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THESIS

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Form, stability and behavior of the solutions of a variety of systems of differences equations

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HAMIDA HAMIOUD

Abstract

Our objective in this thesis is to study the qualitative behavior and the solvability of some systems of difference equations. In the first two chapters, we will study a nonautonomous fourth-order system of difference equations and a general second order system defined by homogeneous functions. More precisely, we will discuss stability of equilibrium points, periodicity and oscillatory behavior and we will support and confirm our results with some examples and applications. In the last chapter, we will establish explicit formulas of well-defined solutions for a two dimensional system of nonlinear difference equations in terms of a generalized Fibonacci sequence, as well as the formulas of the well-defined solutions of its corresponding three-dimensional case and some more general systems.

Keywords : *System of difference equations, stability of the equilibrium points, periodic and oscilatory solutions, homogeneous functions, explicit formulas of the solutions.*

ملخص

هدفنا في هذه الأطروحة هو دراسة سلوك و قابلية الحل لبعض جمل معادلات الفروق. في الفصلين الأوليين، سوف ندرس جملة غير مستقلة لمعادلات الفروق من الرتبة الرابعة وجملة عامة من الرتبة الثانية معرفة بواسطة دوال متجانسة. بتعبير أدق، سنناقش استقرار نقاط التوازن، الدورية واالتذبذب كما ندعم نتائجنا و نعززها ببعض الأمثلة والتطبيقات. في الفصل الأخير، سنعطي صيغا صريحة للحلول المعرفة بشكل جيد لجملة معادلات الفروق غير الخطية بمتغيرين وذلك باستخدام متتالية فيبوناشي المعممة، سنعطي كذلك صيغ صريحة لحلول معرفة جيدا للجملة المثيلة بثلاث متغيرات و أخرى عامة.

الكلمات المفتاحية: جمل معادلات الفروق، استقرار نقاط التوازن، الحلول الدورية والتذبذب، الدوال المتجانسة، الصيغ الصريحة للحلول.

Résumé

Notre objectif dans cette thèse est d'étudier le comportement et la solvabilité de quelques systèmes d'équations aux différences. Dans les deux premiers chapitres, nous étudierons un système d'équations aux différences non autonome du quatrième ordre et un système général du second ordre défini par des fonctions homogènes. Plus précisément, nous discuterons la stabilité des points d'équilibre, la périodicité et le comportement oscillatoire et nous confirmerons nos résultats avec quelques exemples et applications. Dans le dernier chapitre, nous établirons des formules explicites de solutions bien définies pour un système d'équations aux différences non linéaire bidimensionnel en termes d'une suite de Fibonacci généralisée, ainsi que les formules des solutions bien définies de son cas tridimensionnel correspondant et quelques systèmes plus généraux.

Mots-clés: *Système d'équations aux différences, stabilité des points d'équilibre, solutions périodiques et oscillatoires, fonctions homogènes, formules explicites des solutions.*

CONTENTS

GENERAL INTRODUCTION

Difference equations and their systems is a very fruitful and important field of research. Difference equations play a key role for modeling various discrete phenomena in applied and natural sciences such as biology, probability theory, ecology, physiology, physics, engineering, economics and so on. This explains the fact that there has been a continuous interest in this subject since the 19th siecle and a lot of works were published on this field, one can consult for example [1]- [57] and the references cited therein.

The present thesis is a contribution to this field of research. Firstly we will focus on the behavior of the solutions of two systems of difference equations, the first one is a fourth order rational nonautonomous (first chapter) and the second one is a general system defined by homogeneous difference equations (second chapter). The obtained results on these systems established conditions for local and global stability of the corresponding equilibrium points, rate of convergence, existence of oscillatory solutions, existence and non existence of periodic solutions.

Secondly we solved in closed form some nonlinear systems in both two and three dimensions (chapter three). Finding explicit formulas of the solutions of nonlinear difference equations and systems is generally difficult and often impossible. For this reason a lot of researchers try to solve such equations and systems. Noting that obtaining explicit formulas will enable us to understand the behavior of the solutions and so the phenomena represented by the corresponding equations or systems.

The first chapter of this thesis [34] is devoted to the qualitative behavior of the solutions of the following system defined by the rational and non-autonomous difference equations

$$x_{n+1} = \frac{p_n + y_n}{p_n + y_{n-3}}, \quad y_{n+1} = \frac{q_n + z_n}{q_n + z_{n-3}}, \quad z_{n+1} = \frac{r_n + x_n}{r_n + x_{n-3}}, \quad n = 0, 1, 2, \ldots,$$

where $\{p_n\}, \{q_n\}, \{r_n\}$ are 3-periodic sequences of positive numbers, and the initial values x_i, y_i, z_i, $i = -3, -2, -1, 0$ are in $[0, \infty)$. To deal with this system, we will first convert it to an equivalent nine-dimensional system with constant coefficients and then we will establish results on boundedness character, local asymptotic and global stability of the unique equilibrium point $(1,1,1)$, non-existence of periodic solutions and the rate of convergence. To confirm our results, several numerical examples were provided.

In the second chapter, we will study the following general system of difference equations of second order defined by

$$x_{n+1} = f(y_n, y_{n-1}), \quad y_{n+1} = g(x_n, x_{n-1}), \quad n \in \mathbb{N}_0,$$

where the initial values x_{-1}, x_0, y_{-1} and y_0 are positive real numbers and the two functions $f,\ g : (0,\infty)^2 \to (0,\infty)$ are continuous and respectively homogeneous of degree zero and degree $s \in \mathbb{R}$. Using the homogeneity of f and g, it is easy to see that our system has only one equilibrium point. We will establish conditions for local asymptotic stability as well as for the global attractivity of the equilibrium point, to do this we will prove some general convergence theorems. Conditions for the existence of periodic and the oscillatory solutions were also provided. As applications of our obtained results, concrete systems were presented.

Our aim in the last chapter [32], [33] of our thesis, is to show the solvability of some (general) systems of difference equations. In the first part of this chapter, we will begin by solving the following system

$$x_{n+1} = \frac{y_n y_{n-1} x_{n-1}^p}{x_n(a_n y_{n-2}^q + b_n y_n y_{n-1})},\ y_{n+1} = \frac{x_n x_{n-1} y_{n-1}^q}{y_n(c_n x_{n-2}^p + d_n x_n x_{n-1})}, \quad n \in \mathbb{N}_0,\ p,\ q \in \mathbb{N}$$

where the parameters $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0}, (d_n)_{n\in\mathbb{N}_0}$ and the initial values $x_{-i}, y_{-i}, i = 0,1,2,$ are non-zero real numbers. After this we will show that the following more general system

$$\begin{aligned} x_{n+1} &= f^{-1}\left(\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)\left[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})\right]}\right), \\ y_{n+1} &= g^{-1}\left(\frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)\left[c_n(f(x_{n-2}))^p + d_n f(x_n)f(x_{n-1})\right]}\right), n \in \mathbb{N}_0, p, q \in \mathbb{N}, \end{aligned} \tag{0.0.1}$$

where $f, g : D \to \mathbb{R}$ are one to one continuous functions on $D \subseteq \mathbb{R}$, the initial values $x_{-i}, y_{-i}, i = 0,1,2,$ are real numbers in D is also solvable in closed form.

In the second part of this chapter and as an extension of the aforementioned general two dimensional system, we will give explicit formulas of the solutions of the following system of difference equations

$$\begin{cases} x_{n+1} = f^{-1}\left(\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})]}\right), \\ y_{n+1} = g^{-1}\left(\frac{h(z_n)h(z_{n-1})(g(y_{n-1}))^q}{g(y_n)[c_n(h(z_{n-2}))^r + d_n h(z_n)h(z_{n-1})]}\right), \\ z_{n+1} = h^{-1}\left(\frac{f(x_n)f(x_{n-1})(h(z_{n-1}))^r}{h(z_n)[s_n(f(x_{n-2}))^p + t_n f(x_n)f(x_{n-1})]}\right), \end{cases}$$

where $n \in \mathbb{N}_0$, $p, q, r \in \mathbb{N}$, $f, g, h : D \to \mathbb{R}$ are continuous one to one functions on $D \subseteq \mathbb{R}$, the coefficients $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0}, (d_n)_{n\in\mathbb{N}_0}, (s_n)_{n\in\mathbb{N}_0}, (t_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers and the initial values $x_{-i}, y_{-i}, z_{-i}, i = 0, 1, 2$, are real numbers. As an application, we will deduce the formulas of solutions of the particular system

$$x_{n+1} = \frac{y_n y_{n-1} x_{n-1}^p}{x_n(a_n y_{n-2}^q + b_n y_n y_{n-1})},\ y_{n+1} = \frac{z_n z_{n-1} y_{n-1}^q}{y_n(c_n z_{n-2}^r + d_n z_n z_{n-1})},\ z_{n+1} = \frac{x_n x_{n-1} z_{n-1}^r}{z_n(s_n x_{n-2}^p + t_n x_n x_{n-1})},$$

obtained from the previous general one by taking $f(x) = g(x) = h(x) = x$.

CHAPTER 1

ON A THREE DIMENSIONAL NONAUTONOMOUS SYSTEM OF DIFFERENCE EQUATIONS

1.1 Introduction

In this chapter, we study the global behavior of the following nonautonomous three dimensional rational system of difference equations of fourth-order defined by

$$x_{n+1} = \frac{p_n + y_n}{p_n + y_{n-3}}, \quad y_{n+1} = \frac{q_n + z_n}{q_n + z_{n-3}}, \quad z_{n+1} = \frac{r_n + x_n}{r_n + x_{n-3}}, \quad n = 0, 1, 2, \ldots, \tag{1.1.1}$$

where $\{p_n\}, \{q_n\}, \{r_n\}$ are 3-periodic sequences of positive numbers, and the initial values $x_i, y_i, z_i \in [0, \infty)$, for $i = -3, -2, -1, 0$. Let

$$p_n = \begin{cases} \alpha_1, & if \quad n = 3k \\ \alpha_2, & if \quad n = 3k+1 \\ \alpha_3, & if \quad n = 3k+2 \end{cases}, \; q_n = \begin{cases} \beta_1, & if \quad n = 3k \\ \beta_2, & if \quad n = 3k+1 \\ \beta_3, & if \quad n = 3k+2 \end{cases}, \; r_n = \begin{cases} \gamma_1, & if \quad n = 3k \\ \gamma_2, & if \quad n = 3k+1 \\ \gamma_3, & if \quad n = 3k+2 \end{cases},$$

where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$ are positive numbers such that

$$\alpha_i \neq \alpha_j, \; \beta_i \neq \beta_j, \; \gamma_i \neq \gamma_j, \; i \neq j, \; i, j = 1, 2, 3.$$

This system has a unique equilibrium point $(\bar{x}, \bar{y}, \bar{z}) = (1, 1, 1)$. In the following, our study focuses on the stability of this equilibrium points and the convergence of solutions a round it.

The first paper motivating us to study this system, is the paper [17] by Dekkar et al., in which the authors studied the system

$$x_{n+1} = \frac{p_n + y_n}{p_n + y_{n-2}},\ y_{n+1} = \frac{q_n + x_n}{q_n + x_{n-2}},\ n = 0, 1, 2, \dots, \tag{1.1.2}$$

where $\{p_n\}$ and $\{q_n\}$ are 2-periodic sequences of positive numbers and the initial values are nonnegative real numbers. Our system (1.1.1) can be seen as the three dimensional extension of the system in [17]. Other non autonomous difference equations and systems that motivated our study, were investigated in the following references [16, 35].

Now, we recall some definitions and well known results, one can consult the references [13, 24, 25, 43].

Consider the autonomous system of difference equations

$$W_{n+1} = \Phi(W_n),\ n \in \mathbb{N}_0. \tag{1.1.3}$$

where $W_0 \in I,\ \mathbb{N}_0 = \{0\} \cup \mathbb{N}, \mathbb{N} = 1, 2, 3, \dots,$ and $\Phi : I \to I, I \subset \mathbb{R}^k, k \in \mathbb{N}$ is a continuous function.

Assume that $\overline{W} \in I$ is an equilibrium point of System (1.1.3), that is a solution of $\overline{W} = \Phi(\overline{W})$.

Let $\|.\|$ be the usual Euclidean norm or any equivalent norm in $\mathbb{R}^k$.

Definition 1.1.1. *The equilibrium point $\overline{W}$ is said to be:*

- *Stable (Localy stable): if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|W_0 - \overline{W}\| < \delta$ implies that $\|W_n - \overline{W}\| < \varepsilon$ for all $n \in \mathbb{N}$.*
- *Unstable: if it is not stable.*
- *Asymptotically stable (Localy asymptotically stable): if it is stable and there exists $\eta > 0$ such that $\|W_0 - \overline{W}\| < \eta$ implies that $\lim_{n\to\infty} W_n = \overline{W}$.*
- *Global attractor: if for evry W_0, we have $\lim_{n\to\infty} W_n = \overline{W}$.*
- *Globally (asymptotically) stable if it is stable and global attractor.*

Now, assume that Φ is C^1 on I, to System (1.1.3), we associate the linear system

$$Z_{n+1} = J_{\Phi}(\overline{W})Z_n,\ n \in \mathbb{N}_0, \tag{1.1.4}$$

where $Z_n = W_n - \overline{W}$ and $J_\Phi(\overline{W})$ is the Jacobian matrix associated to the function Φ evaluated at the equilibrium point $\overline{W}$.

The following well known theorem will be very useful for studying the stability of the equilibrium point $\overline{W}$.

Theorem 1.1.1. *Assume that $\overline{W}$ is an equilibrium point. Then we have the following statements:*

(i) If all the eigenvalues of $J_\Phi(\overline{W})$ lie in the open unit disk $D := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, then the equilibrium point $\overline{W}$ is asymptotically stable.

(ii) If at least one of the eigenvalues of $J_\Phi(\overline{W})$ has absolute value greater than one, then the equilibrium point $\overline{W}$ is unstable.

1.2 The equivalent autonomous system and the boundedness of the solutions

First, we will convert System (1.1.1) into an equivalent nine-dimensional system with constant coefficients.

To do this, let us consider the following changes of variables:

$$\begin{aligned}
&\overset{1}{u}_n = x_{3n-2},\ \overset{2}{u}_n = x_{3n-1},\ \overset{3}{u}_n = x_{3n},\\
&\overset{1}{v}_n = y_{3n-2},\ \overset{2}{v}_n = y_{3n-1},\ \overset{3}{v}_n = y_{3n},\\
&\overset{1}{w}_n = z_{3n-2},\ \overset{2}{w}_n = z_{3n-1},\ \overset{3}{w}_n = z_{3n},\ \ n = 0, 1, ...,
\end{aligned}$$

with

$$\overset{3}{u}_{-1} = x_{-3},\ \overset{3}{v}_{-1} = y_{-3},\ \overset{3}{w}_{-1} = z_{-3}.$$

Now, with these changes of variables and the periodicity of the sequences $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ we get the following equivalent autonomous system

$$\begin{cases}
\overset{1}{u}_{n+1} = \dfrac{\alpha_1 + \overset{3}{v}_n}{\alpha_1 + \overset{3}{v}_{n-1}}, \\
\overset{2}{u}_{n+1} = \dfrac{\alpha_2\beta_1 + \alpha_2\overset{3}{w}_{n-1} + \beta_1 + \overset{3}{w}_n}{(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1})}, \\
\overset{3}{u}_{n+1} = \dfrac{\alpha_3(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1}) + \beta_2\gamma_1 + \beta_2\overset{3}{u}_{n-1} + \gamma_1 + \overset{3}{u}_n}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}, \\
\overset{1}{v}_{n+1} = \dfrac{\beta_1 + \overset{3}{w}_n}{\beta_1 + \overset{3}{w}_{n-1}}, \\
\overset{2}{v}_{n+1} = \dfrac{\beta_2\gamma_1 + \beta_2\overset{3}{u}_{n-1} + \gamma_1 + \overset{3}{u}_n}{(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}, & n = 0, 1, \ldots, \\
\overset{3}{v}_{n+1} = \dfrac{\beta_3(\gamma_2 + \overset{1}{u}_n)(\alpha_1 + \overset{3}{v}_{n-1}) + \gamma_2\alpha_1 + \gamma_2\overset{3}{v}_{n-1} + \alpha_1 + \overset{3}{v}_n}{(\beta_3 + \overset{2}{w}_n)(\gamma_2 + \overset{1}{u}_n)(\alpha_1 + \overset{3}{v}_{n-1})}, \\
\overset{1}{w}_{n+1} = \dfrac{\gamma_1 + \overset{3}{u}_n}{\gamma_1 + \overset{3}{u}_{n-1}}, \\
\overset{2}{w}_{n+1} = \dfrac{\gamma_2\alpha_1 + \gamma_2\overset{3}{v}_{n-1} + \alpha_1 + \overset{3}{v}_n}{(\gamma_2 + \overset{1}{u}_n)(\alpha_1 + \overset{3}{v}_{n-1})}, \\
\overset{3}{w}_{n+1} = \dfrac{\gamma_3(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1}) + \alpha_2\beta_1 + \alpha_2\overset{3}{w}_{n-1} + \beta_1 + \overset{3}{w}_n}{(\gamma_3 + \overset{2}{u}_n)(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1})},
\end{cases} \tag{1.2.1}$$

where $\overset{1}{u}_0 = x_{-2}$, $\overset{2}{u}_0 = x_{-1}$, $\overset{3}{u}_0 = x_0$, $\overset{1}{v}_0 = y_{-2}$, $\overset{2}{v}_0 = y_{-1}$, $\overset{3}{v}_0 = y_0$, $\overset{1}{w}_0 = z_{-2}$, $\overset{2}{w}_0 = z_{-1}$, $\overset{3}{w}_0 = z_0$.

From here on, we will work on the equivalent autonomous System (1.2.1).

In the following first result, we prove that all positive solutions of System (1.2.1) are bounded and persists.

Theorem 1.2.1. *Every positive solution* $\{(\overset{1}{u}_n, \overset{2}{u}_n, \overset{3}{u}_n, \overset{1}{v}_n, \overset{2}{v}_n, \overset{3}{v}_n, \overset{1}{w}_n, \overset{2}{w}_n, \overset{3}{w}_n)\}$ *of System (1.2.1) is bounded and persists.*

Proof. We have for all $n \geq 1$

$$\begin{aligned}
\overset{3}{u}_{n+1} &= \frac{\alpha_3(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1}) + \beta_2\gamma_1 + \beta_2\overset{3}{u}_{n-1} + \gamma_1 + \overset{3}{u}_n}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})} \\
&= \frac{\alpha_3}{\alpha_3 + \overset{2}{v}_n} + \frac{\beta_2}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)} + \frac{\gamma_1 + \overset{3}{u}_n}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})} \\
&= \frac{\alpha_3}{\alpha_3 + \overset{2}{v}_n} + \frac{\beta_2}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)} + \frac{\gamma_1}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})} \\
&\quad + \frac{\alpha_3 + \overset{2}{v}_n}{(\alpha_3 + \overset{2}{v}_{n-1})(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}
\end{aligned}$$

$$\leq 1+\frac{1}{\alpha_3}+\frac{1}{\alpha_3\beta_2}+\frac{1}{\alpha_3\beta_2\gamma_1},$$

from which it follows that for all $n \geq 2$

$$\overset{3}{u}_n \leq 1+\frac{1}{\alpha_3}+\frac{1}{\alpha_3\beta_2}+\frac{1}{\alpha_3\beta_2\gamma_1}.$$

From System (1.2.1), we have

$$\overset{1}{w}_{n+1}=\frac{\gamma_1+\overset{3}{u}_n}{\gamma_1+\overset{3}{u}_{n-1}} \leq 1+\frac{1}{\gamma_1}\overset{3}{u}_n \leq 1+\frac{1}{\gamma_1}+\frac{1}{\alpha_3\gamma_1}+\frac{1}{\alpha_3\beta_2\gamma_1}+\frac{1}{\alpha_3\beta_2\gamma_1^2},$$

from which we obtain that for all $n \geq 3$

$$\overset{1}{w}_n \leq 1+\frac{1}{\gamma_1}+\frac{1}{\alpha_3\gamma_1}+\frac{1}{\alpha_3\beta_2\gamma_1}+\frac{1}{\alpha_3\beta_2\gamma_1^2}.$$

Also, we have

$$\overset{2}{v}_{n+1}=\frac{\beta_2+\overset{1}{w}_{n+1}}{\beta_2+\overset{1}{w}_n} \leq 1+\frac{1}{\beta_2}\overset{1}{w}_{n+1} \leq 1+\frac{1}{\beta_2}+\frac{1}{\beta_2\gamma_1}+\frac{1}{\alpha_3\beta_2\gamma_1}+\frac{1}{\alpha_3\beta_2^2\gamma_1}+\frac{1}{\alpha_3\beta_2^2\gamma_1^2},$$

which implies that for all $n \geq 3$,

$$\overset{2}{v}_n \leq 1+\frac{1}{\beta_2}+\frac{1}{\beta_2\gamma_1}+\frac{1}{\alpha_3\beta_2\gamma_1}+\frac{1}{\alpha_3\beta_2^2\gamma_1}+\frac{1}{\alpha_3\beta_2^2\gamma_1^2}.$$

Furthermore, for all $n \geq 3$, we have

$$\begin{aligned}\overset{3}{u}_{n+1}=\frac{\alpha_3+\overset{2}{v}_{n+1}}{\alpha_3+\overset{2}{v}_n} \geq \frac{\alpha_3}{\alpha_3+\overset{2}{v}_n} &\geq \frac{\alpha_3}{\alpha_3+1+\frac{1}{\beta_2}+\frac{1}{\beta_2\gamma_1}+\frac{1}{\alpha_3\beta_2\gamma_1}+\frac{1}{\alpha_3\beta_2^2\gamma_1}+\frac{1}{\alpha_3\beta_2^2\gamma_1^2}}\\ &=\frac{\alpha_3^2\beta_2^2\gamma_1^2}{\alpha_3^2\beta_2^2\gamma_1^2+\alpha_3\beta_2^2\gamma_1^2+\alpha_3\beta_2\gamma_1^2+\alpha_3\beta_2\gamma_1+\beta_2\gamma_1+\gamma_1+1},\end{aligned}$$

so, for all $n \geq 4$, we obtain

$$\overset{3}{u}_n \geq \frac{\alpha_3^2\beta_2^2\gamma_1^2}{\alpha_3^2\beta_2^2\gamma_1^2+\alpha_3\beta_2^2\gamma_1^2+\alpha_3\beta_2\gamma_1^2+\alpha_3\beta_2\gamma_1+\beta_2\gamma_1+\gamma_1+1},$$

then for all $n \geq 3$

$$\overset{1}{w}_{n+1}=\frac{\gamma_1+\overset{3}{u}_n}{\gamma_1+\overset{3}{u}_{n-1}} \geq \frac{\gamma_1}{\gamma_1+\overset{3}{u}_{n-1}} \geq \frac{\gamma_1}{\gamma_1+1+\frac{1}{\alpha_3}+\frac{1}{\alpha_3\beta_2}+\frac{1}{\alpha_3\beta_2\gamma_1}}$$

$$= \frac{\alpha_3\beta_2\gamma_1^2}{\alpha_3\beta_2\gamma_1^2+\alpha_3\beta_2\gamma_1+\beta_2\gamma_1+\gamma_1+1},$$

which implies that for all $n \geq 4$

$$\overset{1}{w}_n \geq \frac{\alpha_3\beta_2\gamma_1^2}{\alpha_3\beta_2\gamma_1^2+\alpha_3\beta_2\gamma_1+\beta_2\gamma_1+\gamma_1+1},$$

and for all $n \geq 3$

$$\overset{2}{v}_{n+1} = \frac{\beta_2+\overset{1}{w}_{n+1}}{\beta_2+\overset{1}{w}_n} \geq \frac{\beta_2}{\beta_2+\overset{1}{w}_n} \geq \frac{\beta_2}{\beta_2+1+\frac{1}{\gamma_1}+\frac{1}{\alpha_3\gamma_1}+\frac{1}{\alpha_3\beta_2\gamma_1}+\frac{1}{\alpha_3\beta_2\gamma_1^2}}$$
$$= \frac{\alpha_3\beta_2^2\gamma_1^2}{\alpha_3\beta_2^2\gamma_1^2+\alpha_3\beta_2\gamma_1^2+\alpha_3\beta_2\gamma_1+\beta_2\gamma_1+\gamma_1+1},$$

from which it follows that for all $n \geq 4$

$$\overset{2}{v}_n \geq \frac{\alpha_3\beta_2^2\gamma_1^2}{\alpha_3\beta_2^2\gamma_1^2+\alpha_3\beta_2\gamma_1^2+\alpha_3\beta_2\gamma_1+\beta_2\gamma_1+\gamma_1+1}.$$

Similarly we get

$$\overset{3}{v}_n \leq 1+\frac{1}{\beta_3}+\frac{1}{\beta_3\gamma_2}+\frac{1}{\beta_3\gamma_2\alpha_1}, \qquad \text{for all } n \geq 2,$$
$$\overset{3}{v}_n \geq \frac{\beta_3^2\gamma_2^2\alpha_1^2}{\beta_3^2\gamma_2^2\alpha_1^2+\beta_3\gamma_2^2\alpha_1^2+\beta_3\gamma_2\alpha_1^2+\beta_3\gamma_2\alpha_1+\gamma_2\alpha_1+\alpha_1+1}, \qquad \text{for all } n \geq 4,$$
$$\overset{1}{u}_n \leq 1+\frac{1}{\alpha_1}+\frac{1}{\beta_3\alpha_1}+\frac{1}{\beta_3\gamma_2\alpha_1}+\frac{1}{\beta_3\gamma_2\alpha_1^2}, \qquad \text{for all } n \geq 3,$$
$$\overset{1}{u}_n \geq \frac{\beta_3\gamma_2\alpha_1^2}{\beta_3\gamma_2\alpha_1^2+\beta_3\gamma_2\alpha_1+\gamma_2\alpha_1+\alpha_1+1}, \qquad \text{for all } n \geq 4,$$
$$\overset{2}{w}_n \leq 1+\frac{1}{\gamma_2}+\frac{1}{\gamma_2\alpha_1}+\frac{1}{\beta_3\gamma_2\alpha_1}+\frac{1}{\beta_3\gamma_2^2\alpha_1}+\frac{1}{\beta_3\gamma_2^2\alpha_1^2}, \qquad \text{for all } n \geq 3,$$
$$\overset{2}{w}_n \geq \frac{\beta_3\gamma_2^2\alpha_1^2}{\beta_3\gamma_2^2\alpha_1^2+\beta_3\gamma_2\alpha_1^2+\beta_3\gamma_2\alpha_1+\gamma_2\alpha_1+\alpha_1+1}, \qquad \text{for all } n \geq 4,$$
$$\overset{3}{w}_n \leq 1+\frac{1}{\gamma_3}+\frac{1}{\gamma_3\alpha_2}+\frac{1}{\gamma_3\alpha_2\beta_1}, \qquad \text{for all } n \geq 2,$$
$$\overset{3}{w}_n \geq \frac{\gamma_3^2\alpha_2^2\beta_1^2}{\gamma_3^2\alpha_2^2\beta_1^2+\gamma_3\alpha_2^2\beta_1^2+\gamma_3\alpha_2\beta_1^2+\gamma_3\alpha_2\beta_1+\alpha_2\beta_1+\beta_1+1}, \qquad \text{for all } n \geq 4,$$
$$\overset{1}{v}_n \leq 1+\frac{1}{\beta_1}+\frac{1}{\gamma_3\beta_1}+\frac{1}{\gamma_3\alpha_2\beta_1}+\frac{1}{\gamma_3\alpha_2\beta_1^2}, \qquad \text{for all } n \geq 3,$$
$$\overset{1}{v}_n \geq \frac{\gamma_3\alpha_2\beta_1^2}{\gamma_3\alpha_2\beta_1^2+\gamma_3\alpha_2\beta_1+\alpha_2\beta_1+\beta_1+1}, \qquad \text{for all } n \geq 4$$

and

$$\overset{2}{u}_n \le 1 + \frac{1}{\alpha_2} + \frac{1}{\alpha_2\beta_1} + \frac{1}{\gamma_3\alpha_2\beta_1} + \frac{1}{\gamma_3\alpha_2^2\beta_1} + \frac{1}{\gamma_3\alpha_2^2\beta_1^2}, \qquad \text{for all } n \ge 3,$$
$$\overset{2}{u}_n \ge \frac{\gamma_3\alpha_2^2\beta_1^2}{\gamma_3\alpha_2^2\beta_1^2 + \gamma_3\alpha_2\beta_1^2 + \gamma_3\alpha_2\beta_1 + \alpha_2\beta_1 + \beta_1 + 1}, \qquad \text{for all } n \ge 4.$$

□

1.3 Stability of the unique equilibrium point

Since System (1.1.1) has a unique equilibrium point $(\overline{x}, \overline{y}, \overline{z}) = (1, 1, 1)$, then the equivalent autonomous System (1.2.1) has the unique equilibrium point $E = (1, 1, 1, 1, 1, 1, 1, 1, 1)$. Now, we will establish conditions for which the equilibrium point E will be globally stable, that is locally stable and globally attractive.

First, let us write System (1.2.1) in the vectorial form (1.1.3). Let $\varPhi : [0, \infty)^{12} \to [0, \infty)^{12}$ be the function defined by

$\varPhi(W) = (f_1(W), f_2(W), f_3(W), u_3, g_1(W), g_2(W), g_3(W), v_3, h_1(W), h_2(W), h_3(W), w_3)$, where

$$W = (u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4) \in [0, \infty)^{12},$$
$$f_1(W) = \frac{\alpha_1 + v_3}{\alpha_1 + v_4},\ f_2(W) = \frac{\alpha_2\beta_1 + \alpha_2 w_4 + \beta_1 + w_3}{(\alpha_2 + v_1)(\beta_1 + w_4)},$$
$$f_3(W) = \frac{\alpha_3(\beta_2 + w_1)(\gamma_1 + u_4) + \beta_2\gamma_1 + \beta_2 u_4 + \gamma_1 + u_3}{(\alpha_3 + v_2)(\beta_2 + w_1)(\gamma_1 + u_4)},$$
$$g_1(W) = \frac{\beta_1 + w_3}{\beta_1 + w_4},\ g_2(W) = \frac{\beta_2\gamma_1 + \beta_2 u_4 + \gamma_1 + u_3}{(\beta_2 + w_1)(\gamma_1 + u_4)},$$
$$g_3(W) = \frac{\beta_3(\gamma_2 + u_1)(\alpha_1 + v_4) + \gamma_2\alpha_1 + \gamma_2 v_4 + \alpha_1 + v_3}{(\beta_3 + w_2)(\gamma_2 + u_1)(\alpha_1 + v_4)},$$
$$h_1(W) = \frac{\gamma_1 + u_3}{\gamma_1 + u_4},\ h_2(W) = \frac{\gamma_2\alpha_1 + \gamma_2 v_4 + \alpha_1 + v_3}{(\gamma_2 + u_1)(\alpha_1 + v_4)},$$
$$h_3(W) = \frac{\gamma_3(\alpha_2 + v_1)(\beta_1 + w_4) + \alpha_2\beta_1 + \alpha_2 w_4 + \beta_1 + w_3}{(\gamma_3 + u_2)(\alpha_2 + v_1)(\beta_1 + w_4)}.$$

It follows that System (1.2.1) takes the vector form

$$W_{n+1} = \varPhi(W_n),\ n = 0, 1, \ldots$$

where

$$W_n = (\overset{1}{u}_n, \overset{2}{u}_n, \overset{3}{u}_n, \overset{3}{u}_{n-1}, \overset{1}{v}_n, \overset{2}{v}_n, \overset{3}{v}_n, \overset{3}{v}_{n-1}, \overset{1}{w}_n, \overset{2}{w}_n, \overset{3}{w}_n, \overset{3}{w}_{n-1})^T.$$

The Jacobian matrix of the function Φ at the equilibrium point

$$\overline{W} = (1,1,1,1,1,1,1,1,1,1,1,1,1),$$

will be

$$J_{\Phi}(\overline{W}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\alpha_1+1} & \frac{-1}{\alpha_1+1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{\alpha_2+1} & 0 & 0 & 0 & 0 & 0 & a_{2,11} & a_{2,12} \\ 0 & 0 & a_{3,3} & a_{3,4} & 0 & \frac{-1}{\alpha_3+1} & 0 & 0 & a_{3,9} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\beta_1+1} & \frac{-1}{\beta_1+1} \\ 0 & 0 & a_{6,3} & a_{6,4} & 0 & 0 & 0 & 0 & \frac{-1}{\beta_2+1} & 0 & 0 & 0 \\ a_{7,1} & 0 & 0 & 0 & 0 & 0 & a_{7,7} & a_{7,8} & 0 & \frac{-1}{\beta_3+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma_1+1} & \frac{-1}{\gamma_1+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{\gamma_2+1} & 0 & 0 & 0 & 0 & 0 & a_{10,7} & a_{10,8} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\gamma_3+1} & 0 & 0 & a_{11,5} & 0 & 0 & 0 & 0 & 0 & a_{11,11} & a_{11,12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where

$$a_{2,11} = \frac{1}{(\beta_1+1)(\alpha_2+1)}, \quad a_{2,12} = \frac{-1}{(\beta_1+1)(\alpha_2+1)}, \quad a_{3,3} = \frac{1}{(\alpha_3+1)(\beta_2+1)(\gamma_1+1)},$$
$$a_{3,4} = \frac{-1}{(\alpha_3+1)(\beta_2+1)(\gamma_1+1)}, \quad a_{3,9} = \frac{-1}{(\alpha_3+1)(\beta_2+1)}, \quad a_{6,3} = \frac{1}{(\beta_2+1)(\gamma_1+1)},$$
$$a_{6,4} = \frac{-1}{(\beta_2+1)(\gamma_1+1)}, \quad a_{7,1} = \frac{-1}{(\beta_3+1)(\gamma_2+1)}, \quad a_{7,7} = \frac{1}{(\alpha_1+1)(\beta_3+1)(\gamma_2+1)},$$
$$a_{7,8} = \frac{-1}{(\alpha_1+1)(\beta_3+1)(\gamma_2+1)}, \quad a_{10,7} = \frac{1}{(\alpha_1+1)(\gamma_2+1)}, \quad a_{10,8} = \frac{-1}{(\alpha_1+1)(\gamma_2+1)},$$
$$a_{11,5} = \frac{-1}{(\alpha_2+1)(\gamma_3+1)}, a_{11,11} = \frac{1}{(\alpha_2+1)(\beta_1+1)(\gamma_3+1)}, a_{11,12} = \frac{-1}{(\alpha_2+1)(\beta_1+1)(\gamma_3+1)}.$$

The characteristic polynomial of the Jacobian matrix $J_{\Phi}(\overline{W})$ is

$$P(\lambda) = P_1(\lambda)P_2(\lambda)P_3(\lambda),$$

where

$$P_1(\lambda) = \lambda^4 + \frac{-\lambda^3+3\lambda^2-3\lambda+1}{(\alpha_3+1)(\beta_2+1)(\gamma_1+1)}, \; P_2(\lambda) = \lambda^4 + \frac{-\lambda^3+3\lambda^2-3\lambda+1}{(\alpha_2+1)(\beta_1+1)(\gamma_3+1)},$$

and

$$P_3(\lambda) = \lambda^4 + \frac{-\lambda^3 + 3\lambda^2 - 3\lambda + 1}{(\alpha_1 + 1)(\beta_3 + 1)(\gamma_2 + 1)}.$$

To establish necessary and sufficient conditions for which the roots of $P(\lambda)$ lie in the open unit disk D, we will use the following theorem.

Theorem 1.3.1. *[13] Let a_0, a_1, a_2, and a_3 are real numbers, we consider the polynomial*

$$P(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0. \tag{1.3.1}$$

Then, all roots of $P(\lambda)$ lie in the open unit disk D if and only if the following conditions are satisfied

$$|a_1 + a_3| < 1 + a_0 + a_2, \quad |a_1 - a_3| < 2(1 - a_0), \quad a_2 - 3a_0 < 3$$

and

$$a_0 + a_2 + a_0^2 + a_1^2 + a_0^2 a_2 + a_0 a_3^2 < 1 + 2a_0 a_2 + a_1 a_3 + a_0 a_1 a_3 + a_0^3.$$

Using Theorem 1.3.1, we have the following result.

Theorem 1.3.2. *Consider the polynomial*

$$P(\lambda) = \lambda^4 - \frac{1}{c}\lambda^3 + \frac{3}{c}\lambda^2 - \frac{3}{c}\lambda + \frac{1}{c}, \tag{1.3.2}$$

where the parameter $c \in (1, +\infty)$. Let λ_i, $i = 1, 2, 3, 4$ be the roots of $P(\lambda)$, then we have

1) $|\lambda_i| < 1,\ i = 1, 2, 3, 4 \Leftrightarrow c > 2 + \sqrt{5}$.

2) $c = 2 + \sqrt{5} \Leftrightarrow \exists \lambda_{i_0},\ i_0 \in \{1, 2, 3, 4\} : P(\lambda_{i_0}) = 0,\ |\lambda_{i_0}| = 1$.

3) $c < 2 + \sqrt{5} \Rightarrow \exists \lambda_{i_0},\ i_0 \in \{1, 2, 3, 4\} : P(\lambda_{i_0}) = 0,\ |\lambda_{i_0}| > 1$ *and the remaining roots are with modulus* $\neq 1$.

Proof. 1) Comparing (1.3.2) with (1.3.1), we obtain $a_0 = \frac{1}{c}$, $a_1 = -3a_0$, $a_2 = 3a_0$, $a_3 = -a_0$, from which it follows that

$$|a_1 + a_3| < 1 + a_0 + a_2 \Leftrightarrow 4a_0 < 1 + 4a_0,$$

which is always satisfied. Also,

$$|a_1 - a_3| < 2(1 - a_0) \Leftrightarrow 2a_0 < 1 \Leftrightarrow c > 2.$$

As $a_2 - 3a_0 = 0$, it follows that $a_2 - 3a_0 < 3$ is satisfied.

Now, the condition

$$a_0 + a_2 + a_0^2 + a_1^2 + a_0^2 a_2 + a_0 a_3^2 < 1 + 2a_0 a_2 + a_1 a_3 + a_0 a_1 a_3 + a_0^3$$

is equivalent to

$$\begin{aligned} &4a_0 + 10a_0^2 + 4a_0^3 < 1 + 9a_0^2 + 4a_0^3 \\ &\Leftrightarrow a_0^2 + 4a_0 - 1 < 0 \\ &\Leftrightarrow -c^2 + 4c + 1 < 0. \end{aligned}$$

Consider the polynomial $Q(c) = -c^2 + 4c + 1$, $c \in \mathbb{R}$. It is clear that Q has the two roots

$$c_1 = 2 - \sqrt{5},\ c_2 = 2 + \sqrt{5}.$$

Then, it follows that

$$-c^2 + 4c + 1 < 0 \Leftrightarrow c \in \left(-\infty, 2 - \sqrt{5}\right) \cup \left(2 + \sqrt{5}, +\infty\right).$$

Now, if we choose $c > 1$, it follows that

$$-c^2 + 4c + 1 < 0 \Leftrightarrow c \in \left(2 + \sqrt{5}, +\infty\right).$$

Therefore, by Theorem 1.3.1, it follows that all roots of (1.3.2) lie in the open unit disk D if and only if $c > 2 + \sqrt{5}$.

