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Solution of the time-dependent coupled oscillators and coherent states for the inverted oscillator

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DEDICATION

I humbly dedicate this work to my parents, my husband, my sister, my brothers: Karim, Abd ennour and Yasser, my grandmother, my cousin: Soumia, my friends: Oumayma , Amina and family members and loved ones.

Thank you.

Contents

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Introduction

Quantum mechanics is a fundamental theory of physics that studies matter on the scale of atomic and subatomic particles and the motion of physical objects in the microscopic world. Since the classical mechanics reached its limits in the 1920s.

Quantum theory has made its first appearance resolving physical problems such as black body radiation and the photoelectric effect.

This theory was based on the fundamental principles called postulates [1]

Postulate 1: **State of a system**

The state of any physical system is specified, at each time t, by a state vector $|\psi(t)\rangle$ in a Hilbert space $\mathcal{H}$; $|\psi(t)\rangle$ contains all the information about the quantum state of the system. Any superposition of state vectors is also a state vector.

Postulate 2: Observables and operators

An observable or dynamical variable in quantum mechanics is a physical quantity A, that can be measured or observed, it is equivalent a linear Hermitian operator A, whose eigenvectors form a complete basis.

Postulate 3: Measurements and eigenvalues of operators

The measurement of an observable A may be formally expressed by the action of A on a state vector $|\psi(t)\rangle$. The only possible result of the outcome of the measurement will be one of the eigenvalues a_n (which are real) of the operator A . If the result of a measurement of A on a state $|\psi(t)\rangle$ is a_n.

The state of the system change in a specific way depending on the measurement $|\psi_n\rangle$:

$$A\,|\psi(t)\rangle = a_n\,|\psi_n\rangle\,,\quad n = 1, 2, 3... \tag{1}$$

where $a_n = \langle\psi_n\,|\psi(t)\rangle$.

Postulate 4: Probabilistic outcome of measurements

• **Discrete spectra**: The possibility of finding one of the nondegenerate eigenvalues a_n of the corresponding operator A, when measuring an observable A of a system in a state $|\psi\rangle$ is given by

$$P_n\,(a_n) = \frac{|\langle\psi_n\,|\psi\rangle|^2}{\langle\psi\,|\psi\rangle} = \frac{|a_n|^2}{\langle\psi\,|\psi\rangle}, \tag{2}$$

where $|\psi_n\rangle$ is the eigenstate of A with eigenvalue a_n. If the eigenvalue a_n is m-degenerate, P_n becomes

$$P_n(a_n) = \sum_{j=1}^{m} \frac{\left|\langle \psi_n^j | \psi \rangle\right|^2}{\langle \psi | \psi \rangle} = \sum_{j=1}^{m} \frac{\left|a_n^{(j)}\right|^2}{\langle \psi | \psi \rangle}. \tag{3}$$

The act of measurement changes the state of the system from $|\psi\rangle$ to $|\psi_n\rangle$. If the system is already in an eigenstate $|\psi_n\rangle$ of A, a measurement of A guarantees to provide the corresponding eigenvalue a_n:

$$A|\psi_n\rangle = a_n |\psi_n\rangle, \quad n = 1, 2, 3..., \tag{4}$$

- **Continuous spectra:** The relation 2, which is valid for discrete spectra, it is possible to extend to get the probability density that a measurement of A yields a value between a and $a + da$ on a system which is initially in a state $|\psi\rangle$:

$$\frac{dP(a)}{da} = \frac{|\psi(a)|^2}{\langle \psi | \psi \rangle} = \frac{|\psi(a)|^2}{\int_{-\infty}^{+\infty} |\psi(a')|^2 \, da'}, \tag{5}$$

for instance, the probability density for finding a particle between x and $x + dx$ is given by

$$\frac{dP(x)}{dx} = \frac{|\psi(x)|^2}{\langle \psi | \psi \rangle}. \tag{6}$$

Postulate 5: Time evolution of a system

The time evolution of the state vector $|\psi(t)\rangle$ of a system is governed by the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle, \tag{7}$$

where H is the Hamiltonian operator corresponding to the total energy of the system.

The quantum mechanics theory has been truly one of the most revolutionary theories in physics in the last century. Its impact has extended beyond the original ideas presented, generating new and exciting domains such as quantum electrodynamics, quantum computation, quantum information theory, quantum optics, the theory of quantum open systems and others. While it is a very efficient model that has greatly facilitated new innovations in science, it certainly didn't lack limitations. For instance, it has been limited to the study of self-adjoint operators (in the sense of Dirac), i.e. Hermitian quantum systems. This constraint was challenged near the end of the twentieth century, when non-Hermitian quantum mechanics theory was developed to study non-Hermitian systems with a real spectrum. All while still fulfilling the criteria (satisfying all the postulates except 4, which has been adjusted since then for this theory) of a physically acceptable quantum theory, which are: an energy spectrum (eigenvalues) that is entirely real and bounded from below, an eigenstates Hilbert space with an orthonormal basis, and a unitary temporal evolution.

In 1998, Bender and his collaborators introduced the notion of $\mathcal{PT}$-symmetry [2, 3, 4, 5, 6, 7, 8] in non-Hermitian quantum mechanics. The initial results obtained in their study are that a family of non-Hermitian potentials $\vartheta(x) = x^2 (ix)^{\vartheta}$, $\vartheta \in R$ has a spectrum that is real for $\vartheta \geq 0$ and complex for $\vartheta \leq 0$. Later, Mostafazadeh has generalized the $\mathcal{PT}$-symmetry to the notion of the pseudo-Hermiticity [9, 10, 11], which proved that all $\mathcal{PT}$-symmetric Hamiltonians are pseudo-Hermitian.

Numerous physical systems, particularly those involving dissipative quantum systems have generally been described by non-Hermitian Hamiltonians. These non-Hermitian Hamiltonians are utilized to model phenomena where energy is not conserved such as Hamiltonians that do not satisfy the unitarity condition can indeed pose challenges to the usual probabilistic interpretation of quantum mechanics. In non-Hermitian quantum mechanics. It has been discovered that a quantum Hamiltonian must have an unbroken $\mathcal{PT}$-symmetry in order to have a real spectrum [2, 5]. The concept of $\mathcal{PT}$-symmetry has found applications in various areas of physics. Once the non-Hermitian Hamiltonian H is invariant under the combined action of $\mathcal{PT}$ (i.e. H commutes with $\mathcal{PT}$) and its eigenvectors are also those of the $\mathcal{PT}$ operator, then the energy eigenvalues E of the system are real and in this case the $\mathcal{PT}$-symmetry is unbroken.

The concept of pseudo-Hermiticity introduced by Mostafazadah [9, 10, 11], also known as "quasi-Hermiticity" or "$\mathcal{PT}$-symmetry", plays a significant role in understanding the spectral properties of non-Hermitian Hamiltonians in quantum mechanics. Pseudo-Hermitian operators are non-Hermitian operators that have a certain symmetry property related to their eigenvalues and eigenstates. An operator H is said to be pseudo-Hermitian if

$$H^{+} = \eta H \eta^{-1}, \tag{8}$$

where the metric operator

$$\eta = \rho^{+}\rho,\ \eta^{-1} = \rho^{-1}\left(\rho^{+}\right)^{-1}, \tag{9}$$

is a linear, invertible and Hermitian operator, the Hamiltonian operator is considered pseudo-Hermitian, if it satisfies relation (8).

The pseudo-Hermiticity allows us to establish a connection between a pseudo-Hermitian Hamiltonian H with an equivalent Hermitian Hamiltonian h

$$h = \rho H \rho^{-1}, \tag{10}$$

where the operator ρ called Dyson operator is linear and invertible. We can also studies the time-dependent quantum mechanics systems in both cases (Hermitian and non-Hermitian).

Therefore, in the first chapter, we introduce the Lewis-Riesenfeld theory in Hermitian quantum mechanics. As illustrative example, we study the generalized harmonic oscillator.

In the second chapter, we study the time-dependent coupled oscillator of a two dimensional (2D) by employing the idea of uncoupling the invariant operator to determine the solution of the time-dependent Schrödinger equation.

In the third chapter, we recall the properties of $\mathcal{PT}$ and $\mathcal{CPT}$-symmetry as well as the pseudo-hermiticity.

In the fourth chapter, we investigate the coherent states of the inverted oscillator, which in anti-$\mathcal{PT}$-symmetric Hamiltonian. Finally, we close this work with a conclusion that explains the most important ideas we relied on.

Chapter 1

Time-dependent quantum systems

1.1 Introduction

In quantum mechanics, it is necessary to search for solutions of the time-dependent Schrödinger equation [12]

$$i\hbar\partial_t |\Psi(t)\rangle = h(t) |\Psi(t)\rangle , \tag{1.1}$$

where $h(t)$ is the time dependent hermitian Hamiltonian operator describing the system and $|\Psi(t)\rangle$ is the quantum state of the system.

In this situation it is difficult to find an exact solution, it is crucial to recourse to the approximations methods (for example : the sudden approximation, the adiabatic approximation, the time-dependent perturbation theory and the Lewis-Riesenfeld). The Lewis-Riesenfeld invariant theory [13] allows us to solve the Schrödinger equation in an exact manner and the solution is expressed as a function of the eigenstates of the invariant operator multiplied by a phase.

1.2 The invariant operator in quantum mechanics

An operator $I_h(t)$ is said to be an invariant operator if it satisfies the Von-Neumann equation

$$\frac{dI_h(t)}{dt} = \frac{\partial I_h(t)}{\partial t} + \frac{1}{i\hbar} [I_h(t), h(t)] = 0. \tag{1.2}$$

The time derivative of the expectation value $\langle I_h(t)\rangle$ is writen as

$$i\hbar\frac{d}{dt}\langle I_h(t)\rangle = i\hbar\left(\frac{d}{dt}\langle\Psi(t)|\right)I_h(t)|\Psi(t)\rangle + i\hbar\langle\Psi(t)|\frac{dI_h(t)}{dt}|\Psi(t)\rangle + i\hbar\langle\Psi(t)|I_h(t)\left(\frac{d}{dt}|\Psi(t)\rangle\right), \tag{1.3}$$

the Schrödinger equation and its adjoint are

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = h|\Psi(t)\rangle, \tag{1.4}$$

$$-i\hbar\frac{\partial}{\partial t}\langle\Psi(t)| = \langle\Psi(t)|h^+, \tag{1.5}$$

from the above equations, we will have the following expression (since $h^+ = h$)

$$\frac{d}{dt}\langle I_h(t)\rangle = \frac{\langle\Psi(t)|\,\partial I_h(t)\,|\Psi(t)\rangle}{\partial t} + \frac{1}{i\hbar}\langle\Psi(t)|\left[I_h(t), H(t)\right]|\Psi(t)\rangle, \tag{1.6}$$

the equation (1.2) allows us to deduce that

$$\frac{d}{dt}\langle I_h(t)\rangle = 0, \tag{1.7}$$

i.e., $I_h(t)$ is a constant of movement.

1.3 The invariants theory in quantum mechanics

We shall use the Lewis-Riesenfeld method [13, 14] in order to obtain the quantum solutions for the time-dependent case. To proceed, it is necessary to find an invariant operator $I_h(t)$ satisfying (1.2).

Clearly, this is equivalent to saying that, if $\left|\psi_{\lambda,\kappa}(t)\right\rangle$ is an eigenfunction of $I_h(t)$ with a time-independent eigenvalue λ,

$$I_h(t)\left|\psi_{\lambda,\kappa}(t)\right\rangle = \lambda\left|\psi_{\lambda,\kappa}(t)\right\rangle, \tag{1.8}$$

we can find a solution of the Schrödinger equation in the following form

$$\left|\psi_{\lambda,\kappa}(t)\right\rangle_\alpha = \exp\left[i\zeta_{\lambda,\kappa}(t)\right]\left|\psi_{\lambda,\kappa}(t)\right\rangle, \tag{1.9}$$

where $\zeta_{\lambda,\kappa}(t)$ is the Lewis-Riesenfeld phase satisfies the following equation

$$\hbar\frac{d\zeta_{\lambda,\kappa}(t)}{dt} = \left\langle\psi_{\lambda,\kappa}(t)\right|\left(i\hbar\partial_t - h(t)\right)\left|\psi_{\lambda,\kappa}(t)\right\rangle. \tag{1.10}$$

The general solution of the Schrödinger equation is written as

$$|\Psi(t)\rangle = \sum_{\lambda,\kappa} C_{\lambda,\kappa}(0) \exp\left[i\zeta_{\lambda,\kappa}(t)\right] \left|\psi_{\lambda,\kappa}(t)\right\rangle, \tag{1.11}$$

where the $C_{\lambda,\kappa}(0)$ are time-independent.

1.4 Application: The generalized harmonic oscillator

We propose to solve the Schrödinger equation associated with the time-dependent generalized harmonic oscillator

$$h(t) = \frac{1}{2}\left[Z(t)\, p^2 + Y(t)\,(px + xp) + X(t)\, x^2\right], \tag{1.12}$$

where x and p are the canonical coordinates operators, $X(t), Y(t)$ and $Z(t)$ are arbitrary time-dependent functions. It is well known that an invariant $I_h(t)$, for Eq. (1.12) reads [13, 15, 16]

$$I_h(t) = \frac{\mu_1(t)}{2} p^2 + \frac{\mu_2(t)}{2}(xp + px) + \frac{\mu_3(t)}{2} x^2, \tag{1.13}$$

while the parameters $\mu_1(t), \mu_2(t)$ and $\mu_3(t)$ are time-dependent functions satisfying the time-dependent differential equations

$$\dot{\mu}_1 = 2\mu_1 Y - 2\mu_2 Z\ , \tag{1.14}$$

$$\dot{\mu}_2 = \mu_1 X - \mu_3 Z, \tag{1.15}$$

$$\dot{\mu}_3 = 2\mu_2 X - 2\mu_3 Y\ , \tag{1.16}$$

by setting $\mu_1 = \sigma^2$, we can obtain the invariant in the form

$$I_h(t) = \frac{1}{2}\sigma^2 p^2 + \frac{\sigma^2 Y - \sigma\dot{\sigma}}{Z}(px + xp) + \frac{1}{\sigma^2}\left[1 + \left(\frac{\sigma^2 Y - \sigma\dot{\sigma}}{Z^2}\right)^2\right] x^2, \tag{1.17}$$

where σ satisfies the Milne-Pinney [17, 18]

$$\ddot{\sigma} - \frac{\dot{Z}}{Z}\dot{\sigma} + \left[\left(XZ - Y^2\right) + \frac{\dot{Z}}{Z}Y - \dot{Y}\right]\sigma = \frac{Z^2}{\sigma^3}. \tag{1.18}$$

To obtain the eigenfunctions of $I_h(t)$, we consider the unitary transformation

$$\phi_n'(x,t) = U\phi_n(x,t), \tag{1.19}$$

where U is defined as

$$U = \exp\left[\frac{i}{2\hbar}\left(\frac{Y\sigma - \dot{\sigma}}{Z\sigma}\right)x^2\right], \tag{1.20}$$

the transformations of p and p^2 are

$$\begin{aligned} UpU^+ &= p - \left(\frac{Y\sigma - \dot{\sigma}}{Z\sigma}\right)x, \\ Up^2U^+ &= p^2 - \left(\frac{Y\sigma - \dot{\sigma}}{Z\sigma}\right)(px + xp) + \left(\frac{Y\sigma - \dot{\sigma}}{Z\sigma}\right)^2 x^2. \end{aligned} \tag{1.21}$$

This unitary transformation leads to the following eigenvalue equation of the invariant operator $I_h'(t)$ is

$$I_h'(t)\phi_n'(x,t) = \lambda_n \phi_n'(x,t), \tag{1.22}$$

where the transformed invariant $I_h'(t)$ is

$$I_h'(t) = UI_hU^+ = \frac{1}{2}\sigma^2 p^2 + \frac{1}{2\sigma^2}x^2. \tag{1.23}$$

If we introduce the new variable $\tilde{\sigma} = \frac{x}{\sigma}$, we can write the eigenvalue equation (1.22) as follows

$$\left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial\tilde{\sigma}^2} + \frac{\tilde{\sigma}^2}{2}\right]\varphi_n(\tilde{\sigma}) = \lambda_n \varphi_n(\tilde{\sigma}), \tag{1.24}$$

with

$$\phi_n'(x,t) = \frac{1}{\sqrt{\sigma}}\varphi_n(\tilde{\sigma}). \tag{1.25}$$

The factor $\sqrt{\sigma}$ is introduced in the equation (1.25) to guarantee the normalization condition

$$\int \phi_n'^*(x,t)\,\phi_n'(x,t)\,dx = \int \varphi_n^*(\tilde{\sigma})\,\varphi_n(\tilde{\sigma})\,d\tilde{\sigma} = 1.$$

The solution of the equation (1.24) is

$$\varphi_n(\tilde{\sigma}) = \left[\frac{1}{n!2^n\sqrt{\pi\hbar}}\right]^{\frac{1}{2}} \exp\left[-\frac{\tilde{\sigma}^2}{2\hbar}\right] H_n\left[\left(\frac{1}{\hbar}\right)^{\frac{1}{2}}\tilde{\sigma}\right], \tag{1.26}$$

where $\lambda_n = \hbar\left(n+\frac{1}{2}\right)$ and H_n is the Hermite polynomial of order n. Then

$$\phi_n(x,t) = U^+ \frac{1}{\sqrt{\sigma}} \varphi_n\left(\tilde{\sigma}\right), \tag{1.27}$$

the eigfunctions of $I_h\left(t\right)$ are

$$\phi_n\ (x,t) = \left[\frac{1}{n!2^n\sigma\sqrt{\pi\hbar}}\right]^{\frac{1}{2}} \exp\left[-\frac{i}{2\hbar}\left(\frac{Y\sigma-\dot{\sigma}}{Z\sigma}-\frac{i}{\sigma^2}\right)x^2\right] H_n\left[\left(\frac{1}{\hbar}\right)^{\frac{1}{2}}\left(\frac{x}{\sigma}\right)\right]. \tag{1.28}$$

The Lewis-Riesenfeld phase

$$\hbar\frac{d\zeta_n\left(t\right)}{dt} = \left\langle\phi_n\right| i\hbar\frac{\partial}{\partial t} - h\left|\phi_n\right\rangle = \frac{1}{\sigma}\left\langle\varphi_n\left(\tilde{\sigma}\right)\right| i\hbar U\frac{\partial}{\partial t}U^+ - UhU^+\left|\varphi_n\left(\tilde{\sigma}\right)\right\rangle, \tag{1.29}$$

is obtained as

$$\zeta_n\left(t\right) = -\left(n+\frac{1}{2}\right)\int_0^t \frac{Z}{\sigma^2}dt'. \tag{1.30}$$

The general solution of the Schrödinger equation is written as

$$\begin{aligned}\Psi\ (x,t) \ &= \ \sum_n C_n\left[\frac{1}{n!2^n\sigma\sqrt{\pi\hbar}}\right]^{\frac{1}{2}} \exp\left[-\frac{i}{2\hbar}\left(\frac{Y\sigma-\dot{\sigma}}{Z\sigma}-\frac{i}{\sigma^2}\right)x^2 - i\left(n+\frac{1}{2}\right)\int_0^t\frac{Z}{\sigma^2}dt'\right] \\ &\quad \times H_n\left[\left(\frac{1}{\hbar}\right)^{\frac{1}{2}}\left(\frac{x}{\sigma}\right)\right].\end{aligned} \tag{1.31}$$

Chapter 2

Time-dependent coupled oscillator

2.1 Introduction

In physics in general and quantum mechanics in particular, the study of time-dependent coupled oscillator is essential [19, 20, 21, 22, 23, 24], since it describes different real systems [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]in addition to trapped atoms [36], nano-optomechanical resonances [37, 38], electromagnetically induced transparency [39], stimulated Raman effects [40], time-dependent Josephson phenomena [41], and systems of three isotropically coupled spins 1/2 [42]. The combination of n simple oscillators gives rise to n-coupled oscillators [43, 44, 45]. Lie symmetries of differential equations with damping and driving forces have considered the resolution of n-coupled harmonic oscillators [46].

