of controlled sweeping processes. On the other hand, we establish a novel necessary optimality condition of the *Volterra type*, which is characteristic for integro-differential sweeping control systems while being particularly useful for calculations of optimal solutions.

To reach our goals, we develop the *method of discrete approximations* in the new setting of dynamical systems governed by controlled sweeping integro-differential inclusions, with justifying the *well-posedness* of discrete approximations in the sense of establishing the $W^{1,2}$-strong approximation of feasible solutions for (P) by their extended discrete counterparts as well as verifying the $W^{1,2}$-strong convergence of discrete optimal solutions to a prescribed local minimizer of (P). The results obtained in this direction are of their own interest (including numerical issues), while they are exploited in the paper as a *driving force* to derive necessary optimality conditions in problem (P) by doing this first for the discrete-time problems and then by passing to the limit from them with the diminishing discretization step. To proceed in such a way, we need—by taking into account the very structure of the integro-differential sweeping dynamics in (1)—to employ appropriate tools of first-order and (mainly) *second-order variational analysis* and *generalized differentiation*. It occurs that the best pick for the needed constructions are those introduced by the third author and then developed in many publications; see Section 4.5 for more discussions and references. Moving in this direction allows us to establish below a comprehensive collection of necessary optimality conditions for problem (P) and its discrete approximations, which are of their independent benefits. The given applications to the formulated *optimal control models* for *non-regular electric circuits* with numerical calculations illustrate the efficiency of the obtained necessary optimality conditions to solve particular control problems of the integro-differential type (P) that naturally appear in practical modeling.

The rest of the chapter is organized as follows. Section 4.1 describes the *standing assumptions* used below. In Section 4.2 we first establish the *existence* of feasible and optimal solutions to problem (P) and then define and discuss the notion of *local minimizers* for which the necessary optimality conditions are derived below.

Section 4.3 is devoted to the construction of *discrete approximations* for controlled integro-differential sweeping processes (1) and to the proof that any *feasible solution* to (1) can be $W^{1,2}$-*strongly approximated* by feasible solutions to discrete problems, which are piecewise linearly extended to the whole interval $[0, T]$. The obtained crucial result goes far beyond optimization and occurs to be useful as an efficient machinery of the qualitative and numerical analysis of discontinuous integro-differential inclusions of the sweeping type.

Section 4.4 continues the discrete approximation developments of Section 4.3 while now concentrating on the approximation of the entire problem (P) and its prescribed *local minimizer* by optimal solutions to discrete-time problems. Here we show that the constructed discrete approximations always admit optimal solutions whose extensions on $[0, T]$ *strongly converge* to the given local minimizer under in the $W^{1,2}$ topology.

To proceed with deriving necessary optimality conditions, we recall in Section 4.5 the basic *generalized differential constructions* of variational analysis that are needed for our study. Although all the mappings involved in the description of (P) are assumed to be smooth, the unavoidable source of *nonsmoothness* comes from the *sweeping dynamics* in (1), which requires the usage of appropriate *second-order* constructions applied to (nonconvex) *graphs* of the normal cones. We review the employed constructions of generalized differentiation and present calculation formulas for them expressed entirely via the given data.

In Section 4.6 we derive *necessary optimality conditions* for the *discrete approximations* of problem (P) by using the generalized differential constructions of Section 4.5, their well-developed calculi, and the second-order computations. The obtained results are important for their own sake as necessary optimality conditions for discrete-time counterparts of the controlled integro-differential sweeping processes. Furthermore, the strong convergence of discrete optimal solutions established in Section 4.4 allows us to view the the obtained necessary optimality conditions for discrete approximations as *suboptimality conditions* for the original problem (P) governed by the sweeping integro-differential inclusions.

Section 4.7 accumulates the developments of all the previous sections and provides *necessary optimality conditions* for the designated class local minimizers of the *original optimal control problem* (P) for the integro-differential sweeping processes (1) with the mixed state-control constraints (4.4). By using the method of discrete approximations together with the aforementioned tools of variational analysis and generalized differentiation, we derive a comprehensive set of necessary optimality conditions of the following two types: those which extend to the integro-differential systems the recently obtained conditions for the sweeping processes governed by differential inclusions [22], and completely novel ones that are specific for the controlled integro-differential sweeping dynamics.

The final Section 4.8 is devoted to applications of the obtained necessary optimality conditions for integro-differential control problems governed by integro-differential sweeping processes to some real-life models appearing in *non-regular electrical circuits*. We formulate two models of this type and present a realistic example showing that the obtained results allow us to determine and fully compute optimal solutions. The novel Volterra type optimality condition occurs to be especially useful for the provided computations.

Chapter 1

Preliminaries

In this chapter we describe the notation, the definitions and basic results that are going to be used throughout the thesis. The reader is referred to the monographs [66, 67, 41, 32, 33, 49, 71, 78, 84] for a deeper understanding of the tools and standard results exposed in this chapter.

Throughout H is a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. As usual, we will denote by $B_H(x, \delta)$ (resp. $B_H[x, \delta]$) the open (resp. closed) ball around $x \in H$ with radius $\delta > 0$. It will be convenient to write $\mathbb{B}_H$ or $\mathbb{B}$ in place of $B[0, 1]$. When there is no risk of ambiguity, we will remove the subscript H.
For a nonempty subset S of H consider the distance function $d_S(x) := \inf_{y \in S} \|x - y\|$ be the projection operator $\text{Proj}_S : H \rightrightarrows S$ by

$$\text{Proj}_S(x) := \{y \in S : \ d_S(x) = \|x - y\|\}, \ \ x \in H.$$

Further, we called indicator and support function of S that note by $\psi(\cdot, S)$ and $\sigma_S(\cdot)$ respictively, the functions defined by

$$\psi(x, S) = \begin{cases} 0 & \text{if} \ \ x \in S \\ +\infty & \text{if} \ \ x \notin S \end{cases}, \quad \text{and} \ \ \sigma(\xi, S) = \sup_{x \in S} \langle \xi, x \rangle, \ \ \forall \, \xi \in H.$$

By $\mathcal{C}([T_0, T], H)$ we denote the space of continuous mappings from $[T_0, T]$ into H equipped with the supremum norm $\| \cdot \|_\infty$, where we recall that $-\infty < T_0 < T < +\infty$. In various cases, it will be convenient to use the notation $I := [T_0, T]$ and put

$$Q_\Delta := \{(t, s) \in I^2 : \ s \leq t\}.$$

1.1 Normal Cones

A generalization of 'outward normal vector' to general closed sets is presented in the following definition (see [84]):

Definition 1.1.1. *We say that a vector $v \in H$ is a proximal normal vector of S at $x \in S$ if and only if there are reals $\sigma \geq 0$ and $\delta > 0$ such that*

$$\langle v, y - x \rangle \leq \sigma \|y - x\|^2 \quad \text{for all } y \in S \cap B(x, \delta). \tag{1.1}$$

The cone of all proximal normal vectors to S at some point $x \in S$ is called the proximal normal cone, and denoted by $N_S^P(x)$.

Remark 1.1.1. *It is worth noting that, whenever $\text{Proj}_S(y) \neq \emptyset$, one has*

$$y - z \in N_S^P(z) \quad \text{for all } \ z \in \text{Proj}_S(y).$$

Definition 1.1.2. *We say that a vector $v \in H$ is a limiting normal vector of S at $x \in S$ if there exist sequences $x_i \xrightarrow{S} x$ and $v_i \to v$ such that*

$$v_i \in N_S^P(x_i) \ \text{ for all } \ i.$$

The cone of all limiting normal vectors to S at $x \in S$ is denoted by $N_S^L(x)$ and known as the limiting normal cone to S at x.

Recall now the notions of convex sets.

Definition 1.1.3. *A subset $S \subset H$ is called convex if and only if*

$$\forall\, a, b \in S, \ \ \forall\, \lambda \in [0, 1], \ \ \lambda a + (1 - \lambda) b \in S.$$

In other words S is convex if it contains all the line segment of these points.

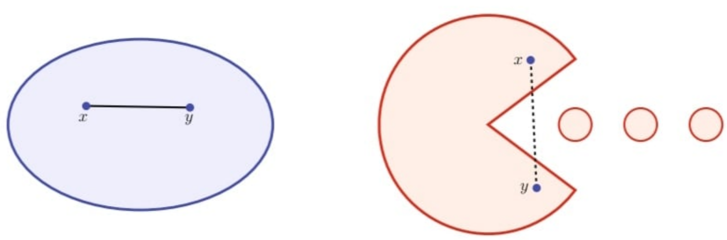

Figure 1.1: Convex and non convex set.

If the set is non convex, we can defined its convex hull as follows:

Definition 1.1.4. *The convex hull of subset $A \subset H$ is intersection of all convex sets containing A. Therfore, is the smallest convex that containing A. and we note $co(A)$.*

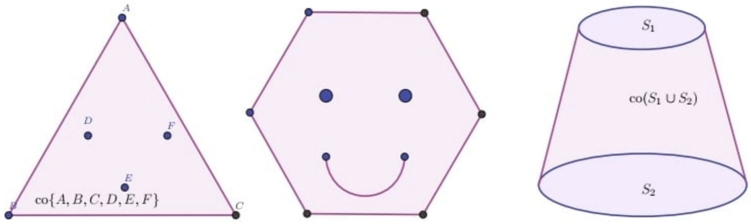

Figure 1.2: Convex hulls in $\mathbb{R}^d$.

In other words we have this characterization

$$co(A) = \left\{ \sum_{i=1}^{n} \beta_i x_i,\ \beta_i \geq 0,\ x_i \in A,\ \sum_{i=1}^{n} \beta_i = 1 \right\}.$$

The definition of closed convex hull is given by

Definition 1.1.5. *The closed convex hull of subset $A \subset H$ is intersection of all closed convex sets containing A. Therfore, is the smallest closed convex that containing A. and we note $\overline{co}(A)$.*

If we deal with convex closed sets S, we recover with Definitions 1.1.1 and 1.1.2 a familiar construction from convex analysis as follows:

Proposition 1.1.1. *Let S be a closed and convex set and $x \in S$, then*

$$N_S^P(x) = N_S^L(x) = \{v \in H : \langle v, y - x \rangle \leq 0,\ \forall\, y \in S\}.$$

In other words the normal cone is the collection of all vectors that not form an acute angle with the vector $\vec{v} = (y - x)$ at the point x.

We list below some properties satisfied by the proximal and the limiting normal cones:

Proposition 1.1.2. *Take a closed subset S of H and a point $x \in S$. Then, the proximal and limiting normal cones have the following properties:*

(a) $N_S^P(x)$ and $N_S^L(x)$ are cones in H, containing $\{0\}$ such that $N_S^P(x) \subset N_S^L(x)$.

(b) If $x \in \operatorname{int}\{S\}$, then $N_S^P(x) = N_S^L(x) = \{0\}$; and if $x \in \operatorname{bdr}\{S\}$, $N_S^L(x)$ has nonzero elements.

(c) $N_S^P(x)$ *is convex (but possibly not closed).*

(d) *The set-valued mapping* $x \rightrightarrows N_S^L(x)$ *has a closed graph, in the sense that, for any sequences* $x_i \xrightarrow{S} x$ *and* $v_i \to v$ *such that* $v_i \in N_S^P(x_i)$ *for all* i*, we have* $v \in N_S^L(x)$*.*

We refer to [84] for a detailed proof of Proposition 1.1.1 and Proposition 1.1.2.

1.2 Subdifferential

In this section, we will give definitions of different types of subdifferentials of non-convex functions and some properties. More details on these definitions can be found in [33, 78, 84] and references therein.

Definition 1.2.1. *Let f be a real function define from H to $\mathbb{R}$. The effective domain of f is the set*

$$\operatorname{dom}(f) = \{x \in H : \ f(x) < \infty\}.$$

Definition 1.2.2. *A function f is said proper if* $\operatorname{dom}(f) \neq \emptyset$*.*

Definition 1.2.3. *Let f be a real function define from H to $\overline{\mathbb{R}}$. The epigraph of f is the set*

$$\operatorname{epi}(f) = \{(x,t) \in H \times \mathbb{R} : \ f(x) \leq t\}.$$

Definition 1.2.4. *The real function f define from H to $\overline{\mathbb{R}}$ is said convex if for all $\beta \in [0;1]$, we have*

$$f(\beta x + (1-\beta)y) \leq \beta f(x) + (1-\beta) f(y), \ \ \forall \beta \in [0,1], \ \ \forall x, y \in \operatorname{dom}(f).$$

Definition 1.2.5. *Let $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function, f is lower semicontinuous at x_0, if for each sequence $(x_k)_{k\in\mathbb{N}}$ in H with $x_k \longrightarrow x_0$ we get*

$$\liminf_{k\to+\infty} f(x_k) \geq f(x_0), \ \text{as}\ k \longrightarrow +\infty.$$

Moreover, f is called lower semicontinuous if it's lower semicontinuous at every point of H.

Definition 1.2.6. *We called that f is upper semicontinuous at x_0 if $-f$ is lower semicontinuous function at x_0. Moreover, recall that f is continuous at the point x_0 if it's lower semicontinuous and upper semicontinuous on x_0.*

Definition 1.2.7. *For a function $f : H \to \mathbb{R} \cup \{+\infty\}$,*

(a) *the proximal subdifferential* $\partial_P f(x)$ *of* f *at* $x \in \operatorname{dom}(f)$ *is defined by*

$$\partial_P f(x) := \{v \in H \ : \ (v, -1) \in N^P(\operatorname{epi} f; (x, f(x)))\}.$$

The set $\partial_P f(x)$ *of all proximal subgradients of* f *at* x *is the proximal subdifferential of* f *at* x.

(b) *the limiting subdifferential* $\partial_L f(x)$ *of* f *at* $x \in \operatorname{dom}(f)$ *is defined by*

$$\partial_L f(x) := \{v \in H \ : \ (v, -1) \in N^L(\operatorname{epi} f; (x, f(x)))\}.$$

The set $\partial_L f(x)$ *of all limiting subgradients of* f *at* x *is the limiting subdifferential of* f *at* x.

Of course, $\partial_P f(x) = \partial_L f(x) = \emptyset$ *if* $f(x) = +\infty$*. Notice that, since* $N_S^P(x) \subset N_S^L(x)$*, we have* $\partial_P f(x) \subset \partial_L f(x)$*.*

We refer to [84] for a detailed proof of the following Proposition.

Proposition 1.2.1. *Let* $f : H \to \mathbb{R} \cup \{+\infty\}$ *be convex proper function, and consider a point* $x \in \operatorname{dom}(f)$*, then*

$$\partial_P f(x) = \partial_L f(x) = \{v \in H : \ \langle v, y - x\rangle \leq f(y) - f(x), \ \ \forall\, y \in H\}.$$

This coincides with the definition of subgradients in the convex analysis sense.

The following figures give the subdifferential of some functions.

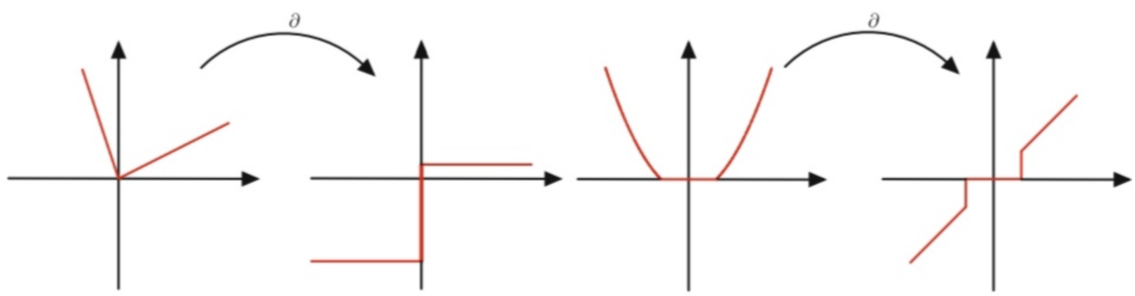

Figure 1.3: Some functions and their subdifferential.

Definition 1.2.8. *Let a function* $f : H \to \mathbb{R} \cup \{+\infty\}$*, and consider a point* $x \in \operatorname{dom}(f)$*,*

(a) *A vector* $v \in H$ *is a proximal subgradient of* f *at* x*, if there exist some reals* $\sigma \geq 0$ *and* $\delta > 0$ *such that*

$$\langle v, y - x\rangle \leq f(y) - f(x) + \sigma\|y - x\|^2 \quad \textit{for all } y \in B(x, \delta).$$

(b) A vector $v \in H$ is a limiting subgradient of f at x, if there exist sequences $x_i \xrightarrow{f} x$ and $v_i \to v$ such that $v_i \in \partial_P f(x_i)$ for all i.

Remark 1.2.1.

(a) There is another link between the proximal (resp., limiting) normal cone and the proximal (resp., limiting) subdifferential, given by $\partial_P \psi_S(x) = N_S^P(x)$ (resp., $\partial_L \psi_S(x) = N_S^L(x)$).

(b) The proximal normal cone is also connected with the distance function to S through the equalities

$$\partial_P d_S(x) = N_S^P(x) \cap \mathbb{B}_H \quad \textit{and} \quad N_S^P(x) = \mathbb{R}_+ \partial_P d_S(x), \textit{ for all } x \in S. \tag{1.2}$$

1.3 Prox-regularity

In this section we give the definition and some property of some classe of set, which generalize the class of convex sets.
We start with a very useful characterization of proximal normal cone. That is, the proximal normal cone can be described in the following geometrical way (see, e.g., [33])

$$N_S^P(x) = \{v \in H : \exists r > 0 \text{ such that } x \in \text{Proj}_S(x + rv)\}, \text{ if } x \in S, \tag{1.3}$$

with $N_S^P(x) = \emptyset$ if $x \notin S$; see Figure 1.4

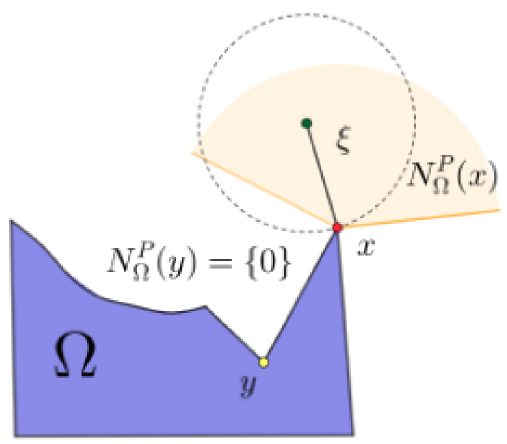

Figure 1.4: Proximal normal cones.

The proximal normal cone is the right concept to use for defining the prox-regularity of a set S by requiring in (1.3) that the constant r be uniform for all the unit proximal normal vectors of S. The sets which satisfy that property are known as (uniformly) prox-regular sets.

Definition 1.3.1. *[67] Given $r \in]0, +\infty]$, the closed subset S is r-*prox-regular *provided that, for every $x \in S$, every unit vector $v \in N_S^P(x)$ with $\|v\| \leq 1$ and every real $t \in]0, r]$ one has $x \in \mathrm{Proj}_S(x + tv)$.*
*Equivalently, S is r-*prox-regular *if for all $x \in S$ and $v \in N_S^P(x)$ with $\|v\| \leq 1$, we have*

$$S \cap B_H(x + rv, r) = \emptyset.$$

In other words, S is r-prox-regular if any external ball with radius smaller than r can be rolled around it; see Figure 1.5

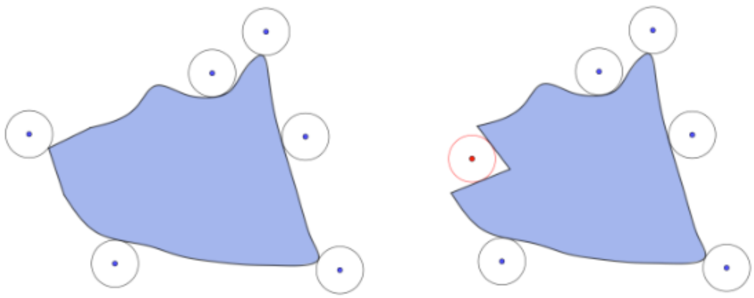

Figure 1.5: Uniform prox-regular set and non uniform prox-regular set.

The next propositions summarize some basic properties of prox-regularity needed in the thesis. For the proof of these results, we refer the reader to [35, 67, 78].

Proposition 1.3.1. *For a given $r \in]0, \infty]$, a closed subset S of the Hilbert space H is uniformly r-prox-regular, or r-prox-regular for short, if and only if for all $x \in S$ and all $0 \neq v \in N_S^P(x)$ one has*

$$\left\langle \frac{v}{\|v\|}, y - x \right\rangle \leq \frac{\|y - x\|^2}{2r}, \quad \forall y \in S. \tag{1.4}$$

Of course, in the latter inequality, $\frac{1}{r} = 0$ for $r = +\infty$ (as usual). It is worth pointing out that for $r = +\infty$, the r-prox-regularity of the closed set S amounts to its convexity.

Proposition 1.3.2. *Let S be a nonempty closed set in H which is uniformly r-prox-regular for some $r \in]0, +\infty]$. Then for any $x_i \in S$, $v_i \in N_S^P(x_i)$ with $i = 1, 2$ one has:*

$$\langle v_2 - v_1, x_2 - x_1 \rangle \geq -\frac{1}{2}\left(\frac{\|v_2\| + \|v_1\|}{r}\right)\|x_2 - x_1\|^2.$$

Proposition 1.3.3. *Let S be a nonempty closed subset in H and let $r \in]0, \infty]$. If the subset S is uniformly r-prox-regular, then the following hold:*

(a) The proximal and limiting normal cones of S coincide at any point in S.

(b) *For all $x \in H$ with $d_S(x) < r$, $\mathrm{Proj}_S(x)$ is nonempty and is a singleton set .*

(c) *The Clarke and the proximal subdifferentials of d_S coincide at all points $x \in H$ with $d_S(x) < r$.*

Remark 1.3.1. *The assertion (a) in Proposition 1.3.3 leads us to put*

$$N_S(x) := N_S^L(x) = N_S^P(x), \tag{1.5}$$

whenever the set S is r-prox-regular.

Proposition 1.3.4. *Let S be a nonempty closed subset in H and let $r \in]0, \infty]$. If the subset S is uniformly r-prox-regular, then the following hold:*

(a) *For any $x \in S$ and any $v \in \partial^P d_S(x)$ one has for any $y \in H$ such that $d_S(y) < r$*

$$\langle v, y - x \rangle \leq \frac{2}{r}\|y - x\|^2 + d_S(y).$$

(b) *For any $x \in H$ with $d_S(x) < r$, the proximal subdifferential $\partial^P d_S(x)$ is a nonempty closed convex subset in H.*

1.4 Hausdorff-Pompeiu distance

Let S, S' be two closed subset of H. We define the *excess* of S over S' as

$$\mathrm{exc}(S, S') := \sup_{x \in S} d_{S'}(x),$$

the excess may well be $+\infty$ (for example, this will occur if S is bounded and S' is unbounded). It is not difficult to prove that

$$\mathrm{exc}(S, S') := \sup_{x \in H}(d_S(x) - d_{S'}(x)).$$

The *Hausdorff-Pompeiu distance* between S and S' is defined by:

$$\mathrm{haus}(S, S') := \max\{\mathrm{exc}(S, S'), \mathrm{exc}(S', S)\}.$$

Furthermore, the Hausdorff distance between S and S' as the uniform distance between $d_S(\cdot)$ and $d_{S'}(\cdot)$ i.e.,

$$\mathrm{haus}(S, S') := \sup_{x \in H} |d_S(x) - d_{S'}(x)|,$$

Let $\rho \in]0, +\infty]$ be an extended real. The *ρ-pseudo excess* of S over S' (also called the pseudo excess of the ρ-truncation of S over S') is defined as the extended real

$$\mathrm{exc}_\rho(S, S') := \sup_{x \in S \cap \rho\mathbb{B}} d_{S'}(x),$$

where by convention $\rho\mathbb{B} = H$ if $\rho = +\infty$. The *Hausdorff-Pompeiu ρ-pseudo distance* between S and S' is then defined as:

$$\mathrm{haus}_\rho(S, S') := \max\{\mathrm{exc}_\rho(S, S'), \mathrm{exc}_\rho(S', S)\}. \tag{1.6}$$

If $\rho = +\infty$, we see that the ρ-pseudo excess of S over S' (resp., the Hausdorff-Pompeiu ρ-pseudo distance between S and S') is the usual excess of S over S' (resp., the usual Hausdorff-Pompeiu distance between S and S'), i.e.,

$$\mathrm{exc}_\infty(S, S') := \sup_{x \in S} d_{S'}(x) := \mathrm{exc}(S, S')$$

$$(\text{resp., } \mathrm{haus}_\infty(S, S') := \max\{\mathrm{exc}(S, S'), \mathrm{exc}(S', S)\} =: \mathrm{haus}(S, S')).$$

It is easily seen that, for every real $\alpha > 0$ such that $\mathrm{exc}_\rho(S, S') < \alpha$, one has

$$S \cap \rho\mathbb{B} \subset S' + \alpha\mathbb{B}.$$

It is also readily seen that

$$d_{S'}(x') \leq \|x - x'\| + \mathrm{exc}_\rho(S, S') \text{ for all } x \in S \cap \rho\mathbb{B}, x' \in H. \tag{1.7}$$

1.5 Some useful results

In this section we give some useful results that will be used in the following chapters.

Definition 1.5.1. *Let a function $x : [T_0, T] \longrightarrow H$, a subinterval $J \subset [T_0, T]$, we define the variation of u on I by the following expression*

$$var(x, J) := \sup\left\{\sum_{i=1}^{n} ||x(t_i) - x(t_{i-1})||,\ n \in \mathbb{N},\ t_i \in J,\ t_0 < t_1 < ... < t_n\right\}.$$

We said that x has a bounded variation on the interval $[T_0, T]$ if $var(x, J) < +\infty$.

Definition 1.5.2. *A function $f : [a; b] \to H$ is said to be absolutely continuous if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $]a_n; b_n[$ are pairwise disjoint subintervals of $[a; b]$*

$$\sum_{n \geq 0}(b_n - a_n) < \delta \Rightarrow \sum_{n \geq 0} \|f(a_n) - f(b_n)\| < \varepsilon.$$

Moreover, The function $f : [a,b] \longrightarrow H$ *is absolutely continuous if and only if*

$$f(b) - f(a) = \int_a^b f'(s)ds.$$

Any absolutely continuous function f *is continuous.*

Lemma 1.5.1. *[47, Lemma 4] Let* $x : [T_0, T] \longrightarrow H$ *be an absolutely continuous function. Then*

- $\frac{1}{2}\left(\frac{d}{dt}||x(t)||^2\right) = \langle \dot{x}(t), x(t)\rangle.$
- $\int_{T_0}^{T} \langle \dot{x}(t), x(t)\rangle = \frac{1}{2}||x(T)||^2 - \frac{1}{2}||x(T_0)||^2.$

We denote by $L^1([T_0,T];H)$ the space of Bochner integrable functions defined over $[T_0,T]$ with respect to the Lebesgue measure. A set $K \subseteq L^1([T_0,T];H)$ is uniformly integrable if there exists $g \in L^1([T_0,T];H)$ such that for all $f \in K$

$$\|f(t)\| \leq g(t) \ \text{ a.e. } \ t \in [T_0,T].$$

Now, we recall the Dunford-Pettis theorem (see [42, Theorem 2.3.24]), which characterizes relatively weakly compact subsets of L^1.

Theorem 1.5.1 (Dunford-Pettis theorem)**.** *A bounded set* $K \subseteq L^1([T_0,T];H)$ *is relatively weakly compact in* $L^1([T_0,T];H)$ *if and only if it is uniformly integrable.*

We recall the classical Arzela-Ascoli theorem (see [42, Theorem 2.3.2]), which characterizes the relatively compact subsets of $\mathcal{C}([T_0,T],H)$.

Theorem 1.5.2 (Arzela-Ascoli)**.** *A set* $K \subset \mathcal{C}([T_0,T];H)$ *is relatively compact if and only if*

1. *for every* $t \in [T_0,T]$*, the set* $K(t) := \{u(t) : u \in K\}$ *is relatively compact in* H*.*
2. K *is uniformly equicontinuous, i.e.,for every* $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$*, such that, if* $t,s \in [T_0,T]$ *and* $|t-s| \leq \delta$*, then*

$$||u(t) - u(s)|| < \varepsilon, \ \forall\, u \in K.$$

The next lemma shows that any weakly convergent sequence in a normed space has a sequence of convex combinations of its members that converges strongly to the same limit (see [19, p. 61]).

Lemma 1.5.2 (Mazur lemma). *Let X be a Banach space, assume that $(x_n)_n$ converges weakly to x on X. Then there exists a sequence $(y_n)_n$ made up of the convex combination of $(x_k)_{k\geq n}$ (i.e., $y_n \in co\{x_k : \ k \geq n\}$) that 's converges strongly to x on X.*

The following Proposition, proved in [64, Proposition 3.2], is a scalar upper semicontinuity property for prox-regular sets.

Proposition 1.5.1. *Let $C : I = [T_0, T] \rightrightarrows H$ be an r-prox-regular valued multimapping for some $r \in]0, +\infty]$. Assume that there exist a positive measure μ on I and $\rho \in]0, +\infty]$ such that for all $s, t \in I$ with $s \leq t$,*

$$\mathrm{exc}_\rho(C(s), C(t)) \leq \mu(]s, t]).$$

Let $\bar{t} \in I$, $\bar{x} \in C(\bar{t}) \cap \rho\mathbb{U}$, $(t_n)_{n\in\mathbb{N}}$ be a sequence of $[\bar{t}, T]$ with $\mu(]\bar{t}, t_n]) \to 0$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence of H with $x_n \to \bar{x}$ and $x_n \in C(t_n)$ for all $n \in \mathbb{N}$. Then, one has

$$\limsup_{n\to+\infty} \sigma(-\partial\, d_{C(t_n)}(x_n), h) \leq \sigma(-\partial\, d_{C(\bar{t})}(\bar{x}), h), \quad \text{for all } \ h \in H.$$

1.6 Control Systems and Differential Inclusions

Differential equations first came into existence with the invention of calculus by Newton and Leibniz in the 17th century. A differential equation to which we associate an initial condition (known as the Cauchy problem)

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0,$$

is a relation between a state x and its rate of change $\dot{x} = \dfrac{dx}{dt}$ dt . It models the evolution of a system and permits to predict its future evolution without changing its behavior. For instance, we can exactly predict time and locations of eclipses but we cannot modify them. A control system is, however, a differential equation involving an external agent, called 'controller', who will affect the evolution of the system. This situation is modeled by the control system below. Namely,

$$\dot{x} = f(t, x, u), \quad u(\cdot) \in \mathcal{U}, \tag{1.8}$$

where $\mathcal{U}$ is a family of admissible control functions defined as

$$\mathcal{U} := \{u : \mathbb{R} \to \mathbb{R}^m; \ u(\cdot) \ \text{ measurable}, \ \ u(t) \in U(t) \ \text{ for a.e. } t\}, \tag{1.9}$$

for a given nonempty multifunction $U(t)$ such that $U(t) \subset \mathbb{R}^m$. In this case, the rate of change $\dot{x}(t)$ depends not only on the state x itself, but also on some external parameters, say $u = (u_1, u_2, ..., u_m)$, which can also vary in time or space. The control function $u(\cdot)$, subject to some constraints, will be chosen by a controller in order to manage, command or regulate the behavior of the system and achieve certain predefined goals, for instance steer the system from one state to another, maximize the terminal value of one of the parameters, minimize or maximize a certain cost functional, etc. We distinguish two types of controls: the time-variable control $t \to u(t)$ and the space-variable control $x \to u(x)$. The first is known as an open loop control while the second is a closed loop control or feedback. In an open loop control system, the control action from the controller is independent of the 'process output'. A good example of this is a central heating boiler controlled only by a timer, so that heat is applied for a constant time, regardless of the temperature of the building. (The control action is the switching on/off of the boiler. The process output is the building temperature). In a closed loop control system, the control action from the controller depends on the process output. Considering the boiler, this would include a temperature thermostat to regulate the building temperature, and thereby feed back a signal to ensure that the controller maintains the temperature set on the thermostat. An open loop control is easier to implement since the only information needed is provided by a clock to measure time. In this work, we are interested in control systems involving time-variable controls.

The dynamics can also be represented as a differential inclusion which is a generalization of the concept of ordinary differential equation:

$$\dot{x} \in F(t,x), \tag{1.10}$$

where the set of velocities is given by

$$F(t,x) := \{y \,|\, y = f(t,x,u), \;\; \text{for some} \;\; u \in U(t)\},$$

and F is a set-valued map, i.e. $F(t,x)$ is a set rather than a single point in $\mathbb{R}^n$.

It is clear that every trajectory for the control system (1.8) is also a solution for the differential inclusion (1.10). The converse is also true under some regularity assumptions on f.

Once these two types of dynamics are defined, we are ready to state optimal control problems which concern the properties of control functions that, when inserted into a differential equation, give solutions which minimize or maximize a certain 'cost' (for the case of control systems) and the properties of state trajectories and the set of velocities F achieving some minimum or maximum 'cost' (for the case of differential inclusions). Let $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be real-valued a cost function. We consider the optimal control problem involving a control system

$$(CSP): \begin{cases} \text{minimize} \quad g(x(S), x(T)) \\ \text{over arcs } \; x(\cdot) \in W^{1,1} \; \text{ and measurable functions } \; u(\cdot) \; \; satisfying \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } \; t \in [S, T], \\ (x(S), x(T)) \in C, \\ u(t) \in U(t) \quad \text{a.e. } \; t \in [S, T]. \end{cases}$$

The data for problem (CSP) involve a closed set C, a set-valued map $t \rightrightarrows U(t) \subset \mathbb{R}^m$ and functions $f : [S,T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : [S,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. We call process a couple $(x(\cdot), u(\cdot))$ such that $u(\cdot)$ is a Lebesgue measurable function satisfying $u(t) \in U(t)$ a.e. $t \in [S,T]$ and $x(\cdot)$ is the solution of the ordinary differential equation

$$\dot{x}(t) = f(t, x(t), u(t)), \quad \text{a.e. } \; t \in [S, T].$$

The process $(x(\cdot), u(\cdot))$ is called feasible if in addition $(x(S), x(T)) \in C$. We say that the process $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a D-local minimizer for (CSP) if, for a given $\varepsilon > 0$

$$g(\bar{x}(S), \bar{x}(T)) \leq g(x(S), x(T)),$$

for every feasible trajectory $(x(\cdot), u(\cdot))$ such that

$$\|x(\cdot) - \bar{x}(\cdot)\|_D \leq \varepsilon.$$

The process is called strong local minimizer when $D = L^\infty([S,T], \mathbb{R}^n)$, and weak local minimizer when $D = W^{1,1}([S,T], \mathbb{R}^n)$, which corresponds to the set of absolutely continuous functions. In some circumstances, we shall emphasize the dependence on the minimizer of " and we would refer to it as a D local "ε-minimizer. Since the set of absolutely continuous functions is larger than the set of L^∞ functions, the $W^{1,1}$-norm is stronger than the L^∞-norm. It follows that the $W^{1,1}$-local minimizers would provide a sharper analysis on the local nature of the optimality conditions than would be the case with L^∞-local minimizers.

An optimal control problem formulated in terms of a differential inclusion is defined as follow

$$(DIP): \begin{cases} \text{minimize} \quad g(x(S), x(T)) \\ \text{over arcs } \; x(\cdot) \in W^{1,1} \; \; satisfying \\ \dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } \; t \in [S, T], \\ (x(S), x(T)) \in C, \end{cases}$$

where $F : [S,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map. A trajectory $x(\cdot)$ which solves the differential inclusion $\dot{x} \in F(t, x)$ is called an F-trajectory.

The next theorem provides conditions for which an optimal control problem (in terms of differential inclusion dynamics) has a minimizer.

Theorem 1.6.1. *[84] Consider the problem (DIP). Assume that*

(a) the multifunction $F : [0,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has closed and nonempty values, $F(\cdot,\cdot)$ is $\mathcal{L} \times \mathcal{B}^n$ and the graph of $F(t,\cdot)$ is closed for a.e. $t \in [S,T]$;

(b) there exist $\alpha \in L^1$ and $\beta \in L^1$ such that

$$F(t,x) \subset (\alpha(t)|x| + \beta(t))\mathbb{B}, \quad \text{for all } (t,x);$$

(c) C is closed and g is a given lower semicontinuous function;

(d) one of these following sets is bounded:

$$C_0 := \{x_0 \in \mathbb{R}^n : (x_0,x_1) \in C \quad \text{for some } x_1 \in \mathbb{R}^n\}.$$

$$C_1 := \{x_1 \in \mathbb{R}^n : (x_0,x_1) \in C \quad \text{for some } x_0 \in \mathbb{R}^n\}.$$

(e) the set of feasible F-trajectories $\{x : \dot{x}(t) \in F(t,x(t)), \quad \text{a.e. } t, \quad \text{and } (x(S),x(T)) \in C\}$ is nonempty;

(f) $F(t,x)$ is convex for each (t,x).

Then (DIP) has a minimizer.

Remark 1.6.1. *When F has no convex values,*

$$\dot{x}(t) \in F(t,x(t)), \quad \text{a.e. } t \in [S,T],$$

will be replaced with

$$\dot{x}(t) \in \operatorname{co} F(t,x(t)), \quad \text{a.e. } t \in [S,T].$$

We end this section by the following (non-dynamic) problem of mathematical programming (MP) with operator, inequality, and geometric constraints to which we can reduce our discrete-time problems of dynamic optimization:

$$(MP) : \begin{cases} \text{minimize} \quad \varphi_0(z) \quad \text{subject to} \\ \varphi_j(z) \le 0, \quad j = 1,...,s, \\ f(z) = 0, \\ z \in E_j \subset Z, \quad j = 1,...,l, \end{cases} \tag{1.11}$$

where φ_j are real-valued functions on Z, where $f : Z \to E$ is a mapping between Banach spaces, and where $E_j \subset Z$. Note that problem (MP) is intrinsically nonsmooth, even in the case of the smooth data f and φ_j in (1.11) and in the generating dynamic problems. Now we derive the necessary optimality conditions for problem (MP) with many geometric constraints.

Proposition 1.6.1. *[52, Proposition 6.16] Let $\bar{z}$ be a local optimal solution to problem* (1.11)*, where the spaces Z and E are Asplund and where the sets E_j are locally closed around $\bar{z}$. Assume also that all φ_i are Lipschitz continuous around $\bar{z}$, that f is generalized Fredholm at $\bar{z}$, and that each E_j is* sequential normal compactness *(SNC) at this point. Then there are real numbers $\{\mu_j \in \mathbb{R} \mid j = 0, ..., s\}$ as well as linear functionals $e^* \in E^*$ and $\{z_j^* \in Z^* \mid j = 1, ..., l\}$, not all zero, such that $\mu_j \geq 0$ for $j = 0, .., s$ and*

$$\mu_j \varphi_j(\bar{z}) = 0, \quad \textit{for } \; j = 1, ..., s,$$

$$z_j^* \in N(\bar{z}, E_j), \quad \textit{for } \; j = 1, ..., l,$$

$$-\sum_{j=1}^{l} z_j^* \in \partial \left(\sum_{j=0}^{s} \mu_j \varphi_j \right) (\bar{z}) + D_N^* f(\bar{z})(e^*).$$

Chapter 2

Nonconvex Integro-Differential Sweeping Process with Applications

Abstract. In this chapter, we analyze and discuss the well-posedness of a new variant of the so-called sweeping process, introduced by J.J. Moreau in the early 70's with motivation in plasticity theory. In this variant, the normal cone to the (mildly non-convex) prox-regular moving set $C(t)$ is supposed to have an absolutely continuous variation, is perturbed by a sum of a Carathéodory mapping and an integral forcing term. The integrand of the forcing term depends on two time-variables, that is, we study a general integro-differential sweeping process of Volterra type. By setting up an appropriate semi-discretization method combined with a new Gronwall-like inequality (differential inequality), we show that the integro-differential sweeping process has one and only one absolutely continuous solution. We also establish the continuity of the solution with respect to the initial value. The results of the chapter are applied to the study of nonlinear integro-differential complementarity systems which are combination of Volterra integro-differential equations with nonlinear complementarity constraints. A second application is concerned with non-regular electrical circuits containing time-varying capacitors and nonsmooth electronic device like diodes. A circuit with transmission line, diode and inductor is also analyzed. Another application to a frictionless mechanical problem is also provided. All these applications represent an additional novelty of our work.

2.1 Gronwall-like differential inequality

We start this section with the following continuous Gronwall's inequality [73, Lemma 4.1, p. 179].

Lemma 2.1.1 (Gronwall's inequality). *Let $T > T_0$ be given reals and $a(\cdot), b(\cdot) \in L^1([T_0,T];\mathbb{R})$ with $b(t) \geq 0$ for almost all $t \in [T_0,T]$. Let the absolutely continuous function $w : [T_0,T] \longrightarrow \mathbb{R}_+$ satisfy*

$$(1-\alpha)w'(t) \leq a(t)w(t) + b(t)w^\alpha(t), \quad a.e.\ t \in [T_0,T],$$

where $0 \leq \alpha < 1$. Then for all $t \in [T_0,T]$, one has

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(T_0)\exp(\int_{T_0}^{t} a(\tau)d\tau) + \int_{T_0}^{t} \exp(\int_{s}^{t} a(\tau)d\tau)b(s)ds.$$

We will need the following lemma which is a straightforward consequence of Gronwall's lemma.

Lemma 2.1.2. *Let $\rho : [T_0,T] \longrightarrow \mathbb{R}$ be a nonnegative absolutely continuous function and let $b_1, b_2, a : [T_0,T] \longrightarrow \mathbb{R}_+$ be non-negative Lebesgue integrable functions. Assume that*

$$\dot\rho(t) \leq a(t) + b_1(t)\rho(t) + b_2(t)\int_{T_0}^{t} \rho(s)\,ds, \quad a.e.\ t \in [T_0,T]. \tag{2.1}$$

Then for all $t \in [T_0,T]$, one has

$$\rho(t) \leq \rho(T_0)\,\exp\left(\int_{T_0}^{t} (b(\tau)+1)\,d\tau\right) + \int_{T_0}^{t} a(s)\,\exp\left(\int_{s}^{t} (b(\tau)+1)\,d\tau\right)ds,$$

where $b(t) := \max\{b_1(t), b_2(t)\}$, a.e. $t \in [T_0,T]$.

Proof. Put $b(t) = \max\{b_1(t), b_2(t)\}$, a.e. $t \in [T_0,T]$. Setting $z(t) = \int_{T_0}^{t} \rho(s)\,ds$ we have $\dot z(t) = \rho(t)$ for all $t \in [T_0,T]$, and $\ddot z(t) = \dot\rho(t)$ for a.e. $t \in [T_0,T]$. Then from (2.1) we see that

$$\ddot z(t) \leq a(t) + b_1(t)\dot z(t) + b_2(t)z(t).$$

Putting $w(t) = \dot z(t) + z(t)$ we have for a.e. $t \in [T_0,T]$

$$\dot w(t) = \ddot z(t) + \dot z(t) \quad \text{and} \quad \dot w(t) \leq a(t) + (b(t)+1)w(t).$$

Applying the Gronwall Lemma 2.1.1 with w, one obtains for all $t \in [T_0,T]$

$$w(t) \leq w(T_0)\exp\big(\int_{T_0}^{t} (b(\tau)+1)\,d\tau\big) + \int_{T_0}^{t} a(s)\,\exp\big(\int_{s}^{t} (b(\tau)+1)\,d\tau\big)\,ds,$$

which gives

$$\rho(t) \leq \dot z(t) + z(t) = w(t) \leq \rho(T_0)\,\exp\left(\int_{T_0}^{t} (b(\tau)+1)\,d\tau\right) + \int_{T_0}^{t} a(s)\,\exp\left(\int_{s}^{t} (b(\tau)+1)\,d\tau\right)ds.$$

∎

We establish now the following new Gronwall-like lemma. A lemma of this type has been previously proved by G. Colombo and C. Kozaily [34]. The lemma in [34] is a little different from Lemma 2.1.3 below. The arguments for the Cauchy property of the sequence of approximate solutions in Theorem 2.2.1 of the next section require the general form of Lemma 2.1.3 below because of the form of the integral $\int_{T_0}^{t} f_2(t,s,x(s))\,ds$ present in the sweeping process (P_{f_1,f_2}).

Lemma 2.1.3 (Gronwall-like differential inequality). *Let $\rho : [T_0,T] \longrightarrow \mathbb{R}$ be a non-negative absolutely continuous function and let $K_1, K_2, \varepsilon : [T_0,T] \longrightarrow \mathbb{R}_+$ be non-negative Lebesgue integrable functions. Suppose for some $\epsilon > 0$*

$$\dot\rho(t) \le \varepsilon(t) + \epsilon + K_1(t)\rho(t) + K_2(t)\sqrt{\rho(t)}\int_{T_0}^{t}\sqrt{\rho(s)}\,ds, \quad a.e.\ t\in[T_0,T]. \tag{2.2}$$

Then for all $t \in [T_0,T]$, one has

$$\begin{aligned}\sqrt{\rho(t)} \le{}& \sqrt{\rho(T_0)+\epsilon}\,\exp\left(\int_{T_0}^{t}(K(s)+1)\,ds\right) + \frac{\sqrt{\epsilon}}{2}\int_{T_0}^{t}\exp\left(\int_{s}^{t}(K(\tau)+1)\,d\tau\right)ds\\ &+2\left(\sqrt{\int_{T_0}^{t}\varepsilon(s)\,ds+\epsilon}-\sqrt{\epsilon}\,\exp\left(\int_{T_0}^{t}(K(\tau)+1)\,d\tau\right)\right)\\ &+2\int_{T_0}^{t}(K(s)+1)\exp\left(\int_{s}^{t}(K(\tau)+1)\,d\tau\right)\sqrt{\int_{T_0}^{s}\varepsilon(\tau)\,d\tau+\epsilon}\;ds,\end{aligned}$$

where $K(t) := \max\left\{\dfrac{K_1(t)}{2}, \dfrac{K_2(t)}{2}\right\}$ for $t\in[T_0,T]$.

Proof. Set $\lambda(t) = \sqrt{\int_{T_0}^{t}\varepsilon(s)\,ds+\epsilon}$ and $z_\varepsilon(t) = \sqrt{\rho(t)+\lambda^2(t)}$ for all $t\in[T_0,T]$.

From (2.2) we have for a.e. $t\in[T_0,T]$

$$\dot\rho(t) \le \varepsilon(t)+\epsilon+K_1(t)(\rho(t)+\lambda^2(t)) + K_2(t)\sqrt{\rho(t)+\lambda^2(t)}\int_{T_0}^{t}\sqrt{\rho(s)+\lambda^2(s)}\,ds \tag{2.3}$$

and

$$\dot z_\varepsilon(t) = \frac{\dot\rho(t)+2\dot\lambda(t)\lambda(t)}{2\sqrt{\rho(t)+\lambda^2(t)}} = \frac{\dot\rho(t)+\varepsilon(t)}{2z_\varepsilon(t)}, \quad \text{or equivalently } \dot\rho(t) = 2z_\varepsilon(t)\dot z_\varepsilon(t)-\varepsilon(t),$$

hence from (2.3)

$$2z_\varepsilon(t)\dot z_\varepsilon(t) \le 2\varepsilon(t)+\epsilon+K_1(t)z_\varepsilon(t)^2 + K_2(t)z_\varepsilon(t)\int_{T_0}^{t}z_\varepsilon(s)\,ds.$$

Therefore, for a.e. $t \in [T_0, T]$ we have

$$\dot{z}_\varepsilon(t) \leq \frac{\varepsilon(t)}{z_\varepsilon(t)} + \frac{\epsilon}{2z_\varepsilon(t)} + \frac{K_1(t)}{2} z_\varepsilon(t) + \frac{K_2(t)}{2} \int_{T_0}^{t} z_\varepsilon(s)\, ds. \tag{2.4}$$

We claim that

$$\dot{z}_\varepsilon(t) \leq 2\dot{\lambda}(t) + \frac{\sqrt{\epsilon}}{2} + \frac{1}{2} \left(K_1(t) z_\varepsilon(t) + K_2(t) \int_{T_0}^{t} z_\varepsilon(s)\, ds \right). \tag{2.5}$$

To argue the latter inequality we note first that

$$\lambda(t) = \sqrt{\int_{T_0}^{t} \varepsilon(s)\, ds + \epsilon} \leq \sqrt{\rho(t) + \int_{T_0}^{t} \varepsilon(s)\, ds + \epsilon} = \sqrt{\rho(t) + \lambda^2(t)} = z_\varepsilon(t),$$

then

$$\frac{1}{z_\varepsilon(t)} \leq \frac{1}{\lambda(t)}, \quad \text{or equivalently } \frac{\varepsilon(t)}{z_\varepsilon(t)} \leq \frac{\varepsilon(t)}{\lambda(t)}.$$

Also we have $\dot{\lambda}(t) = \dfrac{\varepsilon(t)}{2\lambda(t)}$. Then $\dfrac{\varepsilon(t)}{z_\varepsilon(t)} \leq 2\dot{\lambda}(t)$, and $\sqrt{\epsilon} \leq \sqrt{\epsilon + \int_{T_0}^{t} \varepsilon(s)\, ds} = \lambda(t) \leq z_\varepsilon(t)$, hence $\dfrac{\epsilon}{2z_\varepsilon(t)} \leq \dfrac{\sqrt{\epsilon}}{2}$. Altogether and (2.4) yield (2.5) as desired.

Letting $K(t) := \max\left\{ \dfrac{K_1(t)}{2}, \dfrac{K_2(t)}{2} \right\}$ and applying the Gronwall Lemma 2.1.2 with z_ε, one obtains for all $t \in [T_0, T]$

$$\begin{aligned} z_\varepsilon(t) &\leq z_\varepsilon(T_0) \exp\big(\int_{T_0}^{t} (K(\tau) + 1)\, d\tau \big) + \int_{T_0}^{t} \exp\big(\int_{s}^{t} (K(\tau) + 1)\, d\tau \big) (2\dot{\lambda}(s) + \frac{\sqrt{\epsilon}}{2})\, ds \\ &= \sqrt{\rho(T_0) + \epsilon}\, \exp(\int_{T_0}^{t} (K(s) + 1)\, ds) + \int_{T_0}^{t} \exp\big(\int_{s}^{t} (K(\tau) + 1)\, d\tau \big) (2\dot{\lambda}(s) + \frac{\sqrt{\epsilon}}{2})\, ds, \end{aligned}$$

or equivalently

$$\begin{aligned} z_\varepsilon(t) \leq{}& \sqrt{\rho(T_0) + \epsilon}\, \exp\big(\int_{T_0}^{t} (K(s) + 1)\, ds \big) + 2 \int_{T_0}^{t} \exp\big(\int_{s}^{t} (K(\tau) + 1)\, d\tau \big) \dot{\lambda}(s)\, ds \\ &+ \frac{\sqrt{\epsilon}}{2} \int_{T_0}^{t} \exp\big(\int_{s}^{t} (K(\tau) + 1)\, d\tau \big)\, ds. \end{aligned}$$

On the other hand, from integration by parts, we note that

$$\int_{T_0}^{t} \exp\Big(\int_{s}^{t}(K(\tau)+1)\,d\tau\Big)\dot{\lambda}(s)\,ds$$

$$= [\exp\Big(\int_{s}^{t}(K(\tau)+1)\,d\tau\Big)\lambda(s)]_{T_0}^{t} + \int_{T_0}^{t}(K(s)+1)\exp\Big(\int_{s}^{t}(K(\tau)+1)\,d\tau\Big)\lambda(s)\,ds$$

$$= \lambda(t) - \exp\Big(\int_{T_0}^{t}(K(\tau)+1)\,d\tau\Big)\sqrt{\epsilon} + \int_{T_0}^{t}(K(s)+1)\exp\Big(\int_{s}^{t}(K(\tau)+1)\,d\tau\Big)\lambda(s)\,ds,$$

which combined with what precedes gives

$$z_\varepsilon(t) \le \sqrt{\rho(T_0)+\epsilon}\,\exp\Big(\int_{T_0}^{t}(K(s)+1)\,ds\Big) + \frac{\sqrt{\epsilon}}{2}\int_{T_0}^{t}\exp\Big(\int_{s}^{t}(K(\tau)+1)\,d\tau\Big)\,ds$$

$$+\,2\lambda(t) - 2\exp\Big(\int_{T_0}^{t}(K(\tau)+1)\,d\tau\Big)\sqrt{\epsilon} + 2\int_{T_0}^{t}(K(s)+1)\exp\Big(\int_{s}^{t}(K(\tau)+1)\,d\tau\Big)\lambda(s)\,ds.$$

Consequently, observing that $\sqrt{\rho(t)} \le \sqrt{\rho(t)+\lambda^2(t)} = z_\varepsilon(t)$ we obtain

$$\sqrt{\rho(t)} \le \sqrt{\rho(T_0)+\epsilon}\,\exp\Big(\int_{T_0}^{t}(K(s)+1)\,ds\Big) + \frac{\sqrt{\epsilon}}{2}\int_{T_0}^{t}\exp\Big(\int_{s}^{t}(K(\tau)+1)\,d\tau\Big)\,ds + 2\lambda(t)$$

$$-\,2\exp\Big(\int_{T_0}^{t}(K(\tau)+1)\,d\tau\Big)\sqrt{\epsilon} + 2\int_{T_0}^{t}(K(s)+1)\exp\Big(\int_{s}^{t}(K(\tau)+1)\,d\tau\Big)\lambda(s)\,ds,$$

which completes the proof of the lemma. ■

2.2 Existence result for the integro-differential sweeping process

In this section, we give and prove our main results in the study of the integro-differential sweeping process

$$(P_{f_1,f_2}) : \begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t,x(t)) + \displaystyle\int_{T_0}^{t} f_2(t,s,x(s))ds, & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0 \in C(T_0). \end{cases}$$

They concern the existence, uniqueness, and continuous dependence of the solution with respect to the initial data. We will have to use the following assumptions:

$(\mathcal{H}_1)$ For each $t \in I := [T_0, T]$, $C(t)$ is a nonempty closed subset of H which is r -prox-regular for some constant $r \in]0, +\infty]$ (see the definition in the next section), and has an absolutely continuous variation, in the sense that there is some absolutely continuous function $\upsilon : [T_0, T] \longrightarrow \mathbb{R}$ such that

$$C(t) \subset C(s) + |\upsilon(t) - \upsilon(s)| B_H[0, 1], \quad \forall\, t, s \in [T_0, T], \tag{2.6}$$

where $B_H[0, \eta]$ denotes the closed ball of H centered at the origin with radius η.

$(\mathcal{H}_2)$ $f_1 : [T_0, T] \times H \longrightarrow H$ is Bochner measurable in time (i.e., $f(\cdot, x)$ is Bochner measurable for each $x \in H$), and such that

$(\mathcal{H}_{2,1})$ there exists a non-negative function $\beta_1(\cdot) \in L^1([T_0, T], \mathbb{R})$ such that

$$\|f_1(t, x)\| \leq \beta_1(t)(1 + \|x\|), \text{ for all } t \in [T_0, T] \text{ and for any } x \in \bigcup_{t \in [T_0, T]} C(t).$$

$(\mathcal{H}_{2,2})$ for each real $\eta > 0$ there exists a non-negative function $L_1^\eta(\cdot) \in L^1([T_0, T], \mathbb{R})$ such that for any $t \in [T_0, T]$ and for any $(x, y) \in B_H[0, \eta] \times B_H[0, \eta]$,

$$\|f_1(t, x) - f_1(t, y)\| \leq L_1^\eta(t) \|x - y\|.$$

$(\mathcal{H}_3)$ $f_2 : [T_0, T]^2 \times H \longrightarrow H$ is Bochner measurable in $(s, t) \in [T_0, T]^2$ (i.e., $f_2(\cdot, \cdot, x)$ is Bochner measurable on $[T_0, T]^2$ for each $x \in H$) and such that

$(\mathcal{H}_{3,1})$ there exists a non-negative function $\beta_2(\cdot, \cdot) \in L^1(Q_\Delta, \mathbb{R})$ such that

$$\|f_2(t, s, x)\| \leq \beta_2(t, s)(1 + \|x\|), \text{ for all } (t, s) \in Q_\Delta \text{ and for any } x \in \bigcup_{t \in [T_0, T]} C(t).$$

$(\mathcal{H}_{3,2})$ for each real $\eta > 0$ there exists a non-negative function $L_2^\eta(\cdot) \in L^1([T_0, T], \mathbb{R})$ such that for all $(t, s) \in Q_\Delta$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$,

$$\|f_2(t, s, x) - f_2(t, s, y)\| \leq L_2^\eta(t) \|x - y\|.$$

We state first in the next proposition a result which will be utilized in our development. Clearly, when the sets $C(t)$ are bounded, the hypothesis $(\mathcal{H}_1)$ is ensured by the usual Hausdorff variation hypothesis

$$\operatorname{haus}\big(C(s), C(t)\big) \leq |\upsilon(s) - \upsilon(t)|, \quad \forall\, t, s \in [T_0, T].$$

According to the known equality $\operatorname{haus}(S, S') = \sup_{y \in H} |d_S(y) - d_{S'}(y)|$ for bounded sets S and S', the above inequality amounts to requiring for the bounded sets $C(s), C(t)$ that

$$|d_{C(s)}(y) - d_{C(t)}(y)| \leq |\upsilon(s) - \upsilon(t)|, \quad \forall\, t, s \in [T_0, T],\ \forall y \in H. \tag{2.7}$$

The following result is proved in [29, 39] under the hypothesis (2.7). Notice that (2.7) is valid even for unbounded sets. Adapting constants, it is easily seen that (2.7) can be given in the more flexible form of hypothesis $(\mathcal{H}_1)$.

Proposition 2.2.1. *Let H be a real Hilbert space, suppose that $C(\cdot)$ satisfies $(\mathcal{H}_1)$. Let $h : [T_0, T] \longrightarrow H$ be a single-valued mapping in $L^1([T_0,T], H)$. Then for any $x_0 \in C(T_0)$ there exists a unique absolutely continuous solution $x(\cdot)$ for the following differential inclusion*

$$(P_h) : \begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + h(t) \ \ a.e \ \ t \in [T_0, T], \\ x(T_0) = x_0. \end{cases}$$

Moreover $x(\cdot)$ satisfies the following inequality

$$\|\dot{x}(t) + h(t)\| \leq \|h(t)\| + |\dot{v}(t)| \ \ a.e. \ t \in [T_0, T]. \tag{2.8}$$

Theorem 2.2.1. *Let H be a real Hilbert space and assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are satisfied. Then for any initial point $x_0 \in H$, with $x_0 \in C(T_0)$ there exists a unique absolutely continuous solution $x : [T_0, T] \longrightarrow H$ of the differential inclusion (P_{f_1,f_2}). This solution satisfies:*

1. *For a.e. $t \in [T_0, T]$*

$$\|\dot{x}(t)+f_1(t,x(t))+\int_{T_0}^{t} f_2(t,s,x(s))\,ds\| \leq |\dot{v}(t)|+\|f_1(t,x(t))\|+\int_{T_0}^{t} \|f_2(t,s,x(s))\|\,ds. \tag{2.9}$$

2. *If $\displaystyle\int_{T_0}^{T}\left[\beta_1(\tau)+\int_{T_0}^{\tau}\beta_2(\tau,s)\,ds\right]d\tau < \frac{1}{4}$, one has*

$$\|f_1(t,x(t))\| \leq (1+M)\beta_1(t), \ \ \text{for all } \ t \in [T_0, T], \tag{2.10}$$

$$\|f_2(t,s,x(s))\| \leq (1+M)\beta_2(t,s), \ \ \text{for all } \ (t,s) \in Q_\Delta, \tag{2.11}$$

and for almost all $t \in [T_0, T]$

$$\left\|\dot{x}(t)+f_1(t,x(t))+\int_{T_0}^{t} f_2(t,s,x(s))\,ds\right\| \leq (1+M)\left(\beta_1(t)+\int_{T_0}^{t}\beta_2(t,s)\,ds\right)+|\dot{v}(t)|, \tag{2.12}$$

where $M := 2\left(\|x_0\|+\displaystyle\int_{T_0}^{T}|\dot{v}(\tau)|\,d\tau+\frac{1}{2}\right)$.

3. Assume the following strengthened form of assumption $(\mathcal{H}_{3,1})$ *on the function* f_2 *holds:* $(\mathcal{H}'_{3,1})$ *: there exist non-negative functions* $\alpha(\cdot) \in L^1([T_0,T],\mathbb{R})$ *and* $g(\cdot) \in L^1(P_\Delta,\mathbb{R})$ *such that*

$$\|f_2(t,s,x)\| \le g(t,s) + \alpha(t)\|x\|, \ \textit{for any } (t,s) \in Q_\Delta \ \textit{and any } x \in \bigcup_{t\in[T_0,T]} C(t).$$

Then one has

$$\|x(t)\| \le \widetilde{M}, \tag{2.13}$$

$$\|f_1(t,x(t))\| \le (1+\widetilde{M})\beta_1(t), \ \textit{for all } t \in [T_0,T], \tag{2.14}$$

$$\|f_2(t,s,x(s))\| \le g(t,s) + \alpha(t)\widetilde{M}, \ \textit{for a.e. } (t,s) \in Q_\Delta, \tag{2.15}$$

and for almost all $t \in [T_0,T]$

$$\|\dot{x}(t) + f_1(t,x(t)) + \int_{T_0}^{t} f_2(t,s,x(s))\,ds\| \le |\dot{v}(t)| + (1+\widetilde{M})\beta_1(t) + \int_{T_0}^{t} g(t,s)\,ds + T\alpha(t)\widetilde{M}, \tag{2.16}$$

where

$$\widetilde{M} := \|x_0\| \exp\left(\int_{T_0}^{T} (b(\tau)+1)\,d\tau\right)$$
$$+ \exp\left(\int_{T_0}^{T} (b(\tau)+1)\,d\tau\right) \int_{T_0}^{T} \left(|\dot{v}(s)| + 2\beta_1(s) + 2\int_{T_0}^{T} g(s,\tau)\,d\tau\right) ds,$$

$$\textit{and } b(t) := 2\max\{\beta_1(t),\alpha(t)\} \ \textit{for all } t \in [T_0,T].$$

Proof. The proof of existence of solution is divided in several steps.

Step 1. Discretization of the interval $I = [T_0,T]$.

For each $n \in \mathbb{N}$, divide the interval I into n intervals of length $h = \frac{T-T_0}{n}$ and define, for all $i \in \{0,\cdots,n-1\}$

$$\begin{cases} t_{i+1}^n := t_i^n + h = T_0 + ih, \\ I_i^n := \left[t_i^n, t_{i+1}^n\right], \end{cases}$$

so that

$$T_0 = t_0^n < t_1^n < \cdots < t_i^n < t_{i+1}^n < \cdots < t_n^n = T.$$

Step 2. Construction of the sequence $x_n(\cdot)$**.**

We construct a sequence of mappings $(x_n(\cdot))_{n\in\mathbb{N}}$ in $\mathcal{C}(I,H)$ which converges uniformly to a solution $x(\cdot)$ of (P_{f_1,f_2}).

Our method consists in establishing a sequence of discrete solutions $(x_k^n(\cdot))_{n\in\mathbb{N}}$ in each interval

$I_k^n := \left[t_k^n, t_{k+1}^n\right]$ $(0 \leq k \leq n-1)$ by using Proposition 2.2.1. Indeed, we proceed as follows. Consider the following problem

$$(P_0): \begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x_0) + \int_{T_0}^{t} f_2(t, s, x_0)\, ds \quad \text{a.e. } t \in [T_0, t_1^n], \\ x(T_0) = x_0. \end{cases}$$

Then (P_0) is a perturbed sweeping process with the perturbation depending only on time as in (P_h).