2) Assume that $c = 2 + \sqrt{5}$, then we get

$$P(\lambda) = \lambda^4 - \frac{1}{2+\sqrt{5}}\lambda^3 + \frac{3}{2+\sqrt{5}}\lambda^2 - \frac{3}{2+\sqrt{5}}\lambda + \frac{1}{2+\sqrt{5}}.$$

The polynomial P has the following four roots

$$\lambda_1 = \frac{3}{4}\sqrt{5} - \frac{5}{4} + \frac{1}{4}\sqrt{102 - 46\sqrt{5}}, \quad \lambda_2 = \frac{3}{4}\sqrt{5} - \frac{5}{4} - \frac{1}{4}\sqrt{102 - 46\sqrt{5}}$$

$$\lambda_3 = -\frac{1}{4}\sqrt{5} + \frac{1}{4} + \frac{1}{4}\sqrt{-10 - 2\sqrt{5}}, \ \lambda_4 = -\frac{1}{4}\sqrt{5} + \frac{1}{4} - \frac{1}{4}\sqrt{-10 - 2\sqrt{5}}.$$

From this, we have

$$|\lambda_3| = |\lambda_4| = \sqrt{(-\frac{1}{4}\sqrt{5}+\frac{1}{4})^2 + (\frac{1}{4}\sqrt{10+2\sqrt{5}})^2} = 1.$$

Now suppose that $\exists\lambda_{i_0},\ i_0 \in \{1,2,3,4\} : P(\lambda_{i_0}) = 0,\ |\lambda_{i_0}| = 1$, then

$$\begin{aligned} P(\lambda) &= (\lambda-\lambda_{i_0})(\lambda-\overline{\lambda_{i_0}})(\lambda^2+a\lambda+b) \\ &= \lambda^4 + (a-2Re(\lambda_{i_0}))\lambda^3 + (b-2Re(\lambda_{i_0})a+1)\lambda^2 + (a-2Re(\lambda_{i_0})b)\lambda + b. \end{aligned}$$

Comparing with (1.3.2) we obtain

$$a - 2Re(\lambda_{i_0}) = -\frac{1}{c} \tag{1.3.3}$$
$$b - 2Re(\lambda_{i_0})a + 1 = \frac{3}{c} \tag{1.3.4}$$
$$a - 2Re(\lambda_{i_0})b = -\frac{3}{c} \tag{1.3.5}$$
$$b = \frac{1}{c} \tag{1.3.6}$$

Subtracting (1.3.3) from (1.3.5) and using (1.3.6), we get

$$Re(\lambda_{i_0}) = \frac{1}{1-c}.$$

From (1.3.3) we obtain

$$a = \frac{3c-1}{(1-c)c}.$$

So, from (1.3.4) we get

$$\frac{c^2-4c-1}{(1-c)^2} = 0.$$

Hence $c^2 - 4c - 1 = 0$, from which it follows that $c = 2+\sqrt{5}$.

3) From the statements 1) and 2) we get

$$c < 2+\sqrt{5} \Rightarrow \exists\lambda_{i_0},\ i_0 \in \{1,2,3,4\} : P(\lambda_{i_0}) = 0,\ |\lambda_{i_0}| > 1$$

and the remaining roots are with modulus $\neq 1$.

□

In the following result, we summarize conditions for local asymptotic stability of the equilibrium point E of System (1.2.1).

Theorem 1.3.3. • *The unique equilibrium point E of System (1.2.1) is locally asymptotically stable if and only if*

$$(\alpha_3+1)(\beta_2+1)(\gamma_1+1) > 2+\sqrt{5}, \quad (\alpha_2+1)(\beta_1+1)(\gamma_3+1) > 2+\sqrt{5}$$

$$\textit{and} \quad (\alpha_1+1)(\beta_3+1)(\gamma_2+1) > 2+\sqrt{5}.$$

- *Assume that $(\alpha_3+1)(\beta_2+1)(\gamma_1+1) < 2+\sqrt{5}$, or $(\alpha_2+1)(\beta_1+1)(\gamma_3+1) < 2+\sqrt{5}$ or $(\alpha_1+1)(\beta_3+1)(\gamma_2+1) < 2+\sqrt{5}$. Then the unique equilibrium point E of System (1.2.1) is unstable.*

Proof. It's clear that the polynomials $P_1(\lambda), P_2(\lambda)$ and $P_3(\lambda)$ are in the form (1.3.2), so by Theorem 1.3.2 we deduce the statements of our result. □

Now, we will investigate the global attractivity of the unique equilibrium point E of System (1.2.1).

The following two theorems provides conditions on the parameters α_i, β_i, γ_i, $i = 1, 2, 3$ that guaranties the global attractivity of the equilibrium point E.

Theorem 1.3.4. *Assume that $\alpha_i, \beta_i, \gamma_i$, $(i = 1, 2, 3) \in [1, +\infty)$ such that*

- *At least one of α_1, β_3, $\gamma_2 \in (1, +\infty)$.*
- *At least one of α_2, β_1, $\gamma_3 \in (1, +\infty)$.*
- *At least one of α_3, β_2, $\gamma_1 \in (1, +\infty)$.*

Then, the equilibrium point E of System (1.2.1) is globally attractive.

Proof. We can write System (1.2.1) as three separated systems as follows

$$\begin{cases} \overset{1}{u}_{n+1} = \dfrac{\alpha_1 + \overset{3}{v}_n}{\alpha_1 + \overset{3}{v}_{n-1}}, \\ \overset{2}{w}_{n+1} = \dfrac{\gamma_2\alpha_1 + \gamma_2\overset{3}{v}_{n-1} + \alpha_1 + \overset{3}{v}_n}{(\gamma_2 + \overset{1}{u}_n)(\alpha_1 + \overset{3}{v}_{n-1})}, \\ \overset{3}{v}_{n+1} = \dfrac{\beta_3(\gamma_2 + \overset{1}{u}_n)(\alpha_1 + \overset{3}{v}_{n-1}) + \gamma_2\alpha_1 + \gamma_2\overset{3}{v}_{n-1} + \alpha_1 + \overset{3}{v}_n}{(\beta_3 + \overset{2}{w}_n)(\gamma_2 + \overset{1}{u}_n)(\alpha_1 + \overset{3}{v}_{n-1})}, \end{cases} \quad n = 0, 1, \ldots, \tag{1.3.7}$$

$$\begin{cases} \overset{1}{v}_{n+1} = & \dfrac{\beta_1 + \overset{3}{w}_n}{\beta_1 + \overset{3}{w}_{n-1}}, \\ \overset{2}{u}_{n+1} = & \dfrac{\alpha_2\beta_1 + \alpha_2\overset{3}{w}_{n-1} + \beta_1 + \overset{3}{w}_n}{(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1})}, \\ \overset{3}{w}_{n+1} = & \dfrac{\gamma_3(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1}) + \alpha_2\beta_1 + \alpha_2\overset{3}{w}_{n-1} + \beta_1 + \overset{3}{w}_n}{(\gamma_3 + \overset{2}{u}_n)(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1})}, \end{cases} \quad n = 0, 1, \ldots, \tag{1.3.8}$$

and

$$\begin{cases} \overset{1}{w}_{n+1} = & \dfrac{\gamma_1 + \overset{3}{u}_n}{\gamma_1 + \overset{3}{u}_{n-1}}, \\ \overset{2}{v}_{n+1} = & \dfrac{\beta_2\gamma_1 + \beta_2\overset{3}{u}_{n-1} + \gamma_1 + \overset{3}{u}_n}{(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}, \\ \overset{3}{u}_{n+1} = & \dfrac{\alpha_3(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1}) + \beta_2\gamma_1 + \beta_2\overset{3}{u}_{n-1} + \gamma_1 + \overset{3}{u}_n}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}, \end{cases} \quad n = 0, 1, \ldots, \tag{1.3.9}$$

Our goal is to prove that, every solution $\left(\overset{1}{u}_n, \overset{2}{w}_n, \overset{3}{v}_n\right)$ of System (1.3.7) converges to $(1,1,1)$, every solution $\left(\overset{1}{v}_n, \overset{2}{u}_n, \overset{3}{w}_n\right)$ of System (1.3.8) converges to $(1,\ 1,\ 1)$ and every solution $\left(\overset{1}{w}_n, \overset{2}{v}_n, \overset{3}{u}_n\right)$ of System (1.3.9) converges to $(1,\ 1,\ 1)$.
Let for all $n = 0, 1, \ldots,$

$$\overset{1}{s}_n = \gamma_2 + \overset{1}{u}_n, \quad \overset{2}{s}_n = \alpha_2 + \overset{1}{v}_n, \quad \overset{3}{s}_n = \beta_2 + \overset{1}{w}_n,$$

$$\overset{1}{t}_n = \beta_3 + \overset{2}{w}_n, \quad \overset{2}{t}_n = \gamma_3 + \overset{2}{u}_n, \quad \overset{3}{t}_n = \alpha_3 + \overset{2}{v}_n,$$

and for all $n = -1, 0, \ldots,$

$$\overset{1}{d}_n = \alpha_1 + \overset{3}{v}_n, \quad \overset{2}{d}_n = \beta_1 + \overset{3}{w}_n, \quad \overset{3}{d}_n = \gamma_1 + \overset{3}{u}_n.$$

Then Systems (1.3.7), (1.3.8) and (1.3.9) will be respectively

$$\begin{cases} \overset{1}{s}_{n+1} & = \gamma_2 + \dfrac{\overset{1}{d}_n}{\overset{1}{d}_{n-1}}, \\ \overset{1}{t}_{n+1} & = \beta_3 + \dfrac{\overset{1}{s}_{n+1}}{\overset{1}{s}_n} = \beta_3 + \dfrac{\gamma_2\overset{1}{d}_{n-1} + \overset{1}{d}_n}{\overset{1}{s}_n\overset{1}{d}_{n-1}}, \\ \overset{1}{d}_{n+1} & = \alpha_1 + \dfrac{\overset{1}{t}_{n+1}}{\overset{1}{t}_n} = \alpha_1 + \dfrac{\beta_3\overset{1}{s}_n\overset{1}{d}_{n-1} + \gamma_2\overset{1}{d}_{n-1} + \overset{1}{d}_n}{\overset{1}{t}_n\overset{1}{s}_n\overset{1}{d}_{n-1}}, \end{cases} \tag{1.3.10}$$

$$\begin{cases} \overset{2}{s}_{n+1} &= \alpha_2 + \dfrac{\overset{2}{d}_n}{\overset{2}{d}_{n-1}}, \\ \overset{2}{t}_{n+1} &= \gamma_3 + \dfrac{\overset{2}{s}_{n+1}}{\overset{2}{s}_n} = \gamma_3 + \dfrac{\alpha_2 \overset{2}{d}_{n-1} + \overset{2}{d}_n}{\overset{2}{s}_n \overset{2}{d}_{n-1}}, \\ \overset{2}{d}_{n+1} &= \beta_1 + \dfrac{\overset{2}{t}_{n+1}}{\overset{2}{t}_n} = \beta_1 + \dfrac{\gamma_3 \overset{2}{s}_n \overset{2}{d}_{n-1} + \alpha_2 \overset{2}{d}_{n-1} + \overset{2}{d}_n}{\overset{2}{t}_n \overset{2}{s}_n \overset{2}{d}_{n-1}}, \end{cases} \tag{1.3.11}$$

$$\begin{cases} \overset{3}{s}_{n+1} &= \beta_2 + \dfrac{\overset{3}{d}_n}{\overset{3}{d}_{n-1}}, \\ \overset{3}{t}_{n+1} &= \alpha_3 + \dfrac{\overset{3}{s}_{n+1}}{\overset{3}{s}_n} = \alpha_3 + \dfrac{\beta_2 \overset{3}{d}_{n-1} + \overset{3}{d}_n}{\overset{3}{s}_n \overset{3}{d}_{n-1}}, \\ \overset{3}{d}_{n+1} &= \gamma_1 + \dfrac{\overset{3}{t}_{n+1}}{\overset{3}{t}_n} = \gamma_1 + \dfrac{\alpha_3 \overset{3}{s}_n \overset{3}{d}_{n-1} + \beta_2 \overset{3}{d}_{n-1} + \overset{3}{d}_n}{\overset{3}{t}_n \overset{3}{s}_n \overset{3}{d}_{n-1}}. \end{cases} \tag{1.3.12}$$

From which it follows that,

$$\left(\overset{1}{u}_n, \overset{2}{w}_n, \overset{3}{v}_n\right) \to (1,1,1) \Leftrightarrow (\overset{1}{s}_n, \overset{1}{t}_n, \overset{1}{d}_n) \to (\gamma_2 + 1, \beta_3 + 1, \alpha_1 + 1),$$

$$\left(\overset{1}{v}_n, \overset{2}{u}_n, \overset{3}{w}_n\right) \to (1,1,1) \Leftrightarrow (\overset{2}{s}_n, \overset{2}{t}_n, \overset{2}{d}_n) \to (\alpha_2 + 1, \gamma_3 + 1, \beta_1 + 1),$$

$$\left(\overset{1}{w}_n, \overset{2}{v}_n, \overset{3}{u}_n\right) \to (1,1,1) \Leftrightarrow (\overset{3}{s}_n, \overset{3}{t}_n, \overset{3}{d}_n) \to (\beta_2 + 1, \alpha_3 + 1, \gamma_1 + 1).$$

Note that Systems (1.3.10), (1.3.11) and (1.3.12) are in the same form, so we will prove only the result for System (1.3.10), and we deduce the results for System (1.3.11) and System (1.3.12) in a similar way.

Let $\left(\overset{1}{s}_n, \overset{1}{t}_n, \overset{1}{d}_n\right)$ be a solution of System (1.3.10), then we get

$$\overset{1}{s}_n = \gamma_2 + \overset{1}{u}_n \le \gamma_2 + 1 + \frac{\beta_3\gamma_2\alpha_1 + \alpha_1\gamma_2 + \alpha_1 + 1}{\beta_3\gamma_2\alpha_1^2}, \ n \ge 3,$$
$$\overset{1}{s}_n \ge \gamma_2 + \frac{\beta_3\gamma_2\alpha_1^2}{\beta_3\gamma_2\alpha_1^2 + \beta_3\gamma_2\alpha_1 + \gamma_2\alpha_1 + \alpha_1 + 1}, \ n \ge 4,$$
$$\overset{1}{t}_n = \beta_3 + \overset{2}{w}_n \le 1 + \beta_3 + \frac{1}{\gamma_2} + \frac{\beta_3\gamma_2\alpha_1 + \alpha_1\gamma_2 + \alpha_1 + 1}{\beta_3\gamma_2^2\alpha_1^2}, \ n \ge 3,$$
$$\overset{1}{t}_n \ge \beta_3 + \frac{\beta_3\gamma_2^2\alpha_1^2}{\beta_3\gamma_2^2\alpha_1^2 + \beta_3\gamma_2\alpha_1^2 + \beta_3\gamma_2\alpha_1 + \gamma_2\alpha_1 + \alpha_1 + 1}, \ n \ge 4,$$
$$\overset{1}{d}_n = \alpha_1 + \overset{3}{v}_n \le 1 + \alpha_1 + \frac{\gamma_2\alpha_1 + \alpha_1 + 1}{\beta_3\gamma_2\alpha_1}, \ n \ge 2,$$
$$\overset{1}{d}_n \ge \alpha_1 + \frac{\beta_3^2\gamma_2^2\alpha_1^2}{\beta_3^2\gamma_2^2\alpha_1^2 + \beta_3\gamma_2^2\alpha_1^2 + \beta_3\gamma_2\alpha_1^2 + \beta_3\gamma_2\alpha_1 + \gamma_2\alpha_1 + \alpha_1 + 1}, \ n \ge 4.$$

Let

$$m_1 := \liminf_{n\to\infty} \overset{1}{s}_n, \quad m_2 := \liminf_{n\to\infty} \overset{1}{t}_n, \quad m_3 := \liminf_{n\to\infty} \overset{1}{d}_n,$$

$$M_1 := \limsup_{n\to\infty} \overset{1}{s}_n, \quad M_2 := \limsup_{n\to\infty} \overset{1}{t}_n, \quad M_3 := \limsup_{n\to\infty} \overset{1}{d}_n.$$

We have

$$0 < m_i \leq M_i,\ i = 1, 2, 3.$$

From (1.3.10), we get

$$m_1 \geq \gamma_2 + \frac{m_3}{M_3}, \qquad M_1 \leq \gamma_2 + \frac{M_3}{m_3}, \tag{1.3.13}$$

$$m_2 \geq \beta_3 + \frac{m_1}{M_1}, \qquad M_2 \leq \beta_3 + \frac{M_1}{m_1}, \tag{1.3.14}$$

$$m_3 \geq \alpha_1 + \frac{m_2}{M_2}, \qquad M_3 \leq \alpha_1 + \frac{M_2}{m_2}, \tag{1.3.15}$$

from which it follows that

$$m_1 M_3 \geq \gamma_2 M_3 + m_3, \tag{1.3.16}$$

$$M_1 m_3 \leq \gamma_2 m_3 + M_3, \tag{1.3.17}$$

$$m_2 M_1 \geq \beta_3 M_1 + m_1, \tag{1.3.18}$$

$$M_2 m_1 \leq \beta_3 m_1 + M_1, \tag{1.3.19}$$

$$m_3 M_2 \geq \alpha_1 M_2 + m_2, \tag{1.3.20}$$

$$M_3 m_2 \leq \alpha_1 m_2 + M_2. \tag{1.3.21}$$

By multiplying equalities (1.3.16) by M_2 and (1.3.19) by M_3, we get

$$m_1 M_3 M_2 \geq \gamma_2 M_3 M_2 + m_3 M_2, \quad M_2 m_1 M_3 \leq \beta_3 m_1 M_3 + M_1 M_3,$$

then

$$\gamma_2 M_3 M_2 + m_3 M_2 \leq \beta_3 m_1 M_3 + M_1 M_3 \tag{1.3.22}$$

Multiplying equalities (1.3.17) by M_2 and (1.3.20) by M_1, we obtain

$$M_1 m_3 M_2 \leq \gamma_2 m_3 M_2 + M_3 M_2, \quad m_3 M_2 M_1 \geq \alpha_1 M_2 M_1 + m_2 M_1$$

then

$$\alpha_1 M_2 M_1 + m_2 M_1 \leq \gamma_2 m_3 M_2 + M_3 M_2. \tag{1.3.23}$$

Multiplying equalities (1.3.18) by M_3 and (1.3.21) by M_1 we get

$$m_2 M_1 M_3 \geq \beta_3 M_1 M_3 + m_1 M_3, \quad M_3 m_2 M_1 \leq \alpha_1 m_2 M_1 + M_2 M_1$$

then

$$\beta_3 M_1 M_3 + m_1 M_3 \leq \alpha_1 m_2 M_1 + M_2 M_1. \tag{1.3.24}$$

From (1.3.22), (1.3.23) and (1.3.24), we get

$$\gamma_2 M_3 M_2 + m_3 M_2 + \beta_3 M_1 M_3 + m_1 M_3 + \alpha_1 M_2 M_1 + m_2 M_1 \leq \beta_3 m_1 M_3 + M_1 M_3 + \alpha_1 m_2 M_1 + M_2 M_1 + \gamma_2 m_3 M_2 + M_3 M_2,$$

which implies that

$$(\gamma_2 - 1) M_2 (M_3 - m_3) + (\alpha_1 - 1) M_1 (M_2 - m_2) + (\beta_3 - 1) M_3 (M_1 - m_1) \leq 0 \tag{1.3.25}$$

- If $\gamma_2 > 1, \alpha_1 > 1$ and $\beta_3 > 1$, then from (1.3.25) we get $M_1 = m_1$, $M_2 = m_2$ and $M_3 = m_3$. So, from (1.3.13)-(1.3.15) it follows that $(m_1, m_2, m_3) = (\gamma_2 + 1, \beta_3 + 1, \alpha_1 + 1)$, that the solution $\left(\overset{1}{s}_n, \overset{1}{t}_n, \overset{1}{d}_n\right)$ of System (1.3.10) converges to $(\gamma_2 + 1, \beta_3 + 1, \alpha_1 + 1)$.
- If $\beta_3, \alpha_1 > 1$ and $\gamma_2 = 1$, then from (1.3.25) we obtain $M_1 = m_1$, $M_2 = m_2$. Clearly, from (1.3.15) we get

  $$\alpha_1 + 1 \leq m_3 \leq M_3 \leq \alpha_1 + 1.$$

  So, $M_3 = m_3 = \alpha_1 + 1$. Now, from (1.3.13) and (1.3.14), we obtain also that $m_1 = \gamma_2 + 1$ and $m_2 = \beta_3 + 1$). Hence the solution $\left(\overset{1}{s}_n, \overset{1}{t}_n, \overset{1}{d}_n\right)$ of System (1.3.10) converges to $(\gamma_2 + 1, \beta_3 + 1, \alpha_1 + 1)$.
- If $\beta_3 > 1$ and $\gamma_2 = \alpha_1 = 1$, then from (1.3.25) we get $M_1 = m_1$. From (1.3.14), we obtain

  $$\beta_3 + 1 \leq m_2 \leq M_2 \leq \beta_3 + 1,$$

  that is $M_2 = m_2 = \beta_3 + 1$. Now, from (1.3.15) we get

  $$\alpha_1 + 1 \leq m_3 \leq M_3 \leq \alpha_1 + 1.$$

So, $M_3 = m_3 = \alpha_1 + 1$. Finally, from (1.3.13), we obtain $M_1 = m_1 = \gamma_2 + 1$. Hence the solution $\left(\overset{1}{s}_n, \overset{1}{t}_n, \overset{1}{d}_n\right)$ of System (1.3.10) converges to $(\gamma_2 + 1, \beta_3 + 1, \alpha_1 + 1)$.

- The proof in the following cases $(\gamma_2, \beta_3 > 1, \alpha_1 = 1)$, $(\gamma_2, \alpha_1 > 1, \beta_3 = 1$, $(\beta_3 = \alpha_1 = 1, \gamma_2 > 1)$, $(\beta_3 = \gamma_2 = 1, \alpha_1 > 1)$ can be done in a similar way.

By the same arguments, we can prove that the solution $\left(\overset{2}{s}_n, \overset{2}{t}_n, \overset{2}{d}_n\right)$ of System (1.3.11) converges to $(\alpha_2 + 1, \gamma_3 + 1, \beta_1 + 1)$ (resp. the solution $\left(\overset{3}{s}_n, \overset{3}{t}_n, \overset{3}{d}_n\right)$ of System (1.3.12) converges to $(\beta_2 + 1, \alpha_3 + 1, \gamma_1 + 1)$) in each of the following cases $(\gamma_3, \alpha_2, \beta_1 > 1)$, $(\alpha_2, \beta_1 > 1, \gamma_3 = 1)$, $(\gamma_3, \beta_1 > 1, \alpha_2 = 1)$, $(\gamma_3, \alpha_2 > 1, \beta_1 = 1)$, $(\gamma_3 = \alpha_2 = 1, \beta_1 > 1)$, $(\gamma_3 = \beta_1 = 1, \alpha_2 > 1)$, $(\alpha_2 = \beta_1 = 1, \gamma_3 > 1)$, (resp. $(\gamma_1, \alpha_3, \beta_2 > 1)$, $(\alpha_3, \beta_2 > 1, \gamma_1 = 1)$, $(\gamma_1, \beta_2 > 1, \alpha_3 = 1)$, $(\gamma_1, \alpha_3 > 1, \beta_2 = 1)$, $(\gamma_1 = \alpha_3 = 1, \beta_2 > 1)$, $(\gamma_1 = \beta_2 = 1, \alpha_3 > 1)$, $(\alpha_3 = \beta_2 = 1, \gamma_1 > 1)$).

Consequently, the unique positive equilibrium of System (1.2.1) is a global attractor. □

Theorem 1.3.5. *Assume that $\alpha_1\beta_3\gamma_2 > 8$, $\alpha_2\beta_1\gamma_3 > 8$ and $\alpha_3\beta_2\gamma_1 > 8$. Then the unique equilibrium point E of System (1.2.1) is globally attractive.*

Proof. We will prove that every solution $(\overset{1}{u}_n, \overset{2}{w}_n, \overset{3}{v}_n)$ of System (1.3.7) converges to $(1,1,1)$, every solution $(\overset{1}{v}_n, \overset{2}{u}_n, \overset{3}{w}_n)$ of System (1.3.8) converges to $(1,1,1)$ and every solution $(\overset{1}{w}_n, \overset{2}{v}_n, \overset{3}{u}_n)$ of System (1.3.9) converges to $(1,1,1)$. We will prove the result only for System (1.3.7), the results for Systems (1.3.8) and (1.3.9) can be done in a similar way.

Consider the functions: $f : [a_3, b_3] \times [a_3, b_3] \longrightarrow [a_1, b_1]$, $g : [a_1, b_1] \times [a_3, b_3] \times [a_3, b_3] \longrightarrow [a_2, b_2]$ and $h : [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \times [a_3, b_3] \longrightarrow [a_3, b_3]$, where

$$f(w,z) = \frac{\alpha_1 + w}{\alpha_1 + z}, \; g(u,w,z) = \frac{\gamma_2\alpha_1 + \gamma_2 z + \alpha_1 + w}{(\gamma_2 + u)(\alpha_1 + z)}$$

and

$$h(u,v,w,z) = \frac{\beta_3(\gamma_2 + u)(\alpha_1 + z) + \gamma_2\alpha_1 + \gamma_2 z + \alpha_1 + w}{(\beta_3 + v)(\gamma_2 + u)(\alpha_1 + z)},$$

with

$$a_1 := \min(A, \overset{1}{u}_0, \overset{1}{u}_1, \overset{1}{u}_2, \overset{1}{u}_3), \quad b_1 := \max(1 + \frac{B}{\alpha_1}, \overset{1}{u}_0, \overset{1}{u}_1, \overset{1}{u}_2),$$

$$a_2 := \min(\frac{\gamma_2 A}{\gamma_2 A + 1}, \overset{2}{w}_0, \overset{2}{w}_1, \overset{2}{w}_3, \overset{2}{w}_4), \quad b_2 := \max(\frac{\alpha_1\gamma_2 + \alpha_1 + B}{\alpha_1\gamma_2}, \overset{2}{w}_0, \overset{2}{w}_1, \overset{2}{w}_2),$$

$$a_3 := \min(\frac{\beta_3\gamma_2 A}{\beta_3\gamma_2 A + \gamma_2 A + 1}, \overset{3}{v}_{-1}, \overset{3}{v}_0, \overset{3}{v}_1, \overset{3}{v}_2, \overset{3}{v}_3), \quad b_3 := \max(B, \overset{3}{v}_{-1}, \overset{3}{v}_0, \overset{3}{v}_1),$$

$$A := \frac{\beta_3\gamma_2\alpha_1^2}{\beta_3\gamma_2\alpha_1^2 + \beta_3\gamma_2\alpha_1 + \gamma_2\alpha_1 + \alpha_1 + 1}, \quad B := \frac{\beta_3\gamma_2\alpha_1 + \gamma_2\alpha_1 + \alpha_1 + 1}{\beta_3\gamma_2\alpha_1}.$$

Then, our System (1.3.7) takes the form

$$\overset{1}{u}_{n+1} = f(\overset{3}{v}_n, \overset{3}{v}_{n-1}),\ \overset{2}{w}_{n+1} = g(\overset{1}{u}_n, \overset{3}{v}_n, \overset{3}{v}_{n-1}),\ \overset{3}{v}_{n+1} = h(\overset{1}{u}_n, \overset{2}{w}_n, \overset{3}{v}_n, \overset{3}{v}_{n-1}),\ n \in \mathbb{N}_0. \quad (1.3.26)$$

Clearly the function $f(w,z)$ is increasing in w and decreasing in z, the function $g(u,w,z)$ is increasing in w and decreasing in both u and z, however the function $h(u,v,w,z)$ is increasing in w and decreasing in u, v and z.

Let $(\overset{1}{u}_n, \overset{2}{w}_n, \overset{3}{v}_n)$ be a solution of System (1.3.26) with $(\overset{1}{u}_0, \overset{2}{w}_0, \overset{3}{v}_0, \overset{3}{v}_{-1}) \in [a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]^2$.

Set

$$m_1^0 := a_1, \quad M_1^0 := b_1, \quad m_2^0 := a_2, \quad M_2^0 := b_2, \quad m_3^0 := a_3, \quad M_3^0 := b_3,$$

and for each $i = 0, 1, 2, \ldots,$

$$\begin{cases} m_1^{i+1} := f(m_3^i, M_3^i), & M_1^{i+1} := f(M_3^i, m_3^i), \\ m_2^{i+1} := g(M_1^i, m_3^i, M_3^i), & M_2^{i+1} := g(m_1^i, M_3^i, m_3^i), \\ m_3^{i+1} := h(M_1^i, M_2^i, m_3^i, M_3^i), & M_3^{i+1} := h(m_1^i, m_2^i, M_3^i, m_3^i). \end{cases} \quad (1.3.27)$$

Using the monotonicity of f, g and h, we get

$$m_1^0 = a_1 \le f(m_3^0, M_3^0) \le f(M_3^0, m_3^0) \le b_1 = M_1^0,$$

$$m_2^0 = a_2 \le g(M_1^0, m_3^0, M_3^0) \le g(m_1^0, M_3^0, m_3^0) \le b_2 = M_2^0,$$

$$m_3^0 = a_3 \le h(M_1^0, M_2^0, m_3^0, M_3^0) \le h(m_1^0, m_2^0, M_3^0, m_3^0) \le b_3 = M_3^0,$$

that is,

$$m_1^0 \le m_1^1 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le M_2^1 \le M_2^0,$$

$$m_3^0 \le m_3^1 \le M_3^1 \le M_3^0.$$

Again, using the monotonicity of f, g and h, we obtain

$$m_1^1 = f(m_3^0, M_3^0) \le f(m_3^1, M_3^1) \le f(M_3^1, m_3^1) \le f(M_3^0, m_3^0) = M_1^1,$$

$$m_2^1 = g(M_1^0, m_3^0, M_3^0) \le g(M_1^1, m_3^1, M_3^1) \le g(m_1^1, M_3^1, m_3^1) \le g(m_1^0, M_3^0, m_3^0) = M_2^1,$$

$$m_3^1 = h(M_1^0, M_2^0, m_3^0, M_3^0) \le h(M_1^1, M_2^1, m_3^1, M_3^1) \le h(m_1^1, m_2^1, M_3^1, m_3^1) \le h(m_1^0, m_2^0, M_3^0, m_3^0) = M_3^1,$$

from which it follows that

$$m_1^0 \le m_1^1 \le m_1^2 \le M_1^2 \le M_1^1 \le M_1^0,$$

$$m_2^0 \le m_2^1 \le m_2^2 \le M_2^2 \le M_2^1 \le M_2^0,$$

$$m_3^0 \le m_3^1 \le m_3^2 \le M_3^2 \le M_3^1 \le M_3^0.$$

By induction, we obtain for $i = 0, 1, ...,$

$$a_1 = m_1^0 \le m_1^1 \le \cdots \le m_1^i \le \cdots \le M_1^i \le \cdots \le M_1^1 \le M_1^0 = b_1,$$

$$a_2 = m_2^0 \le m_2^1 \le \cdots \le m_2^i \le \cdots \le M_2^i \le \cdots \le M_2^1 \le M_2^0 = b_2,$$

$$a_3 = m_3^0 \le m_3^1 \le \cdots \le m_3^i \le \cdots \le M_3^i \le \cdots \le M_3^1 \le M_3^0 = b_3.$$

Now, we have

$$m_1^0 \le \overset{1}{u}_0 \le M_1^0, \quad m_2^0 \le \overset{2}{w}_0 \le M_2^0, \quad m_3^0 \le \overset{3}{v}_0 \le M_3^0, \quad m_3^0 \le \overset{3}{v}_{-1} \le M_3^0.$$

The monotonicity of the functions f, g, and h yields to

$$m_1^1 = f(m_3^0, M_3^0) \le \overset{1}{u}_1 = f(\overset{3}{v}_0, \overset{3}{v}_{-1}) \le f(M_3^0, m_3^0) = M_1^1,$$

$$m_2^1 = g(M_1^0, m_3^0, M_3^0) \le \overset{2}{w}_1 = g(\overset{1}{u}_0, \overset{3}{v}_0, \overset{3}{v}_{-1}) \le g(m_1^0, M_3^0, m_3^0) = M_2^1,$$

$$m_3^1 = h(M_1^0, M_2^0, m_3^0, M_3^0) \le \overset{3}{v}_1 = h(\overset{1}{u}_0, \overset{2}{w}_0, \overset{3}{v}_0, \overset{3}{v}_{-1}) \le h(m_1^0, m_2^0, M_3^0, m_3^0) = M_3^1,$$

$$m_1^1 = f(m_3^0, M_3^0) \le f(m_3^1, M_3^0) \le \overset{1}{u}_2 = f(\overset{3}{v}_1, \overset{3}{v}_0) \le f(M_3^1, m_3^0) \le f(M_3^0, m_3^0) = M_1^1,$$

$$\begin{aligned} m_2^1 = g(M_1^0, m_3^0, M_3^0) \le g(M_1^1, m_3^1, M_3^0) \le \overset{2}{w}_2 = g(\overset{1}{u}_1, \overset{3}{v}_1, \overset{3}{v}_0) \quad &\le g(m_1^1, M_3^1, m_3^0) \\ &\le g(m_1^0, M_3^0, m_3^0) = M_2^1, \end{aligned}$$

$$\begin{aligned} m_3^1 = h(M_1^0, M_2^0, m_3^0, M_3^0) \le h(M_1^1, M_2^1, m_3^1, M_3^0) \le \overset{3}{v}_2 = h(\overset{1}{u}_1, \overset{2}{w}_1, \overset{3}{v}_1, \overset{3}{v}_0) &\le h(m_1^1, m_2^1, M_3^1, m_3^0) \\ &\le h(m_1^0, m_2^0, M_3^0, m_3^0) \\ &= M_3^1, \end{aligned}$$

$$m_1^2 = f(m_3^1, M_3^1) \leq \overset{1}{u}_3 = f(\overset{3}{v}_2, \overset{3}{v}_1) \leq f(M_3^1, m_3^1) = M_1^2,$$

$$m_2^2 = g(M_1^1, m_3^1, M_3^1) \leq \overset{2}{w}_3 = g(\overset{1}{u}_2, \overset{3}{v}_2, \overset{3}{v}_1) \leq g(m_1^1, M_3^1, m_3^1) = M_2^2,$$

$$m_3^2 = h(M_1^1, M_2^1, m_3^1, M_3^1) \leq \overset{3}{v}_3 = h(\overset{1}{u}_2, \overset{2}{w}_2, \overset{3}{v}_2, \overset{3}{v}_1) \leq h(m_1^1, m_2^1, M_3^1, m_3^1) = M_3^2.$$

By induction, it follows that for $i = 1, 2, ...$, we have

$$\begin{cases} m_1^i \leq \overset{1}{u}_k \leq M_1^i, \\ m_2^i \leq \overset{2}{w}_k \leq M_2^i, \quad & for\ k \geq 2i-1. \\ m_3^i \leq \overset{3}{v}_k \leq M_3^i, \end{cases}$$

Let

$$\begin{aligned} m_1 &:= \lim_{i\to\infty} m_1^i, \quad m_2 := \lim_{i\to\infty} m_2^i, \quad m_3 := \lim_{i\to\infty} m_3^i, \\ M_1 &:= \lim_{i\to\infty} M_1^i, \quad M_2 := \lim_{i\to\infty} M_2^i, \quad M_3 := \lim_{i\to\infty} M_3^i. \end{aligned}$$

Then, we obtain

$$m_1 \leq \liminf_{i\to\infty} \overset{1}{u}_i \leq \limsup_{i\to\infty} \overset{1}{u}_i \leq M_1,$$

$$m_2 \leq \liminf_{i\to\infty} \overset{2}{w}_i \leq \limsup_{i\to\infty} \overset{2}{w}_i \leq M_2,$$

$$m_3 \leq \liminf_{i\to\infty} \overset{3}{v}_i \leq \limsup_{i\to\infty} \overset{3}{v}_i \leq M_3.$$

The continuity of the functions f, g, h and (1.3.27) implies that

$$\begin{cases} m_1 = f(m_3, M_3), & M_1 = f(M_3, m_3), \\ m_2 = g(M_1, m_3, M_3), & M_2 = g(m_1, M_3, m_3), \\ m_3 = h(M_1, M_2, m_3, M_3), & M_3 = h(m_1, m_2, M_3, m_3). \end{cases} \tag{1.3.28}$$

System (1.3.28) is equivalent to the system

$$\begin{cases} m_1 = \dfrac{\alpha_1 + m_3}{\alpha_1 + M_3}, \\ M_1 = \dfrac{\alpha_1 + M_3}{\alpha_1 + m_3}, \\ m_2 = \dfrac{\gamma_2\alpha_1 + \gamma_2 M_3 + \alpha_1 + m_3}{(\gamma_2 + M_1)(\alpha_1 + M_3)} = \dfrac{\gamma_2 + m_1}{\gamma_2 + M_1}, \\ M_2 = \dfrac{\gamma_2\alpha_1 + \gamma_2 m_3 + \alpha_1 + M_3}{(\gamma_2 + m_1)(\alpha_1 + m_3)} = \dfrac{\gamma_2 + M_1}{\gamma_2 + m_1}, \\ m_3 = \dfrac{\beta_3(\gamma_2 + M_1)(\alpha_1 + M_3) + \gamma_2\alpha_1 + \gamma_2 M_3 + \alpha_1 + m_3}{(\beta_3 + M_2)(\gamma_2 + M_1)(\alpha_1 + M_3)} = \dfrac{\beta_3 + m_2}{\beta_3 + M_2}, \\ M_3 = \dfrac{\beta_3(\gamma_2 + m_1)(\alpha_1 + m_3) + \gamma_2\alpha_1 + \gamma_2 m_3 + \alpha_1 + M_3}{(\beta_3 + m_2)(\gamma_2 + m_1)(\alpha_1 + m_3)} = \dfrac{\beta_3 + M_2}{\beta_3 + m_2}. \end{cases} \tag{1.3.29}$$

From (1.3.29), we have

$$\begin{aligned} M_1 - m_1 = \frac{\alpha_1 + M_3}{\alpha_1 + m_3} - \frac{\alpha_1 + m_3}{\alpha_1 + M_3} &= \frac{(\alpha_1 + M_3)^2 - (\alpha_1 + m_3)^2}{(\alpha_1 + m_3)(\alpha_1 + M_3)} \\ &= \frac{(M_3 - m_3)(\alpha_1 + M_3 + \alpha_1 + m_3)}{(\alpha_1 + m_3)(\alpha_1 + M_3)} \\ &= (M_3 - m_3)(\frac{1}{\alpha_1 + m_3} + \frac{1}{\alpha_1 + M_3}) \\ &\leq (M_3 - m_3)\frac{2}{\alpha_1 + m_3} \leq \frac{2}{\alpha_1}(M_3 - m_3). \end{aligned}$$

Similarly, we obtain

$$M_2 - m_2 \leq (M_1 - m_1)\frac{2}{\gamma_2 + m_1} \leq \frac{2}{\gamma_2}(M_1 - m_1),$$

$$M_3 - m_3 \leq (M_2 - m_2)\frac{2}{\beta_3 + m_2} \leq \frac{2}{\beta_3}(M_2 - m_2),$$

hence, we get

$$M_1 - m_1 \leq \frac{8}{\alpha_1 \beta_3 \gamma_2}(M_1 - m_1),$$

from which it follows that

$$(1 - \frac{8}{\alpha_1 \beta_3 \gamma_2})(M_1 - m_1) \leq 0.$$

Since $\alpha_1 \beta_3 \gamma_2 > 8$, then $M_1 = m_1$, therefore $M_2 = m_2$ and $M_3 = m_3$. Thus

$$\begin{cases} m_1 &= f(m_3, m_3) = 1 = \bar{x}, \\ m_2 &= g(m_1, m_3, m_3) = 1 = \bar{y}, \\ m_3 &= h(m_1, m_2, m_3, m_3) = 1 = \bar{z}. \end{cases}$$

Then, every solution of System (1.3.26) converges to $(1, 1, 1)$. Hence every solution $\{(\overset{1}{u}_n, \overset{2}{w}_n, \overset{3}{v}_n)\}$ of System (1.3.7) tends to $(1, 1, 1)$ and similarly we can prove that every solution $\{(\overset{1}{v}_n, \overset{2}{u}_n, \overset{3}{w}_n)\}$ (resp. $\{(\overset{1}{w}_n, \overset{2}{v}_n, \overset{3}{u}_n)\}$ of System (1.3.8)(resp (1.3.9)) tends to $(1, 1, 1)$. Consequently, the equilibrium point E of System (1.2.1) is globally attractive. □

In the next theorem, we summarize the behavior of the solutions of the system (1.1.1).

Theorem 1.3.6. *Consider System (1.1.1). Then, the following statements holds:*

(a) Every solution $\{(x_n, y_n, z_n)\}_{n \geq -3}$ of System (1.1.1) is bounded and persists.

(b) The unique equilibrium point of System (1.1.1) is locally asymptotically stable if and only if $(\alpha_3+1)(\beta_2+1)(\gamma_1+1) > 2+\sqrt{5}$, $(\alpha_2+1)(\beta_1+1)(\gamma_3+1) > 2+\sqrt{5}$ *and* $(\alpha_1+1)(\beta_3+1)(\gamma_2+1) > 2+\sqrt{5}$.