Indeed, as a result of the remarkable attention shed on the study of time-dependent Hamiltonian systems, a variety of techniques have been developed: adiabatic approximation, sudden approximation, the perturbation theory and the Lewis-Reisenfeld invariants theory [13, 47, 48]. These methods did not only focus on Hermitian Hamiltonian systems [13] but extended their study to non-Hermitian ones, where the solutions to the system are established in terms of the eigenstates of a pseudo-Hermitian invariant operator [49, 50, 51].

2.2 Quantum invariant operator and the auxiliary condition

We consider the time-dependent Hamiltonian

$$H(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\frac{p_i^2}{m_i\left(t\right)} + c_i(t)x_i^2\right] + \frac{1}{2}c_3(t)x_1x_2, \tag{2.1}$$

where $m_i\left(t\right),\ c_i(t)$and $c_3(t)$ are time-dependent functions. We propose a quantum invariant operator as follows

$$I\left(t\right) = \frac{1}{2}\sum_{i=1}^{2}\left[\alpha_i\left(t\right)p_i^2 + \beta_i\left(t\right)\left(x_ip_i + p_ix_i\right) + \gamma_i\left(t\right)x_i^2\right] + \frac{1}{2}\eta\left(t\right)x_1x_2, \tag{2.2}$$

where $\alpha_i\left(t\right), \beta_i\left(t\right), \gamma_i\left(t\right)$ $(i = 1, 2)$ and $\eta\left(t\right)$ are the real parameters and differentiable functions of time. The substitution of (2.1) and (2.2) into the Von-Neumann equation (1.2) allows us to give the auxiliary equations as follows

$$\dot{\alpha}_i\left(t\right) = \frac{-2\beta_i(t)}{m_i\left(t\right)}, \tag{2.3}$$

$$\dot{\beta}_i\left(t\right) = c_i\left(t\right)\alpha_i\left(t\right) - \frac{\gamma_i\left(t\right)}{m_i\left(t\right)}, \tag{2.4}$$

$$\dot{\gamma}_i\left(t\right) = 2c_i\left(t\right)\beta_i(t), \tag{2.5}$$

$$\dot{\eta}\left(t\right) = c_3\left(t\right)\left[\beta_1(t) + \beta_2(t)\right], \tag{2.6}$$

and

$$\frac{\eta\left(t\right)}{c_3\left(t\right)} = \alpha_1\left(t\right)m_1\left(t\right) = \alpha_2\left(t\right)m_2\left(t\right). \tag{2.7}$$

Now, noting

$$\alpha_i\left(t\right)\gamma_i(t) - \beta_i^2(t) = \delta_i, \tag{2.8}$$

with δ_i being a real constant, we set

$$\alpha_i\left(t\right) = \rho_i^2, \tag{2.9}$$

using (2.3),(2.4) a simple calculation, lead to

$$\beta_i(t) = -m_i\rho_i\dot{\rho}_i, \tag{2.10}$$

and

$$\gamma_i(t) = c_i(t) m_i \rho_i^2 + m_i \dot{m}_i \rho_i \dot{\rho}_i + m_i^2 \dot{\rho}_i^2 + m_i^2 \rho_i \ddot{\rho}_i, \tag{2.11}$$

from the above equations (2.10), (2.11) and (2.8), we obtian the auxiliary equation for ρ_i

$$\ddot{\rho}_i + \frac{\dot{m}_i}{m_i}\dot{\rho}_i = \frac{\delta_i}{m_i^2 \rho_i^3} - \frac{c_i(t)\rho_i}{m_i}, \tag{2.12}$$

and since

$$\rho_1^2 m_1 = \rho_2^2 m_2, \tag{2.13}$$

we can write the following expression for $\eta(t)$

$$\eta(t) = -\int^t c_3(t') m_1 \rho_1 \left(\dot{\rho}_1 + \rho_1 \frac{\dot{\rho}_2}{\rho_2}\right) dt'. \tag{2.14}$$

Consequently, the invariant (2.2) is represented as

$$I(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\rho_i^2 p_i^2 - m_i \rho_i \dot{\rho}_i (x_i p_i + p_i x_i) + \left(\frac{\delta_i}{\rho_i^2} + m_i^2 \dot{\rho}_i^2\right) x_i^2\right]$$
$$-\frac{1}{2}\left[\int^t c_3(t') m_1 \rho_1 \left(\dot{\rho}_1 + \rho_1 \frac{\dot{\rho}_2}{\rho_2}\right) dt'\right] x_1 x_2. \tag{2.15}$$

By using the Lewis-Riesenfeld theory [13], the invariant operator (2.15) has a time-independent eigenvalues $\lambda_{n,m}$

$$I(t)\left|\varphi_{n_1,n_2}\right\rangle = \lambda_{n_1,n_2}\left|\varphi_{n_1,n_2}\right\rangle, \tag{2.16}$$

where its eigenfunctions $\left|\varphi_{n_1,n_2}\right\rangle$ multiplied by a suitable phases

$$\exp\left[i\zeta_{n_1,n_2}(t)\right] = \exp\left[i\int_0^t \left\langle\varphi_{n_1,n_2}\right|\left(i\hbar\frac{\partial}{\partial t} - H(t)\right)\left|\varphi_{n_1,n_2}\right\rangle\right], \tag{2.17}$$

are solutions of the time-dependent Shrödinger equation (1.1).

2.3 Exact solution and geometric phase

For solving the eigenvalues equation (2.16), we define a unitary transformation U such that

$$\left|\varphi'_{n_1,n_2}\right\rangle = U\left|\varphi_{n_1,n_2}\right\rangle = U_1 U_2 \left|\varphi_{n_1,n_2}\right\rangle, \tag{2.18}$$

where

$$U_1 = \prod_{i=1}^{2} \exp\left[\frac{i}{2\hbar}\left(x_i p_i + p_i x_i\right)\ln\sqrt{\alpha_i}\right], \tag{2.19}$$

$$U_2 = \sum_{i=1}^{2} \exp\left[\frac{i}{2\hbar}\frac{\beta_i}{\alpha_i}x_i^2\right], \tag{2.20}$$

applying of Baker–Campbell–Hausdorff's formula

$$\exp[A]B\exp[-A] = B + [A,B] + \frac{1}{2!}[A,[A,B]] + ..., \tag{2.21}$$

we obtain the following transformation of x_i and p_i

$$U_1 x_i U_1^+ = \sqrt{\alpha_i}x_i, \tag{2.22}$$

$$U_1 p_i U_1^+ = \frac{1}{\sqrt{\alpha_i}}p_i,$$

and

$$U_2 x_i U_2^+ = x_i, \tag{2.23}$$

$$U_2 p_i U_2^+ = p_i - \frac{\beta_i}{\alpha_i}x_i. \tag{2.24}$$

Finally

$$U x_i U^+ = \sqrt{\alpha_i}x_i, \tag{2.25}$$

$$U p_i U^+ = \frac{1}{\sqrt{\alpha_i}}p_i - \frac{\beta_i}{\sqrt{\alpha_i}}x_i, \tag{2.26}$$

and consequently the invariant (2.2) becomes

$$I' = \frac{1}{2}\sum_{i=1}^{2}\left[p_i^2 + \left(\gamma_i\alpha_i - \beta_i^2\right)x_i^2\right] + \frac{1}{2}\eta\sqrt{\alpha_1\alpha_2}x_1x_2. \tag{2.27}$$

We simplify the invariant operator I' by the unitary operator U_3

$$U_3 = \exp\left[\frac{i\theta}{2\hbar}\left(p_2x_1 - p_1x_2\right)\right], \tag{2.28}$$

x_i and p_i transform into

$$U_3x_1U_3^+ = \cos\left(\frac{\theta}{2}\right)x_1 - \sin\left(\frac{\theta}{2}\right)x_2, \quad U_3x_2U_3^+ = \cos\left(\frac{\theta}{2}\right)x_2 + \sin\left(\frac{\theta}{2}\right)x_1,$$

$$U_3p_1U_3^+ = \cos\left(\frac{\theta}{2}\right)p_1 - \sin\left(\frac{\theta}{2}\right)p_2, \quad U_3p_2U_3^+ = \cos\left(\frac{\theta}{2}\right)p_2 + \sin\left(\frac{\theta}{2}\right)p_1. \tag{2.29}$$

Therefore, the invariant (2.27) is written as

$$
\begin{aligned}
I'' = U_3 I' U_3^{+} = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{1}{2}\left[\delta_1 \cos^2\left(\frac{\theta}{2}\right) + \delta_2 \sin^2\left(\frac{\theta}{2}\right) + \frac{\eta\sqrt{\alpha_1\alpha_2}}{2}\sin\theta\right] x_1^2 \\
+ \frac{1}{2}\left[\delta_1 \sin^2\left(\frac{\theta}{2}\right) + \delta_2 \cos^2\left(\frac{\theta}{2}\right) - \frac{\eta\sqrt{\alpha_1\alpha_2}}{2}\sin\theta\right] x_2^2 \\
+ \frac{1}{2}\left[\eta\sqrt{\alpha_1\alpha_2}\cos\theta - (\delta_1 - \delta_2)\sin\theta\right] x_1 x_2,
\end{aligned} \tag{2.30}
$$

from Eq. (2.30), the separation of variables is complete for

$$\eta\sqrt{\alpha_1\alpha_2}\cos\theta - [\delta_1 - \delta_2]\sin\theta = 0, \tag{2.31}$$

the invariant operator I'' becomes

$$I'' = \frac{1}{2}\sum_{i=0}^{2}\left(p_i^2 + \widetilde{\Omega}_i^2 x_i^2\right), \tag{2.32}$$

where

$$\widetilde{\Omega}_1^2 = \delta_1 \cos^2\left(\frac{\theta}{2}\right) + \delta_2 \sin^2\left(\frac{\theta}{2}\right) + \frac{\eta\sqrt{\alpha_1\alpha_2}}{2}\sin\theta, \tag{2.33}$$

$$\widetilde{\Omega}_2^2 = \delta_1 \sin^2\left(\frac{\theta}{2}\right) + \delta_2 \cos^2\left(\frac{\theta}{2}\right) - \frac{\eta\sqrt{\alpha_1\alpha_2}}{2}\sin\theta. \tag{2.34}$$

$\widetilde{\Omega}_1^2, \widetilde{\Omega}_2^2$ are constants. From the Eq.(2.31), we deduce

$$\tan(\theta) = \frac{\eta\sqrt{\alpha_1\alpha_2}}{[\delta_1 - \delta_2]}, \tag{2.35}$$

where

$$\theta = \arctan(\eta\sqrt{\alpha_1\alpha_2}.\,[\delta_1 - \delta_2]^{-1}), \tag{2.36}$$

is constants. To see order this we calculated the time the derivative of θ

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial t}\left[\arctan\left(\frac{\eta\sqrt{\alpha_1\alpha_2}}{\delta_1 - \delta_2}\right)\right], \tag{2.37}$$

by replacing the Eqs.(2.7) and (2.9) in Eq.(2.37), we find

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial t}\left[\arctan\left(\frac{c_3\rho_1^3\rho_2 m_1}{\delta_1 - \delta_2}\right)\right], \tag{2.38}$$

therefore

$$\frac{\partial\left[c_3\rho_1^3\rho_2 m_1\right]}{\partial t}=\dot{c}_3\rho_1^3\rho_2 m_1+3c_3\dot{\rho}_1\rho_1^2\rho_2 m_1+c_3\rho_1^3\dot{\rho}_2 m_1+c_3\rho_1^3\rho_2\dot{m}_1, \tag{2.39}$$

from Eqs.(2.7), (2.88) and (2.9) that means

$$\frac{\eta}{c_3}=m_1\rho_1^2, \tag{2.40}$$

the derivative of (2.40) is given as

$$\eta\frac{\dot{c}_3}{c_3}=\frac{\dot{\eta}}{c_3}-2\dot{\rho}_1\rho_1 m_1-\dot{m}_1\rho_1^2, \tag{2.41}$$

if we replace the expression of $\dot{\eta}$ that is defined in Eq.(2.6), and $\eta = c_3 m_1\rho_1^2$, the Eq.(2.41) becomes as

$$\frac{\dot{c}_3}{c_3}=\frac{\left[\beta_1+\beta_2\right]}{m_1\rho_1^2}-\frac{2\dot{\rho}_1}{\rho_1}-\frac{\dot{m}_1}{m_1}, \tag{2.42}$$

we have that $\beta_i = -m_i\dot{\rho}_i\rho_i^2$, then the above equation (2.42) is written as

$$\frac{\dot{c}_3}{c_3}=-\left[\frac{3\dot{\rho}_1}{\rho_1}+\frac{\dot{m}_1}{m_1}+\frac{\dot{\rho}_2}{\rho_2}\right], \tag{2.43}$$

by using the differential equation(2.43), we deduce that

$$\begin{aligned}\frac{\partial\left[c_3\rho_1^3\rho_2 m_1\right]}{\partial t}&=-\left[\frac{3\dot{\rho}_1}{\rho_1}+\frac{\dot{m}_1}{m_1}+\frac{\dot{\rho}_2}{\rho_2}\right]\rho_1^3\rho_2 m_1 c_3+3c_3\dot{\rho}_1\rho_1^2\rho_2 m_1\\&+c_3\rho_1^3\dot{\rho}_2 m_1+c_3\rho_1^3\rho_2\dot{m}_1\\&=0.\end{aligned} \tag{2.44}$$

Knowing that δ_1, δ_2 are constants, we confirm that the derivative of θ must be equal to zero i.e $\frac{\partial\theta}{\partial t}=0$. Hence that θ and the frenquencies $\left(\widetilde{\Omega}_1^2, \widetilde{\Omega}_2^2\right)$ are time-independent. As recalled above, the invariant operator $I(t)$ should have time-independent eigenvalues, consequently the frequencies $\widetilde{\Omega}_i$ are time-independent.

After decoupling, the invariant (2.32) is the sum of two invariants representing simple harmonic oscillators with time-independent frequencies $\widetilde{\Omega}_i$ and unit masses, whose eigenstates are represented as

$$\left|\varphi_{n_1,n_2}\right\rangle = \prod_{i=1}^{2}\left(\frac{\sqrt{\widetilde{\Omega}_i}}{(\pi\hbar)^{1/2}\, n_i! 2^{n_i}}\right)^{1/2} H_{n_i}\left(\sqrt{\frac{\widetilde{\Omega}_i}{\hbar}} x_i\right)\exp\left[-\frac{i\widetilde{\Omega}_i}{2\hbar}x_i^2\right], \tag{2.45}$$

where H_{n_i} are the Hermite polynomials.