Let $h_0 : [T_0, t_1^n] \to H$ be defined by $h_0(t) := f_1(t, x_0) + \int_{T_0}^{t} f_2(t, s, x_0)\, ds$ for all $t \in [T_0, t_1^n]$. We notice by the measurability assumptions in $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ that h is Bochner measurable (for $[T_0, t_1^n]$ endowed with its Borel σ-field) and we see by the integrable linear growth conditions $(\mathcal{H}_{2,1})$ and $(\mathcal{H}_{3,1})$ that

$$\int_{T_0}^{T} \|h_0(t)\| dt \leq (1 + \|x_0\|) \int_{T_0}^{T} \beta_1(t) dt + (1 + \|x_0\|) \int_{T_0}^{T} \int_{T_0}^{t} \beta_2(t, s) ds\, dt,$$

and since $\beta_1(\cdot) \in L^1([T_0, T], \mathbb{R}_+)$ and $\beta_2(\cdot) \in L^1(Q_\Delta, \mathbb{R}_+)$, then $h_0(\cdot)$ is Bochner integrable on $[T_0, t_1^n]$ with respect to the Lebesgue measure. Therefore, by Proposition 2.2.1 the differential inclusion (P_0) has a unique absolutely continuous solution denoted by

$$x_0^n(\cdot) : [T_0, t_1^n] \longrightarrow H,$$

satisfying the following inequality

$$\left\| \dot{x}_0^n(t) + f_1(t, x_0) + \int_{T_0}^{t} f_2(t, s, x_0)\, ds \right\| \leq \left\| f_1(t, x_0) + \int_{T_0}^{t} f_2(t, s, x_0)\, ds \right\| + |\dot{v}(t)|$$

for a.e. $t \in [T_0, t_1^n]$.

Next, let us consider the following problem

$$(P_1): \begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x_0^n(t_1^n)) + \int_{T_0}^{t_1^n} f_2(t, s, x_0)\, ds \\ + \int_{t_1^n}^{t} f_2(t, s, x_0^n(t_1^n))\, ds \quad \text{a.e. } t \in [t_1^n, t_2^n], \\ x(t_1^n) = x_0^n(t_1^n). \end{cases}$$

Let $h_1 : [t_1^n, t_2^n] \to H$ be defined by

$$h_1(t) := f_1(t, x_0^n(t_1^n)) + \int_{T_0}^{t_1^n} f_2(t, s, x_0)\, ds + \int_{t_1^n}^{t} f_2(t, s, x_0^n(t_1^n))\, ds \text{ for all } t \in [t_1^n, t_2^n].$$

As above h_1 is Bochner measurable on $[t_1^n, t_2^n]$ and we can see by integrable linear growth conditions that

$$\int_{T_0}^{T} \|h_1(t)\| dt \le (1 + \|x_0^n(t_1^n)\|) \int_{T_0}^{T} \beta_1(t) dt$$

$$+ (1 + \|x_0\|) \int_{T_0}^{T} \int_{T_0}^{t_1^n} \beta_2(t,s) ds\, dt + (1 + \|x_0^n(t_1^n)\|) \int_{T_0}^{T} \int_{t_1^n}^{t} \beta_2(t,s) ds\, dt$$

$$\le (1 + \max\{\|x_0^n(t_1^n)\|, \|x_0\|\}) \Big(\int_{T_0}^{T} \beta_1(t) dt + \int_{T_0}^{T} \int_{T_0}^{t} \beta_2(t,s) ds\, dt \Big).$$

We know from the above problem (P_0) that the mapping $x_0^n(\cdot)$ is absolutely continuous on $[T_0, t_1^n]$, then in particular bounded on $[T_0, t_1^n]$. Furthermore, since we have $\beta_1(\cdot) \in L^1([T_0, T], \mathbb{R}_+)$ along with $\beta_2(\cdot) \in L^1(Q_\Delta, \mathbb{R}_+)$, then $h_1(\cdot)$ is Bochner integrable. The same arguments as above show that (P_1) has a unique absolutely continuous solution denoted by

$$x_1^n(\cdot) : [t_1^n, t_2^n] \longrightarrow H,$$

and this solution satisfies the following inequality

$$\left\| \dot{x}_1^n(t) + f_1(t, x_0^n(t_1^n)) + \int_{T_0}^{t_1^n} f_2(t,s,x_0)\, ds + \int_{t_1^n}^{t} f_2(t,s,x_0^n(t_1^n))\, ds \right\|$$

$$\le \left\| f_1(t, x_0^n(t_1^n)) + \int_{T_0}^{t_1^n} f_2(t,s,x_0)\, ds + \int_{t_1^n}^{t} f_2(t,s,x_0^n(t_1^n))\, ds \right\| + |\dot{v}(t)| \quad \text{a.e. } t \in [t_1^n, t_2^n].$$

Successively, for each n, we have a finite sequence of absolutely continuous mappings $(x_k^n(\cdot))_{0 \le k \le n-1}$ with for each $k \in \{0, \cdots, n-1\}$

$$x_k^n(\cdot) : \left[t_k^n, t_{k+1}^n\right] \longrightarrow H$$

such that

$$(P_k) : \begin{cases} -\dot{x}_k^n(t) \in N_{C(t)}(x_k^n(t)) + f_1(t, x_{k-1}^n(t_k^n)) + \displaystyle\sum_{j=0}^{k-1} \int_{t_j^n}^{t_{j+1}^n} f_2(t,s,x_{j-1}^n(t_j^n))\, ds \\ + \displaystyle\int_{t_k^n}^{t} f_2(t,s,x_{k-1}^n(t_k^n))\, ds \quad \text{a.e. } t \in \left[t_k^n, t_{k+1}^n\right] \\ x_k^n(t_k^n) = x_{k-1}^n(t_k^n), \end{cases} \tag{2.17}$$

where for $k=0$ we put $x_{-1}^n(T_0) := x_0$. Moreover, for a.e. $t \in \left[t_k^n, t_{k+1}^n\right]$

$$\begin{aligned} &\left\|\dot{x}_k^n(t) + f_1(t, x_{k-1}^n(t_k^n)) + \sum_{j=0}^{k-1} \int_{t_j^n}^{t_{j+1}^n} f_2(t,s,x_{j-1}^n(t_j^n))\,ds + \int_{t_k^n}^{t} f_2(t,s,x_{k-1}^n(t_k^n))\,ds\right\| \\ &\leq \left\|f_1(t, x_{k-1}^n(t_k^n)) + \sum_{j=0}^{k-1} \int_{t_j^n}^{t_{j+1}^n} f_2(t,s,x_{j-1}^n(t_j^n))\,ds + \int_{t_k^n}^{t} f_2(t,s,x_{k-1}^n(t_k^n))\,ds\right\| + |\dot{v}(t)|. \end{aligned} \tag{2.18}$$

Defining for each $k \in \{0,1,\cdots,n-1\}$ the mapping $h_k : [t_k^n, t_{k+1}^n] \to H$ by

$$h_k(t) := f_1(t, x_{k-1}^n(t_k^n)) + \sum_{j=0}^{k-1} \int_{t_j^n}^{t_{j+1}^n} f_2(t,s,x_{j-1}^n(t_j^n))\,ds + \int_{t_k^n}^{t} f_2(t,s,x_{k-1}^n(t_k^n))\,ds,$$

for all $t \in \left[t_k^n, t_{k+1}^n\right]$. Clearly, by integrable linear growth conditions $(\mathcal{H}_{2,1})$ and $(\mathcal{H}_{3,1})$ we have

$$\begin{aligned} \int_{T_0}^{T} \|h_k(t)\|dt &\leq (1 + \|x_{k-1}^n(t_k^n)\|) \int_{T_0}^{T} \beta_1(t)dt + \sum_{j=0}^{k-1} (1 + \|x_{j-1}^n(t_j^n)\|) \int_{t_j^n}^{t_{j+1}^n} \beta_2(t,s)ds\,dt \\ &\quad + (1 + \|x_{k-1}^n(t_k^n)\|) \int_{T_0}^{T} \int_{t_k^n}^{t} \beta_2(t,s)ds\,dt \\ &\leq (1 + \max_{0\leq j\leq k-1} \left\|x_{j-1}^n(t_j^n)\right\|) \Big(\int_{T_0}^{T} \beta_1(t)dt + \int_{T_0}^{T}\int_{T_0}^{t} \beta_2(t,s)ds\,dt \Big). \end{aligned}$$

We know from the above problems $(P_j)_{0\leq j\leq k-1}$ that the mapping $x_{k-1}^n(\cdot)$ is absolutely continuous on $\left[t_{k-1}^n, t_k^n\right]$, then in particular bounded on $\left[t_{k-1}^n, t_k^n\right]$. Further, since $\beta_1(\cdot) \in L^1([T_0,T], \mathbb{R}_+)$ and $\beta_2(\cdot) \in L^1(P_\Delta, \mathbb{R}_+)$, the mapping $h_k(\cdot)$ is integrable on $[t_k^n, t_{k+1}^n]$.

Now, we define the sequence $(x_n(\cdot))_n$ from the discrete sequences $(x_k^n(.))$ as follows. For each $n \in \mathbb{N}$, let $x_n(\cdot) : [T_0, T] \longrightarrow H$ be such that

$$x_n(t) := x_k^n(t), \text{ if } t \in \left[t_k^n, t_{k+1}^n\right]. \tag{2.19}$$

It is obvious from this definition that $x_n(\cdot)$ is absolutely continuous.
Let $\theta_n(\cdot) : [T_0,T] \longrightarrow [T_0,T]$ be defined by

$$\begin{cases} \theta_n(T_0) := T_0, \\ \theta_n(t) := t_k^n, \text{ if } t \in \left]t_k^n, t_{k+1}^n\right]. \end{cases} \tag{2.20}$$

We obtain from (2.17), (2.18), (2.19) and (2.20), that

$$\begin{cases} -\dot{x}_n(t) \in N_{C(t)}(x_n(t)) + f_1(t, x_n(\theta_n(t))) + \int_{T_0}^{t} f_2(t,s,x_n(\theta_n(s)))\,ds \ \text{ a.e. } t \in [T_0,T] \\ x_n(T_0) = x_0, \end{cases} \tag{2.21}$$

and for a.e. $t \in [T_0, T]$ we have

$$\left\| \dot{x}_n(t) + f_1(t, x_n(\theta_n(t))) + \int_{T_0}^{t} f_2(t, s, x_n(\theta_n(s)))\, ds \right\| \leq \left\| f_1(t, x_n(\theta_n(t))) + \int_{T_0}^{t} f_2(t, s, x_n(\theta_n(s)))\, ds \right\| + |\dot{v}(t)|. \tag{2.22}$$

Step 3. We show that the sequence $(\dot{x}_n(\cdot))$ is uniformly dominated by an integrable function.

Since $\beta_1(\cdot) \in L^1([T_0, T], \mathbb{R}_+)$ and $\beta_2(\cdot, \cdot) \in L^1(P_\Delta, \mathbb{R}_+)$ we suppose without loss of generality that

$$\int_{T_0}^{T} \left[\beta_1(\tau) + \int_{T_0}^{\tau} \beta_2(\tau, s)\, ds \right] d\tau < \frac{1}{4}. \tag{2.23}$$

By construction we have for each $i \in \{0, \cdots, n-1\}$ and for a.e. $t \in \left[t_i^n, t_{i+1}^n\right]$

$$\left\| \dot{x}_n(t) + f_1(t, x_n(t_i^n)) + \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f_2(t, s, x_n(t_j^n))\, ds + \int_{t_i^n}^{t} f_2(t, s, x_n(t_i^n))\, ds \right\|$$

$$\leq \left\| f_1(t, x_n(t_i^n)) + \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f_2(t, s, x_n(t_j^n))\, ds + \int_{t_i^n}^{t} f_2(t, s, x_n(t_i^n))\, ds \right\| + |\dot{v}(t)|.$$

According to $(\mathcal{H}_{2,1})$ and $(\mathcal{H}_{3,1})$ it ensues that for a. e. $t \in [t_i^n, t_{i+1}^n]$

$$\|\dot{x}_n(t)\| \leq 2 \|f_1(t, x_n(t_i^n))\| + 2 \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} \left\| f_2(t, s, x_n(t_j^n)) \right\| ds + 2 \int_{t_i^n}^{t} \|f_2(t, s, x_n(t_i^n))\|\, ds + |\dot{v}(t)|$$

$$\leq 2(1 + \max_{0 \leq k \leq n} \|x_n(t_k^n)\|)\beta_1(t) + 2(1 + \max_{0 \leq k \leq n} \|x_n(t_k^n)\|) \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} \beta_2(t, s)\, ds$$

$$+ 2(1 + \max_{0 \leq k \leq n} \|x_n(t_k^n)\|) \int_{t_i^n}^{t} \beta_2(t, s)\, ds + |\dot{v}(t)|$$

$$= |\dot{v}(t)| + 2(1 + \max_{0 \leq k \leq n} \|x_n(t_k^n)\|)\beta_1(t) + 2(1 + \max_{0 \leq k \leq n} \|x_n(t_k^n)\|) \int_{T_0}^{t} \beta_2(t, s)\, ds,$$

and hence

$$\left\| x_n(t_{i+1}^n) \right\| \leq \|x_n(t_i^n)\| + \int_{t_i^n}^{t_{i+1}^n} |\dot{v}(\tau)|\, d\tau + 2(1 + \max_{0 \leq k \leq n} \|x_n(t_k^n)\|) \int_{t_i^n}^{t_{i+1}^n} \beta_1(\tau)\, d\tau$$

$$+ 2(1 + \max_{0 \leq k \leq n} \|x_n(t_k^n)\|) \int_{t_i^n}^{t_{i+1}^n} \int_{T_0}^{\tau} \beta_2(\tau, s)\, ds\, d\tau.$$

Iterating, it follows that

$$\left\|x_n(t_{i+1}^n)\right\| \leq \|x_0\| + \sum_{k=0}^{i} \int_{t_k^n}^{t_{k+1}^n} |\dot{v}(\tau)|\, d\tau + 2(1 + \max_{0\leq j\leq n} \left\|x_n(t_j^n)\right\|) \sum_{k=0}^{i} \int_{t_k^n}^{t_{k+1}^n} \beta_1(\tau)\, d\tau$$

$$+ 2(1 + \max_{0\leq j\leq n} \left\|x_n(t_j^n)\right\|) \sum_{k=0}^{i} \int_{t_k^n}^{t_{k+1}^n} \int_{T_0}^{\tau} \beta_2(\tau, s)\, ds\, d\tau.$$

This yields the inequality

$$\begin{aligned}\left\|x_n(t_{i+1}^n)\right\| \leq \|x_0\| + \int_{T_0}^{t_{i+1}^n} |\dot{v}(\tau)|\, d\tau + 2(1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|) \int_{T_0}^{t_{i+1}^n} \beta_1(\tau)\, d\tau \\ + 2(1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|) \int_{T_0}^{t_{i+1}^n} \int_{T_0}^{\tau} \beta_2(\tau, s)\, ds\, d\tau.\end{aligned} \tag{2.24}$$

The inequality (2.24) being true for all $i \in \{0, \cdots, n-1\}$, we have

$$\max_{0\leq k\leq n} \|x_n(t_k^n)\| \leq \|x_0\| + \int_{T_0}^{T} |\dot{v}(\tau)|\, d\tau + 2(1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|) \int_{T_0}^{T} \beta_1(\tau)\, d\tau$$

$$+ 2(1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|) \int_{T_0}^{T} \int_{T_0}^{\tau} \beta_2(\tau, s)\, ds\, d\tau,$$

which gives by (2.23)

$$\max_{0\leq k\leq n} \|x_n(t_k^n)\| \leq \|x_0\| + \int_{T_0}^{T} |\dot{v}(\tau)|\, d\tau + \frac{1}{2}(1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|).$$

This can be rewritten as

$$\max_{0\leq k\leq n} \|x_n(t_k^n)\| \leq M, \tag{2.25}$$

where $M := 2\left(\|x_0\| + \int_{T_0}^{T} |\dot{v}(\tau)|\, d\tau + \frac{1}{2}\right)$.

On one hand, by the growth conditions $(\mathcal{H}_{2,1})$ and $(\mathcal{H}_{3,1})$ of f_1, f_2 and (2.25) we have for all n

$$\|f_1(t, x_n(\theta_n(t)))\| \leq \beta_1(t)(1 + \|x_n(\theta_n(t))\|) \leq (1 + M)\beta_1(t) \text{ for all } t \in [T_0, T], \tag{2.26}$$

$$\|f_2(t, s, x_n(\theta_n(s)))\| \leq \beta_2(t, s)(1 + \|x_n(\theta_n(s))\|) \leq (1 + M)\beta_2(t, s) \text{ for all } (t, s) \in Q_\Delta. \tag{2.27}$$

Consequently, (2.22) implies for almost all t and for all n

$$\begin{aligned}\left\|\dot{x}_n(t) + f_1(t, x_n(\theta_n(t))) + \int_{T_0}^{t} f_2(t, s, x_n(\theta_n(s)))\, ds\right\| \\ \leq (1 + M)\left(\beta_1(t) + \int_{T_0}^{t} \beta_2(t, s)\, ds\right) + |\dot{v}(t)|,\end{aligned} \tag{2.28}$$

and hence

$$\|\dot{x}_n(t)\| \leq 2(1+M)\Big(\beta_1(t) + \int_{T_0}^{t} \beta_2(t,s)\,ds\Big) + |\dot{v}(t)|. \tag{2.29}$$

Step 4. We show that $x_n(\cdot)$ converges.

It suffices to show that $x_n(\cdot)$ is a Cauchy sequence in the Banach space $(\mathcal{C}(I,H), \|\cdot\|_\infty)$. Let $m,n \in \mathbb{N}$. For almost all $t \in [T_0,T]$, we have

$$\begin{cases} -\dot{x}_n(t) - f_1(t, x_n(\theta_n(t))) - \displaystyle\int_{T_0}^{t} f_2(t,s,x_n(\theta_n(s)))\,ds \in N_{C(t)}(x_n(t)), \\ -\dot{x}_m(t) - f_1(t, x_m(\theta_m(t))) - \displaystyle\int_{T_0}^{t} f_2(t,s,x_m(\theta_m(s)))\,ds \in N_{C(t)}(x_m(t)). \end{cases} \tag{2.30}$$

Let us set

$$\varphi(t) := (1+M)\Big(\beta_1(t) + \int_{T_0}^{t} \beta_2(t,s)\,ds\Big) + |\dot{v}(t)|, \tag{2.31}$$

$$\gamma(t) := 2(1+M)\Big(\beta_1(t) + \int_{T_0}^{t} \beta_2(t,s)\,ds\Big) + |\dot{v}(t)|. \tag{2.32}$$

The absolute continuity of $x_n(\cdot)$ gives by (2.29)

$$\|x_n(t)\| \leq \|x_0\| + \int_{T_0}^{t} \|\dot{x}_n(s)\|\,ds \leq \eta, \quad \text{for all } t \in [T_0,T], \tag{2.33}$$

with

$$\eta := \|x_0\| + \int_{T_0}^{T} \gamma(s)\,ds.$$

Using (2.28), (2.31) and the hypomonotonicity of the normal cone $N_{C(t)}(\cdot)$ due to Proposition 1.3.2, we get that

$$\Big\langle \dot{x}_n(t) + f_1(t,x_n(\theta_n(t))) + \int_{T_0}^{t} f_2(t,s,x_n(\theta_n(s)))\,ds - \dot{x}_m(t) - f_1(t,x_m(\theta_m(t)))$$
$$- \int_{T_0}^{t} f_2(t,s,x_m(\theta_m(s)))\,ds, x_n(t) - x_m(t) \Big\rangle \leq \frac{\varphi(t)}{r}\|x_n(t) - x_m(t)\|^2,$$

or equivalently

$$\langle \dot{x}_n(t) - \dot{x}_m(t), x_n(t) - x_m(t)\rangle \leq \frac{\varphi(t)}{r}\|x_n(t) - x_m(t)\|^2$$
$$+ \langle f_1(t,x_n(\theta_n(t))) - f_1(t,x_m(\theta_m(t))), x_m(t) - x_n(t)\rangle$$
$$+ \Big\langle \int_{T_0}^{t} f_2(t,s,x_n(\theta_n(s)))\,ds - \int_{T_0}^{t} f_2(t,s,x_m(\theta_m(s)))\,ds, x_m(t) - x_n(t) \Big\rangle.$$

Applying Lemma 1.5.1 and the Lipschitz continuity of $f_1(t,\cdot)$ and $f_2(t,s,\cdot)$ with Lipschitz radius $L_1^\eta(\cdot), L_2^\eta(\cdot) \in L^1(I,\mathbb{R}_+)$ on the bounded subset $B[0,\eta]$ (see the assumptions $(\mathcal{H}_{2,2})$ and $(\mathcal{H}_{3,2})$), it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|x_n(t)-x_m(t)\|^2 \leq \frac{\varphi(t)}{r}\|x_n(t)-x_m(t)\|^2 \\
&+ L_1^\eta(t)\|x_n(t)-x_m(t)\|\Big(\|x_n(\theta_n(t))-x_n(t)\| + \|x_n(t)-x_m(t)\| + \|x_m(t)-x_m(\theta_m(t))\|\Big) \\
&+ L_2^\eta(t)\|x_n(t)-x_m(t)\|\Bigg(\int_{T_0}^{t}\|x_n(\theta_n(s))-x_n(s)\|\,ds + \int_{T_0}^{t}\|x_n(s)-x_m(s)\|\,ds \\
&+ \int_{T_0}^{t}\|x_m(t)-x_m(\theta_m(t))\|\,ds\Bigg)
\end{aligned}$$

By (2.29) and (2.32), we also have for each $n\in\mathbb{N}$ and for all $t\in[T_0,T]$

$$\|x_n(t)-x_n(\theta_n(t))\| = \left\|\int_{\theta_n(t)}^{t}\dot{x}_n(\tau)d\tau\right\| \leq \int_{\theta_n(t)}^{t}\|\dot{x}_n(\tau)\|\,d\tau \leq \int_{\theta_n(t)}^{t}\gamma(\tau)d\tau.$$

Therefore, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|x_n(t)-x_m(t)\|^2 \leq \frac{\varphi(t)}{r}\|x_n(t)-x_m(t)\|^2 + L_1^\eta(t)\|x_n(t)-x_m(t)\|^2 \\
&+ L_1^\eta(t)\|x_n(t)-x_m(t)\|\Big(\int_{\theta_n(t)}^{t}\gamma(\tau)d\tau + \int_{\theta_m(t)}^{t}\gamma(\tau)d\tau\Big) \\
&+ L_2^\eta(t)\|x_n(t)-x_m(t)\|\Big(\int_{T_0}^{t}\int_{\theta_n(s)}^{s}\gamma(\tau)d\tau\,ds + \int_{T_0}^{t}\int_{\theta_m(s)}^{s}\gamma(\tau)d\tau\,ds\Big) \\
&+ L_2^\eta(t)\|x_n(t)-x_m(t)\|\int_{T_0}^{t}\|x_n(s)-x_m(s)\|\,ds.
\end{aligned}$$

Moreover, noting by (2.33) that

$$\|x_n(t)-x_m(t)\| \leq \|x_n(t)\| + \|x_m(t)\| \leq 2\eta,$$

we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|x_n(t)-x_m(t)\|^2 \leq \frac{\varphi(t)}{r}\|x_n(t)-x_m(t)\|^2 + L_1^\eta(t)\|x_n(t)-x_m(t)\|^2 \\
&+ 2\eta L_1^\eta(t)\Big(\int_{\theta_n(t)}^{t}\gamma(\tau)d\tau + \int_{\theta_m(t)}^{t}\gamma(\tau)d\tau\Big) + 2\eta L_2^\eta(t)\Big(\int_{T_0}^{t}\Big[\int_{\theta_n(s)}^{s}\gamma(\tau)d\tau + \int_{\theta_m(s)}^{s}\gamma(\tau)d\tau\Big]\,ds\Big) \\
&+ L_2^\eta(t)\|x_n(t)-x_m(t)\|\int_{T_0}^{t}\|x_n(s)-x_m(s)\|\,ds.
\end{aligned} \tag{2.34}$$

Let us put

$$G_{n,m}(t) := 2\eta L_1^{\eta}(t)\Big(\int_{\theta_n(t)}^{t} \gamma(\tau)d\tau + \int_{\theta_m(t)}^{t} \gamma(\tau)d\tau\Big),$$

$$\widetilde{G}_{n,m}(s) := \int_{\theta_n(s)}^{s} \gamma(\tau)d\tau + \int_{\theta_m(s)}^{s} \gamma(\tau)d\tau.$$

Since $\gamma(\cdot) \in L^1(I, \mathbb{R}_+)$ and for each $t \in I$, we have $\theta_n(t), \theta_m(t) \longrightarrow t$, then

$$\lim_{n,m\to+\infty} G_{n,m}(t) = 0 \quad \text{and} \quad \lim_{n,m\to+\infty} \widetilde{G}_{n,m}(t) = 0. \tag{2.35}$$

On the other hand, for each $n \in \mathbb{N}$ writing

$$\int_{\theta_n(t)}^{t} \gamma(s)ds \le \int_{T_0}^{T} \gamma(s)ds,$$

we see that

$$|G_{n,m}(t)| \le 4\eta L_1^{\eta}(t) \int_{T_0}^{T} \gamma(s)ds \quad \text{and} \quad \left|\widetilde{G}_{n,m}(s)\right| \le 2 \int_{T_0}^{T} \gamma(s)ds.$$

Therefore, for all $t \in [T_0, T]$ by (2.35) and the dominated convergence theorem, we obtain

$$\lim_{n,m\to+\infty} \int_{T_0}^{T} G_{n,m}(t)dt = 0, \tag{2.36}$$

$$\lim_{n,m\to+\infty} \int_{T_0}^{T} \widetilde{G}_{n,m}(s)ds = 0. \tag{2.37}$$

Note also by (2.34) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|x_n(t) - x_m(t)\|^2 \\
&\le \left(\frac{\varphi(t)}{r} + L_1^{\eta}(t)\right)\|x_n(t) - x_m(t)\|^2 + G_{n,m}(t) + 2\eta L_2^{\eta}(t)\int_{T_0}^{T} \widetilde{G}_{n,m}(s)\, ds \\
&+ L_2^{\eta}(t)\,\|x_n(t) - x_m(t)\| \int_{T_0}^{t} \|x_n(s) - x_m(s)\|\, ds.
\end{aligned}$$

Applying Lemma 2.1.3 with

$$\rho(t) = \|x_n(t) - x_m(t)\|^2,\ K_1(t) = 2\Big(\frac{\varphi(t)}{r} + L_1^{\eta}(t)\Big),\ K_2(t) = 2L_2^{\eta}(t)$$

$$\varepsilon(t) := \varepsilon_{n,m}(t) = 2G_{n,m}(t) + 4\eta L_2^{\eta}(t)\int_{T_0}^{T} \widetilde{G}_{n,m}(s)\, ds \ \text{ and } \ \epsilon > 0,$$

we then see that

$$
\begin{aligned}
\|x_n(t)-x_m(t)\| \leq & \sqrt{\|x_n(T_0)-x_m(T_0)\|^2+\epsilon}\, \exp\left(\int_0^t (K(s)+1)\, ds\right) \\
& + \frac{\sqrt{\epsilon}}{2}\int_{T_0}^t \exp\left(\int_s^t (K(\tau)+1)\, d\tau\right) ds \\
& + 2\left(\sqrt{\int_{T_0}^t \varepsilon_{n,m}(s)\, ds+\epsilon} - \exp\left(\int_{T_0}^t (K(\tau)+1)\, d\tau\right)\sqrt{\epsilon}\right) \\
& + 2\int_{T_0}^t (K(s)+1) \exp\left(\int_s^t (K(\tau)+1)\, d\tau\right)\sqrt{\int_{T_0}^s \varepsilon_{n,m}(\tau)\, d\tau+\epsilon}\; ds,
\end{aligned}
$$

where $K(t) := \max\left\{\frac{\varphi(t)}{r} + L_1^\eta(t), L_2^\eta(t)\right\}$ for all $t \in [T_0, T]$.
This, along with the fact that $\lim_{n,m\to+\infty} \varepsilon_{n,m}(t) = 0$ (by (2.35) and (2.37)), $\|x_n(T_0) - x_m(T_0)\| = 0$ and taking $\epsilon \to 0$, we get

$$\lim_{n,m\to+\infty} \|x_n(\cdot) - x_m(\cdot)\|_\infty = 0.$$

Therefore, the sequence $(x_n(\cdot))$ is a Cauchy sequence in $(\mathcal{C}([T_0,T], H), \|\cdot\|_\infty)$ and then converges uniformly on $[T_0,T]$ to a mapping $x(\cdot) \in \mathcal{C}([T_0,T], H)$. Furthermore, using this and the inclusion $x_n(t) \in C(t)$ we see that $x(t) \in C(t)$ for all $t \in [T_0, T]$.

Step 5. We show that $x(\cdot)$ is absolutely continuous.

We have for almost all $t \in I$ and for all n,

$$\|\dot{x}_n(t)\| \leq \gamma(t).$$

So, by Dunford-Pettis-theorem (Theorem 1.5.1), we can extract a subsequence of $(\dot{x}_n(\cdot))$ (that, without loss of generality, we do not relabel) which converges weakly in $L^1(I,H)$ to a mapping $g(\cdot) \in L^1(I,H)$. This means that

$$\int_{T_0}^T \langle \dot{x}_n(s), h(s)\rangle\, ds \longrightarrow \int_{T_0}^T \langle g(s), h(s)\rangle\, ds, \forall\, h \in L^\infty(I,H).$$

Now fix any $t \in [T_0, t]$. We observe that for all $z \in H$

$$\int_{T_0}^T \left\langle \dot{x}_n(s), z\cdot 1_{[T_0,t]}(s)\right\rangle ds = \int_{T_0}^t \langle \dot{x}_n(s), z\rangle\, ds = \langle z, \int_{T_0}^t \dot{x}_n(s) ds\rangle,$$

and

$$\int_{T_0}^T \left\langle g(s), z\cdot 1_{[T_0,t]}(s)\right\rangle ds = \int_{T_0}^t \langle g(s), z\rangle\, ds = \langle z, \int_{T_0}^t g(s) ds\rangle.$$

So from the weak convergence we deduce that

$$\int_{T_0}^t \dot{x}_n(s) ds \longrightarrow \int_{T_0}^t g(s)\, ds \quad \text{weakly in } H.$$

This implies that

$$x_n(T_0) + \int_{T_0}^{t} \dot{x}_n(s)ds \longrightarrow x(T_0) + \int_{T_0}^{t} g(s)\, ds \quad \text{weakly in } \ H.$$

Since $x_n(\cdot)$ is absolutely continuous, we obtain

$$x_n(t) = x_n(T_0) + \int_{T_0}^{t} \dot{x}_n(s)ds \longrightarrow x(T_0) + \int_{T_0}^{t} g(s)\, ds \quad \text{weakly in } \ H.$$

On the other hand, we have

$$x_n(t) \longrightarrow x(t) \quad \text{strongly in} \quad H,$$

hence we get

$$x(t) = x(T_0) + \int_{T_0}^{t} g(s)\, ds.$$

Therefore, $x(\cdot)$ is absolutely continuous and $\dot{x}(t) = g(t)$ for a.e. $t \in [T_0, T]$, so in particular

$$\|x(t)\| \leq \widetilde{\eta} \quad \text{for all} \quad t \in [T_0, T]\,, \tag{2.38}$$

with

$$\widetilde{\eta} := \|x_0\| + \int_{T_0}^{T} \|g(s)\|\, ds.$$

Step 6. We show that $x(\cdot)$ is a solution of (P_{f_1,f_2}) .

For each $t \in I$, since $\theta_n(t) \longrightarrow t$ and $x_n(\cdot)$ converges uniformly to $x(\cdot)$, we have $x_n(\theta_n(t)) \longrightarrow x(t)$. For each $t \in [T_0, T]$ we also note by $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ that the mappings $s \mapsto f_2(t, s, x_n(\theta_n(s))$ and $s \mapsto f_2(t, s, x(s))$ are Bochner integrable on I, so we can set

$$y_n(t) := \int_{T_0}^{t} f_2(t, s, x_n(\theta_n(s)))\, ds, \ \text{ and } y(t) := \int_{T_0}^{t} f_2(t, s, x(s))\, ds.$$

We have shown in the above **Step 5** that $(\dot{x}_n(\cdot))_n$ converges weakly to $\dot{x}(\cdot)$ in $L^1(I, H)$. Moreover, by (2.33) and (2.38) we can choose some real constant $c > 0$ such that, for each n, we have $\|x_n(\theta_n(t))\| \leq c$ and $\|x(t)\| \leq c$ for all $t \in [T_0, T]$. Therefore, by the assumptions $(\mathcal{H}_{2,2})$ and $(\mathcal{H}_{3,2})$ there exist $L_1^c(\cdot)$ and $L_2^c(\cdot)$ in $L^1([T_0, T], \mathbb{R}_+)$ such that $f_1(t, \cdot)$ and $f_2(t, s, \cdot)$ are $L_1^c(t)$-Lipschitz and $L_2^c(t)$-Lipschitz respectively on $B[0, c]$. It follows that

$$\int_{T_0}^{T} \|f_1(t, x_n(\theta_n(t))) - f_1(t, x(t))\|\, dt \leq \int_{T_0}^{T} L_1^c(t)\, \|x_n(\theta_n(t)) - x(t)\| \; dt, \tag{2.39}$$

$$\int_{T_0}^{T} \|y_n(t) - y(t)\|\, dt \leq \int_{T_0}^{T} L_2^c(t) \int_{T_0}^{t} \|x_n(\theta_n(s)) - x(s)\|\, ds\, dt. \tag{2.40}$$

Note that for every $(t,s) \in Q_\Delta$

$$L_1^c(t)\,\|x_n(\theta_n(t)) - x(t)\| \le 2cL_1^c(t),$$

$$L_2^c(t)\int_{T_0}^{t} \|x_n(\theta_n(s)) - x(s)\|\,ds \le 2c(T-T_0)L_2^c(t).$$

Then by (2.39), (2.40) and by the Lebesgue dominated convergence theorem

$$f_1(\cdot, x_n(\theta_n(\cdot))) \longrightarrow f_1(\cdot, x(\cdot)) \text{ strongly in } L^1(I,H),$$

$$y_n(\cdot) \longrightarrow y(\cdot) \text{ strongly in } L^1(I,H).$$

This implies that

$$\zeta_n(\cdot) := \dot{x}_n(\cdot) + f_1(\cdot, x_n(\theta_n(\cdot))) + y_n(\cdot) \longrightarrow \zeta(\cdot) := \dot{x}(\cdot) + f_1(\cdot, x(\cdot)) + y(\cdot)$$

weakly in $L^1(I,H)$. By Mazur's lemma (Lemma 1.5.2) we can find a convex combination $\sum_{k=n}^{r(n)} S_{k,n}\zeta_k(\cdot)$, with $\sum_{k=n}^{r(n)} S_{k,n} = 1$ and $S_{k,n} \in [0,1]$ for all k, n, which converges strongly in $L^1(I,H)$ to $\zeta(\cdot)$. Extracting a subsequence, we may suppose that $\sum_{k=n}^{r(n)} S_{k,n}\zeta_k(\cdot)$ converges almost everywhere on I to the mapping $\zeta(\cdot)$. Then there is a Borel set $N \subset I$ with null Lebesgue measure such that for each $t \in I \setminus N$ we have $\sum_{k=n}^{r(n)} S_{k,n}\zeta_k(t) \to \zeta(t)$ and such that for all $n \in \mathbb{N}$

$$-\zeta_n(t) := -\dot{x}_n(t) - f_1(t, x_n(\theta_n(t))) - \int_{T_0}^{t} f_2(t,s,x_n(\theta_n(s)))\,ds \in N_{C(t)}(x_n(t)).$$

Fix any $t \in I \setminus N$ and any $n \in \mathbb{N}$. Since $C(t)$ is r-prox-regular we have that for every $z \in C(t)$ (see (1.4))

$$\langle -\zeta_n(t), z - x_n(t)\rangle \le \frac{\varphi(t)}{2r}\|z - x_n(t)\|^2, \quad \text{for all } z \in C(t),$$

hence

$$\langle -\zeta_n(t), z - x_n(t)\rangle \le \frac{\varphi(t)}{2r}(\|z - x(t)\| + \|x(t) - x_n(t)\|)^2 := \lambda_n(t), \tag{2.41}$$

with $\lim_{n\longrightarrow\infty} \lambda_n(t) = \frac{\varphi(t)}{2r}\|z - x(t)\|^2$. Let us fix any $z \in C(t)$ and let us write

$$\begin{aligned}\langle -\zeta(t), z - x(t)\rangle &= \langle -\zeta(t) + \sum_{k=n}^{r(n)} S_{k,n}\zeta_k(t), z - x(t)\rangle + \sum_{k=n}^{r(n)} S_{k,n}\langle -\zeta_k(t), z - x_k(t)\rangle \\ &+ \sum_{k=n}^{r(n)} S_{k,n}\langle -\zeta_k(t), -x(t) + x_k(t)\rangle.\end{aligned}$$

The first expression of the second member of the latter equality tends to zero by what precedes, and keeping in mind that $|\zeta_k(t)| \le \varphi(t)$, we also see that the third expression tends to zero.

Concerning the second expression, thanks to (2.41), it satisfies the estimate

$$\sum_{k=n}^{r(n)} S_{k,n}\langle -\zeta_k(t), z - x_k(t)\rangle \leq \sum_{k=n}^{r(n)} S_{k,n}\lambda_k(t).$$

Thus, passing to the limit we obtain

$$\langle -\zeta(t), z - x(t)\rangle \leq \frac{\varphi(t)}{2r}\|z - x(t)\|^2, \quad \forall\, z \in C(t).$$

This shows by (1.1) and (1.5) that

$$-\dot{x}(t) - f_1(t, x(t)) - \int_{T_0}^{t} f_2(t, s, x(s))ds \in N_{C(t)}(x(t)), \;\; a.e. \;\; t \in [T_0, T],$$

and thus

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))ds, \;\; a.e. \;\; t \in [T_0, T].$$

Now consider the situation when

$$\int_{T_0}^{T} \left[\beta_1(\tau) + \int_{T_0}^{\tau} \beta_2(\tau, s)\, ds\right] d\tau \geq \frac{1}{4}.$$

We fix a subdivision of $[T_0, T]$ given by $T_0, T_1, ..., T_k = T$ such that, for any $0 \leq i \leq k-1$,

$$\int_{T_i}^{T_{i+1}} \left[\beta_1(\tau) + \int_{T_0}^{\tau} \beta_2(\tau, s)\, ds\right] d\tau < \frac{1}{4}.$$

Then, by what precedes, there exists an absolutely continuous mapping $x_0 : [T_0, T_1] \longrightarrow H$ such that $x_0(T_0) = x_0$, $x_0(t) \in C(t)$ for all $t \in [T_0, T_1]$, and

$$-\dot{x}_0(t) \in N_{C(t)}(x_0(t)) + f_1(t, x_0(t)) + \int_{T_0}^{t} f_2(t, s, x_0(s))\, ds, \;\; a.e. \;\; t \in [T_0, T_1].$$

Similarly, there is an absolutely continuous mapping $x_1 : [T_1, T_2] \longrightarrow H$ such that $x_1(T_1) = x_0(T_1)$, $x_1(t) \in C(t)$ for all $t \in [T_1, T_2]$, and

$$-\dot{x}_1(t) \in N_{C(t)}(x_1(t)) + f_1(t, x_1(t)) + \int_{T_0}^{t} f_2(t, s, x_1(s))\, ds, \;\; a.e. \;\; t \in [T_1, T_2].$$

By induction, we obtain for each $0 \leq i \leq k-1$ a finite sequence of absolutely continuous mappings $x_i : [T_i, T_{i+1}] \longrightarrow H$ such that for each $0 \leq i \leq k-1$, $x_i(T_i) = x_{i-1}(T_i)$ and $x_i(t) \in C(t)$ for all $t \in [T_i, T_{i+1}]$, and

$$-\dot{x}_i(t) \in N_{C(t)}(x_i(t)) + f_1(t, x_i(t)) + \int_{T_0}^{t} f_2(t, s, x_i(s))\, ds, \;\; a.e. \;\; t \in [T_i, T_{i+1}].$$

We set $x_{-1}(0) = x_0$ and define the mapping $x : [T_0, T] \longrightarrow H$ given by

$$x(t) = x_i(t), \text{ if } t \in [T_i, T_{i+1}], \ 0 \le i \le k-1.$$

Obviously, $x(\cdot)$ is an absolutely continuous mapping satisfying $x(T_0) = x_0$, $x(t) \in C(t)$ for all $t \in [T_0, T]$ and

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))\, ds, \ \ a.e.\ t \in [T_0, T], \tag{2.42}$$

which means that $x(\cdot)$ is a solution of (P_{f_1,f_2}).

Step 7. We prove the estimations.

Let $x(\cdot)$ be a solution of (P_{f_1,f_2}). Take a Borel set $N \subset [T_0, T]$ with null Lebesgue measure such that the inclusion (2.42) holds for every $t \in [T_0, T] \setminus N$. Fix any $t \in [T_0, T] \setminus N$. By (1.1) and (1.5) there is some real $a_0 > 0$ such that for any $a \in]0, a_0]$

$$x(t) \in \mathrm{Proj}_{C(t)}\Big(x(t) - a\dot{x}(t) - af_1(t, x(t)) - a\int_{T_0}^{t} f_2(t, s, x(s))\, ds\Big).$$

We derive from the latter inclusion that

$$\begin{aligned}
& a\Big\|\dot{x}(t) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))\, ds\Big\| \\
& = d_{C(t)}\Big(x(t) - a\dot{x}(t) - a\varepsilon f_1(t, x(t)) - a\int_{T_0}^{t} f_2(t, s, x(s))\, ds\Big) \\
& \le |\upsilon(t) - \upsilon(\tau)| + \Big\|x(t) - x(\tau) - a\dot{x}(t) - af_1(t, x(t)) - a\int_{T_0}^{t} f_2(t, s, x(s))\, ds\Big\|,
\end{aligned}$$

since $x(\tau) \in C(\tau)$ for all $\tau \in [T_0, T]$. For any $\tau \in]T_0, t[$ with $t - a_0 < \tau < t$, taking $a = t - \tau$ one obtains

$$\begin{aligned}
& \Big\|\dot{x}(t) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))\, ds\Big\| \\
& \le \frac{|\upsilon(t) - \upsilon(\tau)|}{t - \tau} + \Big\|\frac{x(t) - x(\tau)}{t - \tau} - \dot{x}(t) - f_1(t, x(t)) - \int_{T_0}^{t} f_2(t, s, x(s))\, ds\Big\|.
\end{aligned}$$

Making $\tau \uparrow t$ yields

$$\begin{aligned}
\Big\|\dot{x}(t) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))\, ds\Big\| & \le |\dot{\upsilon}(t)| + \Big\| - f_1(t, x(t)) - \int_{T_0}^{t} f_2(t, s, x(s))\, ds\Big\| \\
& \le |\dot{\upsilon}(t)| + \|f_1(t, x(t))\| + \int_{T_0}^{t} \|f_2(t, s, x(s))\|\, ds.
\end{aligned} \tag{2.43}$$

This justifies (2.9).

Now assume

$$\int_{T_0}^{t}\left[\beta_1(\tau)+\int_{T_0}^{\tau}\beta_2(\tau,s)\,ds\right]d\tau<\frac{1}{4}.$$

We have from (2.26), (2.27) and (2.28) that the estimates (2.10), (2.11) and (2.12) are obviously fulfilled.

If in addition if

$$\|f_2(t,s,x)\|\le g(t,s)+\alpha(t)\|x\|,$$

we have from (2.43) that

$$\begin{aligned}\|\dot{x}(t)\|\le&|\dot{v}(t)|+2\|f_1(t,x(t))\|+2\int_{T_0}^{t}\|f_2(t,s,x(s))\|\,ds\\ \le&|\dot{v}(t)|+2\beta_1(t)(1+\|x(t)\|)+2\int_{T_0}^{t}g(t,s)\,ds+2\alpha(t)\int_{T_0}^{t}\|x(s)\|\,ds \qquad (2.44)\\ =&|\dot{v}(t)|+2\beta_1(t)+2\int_{T_0}^{t}g(t,s)\,ds+2\beta_1(t)\|x(t)\|+2\alpha(t)\int_{T_0}^{t}\|x(s)\|\,ds.\end{aligned}$$

Putting $\rho(t):=\|x_0\|+\int_{T_0}^{t}\|\dot{x}(s)\|\,ds$ and noting that for a.e. $t\in[T_0,T]$, $\|x(t)\|\le\rho(t)$, the inequality (2.44) ensures that

$$\dot{\rho}(t)\le|\dot{v}(t)|+2\beta_1(t)+2\int_{T_0}^{t}g(t,s)\,ds+2\beta_1(t)\rho(t)+2\alpha(t)\int_{T_0}^{t}\rho(s)\,ds.$$

Applying Gronwall Lemma 2.1.2 with $\rho(\cdot)$, one obtains

$$\begin{aligned}\|x(t)\|\le\rho(t)\le\|x_0\|\exp\left(\int_{T_0}^{t}(b(\tau)+1)\,d\tau\right)\\ +\int_{T_0}^{t}\left(|\dot{v}(s)|+2\beta_1(s)+2\int_{T_0}^{s}g(s,\tau)\,d\tau\right)\exp\left(\int_{s}^{t}(b(\tau)+1)\,d\tau\right)ds,\end{aligned}$$

where $b(\tau):=2\max\{\beta_1(\tau),\alpha(\tau)\}$ for almost all $\tau\in[T_0,T]$. This yields the validity of (2.13), (2.14), (2.15) and (2.16).

Step 8. Uniqueness.

Now, we turn to the uniqueness. If $x_1(\cdot),x_2(\cdot)$ are two solutions, the hypo-monotonicity property of the normal cone in Proposition 1.3.2 yields for almost all $t \in [T_0, T]$

$$\langle -\dot{x}_1(t) - f_1(t, x_1(t)) - \int_{T_0}^{t} f_2(t, s, x_1(s))\,ds + \dot{x}_2(t) + f_1(t, x_2(t)) + \int_{T_0}^{t} f_2(t, s, x_2(s))\,ds,$$

$$x_2(t) - x_1(t)\rangle \leq \frac{1}{2r}\|x_2(t) - x_1(t)\|^2 \sum_{i=1}^{2} \Big(\|\dot{x}_i(t)\| + \|f_1(t, x_i(t))\| + \int_{T_0}^{t} \|f_2(s, x_i(s))\|\,ds \Big),$$

from which we obtain

$$\begin{aligned} &\langle \dot{x}_2(t) - \dot{x}_1(t), x_2(t) - x_1(t)\rangle \\ &\leq \frac{1}{2r}\|x_2(t) - x_1(t)\|^2 \sum_{i=1}^{2} \Big(\|\dot{x}_i(t)\| + \|f_1(t, x_i(t))\| + \int_{T_0}^{t} \|f_2(t, s, x_i(s))\|\,ds \Big) \\ &+ \langle f_1(t, x_1(t)) - f_1(t, x_2(t)), x_2(t) - x_1(t)\rangle \\ &+ \langle \int_{T_0}^{t} f_2(t, s, x_1(s))\,ds - \int_{T_0}^{t} f_2(t, s, x_2(s))\,ds, x_2(t) - x_1(t)\rangle. \end{aligned}$$

Since the absolutely continuous mappings $x_1(\cdot)$ and $x_2(\cdot)$ are in particular bounded on $[T_0, T]$, we can choose some real $\eta > 0$ such that, for each $i = 1, 2$, $\|x_i(t)\| \leq \eta$ for all $t \in [T_0, T]$. The latter inequality assures us that

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\|x_2(t) - x_1(t)\|^2 \leq L_2^{\eta}(t)\|x_2(t) - x_1(t)\| \int_{T_0}^{t} \|x_2(s) - x_1(s)\|\,ds \\ &+ \Big(L_1^{\eta}(t) + \frac{1}{2r} \sum_{i=1}^{2} \Big(\|\dot{x}_i(t)\| + \|f_1(t, x_i(t))\| + \int_{T_0}^{t} \|f_2(t, s, x_i(s))\|\,ds \Big) \Big) \|x_2(t) - x_1(t)\|^2. \end{aligned}$$

Finally, setting $\rho(t) := \|x_2(t) - x_1(t)\|^2$ we get

$$\begin{aligned} \dot{\rho}(t) \leq &\Big(2L_1^{\eta}(t) + \frac{1}{r} \sum_{i=1}^{2} \Big(\|\dot{x}_i(t)\| + \|f_1(t, x_i(t))\| + \int_{T_0}^{t} \|f_2(t, s, x_i(s))\|\,ds \Big) \Big) \rho(t) \\ &+ 2L_2^{\eta}(t)\sqrt{\rho(t)} \int_{T_0}^{t} \sqrt{\rho(s)}\,ds, \end{aligned}$$

hence it suffices to invoke Lemma 2.1.3 with $\varepsilon(\cdot), \epsilon > 0$ arbitrary. Then the proof of the theorem is complete . ■

Now, we give the following stability result, if the initial data of the prolem x_0 change slightly, then the corrosponding solutions would not differ much. Mor precisely we have the following proposition.

Proposition 2.2.2. *Assume that the assumptions of Theorem 2.2.1 (in case 3) holds. For each $a \in C(T_0)$, denote by $x_a(\cdot)$ the unique solution of the integro-differential sweeping process*

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int\limits_{T_0}^{t} f_2(t, s, x(s))\, ds \quad a.e \;\; in \;\; [T_0, T], \\ x(T_0) = a \in C(T_0). \end{cases}$$

Then, the map $\psi : a \longrightarrow x_a(\cdot)$ from $C(T_0)$ to the space $\mathcal{C}([T_0, T], H)$ endowed with the supremum norm (of uniform convergence) is Lipschitz on any bounded subset of $C(T_0)$.

Proof. Let M be any fixed positive real number. We are going to prove that ψ is Lipschitz on $C(T_0) \cap B[0, M]$. According to Theorem 2.2.1 (case 3), there exists a real number M_1 depending only on M such that, for all $z \in C(T_0) \cap B[0, M]$ and for almost all $(t, s) \in Q_\Delta$

$$\begin{aligned} &\|\dot{x}_z(t) + f_1(t, x_z(t)) + \int\limits_{T_0}^{t} f_2(t, s, x_z(s))\, ds\| \\ &\leq \varphi(t) := |\dot{v}(t)| + (1 + M_1)\beta_1(t) + \int\limits_{T_0}^{t} g(t, s)\, ds + T\alpha(t) M_1. \end{aligned}$$

Thanks to this last inequality, for some $\eta > 0$ depending only on M, for all $z \in C(T_0) \cap B[0, M]$ and for all $t \in [T_0, T]$, we have

$$x_z(t) \in B[0, \eta]. \tag{2.45}$$

Fix any $a, b \in C(T_0) \cap M\mathbb{B}$. By the hypomonotonicity property of the normal cone in Proposition 1.3.2 we have for almost all $(t, s) \in Q_\Delta$

$$\begin{aligned} &\Big\langle -\dot{x}_a(t) - f_1(t, x_a(t)) - \int\limits_{T_0}^{t} f_2(t, s, x_a(s))\, ds + \dot{x}_b(t) + f_1(t, x_b(t)) + \int\limits_{T_0}^{t} f_2(t, s, x_b(s))\, ds, x_2(t) - x_1(t) \Big\rangle \\ &\leq \frac{\varphi(t)}{r} \|x_b(t) - x_a(t)\|^2, \end{aligned}$$

from which we obtain

$$\begin{aligned} &\langle \dot{x}_b(t) - \dot{x}_a(t), x_b(t) - x_a(t) \rangle \\ &\leq \frac{\varphi(t)}{r} \|x_b(t) - x_a(t)\|^2 + \langle f_1(t, x_a(t)) - f_1(t, x_b(t)), x_b(t) - x_a(t) \rangle \\ &+ \Big\langle \int\limits_{T_0}^{t} f_2(t, s, x_a(s))\, ds - \int\limits_{T_0}^{t} f_2(t, s, x_b(s))\, ds, x_b(t) - x_a(t) \Big\rangle. \end{aligned}$$

Since, by the assumptions $(\mathcal{H}_{2,2})$ and $(\mathcal{H}_{3,2})$, there are non-negative functions $L_1^\eta(\cdot)$ and $L_2^\eta(\cdot)$ in $L^1([T_0, T], \mathbb{R})$ such that $f_1(t, \cdot)$ and $f_2(t, s, \cdot)$ are $L_1^\eta(t)$-Lipschitz and $L_2^\eta(t)$-Lipschitz respectively on $B[0,\eta]$, the above inequality along with (2.45), entails that for almost all $t \in [T_0, T]$,

$$\frac{d}{dt}\|x_b(t)-x_a(t)\|^2 \leq 2\left(L_1^\eta(t)+\frac{\varphi(t)}{r}\right)\|x_b(t)-x_a(t)\|^2 +2L_2^\eta(t)\|x_b(t)-x_a(t)\| \int_{T_0}^{t} \|x_b(s) - x_a(s)\| \, ds.$$

Applying the Gronwall-like differential inequality in Lemma 2.1.3 , it results that

$$\sup_{t\in[0,T]} \|x_b(t) - x_a(t)\| \leq \|b - a\| \exp\left(\int_{T_0}^{t} (K(s) + 1) \, ds\right),$$

where $K(t) := \max\left\{L_1^\eta(t) + \dfrac{\varphi(t)}{r}, L_2^\eta(t)\right\}$ for all $t \in [T_0, T]$. The proof is then complete. ■

2.3 Nonlinear integro-differential complementarity systems

In the present section, as a consequence of Theorem 2.2.1, we obtain the existence and uniqueness of solutions for nonlinear integro-differential complementarity systems. Our results generalize those from [3].

Let $T > T_0$ be real numbers, $I = [T_0, T]$, $n, m \in \mathbb{N}$, $f_1 : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $f_2 : I^2 \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be given mappings. For $u_1, u_2 \in \mathbb{R}^m$ we will write $0 \leq u_1 \perp u_2 \leq 0$ to mean that $u_1 \in \mathbb{R}^m_+$, $u_2 \in -\mathbb{R}^m_+$ and $\langle u_1, u_2\rangle = 0$, where $\langle \cdot, \cdot\rangle$ is the canonic scalar product in $\mathbb{R}^m$. Assuming that $g(t, \cdot)$ is differentiable for each $t \in I$ and denoting $\nabla_2 g(t, y)$ the gradient of $g(t, \cdot)$ at y, the NIDCS (associated with f_1, f_2 and g) can be described as

$$(\text{ NIDCS }) : \begin{cases} -\dot{x}(t) = f_1(t, x(t)) + \displaystyle\int_{T_0}^{t} f_2(t, s, x(s)) \, ds + \nabla_2 \, g(t, x(t))^T z(t) \\ 0 \leq z(t) \perp g(t, x) \leq 0, \end{cases}$$

where $z : I \longrightarrow \mathbb{R}^m$ is unknown mapping. The term $\nabla_2 \, g(t, x(t))^T z(t)$ can be seen as the generalized reactions due to the constraints in mechanics.

For a mapping $z : [T_0, T] \to \mathbb{R}^m$ we note that

$$z(t) \in \mathbb{R}_m^+ \quad \text{and} \quad \langle z(t), g(t, x)\rangle = 0 \iff z(t) \in N_{\mathbb{R}^m_+}(g(t, x)).$$

So, proceeding as, for example, in [3, Section 9.2] with $9.2 - 9.3$ therein, (NIDCS) is equivalent to

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))\, ds, \tag{2.46}$$

where

$$C(t) := \{x \in \mathbb{R}^n \ : \ g_1(t, x) \leq 0,\ g_2(t, x) \leq 0,\ ...,\ g_m(t, x) \leq 0\}, \tag{2.47}$$

and where we set $g(t, \cdot) = (g_1(t, \cdot), g_2(t, \cdot), ..., g_m(t, \cdot))$ for each $t \in I$.

Theorem 2.3.1. *[3] Let $C(t)$ be defined as in (2.47) and assume that, there exists an extended real $\rho \in]0, \infty]$ such that*

1. *for all $t \in I$, for all $k \in \{1, ..., m\}$, $g_k(t, \cdot)$ is continuously differentiable on $U_\rho(C(t))$ where*
$$U_\rho(C(t)) := \{y \in \mathbb{R}^n \ : \ d_{C(t)}(y) < \rho\}.$$
2. *there exists a real $\gamma > 0$ such that, for all $t \in I$, for all $k \in \{1, ..., m\}$, and for all $x, y \in U_\rho(C(t))$*
$$\langle \nabla_2\, g_k(t, x) - \nabla_2\, g_k(t, y), x - y\rangle \geq -\gamma |x - y|^2,$$
that is, $\nabla_2\, g_k(t, \cdot)$ is γ-hypomonotone on $U_\rho(C(t))$.
3. *there is a real $\delta > 0$ such that for all $(t, x) \in I \times \mathbb{R}^n$ with $x \in \operatorname{bdry}(C(t))$, there exists $\bar{v} \in B[0, 1]$ satisfying, for all $k \in \{1, ..., m\}$*
$$\langle \nabla_2\, g(t, x), \bar{v}\rangle \leq -\delta. \tag{2.48}$$

Then for all $t \in I$, the set $C(t)$ is r-prox-regular with $r = \min\{\rho, \dfrac{\delta}{\gamma}\}$.

The nonlinear differential complementarity system (NDCS) (i.e., (NIDCS) with $f_2 \equiv 0$) was studied in [3], where the authors transform the (NDCS) involving inequality constraints $C(t)$ to a perturbed sweeping process . We extend this approach by employing the above transformation of (NIDCS) into an integro-differential sweeping process of the form (P_{f_1, f_2}). Also, in contrast to [3], we do not assume that the moving set $C(t)$ described by a finite number of inequalities is absolutely continuous with respect to the Hausdorff distance. Rather, we provide sufficient verifiable conditions ensuring this type of regularity needed on $C(\cdot)$.

Proposition 2.3.1. *Let $C(t)$ be defined as in (2.47). Assume that there exist an absolutely continuous function w, a real $\delta > 0$ and a vector $y \in \mathbb{R}^n$ with $\|y\| = 1$ such that for any $i = 1, ..., m$ and any $s, t \in I$*

$$g_i(t, x) \leq g_i(s, x) + |w(t) - w(s)|, \quad \textit{for all} \quad x \in U_r(C(s)), \tag{2.49}$$

$$\langle \nabla_2 g_i(t,x), y\rangle \leq -\delta, \quad \textit{for all} \quad t \in I,\, x \in U_r(C(t)), \tag{2.50}$$

where r denotes the prox-regularity constant of all sets $C(t)$. Then $C(\cdot)$ is absolutely continuous on I in the sense of (2.6) *with $\upsilon(\cdot) := \delta^{-1}w(\cdot)$.*

Proof. Let δ, y and $w(\cdot)$ be as given in the statement. Let $s,t \in I$, let $x \in C(s)$ and choose a subdivision $T_0 < T_1 < ... < T_p = T$ such that $\int_{T_{k-1}}^{T_k} |\dot{\upsilon}(\tau)|\, d\tau < r$ for every $k = 1, ..., p$. Fix any $k = 1, ..., p$ and $s, t \in [T_{k-1}, T_k]$. Take any $i = 1, ..., m$ and note that

$$\begin{aligned} g_i(t, x + |\upsilon(t) - \upsilon(s)|y) &= (g_i(t, x + |\upsilon(t) - \upsilon(s)|y) - g_i(s, x + |\upsilon(t) - \upsilon(s)|y)) \\ &\quad + g_i(s, x + |\upsilon(t) - \upsilon(s)|y) \\ &\leq |w(t) - w(s)| + g_i(s, x + |\upsilon(t) - \upsilon(s)|y) \\ &= |w(t) - w(s)| + g_i(s, x) \\ &\quad + \int_0^1 \langle \nabla_2 g_i(s, x + \theta y|\upsilon(t) - \upsilon(s)|), y|\upsilon(t) - \upsilon(s)|\rangle d\,\theta. \end{aligned}$$

According to (2.50) and to the inclusion $x \in C(s)$ it ensues that

$$g_i(t, x + |\upsilon(t) - \upsilon(s)|y) \leq |w(t) - w(s)| - \delta|\upsilon(t) - \upsilon(s)| \leq 0.$$

This being true for every $i = 1, ..., m$, it follows that $x+|\upsilon(t)-\upsilon(s)|y$ belongs to $C(t)$, otherwise stated, $x \in C(t) + |\upsilon(t) - \upsilon(s)|(-y)$. It results that $C(s) \subset C(t) + |\upsilon(t) - \upsilon(s)|B[0, 1]$. Since the variables s and t play symmetric roles, the set-valued mapping $C(\cdot)$ has an absolutely continuous variation on $[T_{k-1}, T_k]$ in the sense of (2.6). From this we clearly derive that $C(\cdot)$ has an absolutely continuous variation on I. ■

Exemple 2.3.1. *Let $m = 1$, $n = 2$, $T_0 = 0$, $T = 1$, $g(t, x) = t^{\frac{1}{3}} - x_1 - x_2^2$, and define*

$$C(t) = \{x \in \mathbb{R}^2 : g(t, x) \leq 0\}.$$

Clearly, $C(t)$ is r-prox-regular, since $g(t, \cdot)$ satisfies all assumptions of Theorem 2.3.1 for all $t \in I$. Now we check (2.49) *and* (2.50)*. Let $x \in \mathbb{R}^2$, $t, s \in I$. Fix any $\delta \in (0, 1]$ and put $y = (1, 0)$. Then for $w(t) := t^{1/3}$ we have*

$$g(t, x) - g(s, x) = t^{\frac{1}{3}} - s^{\frac{1}{3}} \leq |t^{\frac{1}{3}} - s^{\frac{1}{3}}| = |w(t) - w(s)|,$$

$$\langle \nabla_2 g(t, x), y\rangle = -1 \leq -\delta.$$

We see that $w(t) = t^{1/3}$ is not Lipschitz on I but it is absolutely continuous there. Then $C(\cdot)$ has an absolutely continuous variation υ on I in the sense of (2.6)*, with $\upsilon(t) = t^{1/3}/\delta$.*

Theorem 2.3.2. *Assume that the assumptions in Theorem 2.3.1, Proposition 2.3.1 and conditions* $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ *are satisfied. Then, for every initial data* x_0 *with* $g(0, x_0) \leq 0$, *problem (NIDCS) has one and only one solution* $x(\cdot)$.

Proof. Like, for example, in [3, Section 9.2], the result follows from the above equivalence between the problem (NIDCS) and the integro-differential sweeping process (2.46) since all assumptions of Theorem 2.2.1 are satisfied according to Theorem 2.3.1 and Proposition 2.3.1.

■

2.4 Applications to non-regular electrical circuits

The aim of this section is to illustrate the integro-differential sweeping process in the theory of non-regular electrical circuits. Electrical devices like diodes are described in terms of Ampere-Volt characteristic which is (possibly) a multifunction expressing the difference of potential v_D across the device as a function of current i_D going through the device [15].

2.4.1 Non-regular electrical circuits with time-varying capacitors

Time-varying capacitors are known to be important for the study of diverse electrical circuits as can be seen in [13, 31, 37, 45]. A time-varying linear capacitor is presented in pages 49-51 of [31]. Moreover, in page 50 of the same book [31] it is emphasized how time-varying capacitors are useful for the study of parametric amplifiers and of diverse *physical and biological systems*. The usefulness of time-varying resistors and time-varying inductors is also discussed in [31].

Consider the electrical system shown in Fig.2.1 that is composed of three resistors $R_1 \geq 0$, $R_2 \geq 0$ with voltage/current laws $V_{R_k} = R_k x_k$ $(k = 1, 2)$, two inductors $L_1 \geq 0$, $L_2 \geq 0$ with voltage/current laws $V_{L_k} = L_k \dot{x}_k$ $(k = 1, 2)$, three capacitors with time-varying capacitances $C_1(t) \neq 0$, $C_2(t) \neq 0$ and $C_3(t) \neq 0$ with voltage/current laws $V_{C_k}(t) = \frac{1}{C_k(t)} \int_0^t x_k(\tau)\, d\tau$, $k = 1, 2, 3$, two ideal diodes with characteristics $0 \leq -V_{D_k} \perp i_k \geq 0$ and an absolutely continuous current source $i : [0, T] \to \mathbb{R}$.