(c) Assume that $(\alpha_3+1)(\beta_2+1)(\gamma_1+1) < 2+\sqrt{5}$ *or* $(\alpha_2+1)(\beta_1+1)(\gamma_3+1) < 2+\sqrt{5}$ *or* $(\alpha_1+1)(\beta_3+1)(\gamma_2+1) < 2+\sqrt{5}$. *Then the unique equilibrium point of System (1.1.1) is unstable.*

(d) Assume that $\alpha_1\beta_3\gamma_2 > 8$, $\alpha_2\beta_1\gamma_3 > 8$ *and* $\alpha_3\beta_2\gamma_1 > 8$. *Then the unique equilibrium point of System (1.1.1) is globally asymptotically stable.*

(e) Assume that α_i, β_i, γ_i, $(i = 1, 2, 3) \in [1, +\infty)$ *such that*

- *At least one of* α_1, β_3, $\gamma_2 \in (1, +\infty)$.
- *At least one of* α_2, β_1, $\gamma_3 \in (1, +\infty)$.
- *At least one of* α_3, β_2, $\gamma_1 \in (1, +\infty)$.

Then the unique positive equilibrium of System (1.1.1) is globally asymptotically stable.

Proof. ♦ Statements (a), (b) and (c) are the results of Theorems 1.2.1 and 1.3.3.

♦ Statement (d): The global attractivity derives from Theorem 1.3.5. Concerning the local asymptotic stability, it follows from the fact that if $\alpha_1\beta_3\gamma_2 > 8$, $\alpha_2\beta_1\gamma_3 > 8$ and $\alpha_3\beta_2\gamma_1 > 8$, then we get $(\alpha_3+1)(\beta_2+1)(\gamma_1+1) > 2+\sqrt{5}$, $(\alpha_2+1)(\beta_1+1)(\gamma_3+1) > 2+\sqrt{5}$ and $(\alpha_1+1)(\beta_3+1)(\gamma_2+1) > 2+\sqrt{5}$.

♦ Statement (e): The global attractivity follows from Theorem 1.3.4. It is clear that in each case the conditions $(\alpha_3+1)(\beta_2+1)(\gamma_1+1) > 2+\sqrt{5}$, $(\alpha_2+1)(\beta_1+1)(\gamma_3+1) > 2+\sqrt{5}$ and $(\alpha_1+1)(\beta_3+1)(\gamma_2+1) > 2+\sqrt{5}$ are verified, consequently the unique equilibrium point of the system (1.1.1) is locally asymptotically stable.

□

1.4 Periodic solutions and rate of convergence

In this part, we will show the non existence of periodic solutions of period three and six. Also, we will use some known results of Perron to establish a result on the rate of convergence of the solutions of System (1.1.1).

1.4.1 Periodic Solutions

Definition 1.4.1. *A solution* $(x_n, y_n, z_n)_{n\geq -3}$ *of System (1.1.1) is said to be periodic of period* $p \in \mathbb{N}$ *if*

$$x_{n+p} = x_n, \quad y_{n+p} = y_n, \quad z_{n+p} = z_n, \quad \textit{for all } n \geq -3.$$

Theorem 1.4.1. *System (1.1.1) has neither prime period-three nor prime period-six solutions.*

Proof. Assume that System (1.1.1) has a prime period-three solution of the form

$$..., (a_1, b_1, c_1), \ (a_2, b_2, c_2), \ (a_3, b_3, c_3), ...,$$

then it follows that

$$\begin{aligned}
a_1 &= \frac{p_n + b_3}{p_n + b_3} = 1, \ b_1 = \frac{q_n + c_3}{q_n + c_3} = 1, \ c_1 = \frac{r_n + a_3}{r_n + a_3} = 1, \\
a_2 &= \frac{p_n + b_1}{p_n + b_1} = 1, \ b_2 = \frac{q_n + c_1}{q_n + c_1} = 1, \ c_2 = \frac{r_n + a_1}{r_n + a_1} = 1, \\
a_3 &= \frac{p_n + b_2}{p_n + b_2} = 1, \ b_3 = \frac{q_n + c_2}{q_n + c_2} = 1, \ c_3 = \frac{r_n + a_2}{r_n + a_2} = 1.
\end{aligned}$$

So, System (1.1.1) has no prime period-three solutions.

Now, to prove that System (1.1.1) has no prime period-six solutions it is sufficient to show that systems (1.3.7), (1.3.8) and (1.3.9) has no prime period-two solutions. For the sake of contradiction, assume that System (1.3.7) has a prime period-two solution of the form

$$..., (a, b, c), \ (A, B, C), (a, b, c), ...,$$

then, we get

$$a = \frac{\alpha_1 + C}{\alpha_1 + c}, \ A = \frac{\alpha_1 + c}{\alpha_1 + C}, \ b = \frac{\gamma_2 + a}{\gamma_2 + A}, \ B = \frac{\gamma_2 + A}{\gamma_2 + a}, \ c = \frac{\beta_3 + b}{\beta_3 + B}, \ C = \frac{\beta_3 + B}{\beta_3 + b}. \quad (1.4.1)$$

From (1.4.1), we obtain that

$$A - a = \frac{(a - A)(2\gamma_2 + a + A)(2\beta_3 + b + B)(2\alpha_1 + c + C)}{(\gamma_2 + a)(\gamma_2 + A)(\beta_3 + b)(\beta_3 + B)(\alpha_1 + c)(\alpha_1 + C)},$$

which implies

$$(A-a)(1+\frac{(2\gamma_2+a+A)(2\beta_3+b+B)(2\alpha_1+c+C)}{(\gamma_2+a)(\gamma_2+A)(\beta_3+b)(\beta_3+B)(\alpha_1+c)(\alpha_1+C)})=0.$$

Hence, $A=a$. So, again from (1.4.1), we get $B=b=C=c=1$ and $A=a=1$. This completes the proof. □

1.4.2 Rate of convergence

Consider the system of difference equations

$$U_{n+1}=(A+B(n))U_n, \quad n=0,1,..., \tag{1.4.2}$$

where U_n is an m-dimensional vector, $A\in C^{m\times m}$ is a constant matrix and $B:\mathbb{Z}^+\to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\|\longrightarrow 0,\ n\to\infty \tag{1.4.3}$$

where $\|.\|$ denotes any matrix norm. The following theorem help to determine the rate of convergence of the solutions of System (1.4.2).

Theorem 1.4.2. *(**Perron's Theorem** [24]) Consider System (1.4.2) where the condition (1.4.3) is verified, and let $(U_n)_{n\geq 0}$ be a solution of (1.4.2), then either $U_n=0$ for all n or*

$$\rho=\lim_{n\to\infty}\frac{\|U_{n+1}\|}{\|U_n\|}\ \left(\rho=\lim_{n\to\infty}(\|U_n\|)^{1/n}\right)$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Now, we will establish a result on the rate of convergence of a solution of System (1.1.1) that converges to the unique equilibrium point. We will work on the equivalent System (1.2.1). Assume that $\{(\overset{1}{u}_n,\overset{2}{u}_n,\overset{3}{u}_n,\overset{1}{v}_n,\overset{2}{v}_n,\overset{3}{v}_n,\overset{1}{w}_n,\overset{2}{w}_n,\overset{3}{w}_n)\}$ is a solution of System (1.2.1) that converges to the unique equilibrium point $E=(1,1,1,1,1,1,1,1,1)$.

For $n\geq 0$, let

$$\overset{1}{e}_n=\overset{1}{u}_n-1,\quad \overset{2}{e}_n=\overset{2}{u}_n-1,\quad \overset{3}{e}_n=\overset{3}{u}_n-1,$$

$$\overset{4}{e}_n=\overset{1}{v}_n-1,\quad \overset{5}{e}_n=\overset{2}{v}_n-1,\quad \overset{6}{e}_n=\overset{3}{v}_n-1,$$

$$\overset{7}{e}_n=\overset{1}{w}_n-1,\quad \overset{8}{e}_n=\overset{2}{w}_n-1,\quad \overset{9}{e}_n=\overset{3}{w}_n-1.$$

We have

$$\overset{1}{u}_{n+1} - 1 = \frac{\overset{3}{v}_n - \overset{3}{v}_{n-1}}{\alpha_1 + \overset{3}{v}_{n-1}} = \frac{1}{\alpha_1 + \overset{3}{v}_{n-1}}(\overset{3}{v}_n - 1) - \frac{1}{\alpha_1 + \overset{3}{v}_{n-1}}(\overset{3}{v}_{n-1} - 1),$$

so,

$$\overset{1}{e}_{n+1} = \frac{1}{\alpha_1 + \overset{3}{v}_{n-1}}\overset{6}{e}_n - \frac{1}{\alpha_1 + \overset{3}{v}_{n-1}}\overset{6}{e}_{n-1}.$$

Also

$$\begin{aligned}\overset{2}{u}_{n+1} - 1 =& \frac{\beta_1 + \overset{3}{w}_n - \overset{1}{v}_n(\beta_1 + \overset{3}{w}_{n-1})}{(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1})} = \frac{\overset{3}{w}_n - \overset{3}{w}_{n-1} - (\overset{1}{v}_n - 1)(\beta_1 + \overset{3}{w}_{n-1})}{(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1})}\\ =& \frac{1}{(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1})}(\overset{3}{w}_n - 1) - \frac{1}{(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1})}(\overset{3}{w}_{n-1} - 1)\\ & - \frac{1}{(\alpha_2 + \overset{1}{v}_n)}(\overset{1}{v}_n - 1),\end{aligned}$$

so,

$$\overset{2}{e}_{n+1} = -\frac{1}{\alpha_2 + \overset{1}{v}_n}\overset{4}{e}_n + \frac{1}{(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1})}\overset{9}{e}_n - \frac{1}{(\alpha_2 + \overset{1}{v}_n)(\beta_1 + \overset{3}{w}_{n-1})}\overset{9}{e}_{n-1}.$$

We have

$$\begin{aligned}\overset{3}{u}_{n+1} - 1 =& \frac{\beta_2(\gamma_1 + \overset{3}{u}_{n-1}) + \gamma_1 + \overset{3}{u}_n - \overset{2}{v}_n(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}\\ =& \frac{(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1}) - \overset{1}{w}_n(\gamma_1 + \overset{3}{u}_{n-1}) + \gamma_1 + \overset{3}{u}_n - \overset{2}{v}_n(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}\\ =& \frac{-(\overset{1}{w}_n - 1)(\gamma_1 + \overset{3}{u}_{n-1}) - \overset{3}{u}_{n-1} + \overset{3}{u}_n - (\overset{2}{v}_n - 1)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}\\ =& -\frac{1}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)}(\overset{1}{w}_n - 1) - \frac{1}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}(\overset{3}{u}_{n-1} - 1)\\ & + \frac{1}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}(\overset{3}{u}_n - 1) - \frac{1}{\alpha_3 + \overset{2}{v}_n}(\overset{2}{v}_n - 1),\end{aligned}$$

then,

$$\begin{aligned}\overset{3}{e}_{n+1} =& \frac{1}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}\overset{3}{e}_n - \frac{1}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}\overset{3}{e}_{n-1} - \frac{1}{\alpha_3 + \overset{2}{v}_n}\overset{5}{e}_n\\ & - \frac{1}{(\alpha_3 + \overset{2}{v}_n)(\beta_2 + \overset{1}{w}_n)}\overset{7}{e}_n.\end{aligned}$$

Similarly, we get

$$\overset{1}{v}_{n+1} - 1 = \frac{1}{\beta_1 + \overset{3}{w}_{n-1}}(\overset{3}{w}_n - 1) - \frac{1}{\beta_1 + \overset{3}{w}_{n-1}}(\overset{3}{w}_{n-1} - 1),$$

thus,

$$\overset{4}{e}_{n+1} = \frac{1}{\beta_1 + \overset{3}{w}_{n-1}}\overset{9}{e}_n - \frac{1}{\beta_1 + \overset{3}{w}_{n-1}}\overset{9}{e}_{n-1}.$$

Also,

$$\overset{2}{v}_{n+1} - 1 = \frac{1}{(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}(\overset{3}{u}_n - 1) - \frac{1}{(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}(\overset{3}{u}_{n-1} - 1) - \frac{1}{\beta_2 + \overset{1}{w}_n}(\overset{1}{w}_n - 1),$$

therefore,

$$\overset{5}{e}_{n+1} = \frac{1}{(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}\overset{3}{e}_n - \frac{1}{(\beta_2 + \overset{1}{w}_n)(\gamma_1 + \overset{3}{u}_{n-1})}\overset{3}{e}_{n-1} - \frac{1}{\beta_2 + \overset{1}{w}_n}\overset{7}{e}_n.$$

We have

$$\begin{aligned}\overset{3}{v}_{n+1} - 1 = &- \frac{1}{(\beta_3 + \overset{2}{w}_n)(\gamma_2 + \overset{1}{u}_n)}(\overset{1}{u}_n - 1) - \frac{1}{(\beta_3 + \overset{2}{w}_n)(\gamma_2 + \overset{1}{u}_n)(\alpha_1 + \overset{3}{v}_{n-1})}(\overset{3}{v}_{n-1} - 1)\\ &+ \frac{1}{(\beta_3 + \overset{2}{w}_n)(\gamma_2 + \overset{1}{u}_n)(\alpha_1 + \overset{3}{v}_{n-1})}(\overset{3}{v}_n - 1) - \frac{1}{\beta_3 + \overset{2}{w}_n}(\overset{2}{w}_n - 1),\end{aligned}$$

then,

$$\begin{aligned}\overset{6}{e}_{n+1} = &- \frac{1}{(\beta_3 + \overset{2}{w}_n)(\gamma_2 + \overset{1}{u}_n)}\overset{1}{e}_n + \frac{1}{(\beta_3 + \overset{2}{w}_n)(\gamma_2 + \overset{1}{u}_n)(\alpha_1 + \overset{3}{v}_{n-1})}\overset{6}{e}_n\\ &- \frac{1}{(\beta_3 + \overset{2}{w}_n)(\gamma_2 + \overset{1}{u}_n)(\alpha_1 + \overset{3}{v}_{n-1})}\overset{6}{e}_{n-1} - \frac{1}{\beta_3 + \overset{2}{w}_n}\overset{8}{e}_n.\end{aligned}$$

In a similar way, we have

$$\overset{1}{w}_{n+1} - 1 = \frac{1}{\gamma_1 + \overset{3}{u}_{n-1}}(\overset{3}{u}_n - 1) - \frac{1}{\gamma_1 + \overset{3}{u}_{n-1}}(\overset{3}{u}_{n-1} - 1),$$

which implies that

$$\overset{7}{e}_{n+1} = \frac{1}{\gamma_1 + \overset{3}{u}_{n-1}}\overset{3}{e}_n - \frac{1}{\gamma_1 + \overset{3}{u}_{n-1}}\overset{3}{e}_{n-1}.$$

Furthermore,

$$\begin{aligned}\overset{2}{w}_{n+1}-1=&\frac{1}{(\gamma_2+\overset{1}{u}_n)(\alpha_1+\overset{3}{v}_{n-1})}(\overset{3}{v}_n-1)-\frac{1}{(\gamma_2+\overset{1}{u}_n)(\alpha_1+\overset{3}{v}_{n-1})}(\overset{3}{v}_{n-1}-1)\\&-\frac{1}{\gamma_2+\overset{1}{u}_n}(\overset{1}{u}_n-1),\end{aligned}$$

thus,

$$\overset{8}{e}_{n+1}=-\frac{1}{\gamma_2+\overset{1}{u}_n}\overset{1}{e}_n+\frac{1}{(\gamma_2+\overset{1}{u}_n)(\alpha_1+\overset{3}{v}_{n-1})}\overset{6}{e}_n-\frac{1}{(\gamma_2+\overset{1}{u}_n)(\alpha_1+\overset{3}{v}_{n-1})}\overset{6}{e}_{n-1}.$$

Finally,

$$\begin{aligned}\overset{3}{w}_{n+1}-1=&-\frac{1}{(\gamma_3+\overset{2}{u}_n)(\alpha_2+\overset{1}{v}_n)}(\overset{1}{v}_n-1)-\frac{1}{(\gamma_3+\overset{2}{u}_n)(\alpha_2+\overset{1}{v}_n)(\beta_1+\overset{3}{w}_{n-1})}(\overset{3}{w}_{n-1}-1)\\&+\frac{1}{(\gamma_3+\overset{2}{u}_n)(\alpha_2+\overset{1}{v}_n)(\beta_1+\overset{3}{w}_{n-1})}(\overset{3}{w}_n-1)-\frac{1}{\gamma_3+\overset{2}{u}_n}(\overset{2}{u}_n-1),\end{aligned}$$

which yields

$$\begin{aligned}\overset{9}{e}_{n+1}=&-\frac{1}{\gamma_3+\overset{2}{u}_n}\overset{2}{e}_n-\frac{1}{(\gamma_3+\overset{2}{u}_n)(\alpha_2+\overset{1}{v}_n)}\overset{4}{e}_n+\frac{1}{(\gamma_3+\overset{2}{u}_n)(\alpha_2+\overset{1}{v}_n)(\beta_1+\overset{3}{w}_{n-1})}\overset{9}{e}_n\\&-\frac{1}{(\gamma_3+\overset{2}{u}_n)(\alpha_2+\overset{1}{v}_n)(\beta_1+\overset{3}{w}_{n-1})}\overset{9}{e}_{n-1}.\end{aligned}$$

Note that

$$\lim_{n\to\infty}\overset{1}{u}_n=\lim_{n\to\infty}\overset{2}{u}_n=\lim_{n\to\infty}\overset{3}{u}_n=1,$$

$$\lim_{n\to\infty}\overset{1}{v}_n=\lim_{n\to\infty}\overset{2}{v}_n=\lim_{n\to\infty}\overset{3}{v}_n=1,$$

$$\lim_{n\to\infty}\overset{1}{w}_n=\lim_{n\to\infty}\overset{2}{w}_n=\lim_{n\to\infty}\overset{3}{w}_n=1.$$

So, we can write

$$\frac{1}{\alpha_1+\overset{3}{v}_{n-1}}=\frac{1}{\alpha_1+1}+b_1(n),\qquad \frac{1}{\alpha_2+\overset{1}{v}_n}=\frac{1}{\alpha_2+1}+b_2(n),$$

$$\frac{1}{(\alpha_2+\overset{1}{v}_n)(\beta_1+\overset{3}{w}_{n-1})}=\frac{1}{(\alpha_2+1)(\beta_1+1)}+b_3(n),$$

$$\frac{1}{(\alpha_3+\overset{2}{v}_n)(\beta_2+\overset{1}{w}_n)(\gamma_1+\overset{3}{u}_{n-1})}=\frac{1}{(\alpha_3+1)(\beta_2+1)(\gamma_1+1)}+b_4(n),$$

$$\frac{1}{\alpha_3+\overset{2}{v}_n}=\frac{1}{\alpha_3+1}+b_5(n),\qquad \frac{1}{(\alpha_3+\overset{2}{v}_n)(\beta_2+\overset{1}{w}_n)}=\frac{1}{(\alpha_3+1)(\beta_2+1)}+b_6(n),$$

$$\frac{1}{\beta_1+\overset{3}{w}_{n-1}}=\frac{1}{\beta_1+1}+b_7(n),\quad \frac{1}{(\beta_2+\overset{1}{w}_n)(\gamma_1+\overset{3}{u}_{n-1})}=\frac{1}{(\beta_2+1)(\gamma_1+1)}+b_8(n),$$
$$\frac{1}{\beta_2+\overset{1}{w}_n}=\frac{1}{\beta_2+1}+b_9(n),\quad \frac{1}{(\beta_3+\overset{2}{w}_n)(\gamma_2+\overset{1}{u}_n)}=\frac{1}{(\beta_3+1)(\gamma_2+1)}+b_{10}(n),$$
$$\frac{1}{(\beta_3+\overset{2}{w}_n)(\gamma_2+\overset{1}{u}_n)(\alpha_1+\overset{3}{v}_{n-1})}=\frac{1}{(\beta_3+1)(\gamma_2+1)(\alpha_1+1)}+b_{11}(n),$$
$$\frac{1}{\beta_3+\overset{2}{w}_n}=\frac{1}{\beta_3+1}+b_{12}(n),\quad \frac{1}{\gamma_1+\overset{3}{u}_{n-1}}=\frac{1}{\gamma_1+1}+b_{13}(n),$$
$$\frac{1}{\gamma_2+\overset{1}{u}_n}=\frac{1}{\gamma_2+1}+b_{14}(n),\quad \frac{1}{(\gamma_2+\overset{1}{u}_n)(\alpha_1+\overset{3}{v}_{n-1})}=\frac{1}{(\gamma_2+1)(\alpha_1+1)}+b_{15}(n),$$
$$\frac{1}{\gamma_3+\overset{2}{u}_n}=\frac{1}{\gamma_3+1}+b_{16}(n),\quad \frac{1}{(\gamma_3+\overset{2}{u}_n)(\alpha_2+\overset{1}{v}_n)}=\frac{1}{(\gamma_3+1)(\alpha_2+1)}+b_{17}(n),$$
$$\frac{1}{(\gamma_3+\overset{2}{u}_n)(\alpha_2+\overset{1}{v}_n)(\beta_1+\overset{3}{w}_{n-1})}=\frac{1}{(\gamma_3+1)(\alpha_2+1)(\beta_1+1)}+b_{18}(n),$$

such that $\lim_{n\to\infty} b_i(n)=0,\ i=1,2,...,18$. Then, we get the system

$$E_{n+1}=(A+B(n))E_n,\quad n=0,1,\ldots,$$

where $E_n=(\overset{1}{e}_n,\overset{2}{e}_n,\overset{3}{e}_n,\overset{3}{e}_{n-1},\overset{4}{e}_n,\overset{5}{e}_n,\overset{6}{e}_n,\overset{6}{e}_{n-1},\overset{7}{e}_n,\overset{8}{e}_n,\overset{9}{e}_n,\overset{9}{e}_{n-1})^T$, the constant matrix A is given by

$$A=\begin{pmatrix}
0&0&0&0&0&0&a_{1,7}&a_{1,8}&0&0&0&0\\
0&0&0&0&a_{2,5}&0&0&0&0&0&a_{2,11}&a_{2,12}\\
0&0&a_{3,3}&a_{3,4}&0&a_{3,6}&0&0&a_{3,9}&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&a_{5,11}&a_{5,12}\\
0&0&a_{6,3}&a_{6,4}&0&0&0&0&a_{6,9}&0&0&0\\
a_{7,1}&0&0&0&0&0&a_{7,7}&a_{7,8}&0&a_{7,10}&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&a_{9,3}&a_{9,4}&0&0&0&0&0&0&0&0\\
a_{10,1}&0&0&0&0&0&a_{10,7}&a_{10,8}&0&0&0&0\\
0&a_{11,2}&0&0&a_{11,5}&0&0&0&0&0&a_{11,11}&a_{11,12}\\
0&0&0&0&0&0&0&0&0&0&1&0
\end{pmatrix}.$$

Where

$$a_{1,7}=\frac{1}{\alpha_1+1},\quad a_{1,8}=\frac{-1}{\alpha_1+1},\quad a_{2,5}=\frac{-1}{\alpha_2+1},\quad a_{2,11}=\frac{1}{(\alpha_2+1)(\beta_1+1)},$$
$$a_{2,12}=\frac{-1}{(\alpha_2+1)(\beta_1+1)},\quad a_{3,3}=\frac{1}{(\alpha_3+1)(\beta_2+1)(\gamma_1+1)},\quad a_{3,4}=\frac{-1}{(\alpha_3+1)(\beta_2+1)(\gamma_1+1)},$$

$a_{3,6} = \frac{-1}{\alpha_3+1}, \quad a_{3,9} = \frac{-1}{(\alpha_3+1)(\beta_2+1)}, \quad a_{5,11} = \frac{1}{\beta_1+1}, \quad a_{5,12} = \frac{-1}{\beta_1+1},$
$a_{6,3} = \frac{1}{(\beta_2+1)(\gamma_1+1)}, a_{6,4} = \frac{-1}{(\beta_2+1)(\gamma_1+1)}, a_{6,9} = -\frac{1}{\beta_2+1}, a_{7,1} = \frac{-1}{(\beta_3+1)(\gamma_2+1)},$
$a_{7,7} = \frac{1}{(\beta_3+1)(\gamma_2+1)(\alpha_1+1)}, \quad a_{7,8} = \frac{-1}{(\beta_3+1)(\gamma_2+1)(\alpha_1+1)}, \quad a_{7,10} = \frac{-1}{\beta_3+1},$
$a_{9,3} = \frac{1}{\gamma_1+1}, \quad a_{9,4} = \frac{-1}{\gamma_1+1}, \quad a_{10,1} = \frac{-1}{(\gamma_2+1)}, \quad a_{10,7} = \frac{1}{(\gamma_2+1)(\alpha_1+1)},$
$a_{10,8} = \frac{-1}{(\gamma_2+1)(\alpha_1+1)}, \quad a_{11,2} = \frac{-1}{\gamma_3+1}, \quad a_{11,5} = \frac{-1}{(\gamma_3+1)(\alpha_2+1)},$
$a_{11,11} = \frac{1}{(\gamma_3+1)(\alpha_2+1)(\beta_1+1)}, \quad a_{11,12} = \frac{-1}{(\gamma_3+1)(\alpha_2+1)(\beta_1+1)},$

and

$$B(n) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & b_1 & -b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_2 & 0 & 0 & 0 & 0 & 0 & b_3 & -b_3 \\ 0 & 0 & b_4 & -b_4 & 0 & -b_5 & 0 & 0 & -b_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_7 & -b_7 \\ 0 & 0 & b_8 & -b_8 & 0 & 0 & 0 & 0 & -b_9 & 0 & 0 & 0 \\ -b_{10} & 0 & 0 & 0 & 0 & 0 & b_{11} & -b_{11} & 0 & -b_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{13} & -b_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_{14} & 0 & 0 & 0 & 0 & 0 & b_{15} & -b_{15} & 0 & 0 & 0 & 0 \\ 0 & -b_{16} & 0 & 0 & -b_{17} & 0 & 0 & 0 & 0 & 0 & b_{18} & -b_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $b_i = b_i(n), \quad i = 1, 2, \dots 18,$ with $\|B(n)\| \longrightarrow 0$ for $n \longrightarrow +\infty$. Using Perron's Theorem (Theorem 1.4.2), we get the following result.

Theorem 1.4.3. *Assume that* $\{(\overset{1}{u}_n, \overset{2}{u}_n, \overset{3}{u}_n, \overset{1}{v}_n, \overset{2}{v}_n, \overset{3}{v}_n, \overset{1}{w}_n, \overset{2}{w}_n, \overset{3}{w}_n)\}$ *is a solution of the System (1.2.1) that converges to the unique equilibrium point E. Then, the error vector*

$$E_n = (\overset{1}{e}_n, \overset{2}{e}_n, \overset{3}{e}_n, \overset{3}{e}_{n-1}, \overset{4}{e}_n, \overset{5}{e}_n, \overset{6}{e}_n, \overset{6}{e}_{n-1}, \overset{7}{e}_n, \overset{8}{e}_n, \overset{9}{e}_n, \overset{9}{e}_{n-1})^T$$

of every solution of System (1.2.1) satisfies both of the asymptotic relations

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \quad \left(\rho = \lim_{n\to\infty} (\|X_n\|)^{1/n},\right)$$

where $\overset{1}{e}_n = \overset{1}{u}_n - 1,\ \overset{2}{e}_n = \overset{2}{u}_n - 1,\ \overset{3}{e}_n = \overset{3}{u}_n - 1,\ \overset{4}{e}_n = \overset{1}{v}_n - 1,\ \overset{5}{e}_n = \overset{2}{v}_n - 1,\ \overset{6}{e}_n = \overset{3}{v}_n - 1,$ $\overset{7}{e}_n = \overset{1}{w}_n - 1,\ \overset{8}{e}_n = \overset{2}{w}_n - 1,\ \overset{9}{e}_n = \overset{3}{w}_n - 1$ *and* ρ *is equal to the modulus of one of the eigenvalues of the Jacobian matrix* A.

1.5 Numerical examples

Here, we provide numerical examples that illustrate our obtained results.

Example 1.5.1. *Consider System (1.1.1) with initial conditions* $x_{-3} = 12,\ x_{-2} = 4,$ $x_{-1} = 11,\ x_0 = 1.5,\ y_{-3} = 2,\ y_{-2} = 5,\ y_{-1} = 0.2,\ y_0 = 0.8,\ z_{-3} = 5,\ z_{-2} = 2,\ z_{-1} = 0.6,$ $z_0 = 3.8,$ *and let be the parameters* $\alpha_1 = 1,\ \alpha_2 = 1.2,\ \alpha_3 = 4,\ \beta_1 = 6,\ \beta_2 = 1,\ \beta_3 = 1.7,$ $\gamma_1 = 10,\ \gamma_2 = 1,$ *and* $\gamma_3 = 2,$ *i.e.,*

$$p_n = \begin{cases} 1, & if\ \ n = 3k \\ 1.2, & if\ \ n = 3k+1 \\ 4, & if\ \ n = 3k+2 \end{cases},\quad q_n = \begin{cases} 6, & if\ \ n = 3k \\ 1, & if\ \ n = 3k+1 \\ 1.7, & if\ \ n = 3k+2 \end{cases}$$

and

$$r_n = \begin{cases} 10, & if\ \ n = 3k \\ 1, & if\ \ n = 3k+1 \\ 2, & if\ \ n = 3k+2 \end{cases}$$

In this case, we have $\alpha_1 = \beta_2 = \gamma_2 = 1$ *and* $\alpha_2,\ \alpha_3,\ \beta_1,\ \beta_3,\ \gamma_1,\ \gamma_3 > 1$. *Thus, from Theorem 1.3.6 the unique equilibrium point of System (1.1.1) is globally asymptotically stable. The plots of* x_n, y_n *and* z_n *are shown in Figure 1.1, Figure 1.2 and Figure 1.3 respectively.*

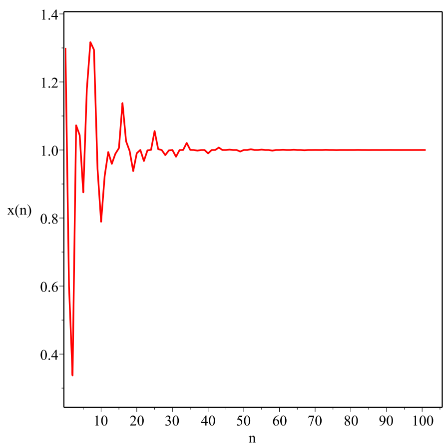

Figure 1.1 – Plot of x_n: Example 1.5.1

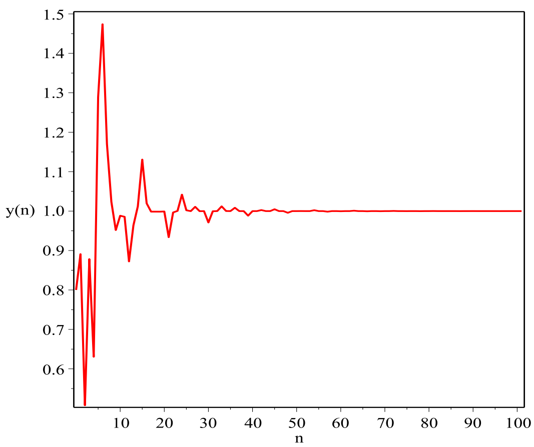

Figure 1.2 – Plot of y_n: Example 1.5.1

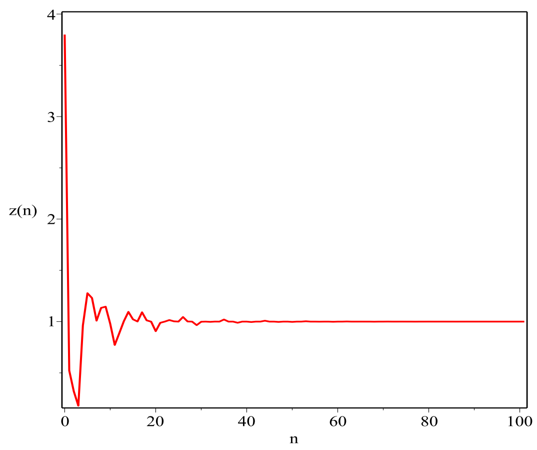

Figure 1.3 – Plot of z_n: Example 1.5.1

Example 1.5.2. *Consider System (1.1.1) with initial conditions* $x_{-3} = 1.6$, $x_{-2} = 0.8$, $x_{-1} = 4.3$, $x_0 = 1.4$, $y_{-3} = 0.8$, $y_{-2} = 3$, $y_{-1} = 0.2$, $y_0 = 2.5$, $z_{-3} = 0.4$, $z_{-2} = 4$, $z_{-1} = 2.5$, *and* $z_0 = 0.8$ *with* $\alpha_1 = 20$, $\alpha_2 = 0.8$, $\alpha_3 = 11$, $\beta_1 = 3$, $\beta_2 = 8$, $\beta_3 = 15$, $\gamma_1 = 4$, $\gamma_2 = 23$, *and* $\gamma_3 = 4.3$, *i.e.,*

$$p_n = \begin{cases} 20, & if \quad n = 3k \\ 0.8, & if \quad n = 3k+1 \\ 11, & if \quad n = 3k+2 \end{cases}, \quad q_n = \begin{cases} 3, & if \quad n = 3k \\ 8, & if \quad n = 3k+1 \\ 15, & if \quad n = 3k+2 \end{cases}$$

and

$$r_n = \begin{cases} 4, & if \quad n = 3k \\ 23, & if \quad n = 3k+1 \\ 4.3, & if \quad n = 3k+2 \end{cases}$$

In this case, we have $\alpha_2\beta_1\gamma_3 > 8, \alpha_1\beta_3\gamma_2 > 8$ *and* $\alpha_3\beta_2\gamma_1 > 8$. *Hence, from Theorem 1.3.6 the unique equilibrium point of System (1.1.1) is globally asymptotically stable. The plots of* x_n, y_n *and* z_n *are shown in Figure 1.4, Figure 1.5* and *Figure 1.6 respectively.*

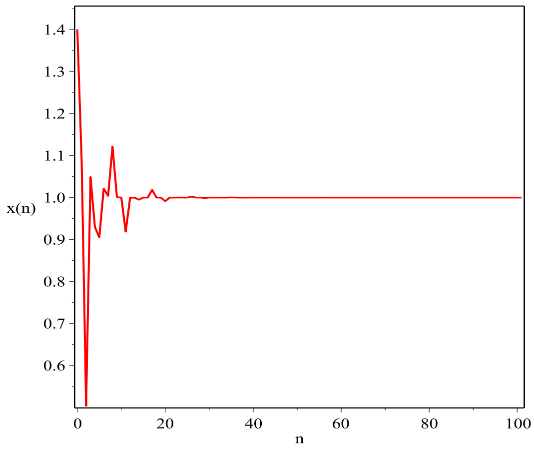

Figure 1.4 – Plot of x_n: Example 1.5.2

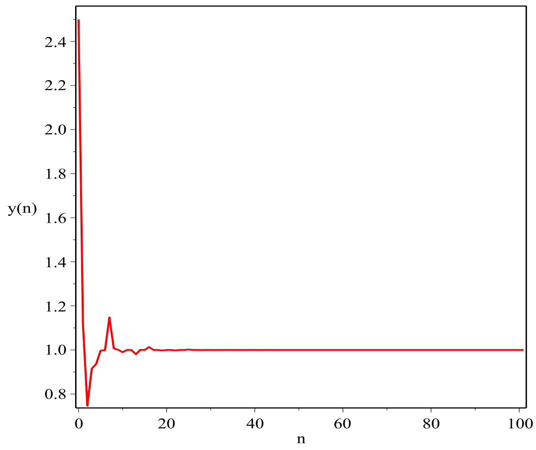

Figure 1.5 – Plot of y_n: Example 1.5.2

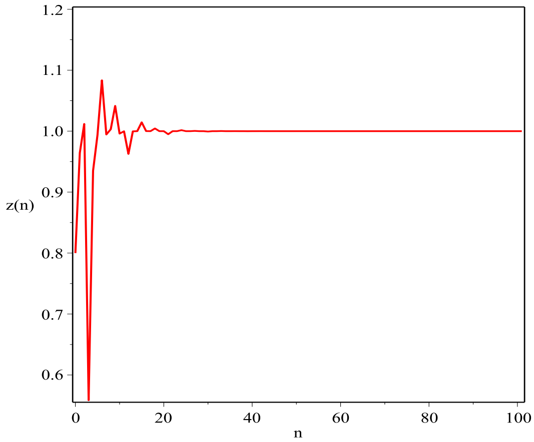

Figure 1.6 – Plot of z_n: Example 1.5.2

Example 1.5.3. *Consider System (1.1.1) with initial conditions* $x_{-3} = 12$, $x_{-2} = 4$, $x_{-1} = 11, x_0 = 1.5$, $y_{-3} = 2$, $y_{-2} = 5$, $y_{-1} = 0.2$, $y_0 = 0.8$, $z_{-3} = 5$, $z_{-2} = 2$, $z_{-1} = 0.6$, $z_0 = 3.8$, *and* $\alpha_1 = 2$, $\alpha_2 = 0.2$, $\alpha_3 = 1.7$, $\beta_1 = 0.4$, $\beta_2 = 3$, $\beta_3 = 2.4$, $\gamma_1 = 6$, $\gamma_2 = 8$, *and* $\gamma_3 = 0.6$, *i.e.,*

$$p_n = \begin{cases} 2, & if \quad n = 3k \\ 0.2, & if \quad n = 3k+1 \\ 1.7, & if \quad n = 3k+2 \end{cases}, \quad q_n = \begin{cases} 0.4, & if \quad n = 3k \\ 3, & if \quad n = 3k+1 \\ 2.4, & if \quad n = 3k+2 \end{cases}$$

and

$$r_n = \begin{cases} 6, & if \quad n = 3k \\ 8, & if \quad n = 3k+1 \\ 0.6, & if \quad n = 3k+2 \end{cases}$$

In this case, we have $(\alpha_2 + 1)(\beta_1 + 1)(\gamma_3 + 1) = 2.688 < 2 + \sqrt{5}$. *Thus, from Theorem*

1.3.6 the unique equilibrium point of System (1.1.1) is unstable. The plots of x_n, y_n *and* z_n *are shown in Figure 1.7, Figure 1.8 and Figure 1.9 respectively.*

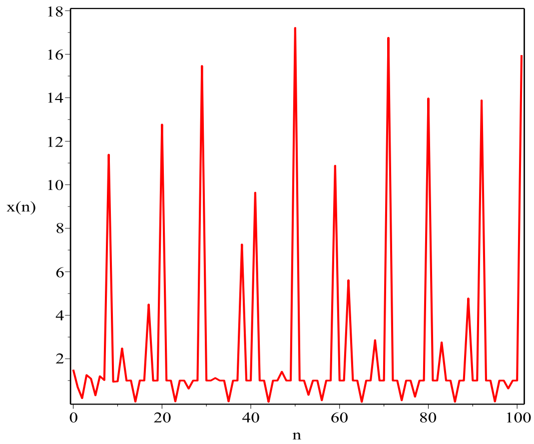

Figure 1.7 – Plot of x_n: Example 1.5.3

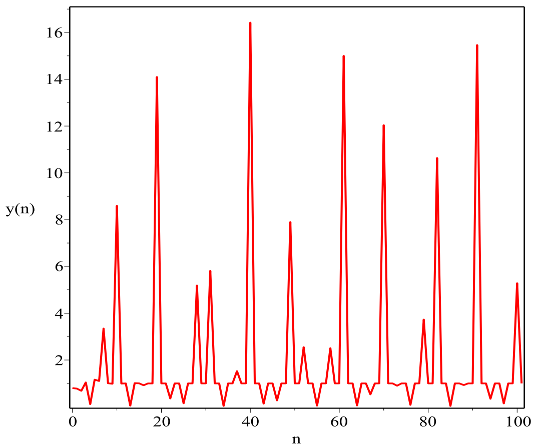

Figure 1.8 – Plot of y_n: Example 1.5.3

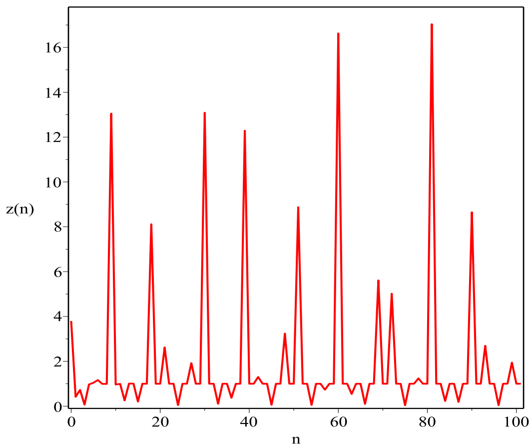

Figure 1.9 – Plot of z_n: Example 1.5.3

Chapter 2

A GENERAL SECOND-ORDER SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS

2.1 Introduction

Recent developments in difference equations, which have been studied with great interest since the 90s, continue with the generalization of some results in the literature. A typical situation is seen in difference equations defined by homogeneous functions of degree zero. The reason for this is that due to their homogeneous function structures of degree zero, results such as stability, periodicity, etc. can be easily obtained on such equations. Also, their equilibrium points, if any, are unique. This advantage can also be used to generalize such equations. For some difference equations constructed with homogeneous functions, see [8, 18, 20, 22, 31]. In [47], the authors considered the general second-order difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n \in \mathbb{N}_0, \tag{2.1.1}$$

where the function $f : (0,\infty)^2 \to (0,\infty)$ is continuous and homogeneous of degree zero and obtained very applicable general results. Developments on the subject continued with the system of general difference equations

$$x_{n+1} = f(y_n, y_{n-1}), \quad y_{n+1} = g(x_n, x_{n-1}), \quad n \in \mathbb{N}_0, \tag{2.1.2}$$

where the functions $f, g : (0, \infty)^2 \to (0, \infty)$ are continuous and homogeneous of degree zero, studied by Touafek in [50], thus yielding more general results, in particular the results of [50] provide conditions for global stability of the equilibrium point of equation (2.1.1), not done in [47]. Further progress in this direction can be found in [11, 12, 55].