Noting that the transformation of the eigenstate $\left|\varphi_{n_1,n_2}\right\rangle$ is

$$\left|\varphi_{n_1,n_2}\right\rangle = U_n^+\left|\varphi''_{n_1,n_2}\right\rangle = U_2^+U_1^+U_3^+\left|\varphi''_{n_1,n_2}\right\rangle, \tag{2.46}$$

by replacing (2.46) in the expression of Lewis-Riesenfeld phase (1.10), we obtian

$$\dot{\zeta}_{n_1,n_2}(t) = \left\langle\varphi''_{n_1,n_2}\right| U_n\left(i\hbar\frac{\partial}{\partial t} - H(t)\right)U_n^+\left|\varphi''_{n_1,n_2}\right\rangle. \tag{2.47}$$

Using the equation (2.7) and the Baker–Campbell–Hausdorff's formula (2.21), we can obtain the expressions

$$\begin{aligned}
U_n\left(i\hbar\frac{\partial}{\partial t}\right)U_n^+ &= U_3U_1U_2\left(i\hbar\frac{\partial}{\partial t}\right)U_2^+U_1^+U_3^+ \\
&= \left[\left(\frac{\alpha_1 c_1}{2} - \frac{\gamma_1}{2m_1} + \frac{c_3}{\eta}\beta_1^2\right)\cos^2\left(\frac{\theta}{2}\right) + \left(\frac{\alpha_2 c_2}{2} - \frac{\gamma_2}{2m_2} + \frac{c_3}{\eta}\beta_2^2\right)\sin^2\left(\frac{\theta}{2}\right)\right]x_1^2 \\
&\quad + \left[\left(\frac{\alpha_1 c_1}{2} - \frac{\gamma_1}{m_1} + \frac{c_3}{\eta}\beta_1^2\right)\sin^2\left(\frac{\theta}{2}\right) + \left(\frac{\alpha_2 c_2}{2} - \frac{\gamma_2}{m_2} + \frac{c_3}{\eta}\beta_2^2\right)\cos^2\left(\frac{\theta}{2}\right)\right]x_2^2 \\
&\quad + \frac{\sin(\theta)}{2}\left[\left(\alpha_2 c_2 - \frac{\gamma_2}{m_2}\right) - \left(\alpha_1 c_1 - \frac{\gamma_1}{m_1}\right) + \frac{2c_3}{\eta}\left(\beta_2^2 - \beta_1^2\right)\right]x_1 x_2 \\
&\quad - \frac{c_3}{2\eta}\left[\beta_1\cos^2\left(\frac{\theta}{2}\right) + \beta_2\sin^2\left(\frac{\theta}{2}\right)\right](x_1p_1 + p_1x_1) \\
&\quad - \frac{c_3}{2\eta}\left[\beta_1\sin^2\left(\frac{\theta}{2}\right) + \beta_2\cos^2\left(\frac{\theta}{2}\right)\right](x_2p_2 + p_2x_2) \\
&\quad + \frac{c_3}{2\eta}(\beta_1 - \beta_2)(x_1p_2 + x_2p_1)\sin(\theta),
\end{aligned} \tag{2.48}$$

and

$$\begin{aligned}
U_n\left[H(t)\right]U_n^+ &= U_3U_1U_2\left[H(t)\right]U_2^+U_1^+U_3^+ \\
&= \frac{c_3}{2\eta}\left(p_1^2+p_2^2\right) \\
&\quad+\left[\left(\frac{c_3}{2\eta}\beta_1^2+\frac{\alpha_1c_1}{2}\right)\cos^2\left(\frac{\theta}{2}\right)+\left(\frac{c_3}{2\eta}\beta_2^2+\frac{\alpha_2c_2}{2}\right)\sin^2\left(\frac{\theta}{2}\right)+\frac{c_3\sqrt{\alpha_1\alpha_2}}{2}\sin\theta\right]x_1^2 \\
&\quad+\left[\left(\frac{c_3}{2\eta}\beta_1^2+\frac{\alpha_1c_1}{2}\right)\sin^2\left(\frac{\theta}{2}\right)+\left(\frac{c_3}{2\eta}\beta_2^2+\frac{\alpha_2c_2}{2}\right)\cos^2\left(\frac{\theta}{2}\right)-\frac{c_3\sqrt{\alpha_1\alpha_2}}{2}\sin\theta\right]x_2^2 \\
&\quad+\frac{1}{2}\left[\frac{c_3}{\eta}\left[\left(\beta_2^2-\beta_1^2\right)-\alpha_1c_1+\alpha_2c_2\right]\sin\left(\theta\right)+c_3\sqrt{\alpha_1\alpha_2}\cos\theta\right]x_1x_2 \\
&\quad-\frac{c_3}{2\eta}\left[\beta_1\cos^2\left(\frac{\theta}{2}\right)+\beta_2\sin^2\left(\frac{\theta}{2}\right)\right]\left(x_1p_1+p_1x_1\right) \\
&\quad-\frac{c_3}{2\eta}\left[\beta_1\sin^2\left(\frac{\theta}{2}\right)+\beta_2\cos^2\left(\frac{\theta}{2}\right)\right]\left(x_2p_2+p_2x_2\right) \\
&\quad+\frac{c_3}{2\eta}\left(\beta_1-\beta_2\right)\left(x_1p_2+x_2p_1\right)\sin\left(\theta\right).
\end{aligned} \tag{2.49}$$

Then, from the equations (2.88) and (2.31), the expression of $\dot{\zeta}_{n_1,n_2}(t)$ is writen as

$$\begin{aligned}
\dot{\zeta}_{n_1,n_2}(t) &= \left\langle\varphi''_{n_1,n_2}\right|U_n\left(i\hbar\frac{\partial}{\partial t}-H(t)\right)U_n^+\left|\varphi''_{n_1,n_2}\right\rangle=-\frac{c_3}{\eta}\left\langle\varphi''_{n_1,n_2}\right|I''\left|\varphi''_{n_1,n_2}\right\rangle \\
&= -\frac{\hbar c_3}{\eta}\left[\widetilde{\Omega}_1\left(n_1+\frac{1}{2}\right)+\widetilde{\Omega}_2\left(n_2+\frac{1}{2}\right)\right],
\end{aligned} \tag{2.50}$$

the Lewis-Riesenfeld phase (2.47) in this case is

$$\zeta_{n_1,n_2}(t)=-\hbar\left(\left[\widetilde{\Omega}_1\left(n_1+\frac{1}{2}\right)+\widetilde{\Omega}_2\left(n_2+\frac{1}{2}\right)\right]\int_0^t\frac{c_3}{\eta}dt'\right), \tag{2.51}$$

so, we find the general solution of the Schrödinger equation

$$\begin{aligned}
\left|\psi(t)\right\rangle &= \sum_n C_n\prod_{i=1}^{2}\left(\frac{\sqrt{\widetilde{\Omega}_i}}{\left(\pi\hbar\right)^{1/2}n_i!2^{n_i}}\right)^{1/2}H_{n_i}\left(\sqrt{\frac{\widetilde{\Omega}_i}{\hbar}}x_i\right) \\
&\quad\exp\left[-\frac{i\widetilde{\Omega}_i}{2\hbar}x_i^2-i\hbar\left(\left[\widetilde{\Omega}_1\left(n_1+\frac{1}{2}\right)+\widetilde{\Omega}_2\left(n_2+\frac{1}{2}\right)\right]\int_0^t\frac{c_3}{\eta}dt'\right)\right].
\end{aligned} \tag{2.52}$$

2.4 The generalisation of the 2D to a 3D coupled oscillator by the method of Hassoul et al [52]

In this section, we show that the analysis of Hassoul et al, in Ref [52] suffers from basic errors. Hassoul et al [52] considered the following Hamiltonian $H(t)$ with different masses $m_i\left(t\right)\left(i=1,2,3\right)$ and frequencies $\omega_i\left(t\right)\left(i=1,2,3\right)$

$$\begin{aligned} H(t) &= \frac{1}{2}\sum_{i=1}^{3}\left[\frac{p_i^2}{m_i\left(t\right)}+m_i\left(t\right)\omega_i^2\left(t\right)x_i^2\right] \\ &+\frac{1}{2}\left[k_{12}\left(t\right)x_1x_2+k_{13}\left(t\right)x_1x_3+k_{23}\left(t\right)x_2x_3\right], \end{aligned} \tag{2.53}$$

where x_i and p_i are the canonical coordinates and momentums, $k_{12}\left(t\right), k_{13}\left(t\right)$ and $k_{23}\left(t\right)$ are coupling parameters respectively.

They choose an invariant operator of the form

$$\begin{aligned} I(t) &= \frac{1}{2}\sum_{i=1}^{3}\left[A_i\left(t\right)p_i^2+B_i\left(t\right)\left(x_ip_i+p_ix_i\right)+C_i\left(t\right)x_i^2\right] \\ &+\frac{1}{2}\left[D_{12}\left(t\right)x_1x_2+D_{13}\left(t\right)x_1x_3+D_{23}\left(t\right)x_2x_3\right], \end{aligned} \tag{2.54}$$

this invariant satisfies the invariance condition (1.2). The derivative of the invariant operator is

$$\frac{\partial I\left(t\right)}{\partial t}=\frac{1}{2}\left[\dot{A}_1\left(t\right)p_1^2+\dot{B}_1\left(t\right)x_1p_1+\dot{C}_1\left(t\right)x_1^2\right]+\frac{1}{2}\left[\dot{A}_2\left(t\right)p_2^2+\dot{B}_2\left(t\right)x_2p_2+\dot{C}_2\left(t\right)x_2^2\right]$$

$$+\frac{1}{2}\left[\dot{A}_3\left(t\right)p_3^2+\dot{B}_3\left(t\right)x_3p_3+\dot{C}_3\left(t\right)x_3^2\right]+\frac{1}{2}\left[\dot{D}_{12}\left(t\right)x_1x_2+\dot{D}_{13}\left(t\right)x_1x_3+\dot{D}_{23}\left(t\right)x_2x_3\right], \tag{2.55}$$

and the expression of the commutation relation between $I\left(t\right)$ and H is

$$
\begin{aligned}
[I(t), H] = {} & \frac{B_1(t)}{2m_1(t)} p_1^2 + \frac{C_1(t)}{m_1(t)} x_1 p_1 + \frac{B_2(t)}{2m_2(t)} p_2^2 + \frac{C_2(t)}{m_2(t)} x_2 p_2 \\
& + \frac{B_3(t)}{2m_3(t)} p_3^2 + \frac{C_3(t)}{m_3(t)} x_3 p_3 + \frac{(D_{12}(t) x_2 p_1 + D_{13}(t) x_3 p_1)}{2m_1(t)} \\
& + \frac{(D_{12}(t) x_1 p_2 + D_{23}(t) x_3 p_2)}{2m_2(t)} + \frac{(D_{13}(t) x_1 p_3 + D_{23}(t) x_2 p_3)}{2m_3(t)} \\
& - m_1(t) \omega_1^2(t) \left(A_1(t) p_1 x_1 + \frac{B_1(t)}{2} x_1^2 \right) - \frac{A_1(t)}{2} \left[k_{12}(t) p_1 x_2 + k_{13}(t) p_1 x_3 \right] \\
& - \frac{B_1(t)}{4} \left(k_{12}(t) x_1 x_2 + k_{13}(t) x_1 x_3 \right) - m_2(t) \omega_2^2(t) \left(A_2(t) p_2 x_2 + \frac{B_2(t)}{2} x_2^2 \right) \\
& - \frac{A_2(t)}{2} \left[k_{12}(t) p_2 x_1 + k_{23}(t) p_2 x_3 \right] - \frac{B_2(t)}{4} \left[k_{12}(t) x_2 x_1 + k_{23}(t) x_2 x_3 \right] \\
& - m_3(t) \omega_3^2(t) \left(A_3(t) p_3 x_3 + \frac{B_3(t)}{2} x_3^2 \right) - \frac{A_3(t)}{2} \left[k_{13}(t) p_3 x_1 + k_{23}(t) p_3 x_2 \right] \\
& - \frac{B_3(t)}{4} \left(k_{13}(t) x_3 x_1 + k_{23}(t) x_3 x_2 \right). \qquad (2.56)
\end{aligned}
$$

By replacing the equations (2.55) and (2.56) in the invariance condition (1.2), we deduce the following auxiliary conditions

$$\dot{A}_i(t) = -\frac{2B_i(t)}{m_i(t)}, \qquad (2.57)$$

$$\dot{B}_i(t) = -\frac{C_i(t)}{m_i(t)} + m_i(t) \omega_i^2(t) A_i(t), \qquad (2.58)$$

$$\dot{C}_i(t) = 2m_i(t) \omega_i^2(t) B_i(t), \qquad (2.59)$$

$$\dot{D}_{12}(t) = \frac{k_{12}(t)}{2} \left[B_1(t) + B_2(t) \right], \qquad (2.60)$$

$$\dot{D}_{13}(t) = \frac{k_{13}(t)}{2} \left[B_1(t) + B_3(t) \right], \qquad (2.61)$$

$$\dot{D}_{23}(t) = \frac{k_{23}(t)}{2} \left[B_2(t) + B_3(t) \right], \qquad (2.62)$$

$$\frac{D_{13}(t)}{D_{12}(t)} = \frac{k_{13}(t)}{k_{12}(t)}, \qquad (2.63)$$

$$\frac{D_{12}(t)}{D_{23}(t)} = \frac{k_{12}(t)}{k_{23}(t)}, \qquad (2.64)$$

$$\frac{D_{23}(t)}{D_{13}(t)} = \frac{k_{23}(t)}{k_{13}(t)}, \qquad (2.65)$$

we note that there are nine auxiliary conditions and the first six equations are similar to the six differential ones given in [52] with a slight difference in the coefficients.

Hassoul et al [52] found that possible solutions of the differential equations (2.57)-(2.62) are given by the following forms

$$A_i(t) = \frac{1}{m_i(t)}, \tag{2.66}$$

$$B_i(t) = \frac{\dot{m}_i(t)}{2m_i(t)}, \tag{2.67}$$

$$C_i(t) = \int \dot{m}_i(t)\, \omega_i^2(t)\, dt, \tag{2.68}$$

$$D_{12}(t) = \int k_{12}(t) \left[\frac{\dot{m}_1(t)}{2m_1(t)} + \frac{\dot{m}_2(t)}{2m_2(t)} \right] dt, \tag{2.69}$$

$$D_{13}(t) = \int k_{13}(t) \left[\frac{\dot{m}_1(t)}{2m_1(t)} + \frac{\dot{m}_3(t)}{2m_3(t)} \right] dt, \tag{2.70}$$

$$D_{23}(t) = \int k_{23}(t) \left[\frac{\dot{m}_2(t)}{2m_2(t)} + \frac{\dot{m}_3(t)}{2m_3(t)} \right] dt, \tag{2.71}$$

These solutions impose a constraint on the system but the authors did not take this into consideration.

From the equation (2.57) and equation (2.66), we find

$$\frac{-\dot{m}_i(t)}{m_i^2(t)} = \frac{-2B_i(t)}{m_i(t)} \quad \Rightarrow \quad B_i(t) = \frac{\dot{m}_i(t)}{2m_i(t)}. \tag{2.72}$$

We assume that the coefficients satisfy this following condition

$$A_i(t)C_i(t) - B_i^2(t) = \delta_i, \tag{2.73}$$

with δ_i is a real constant. This condition (2.73) is not mentioned in [52].

It is clear that

$$C_i(t) = \frac{\delta_i + B_i^2(t)}{A_i(t)}, \tag{2.74}$$

substituting the Eq (2.66) and (2.67), the expression of $C_i(t)$ is

$$C_i(t) = \delta_i . m_i(t) + \frac{\dot{m}_i^2(t)}{4m_i(t)}. \tag{2.75}$$

Now, deriving the equation (2.67), we obtain

$$\dot{B}_i(t) = \frac{\ddot{m}_i(t)}{2m_i(t)} - \frac{\dot{m}_i^2(t)}{2m_i^2(t)}, \tag{2.76}$$

replacing the equations (2.66) and (2.75) in the equation (2.58), we find

$$\dot{B}_i(t) = \omega_i^2(t) - \delta_i - \frac{\dot{m}_i^2(t)}{4m_i^2(t)}. \tag{2.77}$$

From the two equations (2.76) and (2.77), we get

$$\ddot{m}_i(t) - \frac{1}{2}\frac{\dot{m}_i^2(t)}{m_i(t)} + 2\left(\delta_i - \omega_i^2(t)\right) m_i(t) = 0, \tag{2.78}$$

when considering $m_i(t) = 1/A_i(t)$, the last equation is the constraint equation which is difficult to solve. The system can not be resolved for any given mass. From here we can deduce that Hassoul et al [52] incorrectly generalized the 2D coupled oscillator to a 3D.

For diagonalizing invariant operator $I(t)$, authors of [52] diagonalize the matrix $\mathbb{k}$ as

$$\mathbb{k} = \begin{pmatrix} \varpi_1^2 & \frac{1}{2}K_{12} & \frac{1}{2}K_{13} \\ \frac{1}{2}K_{12} & \varpi_2^2 & \frac{1}{2}K_{23} \\ \frac{1}{2}K_{13} & \frac{1}{2}K_{23} & \varpi_3^2 \end{pmatrix}, \tag{2.79}$$

where the expression of ϖ_i^2 is

$$\varpi_i^2 = \frac{\int_0^t \omega_i^2(t)\dot{m}_i(t)dt}{m_i(t)} - \left[\frac{\dot{m}_i(t)}{4m_i(t)}\right]^2, \tag{2.80}$$

and the expressions of K_{12}, K_{13} and K_{23} are defined as

$$K_{12} = \frac{\int_0^t k_{12}\left(\dot{m}_1(t)/m_1(t) + \dot{m}_2(t)/m_2(t)\right)dt}{\sqrt{m_1(t)m_2(t)}}, \tag{2.81}$$

$$K_{13} = \frac{\int_0^t k_{13}\left(\dot{m}_1(t)/m_1(t) + \dot{m}_3(t)/m_3(t)\right)dt}{\sqrt{m_1(t)m_3(t)}}, \tag{2.82}$$

$$K_{23} = \frac{\int_0^t k_{23}\left(\dot{m}_2(t)/m_2(t) + \dot{m}_3(t)/m_3(t)\right)dt}{\sqrt{m_2(t)m_3(t)}}. \tag{2.83}$$

By using this formula [52]

$$\mathbb{R} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \lambda_+\left[\frac{1}{2}\left(K_{12} - K_{23}\right) - \Omega^2\right] & \lambda_-\left[\frac{1}{2}\left(K_{12} - K_{23}\right) + \Omega^2\right] \\ \frac{1}{\sqrt{3}} & \lambda_+\left[\frac{1}{2}\left(K_{23} - K_{12}\right) + \Omega^2\right] & \lambda_-\left[\frac{1}{2}\left(K_{23} - K_{12}\right) - \Omega^2\right] \\ \frac{1}{\sqrt{3}} & \lambda_+\left[\frac{1}{2}\left(K_{23} - K_{13}\right)\right] & \lambda_-\left[\frac{1}{2}\left(K_{23} - K_{13}\right)\right] \end{pmatrix}, \tag{2.84}$$

where

$$\lambda_{\pm} = \frac{1}{(K_{23} - K_{13})} \left[\frac{2}{3} \pm \frac{K_{12} - (K_{23} + K_{13})/2}{3\Omega^2} \right], \tag{2.85}$$

and

$$\Omega^2 = \frac{1}{2} \left[K_{12}^2 + K_{13}^2 + K_{23}^2 - (K_{12}K_{13} + K_{12}K_{23} + K_{13}K_{23}) \right], \tag{2.86}$$

we obtain

$$\mathbb{R}^{-1}\mathbb{k}\mathbb{R} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}. \tag{2.87}$$