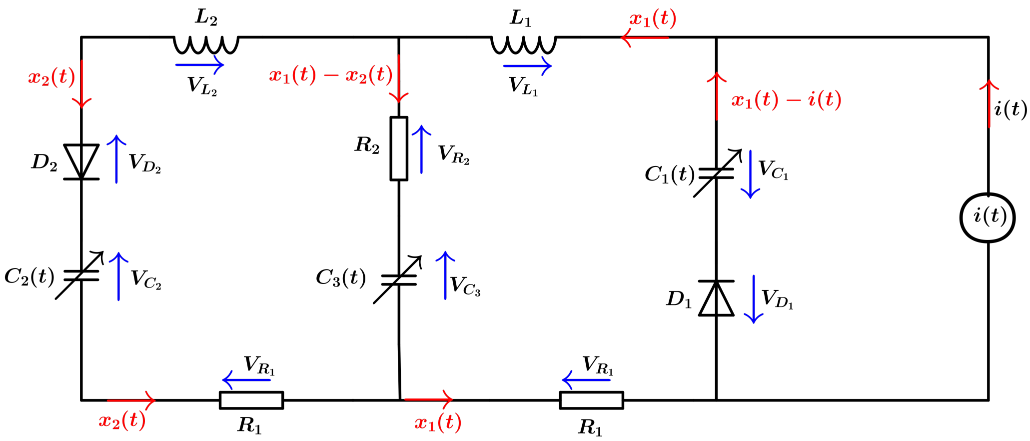

Figure 2.1: Electrical circuit with resistors, inductances, time-varying capacitors and ideal diodes.

We refer to [13, 31, 37, 45] for diverse systems with time-varying capacitors. For the system in Fig.2.1 using Kirchhoff's laws, we have

$$\begin{cases} V_{R_1}+V_{R_2}+V_{L_1}+V_{C_1}+V_{C_3}=-V_{D_1}\in -N_{\mathbb{R}_+}(x_1-i)\\ V_{R_1}-V_{R_2}+V_{L_2}+V_{C_2}-V_{C_3}=-V_{D_2}\in -N_{\mathbb{R}_+}(x_2). \end{cases}$$

Therefore the dynamics of this circuit is given by

$$\begin{aligned} &\overbrace{\begin{pmatrix}-\dot{x}_1(t)\\ -\dot{x}_2(t)\end{pmatrix}}^{-\dot{x}(t)} \in N_{[i(t),+\infty[\times[0,+\infty[}(x(t)) + \overbrace{\begin{pmatrix}\frac{R_1+R_2}{L_1} & -\frac{R_2}{L_1}\\ -\frac{R_2}{L_2} & \frac{R_1+R_2}{L_2}\end{pmatrix}}^{A_1}\overbrace{\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}}^{x(t)}\\ &\quad+\int_0^t\Bigg[\overbrace{\begin{pmatrix}\frac{1}{L_1C_1(t)}+\frac{1}{L_1C_3(t)} & -\frac{1}{L_1C_3(t)}\\ -\frac{1}{L_2C_3(t)} & \frac{1}{L_2C_2(t)}+\frac{1}{L_2C_3(t)}\end{pmatrix}}^{A_2}\overbrace{\begin{pmatrix}x_1(s)\\ x_2(s)\end{pmatrix}}^{x(s)}+\begin{pmatrix}\frac{1}{L_1C_1(t)}i(s)\\ 0\end{pmatrix}\Bigg]ds. \end{aligned} \tag{2.51}$$

Proposition 2.4.1. *Assume that* $i:[0,T]\longrightarrow\mathbb{R}$ *is an absolutely continuous function and* $C_k:[0,T]\longrightarrow\mathbb{R}\setminus\{0\}$, $k=1,2,3$ *are continuous functions. Then for any initial condition* $x(0)=x_0\in C(0)$, *problem* (2.51) *has one and only one absolutely continuous solution* $x(\cdot)$.

Proof. Put $w(t)=(i(t),0)^T$, $C(t):=w(t)+[0,+\infty[\times[0,+\infty[$, $f_1(t,x)=A_1x$, $f_2(t,s,x)=A_2(t)x+\dfrac{1}{L_1C_1(t)}w(s)$. With this (2.51) can be rewritten in the frame of our problem (P_{f_1,f_2}) as

$$\begin{cases} -\dot{x}(t)\in N_{C(t)}(x(t))+f_1(t,x(t))+\int_0^t f_2(t,s,x(s))ds \quad \text{a.e. } in\ [0,T],\\ x(0)=x_0\in C(0). \end{cases}$$

Then the above data satisfy all the assumptions of Theorem 2.2.1 (precisely case 3), with

$$v(t) = \int_0^t \|\dot{w}(s)\|\,ds, \;\; \beta_1(t) = \|A_1\|, \;\; g(t,s) = \frac{1}{L_1 C_1(t)}\|w(s)\|, \;\; \alpha(t) = \|A_2(t)\|.$$

This finishes the proof. ■

This example is another illustration of the applicability of the above developments. It is worth noticing that the above integrand perturbation $f_2(\cdot,\cdot)$ is not uniformly bounded, then the existence result of [34] is not applicable here, since it is assumed in (f_1) in [34, p. 232] that $\|f_{0,2}(s,x)\| \leq M$. However, according to (P_φ) (with $\varphi \equiv 1$) the above example can be treated by reduction to a classical perturbed sweeping process (PSP). So, we provide next another example of circuit for which such a reduction is no longer applicable.

2.4.2 Non-regular electrical circuit with tranmission line, diode and inductor

We pass now to diode and inductor models which are connected to transmission lines. Consider first a *transmission line* as presented in Figure 1.9 of D.E. Stewart's book [76, p. 12] with an inductor with inductance L_0, a resistor with resistance R_0, a capacitor with capacitance C_0 and a leakage conductor with leakage conductance G_0 per unit length of the transmission line. Using Laplace transform, Stewart [76, p 13] obtained

$$\begin{cases} V(t,0) = (k * I(\cdot,0))(t) + q(t), \\ k(t) = \frac{L_0}{C_0}\delta(t) + k_1(t), \end{cases} \tag{2.52}$$

where $*$ denotes the convolution product, k_1 is bounded on $[0,T]$ and δ is the Dirac function.

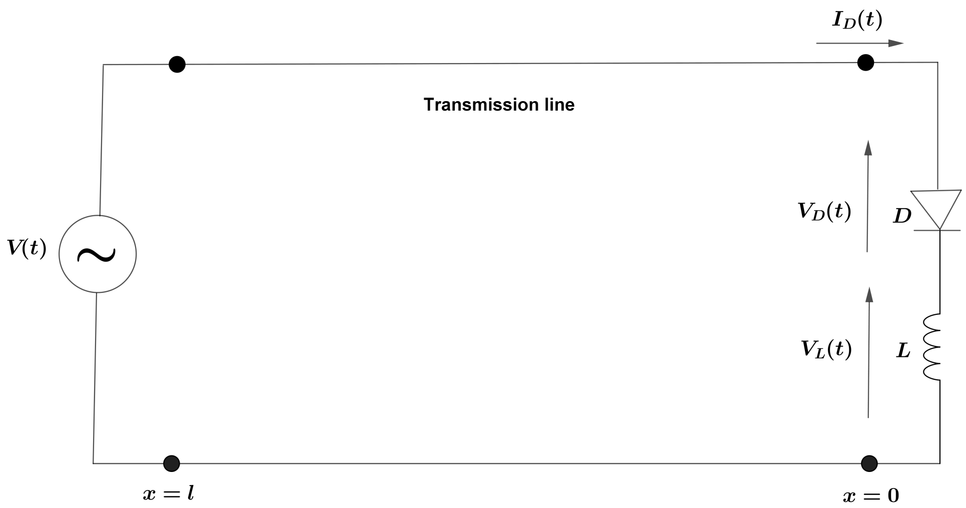

Figure 2.2: Circuit with diode, inductor and transmission line

Consider now the circuit in Fig.2.2 with the above transmission line, an ideal diode D and an inductor with inductance L. The dynamic of this circuit is then

$$\begin{cases} L\dot{I}(t,0) + V_D(t) = V(t,0) + R_D I(t,0) = (k * I(\cdot,0))(t) + R_D I(t,0) \\ I_D(t) = I(t,0), \\ V_D(t) \in N_{[0,+\infty[}(I_D(t)), \end{cases} \tag{2.53}$$

or equivalently (noting as in [76, p.142] that $(\delta * I(\cdot,0))(t) = I(t,0)$)

$$\begin{cases} L\dot{I}(t,0) + V_D(t) = \int\limits_0^t k_1(t-s)I(s,0)ds + \frac{L_0}{C_0}I(t,0) + q(t) + R_D I(t,0), \\ I_D(t) = I(t,0), \\ V_D(t) \in N_{[0,+\infty[}(I_D(t)). \end{cases} \tag{2.54}$$

Put $x(t) := I_D(t)$, $C(t) := [0,+\infty[$, $f_1(t,x) := \frac{1}{L}q(t) + \big(\frac{L_0}{C_0 L} + \frac{R_D}{L}\big)x$, $f_2(t,s,x) := \frac{1}{L}k_1(t-s)x$. Then (2.54) can be rewritten in the frame of our problem (P_{f_1,f_2}) as

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(x(t)) + f_1(t,x(t)) + \int\limits_0^t f_2(t,s,x(s))ds \quad \text{a.e. } in\ [0,T], \\ x(0) = x_0 \in C(0). \end{cases}$$

All the conditions in Theorem 2.2.1 are satisfied, so the following proposition is directly derived.

Proposition 2.4.2. *The above problem* (2.54) *has one and only one absolutely continuous solution* $x(\cdot)$.

It is clear that the foregoing time-dependent variational inequality (2.52) does not involve the derivative of the unknown function, so it does not cover the integro-differential model (2.54). In Subsection 4.6.1 of Stewart's monograph [76] the system (2.52) is solved by means of an iteration technique and fixed point argument. We mention that no solution of model (2.54) is provided either in [76] or (to the best of our knowledge) in the literature. Further, from the form $\int_0^t f_2(t,s,x(s))\,ds = \int_0^t k_1(t-s)x(s)\,ds$ in (2.54) we clearly see that (2.54) is an inclusion relating to the general integro-differential sweeping process developed in Section 2.2.

We emphasize that due to the above form $f_2(t,s,x(s)) = k_1(t-s)x(s)$ neither the result nor the approach in [34] is capable to deal with (2.53)-(2.54) (we remind that the integral perturbation in [34] is of the form $\int_0^t f_{0,2}(s,x(s))\,ds$ not suitable here).

To end the section, it must be said that constraints coming from electric circuit are generally convex. However, nonconvex prox-regular constraint cases after certain transformations of equations for electric circuits was efficiently utilized in the analysis of such problems, e.g., in [16, Section 4].

2.5 An integro-differential sweeping process approach to a frictionless contact problem

This section provides another application of our results. We consider a quasistatic problem which models the contact between a deformable body and an obstacle, the so-called foundation. The material is assumed to have a viscoelastic behavior which is modeled by a constitutive law with long-term memory, thus, at each moment of time, the stress tensor depends not only on the present strain tensor, but also on its whole history. The contact is frictionless and is modeled by the well-known Signorini conditions. We refer to [44, 48, 72, 75] for the modeling details of this kind of problem. For our purpose of motivation, the main concern is to derive a formulation of the problem, expressed in terms of integro-differential sweeping process, and to prove its unique solvability under appropriate regularity hypotheses.

Functions spaces.

First we introduce notation which will be employed in the description of the contact problem. Let $d \in \{1,2,3\}$ and let $\mathbb{S}^d$ denote the space of second-order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order d. As usual for mechanical contact problems, generic vectors and tensors in $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by boldface characters, and index notation will be utilized for their components, so $\boldsymbol{\zeta} \in \mathbb{R}^d$ and $\boldsymbol{\alpha} \in \mathbb{S}^d$ can be written as $\boldsymbol{\zeta} = (\zeta_i)$ and $\boldsymbol{\alpha} = (\alpha_{ij})$. The zero elements of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted $\mathbf{0}_{\mathbb{R}^d}$ and $\mathbf{0}_{\mathbb{S}^d}$ respectively. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are canonically defined by

$$\boldsymbol{\zeta} \cdot \boldsymbol{\xi} = \sum_i \zeta_i \cdot \xi_i, \quad \|\boldsymbol{\zeta}\| = (\boldsymbol{\zeta} \cdot \boldsymbol{\zeta})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\zeta} = (\zeta_i),\, \boldsymbol{\xi} = (\xi_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \sum_{i,j} \alpha_{ij} \cdot \beta_{ij}, \quad \|\boldsymbol{\alpha}\| = (\boldsymbol{\alpha} \cdot \boldsymbol{\alpha})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\alpha} = (\alpha_{ij}),\, \boldsymbol{\beta} = (\beta_{ij}) \in \mathbb{S}^d,$$

where the indices i, j in the above sums run from 1 to d. Here it is convenient to denote $\boldsymbol{\zeta} \cdot \boldsymbol{\xi}$ the inner product instead of $\langle \boldsymbol{\zeta}, \boldsymbol{\xi} \rangle$.

We consider a viscoelastic body which occupies a domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary Γ. We denote by $\overline{\Omega} = \Omega \cup \Gamma$ the closure of Ω in $\mathbb{R}^d$. The boundary Γ is decomposed into three parts $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ with Γ_1, Γ_2 and Γ_3 being relatively open and mutually disjoint and, moreover, the (area/surface) measure $\text{meas}(\Gamma_1)$ relative to Γ is positive, i.e., $\text{meas}(\Gamma_1) > 0$.

As usual, $H^1(\Omega)$ is the Sobolev space of real-valued functions in $L^2(\Omega)$ with first order distributional derivatives in $L^2(\Omega)$ as well. Denoting $H^1(\Omega)^d$ the space of mappings $\boldsymbol{v} : \Omega \to \mathbb{R}^d$ with $v_i \in H^1(\Omega)$, $i = 1, \cdots, d$, we will use the spaces

$$V = \{\boldsymbol{v} \in H^1(\Omega)^d \ : \ \boldsymbol{v} = 0 \ \text{ on } \Gamma_1\},$$

$$Q = \{\boldsymbol{\theta} = (\theta_{ij}) \ : \ \theta_{ij} = \theta_{ji} \in L^2(\Omega)\}.$$

The spaces Q and V are endowed with the canonical inner products given by

$$(\boldsymbol{\theta}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\theta} \cdot \boldsymbol{\tau}\, dx, \quad (\boldsymbol{u}, \boldsymbol{v})_{\mathcal{E}} = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q.$$

Here $\boldsymbol{\varepsilon}$ represents the deformation operator, that is,

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \ldots, d,$$

and the index "$\mathcal{E}$" is utilized to emphasize that the inner product $(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{E}}$ is constructed by means of the function $\boldsymbol{\varepsilon}(\cdot)$. Put $\|\boldsymbol{\tau}\| = (\boldsymbol{\tau}, \boldsymbol{\tau})_Q^{1/2}$ and $\|\boldsymbol{v}\|_{\mathcal{E}} = (\boldsymbol{v}, \boldsymbol{v})_{\mathcal{E}}^{1/2}$. The space Q endowed with the inner product $(\cdot, \cdot)_Q$ and the associated norm $\|\cdot\|_Q$ is clearly a Hilbert space. Regarding V, by the assumption $\text{meas}(\Gamma_1) > 0$ Korn's inequality (see, e.g., [46, Lemma 6.2, p 115]) tells us that for some constant $\kappa > 0$ we have $\kappa \|\boldsymbol{v}\|_{H^1(\Omega)^d} \leq \|\boldsymbol{v}\|_{\mathcal{E}}$ for all $\boldsymbol{v} \in V$, and from this and the definition of $\|\cdot\|_{\mathcal{E}}$ we see that $\|\cdot\|_{\mathcal{E}}$ is a norm on V which is equivalent to $\|\cdot\|_{H^1(\Omega)^d}$ on V. Therefore, the space V endowed with the inner product $(\cdot, \cdot)_{\mathcal{E}}$ and the associated norm $\|\cdot\|_{\mathcal{E}}$ is also a Hilbert space.

For a vector $\boldsymbol{v} \in V$, its normal and tangential components are $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$, respectively, where $\boldsymbol{\nu}$ denotes the outward unit normal vector to the boundary Γ. The normal and tangential components of the stress tensor $\boldsymbol{\sigma}$ on the boundary Γ are denoted by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$, respectively. In addition, we recall that the Sobolev trace theorem yields

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c\|\boldsymbol{v}\|_E \quad \text{for all } \boldsymbol{v} \in V, \tag{2.55}$$

c being a positive constant which depends on Ω, Γ_1 and Γ_3.
Next, we recall that the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx + \int_\Omega \operatorname{Div}\boldsymbol{\sigma}\cdot\boldsymbol{v}\,dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,da \quad \text{for all } \boldsymbol{v} \in H^1(\Omega)^d, \tag{2.56}$$

where Div denotes the divergence operator given by $\operatorname{Div}\boldsymbol{\sigma} = (\sum_j \frac{\partial\sigma_{ij}}{\partial x_j})$, that is, the sum $\sum_j \frac{\partial\sigma_{ij}}{\partial x_j}$ is the i-th component of $\operatorname{Div}\boldsymbol{\sigma}$.
Let $\boldsymbol{Q}_\infty$ be the space of fourth order tensor fields given by

$$\boldsymbol{Q}_\infty = \{\boldsymbol{e} = (e_{ijkh}) \;:\; e_{ijkh} = e_{jikh} = e_{khij} \in L^\infty(\Omega),\ 1 \le i,j,k,h \le d\}.$$

It is easy to see that $\boldsymbol{Q}_\infty$ is a real Banach space with the norm

$$\|\boldsymbol{e}\|_{\boldsymbol{Q}_\infty} = \max_{1\le i,j,k,h\le d} \|e_{ijkh}\|_{L^\infty(\Omega)},$$

and, moreover,

$$\|\boldsymbol{e}\boldsymbol{\tau}\|_Q \le d\|\boldsymbol{e}\|_{\boldsymbol{Q}_\infty}\|\boldsymbol{\tau}\|_Q \quad \text{for all } \boldsymbol{e} \in \boldsymbol{Q}_\infty,\ \boldsymbol{\tau} \in Q, \tag{2.57}$$

where $\boldsymbol{e}\boldsymbol{\tau}$ is the tensor function in Q given by its i,j components as $\boldsymbol{e}\boldsymbol{\tau} = (\sum_{k,h} e_{ijkh}\tau_{kh})$. More on actions of tensors on vectors and matrices can be found, e.g., in [38].

Classically, for $\boldsymbol{u}\colon \Omega\times[0,T] \to \mathbb{R}^d$ and $\boldsymbol{\sigma}\colon \Omega\times[0,T] \to \mathbb{S}^d$ it can be convenient (as below) to denote by $\boldsymbol{u}(t)$ and $\boldsymbol{\sigma}(t)$ the mappings $\boldsymbol{u}(\cdot,t)$ and $\boldsymbol{\sigma}(\cdot,t)$.

The formulation of the problem is as follows.

Problem 1. *Find $\boldsymbol{u}\colon \Omega\times[0,T] \to \mathbb{R}^d$ and $\boldsymbol{\sigma}\colon \Omega\times[0,T] \to \mathbb{S}^d$ with $u_i(\cdot,t)$ and $\sigma_{ij}(\cdot,t)$ in $H^1(\Omega)$ such that for a.e. $t \in]0,T[$*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \quad \text{in} \quad \Omega, \tag{2.58}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{2.59}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) \quad \text{on} \quad \Gamma_2, \tag{2.60}$$

$$u_\nu(t) \le 0,\ \ \sigma_\nu(t) \le 0,\ \ \sigma_\nu(t)u_\nu(t) = 0,\ \ \boldsymbol{\sigma}_\tau(t) = 0 \quad \text{on} \quad \Gamma_3, \tag{2.61}$$

and

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1\times]0,T[, \tag{2.62}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega. \tag{2.63}$$

Here, $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$, $\mathcal{R} : [0,T] \to \mathbf{Q}_\infty$, $\mathcal{B} : [0,T] \times Q \to Q$ are prescribed mappings, and $\mathcal{B}$ is defined in the form $\mathcal{B}(t,\boldsymbol{\theta})(x) = \mathcal{B}_0(x,t,\boldsymbol{\theta}(x))$ for all $x \in \Omega$, where $\mathcal{B}_0 : \Omega \times]0,T[\times \mathbb{S}^d \to \mathbb{S}^d$.

Now, we present a short description of the conditions in Problem 1. Equation (2.58) represents the viscoelastic constitutive law with long memory in which $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{R}$ denote the viscosity, elasticity and relaxation operators, respectively. Equations of type (2.58) are related to the Kelvin-Voigt law, so when $\mathcal{R}$ vanishes, (2.58) reduces to the well known Kelvin-Voigt constitutive law extensively studied in the literature, see Shillor, Sofonea and Telega [72, Chapter 8], and the references therein. Equation (2.59) is the equilibrium equation, while conditions (2.62) and (2.60) are the displacement and traction boundary conditions, respectively. Conditions (2.61) represent the frictionless Signorini contact conditions in which u_ν denotes the normal displacement, σ_ν represents the normal stress, and $\boldsymbol{\sigma}_\tau$ is the tangential stress on the potential contact surface. Finally, (2.63) represents the initial condition in which $\boldsymbol{u}_0$ is the initial displacement field.
We consider the following usual hypotheses (see, e.g., [74]):

$\big(\mathcal{H}(\mathcal{A})\big)$: We assume that the viscosity tensor $\mathcal{A} = (a_{ijkh})\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the natural properties of symmetry and ellipticity :

(a) $a_{ijkh} \in L^\infty(\Omega)$.

(b) $\mathcal{A}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{A}\boldsymbol{\tau}$ for all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d$, a.e. in Ω.

(c) $\exists\, m_{\mathcal{A}} > 0\colon$ $\mathcal{A}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq m_{\mathcal{A}} \|\boldsymbol{\tau}\|^2_{\mathbb{S}^d}$ for all $\boldsymbol{\tau} \in \mathbb{S}^d$, a.e. in Ω.

We recall that the i,j component of the tensor function $\mathcal{A}\boldsymbol{\tau}$ is $\sum\limits_{kh} a_{ijkh}\tau_{kh}$.

$\big(\mathcal{H}(\mathcal{B})\big)$: $\mathcal{B}_0\colon \Omega \times]0,T[\times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(a) There is $L_{\mathcal{B}} \geq 0$ such that

$$\|\mathcal{B}_0(\boldsymbol{x},t,\boldsymbol{\alpha}_1) - \mathcal{B}_0(\boldsymbol{x},t,\boldsymbol{\alpha}_2)\|_{\mathbb{S}^d} \leq L_{\mathcal{B}} \|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\|_{\mathbb{S}^d},$$

for all $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{S}^d$ and a.e. $(\boldsymbol{x},t) \in \Omega \times]0,T[$.

(b) $\mathcal{B}_0(\cdot,\cdot,\boldsymbol{\varepsilon})$ is Borel measurable on $\Omega \times]0,T[$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$.

(c) $\mathcal{B}_0(\cdot,t,\mathbf{0}_{\mathbb{S}^d})$ belongs to Q for all $t \in]0,T[$.

$\big(\mathcal{H}(\mathcal{R}, \boldsymbol{f}_0, \boldsymbol{f}_N)\big)$: The prescribed relaxation tensor $\mathcal{R}$ and densities of body forces $\boldsymbol{f}_0$ and surface tractions $\boldsymbol{f}_N$ are such that

(a) $\mathcal{R} \in \mathcal{C}([0,T], \boldsymbol{Q}_\infty)$.

(b) $\boldsymbol{f}_0 \in \mathcal{C}([0,T], L^2(\Omega)^d)$.

(c) $\boldsymbol{f}_N \in \mathcal{C}([0,T], L^2(\Gamma_2)^d)$.

Now, we turn to an analysis of any eventual solution of Problem 1 (if any).
To this end we assume in what follows that the viscosity and elasticity operators satisfy assumptions $\big(\mathcal{H}(\mathcal{A})\big)$ and $\big(\mathcal{H}(\mathcal{B})\big)$, respectively. The relaxation operator, the densities of body forces and the surface tractions satisfy the assumption $\big(\mathcal{H}(\mathcal{R}, \boldsymbol{f}_0, \boldsymbol{f}_N)\big)$.
We also introduce the set of admissible displacements fields, defined by

$$U = \{\boldsymbol{v} \in V \ : \ v_\nu \leq 0 \quad \text{a.e. on } \Gamma_3\}, \tag{2.64}$$

and we note that U is a closed convex subset of V such that $\mathbf{0}_V \in U$. And, finally, the initial displacement satisfies $\boldsymbol{u}_0 \in U$. For $\boldsymbol{u}, \boldsymbol{v} \in V$ let

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \quad \|\boldsymbol{u}\|_V = (\boldsymbol{u}, \boldsymbol{u})_V^{\frac{1}{2}}. \tag{2.65}$$

Using the assumption $\big(\mathcal{H}(\mathcal{A})\big)$ we obtain that $(\cdot,\cdot)_V$ is an inner product on V and $\|\cdot\|_V$ and $\|\cdot\|_\mathcal{E}$ are equivalent norms on V. Therefore, $(V, \|\cdot\|_V)$ is a real Hilbert space.
Next, with the volume measure dx and the area/surface measure da on Γ, we notice that

$$\boldsymbol{v} \mapsto \int\limits_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int\limits_{\Gamma_2} \boldsymbol{f}_N(t) \cdot \boldsymbol{v}\, da \ \ \forall \boldsymbol{v} \in V,\ t \in [0,T],$$

is a continuous linear functional on the space V. Therefore, we may apply the Riesz representation theorem to define the element $\boldsymbol{f}(t) \in V$ by the equality

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int\limits_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\, dx + \int\limits_{\Gamma_2} \boldsymbol{f}_N(t) \cdot \boldsymbol{v}\, da \ \ \forall \boldsymbol{v} \in V,\ t \in [0,T]. \tag{2.66}$$

Let $(\boldsymbol{u}, \boldsymbol{\sigma})$ be a pair of feasible functions, satisfying (2.58)-(2.63). Fix any t in a suitable (full Lebesgue measure) subset of $]0,T[$ over which (2.58)-(2.61) hold. Let $\boldsymbol{v} \in U$. Using the Green formula (2.56) and using (2.59) we have

$$\int\limits_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\, dx = \int\limits_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, da + \int\limits_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, dx.$$

From the boundary conditions (2.62), (2.60) and the following decomposition formula $\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) = \sigma_\nu(t)(v_\nu - u_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))$ on Γ_3, it ensues that

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\, dx = \int_{\Gamma_2} \boldsymbol{f}_N(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, da + \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - u_\nu(t))\, da \\ + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\, da + \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, dx. \end{aligned} \tag{2.67}$$

Using (2.64) and putting (2.61) into (2.67) we obtain

$$\int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\, dx \geq \int_{\Gamma_2} \boldsymbol{f}_N(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, da + \int_\Omega \boldsymbol{f}_0(t) \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, dx. \tag{2.68}$$

The inequality (2.68), the constitutive law (2.58) and the initial conditions (2.63) yield that any solution of Probem 1 is a solution of the following Problem 2.

Problem 2. *Find the displacement field* $\boldsymbol{u}\colon [0,T] \to V$, *such that*

$$\begin{cases} \big(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\big)_Q + \big(\mathcal{B}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\big)_Q \\ + \bigg(\displaystyle\int_0^t \mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\bigg)_Q \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \forall \boldsymbol{v} \in U, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{u}(t) \in U \ \textit{for}\ t \in [0,T]. \end{cases} \tag{2.69}$$

We will see below that Problem 2 has one and only one solution. Thus, consider the unique solution $\boldsymbol{u}$ of Problem 2. Let $D := D(\Omega; \mathbb{R}^d) = C_0^\infty(\Omega; \mathbb{R}^d)$ denote the space of all mappings defined on Ω with values in $\mathbb{R}^d$ which are infinitely differentiable and have compact support in Ω. Consider any $\varphi \in D$ and take $\boldsymbol{v} := \boldsymbol{u}(t) + \boldsymbol{\varphi}$. Clearly, $\boldsymbol{v} \in U$ since $D(\Omega; \mathbb{R}^d) \subset U$. Then by the inequality in (2.69), by (2.58) and (2.66) we have in the sense of distribution that

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \rangle_{D' \times D} \geq \langle \boldsymbol{f}_0(t), \boldsymbol{\varphi} \rangle_{D' \times D} \quad \text{for all } \boldsymbol{\varphi} \in D(\Omega; \mathbb{R}^d) \ \text{ a.e. } t \in [0,T].$$

We perform integrations by parts to obtain that

$$\langle -\mathrm{Div}\, \boldsymbol{\sigma}(t), \boldsymbol{\varphi} \rangle_{D' \times D} \geq \langle \boldsymbol{f}_0(t), \boldsymbol{\varphi} \rangle_{D' \times D} \quad \text{for all } \boldsymbol{\varphi} \in D(\Omega; \mathbb{R}^d).$$

Similarly, taking $\boldsymbol{v} := \boldsymbol{u}(t) - \boldsymbol{\varphi}$ and using the same arguments we also have that

$$\langle -\mathrm{Div}\, \boldsymbol{\sigma}(t), \boldsymbol{\varphi} \rangle_{D' \times D} \leq \langle \boldsymbol{f}_0(t), \boldsymbol{\varphi} \rangle_{D' \times D} \quad \text{for all } \boldsymbol{\varphi} \in D(\Omega; \mathbb{R}^d).$$

So, it follows that

$$\mathrm{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = 0.$$

On the other hand, it is clear by definition of the spaces V and U that

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1 \times [0,T],$$

and for a.e. $t \in]0, T[$

$$u_\nu(t) \leq 0, \quad \text{on} \quad \Gamma_3.$$

Notice also that $\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega.$

Suppose in addition that the mapping $\boldsymbol{u}$ is smooth, in the sense that $u(\cdot, t) \in \mathcal{C}^2(\Omega)$, and that Γ_2 and Γ_3 are $\mathcal{C}^\infty$-smooth for example. Then Theorem 6.3 in the book [46] of Kikuchi and Oden along with the comments subsequent to that theorem in that book, ensue that for a.e. $t \in]0, T[$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) \quad \text{on} \quad \Gamma_2,$$

$$\sigma_\nu(t) \leq 0, \quad \sigma_\nu(t)u_\nu(t) = 0, \quad \boldsymbol{\sigma}_\tau(t) = 0 \quad \text{on} \quad \Gamma_3.$$

So, under the above smoothness conditions, $\boldsymbol{u}$ is solution of Problem 1.

To summarize, Problem 2 admits one and only one solution (as we will see below), and any solution of Problem 1 (if any) coincides with the solution of Problem 2. Furthermore, under the above regularity of Γ_2 and Γ_3, if the unique solution $\boldsymbol{u}$ of Problem 2 possesses the regularity $\boldsymbol{u}(\cdot, t) \in \mathcal{C}^2(\Omega)$, then it is a solution of Problem 1. The conclusion is that the unique solution $\boldsymbol{u}$ of Problem 2 (furnished by the next theorem) is a right weak solution for the concerned Problem 1.
After the preceding analysis, we present our existence and uniqueness result for Problem 2.

Theorem 2.5.1. *Under the above assumptions, for each $\boldsymbol{u}_0 \in U$, Problem 2 has a unique absolutely continuous solution $\boldsymbol{u}$.*

Proof. The proof consists of two parts in which we rewrite Problem 2 in an equivalent form of integro-differential sweeping process and apply the result of Theorem 2.2.1. To this end, denoting by $\mathcal{L}(V)$ the space of continuous linear operators from V into itself, we apply the Riesz representation theorem to define the operators $B\colon [0, T] \times V \to V$ and $R\colon [0, T] \to \mathcal{L}(V)$ by

$$(B(t, \boldsymbol{v}), \boldsymbol{w})_V = \big(\mathcal{B}(t, \boldsymbol{\varepsilon}(\boldsymbol{v})), \boldsymbol{\varepsilon}(\boldsymbol{w})\big)_Q \quad (R(t)\boldsymbol{v}, \boldsymbol{w})_V = (\mathcal{R}(t)\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q, \tag{2.70}$$

for all $\boldsymbol{v}, \boldsymbol{w} \in V,\ t \in [0, T]$. Moreover, using (2.65) and inequality (2.69), we derive the following variational inequality for a.e. $t \in]0, T[$

$$\begin{cases} \big(\dot{\boldsymbol{u}}(t), \boldsymbol{v} - \boldsymbol{u}(t)\big)_V + \big(B(t, \boldsymbol{u}(t)), \boldsymbol{v} - \boldsymbol{u}(t)\big)_V \\ + \Big(\displaystyle\int_0^t R(t-s)\boldsymbol{u}(s)\, ds, \boldsymbol{v} - \boldsymbol{u}(t)\Big)_V \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \quad \text{for all } \boldsymbol{v} \in U, \end{cases} \tag{2.71}$$

along with $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\boldsymbol{u}(t) \in U$. Then, the variational inequality (2.71) subject to the latter conditions is equivalent to the following integro-differential inclusion

$$\begin{cases} -\dot{\boldsymbol{u}}(t) \in N_U(\boldsymbol{u}(t)) + B(t,\boldsymbol{u}(t)) - \boldsymbol{f}(t) + \int\limits_0^t R(t-s)\boldsymbol{u}(s)\,ds \quad \text{a.e. } t \in [0,T], \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \in U. \end{cases} \tag{2.72}$$

Now, we prove the existence and uniqueness result for problem (2.72), by applying Theorem 2.2.1. In what follows, we will verify that the data of problem (2.72) satisfy hypotheses of Theorem 2.2.1 on the space $H = V$.

(I). Clearly, $C(\cdot) = U$ satisfy $(\mathcal{H}_1)$ since U is a fixed nonempty closed convex subset of V.

(II). The function f_1 defined by $f_1(t,\boldsymbol{v}) = B(t,\boldsymbol{v}) - \boldsymbol{f}(t)$ for all $t \in [0,T]$ and all $\boldsymbol{v} \in V$ satisfies for some real constant $k > 0$ the hypothesis $(\mathcal{H}_2)$ with $\beta_1(t) = \max(k^2 L_{\mathcal{B}}, k\|\mathcal{B}(t,\boldsymbol{0}_{\mathbb{S}^d})\|_Q + \|\boldsymbol{f}(t)\|_V)$ and $L_1(t) = k^2 L_{\mathcal{B}}$ for all $t \in [0,T]$.

Indeed, by definition of operator B in (2.70) we see for all $\boldsymbol{v} \in V$ that

$$\begin{aligned} \|B(t,\boldsymbol{v})\|_V^2 &= \big(B(t,\boldsymbol{v}), B(t,\boldsymbol{v})\big)_V = \big(\mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v})), \boldsymbol{\varepsilon}(B(t,\boldsymbol{v}))\big)_Q \\ &\le \|\mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v}))\|_Q \cdot \|\boldsymbol{\varepsilon}(B(t,\boldsymbol{v}))\|_Q = \|\mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v}))\|_Q \cdot \|B(t,\boldsymbol{v})\|_{\mathcal{E}} \\ &\le k\|\mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v}))\|_Q \cdot \|B(t,\boldsymbol{v})\|_V, \quad \text{for some constant } k > 0, \end{aligned}$$

recall that $\|\cdot\|_V$ and $\|\cdot\|_{\mathcal{E}}$ are equivalent norms on V. On the other hand, using $\big(\mathcal{H}(\mathcal{B})\big)$ yields

$$\begin{aligned} \|B(t,\boldsymbol{v})\|_V &\le k\|\mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v}))\|_Q \le k(\|\mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v})) - \mathcal{B}(t,\boldsymbol{0}_{\mathbb{S}^d})\|_Q + \|\mathcal{B}(t,\boldsymbol{0}_{\mathbb{S}^d})\|_Q) \\ &\le k(L_{\mathcal{B}}\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q + \|\mathcal{B}(t,\boldsymbol{0}_{\mathbb{S}^d})\|_Q) = k(L_{\mathcal{B}}\|\boldsymbol{v}\|_E + \|\mathcal{B}(t,\boldsymbol{0}_{\mathbb{S}^d})\|_Q) \\ &\le k(kL_{\mathcal{B}}\|\boldsymbol{v}\|_V + \|\mathcal{B}(t,\boldsymbol{0}_{\mathbb{S}^d}\|_Q). \end{aligned}$$

We conclude that

$$\begin{aligned} \|f_1(t,\boldsymbol{v})\|_V &\le \|B(t,\boldsymbol{v})\|_V + \|\boldsymbol{f}(t)\|_V \le k(kL_{\mathcal{B}}\|\boldsymbol{v}\|_V + \|\mathcal{B}(t,\boldsymbol{0}_{\mathbb{S}^d}\|_Q) + \|\boldsymbol{f}(t)\|_V \\ &\le \max(k^2 L_{\mathcal{B}}, k\|\mathcal{B}(t,\boldsymbol{0}_{\mathbb{S}^d})\|_Q + \|\boldsymbol{f}(t)\|_V)(1 + \|\boldsymbol{v}\|_V). \end{aligned}$$

Similarly, given $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ we have by the way that $B(t,\boldsymbol{v})$ has been defined

$$\begin{aligned} \|B(t,\boldsymbol{v}_1) - B(t,\boldsymbol{v}_2)\|_V &= \sup_{\|\boldsymbol{w}\|_V \le 1} (\mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v}_1)) - \mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v}_2)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \\ &\le \sup_{\|\boldsymbol{w}\|_V \le 1} \|\mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v}_1)) - \mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v}_2))\|_Q \|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_Q \\ &\le k\|\mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v}_1)) - \mathcal{B}(t,\boldsymbol{\varepsilon}(\boldsymbol{v}_2))\|_Q. \end{aligned}$$

From this and $\big((H)(\mathcal{B})\big)$ we obtain for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$

$$\begin{aligned} \|B(t,\boldsymbol{v}_1) - B(t,\boldsymbol{v}_2)\|_V &\le kL_{\mathcal{B}}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \boldsymbol{\varepsilon}(\boldsymbol{v}_2)\|_Q = kL_{\mathcal{B}}\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{\mathcal{E}} \\ &\le k^2 L_{\mathcal{B}}\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V. \end{aligned}$$

(III). The function $f_2(t,s,\boldsymbol{v}) = R(t-s)\boldsymbol{v}$ for all $(t,s) \in Q_\Delta$ and $\boldsymbol{v} \in V$ satisfies, for some real constant $k > 0$, the hypothesis $(\mathcal{H}_3)$ with $\beta_2(t,s) = k^2 d\|\mathcal{R}(t-s)\|_{Q_\infty}$ and $L_2(t) = k^2 d \sup_{t\in[0,T]} \|\mathcal{R}(t)\|_{Q_\infty}$ for all $(t,s) \in Q_\Delta$.
Indeed, by definition of operator R in (2.70) we have for all $\boldsymbol{v} \in V$ and all $(t,s) \in Q_\Delta$ that

$$\begin{aligned}
\|R(t-s)\boldsymbol{v}\|_V^2 &= (R(t-s)\boldsymbol{v}, R(t-s)\boldsymbol{v})_V = (\mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(R(t-s)\boldsymbol{v}))_Q \\
&\le \|\mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \cdot \|\boldsymbol{\varepsilon}(R(t-s)\boldsymbol{v})\|_Q = \|\mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \cdot \|R(t-s)\boldsymbol{v}\|_{\mathcal{E}} \\
&\le k\|\mathcal{R}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \cdot \|R(t-s)\boldsymbol{v}\|_V.
\end{aligned}$$

Next, using assumptions $\big(\mathcal{H}(\mathcal{R}, \boldsymbol{f}_0, \boldsymbol{f}_N)\big)$-$(a)$ and the inequality (2.57), we obtain

$$\begin{aligned}
\|f_2(t,s,\boldsymbol{v})\|_V = \|R(t-s)\boldsymbol{v}\|_V &\le kd\|\mathcal{R}(t-s)\|_{Q_\infty} \cdot \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q = kd\|\mathcal{R}(t-s)\|_{Q_\infty} \cdot \|\boldsymbol{v}\|_{\mathcal{E}} \\
&\le k^2 d\|\mathcal{R}(t-s)\|_{Q_\infty} \cdot \|\boldsymbol{v}\|_V \\
&\le k^2 d\|\mathcal{R}(t-s)\|_{Q_\infty} \cdot (1 + \|\boldsymbol{v}\|_V).
\end{aligned}$$

Further, by (2.73) we have for any $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and $(t,s) \in Q_\Delta$ that

$$\begin{aligned}
\|f_2(t,s,\boldsymbol{v}_1) - f_2(t,s,\boldsymbol{v}_2)\|_V &= \|R(t-s)\boldsymbol{v}_1 - R(t-s)\boldsymbol{v}_2\|_V = \|R(t-s)(\boldsymbol{v}_1 - \boldsymbol{v}_2)\|_V \\
&\le k^2 d\|\mathcal{R}(t-s)\|_{Q_\infty} \cdot \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \\
&\le k^2 d \sup_{t\in[0,T]} \|\mathcal{R}(t)\|_{Q_\infty} \cdot \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V.
\end{aligned}$$

We have verified that all hypotheses of Theorem 2.2.1 are satisfied. Hence, we deduce that problem (2.72) has a unique absolutely continuous solution $\boldsymbol{u}$, so Problem 2 has a unique solution. The proof of the Theorem is then complete. ■

Chapter 3

On the Discretization of Truncated Integro-Differential Sweeping Process and Optimal Control

Abstract. We consider the Volterra integro-differential equation with a time-dependent prox-regular constraint that changes in an absolutely continuous way in time (a Volterra absolutely continuous time-dependent sweeping process). The aim of our chapter is twofold. The first one is to show the solvability of the initial value problem by setting up an appropriate catching-up algorithm (full discretization). This part is a continuation of Chapter 2 where we used a semi-discretization method. Obviously, strong solutions and convergence of full discretization scheme are desirable properties, especially for numerical simulations. Applications to non-regular electrical circuits are provided. The second aim is to establish the existence of optimal solution to an optimal control problem involving the Volterra integro-differential sweeping process.

3.1 A full discretization for the integro-differential sweeping process

Given in all the sequel, for each $t \in [T_0, T]$, a nonempty closed subset $C(t)$ of H which is r-prox-regular for some extended real $r \in]0, +\infty]$ and given $x_0 \in C(T_0)$, our main results, in this section, are stated under the following assumptions:

$(\mathcal{H}_1)$ There are an absolutely continuous function $\upsilon : [T_0, T] \longrightarrow \mathbb{R}$ and an extended real

$\rho \in]0, +\infty]$ with $\rho \geq 2\left(\|x_0\| + \int_{T_0}^{T} \dot{v}(s)\,\mathrm{d}s + \frac{1}{2}\right)$ such that

$$\mathrm{haus}_\rho(C(t), C(s)) \leq |v(t) - v(s)|, \quad \forall\, s, t \in [T_0, T],$$

where $\mathrm{haus}_\rho(\cdot,\cdot)$ denotes the Hausdorff-Pompeiu ρ-pseudo distance, see (1.6) for the definition.

$(\mathcal{H}_2)$ The mapping $f_1 : [T_0, T] \times H \longrightarrow H$ is (Bochner) measurable in time (i.e., $f(\cdot, x)$ is (Bochner) measurable for each $x \in H$), and uniformly continuous in the state on bounded sets (i.e., $f(t, \cdot)$ is uniformly continuous on bounded sets), and such that

$(\mathcal{H}_{2,1})$ there exists a non-negative function $\beta_1(\cdot) \in L^1([T_0, T], \mathbb{R})$ such that

$$\|f_1(t, x)\| \leq \beta_1(t)(1 + \|x\|), \text{ for any } t \in [T_0, T] \text{ and any } x \in \bigcup_{t\in[T_0,T]} C(t),$$

$(\mathcal{H}_{2,2})$ for each real $\eta > 0$ there exists a non-negative function $L_1^\eta(\cdot) \in L^1([T_0, T], \mathbb{R})$ such that for any $t \in [T_0, T]$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$,

$$\langle f_1(t, x) - f_1(t, y), x - y\rangle \geq -L_1^\eta(t)\|x - y\|^2.$$

$(\mathcal{H}_3)$ For $Q_\Delta := \{(t, s) \in [T_0, T] \times [T_0, T] : s \leq t\}$ the mapping $f_2 : Q_\Delta \times H \longrightarrow H$ is (Bochner) measurable in (t, s) (i.e., $f(\cdot, \cdot, x)$ is (Bochner) measurable for any $x \in H$) and such that

$(\mathcal{H}_{3,1})$ there exists a non-negative function $\beta_2(\cdot,\cdot) \in L^1(Q_\Delta, \mathbb{R})$ such that

$$\|f_2(t, s, x)\| \leq \beta_2(t, s)(1 + \|x\|), \text{ for any } (t, s) \in Q_\Delta \text{ and any } x \in \bigcup_{t\in[T_0,T]} C(t),$$

$(\mathcal{H}_{3,2})$ for each real $\eta > 0$ there exists a non-negative function $L_2^\eta(\cdot) \in L^1([T_0, T], \mathbb{R})$ such that for any $(t, s) \in Q_\Delta$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$,

$$\|f_2(t, s, x) - f_2(t, s, y)\| \leq L_2^\eta(t)\|x - y\|.$$

Now, we are ready to state the main result.

Theorem 3.1.1. *Let H be a real Hilbert space, let $C(t)$ be an r-prox-regular set in H for each $t \in [T_0, T]$ and let $x_0 \in C(T_0)$. Assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are satisfied. Then there exists a unique absolutely continuous solution $x : [T_0, T] \longrightarrow H$ of the Volterra integro-differential inclusion:*

$$(P_{f_1,f_2}) : \begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))\,\mathrm{d}s, & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0). \end{cases} \tag{3.1}$$

Furthermore, this solution satisfies:

(a) For a.e. $t \in [T_0, T]$

$$\|\dot{x}(t)+f_1(t,x(t))+\int_{T_0}^{t} f_2(t,s,x(s))\,\mathrm{d}s\| \le |\dot{v}(t)|+\|f_1(t,x(t))\|+\int_{T_0}^{t}\|f_2(t,s,x(s))\|\,\mathrm{d}s. \quad (3.2)$$

(b) If $\int_{T_0}^{T}\left[\beta_1(\tau)+\int_{T_0}^{\tau}\beta_2(\tau,s)\,\mathrm{d}s\right]\mathrm{d}\tau < \frac{1}{4}$, *one has*

$$\|f_1(t,x(t))\| \le l\beta_1(t), \ \textit{for all}\ t\in[T_0,T], \quad (3.3)$$

$$\|f_2(t,s,x(s))\| \le l\beta_2(t,s), \ \textit{for all}\ (t,s)\in Q_\Delta, \quad (3.4)$$

and for almost all $t \in [T_0, T]$

$$\left\|\dot{x}(t)+f_1(t,x(t))+\int_{T_0}^{t} f_2(t,s,x(s))\,\mathrm{d}s\right\| \le (l+1)\left(\beta_1(t)+\int_{T_0}^{t}\beta_2(t,s)\,\mathrm{d}s\right)+|\dot{v}(t)|, \quad (3.5)$$

where $l = 2\left(\|x_0\|+\int_{T_0}^{T}|\dot{v}(\tau)|\,\mathrm{d}\tau+1\right)$.

(c) Assume the following strengthened form of assumption $(\mathcal{H}_{3,1})$ *on the function* f_2 *holds:* $(\mathcal{H}'_{3,1})$ *: there exist non-negative functions* $\alpha(\cdot)\in L^1([T_0,T],\mathbb{R})$ *and* $g(\cdot)\in L^1(Q_\Delta,\mathbb{R})$ *such that*

$$\|f_2(t,s,x)\| \le g(t,s)+\alpha(t)\|x\|, \ \textit{for any}\ (t,s)\in Q_\Delta \ \textit{and any}\ x\in \bigcup_{t\in[T_0,T]} C(t).$$

Then we have

$$\|f_1(t,x(t))\| \le (\widetilde{l}+1)\beta_1(t), \quad \textit{for all}\, t\in[T_0,T], \quad (3.6)$$

$$\|f_2(t,s,x(s))\| \le g(t,s)+\alpha(t)\widetilde{l}, \quad \textit{a.e.}\ (t,s)\in Q_\Delta, \quad (3.7)$$

and for almost all $t \in [T_0, T]$

$$\|\dot{x}(t)+f_1(t,x(t))+\int_{T_0}^{t} f_2(t,s,x(s))\,\mathrm{d}s\| \le |\dot{v}(t)|+(\widetilde{l}+1)\beta_1(t)+\int_{T_0}^{t} g(t,s)\,\mathrm{d}s+T\alpha(t)\widetilde{l}, \quad (3.8)$$

where

$$\widetilde{l} = \left(\|x_0\|+\int_{T_0}^{T}\left(|\dot{v}(s)|+2\beta_1(s)+2\int_{T_0}^{T} g(s,\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\right)\exp\left(\int_{T_0}^{T}(b(\tau)+1)\,\mathrm{d}\tau\right),$$

and

$$b(t) = 2\max\{\beta_1(t),\alpha(t)\} \quad \textit{for all} \quad t\in[T_0,T].$$

Proof. We are going to construct a sequence of maps $x_n(\cdot) \in \mathcal{C}([T_0, T], H)$ which has a subsequence converging to a solution of (P_{f_1,f_2}). Note by $(\mathcal{H}_1)$ that, replacing $\dot{v}(t)$ by $|\dot{v}(t)|$, we may suppose (without loss of generality) that $\dot{v}(t) \geq 0$ for all $t \in [T_0, T]$.

Case 1 : First, let us suppose that

$$\int_{T_0}^{T} \left[\beta_1(\tau) + \int_{T_0}^{\tau} \beta_2(\tau, s)\, \mathrm{d}s\right] \mathrm{d}\tau < \frac{1}{4}. \tag{3.9}$$

We develop that case through 6 steps.

Step 1. Discretization of the interval $I = [T_0, T]$**.**

First, we divide I in two intervals with the same length $(T - T_0)/2$, then we divide each of the 2^1 obtained intervals in two intervals with the same length $(T - T_0)/2^2$, etc. Otherwise stated, for each integer $n \geq 1$ we consider the partition of the time interval $[T_0, T]$ associated with

$$t_i^n = T_0 + ih_n, \quad 0 \leq i \leq 2^n, \quad h_n = \frac{T - T_0}{2^n}.$$

Then, for any $t \in]T_0, T]$ and any integers $m > n \geq 1$ there is one and only one pair (i, j) with $0 \leq i \leq 2^n - 1$ and $0 \leq j \leq 2^m - 1$ such that

$$t \in]t_j^m, t_{j+1}^m] \subset]t_i^n, t_{i+1}^n]. \tag{3.10}$$

Let us also set for each $n \in \mathbb{N}$ and each $0 \leq i \leq 2^n - 1$

$$k_i^n = \int_{t_i^n}^{t_{i+1}^n} \dot{v}(s)\, \mathrm{d}s, \quad \mu_i^n = \int_{t_i^n}^{t_{i+1}^n} \beta_1(s)\, \mathrm{d}s, \quad \alpha_i^n = \int_{t_i^n}^{t_{i+1}^n} \int_{T_0}^{\tau} \beta_2(\tau, s) \mathrm{d}s\, \mathrm{d}\tau,$$

and

$$\varepsilon_n = \max_{0 \leq i \leq n-1} \{h_n + \mu_i^n + k_i^n + \alpha_i^n\}.$$

Note that $\varepsilon_n \longrightarrow 0$, since $h_n \longrightarrow 0$ and the functions $\dot{v}$, β_1, β_2 are integrable. Then we can choose an integer $n_0 \geq 1$ satisfying for every $n \geq n_0$

$$\varepsilon_n \leq \frac{r}{3(l+1)}, \quad \text{where } l := 2\left(\|x_0\| + \int_{T_0}^{T} \dot{v}(s)\, \mathrm{d}s + 1\right).$$

Step 2. Construction of finite sequences $(x_i^n)_{0 \leq i \leq 2^n}$**.**

Fix any $n \geq n_0$. Put $x_0^n = x_0$. By induction, let us construct a sequence $(x_i^n)_{1 \leq i \leq 2^n}$ and a sequence $(y_i^n)_{1 \leq i \leq 2^n}$ such that for each $i \in \{1, ..., 2^n\}$

$$y_i^n = x_{i-1}^n - \int_{t_{i-1}^n}^{t_i^n} f_1(s, x_{i-1}^n)\, \mathrm{d}s - \int_{t_{i-1}^n}^{t_i^n} \left\{ \sum_{j=1}^{i-1} \int_{t_{j-1}^n}^{t_j^n} f_2(\tau, s, x_{j-1}^n) \mathrm{d}s + \int_{t_{i-1}^n}^{\tau} f_2(\tau, s, x_{i-1}^n) \mathrm{d}s \right\} \mathrm{d}\tau \tag{3.11}$$

$$x_i^n = \mathrm{Proj}_{C(t_i^n)}(y_i^n), \tag{3.12}$$

$$\max_{0\le j\le i}\|x_j^n\| \le 2\left(\|x_0\| + \int_{T_0}^{T} \dot{v}(s)\,\mathrm{d}s + \frac{1}{2}\right) = l-1 \tag{3.13}$$

and

$$\|x_i^n - y_i^n\| \le (l+1)(k_{i-1}^n + \mu_{i-1}^n + \alpha_{i-1}^n), \tag{3.14}$$

along with for each $i \in \{0, \cdots, 2^n - 1\}$

$$\|x_{i+1}^n\| \le \|x_i^n\| + k_i^n + 2(1+\max_{0\le j\le i}\|x_j^n\|)\left(\int_{t_i^n}^{t_{i+1}^n} \beta_1(s)\,\mathrm{d}s + \int_{t_i^n}^{t_{i+1}^n}\int_{T_0}^{\tau} \beta_2(\tau,s)\mathrm{d}s\,\mathrm{d}\tau\right). \tag{3.15}$$

Using the inequality $\|x_0^n\| \le \rho$ and the fact that $x_n(T_0) \in C(T_0)$, one has

$$\begin{aligned}
&d_{C(t_1^n)}\left(x_0^n - \int_{t_0^n}^{t_1^n} f_1(s,x_0^n)\,\mathrm{d}s - \int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} f_2(\tau,s,x_0^n)\mathrm{d}s\,\mathrm{d}\tau\right)\\
&\le d_{C(t_1^n)}(x_0^n) + \int_{t_0^n}^{t_1^n} \|f_1(s,x_0^n)\|\,\mathrm{d}s + \int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} \|f_2(\tau,s,x_0^n)\|\mathrm{d}s\,\mathrm{d}\tau\\
&\le \sup_{x\in C(T_0)\cap\rho\mathbb{B}} d_{C(t_1^n)}(x) + \int_{t_0^n}^{t_1^n} \|f_1(s,x_0^n)\|\,\mathrm{d}s + \int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} \|f_2(\tau,s,x_0^n)\|\mathrm{d}s\,\mathrm{d}\tau,
\end{aligned}$$

hence by $(\mathcal{H}_1)$

$$\begin{aligned}
&d_{C(t_1^n)}\left(x_0^n - \int_{t_0^n}^{t_1^n} f_1(s,x_0^n)\,\mathrm{d}s - \int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} f_2(\tau,s,x_0^n)\mathrm{d}s\,\mathrm{d}\tau\right)\\
&\le \mathrm{haus}_\rho(C(T_0), C(t_1^n)) + \int_{t_0^n}^{t_1^n} \|f_1(s,x_0^n)\|\,\mathrm{d}s + \int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} \|f_2(\tau,s,x_0^n)\|\mathrm{d}s\,\mathrm{d}\tau\\
&\le |v(t_1^n) - v(T_0)| + \int_{t_0^n}^{t_1^n} \|f_1(s,x_0^n)\|\,\mathrm{d}s + \int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} \|f_2(\tau,s,x_0^n)\|\mathrm{d}s\,\mathrm{d}\tau\\
&= k_0^n + \int_{t_0^n}^{t_1^n} \|f_1(s,x_0^n)\|\,\mathrm{d}s + \int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} \|f_2(\tau,s,x_0^n)\|\mathrm{d}s\,\mathrm{d}\tau.
\end{aligned} \tag{3.16}$$

On the other hand, one has by $(\mathcal{H}_2)$

$$\int_{t_0^n}^{t_1^n} \|f_1(s,x_0^n)\|\,\mathrm{d}s \le (1+\|x_0^n\|)\int_{t_0^n}^{t_1^n} \beta_1(s)\,\mathrm{d}s \le l\mu_0^n, \tag{3.17}$$

and

$$\int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} \|f_2(\tau,s,x_0^n)\|\mathrm{d}s\,\mathrm{d}\tau \le (1+\|x_0^n\|)\int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} \beta_2(\tau,s)\mathrm{d}s\,\mathrm{d}\tau \le l\alpha_0^n. \tag{3.18}$$

We deduce with

$$y_1^n := x_0^n - \int_{t_0^n}^{t_1^n} f_1(s, x_0^n)\,\mathrm{d}s - \int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} f_2(\tau, s, x_0^n)\mathrm{d}s\,\mathrm{d}\tau$$

that

$$d_{C(t_1^n)}(y_1^n) \le (l+1)(k_0^n + \mu_0^n + \alpha_0^n) \le (l+1)\varepsilon_n \le \frac{r}{3} < r. \tag{3.19}$$

The r-prox-regularity of the set $C(t_1^n)$ and Proposition 1.3.3 ensure the existence and the uniqueness of the projection $\mathrm{Proj}_{C(t_1^n)}(y_1^n)$ and then we can define

$$x_1^n = \mathrm{Proj}_{C(t_1^n)}(y_1^n).$$

Then, coming back to (3.19), we obtain that

$$\|x_1^n - y_1^n\| \le (l+1)(k_0^n + \mu_0^n + \alpha_0^n).$$

According to the estimates (3.16), (3.17) and (3.18), we have

$$\begin{aligned}\|x_1^n\| &\le \|x_0^n\| + k_0^n + 2\int_{t_0^n}^{t_1^n} \|f_1(s, x_0^n)\|\,\mathrm{d}s + 2\int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} \|f_2(\tau, s, x_0^n)\|\mathrm{d}s\,\mathrm{d}\tau \\ &\le \|x_0^n\| + k_0^n + 2(1+\|x_0^n\|)\left(\int_{t_0^n}^{t_1^n} \beta_1(s)\,\mathrm{d}s + \int_{t_0^n}^{t_1^n}\int_{t_0^n}^{\tau} \beta_2(\tau, s)\mathrm{d}s\,\mathrm{d}\tau\right),\end{aligned}$$

thus

$$\|x_1^n\| \le \|x_0^n\| + \int_{T_0}^{T} \dot{v}(s)\,\mathrm{d}s + 2(1+\|x_0^n\|)\left(\int_{T_0}^{T} \beta_1(s)\,\mathrm{d}s + \int_{T_0}^{T}\int_{T_0}^{\tau} \beta_2(\tau, s)\mathrm{d}s\,\mathrm{d}\tau\right).$$

It results, thanks to (3.9), that

$$\|x_1^n\| \le \|x_0^n\| + \int_{T_0}^{T} \dot{v}(s)\,\mathrm{d}s + \frac{1}{2}(1+\|x_0^n\|),$$

thus

$$\max\{\|x_1^n\|, \|x_0^n\|\} \le 2\left(\|x_0^n\| + \int_{T_0}^{T} \dot{v}(s)\,\mathrm{d}s + \frac{1}{2}\right) = l - 1,$$

where the constant l is as defined in the statement of the theorem. Now, fix any $i \in \{1, ..., n-1\}$ and suppose that we have constructed $x_1^n, ..., x_i^n$ and $y_1^n, \cdots, y_i^n$ such that for each $q \in \{1, ..., i\}$

$$y_q^n = x_{q-1}^n - \int_{t_{q-1}^n}^{t_q^n} f_1(s, x_{q-1}^n)\,\mathrm{d}s - \int_{t_{q-1}^n}^{t_q^n} \left\{\sum_{j=1}^{q-1}\int_{t_{j-1}^n}^{t_j^n} f_2(\tau, s, x_{j-1}^n)\mathrm{d}s + \int_{t_{q-1}^n}^{\tau} f_2(\tau, s, x_{q-1}^n)\mathrm{d}s\right\}\mathrm{d}\tau,$$

and

$$x_q^n = \mathrm{Proj}_{C(t_q^n)}(y_q^n), \tag{3.20}$$

along with

$$\max_{0\le j\le q}\|x_j^n\| \le 2\left(\|x_0\| + \int\limits_{T_0}^{T} \dot{\upsilon}(s)\,\mathrm{d}s + \frac{1}{2}\right) = l-1, \tag{3.21}$$

$$\|x_q^n - y_q^n\| \le (l+1)(k_{q-1}^n + \mu_{q-1}^n + \alpha_{q-1}^n),$$

and

$$\|x_q^n\| \le \|x_{q-1}^n\| + k_{q-1}^n + 2(1 + \max_{0\le j\le q-1}\|x_j^n\|)\left(\int\limits_{t_{q-1}^n}^{t_q^n} \beta_1(s)\,\mathrm{d}s + \int\limits_{t_{q-1}^n}^{t_q^n}\int\limits_{T_0}^{\tau} \beta_2(\tau,s)\mathrm{d}s\,\mathrm{d}\tau\right).$$

We will define x_{i+1}^n and y_{i+1}^n as follows. From (3.20) and (3.21) and from the assumption on ρ we note that $\|x_j^n\| \le \rho$ and $x_j^n \in C(t_j^n)$ for each $j \in \{0, ..., i\}$, so according to the assumption $(\mathcal{H}_1)$ one obtains

$$\begin{aligned}
&d_{C(t_{i+1}^n)}\left(x_i^n - \int\limits_{t_i^n}^{t_{i+1}^n} f_1(s,x_i^n)\,\mathrm{d}s - \int\limits_{t_i^n}^{t_{i+1}^n}\left\{\sum_{j=0}^{i-1}\int\limits_{t_j^n}^{t_{j+1}^n} f_2(\tau,s,x_j^n)\mathrm{d}s + \int\limits_{t_i^n}^{\tau} f_2(\tau,s,x_i^n)\mathrm{d}s\right\}\mathrm{d}\tau\right)\\
&\le d_{C(t_{i+1}^n)}(x_i^n) + \int\limits_{t_i^n}^{t_{i+1}^n}\|f_1(s,x_i^n)\|\mathrm{d}s + \int\limits_{t_i^n}^{t_{i+1}^n}\left\{\sum_{j=0}^{i-1}\int\limits_{t_j^n}^{t_{j+1}^n}\|f_2(\tau,s,x_j^n)\|\mathrm{d}s + \int\limits_{t_i^n}^{\tau}\|f_2(\tau,s,x_i^n)\|\mathrm{d}s\right\}\mathrm{d}\tau\\
&\le \mathrm{haus}_\rho(C(t_i^n), C(t_{i+1}^n)) + \int\limits_{t_i^n}^{t_{i+1}^n}\|f_1(s,x_i^n)\|\mathrm{d}s\\
&\quad + \int\limits_{t_i^n}^{t_{i+1}^n}\left\{\sum_{j=0}^{i-1}\int\limits_{t_j^n}^{t_{j+1}^n}\|f_2(\tau,s,x_j^n)\|\mathrm{d}s + \int\limits_{t_i^n}^{\tau}\|f_2(\tau,s,x_i^n)\|\mathrm{d}s\right\}\mathrm{d}\tau,
\end{aligned}$$

hence

$$\begin{aligned}
&d_{C(t_{i+1}^n)}\left(x_i^n - \int\limits_{t_i^n}^{t_{i+1}^n} f_1(s,x_i^n)\,\mathrm{d}s - \int\limits_{t_i^n}^{t_{i+1}^n}\left\{\sum_{j=0}^{i-1}\int\limits_{t_j^n}^{t_{j+1}^n} f_2(\tau,s,x_j^n)\mathrm{d}s + \int\limits_{t_i^n}^{\tau} f_2(\tau,s,x_i^n)\mathrm{d}s\right\}\mathrm{d}\tau\right)\\
&\le |\upsilon(t_{i+1}^n) - \upsilon(t_i^n)| + \int\limits_{t_i^n}^{t_{i+1}^n}\|f_1(s,x_i^n)\|\mathrm{d}s\\
&\quad + \int\limits_{t_i^n}^{t_{i+1}^n}\left\{\sum_{j=0}^{i-1}\int\limits_{t_j^n}^{t_{j+1}^n}\|f_2(\tau,s,x_j^n)\|\mathrm{d}s + \int\limits_{t_i^n}^{\tau}\|f_2(\tau,s,x_i^n)\|\mathrm{d}s\right\}\mathrm{d}\tau \qquad (3.22)\\
&= k_i^n + \int\limits_{t_i^n}^{t_{i+1}^n}\|f_1(s,x_i^n)\|\mathrm{d}s + \int\limits_{t_i^n}^{t_{i+1}^n}\left\{\sum_{j=0}^{i-1}\int\limits_{t_j^n}^{t_{j+1}^n}\|f_2(\tau,s,x_j^n)\|\mathrm{d}s + \int\limits_{t_i^n}^{\tau}\|f_2(\tau,s,x_i^n)\|\mathrm{d}s\right\}\mathrm{d}\tau.
\end{aligned}$$

On the other hand,

$$\int_{t_i^n}^{t_{i+1}^n} \|f_1(s, x_i^n)\| \, \mathrm{d}s \leq (1 + \max_{0\leq j\leq i}\|x_j^n\|) \int_{t_i^n}^{t_{i+1}^n} \beta_1(s) \, \mathrm{d}s \leq l \int_{t_i^n}^{t_{i+1}^n} \beta_1(s) \, \mathrm{d}s = l\mu_i^n,$$

and

$$\begin{aligned}
&\int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} \|f_2(\tau, s, x_j^n)\|\mathrm{d}s + \int_{t_i^n}^{\tau} \|f_2(\tau, s, x_i^n)\|\mathrm{d}s \right\} \mathrm{d}\tau \\
&\leq (1 + \max_{0\leq j\leq i}\|x_j^n\|) \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} \|\beta_2(\tau, s)\|\mathrm{d}s + \int_{t_i^n}^{\tau} \|\beta_2(\tau, s)\|\mathrm{d}s \right\} \mathrm{d}\tau \\
&\leq l \int_{t_i^n}^{t_{i+1}^n} \int_{T_0}^{\tau} \|\beta_2(\tau, s)\|\mathrm{d}s \, \mathrm{d}\tau = l\alpha_i^n.
\end{aligned}$$

Therefore, defining

$$y_{i+1}^n = x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_1(s, x_i^n) \, \mathrm{d}s - \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f_2(\tau, s, x_j^n)\mathrm{d}s + \int_{t_i^n}^{\tau} f_2(\tau, s, x_i^n)\mathrm{d}s \right\} \mathrm{d}\tau,$$

we obtain

$$d_{C(t_{i+1}^n)}(y_{i+1}^n) \leq k_i^n + l\mu_i^n + l\alpha_i^n \leq (l+1)(k_i^n + \mu_i^n + \alpha_i^n) \leq (l+1)\varepsilon_n \leq \frac{r}{3} < r, \tag{3.23}$$

which implies by the prox-regularity of the set $C(t_{i+1}^n)$ and Proposition 1.3.3 the existence and the uniqueness of the projection $\mathrm{Proj}_{C(t_{i+1}^n)}(y_{i+1}^n)$ and hence we can define

$$x_{i+1}^n = \mathrm{Proj}_{C(t_{i+1}^n)}(y_{i+1}^n). \tag{3.24}$$

Note that by (3.23) and (3.24) we have

$$\|x_{i+1}^n - y_{i+1}^n\| \leq (l+1)(k_i^n + \mu_i^n + \alpha_i^n). \tag{3.25}$$

Further, by (3.22) and (3.24)

$$\begin{aligned}
\|x_{i+1}^n\| &\leq \|x_i^n\| + k_i^n \\
&\quad + 2\Big(\int_{t_i^n}^{t_{i+1}^n} \|f_1(s, x_i^n)\|\mathrm{d}s + \int_{t_{q-1}^n}^{t_q^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} \|f_2(\tau, s, x_j^n)\|\mathrm{d}s + \int_{t_i^n}^{\tau} \|f_2(\tau, s, x_i^n)\|\mathrm{d}s \right\} \mathrm{d}\tau \Big) \\
&\leq \|x_i^n\| + k_i^n + 2(1 + \max_{0\leq j\leq i}\|x_j^n\|)\Big(\int_{t_i^n}^{t_{i+1}^n} \beta_1(s) \, \mathrm{d}s + \int_{t_i^n}^{t_{i+1}^n} \int_{T_0}^{\tau} \|\beta_2(\tau, s)\|\mathrm{d}s \, \mathrm{d}\tau \Big).
\end{aligned}$$

Now, by the induction assumption, we get

$$\|x_{i+1}^n\| \le \|x_0\| + \sum_{p=0}^{i} k_p^n + 2(1+\max_{0\le j\le i}\|x_j^n\|)\Big(\sum_{p=0}^{i}\int_{t_p^n}^{t_{p+1}^n}\beta_1(s)\,ds + \sum_{p=0}^{i}\int_{t_p^n}^{t_{p+1}^n}\int_{T_0}^{\tau}\|\beta_2(\tau,s)\|\mathrm{d}s\,\mathrm{d}\tau\Big)$$

$$\le \|x_0\| + \int_{T_0}^{t_{i+1}^n}\dot{v}(s)\,\mathrm{d}s + 2(1+\max_{0\le j\le i}\|x_j^n\|)\Big(\int_{T_0}^{t_{i+1}^n}\beta_1(s)\,\mathrm{d}s + \int_{T_0}^{t_{i+1}^n}\int_{T_0}^{\tau}\|\beta_2(\tau,s)\|\mathrm{d}s\,\mathrm{d}\tau\Big),$$

thus

$$\|x_{i+1}^n\| \le \|x_0\| + \int_{T_0}^{T}\dot{v}(s)\,\mathrm{d}s + 2(1+\max_{0\le j\le i+1}\|x_j^n\|)\Big(\int_{T_0}^{T}\beta_1(s)\,\mathrm{d}s + \int_{T_0}^{T}\int_{T_0}^{\tau}\|\beta_2(\tau,s)\|\mathrm{d}s\,\mathrm{d}\tau\Big).$$

It results, thanks to (3.9), that

$$\max_{0\le j\le i+1}\|x_j^n\| \le \|x_0\| + \int_{T_0}^{T}\dot{v}(s)\,\mathrm{d}s + \frac{1}{2}(1+\max_{0\le j\le i+1}\|x_j^n\|),$$

or equivalently

$$\max_{0\le j\le i+1}\|x_j^n\| \le 2\Big(\|x_0\| + \int_{T_0}^{T}\dot{v}(s)\,\mathrm{d}s + \frac{1}{2}\Big) = l-1. \tag{3.26}$$

The induction is then complete.