Another difference equation type is those constructed with homogeneous functions of order $\gamma \in \mathbb{R}$. To obtain more general results on such equations, Moaaz considered the higher-order general difference equation

$$x_{n+1} = f(x_{n-l}, x_{n-k}), \quad n \in \mathbb{N}_0, \tag{2.1.3}$$

where the function $f : (0, \infty)^2 \to (0, \infty)$ is continuous and homogeneous of degree γ in [46].

The studies mentioned above motivate researchers to further develop the results on the subject. In this context, the aims of this part is to study the following general systems of second order defined by homogeneous functions

$$x_{n+1} = f(y_n, y_{n-1}), \quad y_{n+1} = g(x_n, x_{n-1}), \quad n \in \mathbb{N}_0, \tag{2.1.4}$$

where the initial values x_{-1}, x_0, y_{-1} and y_0 are positive real numbers, the function $f : (0, \infty)^2 \to (0, \infty)$ is continuous and homogeneous of degree zero and $g : (0, \infty)^2 \to (0, \infty)$ is continuous and homogeneous of degree $s \in \mathbb{R}$. More precisely, in parallel with the paper [50], we obtain sufficient conditions for the stability of the unique equilibrium point of System (2.1.4) and also study the periodicity and oscillation of the solutions. By studying the system in this way, we considerably generalize the results of Touafek [50]. In addition, we demonstrate the consistency of our claims with appropriate concrete examples and simulate them with numerical values.

If s=1 is taken, it is easily seen that the results of this study are reduced to the results of the paper [50]. Now, we present some definitions and results that are preliminary to our study. For more detailed results on the subject, refer to [24, 41, 43]. The following definition and theorem are extracted from [45].

Definition 2.1.1. *A function $\phi : (0, \infty)^2 \longrightarrow (0, \infty)$ is said to be homogeneous of degree $s \in \mathbb{R}$ if for all $(\alpha, \beta) \in (0, \infty)^2$ and for all $\lambda > 0$ we have,*

$$\phi(\lambda\alpha, \lambda\beta) = \lambda^s \phi(\alpha, \beta).$$

Theorem 2.1.1. *Let* $\phi : (0,\infty)^2 \longrightarrow (0,\infty)$ *be a* C^1 *function on* $(0,\infty)^2$.

- *Then* ϕ *is homogeneous of degree* s *if and only if*

$$\alpha \frac{\partial \phi}{\partial \alpha}(\alpha,\beta) + \beta \frac{\partial \phi}{\partial \beta}(\alpha,\beta) = s\phi(\alpha,\beta),\ (\alpha,\beta) \in (0,\infty)^2.$$

(This statement, is usually called ***Euler's Theorem****).*

- *If* ϕ *is homogeneous of degree* s *on* $(0,\infty)^2$*, then* $\frac{\partial \phi}{\partial \alpha}$ *and* $\frac{\partial \phi}{\partial \beta}$ *are homogeneous of degree* $s-1$ *on* $(0,\infty)^2$.

2.2 Local Stability

In this section, we conduct our study on the local stability of the unique equilibrium point of System (2.1.4).

Using the assumption that f and g are homogeneous functions of degree zero and s, respectively, one can easily see that

$$(\bar{x},\bar{y}) = (f(1,1), (f(1,1))^s\, g(1,1))$$

is the unique equilibrium point of System (2.1.4) in $(0,\infty)^2$.

To generate the corresponding linearized form of System (2.1.4) about the unique equilibrium point, let us consider the function $F : (0,\infty)^4 \to (0,\infty)^4$ defined by

$$F(X) = (f_1(X), f_2(X), g_1(X), g_2(X))\,, \quad X = (u,v,w,t),$$

with

$$f_1(X) = f(w,t), \quad f_2(X) = u, \quad g_1(X) = g(u,v), \quad g_2(X) = w.$$

Then, System (2.1.4) can be expressed as

$$X_{n+1} = F(X_n),\ X_n = (x_n, x_{n-1}, y_n, y_{n-1})^T, \quad n \in \mathbb{N}_0. \tag{2.2.1}$$

Hence we can say that if $(\bar{x},\bar{y}) = (f(1,1),(f(1,1))^s g(1,1))$ is an equilibrium point of System (2.1.4), then the point

$$\overline{X} = (\bar{x},\bar{x},\bar{y},\bar{y}) = (f(1,1), f(1,1), (f(1,1))^s g(1,1), (f(1,1))^s g(1,1))$$

is an equilibrium point of System (2.2.1).

We assume that the functions f and g are C^1 on $(0,\infty)^2$. The linearized system of (2.1.4) about the equilibrium $\overline{X}$ is

$$Z_{n+1} = J_F Z_n, \quad n \in \mathbb{N}_0$$

and so the Jacobian matrix J_F associated to F evaluated at $\overline{X}$ is given by

$$\begin{pmatrix} 0 & 0 & \frac{\partial f}{\partial w}(\bar{y},\bar{y}) & \frac{\partial f}{\partial t}(\bar{y},\bar{y}) \\ 1 & 0 & 0 & 0 \\ \frac{\partial g}{\partial u}(\bar{x},\bar{x}) & \frac{\partial g}{\partial v}(\bar{x},\bar{x}) & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since f is homogeneous of degree 0, from Theorem 2.1.1-(1), we obtain

$$\bar{y}\frac{\partial f}{\partial w}(\bar{y},\bar{y}) + \bar{y}\frac{\partial f}{\partial t}(\bar{y},\bar{y}) = 0,$$

which implies that

$$\frac{\partial f}{\partial t}(\bar{y},\bar{y}) = -\frac{\partial f}{\partial w}(\bar{y},\bar{y}).$$

Similarly, since g is homogeneous of degree s, from Theorem 2.1.1-(1), we obtain

$$\bar{x}\frac{\partial g}{\partial u}(\bar{x},\bar{x}) + \bar{x}\frac{\partial g}{\partial v}(\bar{x},\bar{x}) = sg(\bar{x},\bar{x}),$$

or after some operations

$$\frac{\partial g}{\partial v}(\bar{x},\bar{x}) = s(f(1,1))^{s-1}g(1,1) - \frac{\partial g}{\partial u}(\bar{x},\bar{x}).$$

Therefore, the matrix J_F takes the form

$$\begin{pmatrix} 0 & 0 & \frac{\partial f}{\partial w}(\bar{y},\bar{y}) & -\frac{\partial f}{\partial w}(\bar{y},\bar{y}) \\ 1 & 0 & 0 & 0 \\ \frac{\partial g}{\partial u}(\bar{x},\bar{x}) & s(f(1,1))^{s-1}g(1,1) - \frac{\partial g}{\partial u}(\bar{x},\bar{x}) & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this case, the characteristic polynomial of the matrix J_F is given by

$$P(\lambda) = \lambda^4 - \frac{\partial g}{\partial u}(\bar{x},\bar{x})\frac{\partial f}{\partial w}(\bar{y},\bar{y})\lambda^2 - \frac{\partial f}{\partial w}(\bar{y},\bar{y})\left(s(f(1,1))^{s-1}g(1,1) - 2\frac{\partial g}{\partial u}(\bar{x},\bar{x})\right)\lambda + \frac{\partial f}{\partial w}(\bar{y},\bar{y})\left(s(f(1,1))^{s-1}g(1,1) - \frac{\partial g}{\partial u}(\bar{x},\bar{x})\right). \quad (2.2.2)$$

Theorem 2.2.1. *Assume that the functions $f(u,v)$ and $g(u,v)$ are $\mathbb{C}^1$ on $(0,\infty)^2$. The equilibrium point $(f(1,1),(f(1,1))^s g(1,1))$ of System (2.1.4) is locally asymptotically stable if*

$$\left|\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial u}(1,1)\right| + \left|\frac{\partial f}{\partial u}(1,1)\right|\left(\left|sg(1,1) - 2\frac{\partial g}{\partial u}(1,1)\right| + \left|sg(1,1) - \frac{\partial g}{\partial u}(1,1)\right|\right) < f(1,1)g(1,1).$$

Proof. We have

$$P(\lambda) = \lambda^4 - \frac{\partial g}{\partial u}(\bar{x},\bar{x})\frac{\partial f}{\partial u}(\bar{y},\bar{y})\lambda^2 - \frac{\partial f}{\partial u}(\bar{y},\bar{y})\left(s(f(1,1))^{s-1}g(1,1) - 2\frac{\partial g}{\partial u}(\bar{x},\bar{x})\right)\lambda$$

$$+\frac{\partial f}{\partial u}(\bar{y},\bar{y})\left(s(f(1,1))^{s-1}g(1,1) - \frac{\partial g}{\partial u}(\bar{x},\bar{x})\right).$$

Let us consider the functions

$$\Phi(\lambda) = \lambda^4$$

and

$$\Psi(\lambda) = -\frac{\partial g}{\partial u}(\bar{x},\bar{x})\frac{\partial f}{\partial u}(\bar{y},\bar{y})\lambda^2 - \frac{\partial f}{\partial u}(\bar{y},\bar{y})\left(s(f(1,1))^{s-1}g(1,1) - 2\frac{\partial g}{\partial u}(\bar{x},\bar{x})\right)\lambda$$

$$+\frac{\partial f}{\partial u}(\bar{y},\bar{y})\left(s(f(1,1))^{s-1}g(1,1) - \frac{\partial g}{\partial u}(\bar{x},\bar{x})\right).$$

We have

$$\begin{aligned}|\Psi(\lambda)| \leq & \left|\frac{\partial g}{\partial u}(\bar{x},\bar{x})\frac{\partial f}{\partial u}(\bar{y},\bar{y})\lambda^2\right| + \left|\left(s(f(1,1))^{s-1}g(1,1)\frac{\partial f}{\partial u}(\bar{y},\bar{y}) - 2\frac{\partial f}{\partial u}(\bar{y},\bar{y})\frac{\partial g}{\partial u}(\bar{x},\bar{x})\right)\lambda\right| \\ & + \left|s(f(1,1))^{s-1}g(1,1)\frac{\partial f}{\partial u}(\bar{y},\bar{y}) - \frac{\partial f}{\partial u}(\bar{y},\bar{y})\frac{\partial g}{\partial u}(\bar{x},\bar{x})\right| \\ = & \left|\frac{\partial g}{\partial u}(\bar{x},\bar{x})\frac{\partial f}{\partial u}(\bar{y},\bar{y})\right| + \left|s(f(1,1))^{s-1}g(1,1)\frac{\partial f}{\partial u}(\bar{y},\bar{y}) - 2\frac{\partial f}{\partial u}(\bar{y},\bar{y})\frac{\partial g}{\partial u}(\bar{x},\bar{x})\right| \\ & + \left|s(f(1,1))^{s-1}g(1,1)\frac{\partial f}{\partial u}(\bar{y},\bar{y}) - \frac{\partial f}{\partial u}(\bar{y},\bar{y})\frac{\partial g}{\partial u}(\bar{x},\bar{x})\right|, \quad |\lambda| = 1.\end{aligned}$$

Using Theorem 2.1.1-(2) and the fact that f and g are homogeneous functions of degree zero and s, respectively, we get that $\frac{\partial f}{\partial u}$ is homogeneous of degree -1 and $\frac{\partial g}{\partial u}$ is homogeneous of degree $s-1$. This yields the following identities between partial derivatives

$$\frac{\partial f}{\partial u}(\bar{y},\bar{y}) = \frac{\frac{\partial f}{\partial u}(1,1)}{\bar{y}} = \frac{1}{(f(1,1))^s g(1,1)}\frac{\partial f}{\partial u}(1,1),$$

$$\frac{\partial g}{\partial u}(\bar{x},\bar{x}) = \bar{x}^{s-1}\frac{\partial g}{\partial u}(1,1) = (f(1,1))^{s-1}\frac{\partial g}{\partial u}(1,1).$$

and

$$\frac{\partial g}{\partial u}(\bar{x},\bar{x})\frac{\partial f}{\partial u}(\bar{y},\bar{y}) = \frac{1}{f(1,1)g(1,1)}\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial u}(1,1).$$

Hence, we arrive at the following inequality

$$\begin{aligned}|\Psi(\lambda)| \leq & \frac{1}{f(1,1)g(1,1)}\left|\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial u}(1,1)\right| + \left|\frac{s}{f(1,1)}\frac{\partial f}{\partial u}(1,1) - 2\frac{1}{f(1,1)g(1,1)}\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial u}(1,1)\right| \\ & + \left|\frac{s}{f(1,1)}\frac{\partial f}{\partial u}(1,1) - \frac{1}{f(1,1)g(1,1)}\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial u}(1,1)\right| \\ = & \frac{1}{f(1,1)g(1,1)}\left[\left|\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial u}(1,1)\right| + \left|\frac{\partial f}{\partial u}(1,1)\right|\left(\left|sg(1,1) - 2\frac{\partial g}{\partial u}(1,1)\right| + \left|sg(1,1) - \frac{\partial g}{\partial u}(1,1)\right|\right)\right] \\ < & 1 = |\Phi(\lambda)|, \quad \text{for all } \lambda \in \mathbb{C} : |\lambda| = 1.\end{aligned}$$

Consequently, from Rouché's Theorem, it follows that if the above inequality is satisfied, then all roots of the characteristic polynomial P lie inside the unit disk. This means that the equilibrium point $(f(1,1), (f(1,1))^s g(1,1))$ is locally asymptotically stable according to Theorem 1.1.1. Thus, the proof is completed. □

2.3 Global attractivity

In this section, we investigate the global attractivity of the unique equilibrium point of System (2.1.4). The next eight theorems are on general convergence results. Theorem 2.3.3-Theorem 2.3.4, Theorem 2.3.7-Theorem 2.3.8, among others, are slight modifications of the theorems proved in [50].

Theorem 2.3.1. *Let $[a_1, b_1]$ and $[a_2, b_2]$ be intervals of real numbers and assume that the function $f : [a_2, b_2]^2 \longrightarrow [a_1, b_1]$ is continuous and homogeneous of degree zero and the function $g : [a_1, b_1]^2 \longrightarrow [a_2, b_2]$ is continuous and homogeneous of degree $s \in \mathbb{R}$. Also assume that f and g satisfy the following conditions:*

H_1 : *$f(u,v)$ is increasing in u and decreasing in v, however $g(w,z)$ is increasing in each of its arguments.*

H_2 : *If $(m,M,r,R) \in [a_1,b_1]^2 \times [a_2,b_2]^2$ is a solution of the system*

$$m = f(r,R), \quad M = f(R,r), \quad r = g(m,m), \quad R = g(M,M),$$

then

$$m = M, \quad r = R.$$

Then every solution of System (2.1.4) converges to the unique equilibrium point

$$(\bar{x},\bar{y}) = (f(1,1),(f(1,1))^s g(1,1)).$$

Proof. Let (x_n,y_n) be a solution of System (2.1.4). Set

$$m_0 := a_1, \qquad M_0 := b_1, \qquad r_0 := a_2, \qquad R_0 := b_2,$$

and for each $i = 0,1,2,\ldots,$

$$\begin{cases} m_{i+1} := f(r_i,R_i), & M_{i+1} := f(R_i,r_i), \\ r_{i+1} := g(m_i,m_i), & R_{i+1} := g(M_i,M_i). \end{cases} \tag{2.3.1}$$

It follows from the assumptions of f and g that

$$m_0 = a_1 \le f(r_0,R_0) \le f(R_0,r_0) \le b_1 = M_0,$$

and

$$r_0 = a_2 \le g(m_0,m_0) \le g(M_0,M_0) \le b_2 = R_0,$$

that is,

$$m_0 \le m_1 \le M_1 \le M_0 \quad \text{and} \quad r_0 \le r_1 \le R_1 \le R_0.$$

Then, we have

$$m_1 = f(r_0,R_0) \le f(r_1,R_1) = m_2 \le f(R_1,r_1) = M_2 \le f(R_0,r_0) = M_1,$$

$$r_1 = g(m_0,m_0) \le g(m_1,m_1) = r_2 \le g(M_1,M_1) = R_2 \le g(M_0,M_0) = R_1,$$

so we see that

$$m_0 \le m_1 \le m_2 \le M_2 \le M_1 \le M_0,$$

and

$$r_0 \le r_1 \le r_2 \le R_2 \le R_1 \le R_0.$$

By induction, we can see that for each $i = 0, 1, ...,$

$$a_1 = m_0 \le m_1 \le \cdots \le m_i \le \cdots \le M_i \le \cdots \le M_1 \le M_0 = b_1,$$

$$a_2 = r_0 \le r_1 \le \cdots \le r_i \le \cdots \le R_i \le \cdots \le R_1 \le R_0 = b_2.$$

Now, we have

$$m_0 = a_1 \le x_n \le b_1 = M_0, \quad r_0 = a_2 \le y_n \le b_2 = R_0, \quad n = 1, 2, \ldots$$

From the monotonicity of f and g, it follows that

$$m_1 = f(r_0, R_0) \le x_{n+1} = f(y_n, y_{n-1}) \le f(R_0, r_0) = M_1,$$

and

$$r_1 = g(m_0, m_0) \le y_{n+1} = g(x_n, x_{n-1}) \le g(M_0, M_0) = R_1,$$

for all $n \ge 2$ and so

$$m_1 \le x_n \le M_1 \quad \text{and} \quad r_1 \le y_n \le R_1,$$

for all $n \ge 3$. We have

$$m_2 = f(r_1, R_1) \le x_{n+1} = f(y_n, y_{n-1}) \le f(R_1, r_1) = M_2,$$

and

$$r_2 = g(r_1, r_1) \le y_{n+1} = g(x_n, x_{n-1}) \le g(M_1, M_1) = R_2,$$

for $n \ge 4$ and so

$$m_2 \le x_n \le M_2 \quad \text{and} \quad r_2 \le y_n \le R_2,$$

for all $n \ge 5$. By applying an induction procedure, for $i = 1, 2, ...,$ we have

$$m_i \le x_n \le M_i \quad \text{and} \quad r_i \le y_n \le R_i,$$

for all $n \geq 2i+1$. Let

$$m := \lim_{i\to\infty} m_i, \quad r := \lim_{i\to\infty} r_i, \quad M := \lim_{i\to\infty} M_i, \quad R := \lim_{i\to\infty} R_i.$$

Then we obtain

$$m \leq x_n \leq M, \quad r \leq y_n \leq R.$$

By the continuity of the functions f, g and (2.3.1), we get

$$\begin{cases} m = f(r,R), & M = f(R,r), \\ r = g(m,m), & R = g(M,M), \end{cases}$$

hence from assumption H_2, we have

$$m = M, \quad r = R.$$

Then

$$\lim_{n\to\infty} x_n = m, \quad \lim_{n\to\infty} y_n = r.$$

Thus, from (2.1.4) and using the fact that f is homogeneous of degree zero and g is homogeneous of degree s, we get that

$$m = f(r,R) = f(1,1), \quad r = g(M,M) = M^s g(1,1) = (f(1,1))^s g(1,1).$$

Consequently, the equilibrium point of System (2.1.4) is globally attractive. □

Theorem 2.3.2. *Let $[a_1,b_1]$ and $[a_2,b_2]$ be intervals of real numbers and assume that the function $f : [a_2,b_2]^2 \longrightarrow [a_1,b_1]$ is continuous and homogeneous of degree zero and the function $g : [a_1,b_1]^2 \longrightarrow [a_2,b_2]$ is continuous and homogeneous of degree $s \in \mathbb{R}$. Also assume that f and g satisfy the following conditions:*

H_1 : *$f(u,v)$ is increasing in u and decreasing in v, however $g(w,z)$ is decreasing in each of its arguments.*

H_2 : *If $(m,M,r,R) \in [a_1,b_1]^2 \times [a_2,b_2]^2$ is a solution of the system*

$$m = f(r,R), \quad M = f(R,r), \quad r = g(M,M), \quad R = g(m,m),$$

then

$$m = M, \quad r = R.$$

Then every solution of System (2.1.4) converges to the unique equilibrium point

$$(\bar{x}, \bar{y}) = (f(1,1), (f(1,1))^s g(1,1)).$$

Proof. Let (x_n, y_n) be a solution of System (2.1.4). Set

$$m_0 := a_1, \qquad M_0 := b_1, \qquad r_0 := a_2, \qquad R_0 := b_2,$$

and for each $i = 0, 1, 2, \ldots,$

$$\begin{cases} m_{i+1} := f(r_i, R_i), & M_{i+1} := f(R_i, r_i), \\ r_{i+1} := g(M_i, M_i), & R_{i+1} := g(m_i, m_i). \end{cases} \tag{2.3.2}$$

It follows from the assumptions of f and g that

$$m_0 = a_1 \le f(r_0, R_0) \le f(R_0, r_0) \le b_1 = M_0,$$

and

$$r_0 = a_2 \le g(M_0, M_0) \le g(m_0, m_0) \le b_2 = R_0,$$

that is,

$$m_0 \le m_1 \le M_1 \le M_0 \quad \text{and} \quad r_0 \le r_1 \le R_1 \le R_0.$$

Then, we have

$$m_1 = f(r_0, R_0) \le f(r_1, R_1) \le f(R_1, r_1) \le f(R_0, r_0) = M_1,$$

$$r_1 = g(M_0, M_0) \le g(M_1, M_1) \le g(m_1, m_1) \le g(m_0, m_0) = R_1,$$

so we see that

$$m_0 \le m_1 \le m_2 \le M_2 \le M_1 \le M_0,$$

and

$$r_0 \le r_1 \le r_2 \le R_2 \le R_1 \le R_0.$$

By induction, we observe that for each $i = 0, 1, ...,$

$$a_1 = m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0 = b_1,$$

$$a_2 = r_0 \leq r_1 \leq \cdots \leq r_i \leq \cdots \leq R_i \leq \cdots \leq R_1 \leq R_0 = b_2.$$

Now, we have

$$m_0 = a_1 \leq x_n \leq b_1 = M_0, \quad r_0 = a_2 \leq y_n \leq b_2 = R_0, \quad n = 1, 2, \ldots$$

The monotonicity of the functions f and g yields to, for all $n \geq 2$

$$m_1 = f(r_0, R_0) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(R_0, r_0) = M_1,$$

and

$$r_1 = g(M_0, M_0) \leq y_{n+1} = g(x_n, x_{n-1}) \leq g(m_0, m_0) = R_1,$$

so

$$m_1 \leq x_n \leq M_1 \quad \text{and} \quad r_1 \leq y_n \leq R_1, \quad \text{for all} \;\; n \geq 3.$$

For $n \geq 4$, we have

$$m_2 = f(r_1, R_1) \leq x_{n+1} = f(y_n, y_{n-1}) \leq f(R_1, r_1) = M_2,$$

and

$$r_2 = g(M_1, M_1) \leq y_{n+1} = g(x_n, x_{n-1}) \leq g(m_1, m_1) = R_2,$$

so

$$m_2 \leq x_n \leq M_2 \quad \text{and} \quad r_2 \leq y_n \leq R_2, \quad \text{for all} \;\; n \geq 5.$$

By induction, it follows that for $i = 1, 2, ...,$ we have

$$m_i \leq x_n \leq M_i \quad \text{and} \quad r_i \leq y_n \leq R_i, \quad \text{for all} \;\; n \geq 2i + 1.$$

Let

$$m := \lim_{i\to\infty} m_i, r := \lim_{i\to\infty} r_i, M := \lim_{i\to\infty} M_i, R := \lim_{i\to\infty} R_i.$$

Then, we obtain

$$m \leq x_n \leq M, \quad r \leq y_n \leq R.$$

From the continuity of the functions f, g and (2.3.2), we get

$$\begin{cases} m = f(r, R), & M = f(R, r), \\ r = g(M, M), & R = g(m, m), \end{cases}$$

hence from the assumption H_2, we have

$$m = M, \quad r = R.$$

Then

$$\lim_{n\to\infty} x_n = m, \quad \lim_{n\to\infty} y_n = r.$$

Thus, from (2.1.4) and using the fact that f is homogeneous of degree zero and g is homogeneous of degree s, we get that

$$m = f(r, R) = f(1, 1), \quad r = g(M, M) = M^s g(1, 1) = (f(1, 1))^s g(1, 1).$$

Consequently, the equilibrium point of System (2.1.4) is globally attractive. □

The next six theorems can be proved in the same way.

Theorem 2.3.3. *Let $[a_1, b_1]$ and $[a_2, b_2]$ be intervals of real numbers and assume that the function $f : [a_2, b_2]^2 \longrightarrow [a_1, b_1]$ is continuous and homogeneous of degree zero and the function $g : [a_1, b_1]^2 \longrightarrow [a_2, b_2]$ is continuous and homogeneous of degree $s \in \mathbb{R}$. Also assume that f and g satisfy the following conditions:*

H_1 : *$f(u, v)$ is increasing in u and decreasing in v, however $g(w, z)$ is decreasing in w and increasing in z.*

H_2 : *If $(m, M, r, R) \in [a_1, b_1]^2 \times [a_2, b_2]^2$ is a solution of the system*

$$m = f(r, R), \quad M = f(R, r), \quad r = g(M, m), \quad R = g(m, M),$$

then

$$m = M, \quad r = R.$$

Then every solution of System (2.1.4) converges to the unique equilibrium point

$$(\bar{x}, \bar{y}) = (f(1,1), (f(1,1))^s g(1,1)).$$

Theorem 2.3.4. *Let* $[a_1, b_1]$ *and* $[a_2, b_2]$ *be intervals of real numbers and assume that the function* $f : [a_2, b_2]^2 \longrightarrow [a_1, b_1]$ *is continuous and homogeneous of degree zero and the function* $g : [a_1, b_1]^2 \longrightarrow [a_2, b_2]$ *is continuous and homogeneous of degree* $s \in \mathbb{R}$. *Also assume that* f *and* g *satisfy the following conditions:*

$H_1:$ $f(u, v)$ *is increasing in* u *and decreasing in* v, *however* $g(w, z)$ *is increasing in* w *and decreasing in* z.

$H_2:$ *If* $(m, M, r, R) \in [a_1, b_1]^2 \times [a_2, b_2]^2$ *is a solution of the system*

$$m = f(r, R), \quad M = f(R, r), \quad r = g(m, M), \quad R = g(M, m),$$

then

$$m = M, \quad r = R.$$

Then every solution of System (2.1.4) converges to the unique equilibrium point

$$(\bar{x}, \bar{y}) = (f(1,1), (f(1,1))^s g(1,1)).$$

Theorem 2.3.5. *Let* $[a_1, b_1]$ *and* $[a_2, b_2]$ *be intervals of real numbers and assume that the function* $f : [a_2, b_2]^2 \longrightarrow [a_1, b_1]$ *is continuous and homogeneous of degree zero and the function* $g : [a_1, b_1]^2 \longrightarrow [a_2, b_2]$ *is continuous and homogeneous of degree* $s \in \mathbb{R}$. *Also assume that* f *and* g *satisfy the following conditions:*

$H_1:$ $f(u, v)$ *is decreasing in* u *and increasing in* v, *however* $g(w, z)$ *is increasing in each of its arguments.*

$H_2:$ *If* $(m, M, r, R) \in [a_1, b_1]^2 \times [a_2, b_2]^2$ *is a solution of the system*

$$m = f(R, r), \quad M = f(r, R), \quad r = g(m, m), \quad R = g(M, M),$$

then

$$m = M, \quad r = R.$$

Then every solution of System (2.1.4) converges to the unique equilibrium point

$$(\bar{x}, \bar{y}) = (f(1,1), (f(1,1))^s g(1,1)).$$

Theorem 2.3.6. *Let* $[a_1, b_1]$ *and* $[a_2, b_2]$ *be intervals of real numbers and assume that the function* $f : [a_2, b_2]^2 \longrightarrow [a_1, b_1]$ *is continuous and homogeneous of degree zero and the function* $g : [a_1, b_1]^2 \longrightarrow [a_2, b_2]$ *is continuous and homogeneous of degree* $s \in \mathbb{R}$. *Also assume that* f *and* g *satisfy the following conditions:*

H_1 : $f(u, v)$ *is decreasing in* u *and increasing in* v*, however* $g(w, z)$ *is decreasing in each of its arguments.*

H_2 : *If* $(m, M, r, R) \in [a_1, b_1]^2 \times [a_2, b_2]^2$ *is a solution of the system*

$$m = f(R, r), \quad M = f(r, R), \quad r = g(M, M), \quad R = g(m, m),$$

then

$$m = M, \quad r = R.$$

Then every solution of System (2.1.4) converges to the unique equilibrium point

$$(\bar{x}, \bar{y}) = (f(1,1), (f(1,1))^s g(1,1)).$$

Theorem 2.3.7. *Let* $[a_1, b_1]$ *and* $[a_2, b_2]$ *be intervals of real numbers and assume that the function* $f : [a_2, b_2]^2 \longrightarrow [a_1, b_1]$ *is continuous and homogeneous of degree zero and the function* $g : [a_1, b_1]^2 \longrightarrow [a_2, b_2]$ *is continuous and homogeneous of degree* $s \in \mathbb{R}$. *Also assume that* f *and* g *satisfy the following conditions:*

H_1 : $f(u, v)$ *is decreasing in* u *and increasing in* v*, however* $g(w, z)$ *is increasing in* w *and decreasing in* z.

H_2 : *If* $(m, M, r, R) \in [a_1, b_1]^2 \times [a_2, b_2]^2$ *is a solution of the system*

$$m = f(R, r), \quad M = f(r, R), \quad r = g(m, M), \quad R = g(M, m),$$

then

$$m = M, \quad r = R.$$

Then every solution of System (2.1.4) converges to the unique equilibrium point

$$(\bar{x}, \bar{y}) = (f(1,1), (f(1,1))^s g(1,1)).$$

Theorem 2.3.8. *Let $[a_1, b_1]$ and $[a_2, b_2]$ be intervals of real numbers and assume that the function $f : [a_2, b_2]^2 \longrightarrow [a_1, b_1]$ is continuous and homogeneous of degree zero and the function $g : [a_1, b_1]^2 \longrightarrow [a_2, b_2]$ is continuous and homogeneous of degree $s \in \mathbb{R}$. Also assume that f and g satisfy the following conditions:*

$H_1 :$ *$f(u, v)$ is decreasing in u and increasing in v, however $g(w, z)$ is decreasing in w and increasing in z.*

$H_2 :$ *If $(m, M, r, R) \in [a_1, b_1]^2 \times [a_2, b_2]^2$ is a solution of the system*

$$m = f(R, r), \quad M = f(r, R), \quad r = g(M, m), \quad R = g(m, M),$$

then

$$m = M, \quad r = R.$$

Then every solution of System (2.1.4) converges to the unique equilibrium point

$$(\bar{x}, \bar{y}) = (f(1,1), (f(1,1))^s g(1,1)).$$

As a consequence of the results of this section, we have the following theorem.

Theorem 2.3.9. *The equilibrium point $(\bar{x}, \bar{y}) = (f(1,1), (f(1,1))^s g(1,1))$ is globally stable if the assumptions of Theorem 2.2.1 and the assumptions of Theorem 2.3.1 or Theorem 2.3.2 or Theorem 2.3.3 or Theorem 2.3.4 or Theorem 2.3.5 or Theorem 2.3.6 or Theorem 2.3.7 or Theorem 2.3.8 are verified.*

2.4 Existence of periodic solutions

In this section, we obtain necessary and sufficient conditions for System (2.1.4) to have a periodic solution with prime period two and prime period three. We recall that a solution $(x_n, y_n)_{n \geq -1}$ of System (2.1.4) is said to be periodic of period $p \in \mathbb{N}$ if

$$x_{n+p} = x_n, \quad y_{n+p} = y_n, \quad \text{for all } n \geq -1.$$

Theorem 2.4.1. *Assume that* $\gamma, \lambda \in (0,1) \cup (1,\infty)$. *Then, System (2.1.4) has a prime period two solution*

$$\ldots, (\gamma p, \lambda q), (p, q), (\gamma p, \lambda q), (p, q), \ldots$$

if and only if

$$f(1,\lambda) = \gamma f(\lambda, 1), \quad g(1,\gamma) = \lambda g(\gamma, 1),$$

where

$$p = f(\lambda, 1), \quad q = (f(\lambda, 1))^s g(\gamma, 1).$$

Proof. Assume that

$$\ldots, (\gamma p, \lambda q), (p, q), (\gamma p, \lambda q), (p, q), \ldots$$

is a solution of System (2.1.4). Then, we can write the equalities

$$\gamma p = f(q, \lambda q) = f(1, \lambda), \tag{2.4.1}$$

$$p = f(\lambda q, q) = f(\lambda, 1), \tag{2.4.2}$$

$$\lambda q = g(p, \gamma p) = p^s g(1, \gamma), \tag{2.4.3}$$

$$q = g(\gamma p, p) = p^s g(\gamma, 1). \tag{2.4.4}$$

From (2.4.1) and (2.4.2), it follows that

$$f(1, \lambda) = \gamma f(\lambda, 1).$$

From (2.4.3) and (2.4.4), it follows that

$$g(1, \gamma) = \lambda g(\gamma, 1).$$

Now, assume that

$$f(1,\lambda) = \gamma f(\lambda, 1), \quad g(1,\gamma) = \lambda g(\gamma, 1)$$

and

$$x_0 = f(1, \lambda), \quad x_{-1} = f(\lambda, 1), \quad y_0 = (f(\lambda, 1))^s g(1, \gamma), \quad y_{-1} = (f(\lambda, 1))^s g(\gamma, 1).$$

Then, we have

$$x_1 = f(y_0, y_{-1}) = f(g(1, \gamma), g(\gamma, 1)) = f(\lambda g(\gamma, 1), g(\gamma, 1)) = f(\lambda, 1) = x_{-1},$$

$$y_1 = g(x_0, x_{-1}) = g(f(1,\lambda), f(\lambda,1)) = g(\gamma f(\lambda,1), f(\lambda,1)) = (f(\lambda,1))^s g(\gamma,1) = y_{-1},$$
$$x_2 = f(y_1, y_0) = f((f(\lambda,1))^s g(\gamma,1), (f(\lambda,1))^s g(1,\gamma)) = f((g(\gamma,1), g(1,\gamma))$$
$$= f(g(\gamma,1), \lambda g(\gamma,1)) = f(1,\lambda) = x_0,$$
$$y_2 = g(x_1, x_0) = g(f(\lambda,1), f(1,\lambda)) = g(f(\lambda,1), \gamma f(\lambda,1)) = f(\lambda,1)^s g(1,\gamma) = y_0.$$

By induction, it follows that

$$x_{2n-1} = x_{-1}, \quad x_{2n} = x_0, \quad y_{2n-1} = y_{-1}, \quad y_{2n} = y_0, \quad n \in \mathbb{N}_0.$$

So, the proof is completed. □

Theorem 2.4.2. *Assume that $\alpha, \beta, \gamma, \lambda \in (0,1) \cup (1,\infty)$. Then, System (2.1.4) has a prime period three solution*

$$\ldots, (\alpha p, \beta q), (\gamma p, \lambda q), (p,q), (\alpha p, \beta q), (\gamma p, \lambda q), (p,q), , \ldots$$

if and only if

$$f(1,\lambda) = \alpha f(\lambda,\beta), \quad f(\beta,1) = \gamma f(\lambda,\beta),$$
$$g(1,\gamma) = \beta g(\gamma,\alpha), \quad g(\alpha,1) = \lambda g(\gamma,\alpha).$$

where

$$p = f(\lambda,\beta), \quad q = (f(\lambda,\beta))^s g(\gamma,\alpha).$$

Proof. Assume that

$$\ldots, (\alpha p, \beta q), (\gamma p, \lambda q), (p,q), (\alpha p, \beta q), (\gamma p, \lambda q), (p,q), , \ldots$$

is a solution for System (2.1.4). Then, we have

$$\alpha p = f(q, \lambda q) = f(1,\lambda), \tag{2.4.5}$$
$$\gamma p = f(\beta q, q) = f(\beta, 1), \tag{2.4.6}$$
$$p = f(\lambda q, \beta q) = f(\lambda, \beta), \tag{2.4.7}$$
$$\beta q = g(p, \gamma p) = p^s g(1,\gamma), \tag{2.4.8}$$
$$\lambda q = g(\alpha p, p) = p^s g(\alpha, 1), \tag{2.4.9}$$
$$q = g(\gamma p, \alpha p) = p^s g(\gamma, \alpha). \tag{2.4.10}$$

From (2.4.5), (2.4.6) and (2.4.7), it follows that

$$f(1,\lambda)=\alpha f(\lambda,\beta),\quad f(\beta,1)=\gamma f(\lambda,\beta).$$

From (2.4.8), (2.4.9) and (2.4.10), it follows that

$$g(1,\gamma)=\beta g(\gamma,\alpha),\quad g(\alpha,1)=\lambda g(\gamma,\alpha),$$

as claimed. Now, let

$$f(1,\lambda)=\alpha f(\lambda,\beta),\quad f(\beta,1)=\gamma f(\lambda,\beta),\quad g(1,\gamma)=\beta g(\gamma,\alpha),\quad g(\alpha,1)=\lambda g(\gamma,\alpha),$$

and also

$$x_0=f(\beta,1),\quad x_{-1}=f(1,\lambda),\quad y_0=(f(\lambda,\beta))^s g(\alpha,1),\quad y_{-1}=(f(\lambda,\beta))^s g(1,\gamma).$$

Then, we have

$$\begin{aligned}
x_1&=f(y_0,y_{-1})=f((f(\lambda,\beta))^s g(\alpha,1),(f(\lambda,\beta))^s g(1,\gamma))=f(g(\alpha,1),g(1,\gamma))\\
&=f(\lambda g(\gamma,\alpha),\beta g(\gamma,\alpha))=f(\lambda,\beta),\\
y_1&=g(x_0,x_{-1})=g(f(\beta,1),f(1,\lambda))=g(\gamma f(\lambda,\beta),\alpha f(\lambda,\beta))=(f(\lambda,\beta))^s g(\gamma,\alpha),\\
x_2&=f(y_1,y_0)=f((f(\lambda,\beta))^s g(\gamma,\alpha),(f(\lambda,\beta))^s g(\alpha,1))=f(g(\gamma,\alpha),g(\alpha,1))\\
&=f(g(\gamma,\alpha),\lambda g(\gamma,\alpha))=f(1,\lambda)=x_{-1},\\
y_2&=g(x_1,x_0)=g(f(\lambda,\beta),f(\beta,1))=g(f(\lambda,\beta),\gamma f(\lambda,\beta))=(f(\lambda,\beta))^s g(1,\gamma)=y_{-1},\\
x_3&=f(y_2,y_1)=f((f(\lambda,\beta))^s g(1,\gamma),(f(\lambda,\beta))^s g(\gamma,\alpha))=f(g(1,\gamma),g(\gamma,\alpha))\\
&=f(\beta g(\gamma,\alpha),g(\gamma,\alpha))=f(\beta,1)=x_0,\\
y_3&=g(x_2,x_1)=g(f(1,\lambda),f(\lambda,\beta))=g(\alpha f(\lambda,\beta),f(\lambda,\beta))=(f(\lambda,\beta))^s g(\alpha,1)=y_0,\\
x_4&=f(y_3,y_2)=f((f(\lambda,\beta))^s g(\alpha,1),(f(\lambda,\beta))^s g(1,\gamma))=f(g(\alpha,1),g(1,\gamma))\\
&=f(\lambda g(\gamma,\alpha),\beta g(\gamma,\alpha))=f(\lambda,\beta)=x_1,\\
y_4&=g(x_3,x_2)=g(f(\beta,1),f(1,\lambda))=g(\gamma f(\lambda,\beta),\alpha f(\lambda,\beta))\\
&=(f(\lambda,\beta))^s g(\gamma,\alpha)=y_1.
\end{aligned}$$

By induction, it follows that

$$x_{3n-1} = x_{-1}, \quad x_{3n} = x_0, \quad x_{3n+1} = x_1, \quad y_{3n-1} = y_{-1}, \quad y_{3n} = y_0, \quad y_{3n+1} = y_1, \quad n \in \mathbb{N}_0.$$

So, the proof is completed. □

2.5 Oscillation of positive solutions

In this section, we establish sufficient conditions that the solutions of System (2.1.4) are oscillatory about the equilibrium point $(f(1,1), (f(1,1))^s g(1,1))$.