A straightforward evaluation of the product $\mathbb{R}^{-1}\mathbb{k}\mathbb{R}$ leads to the following elements

$$M_{11} = \frac{1}{2}\left(\varpi_1^2 + \varpi_2^2\right) + \frac{1}{4}(K_{13} + K_{23}) + \frac{1}{2}K_{12}, \tag{2.88}$$

$$M_{12} = \frac{\sqrt{3}}{2} \left[\frac{\lambda_+ (\varpi_1^2 - \varpi_2^2)}{2} \left(K_{12} - K_{23} - 2\Omega^2\right) + \frac{\lambda_+}{4}\left(K_{23}^2 - K_{13}^2\right) \right], \tag{2.89}$$

$$M_{13} = \frac{\sqrt{3}}{2} \left[\frac{\lambda_- (\varpi_1^2 - \varpi_2^2)}{2} \left(K_{12} - K_{23} + 2\Omega^2\right) + \frac{\lambda_-}{4}\left(K_{23}^2 - K_{13}^2\right) \right], \tag{2.90}$$

$$\begin{aligned} M_{21} = {} & \frac{1}{4\sqrt{3}\lambda_+\Omega^2} \left(\frac{K_{23} - K_{12} - 2\Omega^2}{K_{23} - K_{13}} \right) \left[\varpi_1^2 + \varpi_2^2 - 2\varpi_3^2 + K_{12} - \frac{K_{13} + K_{23}}{2} \right] \\ & + \frac{1}{4\sqrt{3}\lambda_+\Omega^2} \left[\varpi_2^2 - \varpi_1^2 + \frac{K_{23} - K_{13}}{2} \right], \end{aligned} \tag{2.91}$$

$$\begin{aligned} M_{22} = {} & \frac{1}{4\Omega^2} \left[\frac{(K_{23} - K_{12})^2 - 4\Omega^4}{K_{23} - K_{13}} \right] \left(\frac{\varpi_2^2 - \varpi_1^2}{2} \right) \\ & + \frac{1}{4\Omega^2} \left[\frac{(\varpi_1^2 + \varpi_2^2) - K_{12}}{2} \left(K_{23} - K_{12} + 2\Omega^2\right) + \frac{(K_{23} - K_{13})^2}{4} \right] \\ & - \frac{(K_{23} - K_{12} - 2\Omega^2)}{4\Omega^2} \left[\frac{K_{23} - K_{12} + 2\Omega^2}{2} + \varpi_3^2 - \frac{K_{23} + K_{13}}{4} \right], \end{aligned} \tag{2.92}$$

$$\begin{aligned} M_{23} = {} & \frac{\lambda_-}{4\lambda_+\Omega^2} \left[\frac{(K_{23} - K_{12} - 2\Omega^2)^2}{K_{23} - K_{13}} \right] \left(\frac{\varpi_2^2 - \varpi_1^2}{2} \right) \\ & + \frac{\lambda_-}{4\lambda_+\Omega^2} \left[\frac{(\varpi_1^2 + \varpi_2^2 - K_{12})}{2} \left(K_{23} - K_{12} - 2\Omega^2\right) + \frac{(K_{23} - K_{13})^2}{4} \right] \\ & - \frac{\lambda_- (K_{23} - K_{12} - 2\Omega^2)}{4\lambda_+\Omega^2} \left[\frac{K_{23} - K_{12} - 2\Omega^2}{2} + \varpi_3^2 - \frac{K_{23} + K_{13}}{4} \right], \end{aligned} \tag{2.93}$$

$$M_{31} = \frac{1}{4\sqrt{3}\lambda_+\Omega^2}\left(\frac{K_{12}-K_{23}-2\Omega^2}{K_{23}-K_{13}}\right)\left[\varpi_1^2+\varpi_2^2-2\varpi_3^2+K_{12}-\frac{K_{13}+K_{23}}{2}\right] + \frac{1}{4\sqrt{3}\lambda_+\Omega^2}\left[\varpi_1^2-\varpi_2^2+\frac{K_{13}-K_{23}}{2}\right], \tag{2.94}$$

$$\begin{aligned} M_{32} = &\frac{\lambda_+}{4\lambda_-\Omega^2}\left[\frac{(K_{12}-K_{23}-2\Omega^2)^2}{K_{23}-K_{13}}\right]\left(\frac{\varpi_1^2-\varpi_2^2}{2}\right) \\ &+\frac{\lambda_+}{4\lambda_-\Omega^2}\left[\frac{(K_1^2+K_2^2-K_{12})}{2}\left(K_{12}-K_{23}-2\Omega^2\right)-\frac{(K_{23}-K_{13})^2}{4}\right] \\ &-\frac{\lambda_+(K_{12}-K_{23}-2\Omega^2)}{4\Omega^2\lambda_-}\left[\frac{K_{23}-K_{12}+2\Omega^2}{2}+\varpi_3^2-\frac{K_{13}+K_{23}}{4}\right], \end{aligned} \tag{2.95}$$

$$\begin{aligned} M_{33} = &\frac{\lambda_+}{4\lambda_-\Omega^2}\left[\frac{(K_{12}-K_{23})^2-4\Omega^4}{K_{23}-K_{13}}\right]\left(\frac{\varpi_1^2-\varpi_2^2}{2}\right) \\ &+\frac{\lambda_+}{4\lambda_-\Omega^2}\left[\frac{(\varpi_1^2+\varpi_2^2)-K_{12}}{2}\left(K_{12}-K_{23}+2\Omega^2\right)-\frac{(K_{23}-K_{13})^2}{4}\right] \\ &-\frac{\lambda_+(K_{12}-K_{23}-2\Omega^2)}{4\lambda_-\Omega^2}\left[\frac{K_{23}-K_{12}-2\Omega^2}{2}+\varpi_3^2-\frac{K_{23}+K_{13}}{4}\right], \end{aligned} \tag{2.96}$$

So, we didn't find the same result in [52] unless the parameters of $\Bbbk$ obey

$$K_{12} = K_{13} = K_{23} \ \text{ and } \ \varpi_1^2 = \varpi_2^2 = \varpi_3^2 \ , \tag{2.97}$$

which implies that $\Omega^2 = 0$ and the eigenvalues Ω_i^2 read

$$\Omega_1^2 = \varpi_1^2 + K_{12}, \tag{2.98}$$

$$\Omega_2^2 = \varpi_1^2 - \frac{K_{12}}{2}, \tag{2.99}$$

$$\Omega_3^2 = \varpi_1^2 - \frac{K_{12}}{2}. \tag{2.100}$$

As stated above, the invariant operator has time-independent eigenvalues [13] contrary to what is found in [52] where the eigenvalues Ω_i^2 are time-dependent.

Neverthemore, the derivative of Ω_i^2 with respect to time gives

$$\frac{d\Omega_1^2}{dt} = \frac{d}{dt}\left(\frac{D_{12}}{\sqrt{m_1 m_2}}\right) + \frac{d}{dt}\left(\frac{D_{13}}{\sqrt{m_1 m_3}}\right), \tag{2.101}$$

$$\frac{d\Omega_2^2}{dt} = -\frac{d}{dt}\left(\frac{D_{23}}{\sqrt{m_2 m_3}}\right) + \frac{d\Omega^2}{dt}, \tag{2.102}$$

$$\frac{d\Omega_3^2}{dt} = -\frac{d}{dt}\left(\frac{D_{23}}{\sqrt{m_2 m_3}}\right) - \frac{d\Omega^2}{dt}, \tag{2.103}$$

one can see that even if $\Omega^2 = 0$, the parameters D_{12}, D_{13}, D_{23} and the masses m_i are defined as time-dependent but the eigenvalues Ω_i^2 are not time-independent in [52] this contradicts the Lewis-Riesenfeld theory.

The formula of the phases in [53, 52] have seemed to be found using the invariant operator rather than the Hamiltonian operator leading to the omission of the term x_1x_2. It should be emphasized that the Hamiltonian operator generates the dynamics of the system and not the invariant operator. It appears as if the authors have taken results of [13, 54, 55, 15] setting $\frac{1}{\rho^2} = m_i(t)$ as if the invariant operator is the generator of the dynamics. They claimed to demonstrate that the solution to the time-dependent Schrödinger equation with the coupled terms x_1x_2 in the Hamiltonian can be reduced to the solution of a time-independent Schrödinger equation that includes the quantum invariant due to the mixed term x_1x_2 in the Hamiltonian make a contribution and not be omitted, we think these results are incorrect.

2.5 Discussion of Hassoul et al method [53] by pointing out the inconsistencies

This section summarizes the basic errors made in [53]. The authors of Ref [53] examined the system (2.1) by choosing the time-dependent invariant operator $I(t)$ (2.2). According to them, they introduce two pairs of creation and annihilation operators uncouples the invariant operator (2.2), so that it becomes the one that describes two independent systems. Hassoul et al [53] found the solution of the equations (2.3-2.6), which they not mention that $\alpha_i(t) = 1/m_i(t)$ must satisfy the following constraint equation

$$\ddot{m}_i(t) - \frac{1}{2}\frac{\dot{m}_i^2(t)}{m_i(t)} + 2\left(\delta_i m_i(t) - c_i(t)\right) = 0, \tag{2.104}$$

which is difficult to solve. Also, the authors of Ref [53] not mention condition (2.7) and decoupled invariant operator by using the canonical transformations. But these results are not valid without both condition $m_1(t) = m_2(t)$ and $\omega_1^2(t) = \omega_2^2(t)$.

Additionally, to our understanding, in the Lewis and Riesenfeld theory [13], the invariant operator possesses time-independent eigenvalues, but the frequencies ω_i^2 are time-dependent which contradicts the claims of Ref [53] that the eigenvalues of invariant operator are time-dependent. As a result, it difficult to calculate the phases. The frequencies mentioned in Ref.[53] are dependent of time.

The expression of ω_1^2 is given by [53]

$$\omega_1^2 = \left(\frac{\int \left[c_1 \frac{\dot{m}_1}{m_1}\right] dt}{m_1} - \frac{\dot{m}_1^2}{4m_1^2}\right)\cos^2\frac{\theta}{2} + \left(\frac{\int \left[c_2 \frac{\dot{m}_2}{m_2}\right] dt}{m_2} - \frac{\dot{m}_2^2}{4m_2^2}\right)\sin^2\frac{\theta}{2} + \frac{\int c_3 \left[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}\right] dt}{2\sqrt{m_1 m_2}}\sin\theta, \tag{2.105}$$

By using(2.105), we calculate time-dependent derivative of ω_1^2

$$\begin{aligned}\frac{d\omega_1^2(t)}{dt} &= \left[\frac{\left[c_1\frac{\dot{m}_1}{m_1}\right]}{m_1} - \frac{\dot{m}_1}{m_1^2}\int_0^t \left[c_1\frac{\dot{m}_1}{m_1}\right] dt - \frac{\ddot{m}_1 m_1 - \dot{m}_1^2}{m_1^2(t)}\left(\frac{\dot{m}_1}{2m_1}\right)\right]\cos^2\left(\frac{\theta}{2}\right) \\ &+ \left[\frac{\left[c_2\frac{\dot{m}_2}{m_2}\right]}{m_2(t)} - \frac{\dot{m}_2}{m_2^2}\int_0^t \left[c_2\frac{\dot{m}_2}{m_2}\right] dt - \frac{\ddot{m}_2 m_2 - \dot{m}_2^2}{m_2^2}\left(\frac{\dot{m}_2}{2m_2}\right)\right]\sin^2\left(\frac{\theta}{2}\right) \\ &+ \frac{\sin\theta}{2}\left(\left[\frac{\int_0^t \left[c_2\frac{\dot{m}_2}{2m_2}\right] dt}{m_2} - \frac{\int_0^t \left[c_1\frac{\dot{m}_1}{2m_1}\right] dt}{m_1}\right] - \left[\frac{\dot{m}_1^2}{4m_1^2} - \frac{\dot{m}_2^2}{4m_2^2}\right]\right)\dot{\theta} \\ &+ \sin(\theta)\left[\frac{c_3\left[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}\right]}{2\sqrt{m_1 m_2}} - \frac{1}{4\sqrt[3]{m_1 m_2}}\left(\frac{\dot{m}_1}{m_1} + \frac{\dot{m}_2}{m_2}\right)\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}\right] dt\right] \\ &+ \cos(\theta)\left(\frac{\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}\right] dt}{2\sqrt{m_1 m_2}}\right)\dot{\theta},\end{aligned} \tag{2.106}$$

from the Eq (2.73), we notice that

$$\frac{c_1(t)\left(\dot{m}_1/m_1\right)}{m_1} - \frac{\dot{m}_1}{m_1^2}\int_0^t c_1(t)\dot{m}_1(t)dt - \frac{\ddot{m}_1 m_1 - \dot{m}_1^2}{m_1^2}\left(\frac{\dot{m}_1}{2m_1}\right) = \alpha_1(t)\dot{\gamma}_1(t) + \dot{\alpha}_1(t)\gamma_1(t) - 2\beta_1(t)\dot{\beta}_1(t) = 0, \tag{2.107}$$

and

$$\frac{c_2\left(\dot{m}_2/m_2\right)}{m_2} - \frac{\dot{m}_2}{m_2^2}\int_0^t c_2(t)\dot{m}_2 dt - \frac{\ddot{m}_2 m_2 - \dot{m}_2^2}{m_2^2}\left(\frac{\dot{m}_2}{2m_2}\right) = \alpha_2(t)\dot{\gamma}_2(t) + \dot{\alpha}_2(t)\gamma_2(t) - 2\beta_2(t)\dot{\beta}_2(t) = 0, \tag{2.108}$$

the equation (2.106), becomes

$$\frac{d\omega_1^2}{dt} = \frac{\sin\theta}{2}\left(\left[\frac{\int_0^t\left[c_2\frac{\dot{m}_2}{2m_2}\right]dt}{m_2} - \frac{\int_0^t\left[c_1\frac{\dot{m}_1}{2m_1}\right]dt}{m_1}\right] - \left[\frac{\dot{m}_1^2}{4m_1^2} - \frac{\dot{m}_2^2}{4m_2^2}\right]\right)\dot{\theta}$$
$$+\cos(\theta)\left(\frac{\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]dt}{2\sqrt{m_1 m_2}}\right)\dot{\theta}$$
$$+\sin(\theta)\left[\frac{c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]}{2\sqrt{m_1 m_2}} - \frac{1}{4\sqrt[3]{m_1 m_2}}\left(\frac{\dot{m}_1}{m_1}+\frac{\dot{m}_2}{m_2}\right)\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]dt\right], \quad (2.109)$$

they have[53]

$$\tan(\theta) = \frac{1}{\sqrt{m_1 m_2}}\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]dt$$
$$\times\left[\left(\frac{1}{m_1}\int_0^t\left[c_1\frac{\dot{m}_1}{m_1}\right]dt - \frac{\dot{m}_1^2}{4m_1^2}\right) - \left(\frac{1}{m_2}\int_0^t\left[c_2\frac{\dot{m}_2}{m_2}\right]dt - \frac{\dot{m}_2^2}{4m_2^2}\right)\right]^{-1}, \quad (2.110)$$

from Eq(2.107),(2.108) and equation (2.110), the derivative of $\theta(t)$ is

$$\dot{\theta} = \cos^2\left(\frac{\theta(t)}{2}\right)\left[\frac{-1}{2\sqrt{m_1 m_2}}\left(\frac{\dot{m}_1}{m_1}+\frac{\dot{m}_2}{m_2}\right)\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]dt + \frac{\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]}{\sqrt{m_1 m_2}}\right]$$
$$\times\left[\left(\frac{1}{m_1}\int_0^t\left[c_1\frac{\dot{m}_1}{m_1}\right]dt - \frac{\dot{m}_1^2}{4m_1^2}\right) - \left(\frac{1}{m_2}\int_0^t\left[c_2\frac{\dot{m}_2}{m_2}\right]dt - \frac{\dot{m}_2^2}{4m_2^2}\right)\right] \neq 0, \quad (2.111)$$

which show that $\theta(t)$ is time-dependent.

Insering (2.111) in equation (2.109), we get

$$\frac{d\omega_1^2}{dt} = \cos^2\left(\frac{\theta}{2}\right)\left[\left(\frac{1}{m_1}\int_0^t\left[c_1\frac{\dot{m}_1}{m_1}\right]dt - \frac{\dot{m}_1^2}{4m_1^2}\right) - \left(\frac{1}{m_2}\int_0^t\left[c_2\frac{\dot{m}_2}{m_2}\right]dt - \frac{\dot{m}_2^2}{4m_2^2}\right)\right]$$
$$\times\left[\frac{-1}{2\sqrt{m_1 m_2}}\left(\frac{\dot{m}_1}{m_1}+\frac{\dot{m}_2}{m_2}\right)\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]dt + \frac{\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]}{\sqrt{m_1 m_2}}\right]$$
$$\times\left[\frac{\sin\theta}{2}\left(\left[\frac{\int_0^t\left[c_2\frac{\dot{m}_2}{m_2}\right]dt}{m_2} - \frac{\int_0^t\left[c_1\frac{\dot{m}_1}{m_1}\right]dt}{m_1}\right] - \left[\frac{\dot{m}_1^2}{4m_1^2} - \frac{\dot{m}_2^2}{4m_2^2}\right]\right) + \cos(\theta)\left(\frac{\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]dt}{2\sqrt{m_1 m_2}}\right)\right]$$
$$+\sin(\theta)\left[\frac{c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]}{2\sqrt{m_1 m_2}} - \frac{1}{4\sqrt[3]{m_1 m_2}}\left(\frac{\dot{m}_1}{m_1}+\frac{\dot{m}_2}{m_2}\right)\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1}+\frac{\dot{m}_2}{2m_2}\right]dt\right] \neq 0, \quad (2.112)$$

if the conditions ($c_1 = c_2$, $c_3 = 0$ and $m_1 = m_2$) are satisfied then the equation (2.112) vanishes.

Chapter 3

$\mathcal{PT}$-symmetry and pseudo-Hermitian quantum mechanics

$\mathcal{PT}$ -symmetry and pseudo-Hermiticity are two fundamental concepts in quantum mechanics, which show that non-Hermitian systems can have real energy spectra. The $\mathcal{PT}$ -symmetry quantum theory was principally developed by Bender and his collaborators [2, 3, 4, 5, 6, 7, 8], who showed that the spectrum of a one-dimensional non-Hermitian Hamiltonian is real, positive and discrete. The reality of this spectrum is a consequence of the $\mathcal{PT}$ -symmetry of the Hamiltonian.

In 2002, Mostafazadeh introduced the concept of pseudo-Hermiticity [9, 10, 11], in order to construct a mathematical relation with the notion of $\mathcal{PT}$-symmetry. He demonstrated that all $\mathcal{PT}$ -symmetric Hamiltonians are pseudo-Hermitian and also showed that any diagonalizable operator is said to be pseudo-Hermitian if its eigenvalues are real.

3.1 $\mathcal{PT}$-symmetry quantum mechanics

As we mentioned previously, $\mathcal{PT}$ -symmetry was first introduced by Bander et al [2, 3, 4, 5, 6, 7, 8]. Let us now briefly present the definitions and the properties of $\mathcal{PT}$ -symmetry.