Step 3. Boundedness of the approximate solutions.

Fix any integer $n \ge n_0$. Let us define the mapping $x_n(\cdot) : [T_0, T] \longrightarrow H$ as follows.

Put $x_n(0) := x_0$, for $t \in]t_i^n, t_{i+1}^n]$, $i \in \{0, ..., n-1\}$, set

$$\begin{aligned} x_n(t) = x_i^n + \frac{a(t)-a(t_i^n)}{k_i^n+\mu_i^n+\alpha_i^n}\Big(x_{i+1}^n - x_i^n + \int_{t_i^n}^{t_{i+1}^n} f_1(s,x_i^n)\,\mathrm{d}s \\ + \int_{t_i^n}^{t_{i+1}^n}\Big\{\sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} f_2(\tau,s,x_j^n)\mathrm{d}s + \int_{t_i^n}^{\tau} f_2(\tau,s,x_i^n)\mathrm{d}s\Big\}\,\mathrm{d}\tau\Big) \\ - \int_{t_i^n}^{t} f_1(s,x_i^n)\,\mathrm{d}s - \int_{t_i^n}^{t}\Big\{\sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} f_2(\tau,s,x_j^n)\mathrm{d}s + \int_{t_i^n}^{\tau} f_2(\tau,s,x_i^n)\mathrm{d}s\Big\}\,\mathrm{d}\tau, \end{aligned} \tag{3.27}$$

where $a(t) = v(t) + \int_{T_0}^{t}\beta_1(s)\,\mathrm{d}s + \int_{T_0}^{t}\int_{T_0}^{\tau}\beta_2(\tau,s)\mathrm{d}s\,\mathrm{d}\tau.$

Notice that $\lim_{t\downarrow t_i^n} x_n(t) = x_i^n$. Using the equality

$$a(t_{i+1}^n) - a(t_i^n) = \int_{t_i^n}^{t_{i+1}^n}\dot{v}(s)\,\mathrm{d}s + \int_{t_i^n}^{t_{i+1}^n}\beta_1(s)\,\mathrm{d}s + \int_{t_i^n}^{t_{i+1}^n}\int_{T_0}^{\tau}\beta_2(\tau,s)\,\mathrm{d}s\mathrm{d}\tau = k_i^n + \mu_i^n + \alpha_i^n,$$

we also notice that $x_n(t_{i+1}^n) = x_{i+1}^n$, hence $x_n(t_i^n) = x_i^n$. Altogether, we obtain that $x_n(\cdot)$ is absolutely continuous on each interval $[t_i^n, t_{i+1}^n]$ as well as on the whole interval $[T_0, T]$. Moreover, (3.27) yields for any $i \in \{0, ..., n-1\}$ and a.e. $t \in]t_i^n, t_{i+1}^n[$

$$\begin{aligned} \dot{x}_n(t) = \frac{\dot{a}(t)}{k_i^n + \mu_i^n + \alpha_i^n} \Bigg(& x_{i+1}^n - x_i^n + \int_{t_i^n}^{t_{i+1}^n} f_1(s, x_i^n)\,\mathrm{d}s \\ & + \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f_2(\tau, s, x_j^n)\mathrm{d}s + \int_{t_i^n}^{\tau} f_2(\tau, s, x_i^n)\mathrm{d}s \right\} \mathrm{d}\tau \Bigg) \\ & - f_1(t, x_i^n) - \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f_2(t, s, x_j^n)\mathrm{d}s - \int_{t_i^n}^{t} f_2(t, s, x_i^n)\mathrm{d}s. \end{aligned} \tag{3.28}$$

Furthermore, from (3.25) and (3.28), we get

$$\begin{aligned} & \left\| \dot{x}_n(t) + f_1(t, x_i^n) + \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f_2(t, s, x_j^n)\mathrm{d}s + \int_{t_i^n}^{t} f_2(t, s, x_i^n)\mathrm{d}s \right\| \\ & = \frac{\dot{a}(t)}{k_i^n + \mu_i^n + \alpha_i^n} \left\| x_{i+1}^n - y_{i+1}^n \right\| \leq (l+1)\dot{a}(t). \end{aligned} \tag{3.29}$$

Then, since

$$\|f_1(t, x_i^n)\| \leq \beta_1(t)(1 + \|x_i^n\|) \leq l\beta_1(t), \tag{3.30}$$

$$\|f_2(t, s, x_i^n)\| \leq \beta_2(t, s)(1 + \|x_i^n\|) \leq l\beta_2(t, s), \tag{3.31}$$

and

$$\begin{aligned} & \left\| \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f_2(t, s, x_j^n)\mathrm{d}s + \int_{t_i^n}^{t} f_2(t, s, x_i^n)\mathrm{d}s \right\| \\ & \leq l \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} \beta_2(t, s)\,\mathrm{d}s + \int_{t_i^n}^{t} \beta_2(t, s)\mathrm{d}s \right\} = l \int_{T_0}^{t} \beta_2(t, s)\mathrm{d}s, \end{aligned}$$

we have for a.e. $t \in [T_0, T]$

$$\|\dot{x}_n(t)\| \leq (l+1)\dot{a}(t) + l\beta_1(t) + l \int_{T_0}^{t} \beta_2(t, s)\mathrm{d}s \leq (l+1)\dot{a}(t) + l\dot{a}(t) = (2l+1)\dot{a}(t). \tag{3.32}$$

From this it follows that

$$\|x_n(t)\| \leq \|x_0\| + (2l+1) \int_{T_0}^{T} \dot{a}(s)\mathrm{d}s = M, \quad \forall\, t \in [T_0, T]. \tag{3.33}$$

Step 4. Convergence of the approximate solutions.

In this step we show for each $t \in [T_0, T]$ that $(x_n(t))_{n \geq n_0}$ is a Cauchy sequence in the Hilbert space H.

Let us define the functions $\theta_n, \eta_n : [T_0, T] \longrightarrow [T_0, T]$ by $\theta_n(T_0) = T_0, \eta_n(T_0) = T_0$, and

$$\theta_n(t) = t_{i+1}^n,\ \ \eta_n(t) = t_i^n,\ \ if\ \ t \in]t_i^n, t_{i+1}^n]\ (0 \le i \le n-1).$$

Observe that, for all $t \in [T_0, T]$,

$$\lim_{n \longrightarrow \infty} |\theta_n(t) - t| = \lim_{n \longrightarrow \infty} |\eta_n(t) - t| = 0.$$

On one hand, the construction of $\eta_n(\cdot)$ and (3.29), (3.30), (3.31) assure us that, for almost all t and for all n,

$$\left\| \dot{x}_n(t) + f_1\big(t, x_n(\eta_n(t))\big) + \int_{T_0}^{t} f_2\big(t, s, x_n(\eta_n(s))\big)\,\mathrm{d}s \right\| \le (l+1)\dot{a}(t), \tag{3.34}$$

$$\big\| f_1(t, x_n\big(\eta_n(t))\big) \big\| \le \beta_1(t)(1 + \|x_n(\eta_n(t))\|) \le l\beta_1(t), \tag{3.35}$$

$$\big\| f_2\big(t, s, x_n(\eta_n(s))\big) \big\| \le \beta_2(t,s)(1 + \|x_n(\eta_n(s))\|) \le l\beta_2(t,s). \tag{3.36}$$

Let us fix $m, n \in \mathbb{N}$ such that $m > n \ge n_0$. Then, by (3.24), the construction of $x_n(\cdot)$, $\theta_n(\cdot)$, $\eta_n(\cdot)$ and the properties of normal cones to subsets (see Remark 1.1.1), we have for any $i \in \{0, ..., n-1\}$ and a.e. $t \in]t_i^n, t_{i+1}^n[$ we have that the vector

$$-\dot{x}_n(t) - f_1\big(t, x_n(\eta_n(t))\big) - \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f_2\big(t, s, x_n(\eta_n(s))\big)\mathrm{d}s - \int_{t_i^n}^{t} f_2\big(t, s, x_n(\eta_n(s))\big)\mathrm{d}s$$

belongs to the normal cone $N_{C(\theta_n(t))}\big(x_n(\theta_n(t))\big)$, otherwise stated

$$-\dot{x}_n(t) \in N_{C(\theta_n(t))}\big(x_n(\theta_n(t))\big) + f_1\big(t, x_n(\eta_n(t))\big) + \int_{T_0}^{t} f_2\big(t, s, x_n(\eta_n(s))\big)\mathrm{d}s.$$

According to (1.2), the latter inclusion and relation (3.34) entail for a.e. $t \in [T_0, T]$

$$-\dot{x}_n(t) - f_1\big(t, x_n(\eta_n(t))\big) - \int_{T_0}^{t} f_2\big(t, s, x_n(\eta_n(s))\big)\mathrm{d}s \in (l+1)\dot{a}(t)\partial d_{C(\theta_n(t))}\big(x_n(\theta_n(t))\big). \tag{3.37}$$

Now, take any $t \in [T_0, T]$ and by (3.10) choose $i \in \{0, ..., n-1\}$ and $j \in \{0, ..., m-1\}$ such that

$$t \in]t_j^m, t_{j+1}^m] \subset]t_i^n, t_{i+1}^n]. \tag{3.38}$$

Then

$$\|x_n(\theta_n(t)) - x_n(t)\| \leq \int_t^{\theta_n(t)} \|\dot{x}_n(\tau)\| \,\mathrm{d}\tau \leq (2l+1) \int_t^{\theta_n(t)} \dot{a}(\tau)\,\mathrm{d}\tau$$
$$= (2l+1) \int_t^{t_{i+1}^n} \left(\dot{v}(\tau) + \beta_1(\tau) + \int_{T_0}^{\tau} \beta_2(\tau,s)\mathrm{d}s \right) \mathrm{d}\tau$$
$$\leq (2l+1) \int_{t_i^n}^{t_{i+1}^n} \left(\dot{v}(\tau) + \beta_1(\tau) + \int_{T_0}^{\tau} \beta_2(\tau,s)\mathrm{d}s \right) \mathrm{d}\tau,$$

hence

$$\|x_n(\theta_n(t)) - x_n(t)\| \leq (2l+1)(k_i^n + \mu_i^n + \alpha_i^n) \leq (2l+1)\varepsilon_n, \tag{3.39}$$

and this inequality allows us to deduce that

$$\begin{aligned} &\|x_n(\theta_n(t)) - x_m(\theta_m(t))\| \\ &\leq \|x_n(\theta_n(t)) - x_n(t)\| + \|x_n(t) - x_m(t)\| + \|x_m(t) - x_m(\theta_m(t))\| \\ &\leq (2l+1)(\varepsilon_n + \varepsilon_m) + \|x_n(t) - x_m(t)\|. \end{aligned} \tag{3.40}$$

Similarly, we get

$$\|x_n(\eta_n(t)) - x_n(t)\| \leq (2l+1)\varepsilon_n, \tag{3.41}$$

and

$$\|x_n(\eta_n(t)) - x_m(\eta_m(t))\| \leq (2l+1)(\varepsilon_n + \varepsilon_m) + \|x_n(t) - x_m(t)\|. \tag{3.42}$$

Observe from (3.24) and (3.26) that

$$x_m(\theta_m(t)) \in C(\theta_m(t)) \cap \rho\mathbb{B} \text{ for all } \ m \geq n_0, \ t \in [T_0, T], \tag{3.43}$$

which implies that (see (1.7)), for all $t \in [T_0, T]$

$$d_{C(\theta_n(t))}(x_m(t)) \leq \|x_m(t) - x_m(\theta_m(t))\| + \mathrm{haus}_\rho(C(\theta_n(t)), C(\theta_m(t)))$$
$$\leq \|x_m(t) - x_m(\theta_m(t))\| + \Big|\int_{\theta_m(t)}^{\theta_n(t)} \dot{v}(s)\,ds\Big|.$$

According to (3.38) and (3.39) we obtain

$$\begin{aligned} d_{C(\theta_n(t))}(x_m(t)) &\leq (2l+1)\varepsilon_m + \Big|\int_{t_{j+1}^m}^{t_{i+1}^n} \dot{v}(s)\,ds\Big| \\ &\leq (2l+1)\varepsilon_m + \int_{t_i^n}^{t_{i+1}^n} \dot{v}(s)\,ds \\ &\leq (2l+1)\varepsilon_m + \varepsilon_m + \varepsilon_n = (2l+2)\varepsilon_m + \varepsilon_n. \end{aligned} \tag{3.44}$$

Thus, for m, n large enough, say $m > n \geq n_1$ with $n_1 \geq n_0$ we have $2(l+1)\varepsilon_m + \varepsilon_n < r$, thus $d_{C(\theta_n(t))}(x_m(t)) < r$, for all $t \in [T_0, T]$. Setting $\delta(t) = (l+1)\dot{a}(t)$, a.e. $t \in [T_0, T]$, and using (3.37) and Proposition 1.3.4 we obtain

$$\begin{aligned}
&\left\langle \dot{x}_n(t) + f_1(t, x_n(\eta_n(t))) + \int_{T_0}^{t} f_2(t, s, x_n(\eta_n(s)))\mathrm{d}s, x_n(\theta_n(t)) - x_m(t) \right\rangle \\
&\leq \frac{2\delta(t)}{r}\|x_n(\theta_n(t)) - x_m(t)\|^2 + \delta(t) d_{C(\theta_n(t))}(x_m(t)) \\
&\leq \frac{2\delta(t)}{r}\big(\|x_n(\theta_n(t)) - x_n(t)\| + \|x_n(t) - x_m(t)\|\big)^2 + \delta(t) d_{C(\theta_n(t))}(x_m(t)),
\end{aligned}$$

and hence by (3.39) and (3.44)

$$\begin{aligned}
&\leq \frac{2\delta(t)}{r}\big((2l+1)\varepsilon_n + \|x_n(t) - x_m(t)\|\big)^2 + \delta(t)\big((2l+2)\varepsilon_m + \varepsilon_n\big) \\
&\leq \frac{4\delta(t)}{r}\|x_n(t) - x_m(t)\|^2 + \frac{4\delta(t)}{r}(2l+1)^2\varepsilon_n^2 + \delta(t)((2l+2)\varepsilon_m + \varepsilon_n).
\end{aligned} \tag{3.45}$$

Then writing

$$\begin{aligned}
&\left\langle \dot{x}_n(t) + f_1(t, x_n(\eta_n(t))) + \int_{T_0}^{t} f_2(t, s, x_n(\eta_n(s)))\mathrm{d}s, x_n(t) - x_m(t) \right\rangle \\
&= \left\langle \dot{x}_n(t) + f_1(t, x_n(\eta_n(t))) + \int_{T_0}^{t} f_2(t, s, x_n(\eta_n(s)))\mathrm{d}s, x_n(t) - x_n(\theta_n(t)) \right\rangle \\
&+ \left\langle \dot{x}_n(t) + f_1(t, x_n(\eta_n(t))) + \int_{T_0}^{t} f_2(t, s, x_n(\eta_n(s)))\mathrm{d}s, x_n(\theta_n(t)) - x_m(t) \right\rangle,
\end{aligned}$$

we see by (3.39) and (3.45)

$$\begin{aligned}
&\left\langle \dot{x}_n(t) + f_1(t, x_n(\eta_n(t))) + \int_{T_0}^{t} f_2(t, s, x_n(\eta_n(s)))\mathrm{d}s, x_n(t) - x_m(t) \right\rangle \\
&\leq \delta(t)(2l+1)\varepsilon_n + \frac{4\delta(t)}{r}\|x_n(t) - x_m(t)\|^2 + \frac{4\delta(t)}{r}(2l+1)^2\varepsilon_n^2 + \delta(t)((2l+2)\varepsilon_m + \varepsilon_n) \\
&\leq \varrho_{n,m}(t) + \frac{4\delta(t)}{r}\|x_n(t) - x_m(t)\|^2,
\end{aligned}$$

where

$$\varrho_{n,m}(t) = \delta(t)(2l+1)(\varepsilon_n + \varepsilon_m) + \frac{4\delta(t)}{r}(2l+1)^2(\varepsilon_n + \varepsilon_m)^2 + 2\delta(t)(l+1)(\varepsilon_m + \varepsilon_n).$$

In the same way, we also have

$$\begin{aligned}
&\left\langle \dot{x}_m(t) + f_1(t, x_m(\eta_m(t))) + \int_{T_0}^{t} f_2(t, s, x_m(\eta_m(s)))\mathrm{d}s, x_m(t) - x_n(t) \right\rangle \\
&\leq \varrho_{n,m}(t) + \frac{4\delta(t)}{r}\|x_n(t) - x_m(t)\|^2.
\end{aligned}$$

It then follows from both last inequalities that

$$\begin{aligned}
&\langle \dot{x}_m(t) - \dot{x}_n(t), x_m(t) - x_n(t)\rangle \\
&\leq \langle f_1(t, x_n(\eta_n(t))) - f_1(t, x_m(\eta_m(t))), x_m(t) - x_n(t)\rangle \\
&+ \left\langle \int_{T_0}^{t} f_2(t, s, x_n(\eta_n(s)))\mathrm{d}s - \int_{T_0}^{t} f_2(t, s, x_m(\eta_m(s)))\mathrm{d}s, x_m(t) - x_n(t) \right\rangle \\
&+ 2\varrho_{n,m}(t) + \frac{8\delta(t)}{r}\|x_n(t) - x_m(t)\|^2,
\end{aligned}$$

hence

$$\begin{aligned}
&\langle \dot{x}_m(t) - \dot{x}_n(t), x_m(t) - x_n(t)\rangle \\
&\leq \langle f_1(t, x_n(\eta_n(t))) - f_1(t, x_m(\eta_m(t))), x_m(t) - x_n(t)\rangle \\
&+ \left\| \int_{T_0}^{t} f_2(t, s, x_n(\eta_n(s)))\mathrm{d}s - \int_{T_0}^{t} f_2(t, s, x_m(\eta_m(s)))\mathrm{d}s \right\| . \|x_m(t) - x_n(t)\| \\
&+ 2\varrho_{n,m}(t) + \frac{8\delta(t)}{r}\|x_n(t) - x_m(t)\|^2.
\end{aligned}$$

Now, for all $t \in [T_0, T]$, all $n, m \in \mathbb{N}$ with $n, m \geq n_1$ set

$$\varphi_{n,m}(t) = \|f_1(t, x_n(\eta_n(t))) - f_1(t, x_n(t))\| . \|x_m(t) - x_n(t)\|. \tag{3.46}$$

Writing for all $t \in [T_0, T]$

$$\begin{aligned}
&\langle f_1(t, x_n(\eta_n(t))) - f_1(t, x_m(\eta_m(t))), x_m(t) - x_n(t)\rangle \\
&= \langle f_1(t, x_n(t)) - f_1(t, x_m(t)), x_m(t) - x_n(t)\rangle \\
&+ \langle f_1(t, x_n(\eta_n(t))) - f_1(t, x_n(t)), x_m(t) - x_n(t)\rangle \\
&+ \langle f_1(t, x_m(t)) - f_1(t, x_m(\eta_m(t))), x_m(t) - x_n(t)\rangle ,
\end{aligned}$$

we can apply assumption $(\mathcal{H}_{2,2})$ with $\eta = M$ (see (3.33)) to get

$$\langle f_1(t, x_n(\eta_n(t))) - f_1(t, x_m(\eta_m(t))), x_m(t) - x_n(t)\rangle \leq \varphi_{m,n}(t) + \varphi_{n,m}(t) + L_1^{\eta}(t)\|x_m(t) - x_n(t)\|^2. \tag{3.47}$$

Further, from the Lipschitz property of f_2 with respect to x and (3.42), we have

$$\begin{aligned}
&\left\| \int_{T_0}^{t} f_2(t, s, x_n(\eta_n(s)))\mathrm{d}s - \int_{T_0}^{t} f_2(t, s, x_m(\eta_m(s)))\mathrm{d}s \right\| \\
&\leq L_2^{\eta}(t) \int_{T_0}^{t} \|x_n(\eta_n(s)) - x_m(\eta_m(s))\| \,\mathrm{d}s \qquad (3.48)\\
&\leq L_2^{\eta}(t) T(2l+1)(\varepsilon_n + \varepsilon_m) + L_2^{\eta}(t) \int_{T_0}^{t} \|x_n(s) - x_m(s)\| \,\mathrm{d}s.
\end{aligned}$$

Thanks to (3.47), (3.48) we have, for almost all $t \in [T_0, T]$

$$
\begin{aligned}
&\langle \dot{x}_m(t) - \dot{x}_n(t), x_m(t) - x_n(t) \rangle \\
&\leq \varphi_{m,n}(t) + \varphi_{n,m}(t) + L_1^\eta(t)\|x_m(t) - x_n(t)\|^2 \\
&+ \left\{ L_2^\eta(t) T(2l+1)(\varepsilon_n + \varepsilon_m) + L_2^\eta(t) \int_{T_0}^{t} \|x_n(s) - x_m(s)\| \, \mathrm{d}s \right\} \|x_m(t) - x_n(t)\| \\
&+ 2\varrho_{n,m}(t) + \frac{8\delta(t)}{r}\|x_n(t) - x_m(t)\|^2,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&\langle \dot{x}_m(t) - \dot{x}_n(t), x_m(t) - x_n(t) \rangle \\
&\leq 2\varrho_{n,m}(t) + \frac{8\delta(t)}{r}\|x_n(t) - x_m(t)\|^2 \\
&+ \varphi_{m,n}(t) + \varphi_{n,m}(t) + L_2^\eta(t) T(2l+1)(\varepsilon_n + \varepsilon_m)\|x_m(t) - x_n(t)\| + L_1^\eta(t)\|x_m(t) - x_n(t)\|^2 \\
&+ L_2^\eta(t)\|x_m(t) - x_n(t)\| \int_{T_0}^{t} \|x_m(s) - x_n(s)\| \, \mathrm{d}s.
\end{aligned}
$$

By (3.33), we get

$$
\begin{aligned}
&\langle \dot{x}_m(t) - \dot{x}_n(t), x_m(t) - x_n(t) \rangle \\
&\leq \varphi_{m,n}(t) + \varphi_{n,m}(t) + \Bigg(2\delta(t)(2l+1) + \frac{8\delta(t)}{r}(2l+1)^2(\varepsilon_n + \varepsilon_m) + 4\delta(t)(l+1) \\
&+ 2ML_2^\eta(t)T(2l+1) \Bigg)(\varepsilon_n + \varepsilon_m) \\
&+ \left(L_1^\eta(t) + \frac{8\delta(t)}{r} \right)\|x_m(t) - x_n(t)\|^2 + L_2^\eta(t)\|x_m(t) - x_n(t)\| \int_{T_0}^{t} \|x_m(s) - x_n(s)\| \, \mathrm{d}s,
\end{aligned}
$$

Lemma 1.5.1 and the latter inequality ensures that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|x_n(t) - x_m(t)\|^2 &= 2\langle \dot{x}_m(t) - \dot{x}_n(t), x_m(t) - x_n(t) \rangle \\
&\leq 2\varepsilon_{n,m}(t) + 2\left(L_1^\eta(t) + \frac{8\delta(t)}{r} \right)\|x_n(t) - x_m(t)\|^2 \\
&+ 2L_2^\eta(t)\|x_n(t) - x_m(t)\| \int_{T_0}^{t} \|x_n(s) - x_m(s)\| \, \mathrm{d}s,
\end{aligned}
\tag{3.49}
$$

where

$$
\begin{aligned}
\varepsilon_{n,m}(t) &= \varphi_{m,n}(t) + \varphi_{n,m}(t) \\
&+ \left(2\delta(t)(2l+1) + \frac{8\delta(t)}{r}(2l+1)^2(\varepsilon_n + \varepsilon_m) + 4\delta(t)(l+1) + 2ML_2^\eta(t)T(2l+1) \right)(\varepsilon_n + \varepsilon_m).
\end{aligned}
$$

Now, note by (3.41) that $x_n(\eta_n(t)) - x_n(t) \longrightarrow 0$. It follows $f_1(t, x_n(\eta_n(t))) - f_1(t, x_n(t)) \longrightarrow 0$ according to the uniform continuity of $f_1(t, \cdot)$ over $B[0, \eta]$. From the definition in (3.46) of $\varphi_{n,m}$

we also have $\varphi_{n,m}(t) \longrightarrow 0$ as $n, m \longrightarrow 0$ since the sequence $(x_n(\cdot))_{n\in\mathbb{N}}$ is uniformly bounded by (3.33).

Further $\varepsilon_{n,m}(\cdot)$ is integrable since $\delta(\cdot)$ and $L_2^\eta(\cdot)$ are integrable, which again confirms that, for almost all $t \in [T_0, T]$

$$\lim_{n,m\longrightarrow\infty} \varepsilon_{n,m}(t) = 0,$$

since $\varepsilon_n \longrightarrow 0$ and $\varepsilon_m \longrightarrow 0$ as $n, m \longrightarrow \infty$.

Applying Lemma 2.1.3 and using estimate (3.49), we then see for any $\epsilon > 0$ that

$$\begin{aligned}
&\|x_n(t) - x_m(t)\| \\
&\leq \sqrt{\|x_n(T_0) - x_m(T_0)\|^2 + \epsilon}\, \exp\left(\int_{T_0}^{t} (K(s)+1)\, \mathrm{d}s \right) + \frac{\sqrt{\epsilon}}{2} \int_{T_0}^{t} \exp\left(\int_{s}^{t} (K(\tau)+1)\, \mathrm{d}\tau \right) \mathrm{d}s \\
&+ 2\left(\sqrt{\int_{T_0}^{t} \varepsilon_{n,m}(s)\, \mathrm{d}s + \epsilon} - \exp\left(\int_{T_0}^{t} (K(\tau)+1)\, \mathrm{d}\tau \right) \sqrt{\epsilon} \right) \\
&+ 2 \int_{T_0}^{t} (K(s)+1) \exp\left(\int_{s}^{t} (K(\tau)+1)\, \mathrm{d}\tau \right) \sqrt{\int_{T_0}^{t} \varepsilon_{n,m}(\tau)\, \mathrm{d}\tau + \epsilon}\, \mathrm{d}s,
\end{aligned} \tag{3.50}$$

where $K(t) = \max\{L_1^\eta(t) + \dfrac{8\delta(t)}{r}, L_2^\eta(t)\}$, for almost all $t \in [T_0, T]$.

Since $\varepsilon_{n,m}(\cdot)$ is Lebesgue integrable and $\lim_{n,m\longrightarrow\infty} \varepsilon_{n,m}(t) = 0$ a.e. $t \in [T_0, T]$, it follows from the dominated convergence theorem that

$$\lim_{n,m\longrightarrow\infty} \int_{T_0}^{t} \varepsilon_{n,m}(s)\, ds = 0.$$

From (3.50) and taking $\epsilon \to 0$ one obtains that $\lim_{n,m\longrightarrow\infty} \|x_n(t) - x_m(t)\| = 0$. For each $t \in [T_0, T]$ the sequence $(x_n(t))_{n\in\mathbb{N}}$ is a Cauchy sequence in H, then $(x_n(\cdot))_{n\in\mathbb{N}}$ converges to a mapping $x(\cdot)$ from $[T_0, T]$ into H.

Step 5. We show that $x(\cdot)$ is absolutely continuous.

By (3.32) we have for almost all $t \in I$ and for any n,

$$\|\dot{x}_n(t)\| \leq (2l+1)\dot{a}(t) = \gamma(t).$$

So we can extract a subsequence of $(\dot{x}_n(\cdot))$ (that, without loss of generality, we do not relabel) which converges weakly in $L^1(I, H)$ to a function $g(\cdot) \in L^1(I, H)$. This means that

$$\int_{T_0}^{T} \langle \dot{x}_n(s), h(s) \rangle\, \mathrm{d}s \longrightarrow \int_{T_0}^{T} \langle g(s), h(s) \rangle\, \mathrm{d}s, \forall\, h \in L^\infty(I, H).$$

Then for every $z \in H$

$$\lim_{n\to\infty} \left\langle z, \int_{T_0}^{t} \dot{x}_n(s)\,\mathrm{d}s \right\rangle = \lim_{n\to\infty} \int_{T_0}^{T} \left\langle \dot{x}_n(s), z \cdot 1_{[T_0,t]}(s) \right\rangle \mathrm{d}s,$$

and

$$\left\langle z, \int_{T_0}^{t} g(s)\,\mathrm{d}s \right\rangle = \int_{T_0}^{T} \left\langle g(s), z \cdot 1_{[T_0,t]}(s) \right\rangle \mathrm{d}s.$$

So, from the weak convergence we deduce that

$$\int_{T_0}^{t} \dot{x}_n(s)\mathrm{d}s \longrightarrow \int_{T_0}^{t} g(s)\,\mathrm{d}s \text{ weakly in } H \text{ as } n \to \infty.$$

This and the absolute continuity of $x_n(\cdot)$ imply that

$$x_n(t) = x_n(T_0) + \int_{T_0}^{t} \dot{x}_n(s)\,\mathrm{d}s \longrightarrow x(T_0) + \int_{T_0}^{t} g(s)\,\mathrm{d}s \text{ weakly in } H \text{ as } n \to \infty.$$

On the other hand, we have for all $t \in [T_0, T]$

$$x_n(t) \longrightarrow x(t) \text{ strongly in } H,$$

hence we get

$$x(t) = x(T_0) + \int_{T_0}^{t} g(s)\mathrm{d}s.$$

Therefore, $x(\cdot)$ is absolutely continuous and $\dot{x}(t) = g(t)$ a.e. $t \in [T_0, T]$, so in particular

$$\|x(t)\| \le \widetilde{M} \text{ for all } t \in [T_0, T]\,, \tag{3.51}$$

with

$$\widetilde{M} = \|x_0\| + \int_{T_0}^{T} g(s)\,\mathrm{d}s.$$

Step 6. We show that $x(\cdot)$ is a solution of (P_{f_1,f_2}).

First, it is obvious that $x(0) = x_0$, and that, for all $t \in I$,

$$\begin{aligned}
\|x_n(\theta_n(t)) - x(t)\| &\le \|x_n(\theta_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\| \\
&\le \int_{t}^{\theta_n(t)} \|\dot{x}_n(s)\|\,\mathrm{d}s + \|x_n(t) - x(t)\| \\
&\le \int_{t}^{\theta_n(t)} \gamma(s)\,\mathrm{d}s + \|x_n(t) - x(t)\|,
\end{aligned}$$

so that, for all $t \in I$,

$$\lim_{n\to\infty} \|x_n(\theta_n(t)) - x(t)\| = 0, \tag{3.52}$$

and similarly

$$\lim_{n\to\infty} \|x_n(\eta_n(t)) - x(t)\| = 0. \tag{3.53}$$

Let us prove that $x(t) \in C(t)$ for all $t \in I$. From (1.7) and (3.43), we have

$$d_{C(t)}(x_m(\theta_m(t))) \leq \text{haus}_\rho(C(t), C(\theta_m(t)) \leq |\int_t^{\theta_m(t)} \dot{v}(s)\,\mathrm{d}s|.$$

Using (3.52), and passing to the limit, in the preceding inequality, we get, thanks to the closedness of $C(t)$ and the convergence $|\int_t^{\theta_m(t)} \dot{v}(s)\,\mathrm{d}s| \to 0$ as $m \to \infty$

$$x(t) \in C(t), \;\text{ for all }\; t \in [T_0, T].$$

Now it remains to prove that

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))\,\mathrm{d}s, \;\text{ a.e }\; t \in I.$$

Let us set for each $t \in I$

$$y_n(t) = \int_{T_0}^{t} f_2(t, s, x_n(\eta_n(s)))\,\mathrm{d}s \quad\text{and}\quad y(t) = \int_{T_0}^{t} f_2(t, s, x(s))\,\mathrm{d}s.$$

It is not difficult to check from the continuity of $f_1(t,\cdot)$, $f_2(t,s,\cdot)$ and the convergence of $x_n(\cdot)$, that

$$\int_{T_0}^{T} \langle f_1(t, x_n(\eta_n(t))), \varphi(t)\rangle\,\mathrm{d}t \longrightarrow \int_{T_0}^{T} \langle f_1(t, x(t)), \varphi(t)\rangle\,\mathrm{d}t \;\;\forall\,\varphi \in L^\infty(I, H),$$

$$\int_{T_0}^{T} \langle y_n(t), \varphi(t)\rangle\,\mathrm{d}t \longrightarrow \int_{T_0}^{T} \langle y(t), \varphi(t)\rangle\,\mathrm{d}t \;\;\forall\,\varphi \in L^\infty(I, H).$$

This implies that

$$\zeta_n(\cdot) := \dot{x}_n(\cdot) + f_1(\cdot, x_n(\eta_n(\cdot))) + y_n(\cdot) \longrightarrow \zeta(\cdot) := \dot{x}(\cdot) + f_1(\cdot, x(\cdot)) + y(\cdot)$$

weakly in $L^1(I, H)$.

Due to Mazur lemma, there is a sequence $(\xi_n(\cdot))_n$ converging strongly in $L^1(I, H)$ to $\zeta(\cdot)$ with

$$\xi_n(\cdot) \in \text{co}\{\zeta_k(\cdot), k \geq n\}. \tag{3.54}$$

Extracting a subsequence, we may suppose that

$$\xi_n(t) \longrightarrow \zeta(t) \text{ a.e. } t \in I.$$

Combining the inclusion (3.54) with the latter convergence, we obtain

$$\zeta(t) \in \bigcap_{n\in\mathbb{N}} \overline{\text{co}}\{\zeta_k(t), k \geq n\} \text{ a.e. } t \in I. \tag{3.55}$$

Coming back to (3.37) and recalling that $\delta(\cdot) = (l+1)\dot{a}(\cdot)$, it follows that, for almost every $t \in I$,

$$\langle z, \zeta_n(t)\rangle \leq \delta(t)\sigma(-\partial d_{C(\theta_n(t))}(x_n(\theta_n(t)), z), \text{ for all } z \in H, \tag{3.56}$$

where $\sigma(-\partial d_{C(\theta_n(t))}(x_n(\theta_n(t)), \cdot)$ is the support function of $-\partial d_{C(\theta_n(t))}(x_n(\theta_n(t))$. On the other hand, by (3.54) and (3.56), for almost all $t \in I$ we have for all $n \in \mathbb{N}$

$$\langle z, \zeta_n(t)\rangle \leq \sup_{k\geq n}\langle z, \zeta_k(t)\rangle \leq \delta(t)\sup_{k\geq n}\sigma(-\partial d_{C(\theta_k(t))}(x_k(\theta_k(t)), z), \text{ for all } z \in H.$$

From (3.55) it ensues that for almost all $t \in I$

$$\langle z, \zeta(t)\rangle \leq \delta(t)\limsup_{n\to+\infty}\sigma(-\partial d_{C(\theta_n(t))}(x_n(\theta_n(t)), z), \text{ for all } z \in H,$$

by using Proposition 1.5.1 and the latter inequality entails for almost every $t \in I$

$$\langle z, \zeta(t)\rangle \leq \delta(t)\sigma(-\partial d_{C(t)}(x(t)), z), \text{ for all } z \in H.$$

This implies by the closedness and convexity of $\partial d_{C(t)}(x(t))$ and by properties of support function that for almost all $t \in I$

$$\zeta(t) \in -\delta(t)\partial d_{C(t)}(x(t)), \text{ hence } -\zeta(t) \in \delta(t)\partial d_{C(t)}(x(t)) \subset N_{C(t)}(x(t)).$$

Consequently, as desired it follows that

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))\mathrm{d}s \qquad \text{a.e. } t \in [T_0, T].$$

Case 2 : Assume that

$$\int_{T_0}^{T}\left[\beta_1(\tau) + \int_{T_0}^{\tau}\beta_2(\tau, s)\,\mathrm{d}s\right]\mathrm{d}\tau \geq \frac{1}{4}.$$

We fix a subdivision of $[T_0, T]$ given by $T_0, T_1, ..., T_k = T$ such that, for any $0 \leq i \leq k-1$,

$$\int_{T_i}^{T_{i+1}}\left[\beta_1(\tau) + \int_{T_0}^{\tau}\beta_2(\tau, s)\,\mathrm{d}s\right]\mathrm{d}\tau < \frac{1}{4}.$$

Then, by what precedes, there exists an absolutely continuous map $x_0 : [T_0, T_1] \longrightarrow H$ such that $x_0(T_0) = x_0$, $x_0(t) \in C(t)$ for all $t \in [T_0, T_1]$, and

$$-\dot{x}_0(t) \in N_{C(t)}(x_0(t)) + f_1(t, x_0(t)) + \int_{T_0}^{t} f_2(t, s, x_0(s))\,\mathrm{d}s, \quad a.e.\ t \in [T_0, T_1].$$

Similarly, there is an absolutely continuous map $x_1 : [T_1, T_2] \longrightarrow H$ such that $x_1(T_1) = x_0(T_1)$, $x_1(t) \in C(t)$ for all $t \in [T_1, T_2]$, and

$$-\dot{x}_1(t) \in N_{C(t)}(x_1(t)) + f_1(t, x_1(t)) + \int_{T_0}^{t} f_2(t, s, x_1(s))\,\mathrm{d}s, \quad a.e.\ t \in [T_1, T_2].$$

By induction, we obtain for each $0 \leq i \leq k-1$ an absolutely continuous mapping $x_i : [T_i, T_{i+1}] \longrightarrow H$ such that $x_i(T_i) = x_{i-1}(T_i)$ for each $1 \leq i \leq k-1$ and such that for each for each $0 \leq i \leq k-1$ one has $x_i(t) \in C(t)$ for all $t \in [T_i, T_{i+1}]$, and

$$-\dot{x}_i(t) \in N_{C(t)}(x_i(t)) + f_1(t, x_i(t)) + \int_{T_0}^{t} f_2(t, s, x_i(s))\,\mathrm{d}s, \quad a.e.\ t \in [T_i, T_{i+1}].$$

We define the mapping $x : [T_0, T] \longrightarrow H$ by

$$x(t) = x_i(t), \ \text{ if } t \in [T_i, T_{i+1}], \ 0 \leq i \leq k-1.$$

Obviously, $x(\cdot)$ is an absolutely continuous mapping satisfying $x(T_0) = x_0$, $x(t) \in C(t)$ for all $t \in [T_0, T]$ and

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int_{T_0}^{t} f_2(t, s, x(s))\,\mathrm{d}s, \quad a.e.\ t \in [T_0, T]. \tag{3.57}$$

Step 7. Estimations. The arguments for the estimations in the statement are similar to those in **Step** 7 of Theorem 2.2.1 in section 2.2.

Step 8. Uniqueness.

Now, we turn to the uniqueness. If $x_1(\cdot), x_2(\cdot)$ are two solutions, the hypo-monotonicity property of the normal cone in Proposition 1.3.2 yields for almost all $t \in [T_0, T]$

$$\begin{aligned}&\Big\langle -\dot{x}_1(t) - f_1(t, x_1(t)) - \int_{T_0}^{t} f_2(t, s, x_1(s))\,\mathrm{d}s + \dot{x}_2(t)\\
&+ f_1(t, x_2(t)) + \int_{T_0}^{t} f_2(t, s, x_2(s))\,\mathrm{d}s, x_2(t) - x_1(t)\Big\rangle\\
&\leq \frac{1}{2r}\|x_2(t) - x_1(t)\|^2 \sum_{i=1}^{2}\Big(\|\dot{x}_i(t)\| + \|f_1(t, x_i(t))\| + \int_{T_0}^{t} \|f_2(t, s, x_i(s))\|\,\mathrm{d}s\Big),\end{aligned}$$

from which we obtain

$$\begin{aligned}
&\langle \dot{x}_2(t) - \dot{x}_1(t), x_2(t) - x_1(t) \rangle \\
&\leq \frac{1}{2r} \|x_2(t) - x_1(t)\|^2 \sum_{i=1}^{2} \left(\|\dot{x}_i(t)\| + \|f_1(t, x_i(t))\| + \int_{T_0}^{t} \|f_2(t, s, x_i(s))\| \, \mathrm{d}s \right) \\
&+ \langle f_1(t, x_1(t)) - f_1(t, x_2(t)), x_2(t) - x_1(t) \rangle \\
&+ \left\langle \int_{T_0}^{t} f_2(t, s, x_1(s)) \, \mathrm{d}s - \int_{T_0}^{t} f_2(t, s, x_2(s)) \, \mathrm{d}s, x_2(t) - x_1(t) \right\rangle .
\end{aligned}$$

Since the absolutely continuous mappings $x_1(\cdot)$ and $x_2(\cdot)$ are in particular bounded on $[T_0, T]$, we can choose some real $\eta > 0$ such that, for each $i = 1, 2$, $\|x_i(t)\| \leq \eta$ for all $t \in [T_0, T]$. Moreover, applying the Lipschitz continuity of $f_2(t, s, \cdot)$ and the hypo-monotonicity of $f_1(t, \cdot)$ we get from the latter inequality that

$$\begin{aligned}
&\frac{d}{dt} \frac{1}{2} \|x_2(t) - x_1(t)\|^2 \leq L_2^{\eta}(t) \|x_2(t) - x_1(t)\| \int_{T_0}^{t} \|x_2(s) - x_1(s)\| \, ds \\
&+ \left(L_1^{\eta}(t) + \frac{1}{2r} \sum_{i=1}^{2} \left(\|\dot{x}_i(t)\| + \|f_1(t, x_i(t))\| + \int_{T_0}^{t} \|f_2(t, s, x_i(s))\| \, ds \right) \right) \|x_2(t) - x_1(t)\|^2.
\end{aligned}$$

Finally, setting $\Theta(t) := \|x_2(t) - x_1(t)\|^2$ we get

$$\begin{aligned}
\dot{\Theta}(t) \leq &\left(2L_1^{\eta}(t) + \frac{1}{r} \sum_{i=1}^{2} \left(\|\dot{x}_i(t)\| + \|f_1(t, x_i(t))\| + \int_{T_0}^{t} \|f_2(t, s, x_i(s))\| \, \mathrm{d}s \right) \right) \Theta(t) \\
&+ 2L_2^{\eta}(t) \sqrt{\Theta(t)} \int_{T_0}^{t} \sqrt{\Theta(s)} \, \mathrm{d}s,
\end{aligned}$$

hence it suffices to invoke Lemma 2.1.3 with $\varepsilon(\cdot), \epsilon > 0$ arbitrary. Then the proof of the theorem is complete . ■

Consider now the hypothesis

[$(\mathcal{H}_1')$] For each $t \in [T_0, T]$ the nonempty closed $C(t)$ of H is r-prox-regular for some extended real $r \in]0, +\infty]$ and there is some absolutely continuous function $\upsilon : [T_0, T] \longrightarrow \mathbb{R}$ such that

$$\text{haus}(C(t), C(s)) \leq |\upsilon(t) - \upsilon(s)|, \quad \forall \, s, t \in [T_0, T].$$

Taking the hypothesis $(\mathcal{H}_1')$ in place of $(\mathcal{H}_1)$, the theorem with $\rho = +\infty$ furnishes the following corollary.

Corollary 3.1.1. *Let H be a real Hilbert space. Assume that $(\mathcal{H}_1')$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ are satisfied. Then for any initial point $x_0 \in H$ with $x_0 \in C(T_0)$ there exists a unique absolutely continuous solution $x : [T_0, T] \longrightarrow H$ of the Volterra integro-differential inclusion (P_{f_1, f_2}).*

The final result of this section extends a topological property in [9] concerning the map $a \mapsto x_a(\cdot)$ which associates with each $a \in C(T_0)$ the unique solution of the foregoing perturbed sweeping process with the initial condition a. For completeness of the paper we sketch the proof.

Proposition 3.1.1. *Assume that the assumptions with $\rho = +\infty$ of Theorem 3.1.1 (in case (c)) holds. For each $a \in C(T_0)$, denote by $x_a(\cdot)$ the unique solution of the Volterra integro-differential sweeping process*

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int\limits_{T_0}^{t} f_2(t, s, x(s))\,\mathrm{d}s \quad a.e \text{ in } [T_0, T], \\ x(T_0) = a \in C(T_0). \end{cases}$$

Then, the map $\Psi : a \longrightarrow x_a(\cdot)$ from $C(T_0)$ to the space $\mathcal{C}([T_0, T], H)$ endowed with the uniform convergence norm is Lipschitz on any bounded subset of $C(T_0)$.

Proof. Let M be any fixed positive real number. We are going to prove that Ψ is Lipschitz on $C(T_0) \cap B[0, M]$. According to Theorem 3.1.1 (case (c)), there exists a real number M_1 depending only on M such that, for all $z \in C(T_0) \cap B[0, M]$ and for almost all $(t, s) \in Q_\Delta$

$$\begin{aligned} &\|\dot{x}_z(t) + f_1(t, x_z(t)) + \int\limits_{T_0}^{t} f_2(t, s, x_z(s))\,\mathrm{d}s\| \\ &\leq \varphi(t) := |\dot{v}(t)| + (1 + M_1)\beta_1(t) + \int\limits_{T_0}^{t} g(t, s)\,\mathrm{d}s + T\alpha(t)M_1. \end{aligned}$$

Thanks to this last inequality, for some $\eta > 0$ depending only on M, for all $z \in C(T_0) \cap B[0, M]$ and for all $t \in [T_0, T]$, we have

$$x_z(t) \in B[0, \eta]. \tag{3.58}$$

Fix any $a, b \in C(T_0) \cap M\mathbb{B}$. By the hypomonotonicity property of the normal cone in Proposition 1.3.2 we have for almost all $(t, s) \in Q_\Delta$

$$\begin{aligned} &\Big\langle -\dot{x}_a(t) - f_1(t, x_a(t)) - \int\limits_{T_0}^{t} f_2(t, s, x_a(s))\,\mathrm{d}s + \dot{x}_b(t) \\ &+ f_1(t, x_b(t)) + \int\limits_{T_0}^{t} f_2(t, s, x_b(s))\,\mathrm{d}s, x_2(t) - x_1(t) \Big\rangle \leq \frac{\varphi(t)}{r}\|x_b(t) - x_a(t)\|^2, \end{aligned}$$

from which we obtain

$$\begin{aligned}
&\langle \dot{x}_b(t) - \dot{x}_a(t), x_b(t) - x_a(t)\rangle \\
&\leq \frac{\varphi(t)}{r}\|x_b(t) - x_a(t)\|^2 + \langle f_1(t, x_a(t)) - f_1(t, x_b(t)), x_b(t) - x_a(t)\rangle \\
&+ \left\langle \int_{T_0}^{t} f_2(t, s, x_a(s))\,\mathrm{d}s - \int_{T_0}^{t} f_2(t, s, x_b(s))\,\mathrm{d}s, x_b(t) - x_a(t) \right\rangle.
\end{aligned}$$

Since, by the assumptions $(\mathcal{H}_{2,2})$ and $(\mathcal{H}_{3,2})$, there are non-negative functions $L_1^\eta(\cdot)$ and $L_2^\eta(\cdot)$ in $L^1([T_0, T], \mathbb{R})$ such that $f_1(t, \cdot)$ (resp., $f_2(t, s, \cdot)$) is $L_1^\eta(t)$-hypomonotone (resp., $L_2^\eta(t)$-Lipschitz) on $B[0, \eta]$, the above inequality along with (3.58), entails that for almost all $t \in [T_0, T]$,

$$\begin{aligned}
\frac{d}{dt}\|x_b(t) - x_a(t)\|^2 \leq 2\left(L_1^\eta(t) + \frac{\varphi(t)}{r}\right)\|x_b(t) - x_a(t)\|^2 \\
+2L_2^\eta(t)\|x_b(t) - x_a(t)\| \int_{T_0}^{t} \|x_b(s) - x_a(s)\|\,\mathrm{d}s.
\end{aligned}$$

Applying the Gronwall-like differential inequality in Lemma 2.1.3, it results that

$$\sup_{t\in[0,T]} \|x_b(t) - x_a(t)\| \leq \|b - a\| \exp\left(\int_{T_0}^{t} (K(s) + 1)\,\mathrm{d}s\right),$$

where $K(t) = \max\left\{L_1^\eta(t) + \frac{\varphi(t)}{r}, L_2^\eta(t)\right\}$ for all $t \in [T_0, T]$. The proof is then complete. ∎

3.2 Applications to non-regular electrical circuits

The aim of this section is to illustrate the integro-differential sweeping process in the theory of non-regular electrical circuits. Electrical devices like diodes are described in terms of Ampere-Volt characteristic which is (possibly) a multifunction expressing the difference of potential v_D across the device as a function of current i_D going through the device, see [12, 4]. Table 3.1 and Fig.3.1 illustrate the ampere-volt characteristic of an ideal diode model. Let us consider the electrical system shown in Fig.3.2 that is composed of two resistors $R_1 \geq 0$, $R_2 \geq 0$ with voltage/current laws $V_{R_k} = \varphi_k(x_k)$ $(k = 1, 2)$, two inductors $L_1 \geq 0$, $L_2 \geq 0$ with voltage/current laws $V_{L_k} = L_k\dot{x}_k$ $(k = 1, 2)$, three capacitors with a time-varying capacitances $C_1(t) \neq 0$, $C_2(t) \neq 0$ and $C_3(t) \neq 0$ with voltage/current laws $V_{C_1} = \frac{1}{C_1(t)}\int_0^t \big(x_1(s) - i(s)\big)\,\mathrm{d}s$, $V_{C_2} = \frac{1}{C_2(t)}\int_0^t x_2(t)\,\mathrm{d}t$ and $V_{C_3} = \frac{1}{C_3(t)}\int_0^t \big(x_1(s) - x_2(s)\big)\,\mathrm{d}s$, two ideal diodes with characteristics $0 \leq -V_{D_1} \perp (x_1 - i) \geq 0$, $0 \leq -V_{D_2} \perp x_2 \geq 0$ and an absolutely continuous current source $i : [0, T] \to R$.

Operation Mode	On (Forward biased)	Off (Reverse biased)
Current Through Voltage / Across	$i_D > 0$, $v_D = 0$	$i_D = 0$, $v_D < 0$
Diode looks like	Short circuit	Open circuit

Table 3.1: Ideal Diode Characteristics.

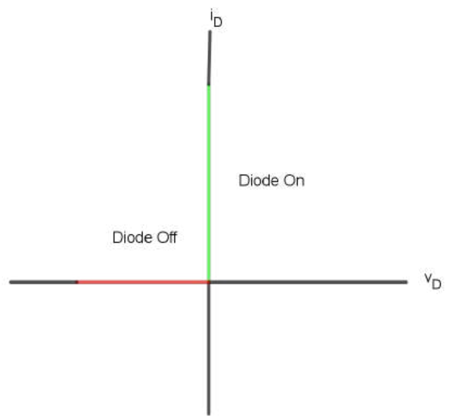

Figure 3.1: Ideal Diode Model.

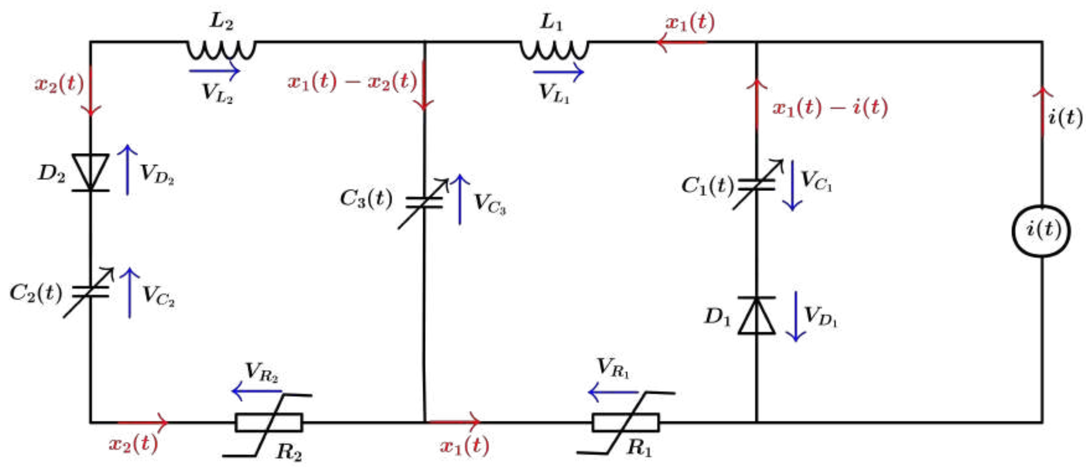

Figure 3.2: Electrical circuit with resistors, inductances, time-varying capacitors and ideal diodes.

Using Kirchhoff's laws, we have

$$\begin{cases} V_{R_1} + V_{L_1} + V_{C_1} + V_{C_3} = -V_{D_1} \in -N(\mathbb{R}_+; x_1 - i) \\ V_{R_2} + V_{L_2} + V_{C_2} - V_{C_3} = -V_{D_2} \in -N(\mathbb{R}_+; x_2). \end{cases}$$

Therefore the dynamics of this circuit is given by

$$\overbrace{\begin{pmatrix} -\dot{x}_1(t) \\ -\dot{x}_2(t) \end{pmatrix}}^{-\dot{x}(t)} \in N_{[i(t),+\infty[\times[0,+\infty[}(x(t)) + \begin{pmatrix} \frac{1}{L_1}\varphi_1(x_1(t)) \\ \frac{1}{L_2}\varphi_2(x_2(t)) \end{pmatrix}$$
$$+\int_0^t \Bigg[\overbrace{\begin{pmatrix} \frac{1}{L_1C_1(t)}+\frac{1}{L_1C_3(t)} & -\frac{1}{L_1C_3(t)} \\ -\frac{1}{L_2C_3(t)} & \frac{1}{L_2C_2(t)}+\frac{1}{L_2C_3(t)} \end{pmatrix}}^{A_2(t)} \overbrace{\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}}^{x(s)} + \begin{pmatrix} \frac{1}{L_1C_1(t)} i(s) \\ 0 \end{pmatrix} \Bigg] \mathrm{d}s. \tag{3.59}$$

Proposition 3.2.1. *Assume that $i : [0,T] \longrightarrow \mathbb{R}$ is an absolutely continuous function, $C_k : [0,T] \longrightarrow \mathbb{R}\setminus\{0\}$, $k = 1,2,3$, are continuous functions and $f_1 := (\frac{1}{L_1}\varphi_1, \frac{1}{L_2}\varphi_2)^t$ satisfies $(\mathcal{H}_2)$. Then for any initial condition $x(0) = x^0$ with $x_1^0 \geq i(0)$ and $x_2^0 \geq 0$, problem (4.165) has one and only one absolutely continuous solution $x(\cdot)$.*

Proof. Put $w(t) = (i(t),0)^t$, $C(t) := w(t)+[0,+\infty[\times[0,+\infty[$, $f_2(t,s,x) = A_2(t)x+\dfrac{1}{L_1C_1(t)}w(s)$. So (4.165) can be rewritten in the framework of our problem (P_{f_1,f_2}) as

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t,x(t)) + \displaystyle\int_0^t f_2(t,s,x(s))\mathrm{d}s \text{ a.e. in } [0,T], \\ x(0) = x_0 \in C(0). \end{cases}$$

Then the above data satisfy all the assumptions of Theorem 3.1.1 (precisely case (c)), with

$$\rho = +\infty,\;\; v(t) = \int_0^t \|\dot{w}(s)\|\,\mathrm{d}s,\;\; g(t,s) = \frac{1}{L_1C_1(t)}\|w(s)\|,\;\; \alpha_2(t) = \|A_2(t)\|.$$

This finishes the proof. ■

3.3 Some numerical experiments

In this section, we will give some numerical simulations to illustrate the theoretical results discussed in Section 3.1. In order to solve numerically problem (P_{f_1,f_2}), we will use the following algorithm discussed in the proof of Theorem 3.1.1. Let us suppose that the dimension of H is finite. For $n \in \mathbb{N}$, let

$$T_0 = t_0^n < t_1^n < \dots < t_n^n = T, \quad h_n = \frac{T-T_0}{2^n}, \quad t_i^n = T_0 + ih_n, \quad 0 \leq i \leq 2^n,$$

be a finite partition of the interval $[0,T]$.

We see that (3.12) is equivalent to solve the following optimization problem :

$$\underset{x\in C(t_{i+1}^n)}{\operatorname{argmin}} \frac{1}{2}\left\| x - x_i^n + \int_{t_i^n}^{t_{i+1}^n} f_1(s,x_i^n)\,\mathrm{d}s + \int_{t_i^n}^{t_{i+1}^n} \left\{ \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f_2(\tau,s,x_j^n)\mathrm{d}s + \int_{t_i^n}^{\tau} f_2(\tau,s,x_i^n)\mathrm{d}s \right\} \mathrm{d}\tau \right\|^2. \tag{3.60}$$

For solving the optimization problem (3.60), we use the trapezoidal rule to calculate the integrals and we can use any nonlinear programming solver (e.g., fmincon : Find minimum of constrained nonlinear multivariable function, in Matlab).

Exemple 3.3.1. *Let* $H = \mathbb{R}^2$, $T = 1$, $T_0 = 0$, $x = (x_1, x_2)$, $x_0 = (x_1^0, x_2^0) = (1,1)$, $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $f_1(t,x) = Ax + (1,0)$, $f_2(t,s,x) = Bx - (2se^{-t}, 0)$.
Put $C(t) := [e^t, e] \times [0, e]$, $\forall\, t \in [0,1]$. *Here the set-valued mapping* $C(\cdot)$ *is absolutely continuous on* $[0,1]$ *with respect to the* Hausdorff *distance where* $\upsilon(t) := e^t$. *Therefore* $x(t) = (e^t, e^t)$ *is the unique solution of*

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t, x(t)) + \int\limits_0^t f_2(t,s,x(s))\mathrm{d}s \qquad a.e.\ t \in [0,1] \\ x(0) = x_0 \in C(0) \end{cases} \tag{3.61}$$

since one has

$$N([e^t, e] \times [0,e]; (e^t, e^t)) = \begin{cases}]-\infty, 0] \times \{0\}, & \text{if } t \in]0,1[\\ ]-\infty, 0] \times \mathbb{R}, & \text{if } t = 0 \\ \mathbb{R} \times [0, +\infty[, & \text{if } t = 1. \end{cases}$$

Using the previous algorithm with $n = 30$ for solving (3.61), we clearly see the convergence of the algorithm in Fig.3.3.