Definition 2.5.1. *(Oscillation). Let $(x_n, y_n)_{n\geq -1}$ be a solution of System (2.1.4). The sequence $(x_n)_{n\geq -1}$ (resp. $(y_n)_{n\geq -1}$) is called non-oscillatory about $\bar{x}$ (resp. $\bar{y}$)if there exists $l \geq -1$ such that either*

$$x_n > \bar{x} \quad or \quad x_n < \bar{x}, \quad for\ all\ n \geq l, \quad (resp.\ y_n > \bar{y} \quad or \quad y_n < \bar{y}, \quad for\ all\ n \geq l,)$$

and it is called oscillatory if it is not nonoscillatory.

Theorem 2.5.1. *Assume that $(x_n, y_n)_{n\geq -1}$ is a solution of System (2.1.4) and that $f(x,y)$ and $g(x,y)$ are decreasing in x for all y and are increasing in y for all x. Then, the following statements are true.*

- *If*

  $$x_{-1} > \bar{x}, \quad x_0 < \bar{x}, \quad y_{-1} > \bar{y}, \quad y_0 < \bar{y},$$

  then we get

  $$x_{2n-1} > \bar{x}, \quad x_{2n} < \bar{x}, \quad y_{2n-1} > \bar{y}, \quad y_{2n} < \bar{y}, \quad n \in \mathbb{N}_0.$$

  That is, the sequence $(x_n)_{n\geq -1}$(resp. $(y_n)_{n\geq -1}$) oscillates about $\bar{x}$ (resp. $\bar{y}$).

- *If*

  $$x_{-1} < \bar{x}, \quad x_0 > \bar{x}, \quad y_{-1} < \bar{y}, \quad y_0 > \bar{y},$$

  then we get

  $$x_{2n-1} < \bar{x}, \quad x_{2n} > \bar{x}, \quad y_{2n-1} < \bar{y}, \quad y_{2n} > \bar{y}, \quad n \in \mathbb{N}_0.$$

That is, the sequence $(x_n)_{n\geq -1}$*(resp.* $(y_n)_{n\geq -1}$*) oscillates about* $\bar{x}$ *(resp.* $\bar{y}$*).*

Proof.

- Assume that

$$x_{-1} > \bar{x}, \quad x_0 < \bar{x}, \quad y_{-1} > \bar{y}, \quad y_0 < \bar{y},$$

then, we obtain

$$\begin{aligned}
x_1 &= f(y_0, y_{-1}) > f(\bar{y}, y_{-1}) > f(\bar{y}, \bar{y}) = f(1,1) = \bar{x},\\
y_1 &= g(x_0, x_{-1}) > g(\bar{x}, x_{-1}) > g(\bar{x}, \bar{x}) = (f(1,1))^s g(1,1) = \bar{y},\\
x_2 &= f(y_1, y_0) < f(\bar{y}, y_0) < f(\bar{y}, \bar{y}) = f(1,1) = \bar{x},\\
y_2 &= g(x_1, x_0) < g(\bar{x}, x_0) < g(\bar{x}, \bar{x}) = (f(1,1))^s g(1,1) = \bar{y}.
\end{aligned}$$

By induction, we get

$$x_{2n} < \bar{x}, \quad x_{2n-1} > \bar{x}, \quad y_{2n} < \bar{y}, \quad y_{2n-1} > \bar{y}, \quad n \in \mathbb{N}_0.$$

- Assume that

$$x_{-1} < \bar{x}, \quad x_0 > \bar{x}, \quad y_{-1} < \bar{y}, \quad y_0 > \bar{y},$$

then, we obtain

$$\begin{aligned}
x_1 &= f(y_0, y_{-1}) < f(\bar{y}, y_{-1}) < f(\bar{y}, \bar{y}) = f(1,1) = \bar{x},\\
y_1 &= g(x_0, x_{-1}) < g(\bar{x}, x_{-1}) < g(\bar{x}, \bar{x}) = (f(1,1))^s g(1,1) = \bar{y},\\
x_2 &= f(y_1, y_0) > f(\bar{y}, y_0) > f(\bar{y}, \bar{y}) = f(1,1) = \bar{x},\\
y_2 &= g(x_1, x_0) > g(\bar{x}, x_0) > g(\bar{x}, \bar{x}) = (f(1,1))^s g(1,1) = \bar{y}.
\end{aligned}$$

By induction, we get

$$x_{2n} < \bar{x}, \quad x_{2n-1} > \bar{x}, \quad y_{2n} < \bar{y}, \quad y_{2n-1} > \bar{y}, \quad n \in \mathbb{N}_0.$$

□

2.6 Applications

In this section, to verify our theoretical results, we give some concrete examples. Also, we consider these examples with numerical values.

Example 2.6.1. *Consider the system*

$$x_{n+1} = A + \frac{y_{n-1}}{By_n + Cy_{n-1}}, \quad y_{n+1} = \left(\frac{cx_n + dx_{n-1}}{ax_n + bx_{n-1}}\right) x_n x_{n-1}, \quad n \in \mathbb{N}_0, \tag{2.6.1}$$

where x_{-i}, y_{-i}, $(i = 0, 1)$ and A, B, C, a, b, c, d are positive real numbers. System (2.6.1) can be written as

$$x_{n+1} = f(y_n, y_{n-1}), \quad y_{n+1} = g(x_n, x_{n-1}), \quad n \in \mathbb{N}_0,$$

where f and g are the functions given by

$$f(u, v) = A + \frac{v}{Bu + Cv}, \quad g(w, z) = \left(\frac{cw + dz}{aw + bz}\right) wz. \tag{2.6.2}$$

It is easy to see that f and g are continuous, and homogeneous of degree zero and degree 2, respectively. Also

$$(\bar{x}, \bar{y}) = (f(1,1), (f(1,1))^s g(1,1) = \left(A + \frac{1}{B+C}, \left(A + \frac{1}{B+C}\right)^2 \frac{c+d}{a+b}\right)$$

is the unique equilibrium point of System (2.6.1)

Theorem 2.6.1. *Every positive solution of System (2.6.1) is bounded and persists.*

Proof. Let $\{(x_n, y_n)\}$ be a solution of System (2.6.1). Then, for all $n \geqslant 0$, We have

$$A \leq x_{n+1} = A + \frac{y_{n-1}}{By_n + Cy_{n-1}} \leq A + \frac{1}{C},$$

from which it follows that

$$A \leq x_n \leq A + \frac{1}{C},$$

for all $n \geq 1$. On the other hand, since

$$min\left\{\frac{c}{a}, \frac{d}{b}\right\} \leq \frac{cx_n + dx_{n-1}}{ax_n + bx_{n-1}} \leq max\left\{\frac{c}{a}, \frac{d}{b}\right\},$$

for all $n \geq 2$, we have

$$min\left\{\frac{c}{a},\frac{d}{b}\right\}A^2 \leq y_{n+1} = \left(\frac{cx_n + dx_{n-1}}{ax_n + bx_{n-1}}\right)x_n x_{n-1} \leq max\left\{\frac{c}{a},\frac{d}{b}\right\}\left(A+\frac{1}{C}\right)^2.$$

This implies that

$$min\left\{\frac{c}{a},\frac{d}{b}\right\}A^2 \leq y_n \leq max\left\{\frac{c}{a},\frac{d}{b}\right\}\left(A+\frac{1}{C}\right)^2,$$

for all $n \geq 3$, as claimed. □

Theorem 2.6.2. *The equilibrium point*

$$(\bar{x},\bar{y}) = \left(A+\frac{1}{B+C},\left(A+\frac{1}{B+C}\right)^2\frac{c+d}{a+b}\right)$$

of System (2.6.1) is locally asymptotically stable if

$$(B-C-A(B+C)^2)(c+d)(a+b)+2B\left|da-cb\right| < 0.$$

Proof. We know from Theorem 2.2.1 that the equilibrium point $(\bar{x},\bar{y})$ is asymptotically stable if

$$\left|\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial w}(1,1)\right|+\left|\frac{\partial f}{\partial u}(1,1)\right|\left(\left|2g(1,1)-2\frac{\partial g}{\partial w}(1,1)\right|+\left|2g(1,1)-\frac{\partial g}{\partial w}(1,1)\right|\right) < f(1,1)g(1,1).$$

Also, we obtain

$$f(1,1) = A+\frac{1}{B+C},\quad \frac{\partial f}{\partial u}(1,1) = \frac{-B}{(B+C)^2},$$
$$g(1,1) = \frac{c+d}{a+b},\quad \frac{\partial g}{\partial w}(1,1) = \frac{ca+db+2cb}{(a+b)^2}.$$

By using these in the above inequality, we get

$$\left|\frac{-B}{(B+C)^2}\frac{ca+db+2cb}{(a+b)^2}\right|+\left|\frac{-B}{(B+C)^2}\right|\left(2\left|\frac{c+d}{a+b}-\frac{ca+db+2cb}{(a+b)^2}\right|+\left|2\frac{c+d}{a+b}-\frac{ca+db+2cb}{(a+b)^2}\right|\right) < \left(A+\frac{1}{B+C}\right)\frac{c+d}{a+b}.$$

After some arrangements, one can find the following inequality

$$(B-C-A(B+C)^2)(c+d)(a+b)+2B\left|da-cb\right| < 0.$$

□

The next theorem is devoted to the global stability of the equilibrium point. For this purpose, we choose the initial values such that

$$x_{-1},\, x_0 \in \left[A, A+\frac{1}{C}\right]$$

and

$$y_{-1},\, y_0,\, y_1,\, y_2, \in \left[min\left\{\frac{c}{a},\frac{d}{b}\right\}A^2, max\left\{\frac{c}{a},\frac{d}{b}\right\}\left(A+\frac{1}{C}\right)^2\right].$$

Theorem 2.6.3. *Assume that*

$$(B-C-A(B+C)^2)(c+d)(a+b)+2B\,|da-cb|<0$$

and

$$4(c+d)\left(A+\frac{1}{C}\right)<(a+b)Cmin\left\{\frac{c}{a},\frac{d}{b}\right\}A^2.$$

Then, the equilibrium point of System (2.6.1) is globally asymptotically stable.

Proof. We know from Theorem 2.6.2 that the equilibrium point $(\bar{x},\bar{y})$ of System (2.6.1) is locally asymptotically stable, and so it suffices to show that $(\bar{x},\bar{y})$ is a global attractor. To prove that we will use Theorem 2.3.5.

Let f and g be as defined in (2.6.2) and define the following parameters:

$$a_1 := A,\; b_1 := A+\frac{1}{C},$$

$$a_2 := min\left\{\frac{c}{a},\frac{d}{b}\right\}A^2,\; b_2 := max\left\{\frac{c}{a},\frac{d}{b}\right\}\left(A+\frac{1}{C}\right)^2.$$

Thus, we have

$$a_1 < f(u,v) < b_1,\; a_2 < g(w,z) < b_2, \text{ for all } (w,z,u,v)\in[a_1,b_1]^2\times[a_2,b_2]^2.$$

That is to say, f and g are bounded. Also, since

$$\frac{\partial f}{\partial u}(u,v)=\frac{-Bv}{(Bu+Cv)^2}<0,\quad \frac{\partial f}{\partial v}(u,v)=\frac{Bu}{(Bu+Cv)^2}>0,$$

$$\frac{\partial g}{\partial w}(w,z)=\frac{bdz^3+2bcwz^2+acw^2z}{(aw+bz)^2}>0,\quad \frac{\partial g}{\partial z}(w,z)=\frac{acw^3+2adzw^2+bdz^2w}{(aw+bz)^2}>0,$$

the condition H_1 of Theorem 2.3.5 is satisfied. It remains to check condition H_2.

To this end, let $(m, M, r, R) \in [a_1, b_1]^2 \times [a_2, b_2]^2$ be a solution of the system

$$\begin{aligned} m &= A + \frac{r}{BR + Cr}, \quad M = A + \frac{R}{Br + CR}, \\ r &= \left(\frac{c+d}{a+b}\right) m^2, \quad R = \left(\frac{c+d}{a+b}\right) M^2. \end{aligned}$$

Suppose that $m \leq M$ and $r \leq R$. Then, we have

$$M - m = \frac{B(R-r)(R+r)}{(Br + CR)(BR + Cr)} \tag{2.6.3}$$

and

$$R - r = \frac{c+d}{a+b}(M-m)(M+m). \tag{2.6.4}$$

From (2.6.3) and (2.6.4), it follows that

$$\begin{aligned} M - m &= \frac{B(c+d)(M+m)(R+r)(M-m)}{(a+b)(Br+CR)(BR+Cr)} \\ &\leq \frac{B(c+d)(M-m)(M+m)(R+r)}{(a+b)CBR^2} \\ &\leq \frac{4MRB(c+d)(M-m)}{(a+b)CBR^2} \\ &\leq \frac{4M(c+d)(M-m)}{(a+b)CR}. \end{aligned}$$

Since $(M, R) \in [a_1, b_1] \times [a_2, b_2]$, we have

$$M - m \leq \frac{4b_1(c+d)(M-m)}{(a+b)Ca_2},$$

thus

$$(M-m)\left(1 - \frac{4b_1(c+d)}{(a+b)Ca_2}\right) \leq 0.$$

By using the fact that

$$4b_1(c+d) < (a+b)Ca_2,$$

we get

$$\left(1 - \frac{4b_1(c+d)}{(a+b)Ca_2}\right) > 0.$$

So, $M = m$ and from (2.6.4) we get $R = r$.

Consequently, condition H_2 is satisfied and the equilibrium point $(\bar{x}, \bar{y})$ of System (2.6.1) is globally attractive. □

Theorem 2.6.4. *Assume that $\gamma, \lambda \in (0,1) \cup (1,\infty)$. Then, System (2.6.1) have a prime period two solution*

$$\ldots, (\gamma p, \lambda q), (p,q), (\gamma p, \lambda q), (p,q), \ldots$$

if and only if

$$(1-\gamma)[(B^2+C^2)A\lambda + (1+\lambda^2)ABC + C\lambda] + (\lambda^2-\gamma)B = 0,$$

and

$$(1-\lambda)(ac+bd)\gamma + (1-\lambda\gamma^2)bc + (\gamma^2-\lambda)ad = 0, \tag{2.6.5}$$

where

$$p = f(\lambda,1), \quad q = (f(\lambda,1))^s g(\gamma,1).$$

Proof. By Theorem 2.4.1, System (2.6.1) have a prime period two solution if and only if

$$f(1,\lambda) = \gamma f(\lambda,1), \quad g(1,\gamma) = \lambda g(\gamma,1), \tag{2.6.6}$$

where

$$f(1,\lambda) = A + \frac{\lambda}{B+C\lambda}, \quad g(1,\gamma) = \frac{c\gamma + d\gamma^2}{a+b\gamma},$$
$$f(\lambda,1) = A + \frac{1}{B\lambda+C}, \quad g(\gamma,1) = \frac{c\gamma^2 + d\gamma}{a\gamma+b}.$$

The first equality of condition (2.6.6) is equivalent to

$$A + \frac{\lambda}{B+C\lambda} = \gamma\left(A + \frac{1}{B\lambda+C}\right),$$

and it yields the result

$$(1-\gamma)[(B^2+C^2)A\lambda + (1+\lambda^2)ABC + C\lambda] + (\lambda^2-\gamma)B = 0,$$

after some elementary operations. Similarly, the second equality of condition (2.6.6) is equivalent to

$$\frac{c\gamma + d\gamma^2}{a+b\gamma} = \lambda\frac{c\gamma^2 + d\gamma}{a\gamma+b},$$

from which it follows that

$$(1-\lambda)(ac+bd)\gamma + (1-\lambda\gamma^2)bc + (\gamma^2-\lambda)ad = 0.$$

Which complete the proof. □

Now we check the results by assigning numerical values to the parameters of System (2.6.1).

Let $\lambda = 3, \gamma = 2$. *Then we obtain*

$$7B - [(B^2 + C^2)3A + 10ABC + 3C] = 0$$

and

$$ad - 4(ac + bd) - 11bc = 0,$$

which are conditions for the periodicity of the solutions. If we choose that the parameters $A = 1/4,\ B = 1,\ C = 1,\ a = 5,\ b = 1,\ c = 1,\ d = 31,$ *then the conditions are satisfied and we have*

$$x_{2n-1} = x_{-1} = f(\lambda, 1) = \frac{1}{2}, \quad y_{2n-1} = y_{-1} = (f(\lambda, 1))^2 g(\gamma, 1) = \frac{3}{2},$$

$$x_{2n} = x_0 = \gamma f(\lambda, 1) = 1, \quad y_{2n} = y_0 = \lambda (f(\lambda, 1))^2 g(\gamma, 1) = \frac{9}{2}.$$

Hence the solution is obtained in the form

$$\left\{ \ldots, \left(\frac{1}{2}, \frac{3}{2}\right), \left(1, \frac{9}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right), \left(1, \frac{9}{2}\right), \ldots \right\}.$$

The plots of x_n *and* y_n *are given by Figure 2.1.*

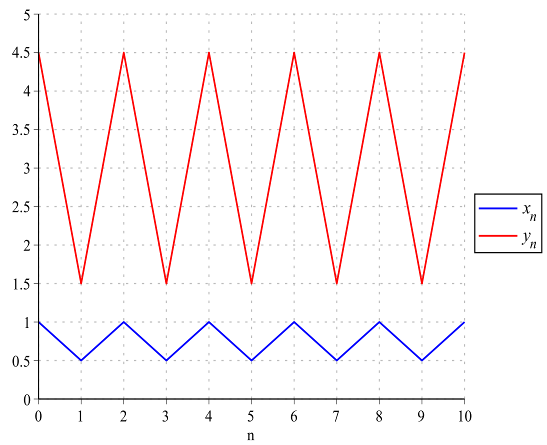

Figure 2.1 – Plots of x_n and y_n.

Example 2.6.2. *Consider the system*

$$x_{n+1} = A + \frac{\alpha y_n + \beta y_{n-1}}{By_n + Cy_{n-1}}, \quad y_{n+1} = \left(a + \frac{x_{n-1}}{bx_n + cx_{n-1}}\right) x_{n-1}, \quad n \in \mathbb{N}_0. \tag{2.6.7}$$

where the initial values $x_{-i}, y_{-i}, i = 0, 1$ and the parameters $A, a, B, b, C, c, \alpha, \beta$ are positive real numbers. System (2.6.7) can be written as

$$x_{n+1} = f(y_n, y_{n-1}), \quad y_{n+1} = g(x_n, x_{n-1}), \quad n \in \mathbb{N}_0,$$

where f and g are the functions defined by

$$f(u, v) = A + \frac{\alpha u + \beta v}{Bu + Cv}, \quad g(w, z) = \left(a + \frac{z}{bw + cz}\right) z. \tag{2.6.8}$$

It is clear that f is continuous and homogeneous of degree zero, g is continuous and homogeneous of degree 1. Also

$$(\bar{x}, \bar{y}) = (f(1,1), (f(1,1))^s g(1,1) = \left(A + \frac{\alpha + \beta}{B + C}, \left(A + \frac{\alpha + \beta}{B + C}\right)\left(a + \frac{1}{b + c}\right)\right)$$

is the unique equilibrium point of System (2.6.7).

Theorem 2.6.5. *Every positive solution $\{(x_n, y_n)\}$ of System (2.6.7) is bounded and persists.*

Proof. Let $\{(x_n, y_n)\}$ be a solution of (2.6.7). Then, for all $n \geqslant 0$, we have

$$A \leq x_{n+1} = A + \frac{\alpha y_n + \beta y_{n-1}}{By_n + Cy_{n-1}} \leq A + \max\left\{\frac{\alpha}{B}, \frac{\beta}{C}\right\},$$

from which it follows that

$$A \leq x_n \leq A + \max\left\{\frac{\alpha}{B}, \frac{\beta}{C}\right\},$$

for all $n \geq 1$. On the other hand, for all $n \geq 2$, we have

$$ax_{n-1} \leq y_{n+1} = \left(a + \frac{x_{n-1}}{bx_n + cx_{n-1}}\right) x_{n-1} \leq \left(a + \frac{1}{c}\right) x_{n-1},$$
$$aA \leq y_{n+1} \leq \left(a + \frac{1}{c}\right)\left(A + \max\left\{\frac{\alpha}{B}, \frac{\beta}{C}\right\}\right),$$

which implies that

$$aA \le y_n \le \left(a + \frac{1}{c}\right)\left(A + \max\left\{\frac{\alpha}{B}, \frac{\beta}{C}\right\}\right),$$

for all $n \ge 3$. □

Theorem 2.6.6. *The equilibrium point*

$$(\bar{x}, \bar{y}) = \left(A + \frac{\alpha+\beta}{B+C}, \left(A + \frac{\alpha+\beta}{B+C}\right)\left(a + \frac{1}{b+c}\right)\right)$$

of System (2.6.7) is locally asymptotically stable if

$$2\left|\alpha C - \beta B\right|(a(b+c)^2 + c + 3b) < (A(B+C) + \alpha + \beta)(a(b+c)+1)(B+C)(b+c).$$

Proof. We know from Theorem 2.2.1 that $(\bar{x}, \bar{y})$ is asymptotically stable if

$$\left|\frac{\partial f}{\partial u}(1,1)\frac{\partial g}{\partial w}(1,1)\right| + \left|\frac{\partial f}{\partial u}(1,1)\right|\left(\left|g(1,1) - 2\frac{\partial g}{\partial w}(1,1)\right| + \left|g(1,1) - \frac{\partial g}{\partial w}(1,1)\right|\right) < f(1,1)g(1,1).$$

Also, we obtain

$$f(1,1) = A + \frac{\alpha+\beta}{B+C}, \quad \frac{\partial f}{\partial u}(1,1) = \frac{\alpha C - \beta B}{(B+C)^2},$$
$$g(1,1) = a + \frac{1}{b+c}, \quad \frac{\partial g}{\partial w}(1,1) = \frac{-b}{(b+c)^2}.$$

By using these in the above inequality, we get
$\left|\frac{\alpha C-\beta B}{(B+C)^2}\frac{-b}{(b+c)^2}\right| + \left|\frac{\alpha C-\beta B}{(B+C)^2}\right|\left(\left|a + \frac{1}{b+c} - 2\frac{-b}{(b+c)^2}\right| + \left|a + \frac{1}{b+c} - \frac{-b}{(b+c)^2}\right|\right) < \left(A + \frac{\alpha+\beta}{B+C}\right)\left(a + \frac{1}{b+c}\right).$
After some arrangements, one can find the following inequality

$$2\left|\alpha C - \beta B\right|(a(b+c)^2 + c + 3b) < (A(B+C) + \alpha + \beta)(a(b+c)+1)(B+C)(b+c).$$

□

In the next theorem, we established conditions for the global attractivity of the equilibrium point. To this end, let choose the initial values such that

$$x_{-1}, x_0 \in \left[A, A + \max\left\{\frac{\alpha}{B}, \frac{\beta}{C}\right\}\right]$$

and

$$y_{-1},\ y_0,\ y_1,\ y_2 \in \left[aA, \left(a + \frac{1}{c}\right)\left(A + \max\left\{\frac{\alpha}{B}, \frac{\beta}{C}\right\}\right)\right].$$

Theorem 2.6.7. *The equilibrium point of System (2.6.7) is globally attractor if one of the following statements holds:*

1) $\alpha C > \beta B$ *and* $2(\alpha C - \beta B)(ac^2 + 3c + b) < BCc^2aA.$

2) $\alpha C < \beta B$ *and* $2(\beta B - \alpha C)(ac^2 + 3c + b) < BCc^2aA.$

Proof. To prove that the equilibrium point

$$(\bar{x}, \bar{y}) = \left(A + \frac{\alpha+\beta}{B+C}, \left(A + \frac{\alpha+\beta}{B+C}\right)\left(a + \frac{1}{b+c}\right)\right)$$

is a global attractor, we will use Theorem 2.3.3 and Theorem 2.3.8.

Let f and g be as defined in (2.6.8) and define the following parameters:

$$a_1 := A,\ b_1 := A + \max\left\{\frac{\alpha}{B}, \frac{\beta}{C}\right\},$$
$$a_2 := aA,\ b_2 := \left(a + \frac{1}{c}\right)\left(A + \max\left\{\frac{\alpha}{B}, \frac{\beta}{C}\right\}\right).$$

Then it follows that

$$a_1 < f(u,v) < b_1,\ a_2 < g(w,z) < b_2,\ \text{for all}\ \ (w,z,u,v) \in [a_1,b_1]^2 \times [a_2,b_2]^2.$$

We have

$$\frac{\partial f}{\partial u}(u,v) = \frac{(\alpha C - \beta B)v}{(Bu+Cv)^2}, \quad \frac{\partial f}{\partial v}(u,v) = \frac{(\beta B - \alpha C)u}{(Bu+Cv)^2},$$
$$\frac{\partial g}{\partial w}(w,z) = \frac{-bz^2}{(bw+cz)^2}, \quad \frac{\partial g}{\partial z}(w,z) = a + \frac{2bzw + cz^2}{(bw+cz)^2}.$$

1) Assume that $\alpha C > \beta B$, (in this case $b_1 := A + \frac{\alpha}{B}$ and $b_2 := \left(a + \frac{1}{c}\right)\left(A + \frac{\alpha}{B}\right)$). Then $f(u,v)$ is increasing in u and decreasing in v, however $g(w,z)$ is decreasing in w and increasing in z.

   So, it follows that the condition H_1 of Theorem 2.3.3 is satisfied. It remains to check condition H_2. To this end, let $(m,M,r,R) \in [a_1,b_1]^2 \times [a_2,b_2]^2$ be a solution of the system

$$m = A + \frac{\alpha r + \beta R}{Br + CR}, \quad M = A + \frac{\alpha R + \beta r}{BR + Cr},$$
$$r = \left(a + \frac{m}{bM + cm}\right)m, \quad R = \left(a + \frac{M}{bm + cM}\right)M,$$

such that $m \leq M$ and $r \leq R$. Then, from the system considered above, we obtain that

$$\begin{aligned} M - m &= \frac{(\alpha C - \beta B)(R + r)(R - r)}{(BR + Cr)(Br + CR)} \\ &\leq \frac{(\alpha C - \beta B)(R + r)(R - r)}{BCR^2} \\ &\leq \frac{2R(\alpha C - \beta B)(R - r)}{BCR^2} \\ &= \frac{2(\alpha C - \beta B)(R - r)}{BCR}. \end{aligned}$$

Since $R \in [a_2, b_2]$, we get

$$M - m \leq \frac{2(\alpha C - \beta B)(R - r)}{BCa_2}. \tag{2.6.9}$$

Similarly, we obtain

$$\begin{aligned} R - r &= \left(a + \frac{b(M^2 + m^2) + (b + c)mM}{(bm + cM)(bM + cm)}\right)(M - m) \\ &= \left(a + \frac{b(M^2 + m^2)}{(bm + cM)(bM + cm)} + \frac{(b + c)mM}{(bm + cM)(bM + cm)}\right)(M - m) \\ &\leq \left(a + \frac{b(M^2 + m^2)}{cbM^2} + \frac{(b + c)mM}{c^2 Mm}\right)(M - m) \\ &\leq \left(a + \frac{2bM^2}{cbM^2} + \frac{b + c}{c^2}\right)(M - m) \\ &= \left(a + \frac{2}{c} + \frac{b + c}{c^2}\right)(M - m). \end{aligned} \tag{2.6.10}$$

Furthermore, from (2.6.9) and (2.6.10), we obtain

$$(M - m)\left(1 - \frac{2(\alpha C - \beta B)}{BCa_2}\left(a + \frac{2}{c} + \frac{b + c}{c^2}\right)\right) \leq 0,$$

under the condition $2(\alpha C - \beta B)(ac^2 + 3c + b) < BCc^2 a_2$, we see that

$$\frac{2(\alpha C - \beta B)(ac^2 + 3c + b)}{BCc^2 a_2} < 1.$$

Hence, $M = m$ and from (2.6.10) we get $R = r$.

Consequently, the condition H_2 is satisfied and so the equilibrium point $(\bar{x}, \bar{y})$ is globally attractor.

2) Assume that $\alpha C < \beta B$, (in this case $b_1 := A + \frac{\beta}{C}$ and $b_2 := \left(a + \frac{1}{c}\right)\left(A + \frac{\beta}{C}\right)$). Then $f(u, v)$ is decreasing in u and increasing in v, however $g(w, z)$ is decreasing in w and increasing in z.

So, it follows that the condition H_1 of Theorem 2.3.8 is satisfied. It remains to check condition H_2. To this end, let $(m, M, r, R) \in [a_1, b_1]^2 \times [a_2, b_2]^2$ be a solution of the system

$$m = A + \frac{\alpha R + \beta r}{BR + Cr}, \quad M = A + \frac{\alpha r + \beta R}{Br + CR},$$
$$r = \left(a + \frac{m}{bM + cm}\right) m, \quad R = \left(a + \frac{M}{bm + cM}\right) M,$$

such that $m \leq M$ and $r \leq R$. Then, from the system considered above, we obtain that

$$\begin{aligned} M - m &= \frac{(\beta B - \alpha C)(R + r)(R - r)}{(BR + Cr)(Br + CR)} \\ &\leq \frac{(\beta B - \alpha C)(R + r)(R - r)}{BCR^2} \\ &\leq \frac{2R(\beta B - \alpha C)(R - r)}{BCR^2} \\ &= \frac{2(\beta B - \alpha C)(R - r)}{BCR}. \end{aligned}$$

Since $R \in [a_2, b_2]$, we get

$$M - m \leq \frac{2(\beta B - \alpha C)(R - r)}{BCa_2}. \tag{2.6.11}$$

Similarly, we obtain

$$R - r \leq \left(a + \frac{2}{c} + \frac{b + c}{c^2}\right)(M - m). \tag{2.6.12}$$

Furthermore, from (2.6.11) and (2.6.12), we obtain

$$(M - m)\left(1 - \frac{2(\beta B - \alpha C)}{BCa_2}\left(a + \frac{2}{c} + \frac{b + c}{c^2}\right)\right) \leq 0,$$

under the condition $2(\beta B - \alpha C)(ac^2 + 3c + b) < BCc^2 a_2$, we see that

$$\frac{2(\beta B - \alpha C)(ac^2 + 3c + b)}{BCc^2 a_2} < 1.$$

Hence, $M = m$ and from (2.6.12) we get $R = r$.

Consequently, the condition H_2 is satisfied and so the equilibrium point $(\bar{x}, \bar{y})$ is globally attractor.

□

Theorem 2.6.8. *The equilibrium point*

$$(\bar{x}, \bar{y}) = \left(A + \frac{\alpha+\beta}{B+C}, \left(A + \frac{\alpha+\beta}{B+C}\right)\left(a + \frac{1}{b+c}\right)\right)$$

of System (2.6.7) is globally asymptotically stable if one of the following statements holds:

(1) $\alpha C > \beta B, \quad 2(\alpha C - \beta B)(ac^2 + 3c + b) < BCc^2aA$ *and*

$$2(\alpha C - \beta B)(a(b+c)^2 + c + 3b) < (A(B+C) + \alpha + \beta)(a(b+c)+1)(B+C)(b+c).$$

(2) $\alpha C < \beta B, \quad 2(\beta B - \alpha C)(ac^2 + 3c + b) < BCc^2aA$ *and*

$$2(\beta B - \alpha C)(a(b+c)^2 + c + 3b) < (A(B+C) + \alpha + \beta)(a(b+c)+1)(B+C)(b+c).$$

Proof. Since the equilibrium point is both locally asymptotically stable and a global attractor, the proof follows from the results of Theorem 2.6.6 and Theorem 2.6.7. □

Theorem 2.6.9. *Assume that $\gamma, \lambda \in (0,1) \cup (1,\infty)$. Then, System (2.6.7) have a prime period two solution*

$$\ldots, (\gamma p, \lambda q), (p, q), (\gamma p, \lambda q), (p, q), \ldots$$

if and only if

$$(1-\gamma)(((B^2+C^2)A + B\alpha + C\beta)\lambda + (1+\lambda^2)BCA) + (1-\lambda^2\gamma)C\alpha + (\lambda^2-\gamma)B\beta = 0$$

and

$$(\gamma-\lambda)(((b^2+c^2)a + c)\gamma + (1+\gamma^2)cba) + b(\gamma^3 - \lambda) = 0,$$

where

$$p = f(\lambda, 1), \quad q = (f(\lambda,1))^s g(\gamma, 1).$$

Proof. From Theorem 2.4.1, it follows that System (2.6.7) have a prime period two solution if and only if

$$f(1,\lambda) = \gamma f(\lambda, 1), \quad g(1,\gamma) = \lambda g(\gamma, 1). \tag{2.6.13}$$

In the case of System (2.6.7), the functions f and g are

$$f(u,v) = A + \frac{\alpha u + \beta v}{Bu + Cv}, \quad g(w,z) = \left(a + \frac{z}{bw + cz}\right) z,$$

and therefore we have

$$f(1,\lambda) = A + \frac{\alpha + \beta\lambda}{B + C\lambda}, \quad f(\lambda,1) = A + \frac{\alpha\lambda + \beta}{B\lambda + C},$$

$$g(1,\gamma) = \left(a + \frac{\gamma}{b + c\gamma}\right)\gamma, \quad g(\gamma,1) = \left(a + \frac{1}{b\gamma + c}\right).$$

In this case, the first equality in (2.6.13) become

$$A + \frac{\alpha + \beta\lambda}{B + C\lambda} = \gamma\left(A + \frac{\alpha\lambda + \beta}{B\lambda + C}\right),$$

and after some operations

$$(1-\gamma)(((B^2 + C^2)A + B\alpha + C\beta)\lambda + (1+\lambda^2)BCA) + (1-\lambda^2\gamma)C\alpha + (\lambda^2 - \gamma)B\beta = 0.$$

Similarly, the second equality in (2.6.13) become

$$\left(a + \frac{\gamma}{b + c\gamma}\right)\gamma = \lambda\left(a + \frac{1}{b\gamma + c}\right),$$

and after some operations

$$(\gamma - \lambda)(((b^2 + c^2)a + c)\gamma + (1+\gamma^2)cba) + b(\gamma^3 - \lambda) = 0,$$

as desired. □

A Numerical Simulation on Theorem 2.6.9: *If we assume that* $\lambda = 5$ *and* $\gamma = 2$, *then the conditions for periodicity with prime period two become*

$$23\beta B - (5((B^2 + C^2)A + \alpha B + \beta C) + 26BCA) - 49C\alpha = 0 \tag{2.6.14}$$

and

$$3b - 3(2((b^2 + c^2)a + c) + 5cba) = 0. \tag{2.6.15}$$

If we choose the parameters as $A = 1, B = 3, C = 1, \alpha = 1, \beta = 3, a = \frac{4}{35}, b = 3, c = \frac{1}{4}$.

Then the conditions (2.6.14) *and* (2.6.15) *are satisfied, from which it follows immediately that*

$$x_{2n-1} = x_{-1} = f(\lambda, 1) = \frac{3}{2}, \quad y_{2n-1} = y_{-1} = f(\lambda, 1)g(\gamma, 1) = \frac{72}{175},$$
$$x_{2n} = x_0 = \gamma f(\lambda, 1) = 3, \quad y_{2n} = y_0 = \lambda(f(\lambda, 1))g(\gamma, 1) = \frac{72}{35}.$$

That is, we obtain the periodic solution as

$$\left\{..., \left(\frac{3}{2}, \frac{72}{175}\right), \left(3, \frac{72}{35}\right), \left(\frac{3}{2}, \frac{72}{175}\right), \left(3, \frac{72}{35}\right), ...\right\}$$

The plots of x_n and y_n are given by Figure 2.2.

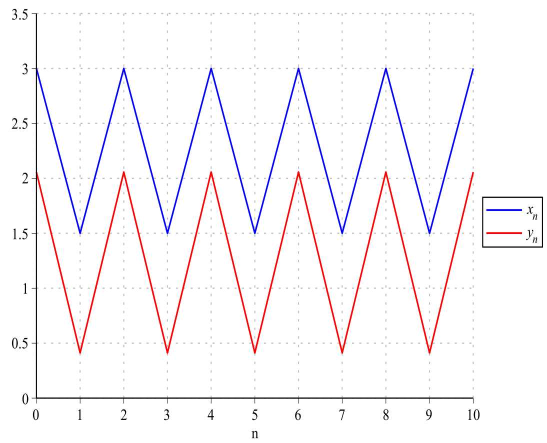

Figure 2.2 – Plots of x_n and y_n.