Definitions and properties

A Hamiltonian H is said to be $\mathcal{PT}$-symmetry if it only satisfies the following relation

$$H = H^{\mathcal{PT}} \Rightarrow H = (\mathcal{PT})H(\mathcal{PT})^{-1}, \tag{3.1}$$

where $\mathcal{T}$ is the time reversal operator and $\mathcal{P}$ is the parity operator. The parity operator $\mathcal{P}$ is linear whereas the time reversal operator $\mathcal{T}$ is antilinear. Furthermore, the square of $\mathcal{P}$ and $\mathcal{T}$ is the identity operator $(\mathcal{PT})^2 = 1$, $\mathcal{P}^2 = \mathcal{T}^2 = 1$ but the two operators are not equal ($\mathcal{P} \neq \mathcal{T}$). The operators $\mathcal{P}$ and $\mathcal{T}$ effect the position operator x, the momentum operator p and the imaginary number i respectively as

$$\mathcal{P}\left\{x \to -x \quad , \quad p \to -p \quad , \quad i \to i\right\}, \tag{3.2}$$

$$\mathcal{T}\left\{x \to x \quad , \quad p \to -p \quad , \quad i \to -i\right\}, \tag{3.3}$$

also the operators $\mathcal{P}$ and $\mathcal{T}$ commute

$$[\mathcal{P}, \mathcal{T}] = 0. \tag{3.4}$$

If the eigenfunctions of the $\mathcal{PT}$-symmetric Hamiltonian H are also eigenfunctions of the $\mathcal{PT}$ operator, the $\mathcal{PT}$-symmetry is unbroken. The $\mathcal{PT}$-symmetry is broken if the eigenfunctions of the $\mathcal{PT}$-symmetric Hamiltonian are not eigenfunctions of the $\mathcal{PT}$ operator.

Therefore to construct a physical quantum theory from the $\mathcal{PT}$-symmetric Hamiltonians, it is necessary that the symmetry is unbroken. With this condition, we can prove the reality of the eigenvalues of a $\mathcal{PT}$-symmetric Hamiltonian.

We can write the eigenvalue equation of the Hamiltonian

$$H\left|\psi_n\right\rangle = E_n\left|\psi_n\right\rangle, \tag{3.5}$$

and the eigenvalue equation of the $\mathcal{PT}$ operator

$$\mathcal{PT}\left|\psi_n\right\rangle = \lambda_n\left|\psi_n\right\rangle, \tag{3.6}$$

where E_n and λ_n are the eigenvalues of H and $\mathcal{PT}$. We have

$$(\mathcal{PT})^2 = 1, \tag{3.7}$$

so

$$\left|\lambda_n\right|^2 = 1, \tag{3.8}$$

and λ_n is a phase which can be absorbed in the eigenfunction $\left|\psi_n\right\rangle$.

The equation (3.4) allows writing

$$\mathcal{PT}H\mathcal{PT}\,|\psi_n\rangle = |\lambda_n|^2\,E_n^*\,|\psi_n\rangle = E_n\,|\psi_n\rangle\,, \tag{3.9}$$

and therefore the eigenvalues E_n of the system are real and in this case the $\mathcal{PT}$-symmetry is unbroken.

Bender [5] introduced an inner product called "$\mathcal{PT}$-inner product" associated with the $\mathcal{PT}$-symmetry of the Hamiltonian, the $\mathcal{PT}$-inner product is defined as

$$\langle\psi_m\,|\psi_n\rangle_{\mathcal{PT}} = \int_c dx\,[\mathcal{PT}\psi_m(x)]\,\psi_n(x) = \int_c dx\,[\psi_m^*(-x)]\,\psi_n(x) = (-1)^n\delta_{mn}, \tag{3.10}$$

the norm of a state is not always positive, which does not verify the postulates of quantum mechanics, and there is a problem that must be solved. To solve this problem, Bender et al. [5] noticed that a $\mathcal{PT}$-symmetric Hamiltonian with unbroken $\mathcal{PT}$-symmetry possesses a hidden symmetry because there are an equal number of states of positive norm and negative norm. Therefore, it is necessary to construct a new inner product where the norm is positive.

3.2 The operator $\mathcal{C}$ and the $\mathcal{CPT}$ -inner product

In quantum mechanics, the norm of a state must be positive. To solve the problem of the negative norm. Bender et al [5] introduced a new lineair operator $\mathcal{C}$, called a charge conjugation operator with the eigenvalues ± 1 and $\mathcal{C}^2 = 1$. This operator $\mathcal{C}$ commutes with the operator $\mathcal{PT}$ and the Hamiltonian H

$$[\mathcal{C},\mathcal{PT}] = 0, \quad [\mathcal{C},H] = 0, \tag{3.11}$$

but not with $\mathcal{P}$ or $\mathcal{T}$ separately

$$[\mathcal{C},\mathcal{P}] \neq 0 \quad , \quad [\mathcal{C},\mathcal{T}] \neq 0. \tag{3.12}$$

The $\mathcal{CPT}$-inner product is defined as follows

$$\langle\varphi_n,\varphi_m\rangle = \int_c dx\,[\mathcal{CPT}\varphi_n(x)]\,\varphi_m(x), \tag{3.13}$$

where

$$\mathcal{CPT}\varphi_n(x) = \int dy\mathcal{C}(x,y)\varphi_n^*(-y), \tag{3.14}$$

so this $\mathcal{CPT}$-inner product is positive, and the eigenfunctions of H are orthonormal.

$$\langle \varphi_n, \varphi_m \rangle_{\mathcal{CPT}} = \int_c dx \left[\mathcal{CPT} \varphi_n(x) \right] \varphi_m(x) = \delta_{mn}. \tag{3.15}$$

3.3 Pseudo-Hermitian Quantum Mechanics

The notion of pseudo-Hermiticity was first introduced by Dirac and Pauli [56, 57, 58, 59], later by Lee and Sudarshan [60, 61], who wanted to solve various problems in several fields of physics, arising from the quantization electrodynamics and other quantum field theories.

Later in 2002, Mostafazadah [9, 10, 11] re-introduced the pseudo-hermiticity, when he showed that all $\mathcal{PT}$-symmetric Hamiltonians are pseudo-Hermitian and also demonstrated that any Hamiltonian with a real spectrum is pseudo-Hermitian. Therefore, the pseudo-Hermitian quantum mechanics is a more general theory than $\mathcal{PT}$-symmetry quantum mechanics. As mentionned in the introduction, an operator H is said to be pseudo-Hermitian if

$$H^{+} = \eta H \eta^{-1}, \tag{3.16}$$

where H^{+} is the adjoint Hamiltonian of H and the metric η is operator that is linear, invertible and Hermitian

$$\eta = \rho^{+} \rho, \ \eta^{-1} = \rho^{-1} \left(\rho^{+}\right)^{-1}, \tag{3.17}$$

The Hamiltonian operator H and its adjoint H^{+} verify the following eigenvalue equations

$$H \left| \psi_n \right\rangle = E_n \left| \psi_n \right\rangle, \tag{3.18}$$

$$H^{+} \left| \phi_n \right\rangle = E_n \left| \phi_n \right\rangle, \tag{3.19}$$

where E_n is real energy of H and H^{+}, the eigenvectors $\{ \left| \psi_n \right\rangle, \left| \phi_n \right\rangle \}$ form a bi-orthonormal basis [10, 62, 63]

$$\left\langle \phi_m \right| \psi_n \rangle = \delta_{mn} \ , \tag{3.20}$$

and verifies the closure relation

$$\sum_n \left| \psi_n \right\rangle \left\langle \phi_n \right| = \sum_n \left| \phi_n \right\rangle \left\langle \psi_n \right| = 1, \tag{3.21}$$

so that H and H^+ can be writen in the following spectral representation

$$H = \sum_n E_n |\psi_n\rangle \langle\phi_n| \quad , \quad H^+ = \sum_n E_n |\phi_n\rangle \langle\psi_n| \,. \tag{3.22}$$

In the case of non-degenerate eigenvalues, the pseudo-metric operator η and its inverse η^{-1} take the form

$$\eta = \sum_n |\phi_n\rangle \langle\phi_n| \quad , \quad \eta^{-1} = \sum_n |\psi_n\rangle \langle\psi_n| \,. \tag{3.23}$$

The eigenvalues of the hermitian Hamiltonian h

$$h |\chi_n\rangle = E_n |\chi_n\rangle \,. \tag{3.24}$$

From equation (2.88), we can determine the eigenfunctions of h and the eigenfunctions of H that are related to each other as

$$|\chi_n\rangle = \rho |\psi_n\rangle \,, \tag{3.25}$$

knowing that the eigenvectors $|\chi_n\rangle$ form an orthonormal basis, i.e. preserve the inner-product definition

$$\langle\chi_m |\chi_n\rangle = \delta_{mn}, \tag{3.26}$$

and substituing the equation (3.25) in (3.26), we obtain

$$\langle\psi_m| \rho^+\rho |\psi_n\rangle = \langle\psi_m| \eta |\psi_n\rangle = \langle\psi_m| \psi_n\rangle_\eta = \delta_{mn}, \tag{3.27}$$

is called pseudo-inner product or η-inner product.

Chapter 4

Coherent states for the harmonic oscillator and for the inverted oscillator

Coherent states have a significant role in numerous areas of physics including quantum optics, nuclear, atomic physics, and solid state. In 1926 Schrödinger introduced the coherent states for the Harmonic oscillator[12], which demonstrated that quantum expectation values of the position $x_c(t)$ and momentum $p_c(t)$ operators of a harmonic oscillator evolve over time in the same way as their classical analogues. Later in the early 1960s the coherent states were discovered by Glauber, Klauder and Sudarshan [69, 70, 71].

4.1 Coherent states of the harmonic oscillator $|\alpha\rangle^{os}$

The coherent states of the harmonic oscillator was introduced by Glauber and defined in three equivalent approaches [12, 54, 69, 70, 72, 73, 74]

First approach

Coherent states $|\alpha\rangle^{os}$ are eigenstates of the annihilation operator a

$$a\,|\alpha\rangle^{os} = \alpha\,|\alpha\rangle^{os}\,, \tag{4.1}$$

where α is a complexe number, the expression for the normalized coherent state in Fock space is given explicitly by

$$|\alpha\rangle^{os} = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle^{os}\,. \tag{4.2}$$

Second approach

Coherent states $|\alpha\rangle^{os}$ are defined by the action of the displacement operator $D^{os}(\alpha)$ on the vacuum state $|0\rangle^{os}$

$$D^{os}(\alpha)|0\rangle^{os} = |\alpha\rangle^{os}, \tag{4.3}$$

where the displacement operator $D^{os}(\alpha)$ is given by the following expression

$$D^{os}(\alpha) = \exp\left(\alpha\, a^{+} - \alpha^{*} a\right). \tag{4.4}$$

The operator $D^{os}(\alpha)$ is unitary

$$D^{+os}(\alpha) = D^{os}(-\alpha) = \left[D^{os}(\alpha)\right]^{-1}, \tag{4.5}$$

$$D^{+os}(\alpha)\, D^{os}(\alpha) = D^{os}(\alpha)\, D^{+os}(\alpha) = 1. \tag{4.6}$$

Therefore, the action of the operator $D^{os}(\alpha)$ on a and a^{+} leads to

$$D^{+os}(\alpha)\, a D^{os}(\alpha) = a + \alpha, \tag{4.7}$$

$$D^{+os}(\alpha)\, a^{+} D^{os}(\alpha) = a^{+} + \alpha^{*}, \tag{4.8}$$

we can also show that

$$D^{os}(\alpha+\beta) = D^{os}(\alpha)\, D^{os}(\beta)\, e^{\mathbf{i}\,\mathrm{Im}\,\alpha\beta}. \tag{4.9}$$

Using the Baker-Hausdorff formula,

$$e^{A+B} = e^{[A,B]/2} e^{B} e^{A}, \tag{4.10}$$

where the both operators A and B commute with $[A, B]$, the operator $D^{os}(\alpha)$ can be expressed in the following two forms

$$D^{os}(\alpha) = e^{\frac{-|\alpha|^2}{2}} e^{\alpha\, a^{+}} e^{-\alpha^{*} a}, \tag{4.11}$$

and

$$D^{os}(\alpha) = e^{\frac{|\alpha|^2}{2}} e^{-\alpha^{*} a} e^{\alpha\, a^{+}}. \tag{4.12}$$

Third approach

The coherent states are defined as states that minimize the Heisenberg uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$, where the expressions of Δx and Δp are give by

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}, \tag{4.13}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}, \tag{4.14}$$

the expection values of x and p can be evaluated from

$$\langle x \rangle = \langle \alpha | \, x \, | \alpha \rangle = \sqrt{\frac{\hbar}{2m\omega}} \, \mathrm{Re}\, \alpha, \tag{4.15}$$

$$\langle p \rangle = \langle \alpha | \, p \, | \alpha \rangle = \sqrt{\frac{\hbar m\omega}{2}} \, \mathrm{Im}\, \alpha, \tag{4.16}$$

therefore, Δx and Δp are expressed as

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}, \tag{4.17}$$

$$\Delta p = \sqrt{\frac{\hbar m\omega}{2}}, \tag{4.18}$$

which shows that these states minimize the Heisenberg uncertainty principle

$$\Delta x \Delta p = \frac{\hbar}{2}. \tag{4.19}$$

4.2 Properties and time evolution of the harmonic oscillator coherent states

Let us briefly introduce the properties of the coherent states $|\alpha\rangle^{os}$

1. Coherent states are not orthogonal between them

$$^{os}\langle \beta | \, \alpha \rangle^{os} = \langle 0 | \, D^{+os}(\beta) \, D^{os}(\alpha) \, | 0 \rangle^{os}, \tag{4.20}$$

   using the equation (4.2), we obtain

$$^{os}\langle \beta | \, \alpha \rangle^{os} = e^{-\frac{|\beta|^2}{2} - \frac{|\alpha|^2}{2} + \beta^* \alpha}, \tag{4.21}$$

   which means the squared modulus $^{os}\langle \beta \, | \alpha \rangle^{os}$ represents the measurement of the distance between the two coherent states.

2. The coherent state $|\alpha\rangle^{os}$ is normalized when $\alpha = \beta$ i.e.

$$^{os}\langle \alpha | \, \alpha \rangle^{os} = I. \tag{4.22}$$

3. The coherent states form an over-complet set of states. The identity operator I is written in terms of coherent states as

$$\frac{1}{\pi}\int |\alpha\rangle^{os}\ {}^{os}\langle\alpha|\, d^2\alpha = I, \tag{4.23}$$

to prove this identity, we put

$$\alpha = re^{i\theta} \text{ and } d^2\alpha = rdrd\theta, \tag{4.24}$$

and using the formula (4.2), we obtain

$$\frac{1}{\pi}\int |\alpha\rangle^{os}\ {}^{os}\langle\alpha|\, d^2\alpha = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty} rdr \int_0^{2\pi}\frac{dr}{\pi}\frac{r^{n+m}}{\sqrt{n!m!}}e^{i(n-m)r}e^{-r^2}\,|n\rangle^{os}\ {}^{os}\langle m|\,, \tag{4.25}$$

a change of variable $r^2 = \bar{u}$, using $\int_0^{2\pi} dre^{i(n-m)r} = 2\pi\delta_{nm}$ and $\int_0^{\infty} d\bar{u}e^{-\bar{u}}\bar{u}^n = n!$, the integral takes the form

$$\frac{1}{\pi}\int |\alpha\rangle^{os}\ {}^{os}\langle\alpha|\, d^2\alpha = \sum_{n=0}^{\infty}\frac{|n\rangle^{os}\ {}^{os}\langle n|}{n!}\int_0^{\infty} d\bar{u}e^{-\bar{u}}\bar{u}^n = I. \tag{4.26}$$

4. The expectation values $\langle x\rangle$, $\langle p\rangle$ and $\langle H^{os}\rangle$ in the states $|\alpha\rangle^{os}$ remain equal to their corrosponding classical quantities.

The evolution of an initial state ($|\psi(0)\rangle^{os} = |n\rangle^{os}$) is given by

$$|\psi(t)\rangle^{os} = U^{os}(t)\,|\psi(0)\rangle^{os} = U^{os}(t)\,|n\rangle^{os} = e^{-i(n+\frac{1}{2})\omega t}\,|n\rangle^{os}\,, \tag{4.27}$$

where $U^{os}(t)$ is the operator of evolution.

The evolution of a coherent state will be given as

$$\begin{aligned} |\alpha(t),t\rangle^{os} &= U^{os}(t)\,|\alpha(0)\rangle^{os} = e^{-\frac{i}{2}\omega t}e^{\frac{-|\alpha(0)|^2}{2}}\sum_{n=0}^{\infty}\frac{\left[\alpha(0)e^{-i\omega t}\right]^n}{\sqrt{n!}}\,|n\rangle^{os} \\ &= e^{-\frac{i}{2}\omega t}\,|\alpha(t)\rangle^{os}\,, \end{aligned} \tag{4.28}$$

with $\alpha(t) = \alpha(0)e^{-i\omega t}$, we multiply the obtained ket by $e^{-\frac{i}{2}\omega t}$ and change $\alpha \to \alpha(0)e^{-i\omega t}$ to go from the state $|\alpha\rangle^{os}$ to its evolved state $|\alpha(t),t\rangle^{os}$ [75].

4.3 The standard harmonic and the inverted oscillators

Consider the ladder operator of the harmonic oscillator

$$H^{os} = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 = \frac{\hbar\omega}{2}\left(a^+ a + a a^+\right), \tag{4.29}$$

where

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\hbar\omega}} \quad , \quad a^+ = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\hbar\omega}}, \tag{4.30}$$

the operators a and a^+ satisfying

$$\left[a, a^+\right] = 1, \tag{4.31}$$

eigenstates of (4.29) in Fock space are the Fock (or number) states $|n\rangle^{os}$ and the eigenvalues $E_n = \omega\left(n + 1/2\right)), $ whith $a\,|n\rangle^{os} = \sqrt{n}\,|n-1\rangle^{os}$, $a^+\,|n\rangle^{os} = \sqrt{n+1}\,|n+1\rangle^{os}$ and $n \in \mathbb{N}$.

We know that the representation of x and p are

$$x = \sqrt{\frac{\hbar}{2\omega m}}\left(a^+ + a\right) \quad , \quad p = i\sqrt{\frac{\hbar\omega m}{2}}\left(a^+ - a\right), \tag{4.32}$$

by understanding the effects of the annihilation and creation operators on the eigenstates of the Hamiltonian, we can compute any expectation values that depend on these quantities.

It is easy to evaluate the energy eigenvalues

$$H^{os}\psi_n^{os}(x) = E_n\psi_n^{os}(x) = \hbar\omega\left(n + \frac{1}{2}\right)\psi_n^{os}(x); \qquad n \in \mathbb{N}, \tag{4.33}$$

and the normalized condition are

$$\langle\psi_m^{os}\,|\psi_n^{os}\rangle = \delta_{mn}. \tag{4.34}$$

The vacuum state of harmonic oscillator is

$$\psi_0^{os}(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\omega m}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-\frac{\omega m}{2\hbar}x^2\right], \tag{4.35}$$

is a very important physics result since it shows us that the energy of a system cannot have zero when it is described by a harmonic oscillator potential.