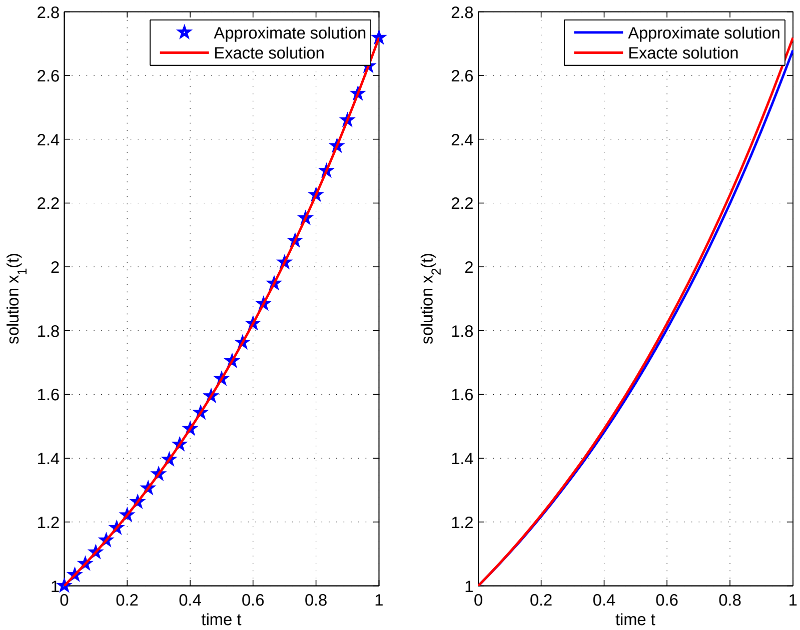

Figure 3.3: Comparison of the exact and the approximate solution of Example 3.3.

Exemple 3.3.2. *Let* $H = \mathbb{R}$, $T = 1$, $T_0 = 0$, $x_0 = a$, $f_1(t, x) = -t + x$, $f_2(t, s, x) = -t(1 + 2t)e^{s(t-s)}x$. *Put* $C(t) := [e^{t^2}, e]$, $\forall t \in [0, 1]$. *Here the set-valued mapping* $C(\cdot)$ *is absolutely continuous on* $[0, 1]$ *with respect to the* Hausdorff *distance where* $\upsilon(t) = e^{t^2}$. *Fig.3.4 presents the approximate solution of problem* (P_{f_1,f_2}) *when the input initial data a varies in* $C(0)$*, it shows the variation of the approximate solution with respect to the input initial data a stated in Proposition 3.1.1.*

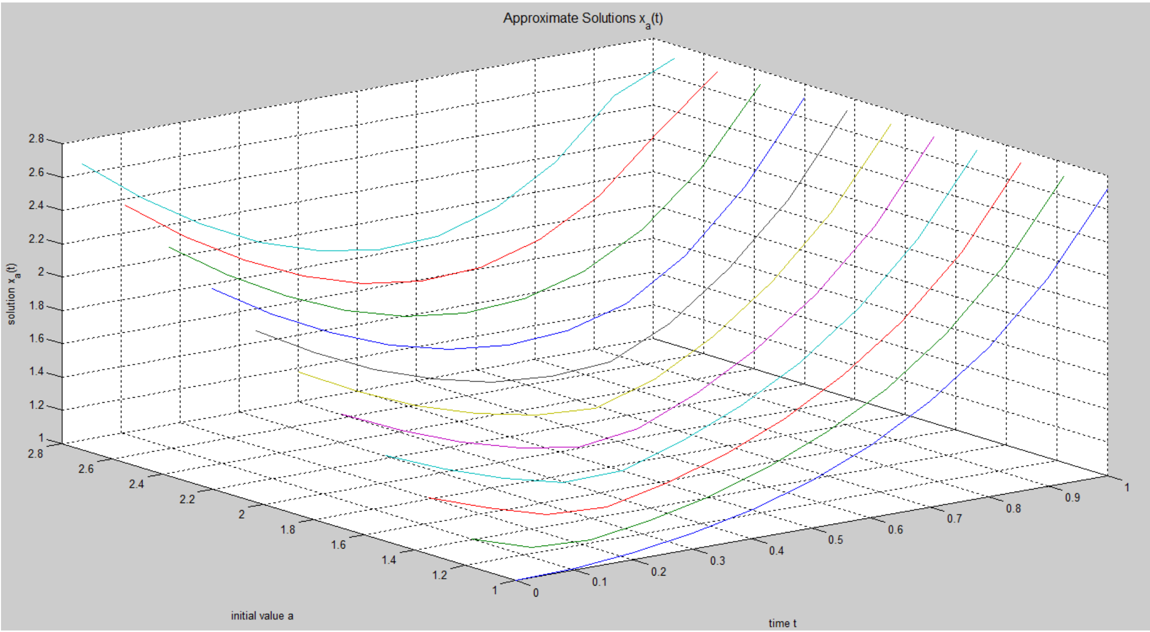

Figure 3.4: Dependence of the solution with respect to the initial data.

We end this section with the following simulation of the non-regular electrical circuit presented in Section 3.2.

Exemple 3.3.3. *We consider the RLC circuit (with* $L_1 = R_1 = 1$, $L_2 = R_2 = 2$, $C_1(t) = C_2(t) = C_3(t) = t + 1$, $i(t) = \sin(2t)$ *and* $\varphi_i(x_i) = R_i\sqrt{x_i}$ *if* $x_1, x_2 \geq 0$ *and* $\varphi_i(x) = 0$ *if either* $x_1 < 0$ *or* $x_2 < 0$, $i = 1, 2$*) coupled to two ideal diodes as shown in Fig.3.2. The network is described by* (3.59)*, where* x_1 *and* x_2 *are the current through the inductors* L_1 *and* L_2 *respectively and* V_{D_1} *and* V_{D_2} *are the voltage across diode* 1 *and* 2 *respectively.*

Depending on whether the diodes are blocking (off) or conducting (on), the system has $2^2 = 4$ *modes:*

- *Mode 1: Both diodes are off, i.e.,* $x_1 = i$ *and* $x_2 = 0$.
- *Mode 2: The first diode is off while the second one is on, i.e.,* $x_1 = i$ *and* $x_2 > 0$ $(V_{D_2} = 0)$.
- *Mode 3: The first diode is on and the second one is off, i.e.,* $x_1 > i$ $(V_{D_1} = 0)$ *and* $x_2 = 0$.
- *Mode 4: Both diodes are conducting, i.e.,* $x_1 > i$ $(V_{D_1} = 0)$ *and* $x_2 > 0$ $(V_{D_2} = 0)$.

Fig.3.5 shows that the mode will vary during the time evolution of the system (3.59) *(diodes go from conducting to blocking or vice versa).*

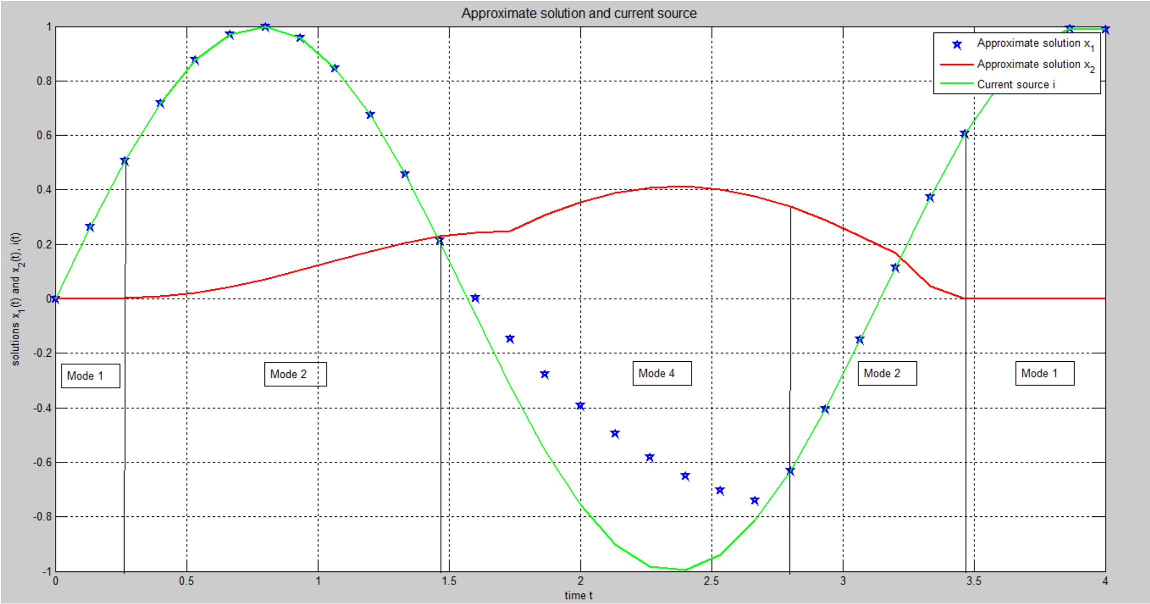

Figure 3.5: Simulation of the circuit Fig.3.2 with the initial conditions $x_1(0) = 0, x_2(0) = 0$ and $L_1 = R_1 = 1$, $L_2 = R_2 = 2$, $C_1(t) = C_2(t) = C_3(t) = t + 1$, $T = 4$. The input current source $i(t) = \sin(2t)$.

3.4 Optimal Control Problem

The aim of this section is to prove the existence of optimal solutions of the optimal control problem (OC) described by a cost function and a nonlinear integro-differential sweeping process of Volterra type

$$(OC): \begin{cases} \text{Minimize } \ \phi(x(T,w)) \\ \text{over processes } \ (x,w) = (x,(u,v)) \ \text{ such that} \\ -\dot{x}(t) \in N_{C(t)}(x(t)) + g_1\big(t,x(t),u(t)\big) + \displaystyle\int_{T_0}^{t} f_2\big(t,s,x(s),v(s)\big)\,\mathrm{d}s, \ \text{a.e } \ t \in I, \qquad (\mathcal{D}_{u,v}) \\ u(\cdot) \text{ and } v(\cdot) \text{ are measurable mappings} \\ (u(t),v(t)) \in U \times V = W, \ \text{a.e } \ t \in I, \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $I = [T_0, T]$ $(T > T_0 > 0)$, $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$, $g_1 : I \times \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$, $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^d$. Suppose that f_2 has the following representation:

$$f_2(t,s,x,v) = \gamma(t,s)\, g_2(s,x,v) \quad \text{for all} \quad (t,s) \in Q_\Delta \text{ and } (x,v) \in \mathbb{R}^n \times \mathbb{R}^d,$$

where $\gamma : Q_\Delta \longrightarrow \mathbb{R}$, $g_2 : I \times \mathbb{R}^{n+d} \longrightarrow \mathbb{R}^n$.

Assume that, the data for the problem (OC) satisfy the following assumptions:

($\mathcal{A}_1$) For each $t \in I$, $C(t)$ is a nonempty closed subset of $\mathbb{R}^n$ which is r-prox-regular for some constant $r \in (0, +\infty]$, and there is a nonnegative and nondecreasing function $\upsilon \in W^{1,2}(I, \mathbb{R})$ and an extended real $\rho \geq 2\left(\|x_0\| + \int_{T_0}^{T} \dot{\upsilon}(s)\, \mathrm{d}s + \frac{1}{2} \right)$ such that

$$\mathrm{haus}_\rho(C(t), C(s)) \leq |\upsilon(t) - \upsilon(s)|, \quad \forall\, s, t \in I.$$

($\mathcal{A}_2$) The functions $g_1 : I \times \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ with $g_1(t, x, u)$ and $g_2 : I \times \mathbb{R}^{n+d} \longrightarrow \mathbb{R}^n$ with $g_2(t, x, \upsilon)$ are (Lebesgue) measurable with respect to t and continuous with respect to u on U and υ on V respectively, and such that

($\mathcal{A}_{2,1}$) there exist real constants $\beta_1, \beta_2 > 0$, such that

$$\|g_1(t, x, u)\| \leq \beta_1(1 + \|x\|),\ \|g_2(t, x, \upsilon)\| \leq \beta_2(1 + \|x\|),$$

for all $t \in I,\ x \in \mathbb{R}^n$ and $(u, \upsilon) \in \mathbb{R}^m \times \mathbb{R}^d$.

($\mathcal{A}_{2,2}$) there exists real constants $L_1, L_2 > 0$ such that for any $t \in I$, any $x, y \in \mathbb{R}^n$ and any $(u, \upsilon) \in \mathbb{R}^m \times \mathbb{R}^d$

$$\|g_1(t, x, u) - g_1(t, y, u)\| \leq L_1\|x - y\|,\ \|g_2(t, x, \upsilon) - g_2(t, y, \upsilon)\| \leq L_2\|x - y\|.$$

($\mathcal{A}_{2,3}$) for each $(t, x) \in I \times \mathbb{R}^n$, the sets

$$g_1(t, x, U) := \{g_1(t, x, u) \ : \ u \in U\}, \quad g_2(t, x, V) := \{g_2(t, x, \upsilon) \ : \ \upsilon \in V\},$$

are convex.

($\mathcal{A}_3$) $\gamma : Q_\Delta \to \mathbb{R}$ is a continuous function such that $\gamma_0 : I \to \mathbb{R}_+$, defined by

$$\gamma_0(t) = \sup_{s \in [0,t]} |\gamma(t, s)|, \quad \text{is bounded on } I.$$

($\mathcal{A}_4$) The sets U and V are compact.

($\mathcal{A}_5$) The cost function ϕ is lower semicontinuous.

Theorem 3.4.1. *Assume that assumptions* ($\mathcal{A}_1$)-($\mathcal{A}_5$) *hold. If there exists at least one admissible solution of* (OC)*, then* (OC) *admits an optimal solution* $\widehat{x}(\cdot) = x(\cdot, \widehat{w})$*, where* $\widehat{w} = (\widehat{u}, \widehat{\upsilon})$.

Proof. Notice first by Theorem 3.1.1 that for each measurable mapping $w(\cdot) = (u(\cdot), \upsilon(\cdot))$ with $w(t) \in U \times V$, the differential inclusion $(\mathcal{D}_{u,\upsilon})$ has a unique solution $x(\cdot, w)$. Consider any minimizing sequence $(w^k)_{k\geq 1}$ of feasible controls of (OC), where $w^k = (u^k, \upsilon^k)$. Denote $x^k(\cdot) := x(\cdot, w_k)$ the solution of $\big(\mathcal{D}_{u^k,\upsilon^k}\big)$. By the proof of Theorem 3.1.1 and assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_{2,1})$, $x^k(\cdot)$ satisfies the following estimates, $\|\dot{x}^k(t)\| \leq \kappa(t)$ a.e. and $\|x^k(t)\| \leq c$, for some nonnegative real function $\kappa \in L^2(I, \mathbb{R})$ and some real constant $c > 0$. Therefore, by the Arzelá-Ascoli theorem (Theorem 1.5.2), $(x^k(\cdot))_{k\geq 1}$ is relatively compact in $\mathcal{C}(I, \mathbb{R}^n)$, and we can extract from this sequence a subsequence, that we do not relabel, which converges uniformly to some mapping $\widehat{x}(\cdot) \in \mathcal{C}(I, \mathbb{R}^n)$. Consider the sequence $(\dot{x}^k)$, (z_1^k), (z_2^k), $(\widetilde{z}_1^k)$, $(\widetilde{z}_2^k)$ where

$$z_1^k(t) = g_1\big(t, \widehat{x}(t), u^k(t)\big), \quad z_2^k(t) = g_2\big(t, \widehat{x}(t), \upsilon^k(t)\big),$$

$$\widetilde{z}_1^k(t) = g_1\big(t, x^k(t), u^k(t)\big), \quad \widetilde{z}_2^k(t) = g_2\big(t, x^k(t), \upsilon^k(t)\big).$$

Under our assumptions, it is easy to see that

$$\int_{T_0}^{T} \|\dot{x}^k(t)\|^2 \,\mathrm{d}t \leq \|\kappa\|^2_{L^2(I,\mathbb{R}^n)}, \quad \max\left\{\int_{T_0}^{T} \|z_1^k(t)\|^2 \,\mathrm{d}t, \int_{T_0}^{T} \|\widetilde{z}_1^k(t)\|^2 \,\mathrm{d}t\right\} \leq T\beta_1^2(1+c)^2,$$

$$\max\left\{\int_{T_0}^{T} \|z_2^k(t)\|^2 \,\mathrm{d}t, \int_{T_0}^{T} \|\widetilde{z}_2^k(t)\|^2 \,\mathrm{d}t\right\} \leq T^2\beta_2^2(1+c)^2.$$

Then, from the sequential weak compactness of the dual ball in $L^2(I, \mathbb{R}^{5n})$ the sequence $(\dot{x}^k, z_1^k, \widetilde{z}_1^k, z_2^k, \widetilde{z}_2^k)$ admits a (non-relabeled) subsequence converging weakly to some $(\xi, \rho_1, \widetilde{\rho}_1, \rho_2, \widetilde{\rho}_2)$ in $L^2(I, \mathbb{R}^{5n})$.

Mazur's theorem (Theorem 1.5.2) asserts the existence of a sequence $(\xi^n, \rho_1^n, \widetilde{\rho}_1^n, \rho_2^n, \widetilde{\rho}_2^n)$ converging to $(\xi, \rho_1, \widetilde{\rho}_1, \rho_2, \widetilde{\rho}_2)$ with respect to the (usual) norm of $L^2(I, \mathbb{R}^{5n})$, where

$$\xi^n(t) = \sum_{k=n}^{r(n)} S_{k,n}\dot{x}^k(t),$$

$$\rho_1^n(t) = \sum_{k=n}^{r(n)} S_{k,n} z_1^k(t) = \sum_{k=n}^{r(n)} S_{k,n} g_1\big(t, \widehat{x}(t), u^k(t)\big),$$

$$\widetilde{\rho}_1^n(t) = \sum_{k=n}^{r(n)} S_{k,n} \widetilde{z}_1^k(t) = \sum_{k=n}^{r(n)} S_{k,n} g_1\big(t, x^k(t), u^k(t)\big),$$

$$\rho_2^n(t) = \sum_{k=n}^{r(n)} S_{k,n} z_2^k(t) = \sum_{k=n}^{r(n)} S_{k,n} g_2\big(t, \widehat{x}(t), \upsilon^k(t)\big),$$

$$\widetilde{\rho}_2^n(t) = \sum_{k=n}^{r(n)} S_{k,n} \widetilde{z}_2^k(t) = \sum_{k=n}^{r(n)} S_{k,n} g_2\big(t, x^k(t), \upsilon^k(t)\big),$$

with $\sum_{k=n}^{r(n)} S_{k,n} = 1$ and $S_{k,n} \in [0,1]$ for all k, n. Then there exists a subsequence of $(\xi^n, \rho_1^n, \widetilde{\rho}_1^n, \rho_2^n, \widetilde{\rho}_2^n)$ (again, we do not relabel) converging to $(\xi, \rho_1, \widetilde{\rho}_1, \rho_2, \widetilde{\rho}_2)$ almost everywhere. From this or the weak convergence in $L^2(I, \mathbb{R}^n)$ of $(\dot{x}^k(\cdot))_k$ to $\xi(\cdot)$, from the uniform convergence of $(x^k(\cdot))_k$ to $\widehat{x}(\cdot)$ and from the above equality $x^k(t) = x^k(T_0) + \int_{T_0}^{t} \dot{x}^k(s)\,\mathrm{d}s$, it follows that $\widehat{x}(t) = x_0 + \int_{T_0}^{t} \xi(s)\,\mathrm{d}s$ for all $t \in I$, and $\dot{\widehat{x}}(t) = \xi(t)$ a.e. $t \in I$. From the convexity of $g_1\big(t, \widehat{x}(t), U\big)$ (by $(\mathcal{A}_{2,3})$) we also have

$$\rho_1^n(t) = \sum_{k=n}^{r(n)} S_{k,n} g_1\big(t, \widehat{x}(t), u^k(t)\big) \in g_1\big(t, \widehat{x}(t), U\big), \text{ a.e. } t \in I.$$

Using the almost everywhere convergence of (ρ_1^n) to ρ_1, we get, thanks to the closedness of $g_1\big(t, \widehat{x}(t), U\big)$

$$\rho_1(t) \in g_1\big(t, \widehat{x}(t), U\big), \text{ a.e. } t \in I.$$

By measurable selection theorems (see Theorem III.38 in [28]) we also assert the existence of a measurable mapping $\widehat{u} : I \to \mathbb{R}^m$ such that $\widehat{u}(t) \in U$ a.e. $t \in I$ and

$$\rho_1(t) = g_1\big(t, \widehat{x}(t), \widehat{u}(t)\big), \text{ a.e. } t \in I.$$

Observing that

$$\begin{aligned}
&\|\widetilde{\rho}_1^n(t) - \rho_1(t)\| \\
&= \left\| \sum_{k=n}^{r(n)} S_{k,n} \Big(g_1\big(t, x^k(t), u^k(t)\big) - g_1\big(t, \widehat{x}(t), u^k(t)\big) + g_1\big(t, \widehat{x}(t), u^k(t)\big) - \rho_1(t) \Big) \right\| \\
&\leq \left\| \sum_{k=n}^{r(n)} S_{k,n} \Big(g_1\big(t, x^k(t), u^k(t)\big) - g_1\big(t, \widehat{x}(t), u^k(t)\big) \Big) \right\| \\
&+ \left\| \sum_{k=n}^{r(n)} S_{k,n} \Big(g_1\big(t, \widehat{x}(t), u^k(t)\big) - \rho_1(t) \Big) \right\|,
\end{aligned}$$

and applying $(\mathcal{A}_{2,2})$ one obtains

$$\|\widetilde{\rho}_1^n(t) - \rho_1(t)\| \leq L_1 \sum_{k=n}^{r(n)} S_{k,n} \|x^k(t) - \widehat{x}(t)\| + \left\| \sum_{k=n}^{r(n)} S_{k,n} \Big(g_1\big(t, \widehat{x}(t), u^k(t)\big) - \rho_1(t) \Big) \right\|.$$

Using the uniform convergence of $x^k(\cdot)$ to $\widehat{x}$ and the almost everywhere convergence of $\rho_1^n(\cdot)$ to $\rho_1(\cdot)$ we obtain that for almost every $t \in I$

$$\lim_{n \to +\infty} \|\widetilde{\rho}_1^n(t) - \rho_1(t)\| = 0.$$

So, by the almost everywhere convergence of $\widetilde{\rho}_1^n(\cdot)$ to $\widetilde{\rho}_1(\cdot)$ and by uniqueness of the limit we deduce that

$$\widetilde{\rho}_1(t) = \rho_1(t) = g_1\big(t, \widehat{x}(t), \widehat{u}(t)\big), \quad \text{a.e. } t \in I.$$

In the same way, we also assert the existence of a measurable mapping $\widehat{\upsilon} : I \to \mathbb{R}^d$ such that $\widehat{\upsilon}(t) \in V$ a.e. $t \in I$ and

$$\lim_{n\to+\infty} \|\widetilde{\rho}_2^n(t) - \rho_2(t)\| = 0,$$

$$\widetilde{\rho}_2(t) = \rho_2(t) = g_2\big(t, \widehat{x}(t), \widehat{\upsilon}(t)\big), \quad \text{a.e. } t \in I. \tag{3.62}$$

Let us set for each $t \in I$

$$y^n(t) = \sum_{k=n}^{r(n)} S_{k,n} \int_{T_0}^{t} \gamma(t,s) g_2\big(s, x^k(s), \upsilon^k(s)\big) \mathrm{d}s \quad \text{and} \quad y(t) = \int_{T_0}^{t} \gamma(t,s) g_2\big(s, \widehat{x}(s), \widehat{\upsilon}(s)\big) \mathrm{d}s.$$

It follows that

$$\begin{aligned}
\|y^n(t) - y(t)\| &= \left\| \sum_{k=n}^{r(n)} S_{k,n} \int_{T_0}^{t} \gamma(t,s) \Big(g_2\big(s, x^k(s), \upsilon^k(s)\big) - g_2\big(s, \widehat{x}(s), \widehat{\upsilon}(s)\big) \Big) \mathrm{d}s \right\| \\
&\le \int_{T_0}^{t} |\gamma(t,s)| \left\| \sum_{k=n}^{r(n)} S_{k,n} g_2\big(s, x^k(s), \upsilon^k(s)\big) - g_2\big(s, \widehat{x}(s), \widehat{\upsilon}(s)\big) \right\| \mathrm{d}s \\
&= \int_{T_0}^{t} |\gamma(t,s)| . \|\widetilde{\rho}_2^n(s) - \widetilde{\rho}_2(s)\| \mathrm{d}s \\
&\le \gamma_0(t) \int_{T_0}^{t} \|\widetilde{\rho}_2^n(s) - \widetilde{\rho}_2(s)\| \mathrm{d}s.
\end{aligned}$$

Then, since $\|x^k(\cdot)\| \le c$, by the almost everywhere convergence of ρ_2^n to ρ_2, (3.62), the assumption $(\mathcal{A}_3)$ on $\gamma(\cdot,\cdot)$ and the Lebesgue dominated convergence theorem, we have

$$y^n(t) \xrightarrow{n\to+\infty} y(t), \quad \text{a.e. } t \in I.$$

Note also that

$$\sum_{k=n}^{r(n)} S_{k,n} \zeta^k(t) \xrightarrow{n\to+\infty} \zeta(t), \quad \text{a.e. } t \in I,$$

where

$$\zeta^k(t) = \dot{x}^k(t) + g_1\big(t, x^k(t), u^k(t)\big) + \int_{T_0}^{t} \gamma(t,s) g_2\big(s, x^k(s), \upsilon^k(s)\big) \, \mathrm{d}s, \quad \text{a.e. } t \in I,$$

and

$$\zeta(t) = \dot{\widehat{x}}(t) + g_1\big(t, \widehat{x}(t), \widehat{u}(t)\big) + \int_{T_0}^{t} \gamma(t,s) g_2\big(s, \widehat{x}(s), \widehat{\upsilon}(s)\big) \, \mathrm{d}s, \quad \text{a.e. } t \in I.$$

Observe that since, for each k, (x^k, w^k) is a feasible solution of (OC), there is a negligible set $N \subset I$ such that for each $t \in I \setminus N$ one has

$$y^n(t) \xrightarrow{n\to+\infty} y(t) \quad \text{and} \quad \sum_{k=n}^{r(n)} S_{k,n}\zeta^k(t) \xrightarrow{n\to+\infty} \zeta(t)$$

along with for all $k \in \mathbb{N}$

$$-\zeta^k(t) \in N_{C(t)}(x^k(t)) \quad \text{for all } k \in I\!N.$$

Fix any $t \in I \setminus N$ and any $k \in \mathbb{N}$. Take any $z \in C(t)$. From Proposition 1.3.1 one has

$$\left\langle -\zeta^k(t), z - x^k(t) \right\rangle \leq \frac{\delta(t)}{2r}\|z - x^k(t)\|^2$$

for some nonnegative real function $\delta \in L^2(I, \mathbb{R})$. Hence

$$\left\langle -\zeta^k(t), z - x^k(t) \right\rangle \leq \frac{\delta(t)}{2r}(\|z - \widehat{x}(t)\| + \|\widehat{x}(t) - x^k(t)\|)^2 := \lambda^k(t), \tag{3.63}$$

with $\lim_{k\longrightarrow\infty} \lambda^k(t) = \dfrac{\delta(t)}{2r}\|z - \widehat{x}(t)\|^2$. Write

$$\begin{aligned}\langle -\zeta(t), z - \widehat{x}(t)\rangle = &\left\langle -\zeta(t) + \sum_{k=n}^{r(n)} S_{k,n}\zeta^k(t), z - \widehat{x}(t) \right\rangle + \sum_{k=n}^{r(n)} S_{k,n}\left\langle -\zeta^k(t), z - x^k(t)\right\rangle \\ &+ \sum_{k=n}^{r(n)} S_{k,n}\left\langle -\zeta^k(t), -\widehat{x}(t) + x^k(t)\right\rangle.\end{aligned}$$

The first expression of the second member of the latter equality tends to zero by what precedes, and keeping in mind that $\|\zeta^k(t)\| \leq \delta(t)$, we also see that the third expression tends to zero. Concerning the second expression, thanks to (3.63), it satisfies the estimate

$$\left\langle -\sum_{k=n}^{r(n)} S_{k,n}\zeta^k(t), z - \sum_{k=n}^{r(n)} S_{k,n}x^k(t) \right\rangle \leq \sum_{k=n}^{r(n)} S_{k,n}\lambda^k(t).$$

Thus, passing to the limit we obtain

$$\langle -\zeta(t), z - \widehat{x}(t)\rangle \leq \frac{\delta(t)}{2r}\|z - \widehat{x}(t)\|^2, \quad \forall\, z \in C(t).$$

This proves that

$$-\dot{\widehat{x}}(t) - g_1\big(t, \widehat{x}(t), \widehat{u}(t)\big) - \int_{T_0}^{t} \gamma(t,s)g_2\big(s, \widehat{x}(s), \widehat{v}(s)\big)\,\mathrm{d}s \in N_{C(t)}(\widehat{x}(t)), \quad a.e.\ t \in I,$$

and thus

$$-\dot{\widehat{x}}(t) \in N_{C(t)}(\widehat{x}(t)) + g_1\big(t, \widehat{x}(t), \widehat{u}(t)\big) + \int_{T_0}^{t} \gamma(t,s)g_2\big(s, \widehat{x}(s), \widehat{v}(s)\big)\,\mathrm{d}s, \quad a.e.\ t \in I.$$

Further, $\widehat{x}(T_0) = x_0$ by the convergence $x^k(T_0) \to \widehat{x}(T_0)$. Since

$$\phi(\widehat{x}(T)) \leq \liminf_{k\to+\infty} \phi(x^k(T)) = \inf_{(x,w)} \phi(x(T)),$$

we conclude that $(\widehat{x}, \widehat{w}) = (\widehat{x}, (\widehat{u}, \widehat{v}))$ is an optimal solution to (OC), proving the Theorem. ■

Chapter 4

Optimal Control of Nonconvex Integro-Differential Sweeping Processes

Abstract. This chapter is devoted to the study, for the first time in the literature, of optimal control problems for sweeping processes governed by integro-differential inclusions of the Volterra type with different classes of control functions acting in nonconvex moving sets, external dynamic perturbations, and integral parts of the sweeping dynamics. We establish the existence of optimal solutions and then obtain necessary optimality conditions for a broad class of local minimizers in such problems. Our approach to deriving necessary optimality conditions is based on the method of discrete approximations married to basic constructions and calculus rules of first-order and second-order variational analysis and generalized differentiation. The obtained necessary optimality conditions are expressed entirely in terms of the problem data and are illustrated by nontrivial examples that include applications to optimal control models of non-regular electrical circuits.

We are interested in the existence of solution and the derivation of necessary conditions for the following Bolza problem

$$(P): \text{ Minimize } \quad J_0[x,u,a,b] := \varphi(x(T)) + \int_0^T l_0\big(t, x(t), u(t), a(t), b(t), \dot{x}(t), \dot{u}(t)\big)\, dt, \tag{4.1}$$

over (u,a,b) such that

$$\begin{cases} -\dot{x}(t) \in N_{C(u(t))}(x(t)) + f_1(a(t), x(t)) + \int_0^t f_2\big(b(s), x(s)\big)\mathrm{d}s, \quad \text{a.e. } [0,T], \\ \big(u(\cdot), a(\cdot), b(\cdot)\big) \in W^{1,2}([0,T], \mathbb{R}^n) \times L^2([0,T], \mathbb{R}^{m+d}), \\ (a(t), b(t)) \in A \times B \subset \mathbb{R}^m \times \mathbb{R}^d \ \text{ a.e. } \ t \in \ [0,T], \\ x(0) = x_0 \in C(0), \end{cases} \tag{4.2}$$

where $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$, $l_0 : [0,T] \times \mathbb{R}^{4n+m+d} \to \mathbb{R}$ and the controlled moving sets are given in the form

$$C(t) := C(u(t)) = C + u(t), \ C := \big\{x \in \mathbb{R}^n \ \big| \ g_i(x) \geq 0, \quad i = 1, \dots, s\big\}. \tag{4.3}$$

As follows from the sweeping inclusion (4.2) and the structure of the controlled moving sets (4.3), problem (P) automatically involves the pointwise *mixed state-control constraints*

$$g_i(x(t) - u(t)) \geq 0 \ \text{ for all } \ t \in [0,T] \ \text{ and } \ i = 1, \dots, s, \tag{4.4}$$

Denoting now the corresponding running cost $l : [0,T] \times \mathbb{R}^{6n+m+d} \to \mathbb{R}$ by

$$l(t, x, y, u, a, b, \dot{x}, \dot{y}, \dot{u}) := l_0(t, x, u, a, b, \dot{x}, \dot{u}), \tag{4.5}$$

we see that problem (P) can be equivalently reformulated in the form:

$$\text{Minimize} \quad J[x,y,u,a,b] := \varphi(x(T)) + \int_0^T l\big(t, x(t), y(t), u(t), a(t), b(t), \dot{x}(t), \dot{y}(t), \dot{u}(t)\big)\, \mathrm{d}t \tag{4.6}$$

over (u,a,b) such that

$$\begin{cases} -\dot{x}(t) \in N_{C(u(t))}(x(t)) + f_1\big(a(t), x(t)\big) + y(t) \quad \text{a.e. } \ t \in [0,T], \\ \dot{y}(t) = f_2\big(b(t), x(t)\big) \quad \text{a.e. } t \in [0,T], \\ \big(u(\cdot), a(\cdot), b(\cdot)\big) \in W^{1,2}([0,T], \mathbb{R}^n) \times L^2([0,T], \mathbb{R}^{m+d}), \\ (a(t), b(t)) \in A \times B \subset \mathbb{R}^m \times \mathbb{R}^d \quad \text{a.e. } t \in \ [0,T], \\ x(0) = x_0 \in C(0), \ \ y(0) = 0. \end{cases} \tag{4.7}$$

Let us start this chapter with listing the *standing assumptions* imposed throughout the chapter unless otherwise stated. On the other hand, in some statements we specify those assumptions from the standing ones, which are actually used in the proof of this particular result

4.1 Standing Assumptions

$(\mathcal{H}_1)$ There are constants $c > 0$ and $M_j > 0$, $j = 1, 2, 3$, and open sets $A_i \subset V_i$ such that $\text{haus}(A_i, \mathbb{R}^n \setminus V_i) > c$ and the functions $g_i(\cdot)$ as $i = 1, \ldots, s$ are twice continuously differentiable satisfying the inequalities

$$M_1 \leq \|\nabla g_i(x)\| \leq M_2, \quad \|\nabla^2 g_i(x)\| \leq M_3 \text{ for all } x \in V_i, \tag{4.8}$$

where "haus" signifies the standard Hausdorff distance between two sets. Furthermore, there are positive numbers β and ρ for which we have the estimate

$$\sum_{i \in I_\rho(x)} \lambda_i \|\nabla g_i(x)\| \leq \beta \| \sum_{i \in I_\rho(x)} \lambda_i \nabla g_i(x)\| \text{ whenever } x \in C \text{ and } \lambda_i \geq 0 \tag{4.9}$$

with the index set for the perturbed constraints defined by

$$I_\rho(x) := \big\{ i \in \{1, \ldots, s\} \;\big|\; g_i(x) \leq \rho \big\}. \tag{4.10}$$

$(\mathcal{H}_2)$ Let $f_1 : \mathbb{R}^{m+n} \to \mathbb{R}^n$, $f_2 : \mathbb{R}^{d+n} \to \mathbb{R}^n$ be continuous mappings satisfying the following properties:

$(\mathcal{H}_{2,1})$ There exist nonnegative constants α_i, $i = 1, 2$, for which we have

$$\|f_1(a(t), x)\| \leq \|a(t)\| + \alpha_1 \|x\| \text{ whenever } t \in [0, T], \ x \in \bigcup_{t \in [0,T]} C(t), \tag{4.11}$$

$$\|f_2(b(t), x)\| \leq \|b(t)\| + \alpha_2 \|x\| \text{ whenever } t \in [0, T], \ x \in \bigcup_{t \in [0,T]} C(t). \tag{4.12}$$

$(\mathcal{H}_{2,2})$ For any real numbers $r_i > 0$ and functions $a(\cdot)$, $b(\cdot) \in L^2([0, T], \mathbb{R}^{m+d})$ there exist nonnegative constants L and L_i as $i = 1, 2$ such that we have the estimates

$$\|f_1(a, x_1) - f_1(a, x_2)\| \leq L \|x_1 - x_2\| \text{ for all } x_1, x_2 \in r_1 \mathbb{B}, \tag{4.13}$$

$$\|f_2(b, x_1) - f_2(b, x_2)\| \leq L_1 \|x_1 - x_2\| \text{ for all } x_1, x_2 \in r_1 \mathbb{B}, \tag{4.14}$$

$$\|f_2(b_1, x) - f_2(b_2, x)\| \leq L_2 \|b_1 - b_2\| \text{ for all } b_1, b_2 \in \mathbb{R}^n, \ x \in r_1 \mathbb{B}. \tag{4.15}$$

$(\mathcal{H}_3)$ The terminal cost $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ is lower semicontinuous (l.s.c.), while and the running cost $l_0 : [0, T] \times \mathbb{R}^{4n+2m+2d} \to \bar{\mathbb{R}}$ is l.s.c. with respect to all but time variable being continuous with respect to t and being majorized by a summable function on $[0, T]$ along reference curves. Furthermore, we assume that $l_0(t, \cdot)$ is bounded from below on bounded sets for a.e. $t \in [0, T]$.

$(\mathcal{H}_4)$ The control sets A and B are closed and bounded in $\mathbb{R}^m$ and $\mathbb{R}^d$, respectively.

It is not hard to verify that if both conditions (4.8) and (4.9) are satisfied, then we have the *positive linear independence* (PLICQ) of the gradients $\nabla g_i(x)$ of the active inequality constraints on C, and that the latter condition is equivalent in our setting to the *Mangasarian-Fromovitz constraint qualification*, which is the basic qualification condition in nonlinear programming.

Putting aside the case of concavity of all the functions g_i in (4.3), it can be seen that the sets C and $C(t)$ therein are *nonconvex*, and hence we need to clarify in which sense the normal cone in (4.2) is understood.

The next result is taken from [83, Proposition 2.9]; see also [6] for more discussions and developments.

Proposition 4.1.1 (prox-regularity of moving sets). *Let assumption* $(\mathcal{H}_1)$ *be satisfied. Then for each* $t \in [0,T]$ *we have that the set* $C(t)$ *is* η*-prox-regular with* $\eta = \dfrac{M_1}{M_3\beta}$.

Indeed, this statement was justified in [83, Proposition 2.9] for the set C in (4.3), and hence it holds for the moving set $C(t) = C + u(t)$ as a translation of C.

We close this section with the following well-known discrete version of Gronwall's inequality; see, e.g., [24] for a short proof and discussions.

Proposition 4.1.2 (discrete Gronwall's inequality)**.** *Let* e_i, ρ_j, γ_j, a_j *be nonnegative numbers with*

$$e_{j+1} \le \sigma_j + \rho_j \sum_{k=0}^{j-1} e_k + (1+\gamma_j)e_j \quad \textit{for all} \quad j \in I\!N.$$

Then whenever $i \in \mathbb{N}$ *we have the estimate*

$$e_i \le (e_0 + \sum_{k=0}^{i-1} \sigma_k) \exp\Big(\sum_{k=0}^{i-1}(k\rho_k + \gamma_k)\Big).$$

4.2 Existence of optimal solutions and local minimizers

In this section we establish the existence of optimal solutions to (P) and describe the type of local minimizers of (P) used below for deriving necessary optimality conditions. But first we present the following *well-posedness* results for the controlled sweeping dynamics (4.2) that ensure, in particular the existence of *feasible* solutions $(x(\cdot), u(\cdot), a(\cdot), b(\cdot))$ in the corresponding $W^{1,2}$ spaces.

Proposition 4.2.1 (well-posedness of integro-differential sweeping processes). *Fix a triple* $(u(\cdot), a(\cdot), b(\cdot)) \in W^{1,2}([0,T], \mathbb{R}^n) \times L^2([0,T], \mathbb{R}^{m+d})$ *and consider the integro-differential sweeping process* (4.2) *under the assumptions in* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$*. Then this control triple generates a unique trajectory* $x(\cdot) \in W^{1,2}([0,T], \mathbb{R}^n)$ *of system* (4.2)*. Denoting further*

$$\beta_1(t) := \max\big\{\|a(t)\|, \alpha_1\big\}, \quad \widetilde{b}(t) := 2\max\big\{\beta_1(t), \alpha_2\big\} \ \textit{for all} \ t \in [0,T],$$

$$\widetilde{l} := \|x(0)\| \exp\Big(\int_0^T (\widetilde{b}(\tau)+1)\,d\tau\Big)$$
$$+ \exp\Big(\int_0^T (\widetilde{b}(\tau)+1)\,d\tau\Big)\int_0^T \Big(\|\dot{u}(s)\| + 2\beta_1(s) + 2\int_0^T \|b(\tau)\|\,d\tau\Big)\,ds,$$

and taking the corresponding arc $y(\cdot)$ from (4.7)*, we have the estimates*

$$\|\dot{x}(t) + f_1(t,x(t)) + y(t)\| \le \|\dot{u}(t)\| + (1+\widetilde{l})\beta_1(t) + \int_0^t \|b(s)\|\,ds + T\alpha_2\widetilde{l}, \quad a.e.\ \ t\in[0,T],$$
$$and \quad \|x(t)\| \le \widetilde{l}. \tag{4.16}$$

$$\|\dot{x}(t)\| \le \|\dot{u}(t)\| + 2(1+\widetilde{l})\beta_1(t) + 2\int_0^t \|\widetilde{b}(s)\|\,ds + 2T\alpha_2\widetilde{l} \ \ a.e.\ \ t\in[0,T], \tag{4.17}$$

$$\|\dot{y}(t)\| \le \|b(t)\| + \alpha_2\widetilde{l} \ \ for\ all\ \ t\in[0,T]. \tag{4.18}$$

Proof. It is not hard to observe from the construction of $C(t)$ in (4.3) that

$$C(u(t)) \subset C(u(s)) + |v(t)-v(s)|\mathbb{B}, \ \ \text{for all}\ \ t,s\in[0,T]\ \ \text{and}\ \ u(t)\in\mathbb{R}^n, \tag{4.19}$$

where $v(t) := \int_0^t \|\dot{u}(s)\|\,ds$. This implies that

$$C(t) \subset C(s) + |v(t)-v(s)|\mathbb{B}\ \ \text{on}\ \ [0,T].$$

Furthermore, by the imposed condition $(\mathcal{H}_1)$ and Proposition 4.1.1 we have that $C(t)$ is η-prox-regular for each $t\in[0,T]$. If in addition $(\mathcal{H}_{2,1})$ holds, then this ensures that

$$\|f_1(a(t),x)\| \le \|a(t)\| + \alpha_1\|x\| \le \max\big\{\|a(t)\|,\alpha_1\big\}(1+\|x\|) = \beta_1(t)(1+\|x\|).$$

Hence all the assumptions of [9, Theorem 4.2] are satisfied, which thus yields the fulfillment of estimates (4.16)–(4.18). Observe finally that $\dot{x}(\cdot),\dot{y}(\cdot)\in L^2([0,T],\mathbb{R}^n)$ since $\beta_1(\cdot)\in L^2([0,T],\mathbb{R}^n)$ by the construction of $\beta_1(\cdot)$. This therefore completes the proof of the proposition. ■

Consider now the set-valued mapping $F_1 : \mathbb{R}^{3n+m} \rightrightarrows \mathbb{R}^n$ given by

$$F_1(x,y,u,a) := N_{C(u)}(x) + f_1(x,a) + y. \tag{4.20}$$

It is easy to deduce from the representation of the set C in (4.3) that

$$F_1(x,y,u,a) := \Big\{-\sum_{i\in I(x-u)} \eta_i\nabla g_i(x-u)\ \Big|\ 0\le\eta_i\Big\} + f_1(x,a) + y, \tag{4.21}$$

(cf. [83, Proposition 2.8]), where the set of *active constraint indices* is

$$I(y) := \big\{i \in \{1, \ldots, s\} \mid g_i(y) = 0\big\} \tag{4.22}$$

Define further the mapping $F : \mathbb{R}^{3n+m+d} \rightrightarrows \mathbb{R}^n$ and $\widetilde{f}_2 : \mathbb{R}^{3n+m+d} \to \mathbb{R}^n$ by

$$F(z) := F(x, y, u, a, b) = F_1(x, y, u, a) = N_{C(u)}(x) + f_1(x, a) + y,$$

$$\widetilde{f}_2(z) := \widetilde{f}_2(x, y, u, a, b) = -f_2(b, x).$$

We can clearly rewrite the problem for the evolution system in (4.7) in terms of $z \in \mathbb{R}^{3n+m+d}$ as

$$\big(-\dot{x}(t), -\dot{y}(t), -\dot{u}(t)\big) \in F\big(z(t)\big) \times \widetilde{f}_2(z(t)) \times \mathbb{R}^n := G(z(t)) \;\; \text{a.e.} \;\; t \in [0, T] \tag{4.23}$$

with the initial conditions

$$x(0) = x_0, \; y_0 = 0 \text{ and } g_i(x_0 - u_0) \geq 0 \;\text{ as }\; i = 1, \ldots, s. \tag{4.24}$$

Before establishing the existence of optimal solutions to (P) in the aforementioned class of feasible solutions, let us reformulate the sweeping differential inclusion (4.2) in a more convenient way. Consider the images of the control sets A and B under the perturbation mappings $f_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $f_2 : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$, respectively, and denote them by $f_1(x, A)$ and $f_2(x, B)$. Then the sweeping inclusion (4.2) with the moving set (4.3) is equivalently represented as

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(x(t), A) + \int_0^t f_2(x(s), B)\,\mathrm{d}s \quad \text{a.e.} \;\; [0, T], \quad x_0 \in C(0). \tag{4.25}$$

To elaborate more rigorously this statement, we need to use standard facts of the theory of measurable multifunctions; see, e.g., [71, Chapter 14]. Recall that the measurability of a closed-valued multifunction $S : [0, T] \rightrightarrows \mathbb{R}^m \times \mathbb{R}^d$ can be described as follows (see [71, Theorem 14.10(b)]: For every $\varepsilon > 0$ there is a closed set $T_\varepsilon \subset [0, T]$ with $\text{mes}([0, T] \setminus T_\varepsilon) < \varepsilon$ such that $S : T_\varepsilon \rightrightarrows \mathbb{R}^m \times \mathbb{R}^d$ is of closed graph. Fix any $x(\cdot) \in W^{1,2}([0, T]; \mathbb{R}^n)$ satisfying (4.25) with some $u(\cdot)$ from (4.3) and define the closed-valued mapping $S : [0, T] \rightrightarrows \mathbb{R}^m \times \mathbb{R}^d$ by

$$S(t) := \left\{(a, b) \in A \times B \mid -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(x(t), a) + \int_0^t f_2(x(s), b)\,\mathrm{d}s, \;\text{ a.e. } t \in [0, T]\right\}. \tag{4.26}$$

Applying to $-\dot{x}(\cdot)$ the classical Luzin property of measurable functions from real analysis, for any $\varepsilon > 0$ we find a closed set $T_\varepsilon \subset [0, T]$ with $\text{mes}([0, T] \setminus T_\varepsilon) < \varepsilon$ for its Lebesgue measure such that $-\dot{x}(\cdot)$ is continuous on T_ε. Using the assumed continuity of $f_1(x, a)$ and $f_2(x, b)$

together with the closed-graph property of the normal cone mapping in (4.26), where $C(t)$ is taken from (4.3), shows that the graph of the mapping $S : T_\varepsilon \rightrightarrows \mathbb{R}^m \times \mathbb{R}^d$ from (4.26) is closed. It tells us that the full mapping $S(\cdot)$ defined in (4.26) for a.e. $t \in [0,T]$ is measurable on $[0,T]$. Employing the measurable selection theorem from [71, Corollary 14.6] ensures the existence of a measurable control $(a(t), b(t)) \in A \times B$ such that the pair $(x(\cdot), a(\cdot), b(\cdot))$, together with the corresponding $u(\cdot)$ generated the moving set $C(t)$ in (4.3), is feasible to (4.2). The converse implication from (4.2) to (4.25) is obvious, and hence we verify the claimed equivalence.

Now we are ready to obtain the existence theorem for optimal solutions to (P) under certain convexity assumptions with respect to velocities. For simplicity, suppose here that the integrand l does not depend on the control variable a and b. If it does, we have to impose the convexity of an extended velocity set that includes the integrand component.

Theorem 4.2.1 (existence of sweeping optimal solutions). *In addition to the standing assumptions* $(\mathcal{H}_1) - (\mathcal{H}_3)$*, suppose that the integrand* l_0 *from* (4.5) *does not depend on the* (a,b)*-variable while being convex with respect of velocity variables* $\dot{x}, \dot{u}$*, and that there is a minimizing sequence* $\{(x^k(\cdot), y^k(\cdot), u^k(\cdot), a^k(\cdot), b^k(\cdot))\}$ *in* (P) *such that* $\{(x^k(\cdot), u^k(\cdot))\}$ *is bounded in* $W^{1,2}([0,T], \mathbb{R}^{2n})$ *and that the sets* $f_1(x^k(t), A)$ *and* $f_2(x^k(t), B)$ *are convex for all* $t \in [0,T]$*, along which the integrand* $l_0(t, \cdot)$ *is majorized by a summable function on* $[0,T]$*. Then problem* (P) *admits an optimal solution belonging to the space* $W^{1,2}([0,T], \mathbb{R}^{3n}) \times L^2([0,T], \mathbb{R}^{m+d})$.

Proof. Proposition 4.2.1 tells us that the set of feasible solutions to problem (P) is nonempty. Let us fix a minimizing sequence of feasible solutions
$z^k(\cdot) = (x^k(\cdot), y^k(\cdot), u^k(\cdot), a^k(\cdot), b^k(\cdot))$ for (P). It follows from the boundedness of $\{u^k(\cdot)\}$ in $W^{1,2}([0,T], \mathbb{R}^n)$ that there exists $u_0 \in \mathbb{R}^n$ such that $u^k(0) \to u_0$ in this space as $k \to \infty$, while the triple (x_0, y_0, u_0) satisfies (4.24). We have that the sequence $\{(\dot{x}^k(\cdot), \dot{u}^k(\cdot)\}$ is bounded in $L^2([0,T], \mathbb{R}^{2n})$. Remembering that in reflexive spaces bounded sets are weakly compact gives us a pair $(v^x(\cdot), v^u(\cdot)) \in L^2([0,T], \mathbb{R}^{2n})$ such that $(\dot{x}_k(\cdot), \dot{u}_k(\cdot)) \to (v^x(\cdot), v^u(\cdot)) \in L^2([0,T], \mathbb{R}^{2n})$ in the weak topology of this space along a subsequence (without relabeling). Define now the functions

$$(\bar{x}(t), \bar{u}(t)) = (x_0, u_0) + \int_0^t (v^x(s), v^u(s))\, ds \quad \text{for all} \quad t \in [0,T]$$

and observe that $(\dot{\bar{x}}(t), \dot{\bar{u}}(t)) = (v^x(t), v^u(t))$ for a.e. $t \in [0,T]$, and thus
$(\bar{x}(\cdot), \bar{u}(\cdot)) \in W^{1,2}([0,T], \mathbb{R}^{2n})$.

The next step is to verify that the limiting quintuple $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot), \bar{b}(\cdot))$ satisfies the combination of the differential inclusion and equation in (4.23), which is equivalent to the original integro-differential inclusion in (4.2). Furthermore, it follows from the closedness and convexity of the normal cone to the moving set $C(t)$ in (4.3) and the assumed convexity of the compact sets $f_1(x^k(t),A)$ and $f_2(x^k(t),B)$ on $[0,T]$ that the right-hand site velocity set in (4.21) (with $y(t)=\int_0^t f_2(x(s),b(s))\,ds$) is convex along the selected minimizing sequence, and we have the inclusion

$$-\dot{\bar{x}}(t)\in N_{\bar{C}(t)}(\bar{x}(t))+f_1(\bar{x}(t),A)+\int_0^t f_2\big(\bar{x}(s),B\big)\,\mathrm{d}s \quad \text{a.e.}\quad [0,T],\quad x_0\in\bar{C}(0),$$

with $\bar{x}(t)\in\bar{C}(t):=C+\bar{u}(t)$ on $[0,T]$, where C taken the form in (4.30). Employing now the aforementioned measurable selection allows us to find a measurable control $\bar{a}(\cdot)$ and $\bar{b}(\cdot)$ such that $\bar{a}(t)\in A$ and $\bar{b}(t)\in B$ and

$$-\dot{\bar{x}}(t)\in N_{\bar{C}(t)}\big(\bar{x}(t)\big)+f_1\big(\bar{x}(t),\bar{a}(t)\big)+\int_0^t f_2\big(\bar{x}(s),\bar{b}(s)\big)\,\mathrm{d}s \quad \text{a.e. } t\in[0,T]$$

To justify further the optimality of $\bar{z}(\cdot)=(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot),\bar{a}(\cdot),\bar{b}(\cdot))$ in (P), it is sufficient to show that

$$J[\bar{x},\bar{y},\bar{u},\bar{a},\bar{b}]=J_0[\bar{x},\bar{u},\bar{a},\bar{b}]\le\liminf_{k\to\infty}J_0[x_k,u_k,a_k,b_k]=\liminf_{k\to\infty}J[x_k,y_k,u_k,a_k,b_k] \qquad (4.27)$$

for the Bolza functionals in (4.1) and (4.6). The latter follows from the assumptions in $(\mathcal{H}_3)$ on the cost functions φ and l_0 due to the Mazur weak closure theorem and the Lebesgue dominated convergence theorem. Indeed, Mazur's theorem ensures that the weak convergence of the $\{(\dot{x}_k(\cdot),\dot{u}_k(\cdot))\}$ to $(\dot{\bar{x}}(\cdot),\dot{\bar{u}}(\cdot))$ in $L^2([0,T],\mathbb{R}^{2n})$ yields the L^2-strong convergence of convex combinations of $(\dot{x}_k(\cdot),\dot{u}_k(\cdot))$ to $(\dot{\bar{x}}(\cdot),\dot{\bar{u}}(\cdot))$, and hence the a.e. convergence of a subsequence of these convex combination on $[0,T]$ to the limiting quadruple. Employing finally the assumed convexity of the running cost l_0 with respect to the velocity variables verifies (4.27). Observe that there is no need to care about the convergence with respect to (a,b)-controls in our setting due to the independence of the integrand l_0 on the (a,b)-components. Thus we complete the proof. ∎

As seen, besides our standing fairly nonrestrictive assumptions, the existence of global minimizers in Theorem 4.2.1 requires the *convexity* of the running cost with respect to *velocity* variables. The obtained results is new, while this convexity phenomenon has been well recognized in the calculus of variations and optimal control of various types of dynamical systems. On the other hand, it has been also well understood in variational theory for continuous-type systems (including sweeping processes) that such problems allows a certain *relaxation* procedure involving the velocity *convexification*, which brings us to relaxed problems, where optimal solutions exist automatically and can be constructively approximated by feasible solutions to the original problems with keeping the same optimal values of the cost functionals. The reader is referred to [22, 33, 26, 36, 39, 51, 49, 79, 84] for various results, discussions, and bibliographies in these directions. Following this line, we construct now the relaxed version of the optimal control problem (P) for the integro-differential sweeping process under consideration.

Taking the integrand l from (4.6) and the velocity mapping G from (4.23), define the *extended running cost*

$$l_G(t,x,y,u,a,b,\upsilon) := l(t,x,y,u,a,b,\upsilon) + \psi_G(\upsilon) \text{ with } G = G(x,y,u,a,b),$$

where the indicator function ψ_G of the set G is given by $\psi_G(\upsilon) := 0$ if $\upsilon \in G$ and $\psi_G(\upsilon) := \infty$ otherwise. Denote further by $\widehat{l}_G(t,x,y,u,a,b,\upsilon)$ the *biconjugate function* to $l_G(t,x,y,u,a,b,\upsilon)$ with respect to the velocity variable $\upsilon = (\dot{x},\dot{u},\dot{y})$, i.e., given by

$$\widehat{l}_G(t,x,y,u,a,b,\upsilon) := (l_G)^{**}(t,x,y,u,a,b,\upsilon).$$

Observe that $\widehat{l}_G(t,x,y,u,a,b,\upsilon)$ is the largest proper, convex, and l.s.c. function with respect to υ, which is majorized by l_G. We clearly have that $\widehat{l}_G = l_G$ if and only if l_G is proper, convex, and l.s.c. with respect to υ.

Now we are ready to define the *relaxed problem* (R) associated with the original optimal control problem (P) for sweeping integro-differential inclusion as follows: minimize

$$\widehat{J}[x,y,u,a,b] := \varphi\big(x(T)\big) + \int_0^T \widehat{l}_G\big(t,x(t),y(t),u(t),a(t),b(t),\dot{x}(t),\dot{y}(t),\dot{u}(t)\big)\,dt \tag{4.28}$$

with $x(0) = x_0 \in C(0)$ and $y(0) = 0$. Theorem 4.2.1 ensures the existence of optimal solutions to (R) under the standing assumptions made. We see furthermore that there is no difference between problems (P) and (R) if the original running cost l_0 is *convex with respect to the velocity* variables. In fact, there exists a deeper connection between (P) and (R) *without any convexity requirements*, which has been well recognized for particular cases of the controlled sweeping processes in [39, 79] by showing that an optimal relaxed solution can be constructively approximated by feasible original ones, and that the optimal cost values for (P) and (R) coincide. Our goal is to investigate this issue for more general cases of integro-differential sweeping control systems in the future research.

We conclude this section by formulating the concept of *local minimizers* in (P) for which we derive necessary optimality conditions without any convexity assumptions.

Definition 4.2.1 (intermediate local minimizers). *Consider problem* (P) *with representation* (4.23) *of the integro-differential inclusion, and let* (4.28) *is the relaxed problem of* (P). *Then we say that:*

(i) *A feasible solution* $\bar{z}(\cdot)$ *to* (P) *is an* INTERMEDIATE LOCAL MINIMIZER *(i.l.m.) of this problem if* $\bar{z}(\cdot) = (\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot),\bar{a}(\cdot),\bar{b}(\cdot)) \in W^{1,2}([0,T];\mathbb{R}^n) \times W^{1,2}([0,T];\mathbb{R}^n) \times W^{1,2}([0,T];\mathbb{R}^n) \times$

$L^2([0,T];\mathbb{R}^m) \times L^2([0,T];\mathbb{R}^d)$ *and there exists* $\varepsilon > 0$ *such that* $J[\bar{z}] \leq J[z]$ *for any feasible solution* $z(\cdot)$ *to* (P) *satisfying*

$$\|(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot)) - (x(\cdot),y(\cdot),u(\cdot))\|_{W^{1,2}} + \|(\bar{a}(\cdot),\bar{b}(\cdot)) - (a(\cdot),b(\cdot))\|_{L^2} < \varepsilon. \tag{4.29}$$

(ii) *A feasible solution* $\bar{z}(\cdot)$ *to* (P) *is a* RELAXED INTERMEDIATE LOCAL MINIMIZER (*r.i.l.m.*) *of this problem if* $J[\bar{z}] = \widehat{J}[\bar{z}]$ *and there exists* $\varepsilon > 0$ *such that* $J[\bar{z}] \leq J[z]$ *for any feasible solution* $z(\cdot)$ *to* (P) *satisfying* (4.29).

Both notions in Definition 4.2.1 were introduced in [51] for Lipschitzian differential inclusions (see also [49, 84] for subsequent studies in the Lipschitzian case), and then they were investigated and developed in [20, 21, 22, 26, 27, 81, 54] for various differential inclusions of the sweeping type. The introduced notions include their *strong* counterpart in which only the first condition from (4.29) is required. In general, the defined "intermediate" notions occupy an intermediate position between strong and weak local minimizers in variational problems. As seen, the only difference between i.l.m. and r.i.l.m. lies in the additional requirement on the *local relaxation stability* $J[\bar{z}] = \widehat{J}[\bar{z}]$, which is often the case (always for the Lipschitzian dynamics [49] and more as in [36, 39, 79]) of *nonconvex* integro-differential systems, particularly for strong minimizers.

4.3 Discrete approximations of integro-differential dynamics

In this section we start developing the *method of discrete approximations* to study controlled integro-differential sweeping processes of type (4.2). The major result obtained here deals, for the fist time in the literature, with the discontinuous integro-differential sweeping dynamics independently of its optimization as in problem (P). We'll later apply it to deriving necessary optimality conditions for intermediate local minimizers of (P), but so far our goal is to construct a well-posed sequence of discrete-time sweeping dynamical systems, which $W^{1,2}$-*strongly* approximates *any feasible* solution to (P). We can do this under the standing assumptions in $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ with the quite natural additional requirement that the velocities $\dot{x}(\cdot)$ and $\dot{u}(\cdot)$ of the given feasible solution to (P) are of *bounded variation* on $[0,T]$. The developed approach allows us to improve the corresponding results of [22] for controlled sweeping processes governed by differential inclusions and extend them to the case of integro-differential dynamics. The obtained results are *fully constructive* and can be treated from both *qualitative* and *numerical* viewpoints as the justification of *finite-dimensional* approximations of infinite-dimensional integro-differential discontinuous systems.