Theorem 2.6.10. *Assume that $\delta, \theta, \gamma, \lambda \in (0, 1) \cup (1, \infty)$. Then, System (2.6.7) have a prime period three solution*

$$\ldots, (\delta p, \theta q), (\gamma p, \lambda q), (p, q), (\delta p, \theta q), (\gamma p, \lambda q), (p, q), , \ldots$$

if and only if

$$[(1-\delta)(B + C\lambda)A + \alpha + \beta\lambda](B\lambda + C\theta) - \delta(\alpha\lambda + \beta\theta)(B + C\lambda) = 0, \tag{2.6.16}$$
$$[(1-\gamma)(B\theta + C)A + \alpha\theta + \beta](B\lambda + C\theta) - \gamma(\alpha\lambda + \beta\theta)(B\theta + C) = 0, \tag{2.6.17}$$
$$[a(\gamma - \theta\delta)(b + c\gamma) + \gamma^2](b\gamma + c\delta) - \theta\delta^2(b + c\gamma) = 0 \tag{2.6.18}$$

and

$$[(1 - \lambda\delta)(b\delta + c)a + 1](b\gamma + c\delta) - (b\delta + c)\lambda\delta^2 = 0. \tag{2.6.19}$$

where $p = f(\lambda, \theta), \quad q = (f(\lambda, \theta))^s g(\gamma, \delta).$

Proof. From Theorem 2.4.2, it follows that System (2.6.7) have a prime period three solution if and only if

$$\begin{aligned} f(1,\lambda) &= \delta f(\lambda,\theta), \quad f(\theta,1) = \gamma f(\lambda,\theta), \\ g(1,\gamma) &= \theta g(\gamma,\delta), \quad g(\delta,1) = \lambda g(\gamma,\delta). \end{aligned} \tag{2.6.20}$$

We have

$$\begin{aligned} &f(1,\lambda) = A + \frac{\alpha+\beta\lambda}{B+C\lambda}, \quad f(\theta,1) = A + \frac{\alpha\theta+\beta v}{B\theta+Cv}, \quad f(\lambda,\theta) = A + \frac{\alpha\lambda+\beta\theta}{B\lambda+C\theta}, \\ &g(1,\gamma) = \left(a + \frac{\gamma}{b+c\gamma}\right)\gamma, \quad g(\delta,1) = \left(a + \frac{1}{b\delta+c}\right), \quad g(\gamma,\delta) = \left(a + \frac{\delta}{b\gamma+c\delta}\right)\delta. \end{aligned}$$

In this case, the first and the second equality in (2.6.20) become

$$\begin{aligned} A + \frac{\alpha+\beta\lambda}{B+C\lambda} &= \delta\left(A + \frac{\alpha\lambda+\beta\theta}{B\lambda+C\theta}\right), \\ A + \frac{\alpha\theta+\beta}{B\theta+C} &= \gamma\left(A + \frac{\alpha\lambda+\beta\theta}{B\lambda+C\theta}\right), \end{aligned}$$

and after some operations, we get

$$\begin{aligned} &[(1-\delta)(B+C\lambda)A + \alpha + \beta\lambda](B\lambda+C\theta) - \delta(\alpha\lambda+\beta\theta)(B+C\lambda) = 0, \\ &[(1-\gamma)(B\theta+C)A + \alpha\theta + \beta](B\lambda+C\theta) - \gamma(\alpha\lambda+\beta\theta)(B\theta+C) = 0. \end{aligned}$$

Similarly, the third and the fourth equality in (2.6.20) become

$$\begin{aligned} \left(a + \frac{\gamma}{b+c\gamma}\right)\gamma &= \left(a + \frac{\delta}{b\gamma+c\delta}\right)\theta\delta, \\ \left(a + \frac{1}{b\delta+c}\right) &= \left(a + \frac{\delta}{b\gamma+c\delta}\right)\lambda\delta, \end{aligned}$$

from which it follows that

$$\begin{aligned} &[a(\gamma-\theta\delta)(b+c\gamma) + \gamma^2](b\gamma+c\delta) - \theta\delta^2(b+c\gamma) = 0, \\ &[(1-\lambda\delta)(b\delta+c)a + 1](b\gamma+c\delta) - (b\delta+c)\lambda\delta^2 = 0. \end{aligned}$$

□

A Numerical Simulation on Theorem 2.6.10: *If we assume that* $\delta = 2, \theta = 7, \lambda = \frac{1}{2}$ *and* $\gamma = 4$*, then the conditions* (2.6.16)*,* (2.6.17)*,* (2.6.18) *for periodicity with prime period three become*

$$\left[\alpha + \frac{1}{2}\beta - \left(B + \frac{1}{2}C\right) A\right] \left(\frac{1}{2}B + 7C\right) - 2\left(\frac{1}{2}\alpha + 7\beta\right)\left(B + \frac{1}{2}C\right) = 0, \tag{2.6.21}$$

$$(7\alpha + \beta - 3(7B + C)A)\left(\frac{1}{2}B + 7C\right) - 4\left(\frac{1}{2}\alpha + 7\beta\right)(7B + C) = 0, \tag{2.6.22}$$

$$(16 - 10a(b + 4c))(4b + 2c) - 28(b + 4c) = 0 \tag{2.6.23}$$

and the condition (2.6.19) *is verified.*

If we choose the parameters as

$$A = \frac{2611}{405}, B = \frac{16}{373}, C = 1, \alpha = 12, \beta = 13, a = \frac{1}{35}, b = 3, c = 1,$$

then the conditions (2.6.21)*,* (2.6.22)*,* (2.6.23) *are satisfied, and the solution take the forme*

$$x_{3n-1} = x_{-1} = \frac{16412}{405}, \quad x_{3n} = x_0 = \frac{32824}{405}, \quad x_{3n+1} = x_1 = \frac{8206}{405},$$

$$y_{3n-1} = y_{-1} = \frac{32824}{675}, \quad y_{3n} = y_0 = \frac{16412}{4725}, \quad y_{3n+1} = y_1 = \frac{32824}{4725}.$$

That is, we obtain the three periodic solution as

$$\left\{..., \left(\frac{16412}{405}, \frac{32824}{675}\right), \left(\frac{32824}{405}, \frac{16412}{4725}\right), \left(\frac{8206}{405}, \frac{32824}{4725}\right), \left(\frac{16412}{405}, \frac{32824}{675}\right), ...\right\}$$

The plots of x_n *and* y_n *are given by Figure* 2.3*.*

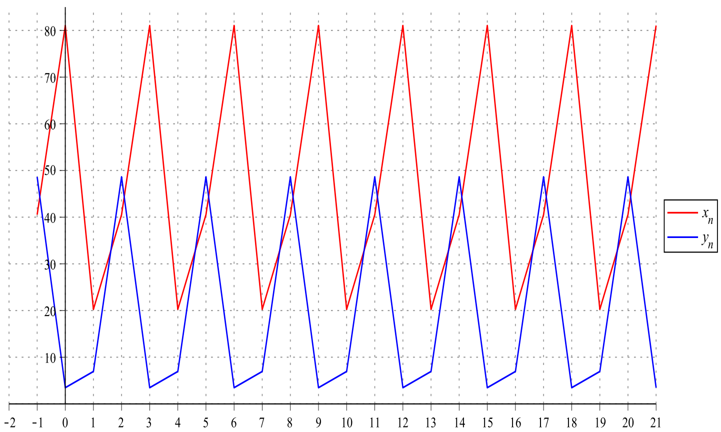

Figure 2.3 – Plots of x_n and y_n.

Theorem 2.6.11. *Let* $(x_n, y_n)_{n\geq -1}$ *be a solution of System (2.6.7). Assume that* $\alpha C < \beta B$*, then the following statements holds :*

1. *Let*

$$x_{-1} > \bar{x}, \quad x_0 < \bar{x}, \quad y_{-1} > \bar{y}, \quad y_0 < \bar{y}.$$

Then the sequence $(x_n)_{n\geq -1}$ *(resp.* $(y_n)_{n\geq -1}$*) oscillates about* $\bar{x}$ *(resp.* $\bar{y}$*).*

2. *Let*

$$x_{-1} < \bar{x}, \quad x_0 > \bar{x}, \quad y_{-1} < \bar{y}, \quad y_0 > \bar{y}.$$

Then the sequence $(x_n)_{n\geq -1}$ *(resp.* $(y_n)_{n\geq -1}$*) oscillates about* $\bar{x}$ *(resp.* $\bar{y}$*).*

Proof. 1. Let

$$x_0 < \bar{x} < x_{-1}, \quad y_0 < \bar{y} < y_{-1}.$$

Then, from System (2.6.7), for $n = 1$, we have

$$\begin{aligned}
x_1 - \bar{x} &= A + \frac{\alpha y_0 + \beta y_{-1}}{B y_0 + C y_{-1}} - \left(A + \frac{\alpha + \beta}{B + C}\right) \\
&= \frac{(\alpha y_0 + \beta y_{-1})(B + C) - (\alpha + \beta)(B y_0 + C y_{-1})}{(B y_0 + C y_{-1})(B + C)} \\
&= \frac{\alpha y_0 B + \alpha y_0 C + \beta y_{-1} B + \beta y_{-1} C - \alpha B y_0 - \beta B y_0 - \alpha C y_{-1} - \beta C y_{-1}}{(B y_0 + C y_{-1})(B + C)} \\
&= \frac{\alpha y_0 C + \beta y_{-1} B - \beta B y_0 - \alpha C y_{-1}}{(B y_0 + C y_{-1})(B + C)} \\
&= \frac{(y_0 - y_{-1})\alpha C + (y_{-1} - y_0)\beta B}{(B y_0 + C y_{-1})(B + C)} \\
&= \frac{(y_0 - y_{-1})(\alpha C - \beta B)}{(B y_0 + C y_{-1})(B + C)}.
\end{aligned}$$

By the inequalities $y_0 < y_{-1}$ and $\alpha C < \beta B$, it is easy to see that

$$x_1 > \bar{x}.$$

Also, since $x_0 < \bar{x} < x_{-1}$, we get

$$\begin{aligned}
y_1 &= \left(a + \frac{x_{-1}}{b x_0 + c x_{-1}}\right) x_{-1} \\
&> \left(a + \frac{x_{-1}}{b x_0 + c x_{-1}}\right) \bar{x} \\
&> \left(a + \frac{x_{-1}}{(b + c) x_{-1}}\right) \bar{x}
\end{aligned}$$

$$> \left(a + \frac{1}{b+c}\right)\left(A + \frac{\alpha+\beta}{B+C}\right) = \bar{y}.$$

Similarly, by following the same steps for $n = 2$, from System (2.6.7), we have that

$$\begin{aligned}
x_2 - \bar{x} &= A + \frac{\alpha y_1 + \beta y_0}{B y_1 + C y_0} - \left(A + \frac{\alpha+\beta}{B+C}\right) \\
&= \frac{(\alpha y_1 + \beta y_0)(B+C) - (\alpha+\beta)(B y_1 + C y_0)}{(B y_1 + C y_0)(B+C)} \\
&= \frac{\alpha y_1 B + \alpha y_1 C + \beta y_0 B + \beta y_0 C - \alpha B y_1 - \beta B y_1 - \alpha C y_0 - \beta C y_0}{(B y_1 + C y_0)(B+C)} \\
&= \frac{\alpha y_1 C + \beta y_0 B - \beta B y_1 - \alpha C y_0}{(B y_1 + C y_0)(B+C)} \\
&= \frac{(y_1 - y_0)\alpha C + (y_0 - y_1)\beta B}{(B y_1 + C y_0)(B+C)} \\
&= \frac{(y_1 - y_0)(\alpha C - \beta B)}{(B y_1 + C y_0)(B+C)}.
\end{aligned}$$

By the inequalities $y_1 > y_0$ and $\alpha C < \beta B$, it is easy to see that

$$x_2 < \bar{x}.$$

Also, since $x_1 > \bar{x} > x_0$, for $n = 2$, we get

$$\begin{aligned}
y_2 &= \left(a + \frac{x_0}{b x_1 + c x_0}\right) x_0 \\
&< \left(a + \frac{x_0}{b x_1 + c x_0}\right) \bar{x} \\
&< \left(a + \frac{x_0}{(b+c) x_0}\right) \bar{x} \\
&< \left(a + \frac{1}{b+c}\right)\left(A + \frac{\alpha+\beta}{B+C}\right) = \bar{y}.
\end{aligned}$$

Consequently, by induction, one can easily see that

$$x_{2n-1} > \bar{x}, \quad x_{2n} < \bar{x}, \quad y_{2n-1} > \bar{y}, \quad y_{2n} < \bar{y}, \quad n \in \mathbb{N}_0.$$

That is to say, the sequence $(x_n)_{n\geq -1}$ (resp. $(y_n)_{n\geq -1}$) oscillates about $\bar{x}$ (resp. $\bar{y}$).

2. Let

$$x_0 > \bar{x} > x_{-1}, \quad y_0 > \bar{y} > y_{-1}.$$

Then, from System (2.6.7), for $n = 1$, we have

$$\begin{aligned}
x_1 - \bar{x} &= A + \frac{\alpha y_0 + \beta y_{-1}}{B y_0 + C y_{-1}} - \left(A + \frac{\alpha + \beta}{B + C}\right) \\
&= \frac{(\alpha y_0 + \beta y_{-1})(B + C) - (\alpha + \beta)(B y_0 + C y_{-1})}{(B y_0 + C y_{-1})(B + C)} \\
&= \frac{\alpha y_0 B + \alpha y_0 C + \beta y_{-1} B + \beta y_{-1} C - \alpha B y_0 - \beta B y_0 - \alpha C y_{-1} - \beta C y_{-1}}{(B y_0 + C y_{-1})(B + C)} \\
&= \frac{\alpha y_0 C + \beta y_{-1} B - \beta B y_0 - \alpha C y_{-1}}{(B y_0 + C y_{-1})(B + C)} \\
&= \frac{(y_0 - y_{-1})\alpha C + (y_{-1} - y_0)\beta B}{(B y_0 + C y_{-1})(B + C)} \\
&= \frac{(y_0 - y_{-1})(\alpha C - \beta B)}{(B y_0 + C y_{-1})(B + C)}.
\end{aligned}$$

By the inequalities $y_0 > y_{-1}$ and $\alpha C < \beta B$, it is easy to see that

$$x_1 < \bar{x}.$$

Also, since $x_0 > \bar{x} > x_{-1}$, for $n = 1$, we get

$$\begin{aligned}
y_1 &= \left(a + \frac{x_{-1}}{b x_0 + c x_{-1}}\right) x_{-1} \\
&< \left(a + \frac{x_{-1}}{b x_0 + c x_{-1}}\right) \bar{x} \\
&< \left(a + \frac{x_{-1}}{(b + c) x_{-1}}\right) \bar{x} \\
&< \left(a + \frac{1}{b + c}\right) \left(A + \frac{\alpha + \beta}{B + C}\right) = \bar{y}.
\end{aligned}$$

Similarly, by following the same steps for $n = 2$, from System (2.6.7), we have that

$$\begin{aligned}
x_2 - \bar{x} &= A + \frac{\alpha y_1 + \beta y_0}{B y_1 + C y_0} - \left(A + \frac{\alpha + \beta}{B + C}\right) \\
&= \frac{(\alpha y_1 + \beta y_0)(B + C) - (\alpha + \beta)(B y_1 + C y_0)}{(B y_1 + C y_0)(B + C)} \\
&= \frac{\alpha y_1 B + \alpha y_1 C + \beta y_0 B + \beta y_0 C - \alpha B y_1 - \beta B y_1 - \alpha C y_0 - \beta C y_0}{(B y_1 + C y_0)(B + C)} \\
&= \frac{\alpha y_1 C + \beta y_0 B - \beta B y_1 - \alpha C y_0}{(B y_1 + C y_0)(B + C)} \\
&= \frac{(y_1 - y_0)\alpha C + (y_0 - y_1)\beta B}{(B y_1 + C y_0)(B + C)} \\
&= \frac{(y_1 - y_0)(\alpha C - \beta B)}{(B y_1 + C y_0)(B + C)}.
\end{aligned}$$

By the inequalities $y_1 < y_0$ and $\alpha C < \beta B$, it is easy to see that

$$x_2 > \bar{x}.$$

Also, since $x_1 < x_0$ and $\bar{x} < x_0$, for $n = 2$, we get

$$\begin{aligned} y_2 &= \left(a + \frac{x_0}{bx_1 + cx_0}\right) x_0 \\ &> \left(a + \frac{x_0}{bx_1 + cx_0}\right) \bar{x} \\ &> \left(a + \frac{x_0}{(b+c)x_0}\right) \bar{x} \\ &> \left(a + \frac{1}{b+c}\right)\left(A + \frac{\alpha+\beta}{B+C}\right) = \bar{y}. \end{aligned}$$

Consequently, by induction, one can easily see that

$$x_{2n-1} < \bar{x}, \quad x_{2n} > \bar{x}, \quad y_{2n-1} < \bar{y}, \quad y_{2n} > \bar{y}, \quad n \in \mathbb{N}_0.$$

That is to say, the sequence $(x_n)_{n\geq -1}$ (resp. $(y_n)_{n\geq -1}$) oscillates about $\bar{x}$ (resp. $\bar{y}$).

□

A Numerical Simulation on Theorem 2.6.11 : *Consider System (2.6.7). Let us choose the parameters as* $A = 1, B = 3, C = 1, \alpha = 1, \beta = 3, a = \frac{4}{3}, b = 3, c = \frac{1}{4}$. *Then system(2.6.7) becomes*

$$x_{n+1} = 1 + \frac{y_n + 3y_{n-1}}{3y_n + 1y_{n-1}}, \quad y_{n+1} = \left(\frac{4}{3} + \frac{x_{n-1}}{3x_n + \frac{1}{4}x_{n-1}}\right) x_{n-1}, \quad n \in \mathbb{N}_0. \qquad (2.6.24)$$

If we choose the initial values such that

$$x_0 = \frac{1}{2} < \bar{x} = 2 < x_{-1} = 3, \quad y_0 = \frac{1}{4} < \bar{y} = \frac{128}{39} \simeq 3.28 < y_{-1} = 5,$$

then the corresponding solution of System 2.6.24 oscillates about the equilibrium point. The plots of x_n *and* y_n *are given by Figure 2.4.*

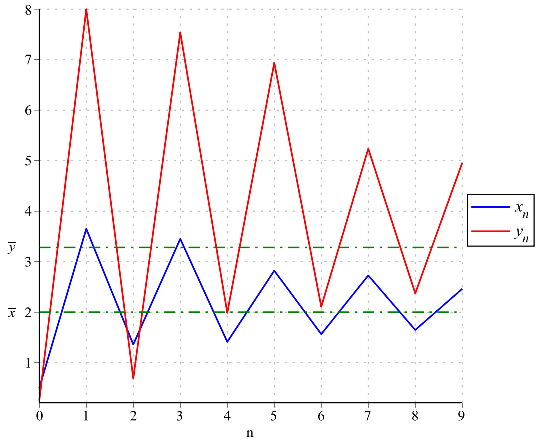

Figure 2.4 – Plots of x_n and y_n.

If we choose the initial values such that

$$x_{-1} = \frac{1}{2} < \bar{x} = 2 < x_0 = 3, \quad y_{-1} = \frac{1}{4} < \bar{y} = \frac{128}{39} \simeq 3.28 < y_0 = 5,$$

then the corresponding solution of System 2.6.24 again oscillates about the equilibrium point. The plots of x_n and y_n are given by Figure 2.5.

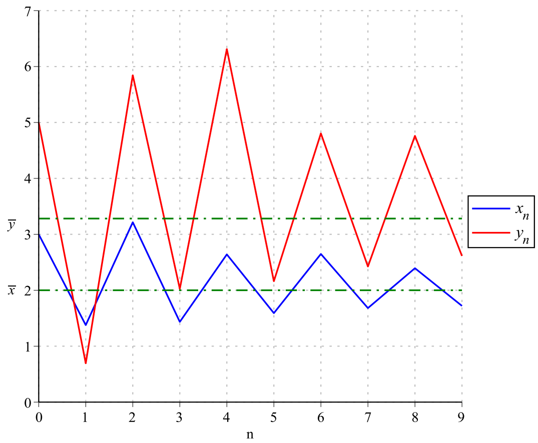

Figure 2.5 – Plots of x_n and y_n.

Chapter 3

SOLVABILITY OF A THIRD-ORDER SYSTEMS OF NONLINEAR DIFFERENCE EQUATIONS VIA A GENERALIZED FIBONACCI SEQUENCE

3.1 Introduction

There has been a noticeable development in the study of solvable systems of difference equations. Although solving explicitly these systems is generally difficult, however in some concrete models and via some convenient transformations this situation can be surmounted. Finding closed-form formulas of the solutions of some systems of difference equations is the subject of this chapter.

As a generalization of the system

$$x_{n+1} = \frac{x_{n-1}y_n}{y_n \pm y_{n-2}}, y_{n+1} = \frac{y_{n-1}x_n}{x_n \pm x_{n-2}}, \quad n \in \mathbb{N}_0. \tag{3.1.1}$$

studied in [53], the authors of [28] considered the following system of difference equations

$$x_{n+1} = \frac{x^p_{n-k+1}y_n}{ay^p_{n-k} + by_n}, y_{n+1} = \frac{y^p_{n-k+1}x_n}{\alpha x_{n-k} + \beta x_n}, \quad n \in \mathbb{N}_0, k, p \in \mathbb{N}, \tag{3.1.2}$$

where the coefficients a, b, α, β and the initial values $x_{-i}, y_{-i}, i \in \{0, 1, ..., k\}$ are real numbers.

Motivating by [28], our goal in this chapter is to find the solution form of some systems of difference equations with powers. The first will be the system

$$x_{n+1}=\frac{y_n y_{n-1} x_{n-1}^p}{x_n(a_n y_{n-2}^q+b_n y_n y_{n-1})},\ y_{n+1}=\frac{x_n x_{n-1} y_{n-1}^q}{y_n(c_n x_{n-2}^p+d_n x_n x_{n-1})},\ n\in\mathbb{N}_0,\ p,\ q\in\mathbb{N} \tag{3.1.3}$$

where the parameters $(a_n)_{n\in\mathbb{N}_0},(b_n)_{n\in\mathbb{N}_0},(c_n)_{n\in\mathbb{N}_0},(d_n)_{n\in\mathbb{N}_0}$ and the initial values $x_{-i},y_{-i},i=0,1,2,$ are non-zero real numbers. Motiviting by [1] and [49], We will also determine the form of the solutions of the following system, which is a generalization of System (3.1.3), defined by

$$\begin{aligned} x_{n+1}&=f^{-1}\left(\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)\left[a_n(g(y_{n-2}))^q+b_n g(y_n)g(y_{n-1})\right]}\right),\\ y_{n+1}&=g^{-1}\left(\frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)\left[c_n(f(x_{n-2}))^p+d_n f(x_n)f(x_{n-1})\right]}\right),n\in\mathbb{N}_0,p,q\in\mathbb{N}, \end{aligned} \tag{3.1.4}$$

where $f,g:D\to\mathbb{R}$ are one to one continuous functions on $D\subseteq\mathbb{R}$, the initial values $x_{-i},y_{-i},i=0,1,2,$ are real numbers in D and the parameters $(a_n)_{n\in\mathbb{N}_0},(b_n)_{n\in\mathbb{N}_0},(c_n)_{n\in\mathbb{N}_0},(d_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers. In the same philosophy, we will solve also in closed form the following system of difference equations

$$\begin{cases} x_{n+1}=f^{-1}\left(\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)[a_n(g(y_{n-2}))^q+b_n g(y_n)g(y_{n-1})]}\right),\\ y_{n+1}=g^{-1}\left(\frac{h(z_n)h(z_{n-1})(g(y_{n-1}))^q}{g(y_n)[c_n(h(z_{n-2}))^r+d_n h(z_n)h(z_{n-1})]}\right),\quad n\in\mathbb{N}_0,\ p,\ q,\ r\ \in\mathbb{N}\\ z_{n+1}=h^{-1}\left(\frac{f(x_n)f(x_{n-1})(h(z_{n-1}))^r}{h(z_n)[s_n(f(x_{n-2}))^p+t_n f(x_n)f(x_{n-1})]}\right), \end{cases} \tag{3.1.5}$$

where $f,\ g,\ h\ :D\to\mathbb{R}$ are continuous one to one functions on $D\subseteq\mathbb{R}$, the coefficients $(a_n)_{n\in\mathbb{N}_0},\ (b_n)_{n\in\mathbb{N}_0},(c_n)_{n\in\mathbb{N}_0},(d_n)_{n\in\mathbb{N}_0},(s_n)_{n\in\mathbb{N}_0},(t_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers and the initial values $x_{-i},y_{-i},z_{-i},i=0,\ 1,\ 2,$ are real numbers in D. As an example, we will apply the obtained results, on the particular system of difference equations

$$x_{n+1}=\frac{y_n y_{n-1} x_{n-1}^p}{x_n(a_n y_{n-2}^q+b_n y_n y_{n-1})},\ y_{n+1}=\frac{z_n z_{n-1} y_{n-1}^q}{y_n(c_n z_{n-2}^r+d_n z_n z_{n-1})},\ z_{n+1}=\frac{x_n x_{n-1} z_{n-1}^r}{z_n(s_n x_{n-2}^p+t_n x_n x_{n-1})}, \tag{3.1.6}$$

which is obtained from the general System (3.1.5) by taking $f(x)=g(x)=h(x)=x$. The following very well known lemma will be used in the resolution of the corresponding linear system.

Lemma 3.1.1. *Let $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and consider* *the linear difference equation*

$$z_{n+k} = p_n z_n + q_n, \quad k = 2, 3, \quad n \in \mathbb{N}_0.$$

Then for $n \in \mathbb{N}_0$ we have

$$z_{kn+i} = \left[\prod_{j=0}^{n-1} p_{kj+i}\right] z_i + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} p_{kj+i}\right] q_{kr+i}, \quad \textit{for } i = 0, 1, \cdots, k-1.$$

Moreover, if $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$ are constants (i.e. $p_n = p, q_n = q$), then

$$z_{kn+i} = \begin{cases} z_i + qn, & a = 1, \\ p^n z_i + \left(\dfrac{p^n - 1}{p-1}\right) q, & \textit{otherwise}, \end{cases} \quad \textit{for } i = 0, 1, \cdots, k-1, \quad n \in \mathbb{N}_0.$$

where, $\prod\limits_{j=i}^{k} A_j = 1$ and $\sum\limits_{j=i}^{k} A_j = 0$, for all $k < i$.

We will see in the sequel that the formulas of well-defined solutions of Systems (3.1.3)-(3.1.6) are presented via a generalized Fibonacci sequence, namely $\{F_{k,n}\}_{n=0}^{\infty}$ defined by

$$F_{k,n+2} = F_{k,n+1} + kF_{k,n}, \quad F_{k,0} = F_{k,1} = 1, \; n \in \mathbb{N}_0, \; k \in \mathbb{R} - \{0\}. \tag{3.1.7}$$

Here are some of its terms:

$$F_{k,0} = 1,$$

$$F_{k,1} = 1,$$

$$F_{k,2} = 1 + k,$$

$$F_{k,3} = 1 + 2k,$$

$$F_{k,4} = 1 + 3k + k^2,$$

$$F_{k,5} = 1 + 4k + 3k^2,$$

$$F_{k,6} = 1 + 5k + 6k^2 + k^3,$$

$$F_{k,7} = 1 + 6k + 10k^2 + 4k^3,$$

$$F_{k,8} = 1 + 7k + 15k^2 + 10k^3 + k^4,$$

$$F_{k,9} = 1 + 8k + 21k^2 + 20k^3 + 5k^4,$$

$$F_{k,10} = 1 + 9k + 28k^2 + 35k^3 + 15k^4 + k^5,$$

$$F_{k,11} = 1 + 10k + 36k^2 + 56k^3 + 35k^4 + 6k^5,$$

$$F_{k,12} = 1 + 11k + 45k^2 + 84k^3 + 70k^4 + 21k^5 + k^6.$$

3.2 The solutions of System (3.1.3)

In this part, we show the solvability of our System (3.1.3). In fact we will give the closed form of the well-defined solutions of our system.

Definition 3.2.1. *A solution $\{x_n, y_n\}_{n\geq -2}$ of System (3.1.3) is said to be well-defined if*

$$x_n\left(a_n y_{n-2}^q + b_n y_n y_{n-1}\right) y_n\left(c_n x_{n-2}^p + d_n x_n x_{n-1}\right) \neq 0,\ n \in \mathbb{N}_0.$$

Now, we will start the resolution of our system. Let $\{x_n, y_n\}_{n\geq -2}$ be a well-defined solution of System (3.1.3). We have

$$\begin{aligned}
x_{n+1} &= \frac{y_n y_{n-1} x_{n-1}^p}{x_n(a_n y_{n-2}^q + b_n y_n y_{n-1})}, & y_{n+1} &= \frac{x_n x_{n-1} y_{n-1}^q}{y_n(c_n x_{n-2}^p + d_n x_n x_{n-1})},\\
\frac{x_{n+1}x_n}{x_{n-1}^p} &= \frac{y_n y_{n-1}}{a_n y_{n-2}^q + b_n y_n y_{n-1}}, & \frac{y_{n+1}y_n}{y_{n-1}^q} &= \frac{x_n x_{n-1}}{c_n x_{n-2}^p + d_n x_n x_{n-1}},\\
\frac{x_{n-1}^p}{x_{n+1}x_n} &= \frac{a_n y_{n-2}^q + b_n y_n y_{n-1}}{y_n y_{n-1}}, & \frac{y_{n-1}^q}{y_{n+1}y_n} &= \frac{c_n x_{n-2}^p + d_n x_n x_{n-1}}{x_n x_{n-1}},\\
\frac{x_{n-1}^p}{x_{n+1}x_n} &= a_n \frac{y_{n-2}^q}{y_n y_{n-1}} + b_n, & \frac{y_{n-1}^q}{y_{n+1}y_n} &= c_n \frac{x_{n-2}^p}{x_n x_{n-1}} + d_n,
\end{aligned}$$

Taking the change of variables

$$u_n = \frac{x_{n-2}^p}{x_n x_{n-1}}, \quad v_n = \frac{y_{n-2}^q}{y_n y_{n-1}}, \tag{3.2.1}$$

System (3.1.3) can be written as

$$u_{n+1} = a_n v_n + b_n, \quad v_{n+1} = c_n u_n + d_n, \quad n \in \mathbb{N}_0. \tag{3.2.2}$$

Hence, we have

$$\begin{aligned}
u_{n+2} = a_{n+1}v_{n+1} + b_{n+1} &= a_{n+1}\left[c_n u_n + d_n\right] + b_{n+1}\\
&= a_{n+1}c_n u_n + (a_{n+1}d_n + b_{n+1}),
\end{aligned}$$

$$v_{n+2} = c_{n+1}u_{n+1} + d_{n+1} = c_{n+1}[a_n v_n + b_n] + d_{n+1}$$
$$= c_{n+1}a_n v_n + (c_{n+1}b_n + d_{n+1}).$$

From this, we get, for all $n \in \mathbb{N}_0$, the following linear second order nonhomogeneous difference equations,

$$\begin{cases} u_{n+2} = a_{n+1}c_n u_n + (a_{n+1}d_n + b_{n+1}), \\ v_{n+2} = c_{n+1}a_n v_n + (c_{n+1}b_n + d_{n+1}). \end{cases} \tag{3.2.3}$$

From Lemma 3.1.1, we have for all $n \in \mathbb{N}_0$ and for $i = 0, 1$, the solutions of equations in (3.2.3) are

$$u_{2n+i} = \left[\prod_{j=0}^{n-1} a_{2j+i+1}c_{2j+i}\right] u_i + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{2j+i+1}c_{2j+i}\right]\left(a_{2r+i+1}d_{2r+i} + b_{2r+i+1}\right), \tag{3.2.4}$$

$$v_{2n+i} = \left[\prod_{j=0}^{n-1} c_{2j+i+1}a_{2j+i}\right] v_i + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{2j+i+1}a_{2j+i}\right]\left(c_{2r+i+1}b_{2r+i} + d_{2r+i+1}\right). \tag{3.2.5}$$

From (3.2.1) and Equations (3.2.4) and (3.2.5), it follows that for all $n \in \mathbb{N}_0$

$$u_{2n} = \left[\prod_{j=0}^{n-1} a_{2j+1}c_{2j}\right] \frac{x_{-2}^p}{x_0 x_{-1}} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{2j+1}c_{2j}\right]\left(a_{2r+1}d_{2r} + b_{2r+1}\right), \tag{3.2.6}$$

$$u_{2n+1} = \frac{\left[\prod_{j=0}^{n-1} a_{2j+2}c_{2j+1}\right]\left[a_0 y_{-2}^q + b_0 y_0 y_{-1}\right]}{y_0 y_{-1}} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{2j+2}c_{2j+1}\right]\left(a_{2r+2}d_{2r+1} + b_{2r+2}\right) \tag{3.2.7}$$

$$v_{2n} = \frac{\left[\prod_{j=0}^{n-1} c_{2j+1}a_{2j}\right] y_{-2}^q}{y_0 y_{-1}} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{2j+1}a_{2j}\right]\left(c_{2r+1}b_{2r} + d_{2r+1}\right), \tag{3.2.8}$$

and

$$v_{2n+1} = \frac{\left[\prod_{j=0}^{n-1} c_{2j+2}a_{2j+1}\right]\left[c_0 x_{-2}^p + d_0 x_0 x_{-1}\right]}{x_0 x_{-1}} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{2j+2}a_{2j+1}\right]\left(c_{2r+2}b_{2r+1} + d_{2r+2}\right). \tag{3.2.9}$$

Now we give the solution form of Equations (3.2.3) when all the coefficients in System (3.1.3) are constant. To do this, we suppose that $a_n = a, b_n = b, c_n = c$ and $d_n = d$, for every $n \in \mathbb{N}_0$. Then Equations (3.2.3) becomes

$$\begin{cases} u_{n+2} = acu_n + ad + b, \\ v_{n+2} = cav_n + cb + d. \end{cases} \tag{3.2.10}$$

From Lemma 3.1.1, we have for all $n \in \mathbb{N}_0$ and for $i = 0, 1$, the solutions of equations in (3.2.10) are

$$u_{2n+i} = \begin{cases} u_i + (ad+b)n, & ac = 1, \\ (ac)^n u_i + \left(\dfrac{(ac)^n - 1}{ac-1}\right)(ad+b), & otherwise, \end{cases} \tag{3.2.11}$$

and

$$v_{2n+i} = \begin{cases} v_i + (csb + ct + d)n, & ac = 1, \\ (ca)^n v_i + \left(\dfrac{(ca)^n - 1}{ca-1}\right)(cb+d), & otherwise, \end{cases} \tag{3.2.12}$$

From (3.2.1) and Equations (3.2.11) and (3.2.12), it follows that for all $n \in \mathbb{N}_0$

$$u_{2n} = \begin{cases} \dfrac{x_{-2}^p}{x_0 x_{-1}} + (ad+b)n, & ac = 1, \\ \\ \dfrac{(ac)^n x_{-2}^p}{x_0 x_{-1}} + \left(\dfrac{(ac)^n - 1}{ac-1}\right)(ad+b), & otherwise, \end{cases} \tag{3.2.13}$$

$$u_{2n+1} = \begin{cases} \dfrac{ay_{-2}^q + by_0 y_{-1}}{y_0 y_{-1}} + (ad+b)n, & ac = 1, \\ \\ \dfrac{(ac)^n (ay_{-2}^q + by_0 y_{-1})}{y_0 y_{-1}} + \left(\dfrac{(ac)^n - 1}{ac-1}\right)(ad+b), & otherwise, \end{cases} \tag{3.2.14}$$

$$v_{2n} = \begin{cases} \dfrac{y_{-2}^q}{y_0 y_{-1}} + (cb+d)n, & ac = 1, \\ \\ \dfrac{(ca)^n y_{-2}^q}{y_0 y_{-1}} + \left(\dfrac{(ca)^n - 1}{ca-1}\right)(cb+d), & otherwise, \end{cases} \tag{3.2.15}$$

$$v_{2n+1} = \begin{cases} \dfrac{cx_{-2}^p + dx_0 x_{-1}}{x_0 x_{-1}} + (cb+d)n, & ac = 1, \\ (ca)^n \dfrac{cx_{-2}^p + dx_0 x_{-1}}{x_0 x_{-1}} + \left(\dfrac{(ca)^n - 1}{ca-1}\right)(cb+d), & otherwise, \end{cases} \tag{3.2.16}$$

Now, from (3.2.1) it follows that

$$x_n = \frac{x_{n-2}^p}{u_n x_{n-1}}, \quad y_n = \frac{y_{n-2}^q}{v_n y_{n-1}}, \tag{3.2.17}$$

So we have,

$$x_0 = \frac{x_{-2}^p}{u_0 x_{-1}},$$

hence

$$x_0 = \frac{x_{-2}^{pF_{p,0}}}{u_0^{F_{p,0}} x_{-1}^{F_{p,1}}}.$$

Moreover,

$$x_1 = \frac{x_{-1}^p}{u_1 x_0} = \frac{u_0 x_{-1} x_{-1}^p}{u_1 x_{-2}^p} = \frac{u_0 x_{-1}^{p+1}}{u_1 x_{-2}^p},$$

so

$$x_1 = \frac{u_0^{F_{p,1}} x_{-1}^{F_{p,2}}}{u_1^{F_{p,0}} x_{-2}^{pF_{p,1}}}$$

and

$$x_2 = \frac{x_0^p}{u_2 x_1} = \frac{x_{-2}^{p^2} u_1 x_{-2}^p}{u_0^p x_{-1}^p u_2 u_0 x_{-1}^{p+1}} = \frac{u_1 x_{-2}^{p^2+p}}{u_2 u_0^{p+1} x_{-1}^{2p+1}},$$

thus

$$x_2 = \frac{u_1^{F_{p,1}} x_{-2}^{pF_{p,2}}}{u_2^{F_{p,0}} u_0^{F_{p,2}} x_{-1}^{F_{p,3}}}.$$

Similarly, we obtain

$$x_3 = \frac{u_2 u_0^{2p+1} x_{-1}^{p^2+3p+1}}{u_3 u_1^{p+1} x_{-2}^{2p^2+p}} = \frac{u_2^{F_{p,1}} u_0^{F_{p,3}} x_{-1}^{F_{p,4}}}{u_3^{F_{p,0}} u_1^{F_{p,2}} x_{-2}^{pF_{p,3}}},$$

$$x_4 = \frac{u_3 u_1^{2p+1} x_{-2}^{p^3+3p^2+p}}{u_4 u_2^{p+1} u_0^{p^2+3p+1} x_{-1}^{3p^2+4p+1}} = \frac{u_3^{F_{p,1}} u_1^{F_{p,3}} x_{-2}^{pF_{p,4}}}{u_4^{F_{p,0}} u_2^{F_{p,2}} u_0^{F_{p,4}} x_{-1}^{F_{p,5}}},$$

$$x_5 = \frac{u_4 u_2^{2p+1} u_0^{3p^2+4p+1} x_{-1}^{p^3+6p^2+5p+1}}{u_5 u_3^{p+1} u_1^{p^2+3p+1} x_{-2}^{3p^3+4p^2+p}} = \frac{u_4^{F_{p,1}} u_2^{F_{p,3}} u_0^{F_{p,5}} x_{-1}^{F_{p,6}}}{u_5^{F_{p,0}} u_3^{F_{p,2}} u_1^{F_{p,4}} x_{-2}^{pF_{p,5}}},$$

$$x_6 = \frac{u_5 u_3^{2p+1} u_1^{3p^2+4p+1} x_{-2}^{p^4+6p^3+5p^2+p}}{u_6 u_4^{p+1} u_2^{p^2+3p+1} u_0^{p^3+6p^2+5p+1} x_{-1}^{4p^3+10p^2+6p+1}} = \frac{u_5^{F_{p,1}} u_3^{F_{p,3}} u_1^{F_{p,5}} x_{-2}^{pF_{p,6}}}{u_6^{F_{p,0}} u_4^{F_{p,2}} u_2^{F_{p,4}} u_0^{F_{p,6}} x_{-1}^{F_{p,7}}},$$

$$x_7 = \frac{u_6^{F_{p,1}} u_4^{F_{p,3}} u_2^{F_{p,5}} u_0^{F_{p,7}} x_{-1}^{F_{p,8}}}{u_7^{F_{p,0}} u_5^{F_{p,2}} u_3^{F_{p,4}} u_1^{F_{p,6}} x_{-2}^{pF_{p,7}}} = \frac{\prod_{i=0}^{3} u_{2i}^{F_{p,2(3-i)+1}} x_{-1}^{F_{p,8}}}{\prod_{i=0}^{3} u_{2i+1}^{F_{p,2(3-i)}} x_{-2}^{pF_{p,7}}},$$

$$x_8 = \frac{u_7^{F_{p,1}} u_5^{F_{p,3}} u_3^{F_{p,5}} u_1^{F_{p,7}} x_{-2}^{pF_{p,8}}}{u_8^{F_{p,0}} u_6^{F_{p,2}} u_4^{F_{p,4}} u_2^{F_{p,6}} u_0^{F_{p,8}} x_{-1}^{F_{p,9}}} = \frac{\prod_{i=0}^{3} u_{2i+1}^{F_{p,2(4-i)-1}} x_{-2}^{pF_{p,8}}}{\prod_{i=0}^{4} u_{2i}^{F_{p,2(4-i)}} x_{-1}^{F_{p,9}}},$$

$$x_9 = \frac{u_8^{F_{p,1}} u_6^{F_{p,3}} u_4^{F_{p,5}} u_2^{F_{p,7}} u_0^{F_{p,9}} x_{-1}^{F_{p,10}}}{u_9^{F_{p,0}} u_7^{F_{p,2}} u_5^{F_{p,4}} u_3^{F_{p,6}} u_1^{F_{p,8}} x_{-2}^{pF_{p,9}}} = \frac{\prod_{i=0}^{4} u_{2i}^{F_{p,2(4-i)+1}} x_{-1}^{F_{p,10}}}{\prod_{i=0}^{4} u_{2i+1}^{F_{p,2(4-i)}} x_{-2}^{pF_{p,9}}},$$

$$x_{10} = \frac{u_9^{F_{p,1}} u_7^{F_{p,3}} u_5^{F_{p,5}} u_3^{F_{p,7}} u_1^{F_{p,9}} x_{-2}^{pF_{p,10}}}{u_{10}^{F_{p,0}} u_8^{F_{p,2}} u_6^{F_{p,4}} u_4^{F_{p,6}} u_2^{F_{p,8}} u_0^{F_{p,10}} x_{-1}^{F_{p,11}}} = \frac{\prod_{i=0}^{4} u_{2i+1}^{F_{p,2(5-i)-1}} x_{-2}^{pF_{p,10}}}{\prod_{i=0}^{5} u_{2i}^{F_{p,2(5-i)}} x_{-1}^{F_{p,11}}}.$$