The inverted oscillator is given by

$$H^r = \frac{1}{2m}p^2 - \frac{1}{2}m\omega^2 x^2 = -\frac{\hbar\omega}{2}\left(a^{+2} + a^2\right), \tag{4.36}$$

which is formally obtainable from (4.29) by the replacement

$$\omega \longrightarrow i\omega, \tag{4.37}$$

in a similar manner, the case $(-i\omega)$ would be useful.

Replacing (4.37) in Eq (4.30) , it gives

$$a \rightarrow A = e^{i\frac{\pi}{4}}(\sqrt{\frac{m\omega}{2\hbar}}x + \frac{p}{\sqrt{2m\omega\hbar}}), \tag{4.38}$$

$$a^{+} \rightarrow \bar{A} = e^{i\frac{\pi}{4}}(\sqrt{\frac{m\omega}{2\hbar}}x - \frac{p}{\sqrt{2m\omega\hbar}}), \tag{4.39}$$

where $\left(A, \bar{A}\right)$ are the pseudo-annihilation and creation operators respectively.

Consequently, the Hamiltonian (4.36) takes the following form

$$H^{r} = \frac{i\hbar\omega}{2}(\bar{A}A + A\bar{A}), \tag{4.40}$$

where the non-Hermitian pseudo-ladder operators $\left(A, \bar{A}\right)$ satisfy the commutation relation $\left[A, \bar{A}\right] = 1$.

Alike the harmonic oscillator, does the inverted oscillator have normalized eigenfunctions as well?. Clearly $\langle\psi_m^r\,|\psi_n^r\rangle \neq \delta_{mn}$ which is easily seen when replacing ω to $i\omega$ in the expression from Eq. (4.35)

$$\psi_0^r(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{i\omega m}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-i\frac{\omega m}{2\hbar}x^2\right]. \tag{4.41}$$

It can be easily verified that the normalization for the pseudo-ground state $\psi_0^r(x)$ diverges as follows

$$\begin{aligned} \langle\psi_0^r\,|\psi_0^r\rangle &= \int_{-\infty}^{+\infty} \psi_0^{*r}(x)\psi_0^r(x)dx \\ &= \frac{1}{2^n n!}\left(\frac{\omega m}{\pi\hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} dx \rightarrow \infty, \end{aligned} \tag{4.42}$$

replacing ω by $i\omega$ is unsuitable, which explains the divergence. We will address this problem in what follows.

4.4 Pseudo-ladder operators in the inverted oscillator

The difficulty lies in establishing a consistent relation between the inverted oscillator H^r and the non-Hermitian Hamiltonian (iH^{os}) in a quantum mechanical framework. Replacing of ω

by $i\omega$ allowed the connection of the the non-Hermitian Hamiltonian (iH^{os}) with the Hermitian Hamiltonian H^r, which is given by

$$\rho^{-1}(iH^{os})\rho = H^r. \tag{4.43}$$

To link the inverted oscillator H^r with the non-Hermitian Hamiltonian (iH^{os}), we use a Dyson operator ρ [76]

$$\begin{aligned}\rho &= \exp\left\{-2\left[\frac{\epsilon}{2}\left(a^+a+\frac{1}{2}\right)+\mu_-\frac{a^2}{2}+\mu_+\frac{a^+}{2}\right]\right\},\\ &= \exp\left[-\vartheta_-\frac{a^2}{2}\right]\exp\left[-\frac{\ln\vartheta_0}{2}\left(a^+a+\frac{1}{2}\right)\right]\exp\left[-\vartheta_+\frac{a^+}{2}\right],\end{aligned} \tag{4.44}$$

where

$$\begin{aligned}\vartheta_+ &= \frac{2\mu_+\sinh\theta}{\theta\cosh\theta-\epsilon\sinh\theta},\\ \vartheta_0 &= \left(\cosh\theta-\frac{\epsilon}{\theta}\sinh\theta\right)^{-2} = \mu_+\mu_- - \chi,\\ \vartheta_- &= \frac{2\mu_-\sinh\theta}{\theta\cosh\theta-\epsilon\sinh\theta},\\ \chi &= -\frac{\cosh\theta+\frac{\epsilon}{\theta}\sinh\theta}{\cosh\theta-\frac{\epsilon}{\theta}\sinh\theta}\quad,\quad \theta=\sqrt{\epsilon^2-4\mu_+\mu_-},\end{aligned} \tag{4.45}$$

and ϵ is a real but μ_+ and μ_- are parameter complex.

Using (2.21), we can easily find the following transformations

$$\left\{\begin{array}{c}\exp\left[\vartheta_-\frac{a^2}{2}\right]\left(a^+a+\frac{1}{2}\right)\exp\left[-\vartheta_-\frac{a^2}{2}\right]=\left(a^+a+\frac{1}{2}\right)+\vartheta_-a^2\\ \exp\left[\vartheta_+\frac{a^+}{2}\right]\left(a^+a+\frac{1}{2}\right)\exp\left[-\vartheta_+\frac{a^+}{2}\right]=\left(a^+a+\frac{1}{2}\right)-\vartheta_+a^+\end{array}\right., \tag{4.46}$$

$$\left\{\begin{array}{c}\exp\left[\frac{\ln\vartheta_0}{2}\left(a^+a+\frac{1}{2}\right)\right]a^2\exp\left[-\frac{\ln\vartheta_0}{2}\left(a^+a+\frac{1}{2}\right)\right]=\frac{a^2}{\vartheta_0}\\ \exp\left[\vartheta_+\frac{a^+}{2}\right]a^2\exp\left[-\vartheta_+\frac{a^+}{2}\right]=a^2-2\vartheta_+\left(a^+a+\frac{1}{2}\right)+\vartheta_+^2a^+\end{array}\right., \tag{4.47}$$

$$\left\{\begin{array}{c}\exp\left[\frac{\ln\vartheta_0}{2}\left(a^+a+\frac{1}{2}\right)\right]a^+\exp\left[-\frac{\ln\vartheta_0}{2}\left(a^+a+\frac{1}{2}\right)\right]=\vartheta_0a^+\\ \exp\left[\vartheta_-\frac{a^2}{2}\right]a^+\exp\left[-\vartheta_-\frac{a^2}{2}\right]=a^++2\vartheta_-\left(a^+a+\frac{1}{2}\right)+\vartheta_-^2a^2\end{array}\right., \tag{4.48}$$

by applying the operator ρ to the harmonic oscillator H^{os}, we find

$$\begin{aligned}\rho^{-1}H^{os}\rho &= \hbar\omega\rho^{-1}\left(a^+a+\frac{1}{2}\right)\rho\\ &= \frac{\hbar\omega}{\vartheta_0}\left\{\left[\vartheta_0-2\vartheta_+\vartheta_-\right]\left(a^+a+\frac{1}{2}\right)+\left[\vartheta_-\vartheta_+^2-\vartheta_0\vartheta_+\right]a^++\vartheta_-a^2\right\}.\end{aligned} \tag{4.49}$$

If we propose the parameters $(\vartheta_+, \vartheta_-, \ \vartheta_0)$ as

$$\vartheta_+ = -i, \ \vartheta_- = \frac{i}{2}, \ \vartheta_0 = 1, \tag{4.50}$$

the Eq.(4.49) and Eq.(4.36) have the same relation.

Then, the simplified form of the Dyson operator Eq.(4.44) is

$$\begin{aligned} \rho &= \exp[-\frac{i}{4}a^2]\exp[\frac{i}{2}a^+], \\ \rho^{-1} &= \exp[-\frac{i}{2}a^+]\exp[\frac{i}{4}a^2], \end{aligned} \tag{4.51}$$

the two Hamiltonians H^{os} and H^r are connected to each other as

$$\begin{aligned} \rho^{-1}H^{os}\rho &= \exp[-\frac{i}{2}a^+]\exp[\frac{i}{4}a^2]\,(H^{os})\exp[-\frac{i}{4}a^2]\exp[\frac{i}{2}a^+] \\ &= \frac{i\hbar\omega}{2}\left(a^+ + a^2\right) \\ &= \frac{i\hbar\omega}{2}(\bar{A}A + A\bar{A}) = -iH^r, \end{aligned} \tag{4.52}$$

and the pseudo-ladder operators $\left(A, \bar{A}\right)$ are linked to the ladder operators (4.30). The transformation of a and a^+ are given by

$$\begin{aligned} A &= \rho^{-1}a\rho \\ &= \exp[-\frac{i}{2}a^+]\exp[\frac{i}{4}a^2]a\exp[-\frac{i}{4}a^2]\exp[\frac{i}{2}a^+] \\ &= \frac{1}{2}\left(a + ia^+\right), \end{aligned} \tag{4.53}$$

and

$$\begin{aligned} \bar{A} &= \rho^{-1}a^+\rho \\ &= \exp[-\frac{i}{2}a^+]\exp[\frac{i}{4}a^2]a^+\exp[-\frac{i}{4}a^2]\exp[\frac{i}{2}a^+] \\ &= \frac{1}{2}\left(a^+ + ia\right), \end{aligned} \tag{4.54}$$

in Fock space, we know that

$$a\left|n\right\rangle^{os} = \sqrt{n}\left|n-1\right\rangle^{os}, \tag{4.55}$$

$$a^+\left|n\right\rangle^{os} = \sqrt{n+1}\left|n+1\right\rangle^{os}. \tag{4.56}$$

By multiplying the above equations from the left by ρ^{-1}, noting that $\rho\rho^{-1} = 1$, we find

$$\rho^{-1}a\left(\rho\rho^{-1}\right)|n\rangle^{os} = \sqrt{n}\rho^{-1}|n-1\rangle^{os}, \tag{4.57}$$

$$\rho^{-1}a^{+}\left(\rho\rho^{-1}\right)|n\rangle^{os} = \sqrt{n+1}\rho^{-1}|n+1\rangle^{os}, \tag{4.58}$$

thus, we can write the following expressions

$$A|n\rangle^{r} = \sqrt{n}|n-1\rangle^{r}, \tag{4.59}$$

$$\bar{A}|n\rangle^{r} = \sqrt{n+1}|n+1\rangle^{r}, \tag{4.60}$$

we can conclude that the eigenstates $|n\rangle^{r}$ are linked to $|n\rangle^{os}$ by the operator ρ as

$$|n\rangle^{r} = \rho^{-1}|n\rangle^{os}. \tag{4.61}$$

The momentum operators (x, p) are related to the ladder operators of the pseudo-Hermitian quadratures (X, P) as follows

$$\begin{aligned} X &= \rho^{-1}x\rho = \sqrt{\frac{\hbar}{2\omega m}}\rho^{-1}\left(a^{+}+a\right)\rho \\ &= \sqrt{\frac{\hbar}{2\omega m}}\left(A+\bar{A}\right), \end{aligned} \tag{4.62}$$

and

$$\begin{aligned} P &= \rho^{-1}p\rho = i\sqrt{\frac{\hbar\omega m}{2}}\rho^{-1}\left(a^{+}-a\right)\rho \\ &= i\sqrt{\frac{\hbar\omega m}{2}}\left(\bar{A}-A\right). \end{aligned} \tag{4.63}$$

Every observable o in the Hermitian system has an equivalent O in the pseudo-Hermitian system provided by

$$O = \rho^{-1}o\rho, \tag{4.64}$$

the expressions of $\left(A, \bar{A}\right)$ are

$$A = \sqrt{\frac{m\omega}{2\hbar}}X + i\frac{1}{\sqrt{2m\hbar\omega}}P, \tag{4.65}$$

$$\bar{A} = \sqrt{\frac{m\omega}{2\hbar}}X - i\frac{1}{\sqrt{2m\hbar\omega}}P. \tag{4.66}$$

We can write the Hamiltonian (4.40) in terms of X and P by

$$H^r = \frac{i}{2}(\frac{P^2}{m} + m\omega^2 X^2). \tag{4.67}$$

Using the commutation relation $[X, P] = i\hbar$, and the Heisenberg equations of motion, we get

$$\begin{aligned} \frac{dX}{dt} &= \frac{1}{i\hbar}\left[X, \frac{i}{2}(\frac{P^2}{m} + m\omega^2 X^2)\right] = i\frac{P}{m}, \\ \frac{dP}{dt} &= \frac{1}{i\hbar}\left[P, \frac{i}{2}(\frac{P^2}{m} + m\omega^2 X^2)\right] = -im\omega^2 X, \end{aligned} \tag{4.68}$$

with the help of the second derivative, we obtain the equation of motion for the inverted oscillator

$$\frac{d^2X}{dt^2} - \omega^2 X = 0. \tag{4.69}$$

4.5 Coherent states and time evolution for the inverted oscillator

We use the pseudo-annihilation $A = \rho^{-1} a \rho$, pseudo-creation operators $\bar{A} = \rho^{-1} a^+ \rho$ and the metric operator $\eta = \rho^+ \rho$ such as $(iH^{os})^+ = \eta\,(iH^{os})\,\eta^{-1}$, i.e. (iH^{os}) is η-pseudo-Hermitian with respect to a positive-definite inner product defined by $\langle .,. \rangle_\eta = \langle .|\eta|. \rangle$

$${}^r \langle n|\, \eta \,|m\rangle^r = {}^{os} \langle n\, |m\rangle^{os} = \delta_{mn}, \tag{4.70}$$

the Fock states of the harmonic oscillator $|n\rangle^{os}$ are connected to the states of the inverted oscillator $|n\rangle^r$ by invertible Dyson operator (Eq4.61).

In addition, the ground state of the inverted oscillator $|0\rangle^r$ $(A\,|0\rangle^r = 0)$.

The vacuum state of the harmonic oscillator $|0\rangle^{os}$ are related to each other as

$$|0\rangle^r = \rho^{-1}\,|0\rangle^{os}.$$

Now, we introduce the coherent states for the inverted oscillator form the coherent states of the harmonic oscillator, by using equation (4.1) and multiplying it from the left by ρ^{-1}, we find

$$\rho^{-1}a\left(\rho\rho^{-1}\right)|\alpha\rangle^{os} = \alpha\rho^{-1}|\alpha\rangle^{os}, \qquad \alpha\in\mathbb{C}, \tag{4.71}$$

the coherent states for the inverted harmonic oscillator in this case is

$$A|\alpha\rangle^{r} = \alpha|\alpha\rangle^{r}, \qquad \alpha\in\mathbb{C}, \tag{4.72}$$

where

$$|\alpha\rangle^{r} = \rho^{-1}|\alpha\rangle^{os}. \tag{4.73}$$

Then, the coherent states for the inverted harmonic oscillator are defined as eigenstates of the pseudo-annihilation operator A.

The normalization condition

$$^{os}\langle\alpha\,|\alpha\rangle^{os} = 1, \tag{4.74}$$

leads to

$$^{r}\langle\alpha|\,\eta\,|\alpha\rangle^{r} = 1, \tag{4.75}$$

and then the integral

$$\frac{1}{\pi}\int_{\mathbb{C}}\rho|\alpha\rangle^{r}\ {}^{r}\langle\alpha|\,\rho^{+}d\alpha^{*}d\alpha = I, \tag{4.76}$$

is the identity operator.

From the coherent states of the oscillator harmonic, we can express the inverted coherent states $|\alpha\rangle^{r}$ in terms of the pseudo-displacement operator $D^{r}(\alpha)$, by multiplying the equation (4.3) on the left by ρ^{-1}, we obtain

$$\begin{aligned}\rho^{-1}|0\rangle^{os} &= \rho^{-1}D^{os}(\alpha)\left(\rho\rho^{-1}\right)|0\rangle^{os}\\ &= \left(\rho^{-1}\exp\left[\alpha a^{+}-\alpha^{*}a\right]\rho\right)\rho^{-1}|0\rangle^{os},\end{aligned} \tag{4.77}$$

in this case, we deduce that the coherent states of the oscillator inverted $|\alpha\rangle^{r}$ can also be generated from by applying a pseudo-displacement $D^{r}(\alpha)$ to the ground states $|0\rangle^{r}$ as follows

$$|\alpha\rangle^{r} = D^{r}(\alpha)|0\rangle^{r} = \exp\left[\alpha\overline{A}-\alpha^{*}A\right]|0\rangle^{r}, \tag{4.78}$$

noting $D^r(\alpha)$ is related to $D^{os}(\alpha)$ as

$$\begin{aligned} D^r(\alpha) &= \rho^{-1} D^{os}(\alpha)\,\rho \\ &= \rho^{-1}\exp\left[\alpha a^+ - \alpha^* a\right]\rho \\ &= \exp\left[\alpha \overline{A} - \alpha^* A\right]. \end{aligned} \tag{4.79}$$

The time evolution of an initial inverted coherent state is given by the action of the evolution operator on the coherent state $|\alpha\rangle^r$

$$\begin{aligned} |\alpha, t\rangle^r &= U^r(t)\,|\alpha\rangle^r \\ &= e^{-\frac{i}{\hbar}H^r t}\,|\alpha\rangle^r, \end{aligned} \tag{4.80}$$

where $U^r(t)$ is the evolution operator.

By the Hamiltonian (4.40), we conclude the evolution for inverted coherent state is

$$\begin{aligned} |\alpha, t\rangle^r &= e^{-\frac{i}{\hbar}H^r t}\,|\alpha\rangle^r \\ &= e^{-\frac{|\alpha|^2}{2}} e^{\frac{\omega t}{2}} e^{\omega \bar{A} A t} \sum_n \frac{(\alpha)^n}{\sqrt{n!}}\,|n\rangle^r. \end{aligned} \tag{4.81}$$

So, the coherent state is written as

$$\begin{aligned} |\alpha, t\rangle^r &= e^{-\frac{\left|\alpha e^{\omega t}\right|^2}{2}} \sum_n \frac{(\alpha e^{\omega t})^n}{\sqrt{n!}}\,|n\rangle^r \\ &= e^{\frac{\omega t}{2}}\left|\alpha e^{\omega t}\right\rangle^r. \end{aligned} \tag{4.82}$$

In order to calculate the product $\Delta X \Delta P$, let's first find the quantities $\langle X\rangle_\eta$, $\langle P\rangle_\eta$, $\langle X^2\rangle_\eta$, $\langle P^2\rangle_\eta$ using (4.64) in the non-Hermitian system. The mean values of the canonical operator O can be calculated as

$$\begin{aligned} \langle O\rangle_\eta &= {}^r\langle \alpha, t|\,\eta O\,|\alpha, t\rangle^r \\ &= {}^r\langle \alpha, t|\,\rho^+ o \rho\;|\alpha, t\rangle^r \\ &= e^{-\frac{\left|\alpha e^{\omega t}\right|^2}{2}} \sum_n \sum_m \frac{(\alpha^* e^{\omega t})^m}{\sqrt{m!}} \frac{(\alpha e^{\omega t})^n}{\sqrt{n!}}\;{}^{os}\langle m|\,o\,|n\rangle^{os}, \end{aligned} \tag{4.83}$$

where $O = X, X^2, P$ and P^2.