For each $k \in I\!N$, consider the *discrete mesh* on $[0,T]$ given by

$$\Delta_k := \left\{0 = t_0^k < t_1^k < \ldots < t_k^k = T\right\} \text{ with } h_j^k := t_{j+1}^k - t_j^k \text{ and } \max_{j=0,\ldots,k-1} h_j^k \le h_k := \frac{T}{k}. \tag{4.30}$$

Theorem 4.3.1 (strong discrete approximation of feasible sweeping solutions)**.** *Under the fulfillment of the assumptions in* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$*, fix any feasible solution* $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot), \bar{b}(\cdot))$ *to* (P) *such that the functions* $\dot{\bar{x}}(\cdot)$, $\dot{\bar{u}}(\cdot)$, $\bar{a}(\cdot)$, *and* $\bar{b}(\cdot)$ *are of bounded variation on* $[0,T]$*, i.e., there is a number* $\mu > 0$ *for which*

$$\max\left\{\text{var}(\dot{\bar{x}}(\cdot),[0,T]),\ \text{var}(\dot{\bar{u}}(\cdot),[0,T]),\ \text{var}(\bar{a}(\cdot),[0,T]),\ \text{var}(\bar{b}(\cdot),[0,T])\right\} \le \mu. \tag{4.31}$$

Then given a discrete mesh Δ_k *in* (4.30)*, there exist sequences of piecewise linear functions* $(x^k(\cdot), y^k(\cdot), u^k(\cdot))$ *and piecewise constant functions* $(a^k(\cdot), b^k(\cdot))$ *on* $[0,T]$ *such that* $(x^k(0), y^k(0), u^k(0)) = (x_0, 0, \bar{u}(0))$ *for all* $k \in \mathbb{N}$*, and it holds*

$$x^k(t) = x^k(t_j^k) + (t - t_j^k)\upsilon_j^k, \quad y^k(t) = y^k(t_j^k) + (t - t_j^k)w_j^k \text{ for } t \in [t_j^k, t_{j+1}^k],\ j = 0, \ldots, k-1,$$

where the discrete velocities υ_j^k *and* w_j^k *satisfy the conditions*

$$-\upsilon_j^k \in F_1(x^k(t_j^k), y^k(t_{j+1}^k), u^k(t_j^k), a^k(t_j^k)), \quad w_j^k = f_2(b^k(t_j^k), x^k(t_j^k)),\ \ j = 0, \ldots, k-1,$$

with F_1 *taken from* (4.20)*. Moreover, we have the convergence* $(x^k(\cdot), y^k(\cdot), u^k(\cdot))$ *to* $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot))$ *in the* $W^{1,2}$*-norm topology on* $[0,T]$*, and* $(a^k(\cdot), b^k(\cdot)) \to (\bar{a}(\cdot), \bar{b}(\cdot))$ *in the* L^2*-norm topology on* $[0,T]$ *as* $k \to \infty$*.*
Furthermore, there exists a constant $\widetilde{\mu} > 0$ *depending by* T*,* L_1*,* L_2*,* μ*, and* $\widetilde{M}$*, where the numbers* μ *and* $\widetilde{M}$ *are taken from* (4.31) *and* (4.40)*, respectively, such that for every* $k \in \mathbb{N}$ *we have the estimates*

$$\text{var}(\dot{u}^k(\cdot),[0,T]) \le \widetilde{\mu} \quad \textit{and} \quad \left\|\frac{u^k(t_1) - u^k(t_0)}{t_1 - t_0}\right\| \le \widetilde{\mu}. \tag{4.32}$$

Proof. Since step functions are dense in $L^2([0,T];\mathbb{R}^m \times \mathbb{R}^d)$, there are sequences $\{a_k(\cdot)\}$ and $\{b_k(\cdot)\}$ with

$$\int_0^T \|a_k(t) - \bar{a}(t)\|^2\,dt \to 0 \quad \text{and} \quad \int_0^T \|b_k(t) - \bar{b}(t)\|^2\,dt \to 0 \quad \text{as} \quad k \to \infty. \tag{4.33}$$

Furthermore, for each $k \in \mathbb{N}$ there exists a partition Δ_k of the interval $[0,T]$ from (4.30) for which the step functions $\{a^k(\cdot)\}$ and $\{b^k(\cdot)\}$ are constant on $[t_j;t_{j+1})$ for $j = 0 \ldots k-1$.

For simplicity, we use the notation $t_j := t_j^k$ for the mesh points as $j = 0, \dots, k$ for each fixed k. Using the recurrent procedure, suppose that $x^k(t_j)$ is given and then define

$$u^k(t_j) := x^k(t_j) - \bar{x}(t_j) + \bar{u}(t_j), \;\; j = 0, \dots, k. \tag{4.34}$$

Remembering that the sets $F_1(z)$ in (4.20) are closed and convex, take the unique projections

$$-\upsilon_j^k := \Pi_{F_{1jk}}(-\dot{\bar{x}}(t_j)) \;\text{ where }\; F_{1jk} := F_1(x^k(t_j), y^k(t_{j+1}), u^k(t_j), a^k(t_j)), \tag{4.35}$$

and deduce from (4.34) with taking into account the constructions of $a^k(\cdot)$ and $b^k(\cdot)$ that

$$\begin{aligned} &F_1(x^k(t_j), y^k(t_{j+1}), u^k(t_j), a^k(t_j)) = N_{C(u^k(t_j))}(x^k(t_j)) + f_1(a^k(t_j), x^k(t_j)) + y^k(t_{j+1}) \\ &= F_1(\bar{x}(t_j), \bar{y}(t_j), \bar{u}(t_j), \bar{a}(t_j)) + f_1(\bar{a}(t_j), x^k(t_j)) - f_1(\bar{a}(t_j), \bar{x}(t_j)) + y^k(t_{j+1}) - \bar{y}(t_j). \end{aligned}$$

For all $j = 0, \dots, k-1$ and $t \in [t_j, t_{j+1}]$, define the vectors and functions

$$w_j^k := f_2(\bar{b}(t_j), x^k(t_j)), \quad x^k(t) := x^k(t_j) + (t - t_j)\upsilon_j^k, \quad y^k(t) = y^k(t_j) + (t - t_j)w_j^k, \tag{4.36}$$

and then use below the notation

$$\begin{aligned} &f_j^x(s) := \|\dot{\bar{x}}(t_j) - \dot{\bar{x}}(s)\|, \;\; f_j^u(s) = \|\dot{\bar{u}}(t_j) - \dot{\bar{u}}(s)\|, \\ &f_j^a(s) := \|\bar{a}(t_j) - \bar{a}(s)\|, \;\; f_j^b(s) := \|\bar{b}(t_j) - \bar{b}(s), \|, \end{aligned}$$

for all $s \in [t_j, t_{j+1})$. Select s_j^x, s_j^u, s_j^a and s_j^b from the subintervals $[t_j, t_{j+1})$ so that

$$\begin{cases} \sup\limits_{s \in [t_j, t_{j+1}]} f_j^x(s) \le \|\dot{\bar{x}}(t_j) - \dot{\bar{x}}(s_j^x)\| + 2^{-k}, \\ \sup\limits_{s \in [t_j, t_{j+1}]} f_j^u(s) \le \|\dot{\bar{u}}(t_j) - \dot{\bar{u}}(s_j^u)\| + 2^{-k}, \\ \sup\limits_{s \in [t_j, t_{j+1}]} f_j^a(s) \le \|\bar{a}(t_j) - \bar{a}(s_j^a)\| + 2^{-k}, \\ \sup\limits_{s \in [t_j, t_{j+1}]} f_j^b(s) \le \|\bar{b}(t_j) - \bar{b}(s_j^b)\| + 2^{-k}. \end{cases} \tag{4.37}$$

It clearly follows from (4.35), (4.36), and (4.37) that

$$\begin{aligned} \|\upsilon_j^k - \dot{\bar{x}}(s)\| &\le \|\upsilon_j^k - \dot{\bar{x}}(t_j)\| + \|\dot{\bar{x}}(t_j) - \dot{\bar{x}}(s)\| \\ &= \text{dist}(-\dot{\bar{x}}(t_j); F_1(x^k(t_j), y^k(t_j), u^k(t_j), a^k(t_j))) + f_j^x(s) \\ &\le \|f_1(\bar{a}(t_j), x^k(t_j)) - f_1(\bar{a}(t_j), \bar{x}(t_j))\| + \|y^k(t_{j+1}) - \bar{y}(t_j)\| + f_j^x(s). \end{aligned}$$

Employing the above constructions leads us to the estimates

$$
\begin{aligned}
&\|y^k(t_{j+1}) - \bar{y}(t_{j+1})\| \\
&= \Big\| y^k(t_j) + h_j f_2(\bar{b}(t_j), x^k(t_j)) - \bar{y}(t_j) - \int_{t_j}^{t_{j+1}} \dot{\bar{y}}(s)\, ds \Big\| \\
&\le \|y^k(t_j) - \bar{y}(t_j)\| + \int_{t_j}^{t_{j+1}} \|f_2(\bar{b}(t_j), x^k(t_j)) - f_2(\bar{b}(s), \bar{x}(s))\|\, ds \\
&\le \|y^k(t_j) - \bar{y}(t_j)\| + \int_{t_j}^{t_{j+1}} \|f_2(\bar{b}(t_j), x^k(t_j)) - f_2(\bar{b}(t_j), \bar{x}(t_j))\|\, ds \\
&+ \int_{t_j}^{t_{j+1}} \|f_2(\bar{b}(t_j), \bar{x}(t_j)) - f_2(\bar{b}(t_j), \bar{x}(s))\|\, ds + \int_{t_j}^{t_{j+1}} \|f_2(\bar{b}(t_j), \bar{x}(s)) - f_2(\bar{b}(s), \bar{x}(s))\|\, ds.
\end{aligned}
$$

The imposed assumptions $(\mathcal{H}_{2,2})$ readily implies that

$$
\begin{aligned}
&\|y^k(t_{j+1}) - \bar{y}(t_{j+1})\| \\
&\le \|y^k(t_j) - \bar{y}(t_j)\| + L_1 h_j \|x^k(t_j) - \bar{x}(t_j)\| + L_1 \int_{t_j}^{t_{j+1}} \|\bar{x}(t_j) - \bar{x}(s)\|\, ds + L_2 \int_{t_j}^{t_{j+1}} \|\bar{b}(t_j) - \bar{b}(s)\|\, ds \\
&\le \|y^k(t_j) - \bar{y}(t_j)\| + L_1 h_j \|x^k(t_j) - \bar{x}(t_j)\| + L_1 \int_{t_j}^{t_{j+1}} \|\bar{x}(t_j) - \bar{x}(s)\|\, ds + L_2 \int_{t_j}^{t_{j+1}} f_j^b(s)\, ds.
\end{aligned}
$$

Proceeding further by induction, we obtain for all $j = 0, \ldots, k-1$ that

$$
\begin{aligned}
&\|y^k(t_{j+1}) - \bar{y}(t_{j+1})\| \\
&\le L_1 \sum_{i=0}^{j} h_i \|x^k(t_i) - \bar{x}(t_i)\| + L_1 \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \|\bar{x}(t_i) - \bar{x}(s)\|\, ds + L_2 \sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} f_i^b(s)\, ds.
\end{aligned} \tag{4.38}
$$

We deduce from (4.37) that

$$
\begin{aligned}
\sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} f_i^b(s)\, ds &\le h_k \sum_{i=0}^{j} \left(\|\bar{b}(t_i) - \bar{b}(s_i^b)\| + \|\bar{b}(s_i^b) - \bar{b}(t_{i+1})\| + 2^{-k} \right) \\
&\le h_k \mathrm{var}(\bar{b}(\cdot), [0,T]) + h_k k 2^{-k} \le h_k \mu + T 2^{-k}.
\end{aligned}
$$

On the other hand, it follows that

$$
\sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \|\bar{x}(t_i) - \bar{x}(s)\|\, ds \le h_k \sum_{i=0}^{j} \sup_{s \in [t_i, t_{i+1}]} \|\bar{x}(t_i) - \bar{x}(s)\|.
$$

Thus we can select a s_i^x from the subintervals $[t_i, t_{i+1})$ such that

$$
\sum_{i=0}^{j} \int_{t_i}^{t_{i+1}} \|\bar{x}(t_i) - \bar{x}(s)\|\, ds \le h_k \sum_{i=0}^{j} \left(\|\bar{x}(t_i) - \bar{x}(s_i^x)\| + 2^{-k} \right).
$$

Since $\bar{x}(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$, we get

$$\begin{aligned}\sum_{i=0}^{j}\int\limits_{t_i}^{t_{i+1}}\|\bar{x}(t_i)-\bar{x}(s)\|\,ds &\le h_k\sum_{i=0}^{j}\int\limits_{t_i}^{s_i^x}\|\dot{\bar{x}}(\tau)\|\,d\tau + h_k k 2^{-k} \le h_k\sum_{i=0}^{j}\int\limits_{t_i}^{t_{i+1}}\|\dot{\bar{x}}(\tau)\|\,d\tau + T2^{-k}\\ &= h_k\int\limits_{0}^{t_{j+1}}\|\dot{\bar{x}}(\tau)\|\,d\tau + T2^{-k} \le h_k\int\limits_{0}^{T}\|\dot{\bar{x}}(\tau)\|\,d\tau + T2^{-k}\\ &\le h_k\sqrt{T}\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)} + T2^{-k}.\end{aligned} \tag{4.39}$$

Substituting the latter estimates into (4.38) gives us

$$\begin{aligned}\|y^k(t_{j+1})-\bar{y}(t_j)\| &\le \|y^k(t_{j+1})-\bar{y}(t_{j+1})\| + \|\bar{y}(t_{j+1})-\bar{y}(t_j)\|\\ &\le L_1\sum_{i=0}^{j}h_i\|x^k(t_i)-\bar{x}(t_i)\| + h_k\left(L_1\sqrt{T}\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)} + L_2\mu\right)\\ &+ (L_1+L_2)T2^{-k} + \int\limits_{t_j}^{t_{j+1}}\|\dot{\bar{y}}(s)\|\,ds\\ &= L_1\sum_{i=0}^{j}h_i\|x^k(t_i)-\bar{x}(t_i)\| + h_k\left(L_1\sqrt{T}\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)} + L_2\mu\right)\\ &+ (L_1+L_2)T2^{-k} + \int\limits_{t_j}^{t_{j+1}}\|f_2(\bar{b}(s),\bar{x}(s))\|\,ds.\end{aligned}$$

From the boundedness of the set B we can select a $M > 0$ such that $\|\bar{b}(t)\| \le M$ a.e. $t \in [0,T]$, which implies in turn by using the imposed assumptions on f_2 and by the first estimate in (4.16) that

$$\begin{aligned}\|y^k(t_{j+1})-\bar{y}(t_j)\| &\le L_1\sum_{i=0}^{j}h_i\|x^k(t_i)-\bar{x}(t_i)\| + h_k\left(L_1\sqrt{T}\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)} + L_2\mu\right)\\ &+ (L_1+L_2)T2^{-k} + h_k(M+\alpha_2\widetilde{l})\\ &= L_1\sum_{i=0}^{j}h_i\|x^k(t_i)-\bar{x}(t_i)\| + h_k\widetilde{M} + (L_1+L_2)T2^{-k},\end{aligned}$$

$$\text{where}\quad \widetilde{M} = L_1\sqrt{T}\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)} + M + \alpha_2\widetilde{l}. \tag{4.40}$$

Substituting the latter into the previous estimate of $\|y^k(t_{j+1})-\bar{y}(t_{j+1})\|$ and employing $(\mathcal{H}_{2,2})$ bring us to

$$\begin{aligned}\|\upsilon_j^k-\dot{\bar{x}}(s)\| &\le L_1\|x^k(t_j)-\bar{x}(t_j)\| + L_1\sum_{i=0}^{j}h_i\|x^k(t_i)-\bar{x}(t_i)\|\\ &+ h_k\widetilde{M} + (L_1+L_2)T2^{-k} + f_j^x(s).\end{aligned} \tag{4.41}$$

We get from the above relationships the following estimates:

$$\begin{aligned}
\|x^k(t_{j+1}) - \bar{x}(t_{j+1})\| &= \|x^k(t_j) + h_k v_j^k - \bar{x}(t_j) - \int\limits_{t_j}^{t_{j+1}} \dot{\bar{x}}(s)\,ds\| \\
&\le \|x^k(t_j) - \bar{x}(t_j)\| + \int\limits_{t_j}^{t_{j+1}} \|v_j^k - \dot{\bar{x}}(s)\|\,ds \\
&\le \|x^k(t_j) - \bar{x}(t_j)\| + L_1 h_k \|x^k(t_j) - \bar{x}(t_j)\| + L_1 h_k \sum_{i=0}^{j} h_i \|x^k(t_i) - \bar{x}(t_i)\| \\
&+ h_k^2 \widetilde{M} + h_k(L_1 + L_2)T2^{-k} + \int\limits_{t_j}^{t_{j+1}} f_j^x(s)\,ds \\
&\le (1 + L_1 h_k)\|x^k(t_j) - \bar{x}(t_j)\| + L_1 h_k^2 \sum_{i=0}^{j-1} \|x^k(t_i) - \bar{x}(t_i)\| \\
&+ h_k^2 \widetilde{M} + h_k(L_1 + L_2)T2^{-k} + \int\limits_{t_j}^{t_{j+1}} f_j^x(s)\,ds.
\end{aligned}$$

Apply now the discrete Gronwall's lemma from Proposition 4.1.2 with the given parameters

$$e_j = \|x^k(t_j) - \bar{x}(t_j)\|, \quad \rho_j = L_1 h_k^2, \quad \gamma_j = L_1 h_k, \ \text{ and}$$

$$\sigma_j = h_k^2 \widetilde{M} + h_k(L_1 + L_2)T2^{-k} + \int\limits_{t_j}^{t_{j+1}} f_j^x(s)\,ds.$$

This ensures the fulfillment of the relationships

$$\begin{aligned}
\sum_{i=0}^{j-1} \sigma_i &= j h_k \left(h_k \widetilde{M} + (L_1 + L_2)T2^{-k} \right) + \sum_{i=0}^{j-1} \int\limits_{t_i}^{t_{i+1}} f_i^x(s)\,ds \\
&\le T \left(h_k \widetilde{M} + (L_1 + L_2)T2^{-k} \right) + \sum_{i=0}^{j-1} \int\limits_{t_i}^{t_{i+1}} f_i^x(s)\,ds.
\end{aligned}$$

We deduce from (4.37) that

$$\begin{aligned}
\sum_{i=0}^{j-1} \int\limits_{t_i}^{t_{i+1}} f_i^x(s)\,ds \quad &\le h_k \sum_{i=0}^{j} \left(\|\dot{\bar{x}}(t_i) - \dot{\bar{x}}(s_i^x)\| + \|\dot{\bar{x}}(s_i^x) - \dot{\bar{x}}(t_{i+1})\| + 2^{-k} \right) \\
&\le h_k \mathrm{var}(\dot{\bar{x}}(\cdot), [0,T]) + h_k k 2^{-k} \le h_k \mu + T2^{-k},
\end{aligned}$$

and therefore arrive at the estimate

$$\begin{aligned}
\sum_{i=0}^{j-1} \sigma_i &\le T \left(h_k \widetilde{M} + (L_1 + L_2)T2^{-k} \right) + T2^{-k} + h_k \mu \\
&= h_k(T\widetilde{M} + \mu) + 2^{-k}((L_1 + L_2)T^2 + T) := c_k.
\end{aligned} \tag{4.42}$$

Having further $\lim_{k\to\infty} c_k = 0$ gives us the conditions

$$\sum_{i=0}^{j-1}(i\rho_k+\gamma_k) = L_1\sum_{i=0}^{j-1}(ih_k^2+h_k) = L_1\Big(h_k^2\frac{j(j-1)}{2}+jh_k\Big) \le L_1T\Big(\frac{T}{2}+1\Big),$$

which imply that for all $j=0,\ldots,k-1$ we get

$$\|x^k(t_j)-\bar{x}(t_j)\| \le c_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right), \tag{4.43}$$

and hence from (4.38), (4.39) and (4.41) we get

$$\begin{aligned}\|y^k(t_j)-\bar{y}(t_j)\| \ &\le L_1Tc_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)+L_1(h_k\sqrt{T}\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)}+T2^{-k})\\ &+L_2\sum_{i=0}^{j}\int_{t_j}^{t_{j+1}} f_j^b(s)\,ds,\end{aligned} \tag{4.44}$$

$$\|\upsilon_j^k-\dot{\bar{x}}(s)\| \le c_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)(L_1+L_1T)+h_k\widetilde{M}+(L_1+L_2)T2^{-k}+f_j^x(s). \tag{4.45}$$

Employing the obtained conditions together with (4.43) and (4.44) tells us that

$$\begin{aligned}&\|x^k(t)-\bar{x}(t)\|\\ &=\|x^k(t_j)+(t-t_j)\upsilon_j^k-\bar{x}(t_j)-\int_{t_j}^{t}\dot{\bar{x}}(s)\,ds\| \le \|x^k(t_j)-\bar{x}(t_j)\|+\int_{t_j}^{t_{j+1}}\|\upsilon_j^k-\dot{\bar{x}}(s)\|\,ds\\ &\le c_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)+h_kc_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)(L_1+L_1T)\\ &+h_k^2\widetilde{M}+h_k(L_1+L_2)T2^{-k}+\int_{t_j}^{t_{j+1}} f_j^x(s)\,ds,\end{aligned}$$

which readily verifies the uniform convergence of the sequence $\{x^k(\cdot)\}$ to $\bar{x}(\cdot)$ as $k\to\infty$. To proceed further, deduce from (4.45) for $j=0,\ldots,k-1$ that

$$\begin{aligned}&\int_0^T\|\dot{x}^k(t)-\dot{\bar{x}}(t)\|^2\,dt\\ &=\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\|\upsilon_j^k-\dot{\bar{x}}(t)\|^2\,dt \le 3h_k\sum_{j=0}^{k-1}c_k^2\exp\left(2L_1T\left(\frac{T}{2}+1\right)\right)(L_1+L_1T)^2\\ &+3h_k\sum_{j=0}^{k-1}\Big(h_k\widetilde{M}+(L_1+L_2)T2^{-k}\Big)^2+3\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(f_j^x(s))^2\,ds.\end{aligned} \tag{4.46}$$

Observe in addition by the constructions and assumptions above that

$$\begin{aligned}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(f_j^x(s))^2\,ds &\le h_k\sum_{j=0}^{k-1}\left(\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\|\dot{\bar{x}}(s_j^x)-\dot{\bar{x}}(t_{j+1})\|+2^{-k}\right)^2\\ &\le 2h_k\sum_{j=0}^{k-1}\left(\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\|\dot{\bar{x}}(s_j^x)-\dot{\bar{x}}(t_{j+1})\|\right)^2+2T4^{-k}\\ &\le 2h_k\Big(\sum_{j=0}^{k-1}\left(\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\|\dot{\bar{x}}(s_j^x)-\dot{\bar{x}}(t_{j+1})\|\right)\Big)^2+2T4^{-k}\\ &\le 2h_k\mu^2+2T4^{-k}.\end{aligned}\tag{4.47}$$

Combining (4.46) and (4.47) gives us the estimates

$$\begin{aligned}\int_0^T\|\dot{x}^k(t)-\dot{\bar{x}}(t)\|^2\,dt &\le 3Tc_k^2\exp\left(2L_1T\left(\frac{T}{2}+1\right)\right)(L_1+L_1T)^2\\ &\quad+3T\left(h_k\widetilde{M}+(L_1+L_2)T2^{-k}\right)^2+6h_k\mu^2+6T4^{-k}.\end{aligned}$$

Note to this end that the terms

$$c_k^2\exp\left(2L_1T\left(\frac{T}{2}+1\right)\right)(L_1+L_1T)^2\quad\text{and}\quad\left(h_k\widetilde{M}+(L_1+L_2)T2^{-k}\right)^2$$

in (4.46) are independent of j, and so we have $kh_k=T$. Remembering that $c_k\to 0$, this verifies the strong convergence of the sequence $\{\dot{x}^k(\cdot)\}$ to $\dot{\bar{x}}(\cdot)$ in the norm topology of $L^2([0,T],\mathbb{R}^n)$ as $k\to\infty$.

Our next step is to justify the $W^{1,2}$-strong convergence of the sequence $\{y^k(\cdot)\}$ to $\bar{y}(\cdot)$. As follows from the above construction, for all $j=0,\ldots,k-1$ and $t\in[t_j,t_{j+1}]$ we have

$$\begin{aligned}&\|\dot{y}^k(t)-\dot{\bar{y}}(t)\|\\ &=\|w_j^k-\dot{\bar{y}}(t)\|=\|f_2(\bar{b}(t_j),x^k(t_j))-f_2(\bar{b}(t),\bar{x}(t))\|\\ &\le\|f_2(\bar{b}(t_j),x^k(t_j))-f_2(\bar{b}(t_j),\bar{x}(t_j))\|\\ &+\|f_2(\bar{b}(t_j),\bar{x}(t_j))-f_2(\bar{b}(t_j),\bar{x}(t))\|+\|f_2(\bar{b}(t_j),\bar{x}(t))-f_2(\bar{b}(t),\bar{x}(t))\|\\ &\le L_1c_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)+\sqrt{h_k}L_1\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)}+L_2f_j^b(t).\end{aligned}$$

Combining the latter with (4.44) gives us the estimates

$$\begin{aligned}&\|y^k(t)-\bar{y}(t)\|\\ &=\|y^k(t_j)+(t-t_j)w_j^k-\bar{y}(t_j)-\int_{t_j}^{t}\dot{\bar{y}}(s)\,ds\|\le\|y^k(t_j)-\bar{y}(t_j)\|+\int_{t_j}^{t_{j+1}}\|w_j^k-\dot{\bar{y}}(s)\|\,ds\\ &\le L_1Tc_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)+L_1(h_k\sqrt{T}\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)}+T2^{-k})+L_2\sum_{i=0}^{j}\int_{t_j}^{t_{j+1}}f_j^b(s)\,ds\\ &+L_1h_kc_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)+h_k\sqrt{h_k}L_1\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)}+L_2\int_{t_j}^{t_{j+1}}f_j^b(s)\,ds,\end{aligned}$$

$$\int_0^T \|\dot{y}^k(t)-\dot{\bar{y}}(t)\|^2\,dt=\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\|\dot{y}^k(t)-\dot{\bar{y}}(t)\|^2\,dt$$

$$\le 2T\left(L_1c_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)+\sqrt{h_k}L_1\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)}\right)^2+2L_2^2\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(f_j^b(s))^2\,ds.$$

Deduce furthermore from the constructions and assumptions above that

$$\begin{aligned}\sum_{i=0}^{j-1}\int_{t_j}^{t_{j+1}}(f_j^b(s))^2\,ds &\le h_k\sum_{i=0}^{j}\left(\|\bar{b}(t_j)-\bar{b}(s_j^b)\|+\|\bar{b}(s_j^b)-\bar{b}(t_{j+1})\|+2^{-k}\right)^2\\ &\le 2h_k\sum_{i=0}^{j}\left(\|\bar{b}(t_j)-\bar{b}(s_j^b)\|+\|\bar{b}(s_j^b)-\bar{b}(t_{j+1})\|\right)^2+2T4^{-k}\\ &\le 2h_k\Big(\sum_{i=0}^{j}\left(\|\bar{b}(t_j)-\bar{b}(s_j^b)\|+\|\bar{b}(s_j^b)-\bar{b}(t_{j+1})\|\right)\Big)^2+2T4^{-k}\\ &\le 2h_k\mu^2+2T4^{-k},\end{aligned}$$

which brings us to the velocity estimate

$$\begin{aligned}&\int_0^T\|\dot{y}^k(t)-\dot{\bar{y}}(t)\|^2\,dt\\ &\le 2T\Big(L_1c_k\exp\Big(L_1T(\frac{T}{2}+1)\Big)+\sqrt{h_k}L_1\|\dot{\bar{x}}\|_{L^2([0,T],\mathbb{R}^n)}\Big)^2+4L_2^2(h_k\mu^2+T4^{-k})\end{aligned}$$

and ensures therefore the strong convergence of $y^k(\cdot)$ to $\bar{y}(\cdot)$ in $W^{1,2}([0,T],\mathbb{R}^n)$.

To complete the proof of the theorem, it remains to verify the $W^{1,2}$-strong convergence of the control sequence $\{u^k(\cdot)\}$ to $\bar{u}(\cdot)$. To this end, observe first that

$$\int_0^T\|\dot{u}^k(t)-\dot{\bar{u}}(t)\|^2\,dt\le 2\int_0^T\Big\|\dot{u}^k(t)-\frac{\bar{u}(t_{j+1})-\bar{u}(t_j)}{h_k}\Big\|^2\,dt+2\int_0^T\Big\|\frac{\bar{u}(t_{j+1})-\bar{u}(t_j)}{h_k}-\dot{\bar{u}}(t)\Big\|^2\,dt \tag{4.48}$$

for all $j=0,\ldots,k-1$ and $t\in[t_j,t_{j+1})$. On the other hand, it follows from (4.34) that

$$\begin{aligned}&\int_0^T\Big\|\dot{u}^k(t)-\frac{\bar{u}(t_{j+1})-\bar{u}(t_j)}{h_k}\Big\|^2\,dt\\ &=\int_0^T\Big\|\frac{u^k(t_{j+1})-u^k(t_j)}{h_k}-\frac{\bar{u}(t_{j+1})-\bar{u}(t_j)}{h_k}\Big\|^2\,dt=\int_0^T\Big\|\frac{u^k(t_{j+1})-\bar{u}(t_{j+1})}{h_k}-\frac{u^k(t_j)-\bar{u}(t_j)}{h_k}\Big\|^2\,dt\\ &=\int_0^T\Big\|\frac{x^k(t_{j+1})-\bar{x}(t_{j+1})}{h_k}-\frac{x^k(t_j)-\bar{x}(t_j)}{h_k}\Big\|^2\,dt=\int_0^T\Big\|\frac{x^k(t_{j+1})-x^k(t_j)}{h_k}-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{h_k}\Big\|^2\,dt\\ &\le 2\int_0^T\Big\|\frac{x^k(t_{j+1})-x^k(t_j)}{h_k}-\dot{\bar{x}}(t)\Big\|^2\,dt+2\int_0^T\Big\|\dot{\bar{x}}(t)-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{h_k}\Big\|^2\,dt\\ &=2\int_0^T\|\dot{x}^k(t)-\dot{\bar{x}}(t)\|^2\,dt+2\int_0^T\Big\|\dot{\bar{x}}(t)-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{h_k}\Big\|^2\,dt,\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\dot{\bar{x}}(t)-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{h_k}\Big\| \le \|\dot{\bar{x}}(t)-\dot{\bar{x}}(t_j)\|+\Big\|\dot{\bar{x}}(t_j)-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{h_k}\Big\|\\
&\le \|\dot{\bar{x}}(t)-\dot{\bar{x}}(t_j)\|+\frac{1}{h_k}\int\limits_{t_j}^{t_{j+1}}\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(t)\|\,dt \le \|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\frac{1}{h_k}\int\limits_{t_j}^{t_{j+1}}\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|\,dt+2^{-k}\\
&\le \|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\|\dot{\bar{x}}(s_j^x)-\dot{\bar{x}}(t_{j+1})\|+2^{-k}.
\end{aligned} \tag{4.49}$$

Hence we arrive at the relationships

$$\begin{aligned}
&\int\limits_0^T \Big\|\dot{\bar{x}}(t)-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{h_k}\Big\|^2 dt\\
&\le \int\limits_0^T \Big(\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\|\dot{\bar{x}}(s_j^x)-\dot{\bar{x}}(t_{j+1})\|+2^{-k}\Big)^2 dt\\
&\le 2\int\limits_0^T \Big(\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\|\dot{\bar{x}}(s_j^x)-\dot{\bar{x}}(t_{j+1})\|\Big)^2 dt+T4^{-k}\\
&=2\sum_{j=0}^{k-1}\int\limits_{t_j}^{t_{j+1}} \Big(\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\|\dot{\bar{x}}(s_j^x)-\dot{\bar{x}}(t_{j+1})\|\Big)^2 dt+T4^{-k+1}\\
&\le 2h_k\Big(\sum_{j=0}^{k-1}\big(\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\|\dot{\bar{x}}(s_j^x)-\dot{\bar{x}}(t_{j+1})\|\big)\Big)^2+T4^{-k+1}\le 2h_k\mu^2+T4^{-k+1},
\end{aligned}$$

which clearly yield the estimate

$$\int\limits_0^T \Big\|\dot{u}^k(t)-\frac{\bar{u}(t_{j+1})-\bar{u}(t_j)}{h_k}\Big\|^2 dt \le 2\int\limits_0^T \|\dot{x}^k(t)-\dot{\bar{x}}(t)\|^2\,dt+8h_k\mu^2+2T4^{-k+1}. \tag{4.50}$$

In the same way we obtain the condition

$$\int\limits_0^T \Big\|\dot{\bar{u}}(t)-\frac{\bar{u}(t_{j+1})-\bar{u}(t_j)}{h_k}\Big\|^2 dt \le 2h_k\mu^2+T4^{-k+1}. \tag{4.51}$$

Substituting (4.50) and (4.51) into (4.48) gives us

$$\int\limits_0^T \|\dot{u}^k(t)-\dot{\bar{u}}(t)\|^2\,dt \le 4\int\limits_0^T \|\dot{x}^k(t)-\dot{\bar{x}}(t)\|^2\,dt+16h_k\mu^2+4T4^{-k+1}+4h_k\mu^2+2T4^{-k+1},$$

which verifies the strong convergence of the sequence $\{\dot{u}^k(\cdot)\}$ to $\dot{\bar{u}}(\cdot)$ in the norm topology of $L^2([0,T],\mathbb{R}^n)$ with $u^k(0)=\bar{u}(0)+x^k(0)-x_0=\bar{u}(0)$. Finally, for all $t\in[0,T]$ we get

$$\|u^k(t)-\bar{u}(t)\| \le \int\limits_0^t \|\dot{u}^k(s)-\dot{\bar{u}}(s)\|\,ds \le \sqrt{T}\Big(\int\limits_0^T \|\dot{u}^k(t)-\dot{\bar{u}}(t)\|^2\,ds\Big)^{1/2} \to 0 \ \text{ as } \ k\to\infty,$$

and thus the sequence $\{u^k(\cdot)\}$ converges to the feasible control $\bar{u}(\cdot)$ strongly in $W^{1,2}([0,T],\mathbb{R}^n)$ as $k\to\infty$.

Our next goal is to verify the claimed representation (4.32). From (4.34) We have

$$\begin{aligned}
&\sum_{j=0}^{k-2}\left\|\frac{u^k(t_{j+2})-u^k(t_{j+1})}{h_k}-\frac{u^k(t_{j+1})-u^k(t_j)}{h_k}\right\|\\
&\le\sum_{j=0}^{k-2}\left\|\frac{\bar{u}(t_{j+2})-\bar{u}(t_{j+1})}{h_k}-\frac{\bar{u}(t_{j+1})-\bar{u}(t_j)}{h_k}\right\|+2\sum_{j=0}^{k-1}\left\|\frac{x^k(t_{j+1})-x^k(t_j)}{h_k}-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{h_k}\right\|\\
&\le\sum_{j=0}^{k-2}\left\|\frac{\bar{u}(t_{j+2})-\bar{u}(t_{j+1})}{h_k}-\dot{\bar{u}}(t_j)\right\|+\sum_{j=0}^{k-2}\left\|\dot{\bar{u}}(t_j)-\frac{\bar{u}(t_{j+1})-\bar{u}(t_j)}{h_k}\right\|\\
&+2\sum_{j=0}^{k-1}\left\|v_j^k-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{h_k}\right\|\\
&\le\frac{1}{h_k}\sum_{j=0}^{k-2}\int_{t_{j+1}}^{t_{j+2}}\|\dot{\bar{u}}(s)-\dot{\bar{u}}(t_j)\|\,ds+\frac{1}{h_k}\sum_{j=0}^{k-2}\int_{t_j}^{t_{j+1}}\|\dot{\bar{u}}(s)-\dot{\bar{u}}(t_j)\|\,ds+2\sum_{j=0}^{k-1}\left\|v_j^k-\dot{\bar{x}}(t)\right\|\\
&+2\sum_{j=0}^{k-1}\left\|\dot{\bar{x}}(t)-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{h_k}\right\|\\
&=\frac{1}{h_k}\sum_{j=0}^{k-2}\int_{t_j}^{t_{j+2}}f_j^u(s)\,ds+2\sum_{j=0}^{k-1}\left\|v_j^k-\dot{\bar{x}}(t)\right\|+2\sum_{j=0}^{k-1}\left\|\dot{\bar{x}}(t)-\frac{\bar{x}(t_{j+1})-\bar{x}(t_j)}{h_k}\right\|.
\end{aligned}$$

It follows from (4.38) and (4.49) that

$$\begin{aligned}
&\sum_{j=0}^{k-2}\left\|\frac{u^k(t_{j+2})-u^k(t_{j+1})}{h_k}-\frac{u^k(t_{j+1})-u^k(t_j)}{h_k}\right\|\\
&\le\frac{1}{h_k}\sum_{j=0}^{k-2}\int_{t_j}^{t_{j+2}}f_j^u(s)\,ds+2\sum_{j=0}^{k-1}\left(\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\|\dot{\bar{x}}(s_j^x)-\dot{\bar{x}}(t_{j+1})\|+2^{-k}\right)\\
&+2\sum_{j=0}^{k-1}\left(c_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)(L_1+L_1T)+h_k\widetilde{M}+(L_1+L_2)T2^{-k}+f_j^x(t)\right).
\end{aligned}$$

We deduce from (4.37) that

$$\begin{aligned}
\sum_{j=0}^{k-2}\int_{t_j}^{t_{j+2}}f_j^u(s)\,ds\quad&\le 2h_k\sum_{j=0}^{k-1}\left(\|\dot{\bar{u}}(t_j)-\dot{\bar{u}}(s_j^u)\|+\|\dot{\bar{u}}(s_j^u)-\dot{\bar{u}}(t_{j+1})\|+2^{-k}\right)\\
&\le 2h_k\mathrm{var}(\dot{\bar{u}}(\cdot),[0,T])+2h_kk2^{-k}\le 2h_k\mu+2T2^{-k}.
\end{aligned}$$

Using the same arguments leads us to the inequalities

$$\sum_{j=0}^{k-1}f_j^x(t)\le\mu+k2^{-k}\quad\text{and}\quad\sum_{j=0}^{k-1}\left(\|\dot{\bar{x}}(t_j)-\dot{\bar{x}}(s_j^x)\|+\|\dot{\bar{x}}(s_j^x)-\dot{\bar{x}}(t_{j+1})\|\right)\le\mu+k2^{-k},$$

and therefore we arrive at the estimate

$$\sum_{j=0}^{k-2}\left\|\frac{u^k(t_{j+2})-u^k(t_{j+1})}{h_k}-\frac{u^k(t_{j+1})-u^k(t_j)}{h_k}\right\|$$
$$\le 6\mu+(8+2(L_1+L_2)T)k2^{-k}+2kc_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)(L_1+L_1T)+2T\widetilde{M}:=\widetilde{\mu}_k.$$

Remembering the definition of c_k in (4.42) and $\lim\limits_{k\to+\infty}k2^{-k}=0$ shows that $\widetilde{\mu}_k\le\widetilde{\mu}$ for some constant $\widetilde{\mu}>0$, which justifies the first estimate in (4.32).

To verify the second estimate therein, we deduce from (4.34) and (4.43) that

$$\begin{aligned}\left\|\frac{u^k(t_1)-u^k(t_0)}{t_1-t_0}\right\| &\le \left\|\frac{u^k(t_1)-\bar{u}(t_1)}{t_1-t_0}\right\|+\left\|\frac{u^k(t_0)-\bar{u}(t_0)}{t_1-t_0}\right\|+\left\|\frac{\bar{u}(t_1)-\bar{u}(t_0)}{t_1-t_0}\right\|\\
&\le \left\|\frac{x^k(t_1)-\bar{x}(t_1)}{t_1-t_0}\right\|+\left\|\frac{x^k(t_0)-\bar{x}(t_0)}{t_1-t_0}\right\|+\frac{1}{h_k}\int_{t_0}^{t_1}\|\dot{u}(s)\|\,ds\\
&\le \frac{2}{h_k}c_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)+\frac{1}{h_k}\int_{t_0}^{t_1}\sup_{s\in[t_0;t_1]}\|\dot{u}(s)\|\,ds\\
&\le \frac{2k}{T}c_k\exp\left(L_1T\left(\frac{T}{2}+1\right)\right)+\sup_{s\in[t_0;t_1]}\|\dot{u}(s)\|\le\widetilde{\mu},\end{aligned}$$

which readily gives us the claimed result in (4.32). This completes the proof of the theorem. ■

4.4 Discrete approximations of local optimal solutions

In this section we continue developing the method of discrete approximations for controlled integro-differential sweeping processes while in a different framework. Our goal is to (strongly in $W^{1,2}$) approximate a prescribed relaxed intermediate local minimizer of the optimal control problem (P) by optimal solutions to discretized sweeping control systems. Given an r.i.l.m. $\bar{z}(\cdot)=(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot),\bar{a}(\cdot),\bar{b}(\cdot))$ of (P) with the number $\varepsilon>0$ from Definition 4.2.1 (ii), consider the mesh Δ_k defined in (4.30) and construct the sequence of discrete-time sweeping optimal control problems (P_k), $k\in I\!N$, as follows:

$$\text{minimize }\ J_k(z^k):=\varphi(x_k^k)+h_k\sum_{j=0}^{k-1}l\left(t_j^k,x_j^k,y_j^k,u_j^k,a_j^k,b_j^k,\frac{x_{j+1}^k-x_j^k}{h_k},\frac{y_{j+1}^k-y_j^k}{h_k},\frac{u_{j+1}^k-u_j^k}{h_k}\right)$$
$$+\frac{1}{2}\sum_{j=0}^{k-1}\int_{t_j^k}^{t_{j+1}^k}\left(\left\|\frac{x_{j+1}^k-x_j^k}{h_k}-\dot{\bar{x}}(t)\right\|^2+\left\|\frac{y_{j+1}^k-y_j^k}{h_k}-\dot{\bar{y}}(t)\right\|^2+\left\|\frac{u_{j+1}^k-u_j^k}{h_k}-\dot{\bar{u}}(t)\right\|^2\right)dt,$$
$$+\frac{1}{2}\sum_{j=0}^{k-1}\int_{t_j^k}^{t_{j+1}^k}\|(a_j^k,b_j^k)-(\bar{a}(t),\bar{b}(t))\|^2\,dt,$$

over the collections $z^k = (x_0^k, \ldots, x_k^k, y_0^k, \ldots, y_k^k, u_0^k, \ldots, u_k^k, a_0^k, \ldots, a_k^k, b_0^k, \ldots, b_k^k)$ subject to the constraints:

$$x_{j+1}^k \in x_j^k - h_k F_h(x_j^k, y_j^k, u_j^k, a_j^k, b_j^k), \quad j = 0, \ldots, k-1, \tag{4.52}$$

where the discrete velocity mapping F_h is given in the form

$$F_h(x_j^k, y_j^k, u_j^k, a_j^k, b_j^k) := F_1(x_j^k, y_{j+1}^k, u_j^k, a_j^k) = N_{C(u_j^k)}(x_j^k) + f_1(x_j^k, a_j^k) + y_{j+1}^k \tag{4.53}$$

due to the definition of F_1 in (4.20), and where $(x_0^k, u_0^k, a_0^k) = (x_0, u_0, \bar{a}(0))$,

$$y_{j+1}^k = y_j^k + h_k f_2(b_j^k, x_j^k) \ \text{ with } \ (y_0^k, b_0^k) = (0, \bar{b}(0)), \tag{4.54}$$

$$g_i(x_k^k - u_k^k) \geq 0, \quad i = 1, \ldots, s, \tag{4.55}$$

$$\|(x_j^k, y_j^k, u_j^k, a_j^k, b_j^k) - (\bar{x}(t_j^k), \bar{y}(t_j^k), \bar{u}(t_j^k), \bar{a}(t_j^k), \bar{b}(t_j^k))\| \leq \frac{\varepsilon}{2}, \quad j = 0, \ldots, k-1, \tag{4.56}$$

$$\begin{aligned}\sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \Big(\Big\| \frac{x_{j+1}^k - x_j^k}{h_k} - \dot{\bar{x}}(t) \Big\|^2 + \Big\| \frac{y_{j+1}^k - y_j^k}{h_k} - \dot{\bar{y}}(t) \Big\|^2 + \Big\| \frac{u_{j+1}^k - u_j^k}{h_k} - \dot{\bar{u}}(t) \Big\|^2 \\ + \|(a_j^k, b_j^k) - (\bar{a}(t), \bar{b}(t))\|^2 \Big) \, dt \leq \frac{\varepsilon}{2}.\end{aligned} \tag{4.57}$$

$$a_j^k \in A, \quad \text{and} \quad b_j^k \in B, \quad j = 0, ..., k-1. \tag{4.58}$$

To proceed further, we first have to make sure that problems (P_k) admit optimal solution.

Proposition 4.4.1 (existence of optimal solutions to discrete approximations). *Suppose that the standing assumptions $(\mathcal{H}_1)-(\mathcal{H}_3)$ are satisfied around the given r.i.l.m. $\bar{z}(\cdot)$. Then each problem (P_k) has an optimal solution provided that $k \in I\!N$ is sufficiently large.*

Proof. Observe that the $W^{1,2}$-strong convergence results of Theorem 4.3.1 and the construction of problems (P_k) allow us to conclude that the sets of feasible solutions to (P_k) are nonempty whenever k is sufficiently large. Now we fix $k \in I\!N$ and show that set of feasible solutions to (P_k) is bounded. Indeed, pick a sequence $z^\nu = (x_0^\nu, \ldots, x_k^\nu, y_0^\nu, \ldots, y_k^\nu, u_0^\nu, \ldots, u_k^\nu, a_0^\nu, \ldots, a_k^\nu, b_0^\nu, \ldots, b_k^\nu)$ of feasible solutions to (P_k) that converges to some $z = (x_0, \ldots, x^k, y_0, \ldots, y^k, u_0, \ldots, u^k, a_0, \ldots, a^k, b_0, \ldots, b^k)$ as $\nu \to 0$ and show that z is feasible to (P_k) as well. Observe that $g_i(x_j - u_j) = \lim_{\nu \to \infty} g_i(x_j^\nu - u_j^\nu) \geq 0$ for all $i = 1, \ldots, s$ and $j = 0, \ldots, k-1$, which ensures that $x_j \in C(u_j)$. It follows from the closed graph property of the normal cone mapping in (4.52) with F_1 taken from (4.20) and the continuity of the functions f_1 and f_2 that

$$-\frac{x_{j+1} - x_j}{h_k} - f_1(a_j, x_j) - y_{j+1} \in N_C(x_j - u_j), \ \ y_{j+1} = y_j + h_k f_2(b_j, x_j),$$

or equivalently, that $x_{j+1} \in x_j + h_k F_1(x_j, y_j, u_j, a_j)$ for all $j = 0, \ldots, k-1$, which verifies the claimed closedness of the feasible solution set. To conclude the proof of the proposition, we notice that the latter set is bounded due to the constraints in (4.56). Thus the existence of optimal solutions in (P_k) is ensured by the classical Weierstrass theorem in finite-dimensional spaces. ■

Now we are ready to establish the desired strong convergence of extended discrete optimal solutions of (P_k) to the prescribed r.i.l.m. of the original problem (P).

Theorem 4.4.1 (strong convergence of extended discrete optimal solutions). *Let $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot), \bar{b}(\cdot))$ be an r.i.l.m. for problem (P). In addition to the standing assumptions $(\mathcal{H}_1) - (\mathcal{H}_3)$ imposed along $\bar{z}(\cdot)$, suppose that the cost functions φ and $l_0 \equiv l$ are continuous at $\bar{x}(T)$ and at $(t, \bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot), \bar{b}(\cdot), \dot{\bar{x}}(\cdot), \dot{\bar{y}}(\cdot), \dot{\bar{u}}(\cdot))$ for a.e. $t \in [0,T]$, respectively. Then any sequence of optimal solutions $\bar{z}^k(\cdot) = (\bar{x}^k(\cdot), \bar{y}^k(\cdot), \bar{u}^k(\cdot), \bar{a}^k(\cdot), \bar{b}^k(\cdot))$ to (P_k), piecewise linearly extended to $[0,T]$ for $(\bar{x}^k(\cdot), \bar{y}^k(\cdot), \bar{u}^k(\cdot))$ and piecewise constantly for $(\bar{a}^k(\cdot), \bar{b}^k(\cdot))$. Is such that the extended sequence of $(\bar{x}^k(\cdot), \bar{y}^k(\cdot), \bar{u}^k(\cdot), \bar{a}^k(\cdot), \bar{b}^k(\cdot))$ converges to $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot), \bar{b}(\cdot))$ strongly in $W^{1,2}([0,T], \mathbb{R}^{3n}) \times L^2([0,T], \mathbb{R}^{m+d})$ as $k \to \infty$.*

Proof. We know from Proposition 4.4.1 that each problem (P_k) admits optimal solutions $\bar{z}^k(\cdot)$ for large $k \in I\!N$. Extend any $\bar{z}^k(\cdot)$ piecewise linearly to the continuous-time interval $[0,T]$. We aim at verifying that

$$\lim_{k\to\infty} \int_0^T \left\| \left(\dot{\bar{x}}^k(t), \dot{\bar{y}}^k(t), \dot{\bar{u}}^k(t), \bar{a}^k(t), \bar{b}^k(t)\right) - \left(\dot{\bar{x}}(t), \dot{\bar{y}}(t), \dot{\bar{u}}(t), \bar{a}(t), \bar{b}(t)\right) \right\|^2 dt = 0, \tag{4.59}$$

which clearly yields the convergence of the quintuple $(\bar{x}^k(\cdot), \bar{y}^k(\cdot), \bar{u}^k(\cdot), \bar{a}^k(\cdot), \bar{b}^k(\cdot))$ to $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot), \bar{b}(\cdot))$ in the norm topology of $W^{1,2}([0,T], \mathbb{R}^{3n}) \times L^2([0,T], \mathbb{R}^{m+d})$ as $k \to \infty$.

To justify (4.59), suppose on the contrary that it fails, i.e., the limit in (4.59), along a subsequence (without relabeling) equals to some $\xi > 0$. The weak compactness of the unit ball in $L^2([0,T], \mathbb{R}^{3n+m+d})$ allows us to find $(v^x(\cdot), v^y(\cdot), v^u(\cdot), \widetilde{a}(\cdot), \widetilde{b}(\cdot))$ such that the quintuple $(\dot{\bar{x}}^k(\cdot), \dot{\bar{y}}^k(\cdot), \dot{\bar{u}}^k(\cdot), \bar{a}^k(\cdot), \bar{b}^k(\cdot))$ converges weakly to $(v^x(\cdot), v^y(\cdot), v^u(\cdot), \widetilde{a}(\cdot), \widetilde{b}(\cdot)) \in L^2([0,T], \mathbb{R}^{3n+m+d})$ as $k \to \infty$. Defining $(\widetilde{x}(\cdot), \widetilde{y}(\cdot), \widetilde{u}(\cdot))$ by

$$(\widetilde{x}(t), \widetilde{y}(t), \widetilde{u}(t)) := (x_0, 0, \bar{u}(0)) + \int_0^t (v^x(s), v^y(s), v^u(s))\, ds, \quad t \in [0,T],$$

we get that $(\dot{\widetilde{x}}(t), \dot{\widetilde{y}}(t), \dot{\widetilde{u}}(t)) = (v^x(t), v^y(t), v^u(t))$ for a.e. $t \in [0,T]$. Arguing now as in the proof of Theorem 4.2.1 shows that the arc $\widetilde{z}(\cdot)$ is feasible to the original problem (P) and hence to the relaxed one (R).

Next we check that $(\widetilde{x}(\cdot),\widetilde{y}(\cdot),\widetilde{u}(\cdot),\widetilde{a}(\cdot),\widetilde{b}(\cdot))$ satisfies the localization conditions in (4.29) relative to $(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot),\bar{a}(\cdot),\bar{b}(\cdot))$. Indeed, the first condition in (4.29) follows directly from the passage to the limit in (4.56) as $k\to\infty$. To justify the second condition in (4.29), we pass to the limit in (4.57) due to the established weak convergence of the derivatives $(\dot{\bar{x}}^k(\cdot),\dot{\bar{y}}^k(\cdot),\dot{\bar{u}}^k(\cdot))\to(\dot{\widetilde{x}}(\cdot),\dot{\widetilde{y}}(\cdot),\dot{\widetilde{u}}(\cdot))$ and the lower semicontinuity of the norm function in L^2. This tells us that

$$\begin{aligned}&\int_0^T \big\|\big(\dot{\widetilde{x}}(t),\dot{\widetilde{y}}(t),\dot{\widetilde{u}}(t)\big)-\big(\dot{\bar{x}}(t),\dot{\bar{y}}(t),\dot{\bar{u}}(t)\big)\big\|\\ &\le \liminf_{k\to\infty}\sum_{j=0}^{k-1}\int_{t_j^k}^{t_{j+1}^k}\Big(\Big\|\frac{x_{j+1}^k-x_j^k}{h_k}-\dot{\bar{x}}(t)\Big\|^2+\Big\|\frac{y_{j+1}^k-y_j^k}{h_k}-\dot{\bar{y}}(t)\Big\|^2+\Big\|\frac{u_{j+1}^k-u_j^k}{h_k}-\dot{\bar{u}}(t)\Big\|^2\Big)\,dt\\ &\le\frac{\varepsilon}{2},\end{aligned}$$

and thus we are done with (4.29). Furthermore, it follows from the construction of the relaxed problem (R) in (4.28) due to the convexity of $\widehat{l}_G$ in the velocity variables, the established weak convergence of the extended discrete derivatives, and applications of the Mazur theorem as in the proof of Theorem 4.3.1 that

$$\begin{aligned}&\int_0^T \widehat{l}_G\big(t,\widetilde{x}(t),\widetilde{y}(t),\widetilde{u}(t),\widetilde{a}(t),\widetilde{b}(t),\dot{\widetilde{x}}(t),\dot{\widetilde{y}}(t),\dot{\widetilde{u}}(t)\big)\,dt\\ &\le\liminf_{k\to\infty}h_k\sum_{j=0}^{k-1} l\Big(t_j^k,\bar{x}_j^k,\bar{y}_j^k,\bar{u}_j^k,\bar{a}_j^k,\bar{b}_j^k,\frac{\bar{x}_{j+1}^k-\bar{x}_j^k}{h_k},\frac{\bar{y}_{j+1}^k-\bar{y}_j^k}{h_k},\frac{\bar{u}_{j+1}^k-\bar{u}_j^k}{h_k}\Big).\end{aligned}$$

We also observe in this way, with taking into account the above definition of ξ, that

$$\begin{aligned}\widehat{J}[\widetilde{z}]+\frac{\xi}{2}&=\varphi(\widetilde{x}(T))+\int_0^T\widehat{l}_G\big(t,\widetilde{x}(t),\widetilde{y}(t),\widetilde{u}(t),\widetilde{a}(t),\widetilde{b}(t),\dot{\widetilde{x}}(t),\dot{\widetilde{y}}(t),\dot{\widetilde{u}}(t)\big)\,dt+\frac{\xi}{2}\\ &\le\liminf_{k\to\infty}J_k(\bar{z}^k).\end{aligned}\tag{4.60}$$

Employing Theorem 4.3.1 gives us a sequence $\{(x^k(\cdot),y^k(\cdot),u^k(\cdot))\}$ and $\{(a^k(\cdot),b^k(\cdot))\}$ of feasible solutions to (P_k) that strongly approximate $(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot))$ and $(\bar{a}(\cdot),\bar{b}(\cdot))$ in $W^{1,2}$ and L^2, which is a feasible solution to (P). The imposed continuity assumptions on φ and l yield

$$\lim_{k\to\infty}J_k[x^k,y^k,u^k,a^k,b^k]=J[\bar{x},\bar{y},\bar{u},\bar{a},\bar{b}].\tag{4.61}$$

On the other hand, it follows from the optimality of $\bar{z}^k(\cdot):=(\bar{x}^k(\cdot),\bar{y}^k(\cdot),\bar{u}^k(\cdot),\bar{a}^k(\cdot),\bar{b}^k(\cdot))$ in (P_k) that

$$J_k[\bar{z}^k]\le J_k[x^k,y^k,u^k,a^k,b^k]\ \text{ for each }\ k\in\mathbb{N}.\tag{4.62}$$

Combining finally the relationships in (4.60)–(4.62), we conclude that

$$\widetilde{J}[\widetilde{x},\widetilde{y},\widetilde{u},\widetilde{a},\widetilde{b}] < \widetilde{J}[\widetilde{x},\widetilde{y},\widetilde{u},\widetilde{a},\widetilde{b}] + \frac{\xi}{2} \le J[\bar{z}] = \widehat{J}[\bar{z}].$$

Due to the choice of $\xi > 0$ above, the latter clearly contradicts the fact that $\bar{z}(\cdot)$ is an r.i.l.m. for problem (P) and thus verifies the limiting condition (4.59). This completes the proof of the theorem. ■

4.5 Generalized differentiation and second-order calculations

This section briefly recalls some tools of first-order and second-order generalized differentiation in variational analysis, which are instrumental in deriving necessary optimality conditions for the discrete-time and continuous-time sweeping control problems formulated above. The reader can find more details and references in the books [49, 52, 71] for the first-order and in [49, 52] for the second-order issues. Observe that, although the initial data of problem (P) and its discrete approximations are smooth and/or convex, the intrinsic source of nonsmoothness comes from the sweeping dynamics and its discretization, which lead us to nonconvex graphical sets and require the usage of robust generalized differentiation with adequate properties. It occurs that the most appropriate constructions for these purposes are the robust nonconvex notions introduced by the third author, while their convexification fails the needed results, particularly of the second-order.

Given a nonempty set $\Omega \subset \mathbb{R}^n$ locally closed around $\bar{x} \in \Omega$, consider the associated projection operator (1.3.3) and recall that the (basic/limiting/Mordukhovich) *normal cone* to Ω at $\bar{x}$ is defined by

$$N_\Omega(\bar{x}) := \big\{ v \in \mathbb{R}^n \big| \, \exists x_k \to \bar{x},\ w_k \in \text{Proj}_\Omega(x_k),\ \alpha_k \ge 0 \ \text{ s.t. } \ \alpha_k(x_k - w_k) \to v \ \text{ as } \ k \to \infty \big\} \tag{4.63}$$

with $N_\Omega(\bar{x}) := \emptyset$ for $\bar{x} \notin \Omega$. If Ω is prox-regular, the normal cone (4.63) agrees with the proximal normal cone (1.3), and they both reduce to the normal cone of convex analysis when Ω is convex. Furthermore, the convex closure of (4.63) gives us the Clarke normal cone [33]. Note that, despite the nonconvexity of (4.63), this normal cone and the associated subdifferential and coderivative constructions for functions and multifunctions are robust and enjoy *full calculi* based on variational/extremal principles of variational analysis. This is not the case for the proximal constructions and may also fail for Clarke's ones without imposing additional interiority-type assumptions.

Considering a set-valued mapping/multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^q$ with $\operatorname{dom} F := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$ and taking a point $(\bar{x}, \bar{y})$ from the graph

$$\operatorname{gph} F := \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^q \;\big|\; y \in F(x)\big\},$$

the *coderivative* $D^*F(\bar{x},\bar{y}) : \mathbb{R}^q \rightrightarrows \mathbb{R}^n$ of F at $(\bar{x},\bar{y})$ is defined by

$$D^*F(\bar{x},\bar{y})(u) := \big\{v \in \mathbb{R}^n \;\big|\; (v,-u) \in N_{\operatorname{gph} F}(\bar{x},\bar{y})\big\} \text{ for all } u \in \mathbb{R}^q, \tag{4.64}$$

where we skip $\bar{y}$ if F is single-valued. If in the latter case F is continuously differentiable around $\bar{x}$, then

$$D^*F(\bar{x})(u) := \big\{\nabla F(\bar{x})^* u\big\} \text{ for all } u \in \mathbb{R}^q.$$

Given further an extended-real-valued function $\varphi : \mathbb{R}^n \to \bar{\mathbb{R}}$ finite at $\bar{x}$, the (first-order) *subdifferential* of φ at $\bar{x}$ is defined geometrically by

$$\partial\varphi(\bar{x}) := \big\{v \in \mathbb{R}^n \;\big|\; (v,-1) \in N_{\operatorname{epi}\varphi}(\bar{x},\varphi(\bar{y}))\big\} \tag{4.65}$$

while admitting various analytic representations given in [52, 71]. Following [50], the *second-order subdifferential* (generalized Hessian) of φ at $\bar{x}$ relative to $\bar{y} \in \varphi(\bar{x})$ is defined by

$$\partial^2\varphi(\bar{x},\bar{y})(u) := \big(D^*\partial\varphi\big)(\bar{x},\bar{y})(u) \text{ for all } u \in \mathbb{R}^q. \tag{4.66}$$

If φ is twice continuously differentiable around $\bar{x}$, then we get

$$\partial^2\varphi(\bar{x})(u) := \big\{\nabla^2\varphi(\bar{x})(u)\big\} \text{ for all } u \in \mathbb{R}^q$$

via the classical (symmetric) Hessian of φ at $\bar{x}$. Note that replacing the limiting normal cone in (4.64) and (4.66) by its convexification (and hence by Clarke's normal cone) to graphical sets *dramatically enlarges* the corresponding constructions and makes them unusable for further analysis and applications. In particular, it follows from [82, Theorem 3.5] that Clarke's normal cone is a linear *subspace* of the maximal dimension if the graph of F is a "Lipschitzian manifold," which is the case of, e.g., subgradient mappings $F = \partial\varphi$ in (4.65) generated by *prox-regular* functions; see [49, 52, 71] for more discussions. In contrast, the second-order subdifferential (4.66) provides an adequate machinery of second-order variational analysis in applications to controlled sweeping processes. We largely use below the following theorem, which gives us a *precise computation* of the velocity mapping associated with the integro-differential sweeping dynamics of our study entirely in terms of the given data of (P). Due to the very structure of the sweeping dynamics, the obtained result contains the computation of the second-order subdifferential of the indicator function in (4.66).

Theorem 4.5.1 (second-order calculation for integro-differential sweeping dynamics). *Define the set-valued mapping* $F_h : \mathbb{R}^{3n+m+d} \rightrightarrows \mathbb{R}^n$ *by*

$$F_h(x,y,u,a,b) := F_1(x,y,u,a) + hf_2(b,x) = N_C(x-u) + f_1(a,x) + y + hf_2(b,x), \tag{4.67}$$

where F_1 *is given in* (4.20) *with* f_1, f_2, *and* C *taken from* (4.2) *and* (4.3), *respectively, and* $h > 0$. *Pick any* $u \in \mathbb{R}^n$ *with* $x - u \in C$, *take* $w \in N_C(x-u) + f_1(a,x) + y + hf_2(b,x)$, *and suppose that* $f_1, f_2 \in \mathcal{C}^1$, $g := (g_1, \ldots, g_s) \in \mathcal{C}^2$ *around the corresponding points with full rank of the Jacobian matrix* $\nabla g(x-u)$. *Let* $\lambda := (\lambda_1, \ldots, \lambda_s) \in \mathbb{R}^s$ *be a unique vector with nonnegative components satisfying the equation*

$$-\nabla g(x-u)^*\lambda = w - f_1(a,x) - y - hf_2(b,x). \tag{4.68}$$

Then the coderivative of F_h *is calculated by the formula*

$$\begin{aligned} &D^*F_h(x,y,u,a,b,w)(z) \\ &= \Big\{\nabla_x f_1(a,x)^*z + h\nabla_x f_2(b,x)^*z - \Big(\sum_{i=1}^s \lambda_i \nabla_x^2 g_i(x-u)\Big)z - \nabla_x g(x-u)^*\sigma, \\ &\Big(\sum_{i=1}^s \lambda_i \nabla_u^2 g_i(y-u)\Big)z + \nabla_u g(y-u)^*\sigma, \nabla_a f_1(a,x)^*z, h\nabla_b f_2(b,x)^*z\Big\} \end{aligned} \tag{4.69}$$

for all $z \in \operatorname{dom} D^*N_C(x-u, w - f_1(a,x) - y - hf_2(b,x))$, *where the coderivative domain is represented as*

$$\begin{aligned} &\operatorname{dom} D^*N_C(x-u, w - f_1(a,x) - y - hf_2(b,x)) \\ &= \Big\{z \in \mathbb{R}^n \;\Big|\; \exists\, \lambda \in \mathbb{R}^s_+ \text{ such that } -\nabla g(x-u)\lambda = w - f_1(a,x) - y - hf_2(b,x), \\ &\lambda_i \langle \nabla\, g_i(x-u), z\rangle = 0 \text{ for all } i = 1, \ldots, s\Big\}, \end{aligned} \tag{4.70}$$

and where we have in (4.69) *that* $\sigma_i = 0$ *if either* $g_i(x-u) > 0$ *or* $\lambda_i = 0$ *with* $\langle \nabla\, g_i(x-u), z\rangle > 0$, *and that* $\sigma_i \geq 0$ *if* $g_i(x-u) = 0$, $\lambda_i = 0$ *with* $\langle \nabla\, g_i(x-u), z\rangle < 0$.

Proof. It follows the lines in the proof of [22, Theorem 6.2] but applying now to the different form of the velocity mapping (4.67) employed in this paper. We skip the details for brevity while mentioning that the coderivative sum and chain rules taken from [49, Theorems 1.62 and 1.66] allow us to derive the claimed equalities in (4.69) and (4.70) to the second-order calculations developed in [80, Theorem 3.3]. ■

4.6 Necessary optimality conditions for discrete problems

In this section we obtain necessary optimality conditions for (global) optimal solutions to the discrete approximation problems (P_k) constructed in Section 4.4 for each $k \in \mathbb{N}$. Due to the

$W^{1,2}$-strong convergence result established in Theorem 4.4.1, the obtained optimality conditions for problems (P_k) with sufficiently large approximation indices k can be viewed as necessary *suboptimality* conditions for the original continuous-time optimal control problem (P) with the integro-differential sweeping dynamics (4.2). In practice, such conditions for (P_k) provide sufficient information for designing numerical algorithms to solve the original problem (P). Nevertheless, in Section 4.7 we furnish the rigorous procedure to derive precise necessary optimality conditions for relaxed intermediate local minimizers in (P) by passing to the limit as $k \to \infty$ from the necessary optimality conditions for (P_k) obtained below.