By induction, it follows that

$$x_{2n} = \frac{\prod_{i=0}^{n-1} u_{2i+1}^{F_{p,2(n-i)-1}} x_{-2}^{pF_{p,2n}}}{\prod_{i=0}^{n} u_{2i}^{F_{p,2(n-i)}} x_{-1}^{F_{p,2n+1}}},$$

$$x_{2n+1} = \frac{\prod_{i=0}^{n} u_{2i}^{F_{p,2(n-i)+1}} x_{-1}^{F_{p,2(n+1)}}}{\prod_{i=0}^{n} u_{2i+1}^{F_{p,2(n-i)}} x_{-2}^{pF_{p,2n+1}}}.$$

Similarly we have,

$$y_0 = \frac{y_{-2}^q}{v_0 y_{-1}},$$

hence

$$y_0 = \frac{y_{-2}^{qF_{q,0}}}{v_0^{F_{q,0}} y_{-1}^{F_{q,1}}}.$$

Moreover,

$$y_1 = \frac{y_{-1}^q}{v_1 y_0} = \frac{v_0 y_{-1} y_{-1}^q}{v_1 y_{-2}^q} = \frac{v_0 y_{-1}^{q+1}}{v_1 y_{-2}^q},$$

so

$$y_1 = \frac{v_0^{F_{q,1}} y_{-1}^{F_{q,2}}}{v_1^{F_{q,0}} y_{-2}^{qF_{q,1}}}$$

and

$$y_2 = \frac{y_0^q}{v_2 y_1} = \frac{y_{-2}^{q^2} v_1 y_{-2}^q}{v_0^q y_{-1}^q v_2 v_0 y_{-1}^{q+1}} = \frac{v_1 y_{-2}^{q^2+q}}{v_2 v_0^{q+1} y_{-1}^{2q+1}},$$

thus

$$y_2 = \frac{v_1^{F_{q,1}} y_{-2}^{qF_{q,2}}}{v_2^{F_{q,0}} v_0^{F_{q,2}} y_{-1}^{F_{q,3}}}.$$

Similarly, we obtain

$$y_3 = \frac{v_2 v_0^{2q+1} y_{-1}^{q^2+3q+1}}{v_3 v_1^{q+1} y_{-2}^{2q^2+q}} = \frac{v_2^{F_{q,1}} v_0^{F_{q,3}} y_{-1}^{F_{q,4}}}{v_3^{F_{q,0}} v_1^{F_{q,2}} y_{-2}^{qF_{q,3}}},$$

$$y_4 = \frac{v_3 v_1^{2q+1} y_{-2}^{q^3+3q^2+q}}{v_4 v_2^{q+1} v_0^{q^2+3q+1} y_{-1}^{3q^2+4q+1}} = \frac{v_3^{F_{q,1}} v_1^{F_{q,3}} y_{-2}^{qF_{q,4}}}{v_4^{F_{q,0}} v_2^{F_{q,2}} v_0^{F_{q,4}} y_{-1}^{F_{q,5}}},$$

$$y_5=\frac{v_4v_2^{2q+1}v_0^{3q^2+4q+1}y_{-1}^{q^3+6q^2+5q+1}}{v_5v_3^{q+1}v_1^{q^2+3q+1}y_{-2}^{3q^3+4q^2+q}}=\frac{v_4^{F_{q,1}}v_2^{F_{q,3}}v_0^{F_{q,5}}y_{-1}^{F_{q,6}}}{v_5^{F_{q,0}}v_3^{F_{q,2}}v_1^{F_{q,4}}y_{-2}^{qF_{q,5}}},$$

$$y_6=\frac{v_5v_3^{2q+1}v_1^{3q^2+4q+1}y_{-2}^{q^4+6q^3+5q^2+q}}{v_6v_4^{q+1}u_2^{q^2+3q+1}v_0^{q^3+6q^2+5q+1}y_{-1}^{4q^3+10q^2+6q+1}}=\frac{v_5^{F_{q,1}}v_3^{F_{q,3}}v_1^{F_{q,5}}y_{-2}^{qF_{q,6}}}{v_6^{F_{q,0}}v_4^{F_{q,2}}v_2^{F_{q,4}}v_0^{F_{q,6}}y_{-1}^{F_{q,7}}},$$

$$y_7=\frac{v_6^{F_{q,1}}v_4^{F_{q,3}}v_2^{F_{q,5}}v_0^{F_{q,7}}y_{-1}^{F_{q,8}}}{v_7^{F_{q,0}}v_5^{F_{q,2}}v_3^{F_{q,4}}v_1^{F_{q,6}}y_{-2}^{pF_{q,7}}}=\frac{\prod\limits_{i=0}^{3}v_{2i}^{F_{q,2(3-i)+1}}y_{-1}^{F_{q,8}}}{\prod\limits_{i=0}^{3}v_{2i+1}^{F_{q,2(3-i)}}y_{-2}^{qF_{q,7}}},$$

$$y_8=\frac{v_7^{F_{q,1}}v_5^{F_{q,3}}v_3^{F_{q,5}}v_1^{F_{q,7}}y_{-2}^{pF_{q,8}}}{v_8^{F_{q,0}}v_6^{F_{q,2}}v_4^{F_{q,4}}v_2^{F_{q,6}}v_0^{F_{q,8}}y_{-1}^{F_{q,9}}}=\frac{\prod\limits_{i=0}^{3}v_{2i+1}^{F_{q,2(4-i)-1}}y_{-2}^{qF_{q,8}}}{\prod\limits_{i=0}^{4}v_{2i}^{F_{q,2(4-i)}}y_{-1}^{F_{q,9}}},$$

$$y_9=\frac{v_8^{F_{q,1}}v_6^{F_{q,3}}v_4^{F_{q,5}}v_2^{F_{q,7}}v_0^{F_{q,9}}y_{-1}^{F_{q,10}}}{v_9^{F_{q,0}}v_7^{F_{q,2}}v_5^{F_{q,4}}v_3^{F_{q,6}}v_1^{F_{q,8}}y_{-2}^{qF_{q,9}}}=\frac{\prod\limits_{i=0}^{4}v_{2i}^{F_{q,2(4-i)+1}}y_{-1}^{F_{q,10}}}{\prod\limits_{i=0}^{4}v_{2i+1}^{F_{q,2(4-i)}}y_{-2}^{qF_{q,9}}},$$

$$y_{10}=\frac{v_9^{F_{q,1}}v_7^{F_{q,3}}v_5^{F_{q,5}}v_3^{F_{q,7}}v_1^{F_{q,9}}y_{-2}^{qF_{q,10}}}{v_{10}^{F_{q,0}}v_8^{F_{q,2}}v_6^{F_{q,4}}v_4^{F_{q,6}}v_2^{F_{q,8}}v_0^{F_{q,10}}y_{-1}^{F_{q,11}}}=\frac{\prod\limits_{i=0}^{4}v_{2i+1}^{F_{q,2(5-i)-1}}y_{-2}^{qF_{q,10}}}{\prod\limits_{i=0}^{5}v_{2i}^{F_{q,2(5-i)}}y_{-1}^{F_{q,11}}}.$$

By induction, it follows that

$$y_{2n}=\frac{\prod\limits_{i=0}^{n-1}v_{2i+1}^{F_{q,2(n-i)-1}}y_{-2}^{qF_{q,2n}}}{\prod\limits_{i=0}^{n}v_{2i}^{F_{q,2(n-i)}}y_{-1}^{F_{q,2n+1}}},$$

$$y_{2n+1}=\frac{\prod\limits_{i=0}^{n}v_{2i}^{F_{q,2(n-i)+1}}y_{-1}^{F_{q,2(n+1)}}}{\prod\limits_{i=0}^{n}v_{2i+1}^{F_{q,2(n-i)}}y_{-2}^{qF_{q,2n+1}}},$$

From the above calculations, we summarize in the following theorem the form of the solutions of System (3.1.3).

Theorem 3.2.1. *Let $\{x_n,y_n\}_{n\geq-2}$ be a well-defined solution of System (3.1.3). Then, for all $n\in\mathbb{N}_0$, we have*

$$x_{2n}=\left(\frac{\prod\limits_{i=0}^{n-1}u_{2i+1}^{F_{p,2(n-i)-1}}}{\prod\limits_{i=0}^{n}u_{2i}^{F_{p,2(n-i)}}}\right)\frac{x_{-2}^{pF_{p,2n}}}{x_{-1}^{F_{p,2n+1}}},$$

$$x_{2n+1}=\left(\frac{\prod\limits_{i=0}^{n}u_{2i}^{F_{p,2(n-i)+1}}}{\prod\limits_{i=0}^{n}u_{2i+1}^{F_{p,2(n-i)}}}\right)\frac{x_{-1}^{F_{p,2(n+1)}}}{x_{-2}^{pF_{p,2n+1}}},$$

$$y_{2n}=\left(\frac{\prod_{i=0}^{n-1}v_{2i+1}^{F_{q,2(n-i)-1}}}{\prod_{i=0}^{n}v_{2i}^{F_{q,2(n-i)}}}\right)\frac{y_{-2}^{qF_{q,2n}}}{y_{-1}^{F_{q,2n+1}}},$$

and

$$y_{2n+1}=\left(\frac{\prod_{i=0}^{n}v_{2i}^{F_{q,2(n-i)+1}}}{\prod_{i=0}^{n}v_{2i+1}^{F_{q,2(n-i)}}}\right)\frac{y_{-1}^{F_{q,2(n+1)}}}{y_{-2}^{qF_{q,2n+1}}}$$

where the terms of the sequences $(u_n)_{n\in\mathbb{N}_0}$ and $(v_n)_{n\in\mathbb{N}_0}$ are given by formulas (2.6)-(2.9) in the case of variables coefficients and (2.13)-(2.16) in the case of constant coefficients.

Remark 3.2.1. *If we take $c_n=a_n$, $d_n=b_n$, $n\in\mathbb{N}_0$ $q=p$ and $y_{-i}=x_{-i}$, $i=0,1,2,$ then, we obtain the one dimensional version of System (3.1.3), that is the equation*

$$x_{n+1}=\frac{x_{n-1}^{p+1}}{a_nx_{n-2}^{p}+b_nx_nx_{n-1}},\quad p\in\mathbb{N},n\in\mathbb{N}_0. \tag{3.2.18}$$

As a consequence the solutions of Equations (3.2.18) can be obtained from Theorem 3.2.1, and their formulas are given in the following result.

Corollary 3.2.1. *Let $\{x_n\}_{n\geq-2}$ be a well-defined solution of Equation (3.2.18), then for $n\in\mathbb{N}_0$ we have*

$$x_{2n}=\left(\frac{\prod_{i=0}^{n-1}u_{2i+1}^{F_{p,2(n-i)-1}}}{\prod_{i=0}^{n}u_{2i}^{F_{p,2(n-i)}}}\right)\frac{x_{-2}^{pF_{p,2n}}}{x_{-1}^{F_{p,2n+1}}},$$

$$x_{2n+1}=\left(\frac{\prod_{i=0}^{n}u_{2i}^{F_{p,2(n-i)+1}}}{\prod_{i=0}^{n}u_{2i+1}^{F_{p,2(n-i)}}}\right)\frac{x_{-1}^{F_{p,2(n+1)}}}{x_{-2}^{pF_{p,2n+1}}},$$

the terms of the sequence $(u_n)_{n\in\mathbb{N}_0}$ are given by formulas (2.6), (2.7) in the case of variables coefficients and (2.13), (2.14) in the case of constant coefficients.

3.3 The form of the solutions of a more general system defined by one to one functions

In this part, we will show the solvability of the following system

$$x_{n+1}=f^{-1}\left(\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)\left[a_n(g(y_{n-2}))^q+b_ng(y_n)g(y_{n-1})\right]}\right),$$

$$y_{n+1} = g^{-1}\left(\frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)\left[c_n(f(x_{n-2}))^p + d_nf(x_n)f(x_{n-1})\right]}\right), n \in \mathbb{N}_0, p, q \in \mathbb{N}, \qquad (3.3.1)$$

where $f, g : D \to \mathbb{R}$ are one to one continuous functions on $D \subseteq \mathbb{R}$, the initial values $x_{-i}, y_{-i}, i = 0, 1, 2,$ are real numbers in D and the parameters $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0},$ $(d_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers.

Definition 3.3.1. *A solution $\{x_n, y_n\}_{n\geq -2}$ of System (3.1.4) is said to be well-defined if for all $n \in \mathbb{N}_0$, we have*

$$f(x_n)\left[a_n(g(y_{n-2}))^q + b_ng(y_n)g(y_{n-1})\right] \neq 0,$$

$$g(y_n)\left[c_n(f(x_{n-2}))^p + d_nf(x_n)f(x_{n-1})\right] \neq 0,$$

$$\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)\left[a_n(g(y_{n-2}))^q + b_ng(y_n)g(y_{n-1})\right]} \in D_{f^{-1}}$$

and

$$\frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)\left[c_n(f(x_{n-2}))^p + d_nf(x_n)f(x_{n-1})\right]} \in D_{g^{-1}}.$$

Since f and g are one to one continuous functions, then we get

$$f(x_{n+1}) = \frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)(a_n(g(y_{n-2}))^q + b_ng(y_n)g(y_{n-1}))},$$

$$g(y_{n+1}) = \frac{f(x_n)f(x_{n-1})(g(y_{n-1}))^q}{g(y_n)(c_n(f(x_{n-2}))^p + d_nf(x_n)f(x_{n-1}))}.$$

Taking the change of variables

$$X_n = f(x_n), \quad Y_n = g(y_n), \quad n \in \mathbb{N}_0 \qquad (3.3.2)$$

it follows that System (3.1.4) can be transformed to the following system

$$X_{n+1} = \frac{Y_nY_{n-1}X^p_{n-1}}{X_n(a_nY^q_{n-2} + b_nY_nY_{n-1})}, \quad Y_{n+1} = \frac{X_nX_{n-1}Y^q_{n-1}}{Y_n(c_nX^p_{n-2} + d_nX_nX_{n-1})}, \quad n \in \mathbb{N}_0,$$

which is in the form (3.1.3). So, using (3.3.2), the fact that

$$x_n = f^{-1}(X_n), \quad y_n = g^{-1}(Y_n), \quad n \in \mathbb{N}_0$$

and Theorem (3.2.1), we get the following result which describes the form of the solutions of System (3.1.4).

Theorem 3.3.1. *Let* $\{x_n, y_n\}_{n\geq -2}$ *be a well-defined solution of System* (3.1.4)*. Then, for all* $n \in \mathbb{N}_0$*, we have*

$$x_{2n} = f^{-1}\left(\left(\frac{\prod_{i=0}^{n-1} u_{2i+1}^{F_{p,2(n-i)-1}}}{\prod_{i=0}^{n} u_{2i}^{F_{p,2(n-i)}}}\right)\frac{(f(x_{-2}))^{pF_{p,2n}}}{(f(x_{-1}))^{F_{p,2n+1}}}\right),$$

$$x_{2n+1} = f^{-1}\left(\left(\frac{\prod_{i=0}^{n} u_{2i}^{F_{p,2(n-i)+1}}}{\prod_{i=0}^{n} u_{2i+1}^{F_{p,2(n-i)}}}\right)\frac{(f(x_{-1}))^{F_{p,2(n+1)}}}{(f(x_{-2}))^{pF_{p,2n+1}}}\right),$$

$$y_{2n} = g^{-1}\left(\left(\frac{\prod_{i=0}^{n-1} v_{2i+1}^{F_{q,2(n-i)-1}}}{\prod_{i=0}^{n} v_{2i}^{F_{q,2(n-i)}}}\right)\frac{(g(y_{-2}))^{qF_{q,2n}}}{(g(y_{-1}))^{F_{q,2n+1}}}\right),$$

and

$$y_{2n+1} = g^{-1}\left(\left(\frac{\prod_{i=0}^{n} v_{2i}^{F_{q,2(n-i)+1}}}{\prod_{i=0}^{n} v_{2i+1}^{F_{q,2(n-i)}}}\right)\frac{(g(y_{-1}))^{F_{q,2(n+1)}}}{(g(y_{-2}))^{qF_{q,2n+1}}}\right),$$

where the terms of the sequences $(u_n)_{n\in\mathbb{N}_0}$ *and* $(v_n)_{n\in\mathbb{N}_0}$ *are given by the following formulas*

$$u_{2n} = \frac{\left[\prod_{j=0}^{n-1} a_{2j+1}c_{2j}\right](f(x_{-2}))^p}{f(x_0)f(x_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{2j+1}c_{2j}\right](a_{2r+1}d_{2r} + b_{2r+1}), \tag{3.3.3}$$

$$u_{2n+1} = \frac{\left[\prod_{j=0}^{n-1} a_{2j+2}c_{2j+1}\right][a_0(g(y_{-2}))^q + b_0 g(y_0)g(y_{-1})]}{g(y_0)g(y_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{2j+2}c_{2j+1}\right](a_{2r+2}d_{2r+1} + b_{2r+2}), \tag{3.3.4}$$

$$v_{2n} = \frac{\left[\prod_{j=0}^{n-1} c_{2j+1}a_{2j}\right](g(y_{-2}))^q}{g(y_0)g(y_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{2j+1}a_{2j}\right](c_{2r+1}b_{2r} + d_{2r+1}), \tag{3.3.5}$$

$$v_{2n+1} = \frac{\left[\prod_{j=0}^{n-1} c_{2j+2}a_{2j+1}\right][c_0(f(x_{-2}))^p + d_0 f(x_0)f(x_{-1})]}{f(x_0)f(x_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{2j+2}a_{2j+1}\right](c_{2r+2}b_{2r+1} + d_{2r+2}). \tag{3.3.6}$$

If the coefficients are variables and by the following formulas

$$u_{2n}=\begin{cases}\dfrac{(f(x_{-2}))^p}{f(x_0)f(x_{-1})}+(ad+b)n, & ac=1,\\ \\ \dfrac{(ac)^n(f(x_{-2}))^p}{f(x_0)f(x_{-1})}+\left(\dfrac{(ac)^n-1}{ac-1}\right)(ad+b), & otherwise,\end{cases} \tag{3.3.7}$$

$$u_{2n+1}=\begin{cases}\dfrac{a(g(y_{-2}))^q+bg(y_0)g(y_{-1})}{g(y_0)g(y_{-1})}+(ad+b)n, & ac=1,\\ \\ \dfrac{(ac)^n(a(g(y_{-2}))^q+bg(y_0)g(y_{-1}))}{g(y_0)g(y_{-1})}+\left(\dfrac{(ac)^n-1}{ac-1}\right)(ad+b), & otherwise,\end{cases} \tag{3.3.8}$$

$$v_{2n}=\begin{cases}\dfrac{(g(y_{-2}))^q}{g(y_0)g(y_{-1})}+(cb+d)n, & ac=1,\\ \\ \dfrac{(ca)^n(g(y_{-2}))^q}{g(y_0)g(y_{-1})}+\left(\dfrac{(ca)^n-1}{ca-1}\right)(cb+d), & otherwise,\end{cases} \tag{3.3.9}$$

$$v_{2n+1}=\begin{cases}\dfrac{c(f(x_{-2}))^p+df(x_0)f(x_{-1})}{f(x_0)f(x_{-1})}+(cb+d)n, & ac=1,\\ \dfrac{(ca)^n(c(f(x_{-2}))^p+df(x_0)f(x_{-1}))}{f(x_0)f(x_{-1})}+\left(\dfrac{(ca)^n-1}{ca-1}\right)(cb+d), & otherwise,\end{cases} \tag{3.3.10}$$

if the coefficients are constants.

Remark 3.3.1. *Clearly if we take the functions f and g such that $f(x)=g(x)=x$, then System (3.1.4) will be nothing other than System (3.1.3).*

3.4 Explicit formulas for the well-defined solutions of System (3.1.5)

Definition 3.4.1. *A solution $\{x_n, y_n, z_n\}_{n\geq -2}$ of System (3.1.5) is said to be well-defined if for all $n \in \mathbb{N}_0$, we have*

$$f(x_n)\left[a_n(g(y_{n-2}))^q+b_ng(y_n)g(y_{n-1})\right]\neq 0,$$

$$g(y_n)\left[c_n(h(z_{n-2}))^r+d_nh(z_n)h(z_{n-1})\right]\neq 0,$$

$$h(z_n)\left[s_n(f(x_{n-2}))^p + t_n f(x_n)f(x_{n-1})\right] \neq 0,$$

$$\frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)\left[a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1})\right]} \in D_{f^{-1}},$$

$$\frac{h(z_n)h(z_{n-1})(g(y_{n-1}))^q}{g(y_n)\left[c_n(h(z_{n-2}))^r + d_n h(z_n)h(z_{n-1})\right]} \in D_{g^{-1}}$$

and

$$\frac{f(x_n)f(x_{n-1})(h(z_{n-1}))^r}{h(z_n)\left[s_n(f(x_{n-2}))^p + t_n f(x_n)f(x_{n-1})\right]} \in D_{h^{-1}}.$$

Let $\{x_n, y_n, z_n\}_{n\geq -2}$ be a well-defined solution to System (3.1.5). Since f, g and h are assumed to be one to one, then it follows from (3.1.5) that

$$f(x_{n+1}) = \frac{g(y_n)g(y_{n-1})(f(x_{n-1}))^p}{f(x_n)(a_n(g(y_{n-2}))^q + b_n g(y_n)g(y_{n-1}))},$$

$$g(y_{n+1}) = \frac{h(z_n)h(z_{n-1})(g(y_{n-1}))^q}{g(y_n)(c_n(h(z_{n-2}))^r + d_n h(z_n)h(z_{n-1}))},$$

$$h(z_{n+1}) = \frac{f(x_n)f(x_{n-1})(h(z_{n-1}))^r}{h(z_n)(s_n(f(x_{n-2}))^p + t_n f(x_n)f(x_{n-1}))}.$$

From which, we get

$$\frac{(f(x_{n-1}))^p}{f(x_{n+1})f(x_n)} = a_n\frac{(g(y_{n-2}))^q}{g(y_n)g(y_{n-1})} + b_n,$$

$$\frac{(g(y_{n-1}))^q}{g(y_{n+1})g(y_n)} = c_n\frac{(h(z_{n-2}))^r}{h(z_n)h(z_{n-1})} + d_n,$$

$$\frac{(h(z_{n-1}))^r}{h(z_{n+1})h(z_n)} = s_n\frac{(f(x_{n-2}))^p}{f(x_n)f(x_{n-1})} + t_n.$$

Consider the following change of variables

$$u_n = \frac{(f(x_{n-2}))^p}{f(x_n)f(x_{n-1})}, \quad v_n = \frac{(g(y_{n-2}))^q}{g(y_n)g(y_{n-1})}, \quad w_n = \frac{(h(z_{n-2}))^r}{h(z_n)h(z_{n-1})}, \tag{3.4.1}$$

then, System (3.1.5) is transformed to the following linear system

$$u_{n+1} = a_n v_n + b_n,\ v_{n+1} = c_n w_n + d_n,\ w_{n+1} = s_n u_n + t_n,\ n \in \mathbb{N}_0. \tag{3.4.2}$$

From (3.4.2), we have

$$\begin{aligned} u_{n+3} = a_{n+2}v_{n+2} + b_{n+2} &= a_{n+2}\left[c_{n+1}w_{n+1} + d_{n+1}\right] + b_{n+2} \\ &= \left[a_{n+2}c_{n+1}w_{n+1} + a_{n+2}d_{n+1}\right] + b_{n+2} \\ &= a_{n+2}c_{n+1}\left[s_n u_n + t_n\right] + a_{n+2}d_{n+1} + b_{n+2} \end{aligned}$$

$$
\begin{aligned}
&= a_{n+2}c_{n+1}s_n u_n + a_{n+2}c_{n+1}t_n + a_{n+2}d_{n+1} + b_{n+2}.\\
v_{n+3} = c_{n+2}w_{n+2} + d_{n+2} &= c_{n+2}[s_{n+1}u_{n+1} + t_{n+1}] + d_{n+2}\\
&= c_{n+2}[s_{n+1}(a_n v_n + b_n) + t_{n+1}] + d_{n+2}\\
&= c_{n+2}[s_{n+1}a_n v_n + s_{n+1}b_n + t_{n+1}] + d_{n+2}\\
&= c_{n+2}s_{n+1}a_n v_n + c_{n+2}s_{n+1}b_n + c_{n+2}t_{n+1} + d_{n+2}.\\
w_{n+3} = s_{n+2}u_{n+2} + t_{n+2} &= s_{n+2}[a_{n+1}v_{n+1} + b_{n+1}] + t_{n+2}\\
&= s_{n+2}[a_{n+1}(c_n w_n + d_n) + b_{n+1}] + t_{n+2}\\
&= s_{n+2}[a_{n+1}c_n w_n + a_{n+1}d_n + b_{n+1}] + t_{n+2}\\
&= s_{n+2}a_{n+1}c_n w_n + s_{n+2}a_{n+1}d_n + s_{n+2}b_{n+1} + t_{n+2}.
\end{aligned}
$$

That is, we have obtained the following three linear third order linear difference equations defined for all $n \in \mathbb{N}_0$ by

$$
\begin{cases}
u_{n+3} = a_{n+2}c_{n+1}s_n u_n + a_{n+2}c_{n+1}t_n + a_{n+2}d_{n+1} + b_{n+2},\\
v_{n+3} = c_{n+2}s_{n+1}a_n v_n + c_{n+2}s_{n+1}b_n + c_{n+2}t_{n+1} + d_{n+2},\\
w_{n+3} = s_{n+2}a_{n+1}c_n w_n + s_{n+2}a_{n+1}d_n + s_{n+2}b_{n+1} + t_{n+2}.
\end{cases} \tag{3.4.3}
$$

Using Lemma(3.1.1), we get for all $n \in \mathbb{N}_0$ and for $i = 0,\ 1,\ 2$ that

$$
\begin{aligned}
u_{3n+i} &= \left[\prod_{j=0}^{n-1} a_{3j+i+2}c_{3j+i+1}s_{3j+i}\right] u_i\\
&+ \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{3j+i+2}c_{3j+i+1}s_{3j+i}\right]\left(a_{3r+i+2}c_{3r+i+1}t_{3r+i} + a_{3r+i+2}d_{3r+i+1} + b_{3r+i+2}\right),
\end{aligned} \tag{3.4.4}
$$

$$
\begin{aligned}
v_{3n+i} &= \left[\prod_{j=0}^{n-1} c_{3j+i+2}s_{3j+i+1}a_{3j+i}\right] v_i\\
&+ \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{3j+i+2}s_{3j+i+1}a_{3j+i}\right]\left(c_{3r+i+2}s_{3r+i+1}b_{3r+i} + c_{3r+i+2}t_{3r+i+1} + d_{3r+i+2}\right),
\end{aligned} \tag{3.4.5}
$$

$$
w_{3n+i} = \left[\prod_{j=0}^{n-1} s_{3j+i+2}a_{3j+i+1}c_{3j+i}\right] w_i
$$

$$+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3j+i+2}a_{3j+i+1}c_{3j+i}\right](s_{3r+i+2}a_{3r+i+1}d_{3r+i}+s_{3r+i+2}b_{3r+i+1}+t_{3r+i+2}). \tag{3.4.6}$$

Now, from (3.4.1), (3.4.4), (3.4.5) and (3.4.6), it follows that for all $n\in\mathbb{N}_0$

$$\begin{aligned} u_{3n} &= \left[\prod_{j=0}^{n-1} a_{3j+2}c_{3j+1}s_{3j}\right]\frac{(f(x_{-2}))^p}{f(x_0)f(x_{-1})} \\ &\quad +\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{3j+2}c_{3j+1}s_{3j}\right](a_{3r+2}c_{3r+1}t_{3r}+a_{3r+2}d_{3r+1}+b_{3r+2}), \end{aligned} \tag{3.4.7}$$

$$\begin{aligned} u_{3n+1} &= \frac{\left[\prod_{j=0}^{n-1} a_{3j+3}c_{3j+2}s_{3j+1}\right][a_0(g(y_{-2}))^q+b_0g(y_0)g(y_{-1})]}{g(y_0)g(y_{-1})} \\ &\quad +\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{3(j+1)}c_{3j+2}s_{3j+1}\right]\left(a_{3r+3}c_{3r+2}t_{3r+1}+a_{3(r+1)}d_{3r+2}+b_{3(r+1)}\right), \end{aligned} \tag{3.4.8}$$

$$\begin{aligned} u_{3n+2} &= \frac{\left[\prod_{j=0}^{n-1} a_{3j+4}c_{3(j+1)}s_{3j+2}\right][a_1c_0(h(z_{-2}))^r+a_1d_0h(z_0)h(z_{-1})+b_1h(z_0)h(z_{-1})]}{h(z_0)h(z_{-1})} \\ &\quad +\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} a_{3j+4}c_{3(j+1)}s_{3j+2}\right]\left(a_{3r+4}c_{3(r+1)}t_{3r+2}+a_{3r+4}d_{3(r+1)}+b_{3r+4}\right), \end{aligned} \tag{3.4.9}$$

$$\begin{aligned} v_{3n} &= \frac{\left[\prod_{j=0}^{n-1} c_{3j+2}s_{3j+1}a_{3j}\right](g(y_{-2}))^q}{g(y_0)g(y_{-1})} \\ &\quad +\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{3j+2}s_{3j+1}a_{3j}\right](c_{3r+2}s_{3r+1}b_{3r}+c_{3r+2}t_{3r+1}+d_{3r+2}), \end{aligned} \tag{3.4.10}$$

$$\begin{aligned} v_{3n+1} &= \frac{\left[\prod_{j=0}^{n-1} c_{3(j+1)}s_{3j+2}a_{3j+1}\right][c_0(h(z_{-2}))^r+d_0h(z_0)h(z_{-1})]}{h(z_0)h(z_{-1})} \\ &\quad +\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{3(j+1)}s_{3j+2}a_{3j+1}\right]\left(c_{3(r+1)}s_{3r+2}b_{3r+1}+c_{3(r+1)}t_{3r+2}+d_{3(r+1)}\right), \end{aligned} \tag{3.4.11}$$

$$\begin{aligned} v_{3n+2} &= \frac{\left[\prod_{j=0}^{n-1} c_{3j+4}s_{3(j+1)}a_{3j+2}\right][c_1s_0(f(x_{-2}))^p+c_1t_0f(x_0)f(x_{-1})+d_1f(x_0)f(x_{-1})]}{f(x_0)f(x_{-1})} \\ &\quad +\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} c_{3j+4}s_{3(j+1)}a_{3j+2}\right]\left(c_{3r+4}s_{3(r+1)}b_{3r+2}+c_{3r+4}t_{3(r+1)}+d_{3r+4}\right) \end{aligned} \tag{3.4.12}$$

and

$$w_{3n} = \frac{\left[\prod_{j=0}^{n-1} s_{3j+2}a_{3j+1}c_{3j}\right](h(z_{-2}))^r}{h(z_0)h(z_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3j+2}a_{3j+1}c_{3j}\right](s_{3r+2}a_{3r+1}d_{3r} + s_{3r+2}b_{3r+1} + t_{3r+2}), \tag{3.4.13}$$

$$w_{3n+1} = \frac{\left[\prod_{j=0}^{n-1} s_{3(j+1)}a_{3j+2}c_{3j+1}\right][s_0(f(x_{-2}))^p + t_0 f(x_0)f(x_{-1})]}{f(x_0)f(x_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3(j+1)}a_{3j+2}c_{3j+1}\right]\left(s_{3(r+1)}a_{3r+2}d_{3r+1} + s_{3(r+1)}b_{3r+2} + t_{3(r+1)}\right), \tag{3.4.14}$$

$$w_{3n+2} = \frac{\left[\prod_{j=0}^{n-1} s_{3j+4}a_{3(j+1)}c_{3j+2}\right][sa(g(y_{-2}))^q + sbg(y_0)g(y_{-1}) + tg(y_0)g(y_{-1})]}{g(y_0)g(y_{-1})} + \sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1} s_{3j+4}a_{3(j+1)}c_{3j+2}\right]\left(s_{3r+4}a_{3(r+1)}d_{3r+2} + s_{3r+4}b_{3(r+1)} + t_{3r+4}\right). \tag{3.4.15}$$

If the coefficients are constant, that is $a_n = a$, $b_n = b$, $c_n = c$, $d_n = d$, $s_n = s$, $t_n = t$, then the linear equations in (3.4.3) becomes

$$\begin{cases} u_{n+3} = acsu_n + act + ad + b, \\ v_{n+3} = csav_n + csb + ct + d, \\ w_{n+3} = sacw_n + sad + sb + t. \end{cases} \tag{3.4.16}$$

Again, from Lemma 3.1.1, we get for all $n \in \mathbb{N}_0$ and for $i = 0, 1, 2$

$$u_{3n+i} = \begin{cases} u_i + (act + ad + b)n, & acs = 1, \\ (acs)^n u_i + \left(\frac{(acs)^n - 1}{acs - 1}\right)(act + ad + b), & otherwise, \end{cases} \tag{3.4.17}$$

$$v_{3n+i} = \begin{cases} v_i + (csb + ct + d)n, & acs = 1, \\ (csa)^n v_i + \left(\frac{(csa)^n - 1}{csa - 1}\right)(csb + ct + d), & otherwise, \end{cases} \tag{3.4.18}$$

and

$$w_{3n+i}=\begin{cases} w_i+(sad+sb+t)n, & acs=1,\\ (sac)^n w_i+\left(\dfrac{(sac)^n-1}{sac-1}\right)(sad+sb+t), & otherwise. \end{cases} \tag{3.4.19}$$

Using, (3.4.1), (3.4.17), (3.4.18) and (3.4.19), it follows that for all $n\in\mathbb{N}_0$

$$u_{3n}=\begin{cases} \dfrac{(f(x_{-2}))^p}{f(x_0)f(x_{-1})}+(act+ad+b)n, & acs=1,\\ \\ \dfrac{(acs)^n(f(x_{-2}))^p}{f(x_0)f(x_{-1})}+\left(\dfrac{(acs)^n-1}{acs-1}\right)(act+ad+b), & otherwise, \end{cases} \tag{3.4.20}$$

$$u_{3n+1}=\begin{cases} \dfrac{a(g(y_{-2}))^q+bg(y_0)g(y_{-1})}{g(y_0)g(y_{-1})}+(act+ad+b)n, & acs=1,\\ \\ \dfrac{(acs)^n(a(g(y_{-2}))^q+bg(y_0)g(y_{-1}))}{g(y_0)g(y_{-1})}+\left(\dfrac{(acs)^n-1}{acs-1}\right)(act+ad+b), & otherwise, \end{cases} \tag{3.4.21}$$

$$u_{3n+2}=\begin{cases} \dfrac{ac(h(z_{-2})^r+adh(z_0)h(z_{-1})+bh(z_0)h(z_{-1})}{h(z_0)h(z_{-1})}+(act+ad+b)n, & acs=1,\\ \dfrac{(acs)^n(ac(h(z_{-2}))^r+adh(z_0)h(z_{-1})+bh(z_0)h(z_{-1}))}{h(z_0)h(z_{-1})}+\left(\dfrac{(acs)^n-1}{acs-1}\right)(act+ad+b), & otherwise, \end{cases} \tag{3.4.22}$$

$$v_{3n}=\begin{cases} \dfrac{(g(y_{-2}))^q}{g(y_0)g(y_{-1})}+(csb+ct+d)n, & acs=1,\\ \\ \dfrac{(aa)^n(g(y_{-2}))^q}{g(y_0)g(y_{-1})}+\left(\dfrac{(acs)^n-1}{acs-1}\right)(csb+ct+d), & otherwise, \end{cases} \tag{3.4.23}$$

$$v_{3n+1}=\begin{cases} \dfrac{c(h(z_{-2})^r+dh(z_0)h(z_{-1})}{h(z_0)h(z_{-1})}+(csb+ct+d)n, & acs=1,\\ \dfrac{(csa)^n(c(h(z_{-2}))^r+dh(z_0)h(z_{-1}))}{h(z_0)h(z_{-1})}+\left(\dfrac{(csa)^n-1}{csa-1}\right)(csb+ct+d), & otherwise, \end{cases} \tag{3.4.24}$$

$$v_{3n+2}=\begin{cases} \dfrac{cs(f(x_{-2}))^p+ctf(x_0)f(x_{-1})+df(x_0)f(x_{-1})}{f(x_0)f(x_{-1})}+(csb+ct+d)n, & acs=1,\\ \dfrac{(csa)^n(cs(f(x_{-2}))^p+ctf(x_0)f(x_{-1})+df(x_0)f(x_{-1}))}{f(x_0)f(x_{-1})}+\left(\dfrac{(csa)^n-1}{csa-1}\right)(csb+ct+d), & otherwise, \end{cases} \tag{3.4.25}$$

$$w_{3n} = \begin{cases} \dfrac{(h(z_{-2}))^r}{h(z_0)h(z_{-1})} + (sad+sb+t)n, & acs = 1, \\ \dfrac{(sac)^n(h(z_{-2}))^r}{h(z_0)h(z_{-1})} + \left(\dfrac{(sac)^n - 1}{sac-1}\right)(sad+sb+t), & otherwise, \end{cases} \tag{3.4.26}$$

$$w_{3n+1} = \begin{cases} \dfrac{s(f(x_{-2}))^p + tf(x_0)f(x_{-1})}{f(x_0)f(x_{-1})} + (sad+sb+t)n, & acs = 1, \\ \dfrac{(sac)^n(s(f(x_{-2}))^p + tf(x_0)f(x_{-1}))}{f(x_0)f(x_{-1})} + \left(\dfrac{(sac)^n - 1}{sac-1}\right)(sad+sb+t), & otherwise, \end{cases} \tag{3.4.27}$$

$$w_{3n+2} = \begin{cases} \dfrac{sa(g(y_{-2}))^q + sbg(y_0)g(y_{-1}) + tg(y_0)g(y_{-1})}{g(y_0)g(y_{-1})} + (sad+sb+t)n, & acs = 1, \\ \dfrac{(sac)^n(sa(g(y_{-2}))^q + sbg(y_0)g(y_{-1}) + tg(y_0)g(y_{-1}))}{g(y_0)g(y_{-1})} + \left(\dfrac{(sac)^n - 1}{sac-1}\right)(sad+sb+t), & otherwise. \end{cases} \tag{3.4.28}$$

Now, using the fact that the functions f, g and h are one to one, it follows from (3.4.1) that

$$x_n = f^{-1}\left(\frac{(f(x_{n-2}))^p}{u_n f(x_{n-1})}\right), \; y_n = g^{-1}\left(\frac{(g(y_{n-2}))^q}{v_n g(y_{n-1})}\right), \; z_n = h^{-1}\left(\frac{(h(z_{n-2}))^r}{w_n h(z_{n-1})}\right), \tag{3.4.29}$$

From which we obtain,

$$x_0 = f^{-1}\left(\frac{(f(x_{-2}))^p}{u_0 f(x_{-1})}\right) = f^{-1}\left(\frac{(f(x_{-2}))^{pF_{p,0}}}{u_0^{F_{p,0}}(f(x_{-1}))^{F_{p,1}}}\right),$$

Moreover,

$$\begin{aligned} x_1 = f^{-1}\left(\frac{(f(x_{-1}))^p}{u_1 f(x_0)}\right) &= f^{-1}\left(\frac{u_0 f(x_{-1})(f(x_{-1}))^p}{u_1(f(x_{-2}))^p}\right) \\ &= f^{-1}\left(\frac{u_0(f(x_{-1}))^{p+1}}{u_1(f(x_{-2}))^p}\right) \\ &= f^{-1}\left(\frac{u_0^{F_{p,1}}(f(x_{-1}))^{F_{p,2}}}{u_1^{F_{p,0}}(f(x_{-2}))^{pF_{p,1}}}\right). \end{aligned}$$