Using (4.83), the mean value of X in the state $|\alpha, t\rangle^r$ is

$$\begin{aligned}\langle X\rangle_\eta &= e^{-|\alpha e^{\omega t}|^2}\sum_n\sum_m \frac{(\alpha^* e^{\omega t})^m}{\sqrt{m!}}\frac{(\alpha e^{\omega t})^n}{\sqrt{n!}}\sqrt{\frac{\hbar}{2\omega m}}\,{}^{os}\langle m|\left(a^+ + a\right)|n\rangle^{os} \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left[\alpha+\alpha^*\right]e^{\omega t}, \qquad (4.84)\end{aligned}$$

and the mean value of P in the state $|\alpha, t\rangle^r$ is easily evaluated

$$\begin{aligned}\langle P\rangle_\eta &= e^{-|\alpha e^{\omega t}|^2}\sum_n\sum_m \frac{(\alpha^* e^{\omega t})^m}{\sqrt{m!}}\frac{(\alpha e^{\omega t})^n}{\sqrt{n!}}i\sqrt{\frac{\hbar\omega m}{2}}\,{}^{os}\langle m|\left(a^+ - a\right)|n\rangle^{os} \\ &= -i\sqrt{\frac{m\omega\hbar}{2}}\left[\alpha-\alpha^*\right]e^{\omega t}, \qquad (4.85)\end{aligned}$$

and follow classical physics; i.e.

$$\langle X\rangle_\eta = x_c, \qquad \langle P\rangle_\eta = p_c, \qquad (4.86)$$

where the letter c stands for classical. We refer to these inverted coherent states as "quasi-classical states" because of this.

Let us now calculate the mean value of X^2 in the state $|\alpha, t\rangle^r$ is

$$\begin{aligned}\left\langle X^2\right\rangle_\eta &= \langle\alpha, t|\,\eta X^2\,|\alpha, t\rangle^r \\ &= \frac{\hbar}{2m\omega}\left[\alpha^2 e^{2\omega t} + \alpha^{*2}e^{2\omega t} + 2(|\alpha|^2 e^{2\omega t} + \frac{1}{2})\right], \qquad (4.87)\end{aligned}$$

and the mean value of P^2 in the state $|\alpha, t\rangle^r$ is

$$\begin{aligned}\left\langle P^2\right\rangle_\eta &= \langle\alpha, t|\,\eta P^2\,|\alpha, t\rangle^r \\ &= \frac{-im\omega\hbar}{2}\left[\alpha^2 e^{2\omega t} + \alpha^{*2}e^{2\omega t} - 2(|\alpha|^2 e^{2\omega t} + \frac{1}{2})\right]. \qquad (4.88)\end{aligned}$$

From Eqs. (4.13), (4.84) and (4.87), we find

$$\Delta X = \sqrt{\frac{\hbar}{2m\omega}}, \qquad (4.89)$$

by using Eqs.(4.13), (4.85) and (4.88), we obtain

$$\Delta P = \sqrt{\frac{m\omega\hbar}{2}}, \qquad (4.90)$$

therefore

$$\Delta X\Delta P = \frac{\hbar}{2}. \qquad (4.91)$$

The original inverted coherent state are a minimum uncertainty states, the time evolution of an initially inverted coherent state can be thought of as the quantum equivalent of a classical trajectory.

Conclusion

This thesis is centered on the following points :

- We have introduced the time-dependent invariant theory for the Hermition systems.

- We used the invariant theory to resolve the time-dependent coupled oscillator of a two dimensional (2D), by appling a unitaty tronformation that allowed us to obtain the eigenfunctions of the invariant operator, hence the exact solutions of the time-dependent Shrödinger equation. FinalIy, we have generalized the 2D system to a 3D coupled oscillator, where we have detected that the method of Hassoul et al [52, 53] contains many errors.

- We have recalled the concepts of $\mathcal{PT}$-symmetry, the pseudo-hermiticity, the $\mathcal{PT}$ and $\mathcal{CPT}$ inner-products with mention of the properties of each one of them.

- We have given the definitions of coherent states for a harmonic oscillator and presented their time evolution.

- We have introduced a method for relating a regular harmonic oscillator to an inverted oscillator, with the help of a time-independent Dyson metric that allowed us to present the pseudo-annihilation operators $A = \rho^{-1} a \rho$ and pseudo-creation $\bar{A} = \rho^{-1} a^{+} \rho$ and constructed coherent states for the inverted oscillator.

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Abstract

In this thesis, we presented the definition of the invariant theory in quantum mechanics that allows us to treat the problems of time-dependent systems and find the solution of the Schrödinger equation. Then, we studied the time-dependent coupled oscillator of a two dimensional (2D), pointing out the errors made by the method of Hassoul et al. Next, we introduced the concepts of PT-symmetry, the pseudo-hermiticity and CPT. Finally, we have constructed the coherent states for the inverted oscillator that minimize the quantum mechanical uncertainty between the position and the momentum.

Key words: PT-symmetry, the pseudo-hermiticity, the time dependent coupled oscillator, the inverted oscillator.

Résumé

Dans cette thèse, nous avons présenté la définition de la théorie des invariants en mécanique quantique qui nous permet de traiter les problèmes des systèmes dépendant du temps et de trouver la solution de l'équation de Schrödinger. Ensuite, nous avons étudié l'oscillateur couplé dépendant du temps d'un bidimensionnel (2D), en soulignant les erreurs commises par la méthode de Hassoul et al. Ensuite, nous avons introduit les notions de PT-symétrique, de pseudo-herméticité et de CPT. Enfin, nous avons construit les états cohérents pour l'oscillateur inversé qui minimisent l'incertitude mécanique quantique entre la position et la quantité de mouvement.

Mots clés : PT-symétrique, la pseudo-herméticité, l'oscillateur couplé dépendant du temps, l'oscillateur inversé.

ملخص

قدمنا في هذه الأطروحة ,تعريف نظرية الثابتة في ميكانيكا الكم التي تسمح لنا بمعالجة مشاكل الأنظمة المعتمدة على الزمن وإيجاد حل معادلة شرودنغر .ثم قمنا بدراسة المذبذب المزدوج المعتمد على الزمن للجسم ثنائي الأبعاد(2D)، مع الإشارة إلى الأخطاء التي وقعت بطريقة حسول وآخرين. بعد ذلك، قدمنا مفاهيم: تناظر شبه التكافؤ, الشبه الهرميتيكية و CPT . أخيرًا، قمنا ببناء الحالات المتماسكة للمذبذب المقلوب الذي يقلل من عدم اليقين في ميكانيكا الكم بين الموضع والزخم.

الكلمات المفتاحية: تناظر شبه التكافؤ, الشبه الهرميتيكية, المذبذب المزدوج المعتمد على الزمن، المذبذب المقلوب.

Inverted oscillator: pseudo hermiticity and coherent states

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It is known that the standard and the inverted harmonic oscillator are different. Replacing thus ω by $\pm i\omega$ in the regular oscillator is necessary going to give the inverted oscillator H^r. This replacement would lead to anti- $\mathcal{PT}$-symmetric harmonic oscillator Hamiltonian $(\mp iH^{os})$. The pseudo-hermiticity relation has been used here to relate the anti-$\mathcal{PT}$-symmetric harmonic Hamiltonian to the inverted oscillator. By using a simple algebra, we introduce the ladder operators describing the inverted harmonic oscillator to reproduce the analytical solutions.We construct the inverted coherent states which minimize the quantum mechanical uncertainty between the position and the momentum.
This paper is dedicated to the memory of Omar Djemli and Nouredinne Mebarki who died due to covid 19.

Keywords: Inverted harmonic oscillator; harmonic Hamiltonian.

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1. Introduction

The inverted oscillator, equipped with a potential exerting a repulsive force on a particle, has been widely studied [1-18]. Such system can be completely solved as the standard harmonic oscillator whose properties are well known.

However, the physics of the inverted harmonic oscillator is different, because its energy spectrum is continuous and its eigenstates are no longer square integrable.The inverted oscillator can be applied to various physical systems such as [1,19-21], the tunneling effects, the mechanism of matter-wave bright solitons, the cosmological model, and the quantum theory of measurement.

In fact, the predominant idea in the literature is that the inverted oscillator is obtainable from the harmonic oscillator by the replacement $\omega \rightarrow \pm i\omega$. Of course, in spite of many useful analogies, it is important to know that the two oscillators (harmonic and inverted) reveal different characteristics. In other words, the inverted oscillator generates a wave packet which are not square integrable and there is no zero-point energy. In comparison with the harmonic oscillator, the physical applications of the inverted harmonic oscillators are limited, since their Hamiltonian is parabolic and the eigenstates are scattering states. The analytic continuation of angular velocity $\omega \rightarrow \pm i\omega$ performs a transformation of a non-Hermitian harmonic oscillator $(\mp iH^{os})$ to inverted one H^r.

In general, non-Hermitian Hamiltonians have been used to describe several physical dissipative systems. Such Hamiltonians do not cause a legitimate probabilistic interpretation due to the shortage of the unitarity condition in their corresponding quantum description. In non-Hermitian quantum mechanics it, was found that the criteria for a quantum Hamiltonian to have a real spectrum is that it possesses an unbroken $\mathcal{PT}$ symmetry ($\mathcal{P}$ is the space-reflectio operator or parity operator, and $\mathcal{T}$ is the time-reversal operator) [22,23]. The concept of $\mathcal{PT}$-symmetry has found applications in several areas of physics. Once the non-Hermitian Hamiltonian H is invariant under the combined action of $\mathcal{PT}$ (*i.e.* H commutes with $\mathcal{PT}$) and its eigenvectors are also those of the $\mathcal{PT}$ operator, then the energy eigenvalues E of the system are real and in this case the $\mathcal{PT}$-symmetry is unbroken.

An alternative approach to explore the basic structure responsible for the reality of the spectrum of a non- Hermitian Hamiltonian is by the notion of the pseudo-hermiticity introduced in Ref. [24]. An operator H is said to be pseudo-Hermitian if

$$H^\dagger = \eta H \eta^{-1}, \tag{1}$$

where the metric operator

$$\eta = \rho^\dagger \rho,\ \eta^{-1} = \left(\rho^\dagger \rho\right)^{-1}, \tag{2}$$

is a linear, invertible and Hermitian operator, we say that the Hamiltonian is pseudo-Hermitian or quasi-Hermitian if it satisfie the relation (1).

The pseudo-Hermiticity allows to link the pseudo-Hermitian Hamiltonian H with an equivalent Hermitian Hamiltonian h

$$h = \rho H \rho^{-1}, \tag{3}$$

where the operator ρ called Dyson operator is linear and invertible. Due to the energy spectrum of $(\pm iH^{os})$ being completely imaginary, we notice that $(\mp iH^{os})$ is anti-$\mathcal{PT}$-symmetric *i.e.*

$$\mathcal{PT}(\pm iH^{os})\mathcal{PT} = (\mp iH^{os}). \tag{4}$$

We recall that a $\mathcal{PT}$-symmetric system can be transformed to an anti-$\mathcal{PT}$-symmetric one by replacing $H^{os} \rightarrow$

$(\pm iH^{os})$ [25-28], which changes the physical structure of the system. In other words, a Hamiltonian H is said to be anti-$\mathcal{PT}$-symmetric if it anticommutes with the $\mathcal{PT}$ operator $\{\mathcal{PT}, H\} = 0$. In analogy with the $\mathcal{PT}$-symmetric case, we call the anti-$\mathcal{PT}$-symmetry of Hamiltonian H unbroken if all of the eigenfunctions of H are eigenfunctions of $\mathcal{PT}$, *i.e.* when the energy spectrum of H is entirely imaginary $E = iE^*$ [29].

In this paper, we generate from the anti-$\mathcal{PT}$-symmetric Hamiltonian $(\pm iH^{os})$ an inverted Hermitian harmonic oscillator-type H^r and also its solution. In Sec. 2, we recall briefl some properties of the standard harmonic and inverted oscillators In Sec. 3, introducing an appropriate quantum metric, we link the anti-$\mathcal{PT}$-symmetric Hamiltonian $(\pm iH^{os})$ to the inverted oscillator Hamiltonian H^r.This procedure allows us to obtain the pseudo-ladder operators, the set of solutions and also to defin the full orthonormalization relation of the eigenstates for inverted harmonic oscillator H^r. In Sec. 4, using the pseudo-ladder operators, we will address the problem constructing of coherent states associated to inverted oscillator H^r. We obtain the mean values of the position and momentum operators in the evolved coherent states and furthermore we calculate the corresponding Heisenberg uncertainty. An outlook over the main results is given in the conclusion.

2. Summary of standard harmonic and the inverted oscillators

Let us recall briefl the ladder operator approach of the usual harmonic oscillator:

$$H^{os} = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 = \frac{\hbar\omega}{2}\left(a^+a + aa^+\right), \quad (5)$$

where

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{p}{\sqrt{2m\hbar\omega}},$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{p}{\sqrt{2m\hbar\omega}}, \quad (6)$$

The operators a and a^+ satisfying the commutation relation

$$\left[a, a^\dagger\right] = 1. \quad (7)$$

Were introduced to facilitate the solution of the eigenvalue problem. Eigenstates of (5) in Fock space are the Fock or number states $|n\rangle^{os}$ with the eigenvalues $\omega\left(n + 1/2\right))$, where $a\,|n\rangle^{os} = \sqrt{n}\,|n-1\rangle^{os}$, $a^\dagger\,|n\rangle^{os} = \sqrt{n+1}\,|n+1\rangle^{os}$ and n is a non-negative integer.

We then have a nice mechanism for computing the eigenstates of the Hamiltonian, but we can also express expectation values using the raising and lowering operators. This leads to the useful representation of x and p:

$$x = \sqrt{\frac{\hbar}{2\omega m}}\left(a^\dagger + a\right), \qquad p = i\sqrt{\frac{\hbar\omega m}{2}}\left(a^\dagger - a\right), \quad (8)$$

such that, we can compute any arbitrary expectation values that depend upon these quantities, merely by knowing the effects of the raising and lowering operators upon the eigenstates of the Hamiltonian.

From this, we can evaluate that the energy eigenvalues

$$H^{os}\psi_n^{os}(x) = E_n\psi_n^{os}(x) = \hbar\omega\left(n + \frac{1}{2}\right)\psi_n^{os}(x); \quad n \in \mathbb{N}, \quad (9)$$

and the normalized condition for the eigenfunctions is verifie

$$\langle\psi_m^{os}\,|\psi_n^{os}\rangle = \delta_{mn}. \quad (10)$$

We see that the energy eigenvalues $E_0 = \hbar\omega/2$ of the ground state

$$\psi_0^{os}(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{\omega m}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-\frac{\omega m}{2\hbar}x^2\right], \quad (11)$$

is a very significan physical result because it tells us that the energy of a system described by a harmonic oscillator potential cannot have zero energy.

In contrast with the harmonic oscillator, the inverted oscillator has a Hamiltonian with the following form:

$$H^r = \frac{1}{2m}p^2 - \frac{1}{2}m\omega^2x^2 = -\frac{\hbar\omega}{2}\left(a^{\dagger 2} + a^2\right). \quad (12)$$

The Hamiltonian (12) is formally obtainable from (5) by the replacement

$$\omega \rightarrow i\omega, \quad (13)$$

similarly, the case $(-i\omega)$ would serve equally well.

On the other hand, for an imaginary frequency, *i.e.* for the inverted harmonic oscillator, we get

$$a \rightarrow A = e^{i\frac{\pi}{4}}\left(\sqrt{\frac{m\omega}{2\hbar}}x + \frac{p}{\sqrt{2m\omega\hbar}}\right), \quad (14)$$

$$a^+ \rightarrow \bar{A} = e^{i\frac{\pi}{4}}\left(\sqrt{\frac{m\omega}{2\hbar}}x - \frac{p}{\sqrt{2m\omega\hbar}}\right), \quad (15)$$

thus, the Hamiltonian (12) can take the following form

$$H^r = \frac{i\hbar\omega}{2}(\bar{A}A + A\bar{A}), \quad (16)$$

where the non-Hermitian pseudo-ladder operators $\left(A, \bar{A}\right)$ are characterized by $\left[A, \bar{A}\right] = 1$ in an analogous way to the ladder operator $\left(a, a^\dagger\right)$ for the harmonic oscillator.

Knowing that the eigenfunctions of the harmonic oscillator are normalized, we ask the question if the inverted oscillator eigenfunctions are also normalized? Clearly, they are not $\langle\psi_m^r\,|\psi_n^r\rangle \neq \delta_{mn}$. This can be seen when calculating the normalization condition for the pseudo-ground state $\psi_0^r(x)$ of the obtained inverted oscillator: from Eq. (11) by changing ω to $i\omega$

$$\psi_0^r(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{i\omega m}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-i\frac{\omega m}{2\hbar}x^2\right]. \quad (17)$$

One can easily verify that the normalization for this state diverges as follows:

$$\langle \psi_0^r \, | \psi_0^r \rangle = \int_{-\infty}^{+\infty} \psi_0^{*r}(x) \psi_0^r(x) dx = \frac{1}{2^n n!} \left(\frac{\omega m}{\pi \hbar} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} dx \to \infty, \tag{18}$$

the reason for this divergence is that the substitution ω by $i\omega$ is unsuitable. we will remedy this inconsistency in what follows.