In what follows we present two sets of necessary optimality conditions in problems (P_k) for each fixed $k \in I\!N$. The first theorem concerns a class of more general problems given in form (P_k) with an arbitrary closed-graph velocity mapping F_h. The obtained results can be treated as extended *discrete-time Euler-Lagrange conditions* for discrete approximations of integro-differential sweeping control problems with the adjoint systems described via the basic normal cone (4.63) to graphs of velocity mappings.

Theorem 4.6.1 (extended Euler-Lagrange conditions for discrete optimal solutions). *Fix $k \in I\!N$ to be sufficiently large and pick any optimal solution $\bar{z}^k := (x_0, \ldots, \bar{x}^k_k, y_0, \ldots, \bar{y}^k_k, \bar{u}^k_0, \ldots, \bar{u}^k_k, a_0, \ldots, \bar{a}^k_k, b_0, \ldots, \bar{b}^k_k)$ to problem (P_k) governed by an arbitrary closed-graph multifunction in (4.52), while not taken from (4.53). Suppose that the cost functions φ and l_0 from (4.1) are locally Lipschitzian around the corresponding components of the optimal solution for all $t \in \Delta_k$ (with the index t dropped below), and that the mappings f_1, f_2, g_i are continuously differentiable around the optimal points. For $j = 0, \ldots, k-1$ denote by*

$$\theta^{xk}_j := \int_{t^k_j}^{t^k_{j+1}} \Big(\frac{\bar{x}^k_{j+1} - \bar{x}^k_j}{h_k} - \dot{\bar{x}}(t)\Big)\, dt, \ \theta^{uk}_j := \int_{t^k_j}^{t^k_{j+1}} \Big(\frac{\bar{u}^k_{j+1} - \bar{u}^k_j}{h_k} - \dot{\bar{u}}(t)\Big)\, dt,$$
$$\theta^{yk}_j := \int_{t^k_j}^{t^k_{j+1}} \Big(\frac{\bar{y}^k_{j+1} - \bar{y}^k_j}{h_k} - \dot{\bar{y}}(t)\Big)\, dt, \tag{4.71}$$

$$\theta^{ak}_j := \int_{t^k_j}^{t^k_{j+1}} (\bar{a}^k_j - \bar{a}(t))\, dt, \ \theta^{bk}_j := \int_{t^k_j}^{t^k_{j+1}} (\bar{b}^k_j - \bar{b}(t))\, dt. \tag{4.72}$$

Then there are $\lambda^k \ge 0$, $\alpha^k = (\alpha^k_1, \ldots, \alpha^k_s) \in \mathbb{R}^s_+$, $p^k_j = (p^{xk}_j, p^{yk}_j, p^{dk}_j, p^{uk}_j) \in \mathbb{R}^{4n}$, $\psi^k_j = (\psi^{ak}_j, \psi^{bk}_j)$, $j = 0, \ldots, k$,

$$(w^{xk}_j, w^{uk}_j, w^{ak}_j, w^{bk}_j, v^{xk}_j, v^{uk}_j) \in \partial l_0\Big(\bar{x}^k_j, \bar{u}^k_j, \bar{a}^k_j, \bar{b}^k_j, \frac{\bar{x}^k_{j+1} - \bar{x}^k_j}{h_k}, \frac{\bar{u}^k_{j+1} - \bar{u}^k_j}{h_k}\Big)$$

such that the following conditions are satisfied:

$$\lambda^k + \|\alpha^k\| + \sum_{j=0}^{k-1}\|p_j^{kx}\| + \sum_{j=0}^{k}\|p_j^{kd}\| + \|p_0^{ky}\| + \|p_0^{ku}\| + \sum_{j=0}^{k-1}\|\psi_j^k\| \neq 0, \tag{4.73}$$

$$\alpha_i^k g_i(\bar{x}_k^k - \bar{u}_k^k) = 0, \;\; i = 1, \ldots, s, \tag{4.74}$$

$$-p_k^x \in \lambda^k \partial\varphi(\bar{x}_k^k) - \sum_{i=1}^{s}\alpha_i^k \nabla_x g_i(\bar{x}^k - \bar{u}^k), \;\; -p_k^{uk} = \sum_{i=1}^{s}\alpha_i^k \nabla g_i(\bar{x}_k^k - \bar{u}_k^k), \tag{4.75}$$

$$p_k^{yk} = 0, \tag{4.76}$$

$$p_{j+1}^y = \lambda^k h_k^{-1}\theta_j^y + h_k^{-1}p_{j+1}^d, \;\; p_{j+1}^u = \lambda^k(v_j^u + h_k^{-1}\theta_j^u), \;\; j = 0, \ldots, k-1, \tag{4.77}$$

Furthermore, for all $j = 0, \ldots, k-1$ we have the extended discrete-time Euler-Lagrange inclusions:

$$\begin{aligned}&\Big(\frac{p_{j+1}^x - p_j^x}{h_k} - \lambda^k w_j^x + h_k^{-1}\xi_j^x p_{j+1}^d, \frac{p_{j+1}^y - p_j^y}{h_k} - \frac{p_{j+1}^d}{h_k}, \frac{p_{j+1}^u - p_j^u}{h_k} - \lambda^k w_j^u, -\frac{1}{h_k}\lambda^k\theta_j^a - \lambda^k w_j^a,\\ &-\frac{1}{h_k}\lambda^k\theta_j^b - \lambda^k w_j^b + \xi_j^b\frac{p_{j+1}^d}{h_k}, p_{j+1}^x - \lambda^k(v_j^x + h_k^{-1}\theta_j^x)\Big)\\ &\in (0,0,0,\frac{1}{h_k}\psi_j^{ka}, \frac{1}{h_k}\psi_j^{kb}) + N_{\operatorname{gph} F_h}\Big(\bar{x}_j^k, \bar{y}_j^k, \bar{u}_j^k, \bar{a}_j^k, \bar{b}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{-h_k}\Big),\end{aligned} \tag{4.78}$$

$$\psi_j^{ak} \in N_A(\bar{a}_j^k) \;\; \textit{and} \;\; \psi_j^{bk} \in N_B(\bar{b}_j^k), \tag{4.79}$$

where the vectors ξ_j^x and ξ_j^b are defined by

$$\xi_j^x := \nabla_x f_2(\bar{b}_j^k, \bar{x}_j^k) \;\; \textit{and} \;\; \xi_j^b := \nabla_b f_2(\bar{b}_j^k, \bar{x}_j^k), \quad j = 0, \ldots, k-1. \tag{4.80}$$

Proof. Denote $z := (x_0, \ldots, x^k, y_0, \ldots, y^k, u_0, \ldots, u^k, a_0, \ldots, a^k, b_0, \ldots, b^k, X_0^k, \ldots, X_{k-1}^k,$ $Y_0, Y_1, \ldots, Y^k, U_0, \ldots, U_{k-1})$, where x_0 is fixed, and consider the problem of mathematical programming (MP): minimize

$$\begin{aligned}\varphi_0(z) &:= \varphi(x_k^k) + h_k\sum_{j=0}^{k-1} l(x_j, y_j, u_j, a_j, b_j, X_j, Y_j, U_j)\\ &+ \frac{1}{2}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}} \|(X_j, Y_j, U_j, a_j^k, b_j^k) - (\dot{\bar{x}}(t), \dot{\bar{y}}(t), \dot{\bar{u}}(t), \bar{a}(t), \bar{b}(t))\|^2\,dt\end{aligned}$$

subject to the equality, inequality, and many geometric constraints of the graphical type defined by

$$e_j^x(z) := x_{j+1} - x_j - h_k X_j = 0, \;\; e_j^y(z) := y_{j+1} - y_j - h_k Y_j = 0, \quad j = 0, \ldots, k-1,$$

$$e_j^u(z) := u_{j+1} - u_j - h_k U_j = 0, \;\; d_j(z) := Y_j - f_2(b_j, x_j) = 0, \quad j = 0, \ldots, k-1,$$

$$c_i(z) := -g_i(x^k - u^k) \le 0, \ \ i = 1, \ldots, s,$$

$$\phi_j(z) := \|(x_j, y_j, u_j, a_j, b_j) - \bar{z}(t_j)\| - \frac{\varepsilon}{2} \le 0, \quad j = 0, \ldots, k-1,$$

$$\phi_k(z) := \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} \|(X_j, Y_j, U_j, a_j^k, b_j^k) - (\dot{\bar{x}}(t), \dot{\bar{y}}(t), \dot{\bar{u}}(t), \bar{a}(t), \bar{b}(t))\|^2 \, dt - \frac{\varepsilon}{2} \le 0,$$

$$z \in \Xi_j := \big\{ z \ \big| \ -X_j \in F_h(x_j, y_j, u_j, a_j, b_j) \big\}, \quad j = 0, \ldots, k-1, \tag{4.81}$$

$$z \in \Xi_k = \big\{ z \ \big| \ x_0 \ \text{ is fixed }, \ y_0 = 0, \ \ (u_0, a_0, b_0) = (\bar{u}(0), \bar{a}(0), \bar{b}(0)) \big\}. \tag{4.82}$$

$$z \in \Omega_j^a := \big\{ z \ \big| \ a_j^k \in A \big\}, \quad j = 0, \ldots, k-1, \tag{4.83}$$

$$z \in \Omega_j^b := \big\{ z \ \big| \ b_j^k \in B \big\}, \quad j = 0, \ldots, k-1, \tag{4.84}$$

Applying now the necessary optimality conditions from Proposition 1.6.1 together with the intersection rule for basic normals taken from [52, Corollary 2.17] to the chosen solution $\bar{z}^k$ of problem (MP), observe first that the inequality constraints in (MP) defined by the functions ϕ_j for $j = 0, \ldots, k$ are inactive for sufficiently large $k \in I\!N$ due to the $W^{1,2} \times L^2$-strong convergence $\bar{z}^k(\cdot) \to \bar{z}(\cdot)$ established above in Theorem 4.4.1. All of this allows us to find $\lambda^k \ge 0$, $\alpha^k = (\alpha_1^k, \ldots, \alpha_s^k) \in \mathbb{R}_+^s$, $p_j = (p_j^x, p_j^y, p_j^u, p_j^d) \in \mathbb{R}^{4n}$ as $j = 1, \ldots, k$, and

$$z_j^* = \big(x_{0j}^*, \ldots, x_{kj}^*, y_{0j}^*, \ldots, y_{kj}^*, u_{0j}^*, \ldots, u_{kj}^*, a_{0j}^*, \ldots, a_{kj}^*, b_{0j}^*, \ldots, b_{kj}^*, X_{0j}^*, \ldots, X_{(k-1)j}^*,$$
$$Y_{0j}^*, \ldots, Y_{(k-1)j}^*, U_{0j}^*, \ldots, U_{(k-1)j}^* \big), \ \ j = 0, \ldots, k-1,$$

which are not zero simultaneously, such that the following conditions are satisfied:

$$z_j^* \in N(\bar{z}^k, \Xi_j \cap \Theta_j), \ \text{where} \ \Theta_j := \Omega_j^a \cap \Omega_j^b \ \ j = 0, \ldots, k-1, \tag{4.85}$$

$$\alpha_i^k c_i(\bar{z}^k) = 0, \ \ i = 1, \ldots, s, \tag{4.86}$$

$$-\sum_{j=0}^{k} z_j^* \in \lambda^k \partial \varphi_0(\bar{z}^k) + \sum_{i=1}^{s} \alpha_i^k \nabla c_i(\bar{z}^k) + \nabla e(\bar{z}^k)^* p, \tag{4.87}$$

where $e(z) := (e_0^x(z), \ldots, e_{k-1}^x(z), e_0^y(z), \ldots, e_{k-1}^y(z), e_0^u(z), \ldots, e_{k-1}^u(z), d_0(z), \ldots, d_{k-1}(z)) \in \mathbb{R}^{4kn}$ and $p = (p_1, \ldots, p_k) \in \mathbb{R}^{4kn}$. The intersection rule from [52, Theorem 2.16] tells us that

$$N(\bar{z}^k, \Xi_j \cap \Theta_j) \subset N(\bar{z}^k, \Xi_j) + N(\bar{z}^k, \Theta_j) \ \ \text{if} \ \ N(\bar{z}^k, \Xi_j) \cap (-N(\bar{z}^k, \Theta_j)) = \{0\}. \tag{4.88}$$

We have to check the fulfillment of the qualification condition in (4.88). To proceed, pick any $\widetilde{z}_j^* \in N(\bar{z}^k, \Xi_j) \cap (-N(\bar{z}^k, \Theta_j))$ and observe by (4.81) that its diagonal components satisfy

$$\big(\widetilde{x}_{jj}^*, \widetilde{u}_{jj}^*, \widetilde{a}_{jj}^*, \widetilde{b}_{jj}^*, -\widetilde{X}_{jj}^*, \widetilde{U}_{jj}^* \big) \in N_{\operatorname{gph} F_h} \Big(\bar{x}_j, \bar{y}_j, \bar{u}_j, \bar{a}_j, \bar{b}_j, \frac{\bar{x}_{j+1} - \bar{x}_j}{-h_k} \Big) \times \{0\}, \tag{4.89}$$

$$-\big(\widetilde{x}_{jj}^*, \widetilde{u}_{jj}^*, \widetilde{X}_{jj}^*, \widetilde{U}_{jj}^*, \widetilde{a}_{jj}^*, \widetilde{b}_{jj}^* \big) \in \{0\} \times \{0\} \times \{0\} \times \{0\} \times N\Big(\bar{a}_j, \bar{b}_j, A \times B \Big), \tag{4.90}$$

for all $j = 0, ..., k-1$ with its other components equal zero. We get from (4.89) and (4.90) that $\widetilde{x}^*_{jj} = 0$, $\widetilde{u}^*_{jj} = 0$, $\widetilde{X}^*_{jj} = 0$, $\widetilde{U}^*_{jj} = 0$. Substituting this into (4.89) and using the coderivative definition (4.64) give us

$$(0, 0, \widetilde{a}^*_{jj}, \widetilde{b}^*_{jj}) \in D^* F_h\Big(\bar{x}_j, \bar{y}_j, \bar{u}_j, \bar{a}_j, \bar{b}_j, \frac{\bar{x}_{j+1} - \bar{x}_j}{-h_k}\Big)(0), \; j = 0, ..., k-1, \tag{4.91}$$

Then we deduce directly from the coderivative estimate (4.69) for the velocity mapping F_h in (4.67) under the imposed PLICQ that $\widetilde{a}^*_{jj} = 0$ and $\widetilde{b}^*_{jj} = 0$ for all $j = 0, ..., k-1$. It shows that $\widetilde{z}^*_j = 0$ for such indices j, and therefore the qualification condition (4.88) is verified. This allows us to use the intersection formula in (4.88) and then, applying it to (4.85) with taking into account the structures of the sets Ω^a_j and Ω^b_j from (4.83) and (4.84), respectively, arrive at the inclusions

$$z^*_j \in N(\bar{z}^k, \Xi_j) + \big[N(\bar{z}^k, \Omega^a_j) + N(\bar{z}^k, \Omega^b_j)\big], \; j = 0, \ldots, k-1. \tag{4.92}$$

Furthermore, by the structures of the sets in (4.81)–(4.84), we find ψ^{ak}_j and ψ^{bk}_j satisfying the normal cone inclusions in (4.79) and such that the obtained inclusions in (4.92) are equivalent to

$$\big(x^*_{jj}, y^*_{jj}, u^*_{jj}, a^*_{jj} - \psi^{ak}_j, b^*_{jj} - \psi^{bk}_j, -X^*_{jj}\big) \in N_{\operatorname{gph} F_h}\Big(\bar{x}_j, \bar{y}_j, \bar{u}_j, \bar{a}_j, \bar{b}_j, \frac{\bar{x}_{j+1} - \bar{x}_j}{-h_k}\Big), \tag{4.93}$$

$j = 0, \ldots, k-1$, while the other components of z^*_j are zero. Similarly we have that the vectors $(x^*_{0k}, y^*_{0k}, u^*_{0k}, a^*_{0k}, b^*_{0k})$ determined by the normal cone to Ξ_k might be the only nonzero components of z^*_k. This gives us the representation

$$\begin{aligned}
-\sum_{j=0}^{k} z^*_j =& (-x^*_{00} - x^*_{0k}, -x^*_{11}, \ldots, -x^*_{(k-1)(k-1)}, 0, -y^*_{00} - y^*_{0k}, -y^*_{11}, \ldots, -y^*_{(k-1)(k-1)}, 0, \\
& - u^*_{00} - u^*_{0k}, -u^*_{11}, \ldots, -u^*_{(k-1)(k-1)}, 0, -a^*_{00} - a^*_{0k}, -a^*_{11}, \ldots, -a^*_{(k-1)(k-1)}, 0, \\
& - b^*_{00} - b^*_{0k}, -b^*_{11}, \ldots, -b^*_{(k-1)(k-1)}, 0, -X^*_{00}, \ldots, -X^*_{(k-1)(k-1)}, 0, \ldots, 0).
\end{aligned}$$

For the other terms in (4.87) we get

$$\begin{aligned}
&\sum_{i=1}^{s} \alpha^k_i \nabla c_i(\bar{z}^k) \\
&= (0, \ldots, -\sum_{i=1}^{d} \alpha_i \nabla_x g_i(\bar{x}^k - \bar{u}^k), 0, \ldots, 0, 0, \ldots, \sum_{i=1}^{d} \alpha_i \nabla_u g_i(\bar{x}^k - \bar{u}^k_k), 0, \ldots, 0, 0 \ldots, 0),
\end{aligned}$$

$$\begin{aligned}
\nabla e(\bar{z}^k)^* p = \big(& - p^x_1 - \xi^x_0 p^d_1, p^x_1 - p^x_2 - \xi^x_1 p^d_2, \ldots, p^x_{k-1} - p^x_k - \xi^x_{k-1} p^d_k, p^x_k, \\
& - p^y_1, p^y_1 - p^y_2, \ldots, p^y_{k-1} - p^y_k, p^y_k, -p^u_1, p^u_1 - p^u_2, \ldots, p^u_{k-1} - p^u_k, p^u_k, \\
& - \xi^b_0 p^d_1, -\xi^b_1 p^d_2, \ldots, -\xi^b_{k-1} p^d_k, 0, -h_k p^x_1, \ldots, -h_k p^x_k, -h_k p^y_1 + p^d_1, \ldots, -h_k p^y_k + p^d_k, \\
& - h_k p^u_1, \ldots, -h_k p^u_k\big).
\end{aligned}$$

Using the summation structure of the function φ_0 and applying there the subdifferential sum rule from [52, Theorem 2.19] lead us the subdifferential inclusion

$$\partial\varphi_0(\bar{z}^k) \subset \partial\varphi(\bar{x}_k^k) + h_k \sum_{j=0}^{k-1} \partial l\big(\bar{x}_j, \bar{y}_j, \bar{u}_j, \bar{a}_j, \bar{b}_j, \bar{X}_j, \bar{Y}_j, \bar{U}_j\big) + \sum_{j=0}^{k-1} \nabla\rho_j(\bar{z}^k),$$

where $\rho_j(y) := \frac{1}{2}\int\limits_{t_j}^{t_{j+1}} \|(X_j, Y_j, U_j, a_j^k, b_j^k) - (\dot{\bar{x}}(t), \dot{\bar{y}}(t), \dot{\bar{u}}(t), \bar{a}(t), \bar{b}(t))\|^2\, dt$. It is easy to see that

$$\nabla\rho_j(\bar{z}^k) = (0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, \theta_j^a, 0 \ldots, 0, 0, \ldots, 0, \theta_j^b, 0 \ldots, 0, 0,$$
$$\theta_j^x, 0, \ldots, 0, 0 \ldots, 0, \theta_j^y, 0, \ldots, 0, 0, \ldots, 0, \theta_j^u, 0, \ldots, 0),$$

$$\partial l(\bar{x}_j, \bar{y}_j, \bar{u}_j, \bar{a}_j, \bar{b}_j, \bar{X}_j, \bar{Y}_j, \bar{U}_j)$$
$$= (0, \ldots, 0, w_j^x, 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, w_j^u, 0, \ldots, 0, 0, \ldots, 0, w_j^a, 0, \ldots, 0, 0, \ldots,$$
$$0, w_j^b, 0, \ldots, 0, 0, \ldots, 0, v_j^x, 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, v_j^u, 0, \ldots, 0).$$

Since $\partial\varphi(\bar{x}_k^k)$ can be written in the form $(0, \ldots, 0, v^k, 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0)$, we arrive at the representation of the term $\lambda^k\partial\varphi_0(\bar{z}^k)$ in (4.87) as

$$\lambda^k(h_k w_0^x, \ldots, h_k w_{k-1}^x, v^k, 0, \ldots, 0, h_k w_0^u, \ldots, h_k w_{k-1}^u, 0, \theta_0^a + h_k w_0^a, \ldots, \theta_{k-1}^a + h_k w_{k-1}^a, 0,$$
$$\theta_0^b + h_k w_0^b, \ldots, \theta_{k-1}^b + h_k w_{k-1}^b, 0, \theta_0^x + h_k v_0^x, \ldots, \theta_{k-1}^x + h_k v_{k-1}^x, \theta_0^y, \ldots, \theta_{k-1}^y,$$
$$\theta_0^u + h_k v_0^u, \ldots, \theta_{k-1}^u + h_k v_{k-1}^u).$$

Combining all the above decomposes the inclusion in (4.87) into the following equalities:

$$-x_{00}^* - x_{0k}^* = \lambda^k h_k w_0^x - p_1^x - \xi_0^x p_1^d,$$

$$-x_{jj}^* = \lambda^k h_k w_j^x + p_j^x - p_{j+1}^x - \xi_j^x p_{j+1}^d, \ \ j = 1, \ldots, k-1, \tag{4.94}$$

$$0 = \lambda v^k - \sum_{i=1}^{s} \alpha_i^k \nabla_x g_i(\bar{x}^k - \bar{u}^k) + p_k^x, \tag{4.95}$$

$$-y_{00}^* = -p_1^y + p_1^d, \quad -y_{jj}^* = p_j^y - p_{j+1}^y + p_{j+1}^d, \ \ j = 1, \ldots, k-1, \quad p_k^y = 0, \tag{4.96}$$

$$-u_{00}^* - u_{0k}^* = \lambda^k h_k w_0^u - p_1^u, \tag{4.97}$$

$$-u_{jj}^* = \lambda^k h_k w_j^u + p_j^u - p_{j+1}^u, \ \ j = 1, \ldots, k-1, \tag{4.98}$$

$$0 = \sum_{i=1}^{s} \alpha_i^k \nabla_u g_i(\bar{x}^k - \bar{u}^k) + p_k^u, \tag{4.99}$$

$$-a_{00}^* = \lambda^k \theta_0^a + \lambda^k h_k w_0^a, \tag{4.100}$$

$$-a_{jj}^* = \lambda^k \theta_j^a + \lambda^k h_k w_j^a, \ \ j = 1, \ldots, k-1, \tag{4.101}$$

$$-b_{00}^* = \lambda^k \theta_0^b + \lambda^k h_k w_0^b - \xi_j^b p_1^d,$$

$$-b^*_{jj} = \lambda^k\theta^b_j + \lambda^k h_k w^b_j - \xi^b_j p^d_{j+1},\ \ j = 1,\ldots,k-1, \quad 0 = p^b_k, \tag{4.102}$$

$$-X^*_{jj} = \lambda^k(h_k v^x_j + \theta^x_j) - h_k p^x_{j+1},\ \ j = 0,\ldots,k-1, \tag{4.103}$$

$$0 = \lambda^k\theta^y_j - h_k p^y_{j+1} + p^d_{j+1},\ \ j = 0,\ldots,k-1, \tag{4.104}$$

$$0 = \lambda^k(h_k v^u_j + \theta^u_j) - h_k p^u_{j+1},\ \ j = 0,\ldots,k-1, \tag{4.105}$$

We deduce from (4.94), (4.96), (4.98), (4.101), (4.102), and (4.103) that

$$\Big(\frac{p^x_{j+1}-p^x_j}{h_k} - \lambda^k w^x_j + h_k^{-1}\xi^x_j p^d_{j+1}, \frac{p^y_{j+1}-p^y_j}{h_k} - \frac{p^d_{j+1}}{h_k}, \frac{p^u_{j+1}-p^u_j}{h_k} - \lambda^k w^u_j, -\frac{1}{h_k}\lambda^k\theta^a_j - \lambda^k w^a_j,$$
$$-\frac{1}{h_k}\lambda^k\theta^b_j - \lambda^k w^b_j + \xi^b_j\frac{p^d_{j+1}}{h_k}, p^x_{j+1} - \lambda^k(v^x_j + h_k^{-1}\theta^x_j)\Big) = h_k^{-1}(x^*_{jj}, y^*_{jj}, u^{k*}_{jj}, a^*_{jj}, b^*_{jj}, -X^*_{jj}).$$

Then the conditions in (4.78) follow from (4.93), and the conditions in (4.74), (4.75),(4.76) (4.77) follow from (4.86), (4.95),(4.96), (4.99), (4.104), (4.105) respectively.

To verify finally the nontriviality condition (4.73) of the theorem, denote $p_0 := (x^*_{0k}, y^*_{0k}, u^*_{0k}, a^*_{0k}, b^*_{0k})$ and suppose by contraposition that $\lambda^k = 0$, $\alpha^k = 0$, $p^y_0 = 0$, $p^u_0 = 0$, $p^a_0 = 0$, $p^b_0 = 0$, $p^x_j = 0$, $\psi^k_j = 0$ for $j = 0,\ldots,k-1$, and $p^d_j = 0$ for $j = 0,\ldots,k$. Then we get the implications

$$(4.95) \Longrightarrow p^x_k = 0,\ \ \text{i.e.,}\ \ p^x_j = 0,\ \ j = 0,\ldots,k,$$
$$(4.94) \Longrightarrow x^*_{jj} = 0\ \ \text{and}\ \ (4.103) \Longrightarrow X^*_{jj} = 0,\ \ j = 0,\ldots,k-1,$$
$$(4.105),(4.104) \Longrightarrow p^u_j = 0, p^a_j = 0, p^b_j = 0, p^y_j = 0,\ \ j = 0,\ldots,k,$$
$$(4.96),(4.98),(4.101),(4.102) \Longrightarrow y^*_{jj} = 0, u^*_{jj} = 0, a^*_{jj} = 0, b^*_{jj} = 0,\ \ j = 0,\ldots,k-1.$$

We know that all the components of z^*_j, which different from $(x^*_{jj}, y^*_{jj}, u^*_{jj}, a^*_{jj}, b^*_{jj}, X^*_{jj})$, are zero for $j = 0,\ldots,k-1$. Hence $z^*_j = 0$ for $j = 0,\ldots,k-1$. Conclude similarly that $z^*_k = 0$ due to

$$(x^*_{0k}, y^*_{0k}, u^*_{0k}, a^*_{0k}, b^*_{0k}) = (p^x_0, p^y_0, p^u_0, p^a_0, p^b_0, p^d_0) = (0,0,0,0,0,0),$$

and thus $z^*_j = 0$ for all $j = 0,\ldots,k$. This contradicts the nontriviality condition in the mathematical program (MP), and therefore verifies the claimed nontriviality (4.73), which completes the proof of the theorem. ■

The next theorem is the main result of this section. It provides necessary optimality conditions for problems (P_k) as formulated in Section 4.3 expressed *entirely in terms of the given data.* We derive this result by combining necessary optimal conditions of Theorem 4.6.1 with the coderivative calculations of Theorem 4.5.1 addressing the specific form of the velocity mapping F_h in (P_k). In this way we obtain a novel *discrete-time Volterra condition* as a part of primal-dual dynamic relationships for discrete approximations of controlled integro-differential sweeping processes.

Theorem 4.6.2 (necessary conditions for discretized integro-differential sweeping control systems). *Let $\bar{z}^k = (\bar{x}^k, \bar{y}^k, \bar{u}^k, \bar{a}^k, \bar{b}^k)$ be an optimal solution to the discrete-time problem (P_k), where $k \in I\!N$ is sufficiently large. In addition to the assumptions of Theorem* 4.6.1*, suppose that the functions g_i, $i = 1, \ldots, s$, are twice continuously differentiable around $\bar{x}^k_j - \bar{u}^k_j$, $j = 0, \ldots, k-1$, with the Jacobian matrix of full rank therein. Then there exist dual elements (λ^k, p^k) as in Theorem* 4.6.1 *together with vectors $\eta^k_j \in \mathbb{R}^s_+$ as $j = 0, \ldots, k$ and $\sigma^k_j \in \mathbb{R}^s$ as $j = 0, \ldots, k-1$ satisfying following conditions:*

Primal-Dual Dynamic Relationships*: for all $j = 0, \ldots, k-1$ we have*

$$\frac{\bar{x}^k_{j+1} - \bar{x}^k_j}{h_k} + f_1(\bar{a}^k_j, \bar{x}^k_j) + \bar{y}^k_j + h_k f_2(\bar{b}^k_j, \bar{x}^k_j) = \sum_{i \in I(\bar{x}^k_j - \bar{u}^k_j)} \eta^k_{ji} \nabla g_i(\bar{x}^k_j - \bar{u}^k_j), \tag{4.106}$$

$$\begin{aligned}
&\frac{p^x_{j+1} - p^x_j}{h_k} - \lambda^k w^x_j + h_k^{-1} \xi^x_j p^d_{j+1} = \nabla_x f_1(\bar{a}^k_j, \bar{x}^k_j)^* (\lambda^k (v^x_j + h_k^{-1} \theta^x_j) - p^x_{j+1}) \\
&+ h_k \nabla_x f_2(\bar{b}^k_j, \bar{x}^k_j)^* (\lambda^k (v^x_j + h_k^{-1} \theta^x_j) - p^x_{j+1}) \\
&- \Big(\sum_{i=1}^d \eta^k_{ji} \nabla^2 g_i(\bar{x}^k_j - \bar{u}^k_j) \Big) (\lambda^k (v^x_j + h_k^{-1} \theta^x_j) - p^x_{j+1}) \\
&- \sum_{i=1}^d \sigma^k_{ji} \nabla g_i(\bar{x}^k_j - \bar{u}^k_j)), \quad \textit{where } \xi^x_j := \nabla_x f_2(\bar{b}^k_j, \bar{x}^k_j),
\end{aligned} \tag{4.107}$$

$$\frac{p^y_{j+1} - p^y_j}{h_k} - \frac{p^d_{j+1}}{h_k} = (\lambda^k (v^x_j + h_k^{-1} \theta^x_j) - p^x_{j+1}), \tag{4.108}$$

$$\frac{p^u_{j+1} - p^u_j}{h_k} - \lambda^k w^u_j = \Big(\sum_{i=1}^d \eta^k_{ji} \nabla^2 g_i(\bar{x}^k_j - \bar{u}^k_j) \Big) (\lambda^k (v^x_j + h_k^{-1} \theta^x_j) - p^x_{j+1}) + \sum_{i=1}^d \sigma^k_{ji} \nabla g_i(\bar{x}^k_j - \bar{u}^k_j)), \tag{4.109}$$

$$-\frac{1}{h_k} \lambda^k \theta^a_j - \lambda^k w^a_j - \frac{1}{h_k} \psi^{ak}_j = \nabla_a f_1(\bar{a}^k_j, \bar{x}^k_j)^* (\lambda^k (v^x_j + h_k^{-1} \theta^x_j) - p^x_{j+1}), \tag{4.110}$$

$$\begin{aligned}
-\frac{1}{h_k} \lambda^k \theta^b_j - \lambda^k w^b_j + h_k^{-1} \xi^b_j p^d_{j+1} - \frac{1}{h_k} \psi^{bk}_j &= h_k \nabla_b f_2(\bar{b}^k_j, \bar{x}^k_j)^* (\lambda^k (v^x_j + h_k^{-1} \theta^x_j) - p^x_{j+1}), \\
&\textit{where } \xi^b_j := \nabla_b f_2(\bar{b}^k_j, \bar{x}^k_j),
\end{aligned} \tag{4.111}$$

$$\psi^{ak}_j \in N_A(\bar{a}^k_j) \quad \textit{and} \quad \psi^{bk}_j \in N_B(\bar{b}^k_j). \tag{4.112}$$

Transversality Conditions*:*

$$-p^x_k \in \lambda^k \partial \varphi(\bar{x}^k_k) - \sum_{i=1}^s \eta^k_{ki} \nabla_x g_i(\bar{x}^k - \bar{u}^k), \tag{4.113}$$

$$-p^{uk}_k = \sum_{i=1}^s \eta^k_{ki} \nabla_u g_i(\bar{x}^k_k - \bar{u}^k_k), \;\; p^{yk}_k = 0. \tag{4.114}$$

Complementary Slackness: *for all $j = 0,\ldots,k-1$ and $i = 1,\ldots,s$ we have the implications*

$$g_i(\bar{x}_j^k - \bar{u}_j^k) > 0 \Longrightarrow \eta_{ji}^k = 0, \tag{4.115}$$

$$\Big[g_i(\bar{x}_j^k - \bar{u}_j^k) > 0, \quad or \quad \eta_{ji}^k = 0 \quad and \quad \langle \nabla g_i(\bar{x}_j^k - \bar{u}_j^k), \lambda^k(v_j^x + h_k^{-1}\theta_j^x) - p_{j+1}^x\rangle > 0\Big] \Longrightarrow \sigma_{ji}^k = 0, \tag{4.116}$$

$$\Big[g_i(\bar{x}_j^k - \bar{u}_j^k) = 0, \;\; \eta_{ji}^k = 0, \;\; \langle \nabla g_i(\bar{x}_j^k - \bar{u}_j^k), \lambda^k(v_j^x + h_k^{-1}\theta_j^x) - p_{j+1}^x\rangle < 0\Big] \Longrightarrow \sigma_{ji}^k \geq 0, \tag{4.117}$$

$$g_i(\bar{x}_k^k - \bar{u}_k^k) > 0 \Longrightarrow \eta_{ki}^k = 0, \tag{4.118}$$

$$\eta_{ji}^k > 0 \Longrightarrow \langle \nabla g_i(\bar{x}_j^k - \bar{u}_j^k), \lambda^k(v_j^x + h_k^{-1}\theta_j^x) - p_{j+1}^x\rangle = 0. \tag{4.119}$$

Nontriviality Condition: *with $\psi^k = (\psi^{ak}, \psi^{bk})$ we have*

$$\lambda^k + \sum_{j=0}^{k}\|p_j^{kd}\| + \|p_0^{ku}\| + \sum_{j=0}^{k-1}\|\psi_j^k\| \neq 0, \tag{4.120}$$

Proof. Using the coderivative construction (4.64), for all $j = 0,\ldots,k-1$ we rewrite the discrete Euler-Lagrange inclusions (4.78) of Theorem 4.6.1 in the form

$$\begin{aligned}
&\Big(\frac{p_{j+1}^x - p_j^x}{h_k} - \lambda^k w_j^x + h_k^{-1}\xi_j^x p_{j+1}^d, \frac{p_{j+1}^y - p_j^y}{h_k} - \frac{p_{j+1}^d}{h_k}, \frac{p_{j+1}^u - p_j^u}{h_k} - \lambda^k w_j^u,\\
&-\frac{1}{h_k}\lambda^k\theta_j^a - \lambda^k w_j^a - \frac{1}{h_k}\psi_j^{ka}, -\frac{1}{h_k}\lambda^k\theta_j^b - \lambda^k w_j^b + h_k^{-1}\xi_j^b p_{j+1}^d - \frac{1}{h_k}\psi_j^{kb}, p_{j+1}^x - \lambda^k(v_j^x + h_k^{-1}\theta_j^x)\Big)\\
&\in D^*F_h\Big(\bar{x}_j^k, \bar{y}_j^k, \bar{u}_j^k, \bar{a}_j^k, \bar{b}_j^k, \frac{\bar{x}_{j+1}^k - \bar{x}_j^k}{-h_k}\Big)(\lambda^k(v_j^x + h_k^{-1}\theta_j^x) - p_{j+1}^x).
\end{aligned} \tag{4.121}$$

It follows from the discrete dynamics (4.52), representation (4.20) of the velocity mapping, and the structure of the moving set in (4.3) that there exist vectors $\eta_j^k \in \mathbb{R}_+^m$ as $j = 0,\ldots,k-1$ such that the conditions in (4.106) and (4.115) are satisfied. Employing further the coderivative calculation in (4.69) and (4.70) of Theorem 4.5.1 with $x := \bar{x}_j^k$, $y := \bar{y}_j^k$, $u := \bar{u}_j^k$, $a := \bar{a}_j^k$, $b := \bar{b}_j^k$, $w := \dfrac{\bar{x}_{j+1}^k - \bar{x}_j^k}{-h_k}$, and $z = (\lambda^k(v_j^x + h_k^{-1}\theta_j^x) - p_{j+1}^x)$ allows us to find σ_j^k, $j = 0,\ldots,k-1$, for which conditions (4.107), (4.108), (4.109), (4.110), (4.111), (4.116), and (4.117) hold. Define $\eta_k^k := \alpha^k$ and observe that $\eta_j^k \in \mathbb{R}_+^m$ for all $j = 0,\ldots,k$. In this way we deduce the transversality conditions (4.113) and (4.114) from (4.75) and (4.76), while (4.118) follows from (4.86) and the definition of η_k^k. Note also that (4.119) follows directly from (4.70).

It remains to verify the fulfillment of the nontriviality condition (4.120). To this end, deduce first from (4.73) and the constructions above that

$$\lambda^k + \|\eta_k^k\| + \sum_{j=0}^{k-1}\|p_j^{kx}\| + \sum_{j=0}^{k}\|p_j^{kd}\| + \|p_0^{ky}\| + \|p_0^{ku}\| + \sum_{j=0}^{k-1}\|\psi_j^k\| \neq 0. \tag{4.122}$$

Assume now that (4.87) is violated, i.e., $\lambda^k = 0$, $p_j^{kd} = 0$ for all $j = 0,\ldots,k$, $\psi_j^k = 0$ for all $j = 0,\ldots,k-1$, and $p_0^{ku} = 0$. Then it follows from (4.77) that $p_j^{uk} = 0$ for all $j = 0,\ldots,k$

and $p_j^{yk} = 0$ for all $j = 1, \ldots, k$. Furthermore, (4.75) yields $\sum\limits_{i=1}^{s} \eta_{ki}^k \nabla_u g_i(\bar{x}_k^k - \bar{u}_k^k) = 0$, and hence $p_k^{xk} = 0$. Employing (4.109) tells us that

$$\Big(\sum_{i=1}^{d} \eta_{ji}^k \nabla_u^2 g_i(\bar{x}_j^k - \bar{u}_j^k)\Big)(\lambda^k(v_j^x + h_k^{-1}\theta_j^x) - p_{j+1}^x) + \sum_{i=1}^{d} \sigma_{ji}^k \nabla_u g_i(\bar{x}_j^k - \bar{u}_j^k)) = 0, \quad j = 0, \ldots, k-1.$$

Combining the latter with (4.107) and $p_k^{xk} = 0$ ensures that $p_j^{xk} = 0$ whenever $j = 0, \ldots, k-1$. To complete the proof of the theorem, we deduce from (4.108) that $p_0^{yk} = 0$. This contradicts the fulfillment of (4.122) and hence verifies the nontriviality condition (4.120). ∎

We conclude this section with presenting *maximization conditions* for optimal solutions that are direct consequences of the normal cone inclusions for adjoint functions under certain additional assumptions.

Corollary 4.6.1 (discrete maximization conditions). *In addition to the assumptions of Theorem* 4.6.2*, suppose that the normal cones* $N_A(\bar{a}_j^k)$ *and* $N_B(\bar{b}_j^k)$ *in* (4.112) *are tangentially generated, i.e., they are dual/polar to some tangent sets as* $N_A(\bar{a}_j^k) = T_A(\bar{a}_j^k)^*$ *and* $N_B(\bar{b}_j^k, B) = T_B(\bar{b}_j^k)^*$ *for all* $j = 0, \ldots, k-1$*. Then for such indices* j *the following maximization conditions are satisfied:*

Local Maximization Conditions*:*

$$\langle \psi_j^{ka}, \bar{a}_j^k \rangle = \max_{v \in T_A(\bar{a}_j^k)} \langle \psi_j^{ka}, v \rangle, \quad \langle \psi_j^{kb}, \bar{b}_j^k \rangle = \max_{v \in T_B(\bar{b}_j^k)} \langle \psi_j^{kb}, v \rangle. \tag{4.123}$$

Moreover, the convexity of A *and* B *ensures the fulfillment of the* Global Maximization Conditions*:*

$$\langle \psi_j^{ka}, \bar{a}_j^k \rangle = \max_{v \in A} \langle \psi_j^{ka}, v \rangle, \quad \langle \psi_j^{kb}, \bar{b}_j^k \rangle = \max_{v \in B} \langle \psi_j^{kb}, v \rangle. \tag{4.124}$$

Proof. The local maximization conditions in (4.124) follow from (4.112) due to the assumed normal-tangent duality. The convexity of the sets A and B yields the global maximization in (4.123), since the limiting normal cone (4.63) reduces to the classical normal cone of convex analysis. ∎

4.7 Necessary conditions for integro-differential processes

This section establishes the main result of the paper providing—*for the first time in the literature*—efficient necessary optimality conditions, expressed entirely via the given data, for local minimizers (in the sense of Definition 4.2.1 (ii)), of the original optimal control problem (P) governed by the sweeping integro-differential inclusions (4.2) with the pointwise mixed state-control constraints (4.4). The derivation of these conditions presented below is based on the results obtained in the previous sections as well as on the appropriate properties of the generalized differential constructions of variational analysis reviewed in Section 4.5 that allow us furnishing the passage to the limit as $k \to \infty$ from the necessary optimality conditions for discrete approximations. For simplicity, we suppose below that the running cost in (4.6) does not depend on t.

Theorem 4.7.1 (optimality conditions for integro-differential sweeping processes). *Consider a relaxed intermediate local minimizer* $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot), \bar{b}(\cdot))$ *of problem* (P) *and suppose in addition to the assumptions of Theorem* 4.6.2 *that the running cost in* (4.6) *is represented as*

$$l_0(\bar{x}, \bar{u}, \bar{a}, \bar{b}, \dot{\bar{x}}, \dot{\bar{u}}) = l_1(\bar{x}, \bar{u}, \bar{a}, \bar{b}, \dot{x}) + l_2(\dot{u}), \tag{4.125}$$

where l_2 *is differentiable on* $\mathbb{R}^n$ *with the estimates*

$$\|\nabla_{\dot{u}} l_2(\dot{u})\| \le L\|\dot{u}\| \quad \text{and} \quad \|\nabla_{\dot{u}} l_2(\dot{u}_1) - \nabla_{\dot{u}} l_2(\dot{u}_2)\| \le L\|\dot{u}_1 - \dot{u}_2\|.$$

Then there exist a multiplier $\lambda \ge 0$ *and functions* $p(\cdot) = (p^x(\cdot), p^y(\cdot), p^u(\cdot)) \in W^{1,2}([0,T], \mathbb{R}^{3n})$, $w(\cdot) = (w^x(\cdot), 0, w^u(\cdot), w^a(\cdot), w^b(\cdot)) \in L^2([0,T], \mathbb{R}^{3n+m+d})$, *and*
$v(\cdot) = (v^x(\cdot), 0, v^u(\cdot)) \in L^2([0,T], \mathbb{R}^{3n})$ *satisfying the subdifferential inclusion*

$$(w(t), v(t)) \in \text{co}\, \partial l(\bar{x}(t), \bar{y}(t), \bar{u}(t), \bar{a}(t), \bar{b}(t), \dot{\bar{x}}(t), \dot{\bar{y}}(t), \dot{\bar{u}}(t)) \quad \text{a.e.} \quad t \in [0,T], \tag{4.126}$$

as well as measure $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathcal{C}([0,T], \mathbb{R}^n)^*$ *for which the following conditions hold:*
Primal-Dual Dynamic Relationships*:*

$$\dot{\bar{x}}(t) + f_1(\bar{a}(t), \bar{x}(t)) + \bar{y}(t) = \sum_{i=1}^{s} \eta_i(t) \nabla g_i(\bar{x}(t) - \bar{u}(t)) \quad \text{a.e.} \quad t \in [0,T]; \tag{4.127}$$

$$g_i(\bar{x}(t) - \bar{u}(t)) > 0 \Longrightarrow \eta_i(t) = 0 \quad \text{a.e.} \quad t \in [0,T]; \tag{4.128}$$

$$\eta_i(t) > 0 \Longrightarrow \langle \nabla\, g_i(\bar{x}(t) - \bar{u}(t)), \lambda v^x(t) - q^x(t)\rangle = 0 \quad \text{a.e.} \quad t \in [0,T], \tag{4.129}$$

where the functions $\eta_i(\cdot) \in L^2([0,T], \mathbb{R}_+)$ *are uniquely determined by representation* (4.127) *for a.e.* $t \in [0,T]$ *while being well-defined at* $t = T$*;*

$$\dot{p}(t) = \lambda(w^x(\cdot), 0, w^u(\cdot)) + \Big(\nabla_x f_1(\bar{a}(t), \bar{x}(t))^*(\lambda v^x(t) - q^x(t)), \lambda v^x(t) - q^x(t), 0\Big), \tag{4.130}$$

$$q^u(t) = \lambda \nabla_{\dot{u}} l_2(\dot{\bar{u}}(t)), \tag{4.131}$$

where $q(\cdot) = (q^x(\cdot), q^y(\cdot), q^u(\cdot)) : [0,T] \to \mathbb{R}^{3n}$ *is of bounded variation on* $[0,T]$*, with* $q^y(\cdot)$ *being absolutely continuous on* $[0,T]$*, such that left-continuous representative of* $q(\cdot)$ *satisfies, for all* $t \in [0,T]$ *except at most a countable subset, the integral equation*

$$q(t) = p(t) - \int\limits_{[t,T]} \Big(-d\gamma(s) - \nabla_x f_2(\bar{b}(s), \bar{x}(s)) q^y(s)\, ds,\ q^y(s)\, ds,\ d\gamma(s)\Big). \tag{4.132}$$

EXTENDED VOLTERRA CONDITION:

$$\dot{q}^y(t) = \lambda v^x(t) - p^x(t) - \int\limits_{[t,T]} d\gamma(s) - \int\limits_{[t,T]} \nabla_x f_2(\bar{b}(s), \bar{x}(s)) q^y(s)\, ds + q^y(t) \quad a.e. \quad t \in [0,T]. \tag{4.133}$$

THE NORMAL CONE ADJOINT INCLUSIONS FOR CONTROL COMPONENTS: *for a.e.* $t \in [0,T]$ *we have*

$$\begin{cases} \psi^a(t) = -\lambda w^a(t) - \nabla_a f_1(\bar{a}(t), \bar{x}(t))^* (\lambda v^x(t) - q^x(t)) \in \operatorname{co} N_A(\bar{a}(t)), \\ \psi^b(t) = -\lambda w^b(t) + \nabla_b f_2(\bar{b}(s), \bar{x}(s)) q^y(s) \in \operatorname{co} N_B(\bar{b}(t)). \end{cases} \tag{4.134}$$

LOCAL MAXIMIZATION CONDITIONS: *assuming that the normal cones in* (4.134) *are tangentially generated, i.e., they are dual/polar to some tangent sets* $N_A(\bar{a}(t)) = T_A(\bar{a}(t))^*$ *and* $N_B(\bar{b}(t)) = T_B(\bar{b}(t))^*$, *yields*

$$\langle \psi^a(t), \bar{a}(t) \rangle = \max_{v \in T_A(\bar{a}(t))} \langle \psi^a(t), v \rangle, \quad \langle \psi^b(t), \bar{b}(t) \rangle = \max_{v \in T_B(\bar{b}(t))} \langle \psi^b(t), v \rangle. \tag{4.135}$$

If furthermore the sets A *and* B *are convex, then we have the*
GLOBAL MAXIMIZATION CONDITIONS :

$$\langle \psi^a(t), \bar{a}(t) \rangle = \max_{v \in A} \langle \psi^a(t), v \rangle, \quad \langle \psi^b(t), \bar{b}(t) \rangle = \max_{v \in B} \langle \psi^b(t), v \rangle. \tag{4.136}$$

RIGHT ENDPOINT CONDITIONS:

$$-p^x(T) + \sum_{i \in I(\bar{x}(T) - \bar{u}(T))} \eta_i(T) \nabla g_i(\bar{x}(T) - \bar{u}(T)) \in \lambda \partial \varphi(\bar{x}(T)), \ p^y(T) = 0, \tag{4.137}$$

$$-p^u(T) = \sum_{i \in I(\bar{x}(T) - \bar{u}(T))} \eta_i(T) \nabla g_i(\bar{x}(T) - \bar{u}(T)), \ p^a(T) = 0, \ p^b(T) = 0, \tag{4.138}$$

$$-\sum_{i \in I(\bar{x}(T) - \bar{u}(T))} \eta_i(T) \nabla g_i(\bar{x}(T) - \bar{u}(T)) \in N_C(\bar{x}(T) - \bar{u}(T)). \tag{4.139}$$

GENERAL NONTRIVIALITY CONDITION:

$$\lambda + \|q^u(0)\| + \|p(T)\| + \int\limits_0^T \|q^y(t)\|\, dt > 0. \tag{4.140}$$

Furthermore, the following implications hold while ensuring the ENHANCED NONTRIVIALITY:

$$\Big[g_i(x_0 - \bar{u}(0)) > 0, \ \ i = 1, \ldots, s \Big] \Longrightarrow \Big[\lambda + \|p(T)\| + \int\limits_0^T \|q^y(t)\| dt > 0 \Big], \tag{4.141}$$

$$\Big[g_i(\bar{x}(T) - \bar{u}(T)) > 0, \ \ i = 1, \ldots, s \Big] \Longrightarrow \Big[\lambda + \|q^u(0)\| + \int\limits_0^T \|q^y(t)\| dt > 0 \Big]. \tag{4.142}$$

Proof. We proceed by passing to the limit as $k \to \infty$ in the necessary optimality conditions for discrete approximation problems obtained in Theorem 4.6.2 with taking into account the strong convergence of discrete optimal solutions established in Theorem 4.4.1. Since some arguments in this procedure are rather similar to those used in [22, Theorem 8.1] and [20, Theorem 4.1] in a more special setting, we skip them for brevity while focusing on significantly new developments. Note, in particular, that the existence of the subgradient functions $(w^x(\cdot), 0, w^u(\cdot), w^a(\cdot), w^b(\cdot), v^x(\cdot), 0, v^u(\cdot))$ satisfying (4.126) can be checked as in [22].

Invoking now the vectors $\eta_j^k \in \mathbb{R}_+^m$ from Theorem 4.6.2, define the piecewise constant functions $\eta^k(\cdot)$ on $[0,T]$ by $\eta^k(t) := \eta_j^k$ as $t \in [t_j^k, t_{j+1}^k)$ with $\eta^k(T) := \eta_k^k$ and deduce from (4.106) for each $k \in I\!N$ we have

$$\dot{\bar{x}}^k(t) + f_1\big(\bar{a}^k(t), \bar{x}^k(t)\big) + \bar{y}^k(t) + h_k f_2\big(\bar{b}^k(t), \bar{x}^k(t)\big) = \sum_{i=1}^{s} \eta_i^k(t) \nabla g_i\big(\bar{x}^k(t) - \bar{u}^k(t)\big), \tag{4.143}$$

if $t \in (t_j^k, t_{j+1}^k)$. The feasibility of $\bar{z}(\cdot)$ in (P) tells us that $-\dot{\bar{x}}(t) \in N_C(\bar{x}(t) - \bar{u}(t)) + f_1(\bar{x}(t), \bar{a}(t)) + \bar{y}(t)$ for a.e. $t \in [0,T]$, where the closed-valued normal cone mapping $N_C(\cdot)$ is measurable by [71, Theorem 14.26]. The classical measurable selection result (see, e.g., [71, Corollary 14.6]) gives us nonnegative measurable functions $\eta_i(\cdot)$ on $[0,T]$ as $i = 1, \ldots, s$ for which the differential equation (4.127) is satisfied.

Let us further verify the dynamic complementarity slackness conditions in (4.128) while remembering that the our stranding assumptions yield the PLICQ; see Section 2. Using (4.143) and (4.127) gives us

$$\begin{aligned}
&\dot{\bar{x}}(t) - \dot{\bar{x}}^k(t) + \bar{y}(t) - \bar{y}^k(t) - h_k f_2\big(\bar{b}^k(t), \bar{x}^k(t)\big) \\
&= \sum_{i=1}^{s} \big[\eta_i(t) \nabla g_i\big(\bar{x}(t) - \bar{u}(t)\big) - \eta_i^k(t) \nabla g_i\big(\bar{x}^k(t) - \bar{u}^k(t)\big)\big] \\
&+ f_1\big(\bar{x}^k(t), \bar{a}^k(t)\big) - f_1\big(\bar{x}(t), \bar{a}(t)\big)
\end{aligned}$$

whenever $t \in (t_j^k, t_{j+1}^k)$ and $j = 0, \ldots, k-1$, which implies the estimate

$$\begin{aligned}
&\Big\| \sum_{i=1}^{s} \big[\eta_i(t) \nabla g_i\big(\bar{x}(t) - \bar{u}(t)\big) - \eta_i^k(t) \nabla g_i\big(\bar{x}^k(t) - \bar{u}^k(t)\big)\big] \Big\|_{L^2} \\
&\le \big\|\dot{\bar{x}}(t) - \dot{\bar{x}}^k(t)\big\|_{L^2} + \big\|\bar{y}(t) - \bar{y}^k(t)\big\|_{L^2} + \big\|f_1\big(\bar{x}(t), \bar{a}(t)\big) - f_1\big(\bar{x}^k(t), \bar{a}^k(t)\big)\big\|_{L^2} \\
&+ h_k \big\|f_2\big(\bar{b}^k(t), \bar{x}^k(t)\big)\big\|_{L^2}
\end{aligned}$$

on (t_j^k, t_{j+1}^k). Employing the strong convergence of $(\bar{x}^k(\cdot), \bar{y}^k(\cdot), \bar{u}^k(\cdot), \bar{a}^k(\cdot), \bar{b}^k(\cdot))$ to $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot), \bar{a}(\cdot), \bar{b}(\cdot))$, the smoothness of f_1 and f_2 and taking into account that $I(\bar{x}^k(\cdot) - \bar{u}^k(\cdot)) \subset I(\bar{x}(\cdot) - \bar{u}(\cdot))$ for $k \in I\!N$ sufficiently large, we get the strong convergence of the sequence

$$\sum_{i \in I(\bar{x}(t) - \bar{u}(t))} \big[\eta_i(t) \nabla g_i\big(\bar{x}(t) - \bar{u}(t)\big) - \eta_i^k(t) \nabla g_i\big(\bar{x}^k(t) - \bar{u}^k(t)\big)\big]$$

to zero in L^2 and thus its a.e. convergence on $[0,T]$ along some subsequence. On the other hand, it follows from (4.8) and (4.9) that

$$\begin{aligned}\eta_i^k(t) &\leq \frac{1}{M_1}\eta_i^k(t)\big\|\nabla g_i\big(\bar{x}^k(t)-\bar{u}^k(t)\big)\big\|\\ &\leq \frac{1}{M_1}\sum_{i\in I(\bar{x}^k(t)-\bar{u}^k(t))}\eta_i^k(t)\big\|\nabla g_i\big(\bar{x}^k(t)-\bar{u}^k(t)\big)\big\|\\ &\leq \frac{\beta}{M_1}\Big\|\sum_{i\in I(\bar{x}^k(t)-\bar{u}(t))}\eta_i^k(t)\nabla g_i\big(\bar{x}^k(t)-\bar{u}^k(t)\big)\Big\|\\ &\leq \frac{\beta}{M_1}\|\dot{\bar{x}}^k(t)\|+\frac{\beta}{M_1}\|\bar{y}^k(t)\|+\frac{\beta}{M_1}\big\|f_1\big(\bar{x}^k(t),\bar{a}^k(t)\big)\big\|+h_k\frac{\beta}{M_1}\big\|f_2\big(\bar{x}^k(t),\bar{b}^k(t)\big)\big\|,\end{aligned}$$

and so $\displaystyle\int_0^T\left[\eta_i^k(t)\right]^2dt\leq M$ for some constant $M>0$, which justifies in turn the boundedness of $\eta_i^k(\cdot)$ in L^2 due to the strong convergence of $(\bar{x}^k(\cdot),\bar{y}^k(\cdot),\bar{u}^k(\cdot),\bar{a}^k(\cdot),\bar{b}^k(\cdot))$ to $(\bar{x}(\cdot),\bar{y}(\cdot),\bar{u}(\cdot),\bar{a}(\cdot),\bar{b}(\cdot))$. It follows from the weak compactness of bounded sets in L^2 that there exists a function $\widetilde{\eta}(\cdot)$ such that a subsequence of $\{\eta_i^k(\cdot)\}$ weakly converges to $\widetilde{\eta}(\cdot)$. Employing again the aforementioned Mazur theorem gives us a sequence of convex combinations of the functions from $\{\eta_i^k(\cdot)\}$, which converges to $\widetilde{\eta}(\cdot)$ strongly in L^2, and hence pointwise for a.e $t\in[0,T)$ along a subsequence. Combining this with the a.e. convergence of $\displaystyle\sum_{i\in I(\bar{x}(t)-\bar{u}(t))}\eta_i^k(t)\nabla g_i\big(\bar{x}^k(t)-\bar{u}^k(t)\big)$ to $\displaystyle\sum_{i\in I(\bar{x}(t)-\bar{u}(t))}\eta_i(t)\nabla g_i\big(\bar{x}(t)-\bar{u}(t)\big)$, and using the positive linear independence of the gradients $\nabla g_i(x)$ on C, which is a consequence of the standing assumptions in Section 2, ensure that $\widetilde{\eta}(t)=\eta(t)$, and that $\eta^k(t)\to\eta(t)$ for a.e. $t\in[0,T)$. Invoking finally the strong convergence results from Theorem 4.4.1 and the complementary slackness condition (4.116) for the discrete problems (P^k), we arrive at the claimed complementary slackness condition in (4.128).

To proceed further, for each $t\in[t_j^k,t_{j+1}^k)$ with $j=0,\ldots,k-1$ consider the quintuples

$$\theta^k(t)=\big(\theta^{kx}(t),\theta^{ky}(t),\theta^{ku}(t),\theta^{ka}(t),\theta^{kb}(t)\big):=\Big(\frac{\theta_j^{kx}}{h_k},\frac{\theta_j^{ky}}{h_k},\frac{\theta_j^{ku}}{h_k},\frac{\theta_j^{ka}}{h_k},\frac{\theta_j^{kb}}{h_k}\Big),\tag{4.144}$$

where the components in (4.144) are defined in (4.71) and (4.72). It easy follows from these constructions and the convergence result of Theorem 4.4.1 that the sequence $\{\theta^k(\cdot)\}$ converges to 0 strongly in $L^2([0,T],\mathbb{R}^{3n+m+d})$, and hence we have that $\theta^k(t)\to 0$ as $k\to\infty$ for a.e. $t\in[0,T)$ along a subsequence (without relabeling).

Having p_j^k with $j=0,\ldots,k$ from Theorem 4.6.2, construct $q^k(\cdot)=(q^{kx}(\cdot),q^{ky}(\cdot),q^{ku}(\cdot))$ by setting $q^k(t_j^k):=p_j^k$ and then extending the quintuples piecewise linearly to $[0,T]$ for all $j=0,\ldots,k$. Define further $\sigma^k(t)$, $\psi^k(\cdot)=(\psi^{ak}(\cdot),\psi^{bk}(\cdot))$ and $\varsigma^k(t)$ on $[0,T]$ by

$$\sigma^k(t):=\sigma_j^k\ \text{ for }\ t\in[t_j^k,t_{j+1}^k),\ j=0,\ldots,k-1,\ \text{ with }\ \sigma^k(t_k^k):=0,\tag{4.145}$$

$$\psi^k(t):=\frac{1}{h_j^k}\psi_j^k\ \text{ for }\ t\in[t_j^k,t_{j+1}^k),\ j=0,\ldots,k-1,\ \text{ with }\ \psi^k(t_k^k):=0,\tag{4.146}$$

$$\varsigma^k(t) := \frac{p^{kd}_{j+1}}{h_k} \ \text{ for } \ t \in [t^k_j, t^k_{j+1}), \ j = 0, \ldots, k-1, \ \text{ with } \ \varsigma^k(t^k_k) := 0. \tag{4.147}$$

Considering the auxiliary functions

$$\vartheta^k(t) := \max\left\{t^k_j \mid t^k_j \le t, \ 0 \le j \le k\right\} \ \text{ for all } \ t \in [0,T], \ \ k \in \mathbb{N}, \tag{4.148}$$

it is obvious to see that $\vartheta^k(t)$ converge to t uniformly in $[0,T]$ as $k \to \infty$. It follows from (4.107)–(4.111) and the constructions above that for all $t \in (t^k_j, t^k_{j+1})$ and $j = 0, \ldots, k-1$ we get

$$\begin{aligned}
&\dot{q}^{kx}(t) - \lambda^k w^{kx}(t)\\
&= -\xi^x(\vartheta^k(t))\varsigma^k(t) + \nabla_x f_1\big(\bar{a}^k(\vartheta^k(t)), \bar{x}^k(\vartheta^k(t))\big)^*\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\big)\\
&+ h_k \nabla_x f_2\big(\bar{b}^k(\vartheta^k(t)), \bar{x}^k(\vartheta^k(t))\big)^*\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\big)\\
&- \sum_{i=1}^s \eta^k_i(t)\nabla^2_x g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\big)\\
&- \sum_{i=1}^s \sigma^k_i(t)\nabla_x g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big).
\end{aligned}$$

$$\dot{q}^{ky}(t) = \varsigma^k(t) + \lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k).$$

$$\begin{aligned}
&\dot{q}^{ku}(t) - \lambda^k w^{ku}(t)\\
&= \sum_{i=1}^s \eta^k_i(t)\nabla^2_u g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\big)\\
&+ \sum_{i=1}^s \sigma^k_i(t)\nabla_u g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big).
\end{aligned}$$

$$\begin{aligned}
&- \lambda^k\theta^{ka}(t) - \lambda^k w^{ka}(t) - \psi^{ka}(t)\\
&= \nabla_a f_1\big(\bar{a}^k(\vartheta^k(t)), \bar{x}^k(\vartheta^k(t))\big)^*\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\big).
\end{aligned}$$

$$\begin{aligned}
&- \lambda^k\theta^{kb}(t) - \lambda^k w^{kb}(t) - \psi^{kb}(t)\\
&= -\xi^b(\vartheta^k(t))\varsigma^k(t) + h_k\nabla_b f_2\big(\bar{a}^k(\vartheta^k(t)), \bar{x}^k(\vartheta^k(t))\big)^*\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\big).
\end{aligned}$$

The next adjoint triple is $p^k(\cdot) = (p^{kx}(\cdot), p^{ky}(\cdot), p^{ku}(\cdot))$ defined by

$$\begin{aligned}
&p^k(t) = q^k(t)\\
&+ \int_t^T \Big(- \sum_{i=1}^s \eta^k_i(\tau)\nabla^2_x g_i\big(\bar{x}^k(\vartheta^k(\tau)) - \bar{u}^k(\vartheta^k(\tau))\big)\big(\lambda^k(v^{kx}(\tau) + \theta^{kx}(\tau)) - q^{kx}(\vartheta^k(\tau) + h_k)\big)\\
&- \sum_{i=1}^s \sigma^k_i(\tau)\nabla_x g_i\big(\bar{x}^k(\vartheta^k(\tau)) - \bar{u}^k(\vartheta^k(\tau))\big) - \xi^x(\vartheta^k(\tau))\varsigma^k(\tau), \ \varsigma^k(\tau),\\
&\sum_{i=1}^s \eta^k_i(\tau)\nabla^2_u g_i\big(\bar{x}^k(\vartheta^k(\tau)) - \bar{u}^k(\vartheta^k(\tau))\big)\big(\lambda^k(v^{kx}(\tau) + \theta^{kx}(\tau)) - q^{kx}(\vartheta^k(\tau) + h_k)\big)\\
&+ \sum_{i=1}^s \sigma^k_i(\tau)\nabla_u g_i\big(\bar{x}^k(\vartheta^k(\tau)) - \bar{u}^k(\vartheta^k(\tau))\big)\Big)\, d\tau
\end{aligned} \tag{4.149}$$

for all $t \in [0, T]$. This gives us $p^k(T) = q^k(T)$ with the pointwise derivative relationship

$$\begin{aligned}
&\dot p^k(t) = \dot q^k(t) \\
&-\Big(-\sum_{i=1}^s \eta_i^k(t)\nabla_x^2 g_i\big(\bar x^k(\vartheta^k(t)) - \bar u^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t)+\theta^{kx}(t)) - q^{kx}(\vartheta^k(t)+h_k)\big) \\
&-\sum_{i=1}^s \sigma_i^k(t)\nabla_x g_i\big(\bar x^k(\vartheta^k(t)) - \bar u^k(\vartheta^k(t))\big) - \xi^x(\vartheta^k(t))\varsigma^k(t),\ \varsigma^k(t), \\
&\sum_{i=1}^s \eta_i^k(t)\nabla_u^2 g_i\big(\bar x^k(\vartheta^k(t)) - \bar u^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t)+\theta^{kx}(t)) - q^{kx}(\vartheta^k(t)+h_k)\big) \\
&+\sum_{i=1}^s \sigma_i^k(t)\nabla_u g_i\big(\bar x^k(\vartheta^k(t)) - \bar u^k(\vartheta^k(t))\big)\Big) \quad \text{a.e.} \quad t \in [0,T].
\end{aligned} \tag{4.150}$$

Furthermore, we deduce from the above that the componentwise equalities

$$\begin{aligned}
&\dot p^{kx}(t) - \lambda^k w^{kx}(t) = \nabla_x f_1\big(\bar a^k(\vartheta^k(t)), \bar x^k(\vartheta^k(t))\big)^*\big(\lambda^k(v^{kx}(t)+\theta^{kx}(t)) - q^{kx}(\vartheta^k(t)+h_k)\big) \\
&\quad + h_k\nabla_x f_2\big(\bar b^k(\vartheta^k(t)), \bar x^k(\vartheta^k(t))\big)^*\big(\lambda^k(v^{kx}(t)+\theta^{kx}(t)) - q^{kx}(\vartheta^k(t)+h_k)\big),
\end{aligned} \tag{4.151}$$

$$\dot p^{ky}(t) = \lambda^k(v^{kx}(t)+\theta^{kx}(t)) - q^{kx}(\vartheta^k(t)+h_k), \tag{4.152}$$

$$\dot p^{ku}(t) - \lambda^k w^{ku}(t) = 0, \tag{4.153}$$

hold for every $t \in (t_j^k, t_{j+1}^k)$, $j = 0, \ldots, k-1$, and $i = 1, \ldots, s$.