Similarly, we obtain

$$x_2 = f^{-1}\left(\frac{u_1(f(x_{-2}))^{p^2+p}}{u_2 u_0^{p+1}(f(x_{-1}))^{2p+1}}\right) = f^{-1}\left(\frac{u_1^{F_{p,1}}(f(x_{-2}))^{pF_{p,2}}}{u_2^{F_{p,0}}u_0^{F_{p,2}}(f(x_{-1}))^{F_{p,3}}}\right),$$

$$x_3 = f^{-1}\left(\frac{u_2 u_0^{2p+1}(f(x_{-1}))^{p^2+3p+1}}{u_3 u_1^{p+1}(f(x_{-2}))^{2p^2+p}}\right) = f^{-1}\left(\frac{u_2^{F_{p,1}}u_0^{F_{p,3}}(f(x_{-1}))^{F_{p,4}}}{u_3^{F_{p,0}}u_1^{F_{p,2}}(f(x_{-2}))^{pF_{p,3}}}\right),$$

$$x_4 = f^{-1}\left(\frac{u_3 u_1^{2p+1}(f(x_{-2}))^{p^3+3p^2+p}}{u_4 u_2^{p+1} u_0^{p^2+3p+1}(f(x_{-1}))^{3p^2+4p+1}}\right) = f^{-1}\left(\frac{u_3^{F_{p,1}} u_1^{F_{p,3}}(f(x_{-2}))^{pF_{p,4}}}{u_4^{F_{p,0}} u_2^{F_{p,2}} u_0^{F_{p,4}}(f(x_{-1}))^{F_{p,5}}}\right),$$

$$x_5 = f^{-1}\left(\frac{u_4 u_2^{2p+1} u_0^{3p^2+4p+1}(f(x_{-1}))^{p^3+6p^2+5p+1}}{u_5 u_3^{p+1} u_1^{p^2+3p+1}(f(x_{-2}))^{3p^3+4p^2+p}}\right) = f^{-1}\left(\frac{u_4^{F_{p,1}} u_2^{F_{p,3}} u_0^{F_{p,5}}(f(x_{-1}))^{F_{p,6}}}{u_5^{F_{p,0}} u_3^{F_{p,2}} u_1^{F_{p,4}}(f(x_{-2}))^{pF_{p,5}}}\right),$$

$$x_6 = f^{-1}\left(\frac{u_5^{F_{p,1}} u_3^{F_{p,3}} u_1^{F_{p,5}}(f(x_{-2}))^{pF_{p,6}}}{u_6^{F_{p,0}} u_4^{F_{p,2}} u_2^{F_{p,4}} u_0^{F_{p,6}}(f(x_{-1}))^{F_{p,7}}}\right),$$

$$x_7 = f^{-1}\left(\frac{u_6^{F_{p,1}} u_4^{F_{p,3}} u_2^{F_{p,5}} u_0^{F_{p,7}}(f(x_{-1}))^{F_{p,8}}}{u_7^{F_{p,0}} u_5^{F_{p,2}} u_3^{F_{p,4}} u_1^{F_{p,6}}(f(x_{-2}))^{pF_{p,7}}}\right),$$

$$x_8 = f^{-1}\left(\frac{u_7^{F_{p,1}} u_5^{F_{p,3}} u_3^{F_{p,5}} u_1^{F_{p,7}}(f(x_{-2}))^{pF_{p,8}}}{u_8^{F_{p,0}} u_6^{F_{p,2}} u_4^{F_{p,4}} u_2^{F_{p,6}} u_0^{F_{p,8}}(f(x_{-1}))^{F_{p,9}}}\right),$$

$$x_9 = f^{-1}\left(\frac{u_8^{F_{p,1}} u_6^{F_{p,3}} u_4^{F_{p,5}} u_2^{F_{p,7}} u_0^{F_{p,9}}(f(x_{-1}))^{F_{p,10}}}{u_9^{F_{p,0}} u_7^{F_{p,2}} u_5^{F_{p,4}} u_3^{F_{p,6}} u_1^{F_{p,8}}(f(x_{-2}))^{pF_{p,9}}}\right),$$

$$x_{10} = f^{-1}\left(\frac{u_9^{F_{p,1}} u_7^{F_{p,3}} u_5^{F_{p,5}} u_3^{F_{p,7}} u_1^{F_{p,9}}(f(x_{-2}))^{pF_{p,10}}}{u_{10}^{F_{p,0}} u_8^{F_{p,2}} u_6^{F_{p,4}} u_4^{F_{p,6}} u_2^{F_{p,8}} u_0^{F_{p,10}}(f(x_{-1}))^{F_{p,11}}}\right),$$

$$x_{11} = f^{-1}\left(\frac{u_{10}^{F_{p,1}} u_8^{F_{p,3}} u_6^{F_{p,5}} u_4^{F_{p,7}} u_2^{F_{p,9}} u_0^{F_{p,11}}(f(x_{-1}))^{F_{p,12}}}{u_{11}^{F_{p,0}} u_9^{F_{p,2}} u_7^{F_{p,4}} u_5^{F_{p,6}} u_3^{F_{p,8}} u_1^{F_{p,10}}(f(x_{-2}))^{pF_{p,11}}}\right),$$

$$x_{12} = f^{-1}\left(\frac{u_{11}^{F_{p,1}} u_9^{F_{p,3}} u_7^{F_{p,5}} u_5^{F_{p,7}} u_3^{F_{p,9}} u_1^{F_{p,11}}(f(x_{-2}))^{pF_{p,12}}}{u_{12}^{F_{p,0}} u_{10}^{F_{p,2}} u_8^{F_{p,4}} u_6^{F_{p,6}} u_4^{F_{p,8}} u_2^{F_{p,10}} u_0^{F_{p,12}}(f(x_{-1}))^{F_{p,13}}}\right).$$

By induction, it follows that

$$x_{6n} = f^{-1}\left(\frac{\left(\prod_{i=0}^{n-1} u_{3(2i)+1}^{F_{p,6(n-i)-1}} u_{3(2i+1)}^{F_{p,6(n-i)-3}} u_{3(2i+1)+2}^{F_{p,6(n-i)-5}}\right)}{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i)+2}^{F_{p,6(n-i)-2}} u_{3(2i+1)+1}^{F_{p,6(n-i)-4}}\right)} \frac{(f(x_{-2}))^{pF_{p,6n}}}{(f(x_{-1}))^{F_{p,6n+1}}}\right), \tag{3.4.30}$$

$$x_{6n+1} = f^{-1}\left(\frac{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i)+2}^{F_{p,6(n-i)-1}} u_{3(2i+1)+1}^{F_{p,6(n-i)-3}}\right)}{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)}^{F_{p,6(n-i)-2}} u_{3(2i+1)+2}^{F_{p,6(n-i)-4}}\right)} \frac{(f(x_{-1}))^{F_{p,6n+2}}}{(f(x_{-2}))^{pF_{p,6n+1}}}\right), \tag{3.4.31}$$

$$x_{6n+2} = f^{-1}\left(\frac{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)}^{F_{p,6(n-i)-1}} u_{3(2i+1)+2}^{F_{p,6(n-i)-3}}\right)}{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+2}} u_{3(2i)+2}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+1}^{F_{p,6(n-i)-2}}\right)} \frac{(f(x_{-2}))^{pF_{p,6n+2}}}{(f(x_{-1}))^{F_{p,6n+3}}}\right), \tag{3.4.32}$$

$$x_{6n+3} = f^{-1}\left(\frac{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+3}} u_{3(2i)+2}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+1}^{F_{p,6(n-i)-1}}\right)}{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)+2}} u_{3(2i+1)}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+2}^{F_{p,6(n-i)-2}}\right)} \frac{(f(x_{-1}))^{F_{p,6n+4}}}{(f(x_{-2}))^{pF_{p,6n+3}}}\right), \tag{3.4.33}$$

$$x_{6n+4}=f^{-1}\left(\frac{\left(\prod_{i=0}^{n}u_{3(2i)+1}^{F_{p,6(n-i)+3}}u_{3(2i+1)}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1}u_{3(2i+1)+2}^{F_{p,6(n-i)-1}}\right)}{\left(\prod_{i=0}^{n}u_{3(2i)}^{F_{p,6(n-i)+4}}u_{3(2i)+2}^{F_{p,6(n-i)+2}}u_{3(2i+1)+1}^{F_{p,6(n-i)}}\right)}\frac{(f(x_{-2}))^{pF_{p,6n+4}}}{(f(x_{-1}))^{F_{p,6n+5}}}\right),\tag{3.4.34}$$

$$x_{6n+5}=f^{-1}\left(\frac{\left(\prod_{i=0}^{n}u_{3(2i)}^{F_{p,6(n-i)+5}}u_{3(2i)+2}^{F_{p,6(n-i)+3}}u_{3(2i+1)+1}^{F_{p,6(n-i)+1}}\right)}{\left(\prod_{i=0}^{n}u_{3(2i)+1}^{F_{p,6(n-i)+4}}u_{3(2i+1)}^{F_{p,6(n-i)+2}}u_{3(2i+1)+2}^{F_{p,6(n-i)}}\right)}\frac{(f(x_{-1}))^{F_{p,6(n+1)}}}{(f(x_{-2}))^{pF_{p,6n+5}}}\right).\tag{3.4.35}$$

Similarly, by following the same steps we obtain

$$y_{6n}=g^{-1}\left(\frac{\left(\prod_{i=0}^{n-1}v_{3(2i)+1}^{F_{q,6(n-i)-1}}v_{3(2i+1)}^{F_{q,6(n-i)-3}}v_{3(2i+1)+2}^{F_{q,6(n-i)-5}}\right)}{\left(\prod_{i=0}^{n}v_{3(2i)}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1}v_{3(2i)+2}^{F_{q,6(n-i)-2}}v_{3(2i+1)+1}^{F_{q,6(n-i)-4}}\right)}\frac{(g(y_{-2}))^{qF_{q,6n}}}{(g(y_{-1}))^{F_{q,6n+1}}}\right),\tag{3.4.36}$$

$$y_{6n+1}=g^{-1}\left(\frac{\left(\prod_{i=0}^{n}v_{3(2i)}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1}v_{3(2i)+2}^{F_{q,6(n-i)-1}}v_{3(2i+1)+1}^{F_{q,6(n-i)-3}}\right)}{\left(\prod_{i=0}^{n}v_{3(2i)+1}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1}v_{3(2i+1)}^{F_{q,6(n-i)-2}}v_{3(2i+1)+2}^{F_{q,6(n-i)-4}}\right)}\frac{(g(y_{-1}))^{F_{q,6n+2}}}{(g(y_{-2}))^{qF_{q,6n+1}}}\right),\tag{3.4.37}$$

$$y_{6n+2}=g^{-1}\left(\frac{\left(\prod_{i=0}^{n}v_{3(2i)+1}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1}v_{3(2i+1)}^{F_{q,6(n-i)-1}}v_{3(2i+1)+2}^{F_{q,6(n-i)-3}}\right)}{\left(\prod_{i=0}^{n}v_{3(2i)}^{F_{q,6(n-i)+2}}v_{3(2i)+2}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1}v_{3(2i+1)+1}^{F_{q,6(n-i)-2}}\right)}\frac{(g(y_{-2}))^{qF_{q,6n+2}}}{(g(y_{-1}))^{F_{q,6n+3}}}\right),\tag{3.4.38}$$

$$y_{6n+3}=g^{-1}\left(\frac{\left(\prod_{i=0}^{n}v_{3(2i)}^{F_{q,6(n-i)+3}}v_{3(2i)+2}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1}v_{3(2i+1)+1}^{F_{q,6(n-i)-1}}\right)}{\left(\prod_{i=0}^{n}v_{3(2i)+1}^{F_{q,6(n-i)+2}}v_{3(2i+1)}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1}v_{3(2i+1)+2}^{F_{q,6(n-i)-2}}\right)}\frac{(g(y_{-1}))^{F_{q,6n+4}}}{(g(y_{-2}))^{qF_{q,6n+3}}}\right),\tag{3.4.39}$$

$$y_{6n+4}=g^{-1}\left(\frac{\left(\prod_{i=0}^{n}v_{3(2i)+1}^{F_{q,6(n-i)+3}}v_{3(2i+1)}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1}v_{3(2i+1)+2}^{F_{q,6(n-i)-1}}\right)}{\left(\prod_{i=0}^{n}v_{3(2i)}^{F_{q,6(n-i)+4}}v_{3(2i)+2}^{F_{q,6(n-i)+2}}v_{3(2i+1)+1}^{F_{q,6(n-i)}}\right)}\frac{(g(y_{-2}))^{qF_{q,6n+4}}}{(g(y_{-1}))^{F_{q,6n+5}}}\right),\tag{3.4.40}$$

$$y_{6n+5}=g^{-1}\left(\frac{\left(\prod_{i=0}^{n}v_{3(2i)}^{F_{q,6(n-i)+5}}v_{3(2i)+2}^{F_{q,6(n-i)+3}}v_{3(2i+1)+1}^{F_{q,6(n-i)+1}}\right)}{\left(\prod_{i=0}^{n}v_{3(2i)+1}^{F_{q,6(n-i)+4}}v_{3(2i+1)}^{F_{q,6(n-i)+2}}v_{3(2i+1)+2}^{F_{q,6(n-i)}}\right)}\frac{(g(y_{-1}))^{F_{q,6(n+1)}}}{(g(y_{-2}))^{qF_{q,6n+5}}}\right),\tag{3.4.41}$$

and

$$z_{6n}=h^{-1}\left(\frac{\left(\prod_{i=0}^{n-1}w_{3(2i)+1}^{F_{r,6(n-i)-1}}w_{3(2i+1)}^{F_{r,6(n-i)-3}}w_{3(2i+1)+2}^{F_{r,6(n-i)-5}}\right)}{\left(\prod_{i=0}^{n}w_{3(2i)}^{F_{r,6(n-i)}}\right)\left(\prod_{i=0}^{n-1}w_{3(2i)+2}^{F_{r,6(n-i)-2}}w_{3(2i+1)+1}^{F_{r,6(n-i)-4}}\right)}\frac{(h(z_{-2}))^{rF_{r,6n}}}{(h(z_{-1}))^{F_{r,6n+1}}}\right),\tag{3.4.42}$$

$$z_{6n+1}=h^{-1}\left(\frac{\left(\prod_{i=0}^{n}w_{3(2i)}^{F_{r,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1}w_{3(2i)+2}^{F_{r,6(n-i)-1}}w_{3(2i+1)+1}^{F_{r,6(n-i)-3}}\right)}{\left(\prod_{i=0}^{n}w_{3(2i)+1}^{F_{r,6(n-i)}}\right)\left(\prod_{i=0}^{n-1}w_{3(2i+1)}^{F_{r,6(n-i)-2}}w_{3(2i+1)+2}^{F_{r,6(n-i)-4}}\right)}\frac{(h(z_{-1}))^{F_{r,6n+2}}}{(h(z_{-2}))^{rF_{r,6n+1}}}\right),\tag{3.4.43}$$

$$z_{6n+2} = h^{-1}\left(\frac{\left(\prod_{i=0}^{n} w_{3(2i)+1}^{F_{r,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)}^{F_{r,6(n-i)-1}} w_{3(2i+1)+2}^{F_{r,6(n-i)-3}}\right)}{\left(\prod_{i=0}^{n} w_{3(2i)}^{F_{r,6(n-i)+2}} w_{3(2i)+2}^{F_{r,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)+1}^{F_{r,6(n-i)-2}}\right)} \frac{(h(z_{-2}))^{rF_{r,6n+2}}}{(h(z_{-1}))^{F_{r,6n+3}}}\right), \quad (3.4.44)$$

$$z_{6n+3} = h^{-1}\left(\frac{\left(\prod_{i=0}^{n} w_{3(2i)}^{F_{r,6(n-i)+3}} w_{3(2i)+2}^{F_{r,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)+1}^{F_{r,6(n-i)-1}}\right)}{\left(\prod_{i=0}^{n} w_{3(2i)+1}^{F_{r,6(n-i)+2}} w_{3(2i+1)}^{F_{r,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)+2}^{F_{r,6(n-i)-2}}\right)} \frac{(h(z_{-1}))^{F_{r,6n+4}}}{(h(z_{-2}))^{rF_{r,6n+3}}}\right), \quad (3.4.45)$$

$$z_{6n+4} = h^{-1}\left(\frac{\left(\prod_{i=0}^{n} w_{3(2i)+1}^{F_{r,6(n-i)+3}} w_{3(2i+1)}^{F_{r,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} w_{3(2i+1)+2}^{F_{r,6(n-i)-1}}\right)}{\left(\prod_{i=0}^{n} w_{3(2i)}^{F_{r,6(n-i)+4}} w_{3(2i)+2}^{F_{r,6(n-i)+2}} w_{3(2i+1)+1}^{F_{r,6(n-i)}}\right)} \frac{(h(z_{-2}))^{rF_{r,6n+4}}}{(h(z_{-1}))^{F_{r,6n+5}}}\right), \quad (3.4.46)$$

$$z_{6n+5} = h^{-1}\left(\frac{\left(\prod_{i=0}^{n} w_{3(2i)}^{F_{r,6(n-i)+5}} w_{3(2i)+2}^{F_{r,6(n-i)+3}} w_{3(2i+1)+1}^{F_{r,6(n-i)+1}}\right)}{\left(\prod_{i=0}^{n} w_{3(2i)+1}^{F_{r,6(n-i)+4}} w_{3(2i+1)}^{F_{r,6(n-i)+2}} w_{3(2i+1)+2}^{F_{r,6(n-i)}}\right)} \frac{(h(z_{-1}))^{F_{r,6(n+1)}}}{(h(z_{-2}))^{rF_{r,6n+5}}}\right). \quad (3.4.47)$$

In summary we have the following result.

Theorem 3.4.1. *Let $\{x_n, y_n, z_n\}_{n\geq -2}$ be a well-defined solution of System (3.1.5). Then, for all $n \in \mathbb{N}_0$, the x_n-component (resp. the y_n component and the z_n-component) are given by equations (3.4.30)-(3.4.35) for x_n (resp. equations (3.4.36)-(3.4.41) for y_n and equations (3.4.42)-(3.4.47) for the z_n), where the sequences $\{u_n\}_{n\in\mathbb{N}_0}$, $\{v_n\}_{n\in\mathbb{N}_0}$ and $\{w_n\}_{n\in\mathbb{N}_0}$ are defined by the formulas (3.4.7)-(3.4.15) in the case of variables coefficients and by formulas (3.4.20)-(3.4.28) in the case of constant coefficients.*

3.5 Formulas of the solutions of System (3.1.6)

If in System (3.1.5), we let $f(x) = g(x) = h(x) = x$, $D = \mathbb{R} - \{0\}$, then we get the following particular system

$$x_{n+1} = \frac{y_n y_{n-1} x_{n-1}^p}{x_n(a_n y_{n-2}^q + b_n y_n y_{n-1})}, \; y_{n+1} = \frac{z_n z_{n-1} y_{n-1}^q}{y_n(c_n z_{n-2}^r + d_n z_n z_{n-1})}, \; z_{n+1} = \frac{x_n x_{n-1} z_{n-1}^r}{z_n(s_n x_{n-2}^p + t_n x_n x_{n-1})}. \quad (3.5.1)$$

Clearly, we have

$$f^{-1}(x) = g^{-1}(x) = h^{-1}(x) = x.$$

In this case a solution $\{x_n, y_n, z_n\}_{n\geq -2}$ of System (3.5.1) is said to be well-defined if for all $n \in \mathbb{N}_0$, we have

$$x_n y_n z_n \left(a_n y_{n-2}^q + b_n y_n y_{n-1}\right)\left(c_n z_{n-2}^r + d_n z_n z_{n-1}\right)\left(s_n x_{n-2}^p + t_n x_n x_{n-1}\right) \neq 0.$$

As a result of Theorem (3.4.1), we have

Corollary 3.5.1. *Let* $\{x_n, y_n, z_n\}_{n\geq -2}$ *be a well-defined solution of System (3.5.1). Then, for* $n = 0,\ 1, \ldots,$ *we have*

$$x_{6n} = \frac{\left(\prod_{i=0}^{n-1} u_{3(2i)+1}^{F_{p,6(n-i)-1}} u_{3(2i+1)}^{F_{p,6(n-i)-3}} u_{3(2i+1)+2}^{F_{p,6(n-i)-5}}\right)}{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i)+2}^{F_{p,6(n-i)-2}} u_{3(2i+1)+1}^{F_{p,6(n-i)-4}}\right)} \frac{x_{-2}^{pF_{p,6n}}}{x_{-1}^{F_{p,6n+1}}},$$

$$x_{6n+1} = \frac{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i)+2}^{F_{p,6(n-i)-1}} u_{3(2i+1)+1}^{F_{p,6(n-i)-3}}\right)}{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)}^{F_{p,6(n-i)-2}} u_{3(2i+1)+2}^{F_{p,6(n-i)-4}}\right)} \frac{x_{-1}^{F_{p,6n+2}}}{x_{-2}^{pF_{p,6n+1}}},$$

$$x_{6n+2} = \frac{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)}^{F_{p,6(n-i)-1}} u_{3(2i+1)+2}^{F_{p,6(n-i)-3}}\right)}{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+2}} u_{3(2i)+2}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+1}^{F_{p,6(n-i)-2}}\right)} \frac{x_{-2}^{pF_{p,6n+2}}}{x_{-1}^{F_{p,6n+3}}},$$

$$x_{6n+3} = \frac{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+3}} u_{3(2i)+2}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+1}^{F_{p,6(n-i)-1}}\right)}{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)+2}} u_{3(2i+1)}^{F_{p,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+2}^{F_{p,6(n-i)-2}}\right)} \frac{x_{-1}^{F_{p,6n+4}}}{x_{-2}^{pF_{p,6n+3}}},$$

$$x_{6n+4} = \frac{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)+3}} u_{3(2i+1)}^{F_{p,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} u_{3(2i+1)+2}^{F_{p,6(n-i)-1}}\right)}{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+4}} u_{3(2i)+2}^{F_{p,6(n-i)+2}} u_{3(2i+1)+1}^{F_{p,6(n-i)}}\right)} \frac{x_{-2}^{pF_{p,6n+4}}}{x_{-1}^{F_{p,6n+5}}},$$

$$x_{6n+5} = \frac{\left(\prod_{i=0}^{n} u_{3(2i)}^{F_{p,6(n-i)+5}} u_{3(2i)+2}^{F_{p,6(n-i)+3}} u_{3(2i+1)+1}^{F_{p,6(n-i)+1}}\right)}{\left(\prod_{i=0}^{n} u_{3(2i)+1}^{F_{p,6(n-i)+4}} u_{3(2i+1)}^{F_{p,6(n-i)+2}} u_{3(2i+1)+2}^{F_{p,6(n-i)}}\right)} \frac{x_{-1}^{F_{p,6(n+1)}}}{x_{-2}^{pF_{p,6n+5}}},$$

$$y_{6n} = \frac{\left(\prod_{i=0}^{n-1} v_{3(2i)+1}^{F_{q,6(n-i)-1}} v_{3(2i+1)}^{F_{q,6(n-i)-3}} v_{3(2i+1)+2}^{F_{q,6(n-i)-5}}\right)}{\left(\prod_{i=0}^{n} v_{3(2i)}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i)+2}^{F_{q,6(n-i)-2}} v_{3(2i+1)+1}^{F_{q,6(n-i)-4}}\right)} \frac{y_{-2}^{qF_{q,6n}}}{y_{-1}^{F_{q,6n+1}}},$$

$$y_{6n+1} = \frac{\left(\prod_{i=0}^{n} v_{3(2i)}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i)+2}^{F_{q,6(n-i)-1}} v_{3(2i+1)+1}^{F_{q,6(n-i)-3}}\right)}{\left(\prod_{i=0}^{n} v_{3(2i)+1}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)}^{F_{q,6(n-i)-2}} v_{3(2i+1)+2}^{F_{q,6(n-i)-4}}\right)} \frac{y_{-1}^{F_{q,6n+2}}}{y_{-2}^{qF_{q,6n+1}}},$$

$$y_{6n+2} = \frac{\left(\prod_{i=0}^{n} v_{3(2i)+1}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)}^{F_{q,6(n-i)-1}} v_{3(2i+1)+2}^{F_{q,6(n-i)-3}}\right)}{\left(\prod_{i=0}^{n} v_{3(2i)}^{F_{q,6(n-i)+2}} v_{3(2i)+2}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)+1}^{F_{q,6(n-i)-2}}\right)} \frac{y_{-2}^{qF_{q,6n+2}}}{y_{-1}^{F_{q,6n+3}}},$$

$$y_{6n+3} = \frac{\left(\prod_{i=0}^{n} v_{3(2i)}^{F_{q,6(n-i)+3}} v_{3(2i)+2}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)+1}^{F_{q,6(n-i)-1}}\right)}{\left(\prod_{i=0}^{n} v_{3(2i)+1}^{F_{q,6(n-i)+2}} v_{3(2i+1)}^{F_{q,6(n-i)}}\right)\left(\prod_{i=0}^{n-1} v_{3(2i+1)+2}^{F_{q,6(n-i)-2}}\right)} \frac{y_{-1}^{F_{q,6n+4}}}{y_{-2}^{qF_{q,6n+3}}},$$

$$y_{6n+4}=\frac{\left(\prod_{i=0}^{n}v_{3(2i)+1}^{F_{q,6(n-i)+3}}v_{3(2i+1)}^{F_{q,6(n-i)+1}}\right)\left(\prod_{i=0}^{n-1}v_{3(2i+1)+2}^{F_{q,6(n-i)-1}}\right)}{\left(\prod_{i=0}^{n}v_{3(2i)}^{F_{q,6(n-i)+4}}v_{3(2i)+2}^{F_{q,6(n-i)+2}}v_{3(2i+1)+1}^{F_{q,6(n-i)}}\right)}\frac{y_{-2}^{qF_{q,6n+4}}}{y_{-1}^{F_{q,6n+5}}},$$

$$y_{6n+5}=\frac{\left(\prod_{i=0}^{n}v_{3(2i)}^{F_{q,6(n-i)+5}}v_{3(2i)+2}^{F_{q,6(n-i)+3}}v_{3(2i+1)+1}^{F_{q,6(n-i)+1}}\right)}{\left(\prod_{i=0}^{n}v_{3(2i)+1}^{F_{q,6(n-i)+4}}v_{3(2i+1)}^{F_{q,6(n-i)+2}}v_{3(2i+1)+2}^{F_{q,6(n-i)}}\right)}\frac{y_{-1}^{F_{q,6(n+1)}}}{y_{-2}^{qF_{q,6n+5}}},$$

$$z_{6n}=\frac{\left(\prod_{i=0}^{n-1}w_{3(2i)+1}^{F_{r,6(n-i)-1}}w_{3(2i+1)}^{F_{r,6(n-i)-3}}w_{3(2i+1)+2}^{F_{r,6(n-i)-5}}\right)}{\left(\prod_{i=0}^{n}w_{3(2i)}^{F_{r,6(n-i)}}\right)\left(\prod_{i=0}^{n-1}w_{3(2i)+2}^{F_{r,6(n-i)-2}}w_{3(2i+1)+1}^{F_{r,6(n-i)-4}}\right)}\frac{z_{-2}^{rF_{r,6n}}}{z_{-1}^{F_{r,6n+1}}},$$

$$z_{6n+1}=\frac{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i)+2}^{F_{r,6(n-i)-1}}w_{3(2i+1)+1}^{F_{r,6(n-i)-3}}\right)}{\prod_{i=0}^{n}\left(w_{3(2i)+1}^{F_{r,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)}^{F_{r,6(n-i)-2}}w_{3(2i+1)+2}^{F_{r,6(n-i)-4}}\right)}\frac{z_{-1}^{F_{r,6n+2}}}{z_{-2}^{rF_{r,6n+1}}},$$

$$z_{6n+2}=\frac{\prod_{i=0}^{n}\left(w_{3(2i)+1}^{F_{r,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)}^{F_{r,6(n-i)-1}}w_{3(2i+1)+2}^{F_{r,6(n-i)-3}}\right)}{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)+2}}w_{3(2i)+2}^{F_{r,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)+1}^{F_{r,6(n-i)-2}}\right)}\frac{z_{-2}^{rF_{r,6n+2}}}{z_{-1}^{F_{r,6n+3}}},$$

$$z_{6n+3}=\frac{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)+3}}w_{3(2i)+2}^{F_{r,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)+1}^{F_{r,6(n-i)-1}}\right)}{\prod_{i=0}^{n}\left(w_{3(2i)+1}^{F_{r,6(n-i)+2}}w_{3(2i+1)}^{F_{r,6(n-i)}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)+2}^{F_{r,6(n-i)-2}}\right)}\frac{z_{-1}^{F_{r,6n+4}}}{z_{-2}^{rF_{r,6n+3}}},$$

$$z_{6n+4}=\frac{\prod_{i=0}^{n}\left(w_{3(2i)+1}^{F_{r,6(n-i)+3}}w_{3(2i+1)}^{F_{r,6(n-i)+1}}\right)\prod_{i=0}^{n-1}\left(w_{3(2i+1)+2}^{F_{r,6(n-i)-1}}\right)}{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)+4}}w_{3(2i)+2}^{F_{r,6(n-i)+2}}w_{3(2i+1)+1}^{F_{r,6(n-i)}}\right)}\frac{z_{-2}^{rF_{r,6n+4}}}{z_{-1}^{F_{r,6n+5}}},$$

$$z_{6n+5}=\frac{\prod_{i=0}^{n}\left(w_{3(2i)}^{F_{r,6(n-i)+5}}w_{3(2i)+2}^{F_{r,6(n-i)+3}}w_{3(2i+1)+1}^{F_{r,6(n-i)+1}}\right)}{\prod_{i=0}^{n}\left(w_{3(2i)+1}^{F_{r,6(n-i)+4}}w_{3(2i+1)}^{F_{r,6(n-i)+2}}w_{3(2i+1)+2}^{F_{r,6(n-i)}}\right)}\frac{z_{-1}^{F_{r,6(n+1)}}}{z_{-2}^{rF_{r,6n+5}}},$$

where the sequences $\{u_n\}_{n\in\mathbb{N}_0}$*,* $\{v_n\}_{n\in\mathbb{N}_0}$ *and* $\{v_n\}_{n\in\mathbb{N}_0}$ *are defined by the formulas*

$$u_{3n}=\frac{\left[\prod_{j=0}^{n-1}a_{3j+2}c_{3j+1}s_{3j}\right]x_{-2}^{p}}{x_0x_{-1}}+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}a_{3j+2}c_{3j+1}s_{3j}\right](a_{3r+2}c_{3r+1}t_{3r}+a_{3r+2}d_{3r+1}+b_{3r+2})$$

$$u_{3n+1}=\frac{\left[\prod_{j=0}^{n-1}a_{3(j+1)}c_{3j+2}s_{3j+1}\right][a_0y_{-2}^{q}+b_0y_0y_{-1}]}{y_0y_{-1}}+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}a_{3(j+1)}c_{3j+2}s_{3j+1}\right]\left(a_{3(r+1)}c_{3r+2}t_{3r+1}+a_{3(r+1)}d_{3r+2}+b_{3(r+1)}\right),$$

$$u_{3n+2}=\frac{\left[\prod_{j=0}^{n-1}a_{3j+4}c_{3(j+1)}s_{3j+2}\right][a_1c_0z^r_{-2}+a_1d_0z_0z_{-1}+b_1z_0z_{-1}]}{z_0z_{-1}}$$
$$+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}a_{3j+4}c_{3(j+1)}s_{3j+2}\right]\left(a_{3r+4}c_{3(r+1)}t_{3r+2}+a_{3r+4}d_{3(r+1)}+b_{3r+4}\right),$$

$$v_{3n}=\frac{\left[\prod_{j=0}^{n-1}c_{3j+2}s_{3j+1}a_{3j}\right]y^q_{-2}}{y_0y_{-1}}+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}c_{3j+2}s_{3j+1}a_{3j}\right](c_{3r+2}s_{3r+1}b_{3r}+c_{3r+2}t_{3r+1}+d_{3r+2}),$$

$$v_{3n+1}=\frac{\left[\prod_{j=0}^{n-1}c_{3(j+1)}s_{3j+2}a_{3j+1}\right][c_0z^r_{-2}+d_0z_0z_{-1}]}{z_0z_{-1}}$$
$$+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}c_{3(j+1)}s_{3j+2}a_{3j+1}\right]\left(c_{3(r+1)}s_{3r+2}b_{3r+1}+c_{3(r+1)}t_{3r+2}+d_{3(r+1)}\right),$$

$$v_{3n+2}=\frac{\left[\prod_{j=0}^{n-1}c_{3j+4}s_{3(j+1)}a_{3j+2}\right][c_1s_0x^p_{-2}+c_1t_0x_0x_{-1}+d_1x_0x_{-1}]}{x_0x_{-1}}$$
$$+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}c_{3j+4}s_{3(j+1)}a_{3j+2}\right]\left(c_{3r+4}s_{3(r+1)}b_{3r+2}+c_{3r+4}t_{3(r+1)}+d_{3r+4}\right)$$

and

$$w_{3n}=\frac{\left[\prod_{j=0}^{n-1}s_{3j+2}a_{3j+1}c_{3j}\right]z^r_{-2}}{z_0z_{-1}}+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}s_{3j+2}a_{3j+1}c_{3j}\right](s_{3r+2}a_{3r+1}d_{3r}+s_{3r+2}b_{3r+1}+t_{3r+2}),$$

$$w_{3n+1}=\frac{\left[\prod_{j=0}^{n-1}s_{3(j+1)}a_{3j+2}c_{3j+1}\right][s_0x^p_{-2}+t_0x_0x_{-1}]}{x_0x_{-1}}$$
$$+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}s_{3(j+1)}a_{3j+2}c_{3j+1}\right]\left(s_{3(r+1)}a_{3r+2}d_{3r+1}+s_{3(r+1)}b_{3r+2}+t_{3(r+1)}\right),$$

$$w_{3n+2}=\frac{\left[\prod_{j=0}^{n-1}s_{3j+4}a_{3(j+1)}c_{3j+2}\right][say^q_{-2}+sby_0y_{-1}+ty_0y_{-1}]}{y_0y_{-1}}$$
$$+\sum_{r=0}^{n-1}\left[\prod_{j=r+1}^{n-1}s_{3j+4}a_{3(j+1)}c_{3j+2}\right]\left(s_{3r+4}a_{3(r+1)}d_{3r+2}+s_{3r+4}b_{3(r+1)}+t_{3r+4}\right),$$

in the case of variables coefficients, and formulas

$$u_{3n} = \begin{cases} \dfrac{x^p_{-2}}{x_0x_{-1}} + (act+ad+b)n, & acs=1, \\ \\ \dfrac{(acs)^n x^p_{-2}}{x_0x_{-1}} + \left(\dfrac{(acs)^n-1}{acs-1}\right)(act+ad+b), & \textit{otherwise}, \end{cases}$$

$$u_{3n+1} = \begin{cases} \dfrac{ay^q_{-2}+by_0y_{-1}}{y_0y_{-1}} + (act+ad+b)n, & acs=1, \\ \\ \dfrac{(acs)^n(ay^q_{-2}+by_0y_{-1})}{y_0y_{-1}} + \left(\dfrac{(acs)^n-1}{acs-1}\right)(act+ad+b), & \textit{otherwise}, \end{cases}$$

$$u_{3n+2} = \begin{cases} \dfrac{acz^r_{-2}+adz_0z_{-1}+bz_0z_{-1}}{z_0z_{-1}} + (act+ad+b)n, & acs=1, \\ \\ (acs)^n\dfrac{acz^r_{-2}+adz_0z_{-1}+bz_0z_{-1}}{z_0z_{-1}} + \left(\dfrac{(acs)^n-1}{acs-1}\right)(act+ad+b), & \textit{otherwise}, \end{cases}$$

$$v_{3n} = \begin{cases} \dfrac{y^q_{-2}}{y_0y_{-1}} + (csb+ct+d)n, & acs=1, \\ \\ \dfrac{(aa)^n y^q_{-2}}{y_0y_{-1}} + \left(\dfrac{(acs)^n-1}{acs-1}\right)(csb+ct+d), & \textit{otherwise}, \end{cases}$$

$$v_{3n+1} = \begin{cases} \dfrac{cz^r_{-2}+dz_0z_{-1}}{z_0z_{-1}} + (csb+ct+d)n, & acs=1, \\ \\ (csa)^n\dfrac{cz^r_{-2}+dz_0z_{-1}}{z_0z_{-1}} + \left(\dfrac{(csa)^n-1}{csa-1}\right)(csb+ct+d), & \textit{otherwise}, \end{cases}$$

$$v_{3n+2} = \begin{cases} \dfrac{csx^p_{-2}+ctx_0x_{-1}+dx_0x_{-1}}{x_0x_{-1}} + (csb+ct+d)n, & acs=1, \\ \\ (csa)^n\dfrac{csx^p_{-2}+ctx_0x_{-1}+dx_0x_{-1}}{x_0x_{-1}} + \left(\dfrac{(csa)^n-1}{csa-1}\right)(csb+ct+d), & \textit{otherwise}, \end{cases}$$

and

$$w_{3n} = \begin{cases} \dfrac{z^r_{-2}}{z_0z_{-1}} + (sad+sb+t)n, & acs=1, \\ \\ (sac)^n\dfrac{z^r_{-2}}{z_0z_{-1}} + \left(\dfrac{(sac)^n-1}{sac-1}\right)(sad+sb+t), & \textit{otherwise}, \end{cases}$$

$$w_{3n+1} = \begin{cases} \dfrac{sx^p_{-2} + tx_0x_{-1}}{x_0x_{-1}} + (sad + sb + t)n, & acs = 1, \\ \\ (sac)^n \dfrac{sx^p_{-2} + tx_0x_{-1}}{x_0x_{-1}} + \left(\dfrac{(sac)^n - 1}{sac - 1}\right)(sad + sb + t), & otherwise, \end{cases}$$

$$w_{3n+2} = \begin{cases} \dfrac{say^q_{-2} + sby_0y_{-1} + ty_0y_{-1}}{y_0y_{-1}} + (sad + sb + t)n, & acs = 1, \\ \\ (sac)^n \dfrac{say^q_{-2} + sby_0y_{-1} + ty_0y_{-1}}{y_0y_{-1}} + \left(\dfrac{(sac)^n - 1}{sac - 1}\right)(sad + sb + t), & otherwise, \end{cases}$$

in the case of constant coefficients.

CONCLUSION AND PERSPECTIVES

In this thesis, we have successfully studied the behavior of the solutions of some systems of difference equations. Results on the stability of the equilibrium points via some convergence theorems, existence (or non-existence) of periodic and oscillatory solutions were been the object of two systems, one was a non-autonomous and the other was defined by homogeneous functions. Others nonlinear systems with powers where some of them are defined with one to one functions were been explicitly solved in the last chapter.

As perspectives, we suggest to investigate the behavior of the following systems of difference equations, which are respectively generalizations of System(1.1.1) , System (2.1.4) and System (3.1.4):

The first system

$$x_{n+1} = \frac{p_n + y_n^a}{p_n + y_{n-k}^a}, \quad y_{n+1} = \frac{q_n + z_n^b}{q_n + z_{n-k}^b}, \quad z_{n+1} = \frac{r_n + x_n^c}{r_n + x_{n-k}^c}, \quad n =\in \mathbb{N}_0, a, b, c = 2, 3, \dots,$$

where the initial values and $\{p_n\}, \{q_n\}, \{r_n\}$ are 3-periodic sequences of positive numbers.

The second syetem

$$x_{n+1} = f^{-1}\left(\frac{g(y_n)g(y_{n-k+1})(f(x_{n-k+1}))^p}{f(x_n)\left[a_n(g(y_{n-k}))^q + b_n g(y_n)g(y_{n-k+1})\right]}\right),$$
$$y_{n+1} = g^{-1}\left(\frac{f(x_n)f(x_{n-k+1})(g(y_{n-k+1}))^q}{g(y_n)\left[c_n(f(x_{n-k}))^p + d_n f(x_n)f(x_{n-k+1})\right]}\right), n \in \mathbb{N}_0, p, q \in \mathbb{N}, k = 3, 4, \dots,$$

where $f, g : D \to \mathbb{R}$ are one to one continuous functions on $D \subseteq \mathbb{R}$, the initial values $x_{-i}, y_{-i}, i = 0, 1, 2,$ are real numbers in D and the parameters $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0}, (d_n)_{n\in\mathbb{N}_0}$ are non-zero real numbers.

The third system

$$x_{n+1} = f(y_n, y_{n-1}, \dots, y_{n-k}), \quad y_{n+1} = g(x_n, x_{n-1}, \dots, x_{n-k}), \quad n \in \mathbb{N}_0,$$

where the initial values x_{-i} and y_{-i}, $i = -k, \ldots, -1, 0$, are positive real numbers, the function $f : (0, \infty)^{k+1} \to (0, \infty)$ is continuous and homogeneous of degree zero and $g : (0, \infty)^{k+1} \to (0, \infty)$ is continuous and homogeneous of degree $s \in \mathbb{R}$.

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