For each $k \in I\!N$ define the vector measure γ^k on $[0,T]$ by

$$\begin{aligned}
\int_B d\gamma^k &:= \int_B \Big(\sum_{i=1}^s \eta_i^k(t)\nabla^2 g_i\big(\bar x^k(\vartheta^k(t)) - \bar u^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t)+\theta^{kx}(t)) - q^{kx}(\vartheta^k(t)+h_k)\big)\Big)\,dt \\
&\quad + \int_B \Big(\sum_{i=1}^s \sigma_i^k(t)\nabla g_i\big(\bar x^k(\vartheta^k(t)) - \bar u^k(\vartheta^k(t))\big)\Big)\,dt
\end{aligned} \tag{4.154}$$

for any Borel subset $B \subset [0,T]$, where the vector function $\sigma^k(t) = (\sigma_1^k(t), \ldots, \sigma_s^k(t))$ is taken from (4.145). Normalizing the nontriviality condition (4.120), we get that

$$\lambda^k + \|p^k(T)\| + \|q^{ku}(0)\| + \sum_{j=0}^k \|p_j^{kd}\| = 1, \quad k \in \mathbb{N}, \tag{4.155}$$

which tells us that all the sequential terms in (4.155) are uniformly bounded. Thus we have without loss of generality that there exists $\lambda \ge 0$ with $\lambda^k \to \lambda$ as $k \to \infty$. It follows from (4.147) that

$$\int_0^T \|\varsigma^k(t)\|\,dt = \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \|\varsigma^k(t)\|\,dt = \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \frac{\|p_{j+1}^{kd}\|}{h_k}\,dt = \sum_{j=0}^{k-1} \|p_{j+1}^{kd}\| = \sum_{j=1}^{k} \|p_j^{kd}\| \le 1, \quad k \in I\!N. \tag{4.156}$$

Arguing further as in the proof of [22, Theorem 8.1] gives us the following:

• The boundedness and the uniform bounded variations of $\{q^k(\cdot)\}$. Hence the Helly theorem ensures the existence of a function $q(\cdot)$ with bounded variation on $[0,T]$ such that $q^k(t) \to q(t)$ as $k \to \infty$ for all $t \in [0,T]$.

• The boundedness of the sequence $\{p^k(\cdot)\}$ in $W^{1,2}([0,T],\mathbb{R}^{3n})$, and hence its weak compactness in this space. It follows therefore from the aforementioned Mazur theorem and basic real analysis that there exists $p(\cdot) \in W^{1,2}([0,T],\mathbb{R}^{3n})$ such that a sequence of convex combinations of $\dot{p}^k(t)$ converges to $\dot{p}(t)$ for a.e. $t \in [0,T]$. Then we arrive at (4.130) by passing to the limit in (4.151)–(4.153) as $k \to \infty$.

Our next goal is to show that the sequence $\{\gamma^k(\cdot)\}$ is bounded in $\mathcal{C}([0,T],\mathbb{R}^n)^*$. To proceed, take any Borel subset $D \subset [0,T]$ and deduce from (4.109) that

$$
\begin{aligned}
&\Big\| \int\limits_D \Big(\sum_{i=1}^s \eta_i^k(t) \nabla_u^2 g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\Big)\, dt \\
&+ \int\limits_D \Big(\sum_{i=1}^s \sigma_i^k(t) \nabla_u g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\, dt \Big\| \\
&\le \Big\| \int\limits_0^T \Big(\sum_{i=1}^s \eta_i^k(t) \nabla_u^2 g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\Big)\, dt \\
&+ \int\limits_0^T \Big(\sum_{i=1}^s \sigma_i^k(t) \nabla_u g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\, dt \Big\| \\
&= \Big\| \sum_{j=0}^{k-1} \int\limits_{t_j}^{t_{j+1}} \Big(\sum_{i=1}^s \eta_i^k(t) \nabla_u^2 g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\Big)\, dt \\
&+ \sum_{j=0}^{k-1} \int\limits_{t_j}^{t_{j+1}} \Big(\sum_{i=1}^s \sigma_i^k(t) \nabla_u g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\, dt \Big\| = \Big\| \sum_{j=0}^{k-1} \int\limits_{t_j}^{t_{j+1}} \Big(\frac{p_{j+1}^u - p_j^u}{h_k} - \lambda^k w_j^u \Big)\, dt \Big\| \\
&= \Big\| \sum_{j=0}^{k-1} (p_{j+1}^u - p_j^u - h_k \lambda^k w_j^u) \Big\| \le \sum_{j=0}^{k-1} \|p_{j+1}^u - p_j^u\| + \lambda^k \sum_{j=0}^{k-1} \|h_k w_j^u\| \\
&= \sum_{j=0}^{k-1} \|q^u(t_{j+1}) - q^u(t_j)\| + \lambda^k \sum_{j=0}^{k-1} \|h_k w_j^u\|.
\end{aligned}
$$

Then from (4.155), the imposed structure of the running cost (4.125) with the Lipschitzian functions therein, and the uniform bounded variations of $\{q^u(\cdot)\}$ we deduce that the sequence of

$$\int_D \Big(\sum_{i=1}^s \eta_i^k(t) \nabla_u^2 g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\Big)\, dt$$

$$+ \int_D \Big(\sum_{i=1}^s \sigma_i^k(t) \nabla_u g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\Big)\, dt$$

is uniformly bounded on $[0, T]$. In the same way it follows from (4.107), (4.155), and the uniform bounded variations of $\{q^x(\cdot)\}$ that the sequence of

$$\int_D \Big(\sum_{i=1}^s \eta_i^k(t) \nabla_x^2 g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\Big)\, dt$$

$$+ \int_D \Big(\sum_{i=1}^s \sigma_i^k(t) \nabla_x g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\Big)\, dt$$

is also uniformly bounded on $[0, T]$. This verifies the boundedness in $\mathcal{C}([0, T], \mathbb{R}^n)^*$ of the sequence $\{\gamma^k\}$.

Thus we get from the weak* sequential compactness of the unit ball in $\mathcal{C}([0, T], \mathbb{R}^n)^*$ that there exists a measure $\gamma \in \mathcal{C}([0, T], \mathbb{R}^n)^*$ such that $\{\gamma^k\}$ weak* converges to γ along a subsequence (without relabeling). This allows us to derive from (4.154) and the construction of the measures γ^k in (4.154) that for all $t \in [0, T]$ the following convergence holds:

$$\int_{[t,T]} \Big(\sum_{i=1}^s \eta_i^k(t) \nabla_x^2 g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\big(\lambda^k(v^{kx}(t) + \theta^{kx}(t)) - q^{kx}(\vartheta^k(t) + h_k)\Big)\, dt$$

$$+ \int_{[t,T]} \Big(\sum_{i=1}^s \sigma_i^k(t) \nabla_x g_i\big(\bar{x}^k(\vartheta^k(t)) - \bar{u}^k(\vartheta^k(t))\big)\Big)\, dt \to \int_{[t,T]} d\gamma(s) \quad \text{as} \quad k \to \infty.$$

Employing now (4.104), the definition of $\varsigma^k(\cdot)$ in (4.147), as well as the convergence $q^{ky}(t) \to q^y(t)$ and $\theta^{ky}(t) \to 0$ for a.e. $t \in [0, T]$ as $k \to \infty$ implies that

$$\varsigma^k(t) \to q^y(t) \quad \text{a.e.} \quad t \in [0, T] \quad \text{as} \quad k \to \infty.$$

Observe by (4.54) and (4.72) that $\dfrac{\theta_j^{ky}}{h_k} = \dfrac{1}{h_k} \displaystyle\int_{t_j^k}^{t_{j+1}^k} (f_2(\bar{b}_j^k, \bar{x}_j^k) - f_2(\bar{b}(t), \bar{x}(t)))\, dt$, which leads us to the estimates

$$\|\varsigma^k(t)\| \le \|q^{ky}(\vartheta^k(t) + h_k)\| + \frac{\lambda^k}{h_k} \int_{t_j^k}^{t_{j+1}^k} \|f_2(\bar{b}_j^k, \bar{x}_j^k) - f_2(\bar{b}(t), \bar{x}(t))\|\, dt$$

$$\le \|q^{ky}(\vartheta^k(t) + h_k)\| + L_1 \frac{\lambda^k}{h_k} \int_{t_j^k}^{t_{j+1}^k} \|\bar{x}_j^k - \bar{x}(t)\|\, dt + L_2 \frac{\lambda^k}{h_k} \int_{t_j^k}^{t_{j+1}^k} \|\bar{b}_j^k - \bar{b}(t)\|\, dt.$$

By the optimality of $\bar{z}^k$ in (P_k) and the constraints in (4.56) we get

$$\|\varsigma^k(t)\| \leq \|q^{ky}(\vartheta^k(t)+h_k)\| + \lambda^k(L_1+L_2)\frac{\varepsilon}{2}.$$

The boundedness of $\{q^k(\cdot)\}$ and the normalization condition (4.155) yield the boundedness of $\{\varsigma^k(\cdot)\}$. Then it follows from the Lebesgue dominate convergence theorem that

$$\int\limits_{[t,T]} \varsigma^k(s)\,ds \to \int\limits_{[t,T]} q^y(s)\,ds \quad \text{as} \quad k\to\infty \quad \text{for all} \quad t\in[0,T], \tag{4.157}$$

and that the function $q^y(\cdot)$ belongs to $L^1([0,T],\mathbb{R}^n)$. Furthermore, for all $t\in[0,T]$ we have

$$\begin{aligned}
&\Big\|\int\limits_{[t,T]} \xi^x(\vartheta^k(s))\varsigma^k(s)\,ds - \int\limits_{[t,T]} \nabla_x f_2(\bar{b}(s),\bar{x}(s))q^y(s)\,ds\Big\| \\
&= \Big\|\int\limits_{[t,T]} \nabla_x f_2(\bar{b}^k(\vartheta^k(s)),\bar{x}^k(\vartheta^k(s)))\varsigma^k(s)\,ds - \int\limits_{[t,T]} \nabla_x f_2(\bar{b}(s),\bar{x}(s))q^y(s)\,ds\Big\| \\
&\leq \Big\|\int\limits_{[t,T]} \nabla_x f_2(\bar{b}^k(\vartheta^k(s)),\bar{x}^k(\vartheta^k(s)))\varsigma^k(s)\,ds - \int\limits_{[t,T]} \nabla_x f_2(\bar{b}(s),\bar{x}(s))\varsigma^k(s)\,ds\Big\| \\
&+ \Big\|\int\limits_{[t,T]} \nabla_x f_2(\bar{b}(s),\bar{x}(s))\varsigma^k(s)\,ds - \int\limits_{[t,T]} \nabla_x f_2(\bar{b}(s),\bar{x}(s))q^y(s)\,ds\Big\|.
\end{aligned}$$

Note first that $\Big\|\int\limits_{[t,T]} \nabla_x f_2(\bar{b}^k(\vartheta^k(s)),\bar{x}^k(\vartheta^k(s)))\varsigma^k(s)\,ds - \int\limits_{[t,T]} \nabla_x f_2(\bar{b}(s),\bar{x}(s))\varsigma^k(s)\,ds\Big\| \to 0$ as $k\to\infty$.

Indeed, the boundedness of $\{\varsigma^k(\cdot)\}$ shows that $\|\varsigma^k(t)\| \leq L_\varsigma$ with some constant $L_\varsigma > 0$ and yields in turn

$$\begin{aligned}
&\Big\|\int\limits_{[t,T]} \nabla_x f_2(\bar{b}^k(\vartheta^k(s)),\bar{x}^k(\vartheta^k(s)))\varsigma^k(s)\,ds - \int\limits_{[t,T]} \nabla_x f_2(\bar{b}(s),\bar{x}(s))\varsigma^k(s)\,ds\Big\| \\
&\leq L_\varsigma \int\limits_{[t,T]} \big\|\nabla_x f_2(\bar{b}^k(\vartheta^k(s)),\bar{x}^k(\vartheta^k(s))) - \nabla_x f_2(\bar{b}(s),\bar{x}(s))\big\|\,ds
\end{aligned} \tag{4.158}$$

To apply the dominated convergence theorem for the second term of the latter inequality, we need to prove the almost everywhere convergence on $[0,T]$ of $\bar{b}^k(\vartheta^k(\cdot))$ to $\bar{b}(\cdot)$. Fixing any $\varepsilon>0$ and using the convergence of $\bar{b}^k(\cdot)$ to $\bar{b}(\cdot)$ in $L^2([0,T];\mathbb{R}^d)$ and the classical Egorov's theorem, find $E_\varepsilon\subset[0,T]$ such that $\text{mes}(E_\varepsilon)<\varepsilon$ for its Lebesgue measure. Hence we can extract a subsequence of $\{\bar{b}^k(\cdot)\}$, which uniformly converges (without relabeling) to $\bar{b}(\cdot)$ in $[0,T]\setminus E_\varepsilon$. Applying to $\bar{b}(\cdot)$ the classical Luzin property of measurable functions from real analysis, we find a closed set $F_\varepsilon\subset[0,T]$ with $\text{mes}([0,T]\setminus F_\varepsilon)<\varepsilon$ such that $\bar{b}(\cdot)$ is continuous on F_ε. Denoting $D_\varepsilon := E_\varepsilon\cup([0,T]\setminus F_\varepsilon)\subset[0,T]$, and employing (4.58) together with the boundedness of the set B allow us to select $M>0$ such that $\|\bar{b}^k(\vartheta^k(t))\|,\ \|\bar{b}(t)\|\le M$ a.e. $t\in[0,T]$. This gives us the relationships

$$\begin{aligned}
&\big\|\bar{b}^k(\vartheta^k)-\bar{b}\big\|^2_{L^2([0,T];\mathbb{R}^d)}=\int\limits_{[0,T]}\|\bar{b}^k(\vartheta^k(t))-\bar{b}(t)\|^2\,dt=\int\limits_{D_\varepsilon}\|\bar{b}^k(\vartheta^k(t))-\bar{b}(t)\|^2\,dt\\
&+\int\limits_{[0,T]\setminus D_\varepsilon}\|\bar{b}^k(\vartheta^k(t))-\bar{b}(t)\|^2\,dt\le 4M^2(\operatorname{mes}(E_\varepsilon)+\operatorname{mes}([0,T]\setminus F_\varepsilon))\\
&+\int\limits_{[0,T]\setminus D_\varepsilon}\|\bar{b}^k(\vartheta^k(t))-\bar{b}(\vartheta^k(t))+\bar{b}(\vartheta^k(t))-\bar{b}(t)\|^2\,dt\\
&\le 8M^2\varepsilon+2\int\limits_{[0,T]\setminus D_\varepsilon}\|\bar{b}^k(\vartheta^k(t))-\bar{b}(\vartheta^k(t))\|^2\,dt+2\int\limits_{[0,T]\setminus D_\varepsilon}\|\bar{b}(\vartheta^k(t))-\bar{b}(t)\|^2\,dt\\
&=8M^2\varepsilon+2\int\limits_{([0,T]\setminus E_\varepsilon)\cap F_\varepsilon}\|\bar{b}^k(\vartheta^k(t))-\bar{b}(\vartheta^k(t))\|^2\,dt+2\int\limits_{([0,T]\setminus E_\varepsilon)\cap F_\varepsilon}\|\bar{b}(\vartheta^k(t))-\bar{b}(t)\|^2\,dt.
\end{aligned}\tag{4.159}$$

Furthermore, the uniform convergence of $\bar{b}^k(\cdot)$ to $\bar{b}(\cdot)$ in $[0,T]\setminus E_\varepsilon$ tells us that

$$\int\limits_{([0,T]\setminus E_\varepsilon)\cap F_\varepsilon}\|\bar{b}^k(\vartheta^k(t))-\bar{b}(\vartheta^k(t))\|^2\,dt\le\int\limits_{([0,T]\setminus E_\varepsilon)\cap F_\varepsilon}\varepsilon^2\,dt\le T\varepsilon^2$$

for all $k\in\mathbb{N}$ sufficiently large. We also have by the continuity of $\bar{b}(\cdot)$ in F_ε and the convergence $\vartheta^k(t)\to t$ as $k\to\infty$ that

$$\int\limits_{([0,T]\setminus E_\varepsilon)\cap F_\varepsilon}\|\bar{b}(\vartheta^k(t))-\bar{b}(t)\|^2\,dt\le\int\limits_{([0,T]\setminus E_\varepsilon)\cap F_\varepsilon}\varepsilon^2\,dt\le T\varepsilon^2$$

for all large $k\in\mathbb{N}$. It follows from the above arguments with the usage of (4.159) that

$$\big\|\bar{b}^k(\vartheta^k)-\bar{b}\big\|^2_{L^2([0,T];\mathbb{R}^d)}\le 8M\varepsilon^2+2T\varepsilon^2,\quad\text{for any}\quad\varepsilon>0,$$

which justifies the convergence of $\bar{b}^k(\vartheta^k(\cdot))$ to $\bar{b}(\cdot)$ in the norm topology of $L^2([0,T];\mathbb{R}^d)$. Then there exists a subsequence of $(\bar{b}^k(\vartheta^k(\cdot)))$ (again, we do not relabel) converging to $\bar{b}(\cdot)$ almost everywhere. Invoking then The imposed assumptions on f_2 give us the boundedness of $\big\{\big\|\nabla_x f_2(\bar{b}^k(\vartheta^k(s)),\bar{x}^k(\vartheta^k(s)))-\nabla_x f_2(\bar{b}(s),\bar{x}(s))\big\|\big\}$ and therefore it follows from the dominated convergence theorem, the uniform convergence $\bar{x}^k(\cdot)\to\bar{x}(\cdot)$ on $[0,T]$ as $k\to\infty$ and the continuous differentiability of $f_2(\cdot,\cdot)$ with the usage of (4.158) that

$$\Big\|\int\limits_{[t,T]}\nabla_x f_2(\bar{b}^k(\vartheta^k(s)),\bar{x}^k(\vartheta^k(s)))\varsigma^k(s)\,ds-\int\limits_{[t,T]}\nabla_x f_2(\bar{b}(s),\bar{x}(s))\varsigma^k(s)\,ds\Big\|\to 0\quad\text{as}\quad k\to\infty.$$

For the second term in the above estimate we get $\Big\|\int\limits_{[t,T]}\nabla_x f_2(\bar{b}(s),\bar{x}(s))\varsigma^k(s)\,ds-\int\limits_{[t,T]}\nabla_x f_2(\bar{b}(s),\bar{x}(s))q^y(s)\,ds\Big\|\to 0$ as $k\to\infty$ by the convergence of $\varsigma^k(\cdot)\to q^y(\cdot)$ as $k\to\infty$ in $L^1([0,T],\mathbb{R}^n)$. Combining this yields

$$\int\limits_{[t,T]}\xi^x(\vartheta^k(s))\varsigma^k(s)\,ds\to\int\limits_{[t,T]}\nabla_x f_2(\bar{b}(s),\bar{x}(s))q^y(s)\,ds\ \ \text{as}\ \ k\to\infty.$$

In the same way we also verify that

$$\int_{[t,T]} \xi^b(\vartheta^k(s))\varsigma^k(s)\,ds \to \int_{[t,T]} \nabla_b f_2(\bar{b}(s),\bar{x}(s))q^y(s)\,ds \quad \text{as} \quad k\to\infty$$

and hence arrive at the adjoint relationships (4.132) by passing to the limit in (4.149) as $k\to\infty$, where the justification of the "except a countable subset" can be done similarly to [84, p. 325]. The passage to the limit in (4.138) as $k\to\infty$ brings us to the integral equation

$$q^y(t)=p^y(t)-\int_{[t,T]} q^y(s)\,ds \quad \text{a.e.} \quad t\in[0,T],$$

which tells us that $q^y(\cdot)$ is an absolutely continuous function satisfying

$$\dot{q}^y(t)=\dot{p}^y(t)+q^y(t) \quad \text{a.e.} \quad t\in[0,T]. \tag{4.160}$$

It follows from (4.130) and (4.132) that

$$\dot{p}^y(t)=\lambda v^x(t)-q^x(t) \quad \text{a.e.} \quad t\in[0,T], \quad \text{and} \tag{4.161}$$

$$q^x(t)=p^x(t)+\int_{[t,T]} d\,\gamma(s)+\int_{[t,T]} \nabla_x f_2(\bar{b}(s),\bar{x}(s))q^y(s)\,ds \quad \text{a.e.} \quad t\in[0,T]. \tag{4.162}$$

Substituting (4.162) into (4.161) and then (4.161) into (4.160), we obtain the extended Volterra condition (4.133).

Let us now show that the triple $\psi(\cdot)=(\psi^a(\cdot),\psi^b(\cdot))$ defined in (4.134) on $[0,T]$ satisfies the normal cone inclusions claimed in those conditions. Indeed, it follows from the construction of $\psi^k(\cdot)$ in (4.146), from the necessary optimality conditions in (4.110)–(4.112) for the discrete problems (P_k), and from the convergence of all the extended functions defining $\psi^k(\cdot)$ in (4.110) and (4.110), which was established in the proof above, that a subsequence $\{\psi^k(\cdot)\}$ weakly converges in $L^2([0,T],\mathbb{R}^{m+d})$. This clearly implies, by using again Mazur's weak closure theorem and passing to the limit in (4.146) for the convexified sequences in both sides therein, that the limiting function $\psi(\cdot)$ satisfies the equations in (4.134). Furthermore, we have the inclusions

$$\psi^{ak}(\vartheta^k(t))\in N_A\big(\bar{a}^k(\vartheta^k(t))\big) \quad \text{and} \quad \psi^{bk}(\vartheta^k(t))\in N_B\big(\bar{b}^k(\vartheta^k(t))\big). \tag{4.163}$$

Passing to the limit in (4.163) as $k\to\infty$ with the usage of Mazur's theorem and the robustness of the limiting normal cone tells us that the limiting function $\psi(t)$ satisfies the convexified inclusions in (4.134) for a.e. $t\in[0,T]$. The local and global maximization conditions (4.135) and (4.136) are derived, under the imposed additional assumptions, from the normal cone inclusions in (4.134) similarly to the case of discrete-time systems in Corollary 4.6.1. Furthermore, it follows from (4.105) and the definition of $\vartheta^k(t)$ in (4.148) that

$$q^{ku}(\vartheta^k(t)+h_k)=\lambda^k(\theta^{ku}(t)+v^{ku}(t)), \quad k\in\mathbb{N}. \tag{4.164}$$

Involving now (4.126), the assumptions on l_2, and the Lebesgue dominated convergence theorem gives us both conditions in (4.131) by passing to the limit in (4.164). The proof of the complementarity slackness conditions (4.128), (4.129) and the endpoint conditions in (4.137)–(4.139) is similar to the one in [22, Theorem 6.1], and we skip it for brevity. Taking into account the convergence in (4.157) and the relationship

$$\int_0^T \|\varsigma^k(t)\|\,dt = \sum_{j=0}^{k-1} \|p^{kd}_{j+1}\|,$$

we arrive at the nontriviality condition (4.140) by passing to the limit in (4.155) as $k \to \infty$. The verification of the enhanced nontriviality conditions in (4.141) and (4.142) under the imposed additional assumptions can be easily proved while arguing by contraposition. This therefore completes the proof of the theorem. ■

4.8 Applications to control of non-regular electrical circuits

This section is entirely devoted to applications of the necessary optimality conditions for integro-differential sweeping control problems obtained in Theorem 4.7.1 to controlled models that appear in non-regular electrical circuits with ideal diodes. However, the dynamics in such models has been described via (uncontrolled) integro-differential sweeping processes (see, e.g., [12, ?]), we are not familiar with formulations of any optimization and/or control problems for such systems. This is done in the first two examples below in different frameworks. The third example of its own interest provides a complete solution of a two-dimensional optimal control problem of the type modeled above by using the obtained necessary optimality conditions.

Exemple 4.8.1 (optimal control of non-regular electrical circuits with controlled current source). *Consider the electrical system depicted in Figure 4.1 that is composed of two resistors* $R_1 \geq 0$ *and* $R_2 \geq 0$ *with voltage/current laws* $V_{R_j} = R_j x_j$, $j = 1, 2$, *two inductors* $L_1 \geq 0$ *and* $L_2 \geq 0$ *with voltage/current laws* $V_{L_j} = L_j \dot{x}_j$ $(j = 1, 2)$, *two capacitors* $C_1 > 0$ *and* $C_2 > 0$ *with voltage/current laws* $V_{C_k} = \frac{1}{C_j}\int x_j(t)\,dt$, $j = 1, 2$, *two ideal diodes with characteristics* $0 \leq -V_{D_j} \perp i_j \geq 0$, *and an absolutely continuous current source* $i : [0, T] \to R$. *Using Kirchhoff's laws, we have*

$$\begin{cases} V_{R_1} + V_{R_2} + V_{L_1} + V_{C_1} = -V_{D_1} \in -N_{\mathbb{R}_+}(x_1 - i), \\ V_{R_1} - V_{R_2} + V_{L_2} + V_{C_2} = -V_{D_2} \in -N_{\mathbb{R}_+}(x_2). \end{cases}$$

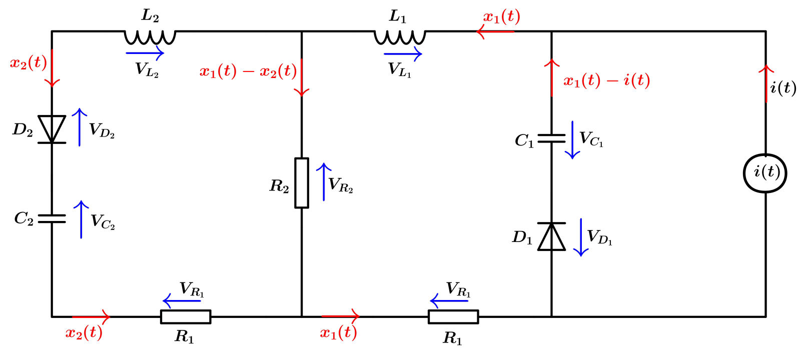

Figure 4.1: Electrical circuit with resistors,Inductances, capacitors and ideal diodes (RLCD).

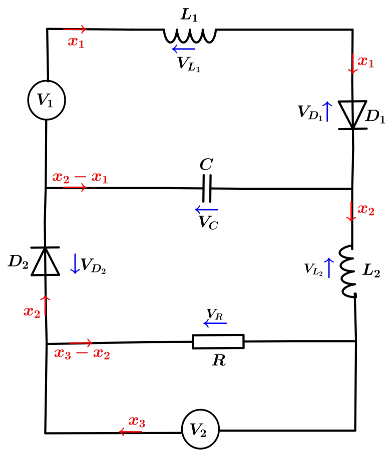

Figure 4.2: (RLCD) circuit with controlled voltage source.

Therefore, the dynamics of this circuit is given by

$$\overbrace{\begin{pmatrix}-\dot{x}_1(t)\\ -\dot{x}_2(t)\end{pmatrix}}^{-\dot{x}(t)} \in N_{C(t)}(x(t)) + \overbrace{\begin{pmatrix}\frac{R_1+R_2}{L_1} & -\frac{R_2}{L_1}\\ -\frac{R_2}{L_2} & \frac{R_1+R_2}{L_2}\end{pmatrix}}^{A_1}\overbrace{\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}}^{x(t)}$$
$$+\int_0^t \left[\overbrace{\begin{pmatrix}\frac{1}{L_1C_1} & 0\\ 0 & \frac{1}{L_2C_2}\end{pmatrix}}^{A_2}\overbrace{\begin{pmatrix}x_1(s)\\ x_2(s)\end{pmatrix}}^{x(s)} + \overbrace{\begin{pmatrix}\frac{1}{L_1C_1}i(s)\\ 0\end{pmatrix}}^{b(s)}\right] ds. \tag{4.165}$$

Put $u(t) := (i(t), 0)^*$, $b(t) = (\frac{1}{L_1C_1}i(t), 0)^*$ *indicating vector columns and denote* $C(t) = C(u(t)) := u(t) + [0, \infty) \times [0, \infty)$, $f_1(x) := A_1x$, $f_2(b, x) := A_2x + b$, *and* $x(0) := (i(0), 0)^*$. *Thus* (4.165) *can be rewritten in the form of the controlled sweeping dynamics* (4.2) *as*

$$-\dot{x}(t) \in N_{C(u(t))}(x(t)) + f_1(x(t)) + \int_0^t f_2(b(s), x(s))\, ds \quad a.e. \;\; t \in [0, T], \; x(0) = x_0 \in C(0)$$

with control functions acting in the moving set and the integral part of the dynamics. The cost functional in this problem is naturally formulated by: minimize

$$J[x,u,b] := \frac{\lambda_T}{2}(x_1(T)-i(0))^2 + \frac{\lambda_P}{2}\int_0^T (x_1(t)-i(0))^2\,dt + \frac{\lambda_I}{2}\int_0^T b_1^2(t)\,dt,$$

where λ_T, λ_Q, and λ_I are nonnegative constants not equal to zero simultaneously. The necessary optimality conditions obtained in Theorem 4.7.1 can be readily applied to this class of optimal control problems.

The next model describes a class of dynamic processes for non-regular electric circuits with an input controlled voltage source, which is dual of the current source considered in Example 4.8.1. In this model we have control actions entering the dynamic perturbation term of the sweeping process.

Exemple 4.8.2 (optimal control of non-regular electrical circuit with controlled voltage source). *Consider the electrical system shown in Figure 4.2 that is composed of a resistor $R \geq 0$ with voltage/current law $V_R = R(x_3 - x_2)$, two inductors $L_1 \geq 0$, $L_2 \geq 0$ with voltage/current laws $V_{L_j} = L_j\dot{x}_j$ $(j = 1, 2)$, a capacitor $C > 0$ with voltage/current law $V_C = \dfrac{1}{C}\displaystyle\int (x_2(t) - x_1(t))\,dt$, two ideal diodes with characteristics $0 \leq -V_{D_j} \perp x_j \geq 0$, and two controlled voltage sources $v_j(t)$, $j = 1, 2$, which have to obey the constraints*

$$\{v_j \in L^2([0,T]) \mid \underline{v}_j \leq v_j(t) \leq \overline{v}_j, a.e., t \in [0,T]\},$$

where $\underline{v}_j < \overline{v}_j$ are fixed real numbers. Employing Kirchhoff's laws tells us that

$$\begin{cases} V_{L_1} - V_C - v_1(t) = -V_{D_1} \in -N_{\mathbb{R}_+}(x_1), \\ V_{L_2} + V_C - V_R = -V_{D_2} \in -N_{\mathbb{R}_+}(x_2), \\ V_R - v_2(t) = 0. \end{cases}$$

In the system above, we substitute the third equation into the second one and get

$$-\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} \in N_{\mathbb{R}^2_+}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \overbrace{\begin{pmatrix}-\frac{1}{L_1} & 0\\ 0 & -\frac{1}{L_2}\end{pmatrix}}^{A_1}\begin{pmatrix}v_1(t)\\ v_2(t)\end{pmatrix} + \int_0^t \overbrace{\begin{pmatrix}\frac{1}{L_1C} & -\frac{1}{L_1C}\\ -\frac{1}{L_2C} & \frac{1}{L_2C}\end{pmatrix}}^{A_2}\begin{pmatrix}x_1(s)\\ x_2(s)\end{pmatrix} ds.$$

The goal is to start from a given state $(x_1(0), x_2(0))$ and to get the other state $(x_1(T), x_2(T))$ as close as possible with minimizing the input energy. Denoting $C := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid g_1(x) = x_1 \geq 0,\ g_2(x) = x_2 \geq 0\}$, $u(t) = (0, 0)$, $x_0 \in C$, $a(t) = (a_1(t), a_2(t))$ with $(a_1(t), a_2(t)) := (v_1(t), v_2(t))$, $f_1(a, x) := A_1 a$, and $f_2(b, x) := A_2 x$, the formulated problem can be written in the form of our basic problem (P) as follows:

$$\text{minimize } J[x,a] := \frac{x_2^2(T)}{2} + \frac{1}{2}\int_0^T [a_1^2(t) + a_2^2(t)]\,dt \tag{4.166}$$

subject to the controlled integro-differential sweeping process

$$-\dot{x}(t) \in N_C(x(t)) + f_1(a(t), x(t)) + \int_0^t f_2(x(s))\, ds \quad \text{a.e. } [0,T],\ x(0) = x_0 \in C. \tag{4.167}$$

It follows from the existence result of Theorem 4.2.1 that the optimal control problem formulated in (4.166) *and* (4.167) *admits an optimal solution. Furthermore, the structure of the cost functional in* (4.166) *together with Proposition 4.2.1 yields the uniqueness of the optimal pair* $(\bar{x}(\cdot), \bar{a}(\cdot))$ *in problem* (4.166), (4.167). *This allows us to find the optimal solution to the formulated optimal control problems by using the necessary optimality conditions Theorem 4.7.1, provided that a feasible solution determined by these conditions is unique.*

Finally, we present a numerical example that demonstrates how to use the necessary optimality conditions of Theorem 4.7.1 to solve a particular case of the optimal control problem formulated in Example 4.8.2 with the given data of controlled integro-differential sweeping inclusion. The corresponding example for the first model described in this section can be constructed similarly, while we skip it for brevity.

Exemple 4.8.3 (calculating optimal solutions of two-dimensional sweeping control model). *Consider the optimal control problem defined in* (4.166) *and* (4.167) *with the following data:*

$$\begin{gathered} T = 1,\ x_0 = (1,1),\ g(x) = (g_1(x), g_2(x)) := (x_1, x_2),\ l_0(x,a) := \frac{a_1^2 + a_2^2}{2}.\ \varphi(x) = \frac{x_2^2}{2}, \\ A_1 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ \text{and}\ A_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \ \text{where}\ |a_1|, |a_2| \le 2. \end{gathered} \tag{4.168}$$

We seek for solutions to problem (4.168) *such that*

$$\bar{x}(t) > 0 \ \text{ for all } \ t \in [0,1) \ \text{ and } \ \bar{x}(1) \in \operatorname{bd} C. \tag{4.169}$$

Depending on whether the diodes are blocking (off) or conducting (on), condition (4.169) *provides the following three possibilities that are listed below as* modes*:*

- Mode 1*:* For all $t \in [0,1)$ both diodes are on*, i.e.,* $x_1 > 0$ $(V_{D_1} = 0)$ *and* $x_2 > 0$ $(V_{D_2} = 0)$. *Furthermore,* at the ending time $T = 1$ the first diode is off while the second one is on*, i.e.,* $x_1(1) = 0$ *and* $x_2(1) > 0$ $(V_{D_2} = 0)$.
- Mode 2*:* For all $t \in [0,1)$ both diodes are on*, i.e.,* $x_1 > 0$ $(V_{D_1} = 0)$ *and* $x_2 > 0$ $(V_{D_2} = 0)$, *while* at the ending time $T = 1$ the first diode is on and the second one is off*, i.e.,* $x_1(1) > 0$ $(V_{D_1} = 0)$ *and* $x_2(1) = 0$.

• MODE 3: For all $t \in [0,1)$ both diodes are on, *i.e.*, $x_1 > 0$ $(V_{D_1} = 0)$ *and* $x_2 > 0$ $(V_{D_2} = 0)$, *while* at the ending time $T = 1$ both diodes are off, *i.e.*, $x_1(1) = 0$ *and* $x_2(1) = 0$.

Applying the necessary optimality conditions of Theorem 4.7.1 gives us a number $\lambda \geq 0$, *functions* $\eta_i(\cdot) \in L^2([0,1],\mathbb{R}_+)$ *as* $i = 1,2$ *well-defined at* $t = 1$, *pairs* $p(\cdot) = (p^x(\cdot), p^y(\cdot)) \in W^{1,2}([0,T];\mathbb{R}^4)$ *and* $q(\cdot) = (q^x(\cdot), q^y(\cdot))$ *with values in* $\mathbb{R}^4$ *and of bounded variations on* $[0,T]$, *as well as pairs* $w(\cdot) = (w^x(\cdot), w^a(\cdot)) \in L^2([0,T],\mathbb{R}^4)$ *and* $v(\cdot) = (v^x(\cdot), v^u(\cdot)) \in L^2([0,T],\mathbb{R}^4)$ *such that*

1. $w(t) = (0, \bar{a}(t)),\ v(t) = (0,0) \quad a.e. \quad t \in [0,1].$

2. $$\begin{cases} \dot{\bar{x}}_1(t) - \bar{a}_1(t) + \int\limits_0^t (\bar{x}_1(s) - \bar{x}_2(s))\,ds = \eta_1(t) \quad a.e. \quad t \in [0,1], \\ \dot{\bar{x}}_2(t) - \bar{a}_2(t) - \int\limits_0^t (\bar{x}_1(s) - \bar{x}_2(s))\,ds = \eta_2(t) \quad a.e. \quad t \in [0,1]. \end{cases}$$

3. $\bar{x}_i(t) > 0 \Longrightarrow \eta_i(t) = 0 \quad a.e. \quad t \in [0,1], \quad i = 1,2.$

4. $\eta_i(t) > 0 \Longrightarrow q_i^x(t) = 0 \quad a.e. \quad t \in [0,1),\ i = 1,2.$

5. $(\dot{p}^x(t), \dot{p}^y(t)) = ((0,0), -q^x(t)) \quad a.e. \quad t \in [0,1].$

6. $q^a(t) = (0,0) \quad a.e. \quad t \in [0,1].$

7. $(q^x(t), q^y(t)) = (p^x(t), p^y(t)) - \int\limits_{[t,1]} (-d\gamma(s) - A_2 q^y(s)\,ds, q^y(s)\,ds) \quad a.e. \quad t \in [0,1].$

8. $$\begin{cases} \dot{q}_1^y(t) = -p_1^x(t) - \int\limits_{[t,1]} d\gamma_1(s) - \int\limits_{[t,1]} (q_1^y(s) - q_2^y(s))\,ds + q_1^y(t) \quad a.e. \quad t \in [0,1], \\ \dot{q}_2^y(t) = -p_2^x(t) - \int\limits_{[t,1]} d\gamma_2(s) + \int\limits_{[t,1]} (q_1^y(s) - q_2^y(s))\,ds + q_2^y(t) \quad a.e. \quad t \in [0,1]. \end{cases}$$

9. $\psi^a(t) = -\lambda(\bar{a}_1(t), \bar{a}_2(t)) + A_1 q^x(t) \quad a.e. \quad t \in [0,1].$

10. $\psi^a(t) \in N_{[-2;2]^2}(\bar{a}(t)) \quad a.e. \quad t \in [0,1].$

11. $\psi_1^a(t)\,\bar{a}_1(t) + \psi_2^a(t)\,\bar{a}_2(t) = \max\limits_{(a_1,a_2)\in[-2;2]\times[-2;2]} \{\psi_1^a(t)\,a_1 + \psi_2^a(t)\,a_2\} \quad a.e. \quad t \in [0,1].$

12. $-p_1^x(1) + \eta_1(1) = 0,\ -p_2^x(1) + \eta_2(1) = \lambda\bar{x}_2(1).$

13. $-\eta(1) \in N_C(\bar{x}(1)).$

14. $p^y(1) = p^a(1) = (0,0).$

15. $\lambda + |\eta_1(1)| + |\eta_2(1)| + \int_0^1 \|q^y(t)\|\,dt > 0.$

It follows directly from items 5–7 that

$$p^x(t) \equiv p^x(1) \quad \textit{for all} \quad t \in [0,1],$$

and proceeding similarly to [21, Example 1] gives us $\int_{[t,1]} d\,\gamma(s) = \gamma(\{1\})$. *Then*

$$q^x(t) = p^x(1) + \gamma(\{1\}) + \int_{[t,1]} A_2 q^y(s)\,ds, \quad t \in [0,1]. \tag{4.170}$$

Solving the integro-differential system in item 8 with $q^y(1) = (0,0)$, *we get on* $[0,1]$ *that*

$$\begin{cases} \displaystyle\int_{[t,1]} q_1^y(s)\,ds = \frac{p_1^x(1) + \gamma_1(\{1\}) + p_2^x(1) + \gamma_2(\{1\})}{2} e^{t-1} \\ \displaystyle + \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{6} e^{-(t-1)} \\ \displaystyle + \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{12} e^{2t-2} - \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{4} \\ \displaystyle - \frac{p_1^x(1) + \gamma_1(\{1\}) + p_2^x(1) + \gamma_2(\{1\})}{2} t, \\ \displaystyle\int_{[t,1]} q_2^y(s)\,ds = \frac{p_1^x(1) + \gamma_1(\{1\}) + p_2^x(1) + \gamma_2(\{1\})}{2} e^{t-1} \\ \displaystyle - \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{6} e^{-(t-1)} \\ \displaystyle - \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{12} e^{2t-2} + \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{4} \\ \displaystyle - \frac{p_1^x(1) + \gamma_1(\{1\}) + p_2^x(1) + \gamma_2(\{1\})}{2} t. \end{cases}$$

Then it follows from the above relationships, equation (4.170), *and the definition of* A_2 *that*

$$\begin{cases} \displaystyle q_1^x(t) = p_1^x(1) + \gamma_1(\{1\}) + \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{3} e^{-(t-1)} \\ \displaystyle + \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{6} e^{2t-2} - \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{2} \\ \displaystyle q_2^x(t) = p_2^x(1) + \gamma_2(\{1\}) - \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{3} e^{-(t-1)} \\ \displaystyle - \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{6} e^{2t-2} + \frac{p_1^x(1) + \gamma_1(\{1\}) - p_2^x(1) - \gamma_2(\{1\})}{2}. \end{cases}$$

Item 9 readily implies that

$$\psi_1^a(t) = -\lambda \bar{a}_1(t) - q_1^x(t) \quad \textit{and} \quad \psi_2^a(t) = -\lambda \bar{a}_2(t) - q_2^x(t) \quad \textit{for all} \quad t \in [0,1].$$

Thus the maximization condition in item 11 tells us that

$$
\begin{aligned}
&-\lambda(\bar a_1^2(t)+\bar a_2^2(t))-(q_1^x(t)\bar a_1(t)+q_2^x(t)\bar a_2(t))\\
&=\max_{|a_1|\le 1\,,\,|a_2|\le 1}\big\{-\lambda(\bar a_1(t)a_1+\bar a_2 a_2(t))-(q_1^x(t)a_1(t)+q_2^x(t)a_2(t))\big\}
\end{aligned}
$$

for all $t\in[0,1]$. *In order to maximize the function*

$$\phi(a_1,a_2):=a_1(-\lambda\bar a_1(t)-q_1^x(t))+a_2(-\lambda\bar a_2(t)-q_2^x(t)$$

with respect to $(a_1,a_2)\in[-2,2]^2$, *we observe that if the optimal value for* a_1, *i.e.,* $\bar a_1(t)$ *is in the interior of* $[-1;1]$, *then* $\dfrac{\partial\phi}{\partial a_1}(\bar a_1(t))=0$, *and the same holds for* a_2. *In other words, we have*

- *if* $|\bar a_1(t)|<2$, *then* $\lambda\bar a_1(t)=-q_1^x(t)$,
- *if* $|\bar a_2(t)|<2$, *then* $\lambda\bar a_2(t)=-q_2^x(t)$,

while if both above cases take place, then

$$
\begin{cases}
\lambda\bar a_1(t)=-q_1^x(t)=-p_1^x(1)-\gamma_1(\{1\})-\dfrac{p_1^x(1)+\gamma_1(\{1\})-p_2^x(1)-\gamma_2(\{1\})}{3}e^{-(t-1)}\\
-\dfrac{p_1^x(1)+\gamma_1(\{1\})-p_2^x(1)-\gamma_2(\{1\})}{6}e^{2t-2}+\dfrac{p_1^x(1)+\gamma_1(\{1\})-p_2^x(1)-\gamma_2(\{1\})}{2},\\
\\
\lambda\bar a_2(t)=-q_2^x(t)=-p_2^x(1)-\gamma_2(\{1\})+\dfrac{p_1^x(1)+\gamma_1(\{1\})-p_2^x(1)-\gamma_2(\{1\})}{3}e^{-(t-1)}\\
+\dfrac{p_1^x(1)+\gamma_1(\{1\})-p_2^x(1)-\gamma_2(\{1\})}{6}e^{2t-2}-\dfrac{p_1^x(1)+\gamma_1(\{1\})-p_2^x(1)-\gamma_2(\{1\})}{2}
\end{cases}
$$

for all $t\in[0,1]$. *Suppose now for simplicity that* $\lambda>0$ *and denote* $v_1:=\dfrac{p_1^x(1)+\gamma_1(\{1\})}{\lambda}$, $v_2:=\dfrac{p_2^x(1)+\gamma_2(\{1\})}{\lambda}$. *Then the nontriviality condition in item 12 holds automatically, and we have*

$$
\begin{cases}
\bar a_1(t)=-v_1-\dfrac{v_1-v_2}{3}e^{-(t-1)}-\dfrac{v_1-v_2}{6}e^{2t-2}+\dfrac{v_1-v_2}{2},\\
\\
\bar a_2(t)=-v_2+\dfrac{v_1-v_2}{3}e^{-(t-1)}+\dfrac{v_1-v_2}{6}e^{2t-2}-\dfrac{v_1-v_2}{2}.
\end{cases}
\tag{4.171}
$$

Substituting these expressions into item 2 and then using item 3 yield

$$
\begin{cases}
\dot{\bar x}_1(t)+v_1+\dfrac{v_1-v_2}{3}e^{-(t-1)}+\dfrac{v_1-v_2}{6}e^{2t-2}-\dfrac{v_1-v_2}{2}+\displaystyle\int_0^t(\bar x_1(s)-\bar x_2(s))\,ds=0,\\
\\
\dot{\bar x}_2(t)+v_2-\dfrac{v_1-v_2}{3}e^{-(t-1)}-\dfrac{v_1-v_2}{6}e^{2t-2}+\dfrac{v_1-v_2}{2}-\displaystyle\int_0^t(\bar x_1(s)-\bar x_2(s))\,ds=0.
\end{cases}
$$

Solving the obtained integro-differential system with $\bar{x}(0) = (1,1)$ *gives us*

$$\begin{cases} \bar{x}_1(t) = 1 - \frac{v_1+v_2}{2}t - \frac{v_1-v_2}{18}e^{2t-2} + \frac{v_1-v_2}{9}e^{1-t} + \frac{1}{3}\cos(\sqrt{2}t)\Big(\frac{v_1-v_2}{6}e^{-2} - \frac{v_1-v_2}{3}e\Big) \\ -\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)\Big(\frac{v_1-v_2}{18}e^{-2} + \frac{2(v_1-v_2)}{9}e\Big), \\ \bar{x}_2(t) = 1 - \frac{v_1+v_2}{2}t + \frac{v_1-v_2}{18}e^{2t-2} - \frac{v_1-v_2}{9}e^{1-t} - \frac{1}{3}\cos(\sqrt{2}t)\Big(\frac{v_1-v_2}{6}e^{-2} - \frac{v_1-v_2}{3}e\Big) \\ +\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)\Big(\frac{v_1-v_2}{18}e^{-2} + \frac{2(v_1-v_2)}{9}e\Big), \\ \bar{x}_3(t) = \bar{a}_2(t) + \bar{x}_2(t). \end{cases} \tag{4.172}$$

By the second condition in (4.169) *we have the following three possibilities:*

(i) *If* $\bar{x}_1(1) = 0$, *then* $v_1 = v_2 - \frac{v_2-1}{c}$, *where* $c := \frac{4}{9} - \frac{\cos(\sqrt{2})}{9}\Big(\frac{e^{-2}}{2} - e\Big) + \frac{\sin(\sqrt{2})}{9\sqrt{2}}\Big(\frac{e^{-2}}{2} + 2e\Big)$. *It is easy to deduce from* (4.171) *and* (4.172) *that*

$$\begin{cases} \bar{a}_1(t) = -v_2 + \frac{v_2-1}{3c}e^{-(t-1)} + \frac{v_2-1}{6c}e^{2t-2} + \frac{v_2-1}{2c}, \\ \bar{a}_2(t) = -v_2 - \frac{v_2-1}{3c}e^{-(t-1)} - \frac{v_2-1}{6c}e^{2t-2} + \frac{v_2-1}{2c}. \end{cases}$$

$$\begin{cases} \bar{x}_1(t) = 1 - v_2 t + \frac{v_2-1}{c}\Big[\frac{t}{2} + \frac{e^{2t-2}}{18} - \frac{e^{1-t}}{9} - \frac{1}{9}\cos(\sqrt{2}t)\Big(\frac{e^{-2}}{2} - e\Big) \\ +\frac{1}{9\sqrt{2}}\sin(\sqrt{2}t)\Big(\frac{e^{-2}}{2} + 2e\Big)\Big], \\ \bar{x}_2(t) = 1 - v_2 t + \frac{v_2-1}{c}\Big[\frac{t}{2} - \frac{e^{2t-2}}{18} + \frac{e^{1-t}}{9} + \frac{1}{9}\cos(\sqrt{2}t)\Big(\frac{e^{-2}}{2} - e\Big) \\ -\frac{1}{9\sqrt{2}}\sin(\sqrt{2}t)\Big(\frac{e^{-2}}{2} + 2e\Big)\Big], \\ \bar{x}_3(t) = \bar{a}_2(t) + \bar{x}_2(t). \end{cases}$$

Observe further that $\bar{x}_2(t) + \bar{x}_1(t) = 2 - 2v_2 t + \frac{v_2-1}{c}t$. *Then we have that*

$\bar{x}_2(1) = 2 - 2v_2 + \dfrac{v_2 - 1}{c}$, *and that the cost functional in* (4.166) *reduces to*

$$J[\bar{x}, \bar{a}] = \frac{1}{2}\Big(2 - 2v_2 + \frac{v_2 - 1}{c}\Big)^2 + \frac{1}{2}\int_0^1 \Big[- v_2 + \frac{v_2 - 1}{c}\Big(\frac{1}{2} + \frac{e^{-(t-1)}}{3} + \frac{e^{2t-2}}{6}\Big)\Big]^2 dt$$
$$+ \frac{1}{2}\int_0^1 \Big[- v_2 + \frac{v_2 - 1}{c}\Big(\frac{1}{2} - \frac{e^{-(t-1)}}{3} - \frac{e^{2t-2}}{6}\Big)\Big]^2 dt$$
$$= v_2^2 + 2(1 - v_2)^2 + \frac{v_2 - 1}{c}\Big[\frac{(3 + 4I)(v_2 - 1)}{4c} - 3v_2 + 2\Big],$$

where $I = \dfrac{e^{-4}}{144}(8e^6 - 16e^3 + 9e^4 - 1)$. *Our goal is to minimize the function* J *under the additional assumptions that* $|\bar{a}_1(t)| < 2$ *and* $|\bar{a}_2(t)| < 2$ *on* $[0, 1]$. *Taking into account the convexity of the function* J *allows us to find its global minimum from* $J'(\overline{v}_2) = 0$ *and thus get* $\overline{v}_2 = 0.5988481275$. *Hence*

$$\begin{cases} \bar{a}_1(t) = -0.5988481275 - \dfrac{0.4011518725}{3c}e^{-(t-1)} - \dfrac{0.4011518725}{6c}e^{2t-2} - \dfrac{0.4011518725}{2c}, \\[2ex] \bar{a}_2(t) = -0.5988481275 + \dfrac{0.4011518725}{3c}e^{-(t-1)} + \dfrac{0.4011518725}{6c}^{2t-2} - \dfrac{0.4011518725}{2c}, \end{cases}$$

and we can directly check that the constraints $|\bar{a}_1(t)| < 2$ *and* $|\bar{a}_2(t)| < 2$ *for all* $t \in [0, 1]$ *are satisfied for* $\overline{v}_2 = 0.5988481275$. *Furthermore, it follows that*

$$\begin{cases} \bar{x}_1(t) = 1 - 0.5988481275.t - \dfrac{0.4011518725}{c}\Big[\dfrac{t}{2} + \dfrac{e^{2t-2}}{18} - \dfrac{e^{1-t}}{9} - \dfrac{1}{9}\cos(\sqrt{2}t)\Big(\dfrac{e^{-2}}{2} - e\Big) \\[1ex] +\dfrac{1}{9\sqrt{2}}\sin(\sqrt{2}t)\Big(\dfrac{e^{-2}}{2} + 2e\Big)\Big], \\[2ex] \bar{x}_2(t) = 1 - 0.5988481275.t - \dfrac{0.4011518725}{c}\Big[\dfrac{t}{2} - \dfrac{e^{2t-2}}{18} + \dfrac{e^{1-t}}{9} + \dfrac{1}{9}\cos(\sqrt{2}t)\Big(\dfrac{e^{-2}}{2} - e\Big) \\[1ex] -\dfrac{1}{9\sqrt{2}}\sin(\sqrt{2}t)\Big(\dfrac{e^{-2}}{2} + 2e\Big)\Big], \\[2ex] \bar{x}_3(t) = \bar{a}_2(t) + \bar{x}_2(t) \ \ \textit{for all} \ \ t \in [0, 1]. \end{cases}$$

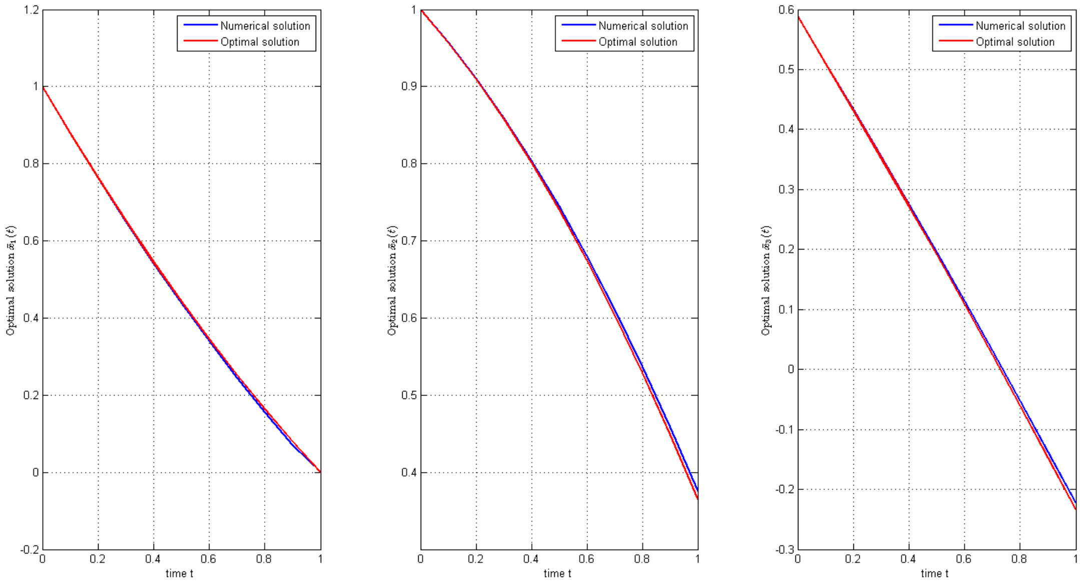

Figure 4.3: Comparison of the exact optimal solution and the numerical solution corresponding to the optimal control.

Fig. 4.3 shows the exact optimal solution computed from the necessary optimality conditions of Theorem 4.7.1 (in red) and the approximate solution corresponding to the optimal control $(\bar{a}_1, \bar{a}_2)$ *for the integro-differential sweeping process* (4.167) *(in blue). To compute the approximate solution, we use the program developed in chapter 3 (section 3.1).*

(ii) *If* $\bar{x}_2(1) = 0,$ *then* $v_1 = v_2 + \dfrac{v_2 - 1}{c},$
where $c := -\dfrac{5}{9} - \dfrac{\cos(\sqrt{2})}{9}\Big(\dfrac{e^{-2}}{2} - e\Big) + \dfrac{\sin(\sqrt{2})}{9\sqrt{2}}\Big(\dfrac{e^{-2}}{2} + 2e\Big)$. *Arguing similar to the above Case (i), we get that*

$$\begin{cases} \bar{a}_1(t) = -1.056787399 - \dfrac{0.056787399}{3c}e^{-(t-1)} - \dfrac{0.056787399}{6c}e^{2t-2} - \dfrac{0.056787399}{2c}, \\[2ex] \bar{a}_2(t) = -1.056787399 + \dfrac{0.056787399}{3c}e^{-(t-1)} + \dfrac{0.056787399}{6c}e^{2t-2} - \dfrac{0.056787399}{2c}, \end{cases}$$

$$\begin{cases} \bar{x}_1(t) = 1 - 1.056787399.t + \dfrac{0.056787399}{c}\Big[-\dfrac{t}{2} - \dfrac{e^{2t-2}}{18} + \dfrac{e^{1-t}}{9} + \dfrac{1}{9}\cos(\sqrt{2}t)\Big(\dfrac{e^{-2}}{2} - e\Big) \\ -\dfrac{1}{9\sqrt{2}}\sin(\sqrt{2}t)\Big(\dfrac{e^{-2}}{2} + 2e\Big)\Big], \\[2ex] \bar{x}_2(t) = 1 - 1.056787399.t + \dfrac{0.056787399}{c}\Big[-\dfrac{t}{2} + \dfrac{e^{2t-2}}{18} - \dfrac{e^{1-t}}{9} - \dfrac{1}{9}\cos(\sqrt{2}t)\Big(\dfrac{e^{-2}}{2} - e\Big) \\ +\dfrac{1}{9\sqrt{2}}\sin(\sqrt{2}t)\Big(\dfrac{e^{-2}}{2} + 2e\Big)\Big], \\[2ex] \bar{x}_3(t) = \bar{a}_2(t) + \bar{x}_2(t) \quad \textit{for all} \;\; t \in [0,1]. \end{cases}$$

(iii) *Let* $\bar{x}_1(1) = 0$ *and* $\bar{x}_2(1) = 0$. *Then it follows from Case (i) that* $v_1 = v_2 - \dfrac{v_2 - 1}{c}$ *and* $0 = 2 - 2v_2 + \dfrac{v_2 - 1}{c}$, *where* $c := \dfrac{4}{9} - \dfrac{\cos(\sqrt{2})}{9}\Big(\dfrac{e^{-2}}{2} - e\Big) + \dfrac{\sin(\sqrt{2})}{9\sqrt{2}}\Big(\dfrac{e^{-2}}{2} + 2e\Big)$. *Hence* $v_1 = v_2 = 1$.

Furthermore, we deduce from (4.171) *and* (4.172) *the following expressions valid for all* $t \in [0, 1]$*:*

$$\begin{cases} \bar{a}_1(t) = \bar{a}_2(t) = -1, \\ \\ \bar{x}_1(t) = \bar{x}_2(t) = 1 - t, \\ \\ \bar{x}_3(t) = \bar{a}_2(t) + \bar{x}_2(t) = -t. \end{cases}$$

Thus the obtained necessary optimality conditions from Theorem 4.7.1 with $\lambda \neq 0$ *allows us to fully compute the unique optimal solution of the integro-differential sweeping control problem formulated in* (4.166) *and* (4.167) *with the initial data given in* (4.168) *such that condition* (4.169) *is satisfied.*

Conclusion

In this thesis, by using tools from nonsmooth and variational analysis, we have studied *Volterra integro-differential inclusions* involving normal cones of nonregular sets in Hilbert spaces. Although the main focus of this thesis has been the sweeping process, the developed methods have allowed us to address several differential inclusions involving normal cones.

- In Chapter 2 we showed the *well-posedness* of the Volterra-type integro-differential perturbed sweeping process (P_{f_1,f_2}) under the absolute continuity in time t of the closed sets $C(t)$ and under their (uniform) prox-regularity.

- In Chapter 3 we have studied the *well-posedness* and the *optimal control* for a Volterra absolutely continuous time-dependent sweeping process where the integral perturbation depends on two time-variables. The main tool is an appropriate *catching-up algorithm*, which is an advantage for implementation in numerical simulations. Applications to non-regular electrical circuits are provided.

- In Chapter 4 we have studied a new class of optimal control problems for sweeping processes governed by integro-differential inclusions of the Volterra type. We establish the *necessary optimality conditions* by using the method of discrete approximations married with appropriate generalized differential tools of modern variational analysis to overcome principal difficulties in passing to the limit from optimality conditions for finite-difference systems. On the other hand, we establish a novel necessary optimality condition of the Volterra type, which is characteristic for integro-differential sweeping control systems while being particularly useful for calculations of optimal solutions. Besides illustrative examples, we apply the obtained results to an optimal control problem associated with of the non-regular electrical circuits, which is formulated in this chapter.

Future Directions

Of course, as it is customary with any topic of research, this study stimulates other questions.

- An existence result with a Lipschitz multimapping $F(t, \cdot)$ in place of the single-valued Lipschitz mapping $f_1(t, \cdot)$ would deserve to be studied as well as the case of subsmooth sets $C(t)$.

- The situation where the prox-regular sets $C(t)$ move in a BV way, i.e., with a bounded variation, would also have a great interest.

- Thus, it would be interesting to get the *necessary optimality conditions* for a general integro-differential sweeping process of Volterra type. (i.e., depends on two time-variables) as (OC) in section 3.4.

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