3. Pseudo-ladder operators in the inverted harmonic oscillator

The Hermitian Hamiltonian H^r and the non-Hermitian Hamiltonian (iH^{os}) are related by a formal replacement $\omega \to i\omega$. The challenge is to establish a consistent relation between the quantum mechanical formalism for the Hermitian Hamiltonian $H^r$and the non-Hermitian one (iH^{os}), we propose that instead of considering this formal transformation, we use the relation that it is valid for any self-adjoint operator, *i.e.* observable, in the Hermitian system to possess a counterpart in the non-Hermitian system given by

$$\rho^{-1}(iH^{os})\rho = H^r. \tag{19}$$

In order to connect the non-Hermitian harmonic oscillator Hamiltonian (iH^{os}) to the Hermitian inverted oscillator H^r, we perform a Dyson type transformation ρ such that [30]

$$\rho = \exp\left\{-2\left[\frac{\epsilon}{2}\left(a^\dagger a + \frac{1}{2}\right) + \mu_- \frac{a^2}{2} + \mu_+ \frac{a^{\dagger 2}}{2}\right]\right\} = \exp\left[-\vartheta_- \frac{a^2}{2}\right] \exp\left[-\frac{\ln \vartheta_0}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] \exp\left[-\vartheta_+ \frac{a^{\dagger 2}}{2}\right], \tag{20}$$

and

$$\vartheta_+ = \frac{2\mu_+ \sinh\theta}{\theta\cosh\theta - \epsilon\sinh\theta}, \qquad \vartheta_0 = \left(\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta\right)^{-2} = \mu_+\mu_- - \chi,$$

$$\vartheta_- = \frac{2\mu_- \sinh\theta}{\theta\cosh\theta - \epsilon\sinh\theta}, \qquad \chi = -\frac{\cosh\theta + \frac{\epsilon}{\theta}\sinh\theta}{\cosh\theta - \frac{\epsilon}{\theta}\sinh\theta}, \qquad \theta = \sqrt{\epsilon^2 - 4\mu_+\mu_-}, \tag{21}$$

where ϵ is a real parameter whereas μ_+ and μ_- are complex ones.

With the help of the following relations

$$\begin{cases} \exp\left[\vartheta_- \frac{a^2}{2}\right]\left(a^\dagger a + \frac{1}{2}\right)\exp\left[-\vartheta_- \frac{a^2}{2}\right] = \left(a^\dagger a + \frac{1}{2}\right) + \vartheta_- a^2 \\ \exp\left[\vartheta_+ \frac{a^{\dagger 2}}{2}\right]\left(a^\dagger a + \frac{1}{2}\right)\exp\left[-\vartheta_+ \frac{a^{\dagger 2}}{2}\right] = \left(a^\dagger a + \frac{1}{2}\right) - \vartheta_+ a^{\dagger 2} \end{cases}, \tag{22}$$

$$\begin{cases} \exp\left[\frac{\ln\vartheta_0}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] a^2 \exp\left[-\frac{\ln\vartheta_0}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] = \frac{a^2}{\vartheta_0} \\ \exp\left[\vartheta_+ \frac{a^{\dagger 2}}{2}\right] a^2 \exp\left[-\vartheta_+ \frac{a^{\dagger 2}}{2}\right] = a^2 - 2\vartheta_+\left(a^\dagger a + \frac{1}{2}\right) + \vartheta_+^2 a^{\dagger 2} \end{cases}, \tag{23}$$

$$\begin{cases} \exp\left[\frac{\ln\vartheta_0}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] a^{\dagger 2} \exp\left[-\frac{\ln\vartheta_0}{2}\left(a^\dagger a + \frac{1}{2}\right)\right] = \vartheta_0 a^{\dagger 2} \\ \exp\left[\vartheta_- \frac{a^2}{2}\right] a^{\dagger 2} \exp\left[-\vartheta_- \frac{a^2}{2}\right] = a^{\dagger 2} + 2\vartheta_-\left(a^\dagger a + \frac{1}{2}\right) + \vartheta_-^2 a^2 \end{cases}, \tag{24}$$

we deduce, under the action of the operator ρ, the transformed Hamiltonian of the harmonic oscillator :

$$\rho^{-1}H^{os}\rho = \hbar\omega\rho^{-1}\left(a^\dagger a + \frac{1}{2}\right)\rho = \frac{\hbar\omega}{\vartheta_0}\left\{\left[\vartheta_0 - 2\vartheta_+\vartheta_-\right]\left(a^\dagger a + \frac{1}{2}\right) + \left[\vartheta_- \vartheta_+^2 - \vartheta_0\vartheta_+\right] a^{\dagger 2} + \vartheta_- a^2\right\}. \tag{25}$$

We notice that Eq. (25) and Eq. (12) have the same structure in their operator content provided that we impose on the parameters $(\vartheta_+, \vartheta_-,\ \vartheta_0)$ the following conditions

$$\vartheta_+ = -i,\ \vartheta_- = \frac{i}{2}, \vartheta_0 = 1, \tag{26}$$

from these constraints, the Dyson operator Eq. (20) takes now the simplifie form^i

$$\rho = \exp\left[-\frac{i}{4}a^2\right]\exp\left[\frac{i}{2}a^{\dagger 2}\right], \qquad \rho^{-1} = \exp\left[-\frac{i}{2}a^{\dagger 2}\right]\exp\left[\frac{i}{4}a^2\right], \tag{27}$$

it follows that the two Hamiltonians H^{os} and H^r are allied to each other as

$$\rho^{-1}H^{os}\rho = i\frac{\hbar\omega}{2}\left(a^{\dagger 2} + a^2\right) = -iH^r. \tag{28}$$

One can verify that in the case of the inverted oscillator, the form of Hamiltonian in the last equation looks like

$$H^r = \frac{i\hbar\omega}{2}(\bar{A}A + A\bar{A}), \tag{29}$$

where the pseudo-ladder operators $\left(A, \bar{A}\right)$ are linked to the ladder operators (6) through the transformation

$$A = \rho^{-1}a\rho = a + ia^{\dagger}, \tag{30}$$

$$\bar{A} = \rho^{-1}a^{\dagger}\rho = \frac{1}{2}\left(a^{\dagger} + ia\right), \tag{31}$$

and satisfy the following commutation relation $\left[A, \bar{A}\right] = 1$.Then, we can deduce that their Fock eigenstates $|n^r\rangle$ are related to $|n^{os}\rangle$ by the invertible operator ρ as

$$|n^r\rangle = \rho^{-1}\,|n^{os}\rangle\,. \tag{32}$$

For instance, the pseudo-Hermitian quadratures (X, P) corresponding in the Hermitian system to the coordinate and momentum operators (x, p) (see Eqs. (8)) respectively, are now

$$X = \rho^{-1}x\rho = \sqrt{\frac{\hbar}{2\omega m}}\rho^{-1}\left(a^{\dagger} + a\right)\rho$$
$$= \sqrt{\frac{\hbar}{2\omega m}}\left(A + \bar{A}\right), \tag{33}$$

$$P = \rho^{-1}p\rho = i\sqrt{\frac{\hbar\omega m}{2}}\rho^{-1}\left(a^{\dagger} - a\right)\rho$$
$$= i\sqrt{\frac{\hbar\omega m}{2}}\left(\bar{A} - A\right). \tag{34}$$

Knowing that any observable o in the Hermitian system possesses a counterpart O in the pseudo-Hermitian system given by

$$O = \rho^{-1}o\rho, \tag{35}$$

one can deduce the useful representation of $\left(A, \bar{A}\right)$ in terms of (X, P) as

$$A = \sqrt{\frac{m\omega}{2\hbar}}X + i\frac{1}{\sqrt{2m\hbar\omega}}P, \tag{36}$$

$$\bar{A} = \sqrt{\frac{m\omega}{2\hbar}}X - i\frac{1}{\sqrt{2m\hbar\omega}}P. \tag{37}$$

Thereby, the Hamiltonian (29) can be written in terms of X and P as

$$H^r = \frac{i}{2}\left(\frac{P^2}{m} + m\omega^2X^2\right). \tag{38}$$

This leads to the equations of motion of the inverted oscillator. Indeed, using the Heisenberg equations of motion and $[X, P] = i\hbar$, we have for X and P:

$$\frac{dX}{dt} = \frac{1}{i\hbar}\left[X, \frac{i}{2}\left(\frac{P^2}{m} + m\omega^2X^2\right)\right] = i\frac{P}{m}.$$
$$\frac{dP}{dt} = \frac{1}{i\hbar}\left[P, \frac{i}{2}\left(\frac{P^2}{m} + m\omega^2X^2\right)\right] = -im\omega^2X. \tag{39}$$

Taking another time derivative of dX/dt, we get the usual equation of motion for the inverted oscillator

$$\frac{d^2X}{dt^2} - \omega^2X = 0, \tag{40}$$

4. Coherent states for the inverted oscillator

The best way to present the inverted coherent states is by translating their definition into the language of the coherent states of the harmonic oscillator which are summarized in what follows. Coherent states, or semi-classic states, are remarkable quantum states that were originally introduced in 1926 by Schrödinger for the Harmonic oscillator [31] where the mean values of the position and momentum operators in these states have properties close to the classical values of the position $x_c(t)$ and the momentum $p_c(t)$. In particular, the coherent states of the harmonic oscillator$|\alpha^{os}\rangle$ [32]- [34] may be obtained in different but equivalent ways:

(i) as eigenstates of the annihilation operator;

$$a\,|\alpha\rangle^{os} = \alpha\,|\alpha\rangle^{os}\,, \tag{41}$$

with eigenvalues $\alpha \in C$.

(ii) as a displacement of the vacuum $|0\rangle^{os}$, where the displacement operator

$$D^{os}\left(\alpha\right) = \exp[\alpha^{*}a^{\dagger} - \alpha a], \tag{42}$$

can be used to generate the coherent state

$$|\alpha\rangle^{os} = D^{os}\left(\alpha\right)|0\rangle^{os}\,, \tag{43}$$

(iii) as states that minimize the Heisenberg uncertainty principle

$$\Delta x\Delta p = \frac{\hbar}{2}. \tag{44}$$

Coherent states form an over-complete set of states. The identity operator I is written in terms of coherent states as

$$\frac{1}{\pi}\int |\alpha\rangle^{os}\,{}^{os}\langle\alpha|\,d^2\alpha = I. \tag{45}$$

The solution for the harmonic oscillator Hamiltonian for an initial coherent state is given in the following simple form

$$|\alpha, t\rangle^{os} = e^{-i\frac{\omega t}{2}}\left|\alpha e^{-i\omega t}\right\rangle^{os}, \tag{46}$$

i.e., a coherent state that rotates with the harmonic oscillator frequency.

In analogy with the usual coherent states, we use the pseudo-annihilation $A = \rho^{-1}a\rho$ and pseudo-creation $\bar{A} = \rho^{-1}a^{\dagger}\rho$ operators which are very convenient to study the inverted coherent states. We emphasize the use of the metric $\eta = \rho^{\dagger}\rho$ operator such as $(iH^{os})^{\dagger} = \eta\,(iH^{os})\,\eta^{-1}$, *i.e.* (iH^{os}) is η-pseudo-Hermitian with respect to a positive-definit inner product define by $\langle .,. \rangle_{\eta} = \langle .|\eta|. \rangle$:

$$^{r}\langle n|\,\eta\,|m\rangle^{r} = {}^{os}\langle n\,|m\rangle^{os} = \delta_{mn}, \tag{47}$$

which indicates that the Fock states are linked to each other as

$$|n\rangle^{r} = \rho^{-1}\,|n\rangle^{os}, \tag{48}$$

additionally, the vacuum state of the inverted oscillator $|0\rangle^{r}$ ($A\,|0\rangle^{r} = 0$) and the vacuum state of the harmonic oscillator $|0\rangle^{os}$ are related as $|0\rangle^{r} = \rho^{-1}\,|0\rangle^{os}$.

The coherent states for the inverted harmonic oscillator are define as eigenstates of the corresponding pseudo-annihilation operator A

$$A\,|\alpha\rangle^{r} = \alpha\,|\alpha\rangle^{r}, \quad \alpha \in \mathbb{C}. \tag{49}$$

with

$$|\alpha\rangle^{r} = \rho^{-1}\,|\alpha\rangle^{os}. \tag{50}$$

Particularly, the normalization condition

$$^{os}\langle \alpha\,|\alpha\rangle^{os} = 1, \tag{51}$$

leads to

$$^{r}\langle \alpha|\,\eta\,|\alpha\rangle^{r} = 1, \tag{52}$$

and then the integral

$$\frac{1}{\pi}\int_{\mathbb{C}} \rho\,|\alpha\rangle^{r}\,{}^{r}\langle \alpha|\,\rho^{+}d\alpha^{*}d\alpha = I, \tag{53}$$

is an identity operator.

These inverted coherent states $|\alpha\rangle^{r}$ can also be generated respectively from the vacuum states $|0\rangle^{r}$ by the action of pseudo-displacement operator $D^{r}(\alpha)$,

$$|\alpha\rangle^{r} = D^{r}(\alpha)\,|0\rangle^{r} = \exp\left[\alpha\bar{A} - \alpha^{*}A\right]|0\rangle^{r}, \tag{54}$$

we note that $D^{r}(\alpha)$ is related to $D^{os}(\alpha)$ as

$$D^{r}(\alpha) = \rho^{-1}D^{os}(\alpha)\,\rho. \tag{55}$$

Using the Hamiltonian (29), we deduce the evolution of an initial inverted coherent state in the following simple form

$$\begin{aligned} |\alpha, t\rangle^{r} &= e^{-i/\hbar H^{r}t}\,|\alpha\rangle^{r} \\ &= e^{-|\alpha|^{2}/2}e^{\omega t/2}e^{\omega\bar{A}At}\sum_{n}\frac{(\alpha)^{n}}{\sqrt{n!}}\,|n\rangle^{r}. \end{aligned} \tag{56}$$

Introducing $e^{\omega\bar{A}At}$ into the sum, and using the fact that the states $|n\rangle^{r}$ are eigenstates of the number operator $\bar{A}A$, we have

$$\begin{aligned} |\alpha, t\rangle^{r} &= e^{-|\alpha e^{\omega t}|^{2}/2}\sum_{n}\frac{(\alpha e^{\omega t})^{n}}{\sqrt{n!}}\,|n\rangle^{r} \\ &= e^{\omega t/2}\left|\alpha e^{\omega t}\right\rangle^{r}. \end{aligned} \tag{57}$$

Since our aim is to compute the Heisenberg uncertainty relations in the position and the momentum, it is required to calculate the expectation values of the canonical variables and their squares in the inverted coherent states. Then, by using the relation (35) in the non-Hermitian system, the expectation value of an arbitrary operator $O = X, X^{2}, P$ and P^{2} can be evaluated from

$$\langle O\rangle_{\eta} = {}^{r}\langle \alpha, t|\,\eta O\,|\alpha, t\rangle^{r} = {}^{r}\langle \alpha, t|\,\rho^{+}o\rho\,|\alpha, t\rangle^{r} = e^{-\frac{|\alpha e^{\omega t}|^{2}}{2}}\sum_{n}\sum_{m}\frac{(\alpha^{*}e^{\omega t})^{m}}{\sqrt{m!}}\frac{(\alpha e^{\omega t})^{n}}{\sqrt{n!}}\,{}^{os}\langle m|\,o\,|n\rangle^{os}. \tag{58}$$

Using the above equation, the expectation values of X and P in the state $|\alpha, t\rangle^{r}$ are easily evaluated:

$$\langle X\rangle_{\eta} = e^{-|\alpha e^{\omega t}|^{2}}\sum_{n}\sum_{m}\frac{(\alpha^{*}e^{\omega t})^{m}}{\sqrt{m!}}\frac{(\alpha e^{\omega t})^{n}}{\sqrt{n!}}\sqrt{\frac{\hbar}{2\omega m}}\,{}^{os}\langle m|\left(a^{\dagger} + a\right)|n\rangle^{os} = \sqrt{\frac{\hbar}{2m\omega}}\left[\alpha + \alpha^{*}\right]e^{\omega t}, \tag{59}$$

$$\langle P\rangle_{\eta} = e^{-|\alpha e^{\omega t}|^{2}}\sum_{n}\sum_{m}\frac{(\alpha^{*}e^{\omega t})^{m}}{\sqrt{m!}}\frac{(\alpha e^{\omega t})^{n}}{\sqrt{n!}}i\sqrt{\frac{\hbar\omega m}{2}}\,{}^{os}\langle m|\left(a^{\dagger} - a\right)|n\rangle^{os} = -i\sqrt{\frac{m\omega\hbar}{2}}\left[\alpha - \alpha^{*}\right]e^{\omega t}, \tag{60}$$

and follow classical physics; *i.e.*

$$\langle X\rangle_{\eta} = x_{c}, \quad \langle P\rangle_{\eta} = p_{c}, \tag{61}$$

where the subscript c indicate classical. This is why we call these inverted coherent states "quasi-classical states".

Let us now evaluate the uncertainty in the position and the momentum.

$$\left\langle X^2\right\rangle_\eta = \langle\alpha,t|\,\eta X^2\,|\alpha,t\rangle^r = \frac{\hbar}{2m\omega}\left[\alpha^2 e^{2\omega t} + \alpha^{*2}e^{2\omega t} + 2\left(|\alpha|^2 e^{2\omega t} + \frac{1}{2}\right)\right], \tag{62}$$

$$\left\langle P^2\right\rangle_\eta = \langle\alpha,t|\,\eta P^2\,|\alpha,t\rangle^r = \frac{-im\omega\hbar}{2}\left[\alpha^2 e^{2\omega t} + \alpha^{*2}e^{2\omega t} - 2\left(|\alpha|^2 e^{2\omega t} + \frac{1}{2}\right)\right]. \tag{63}$$

It is well known that the position uncertainty can be derived from $\Delta X = \sqrt{\langle X^2\rangle_\eta - \langle X\rangle_\eta^2}$. Then using (59) and (62), we have

$$\Delta X = \sqrt{\frac{\hbar}{2m\omega}}.$$

Similarly, from Eqs. (60) and (63), we also have the momentum uncertainty such that

$$\Delta P = \sqrt{\frac{m\omega\hbar}{2}}.$$

Thus, the uncertainty product for canonical variables X and P is given by

$$\Delta X \Delta P = \frac{\hbar}{2}.$$

Therefore, the inverted coherent states are a minimum-uncertainty states and the time evolution of an initially inverted coherent state can be regarded as the quantum analog of a classical trajectory.

5. Conclusion

We have briefl summarized in Sec. 2, some properties of the standard harmonic and inverted oscillators.

We have proposed a scheme that permits relating a regular harmonic oscillator to an inverted oscillator by using a time-independent Dyson metric which allowed us to introduce the pseudo-annihilation $A = \rho^{-1}a\rho$ and pseudo-creation $\bar{A} = \rho^{-1}a^\dagger\rho$ operators associated to the inverted harmonic oscillator. These operators are the basis of the definitio of coherent states for inverted oscillator and their corresponding eigenstates and eigenvalues. Once the Dyson operator has been introduced, and therefore the metric operators, it is straightforward to extend these considerations to the associated eigenstates and inner product structures on the physical Hilbert space. Some of the finding are treated by the Gaussian wave packet (in the x-representation) associated to the generalized coherent state in Ref. [35].

Coherent states of the inverted harmonic oscillator are constructed in different forms:

(1) as eigenstates of the pseudo-annihilation operator A;

(2) as a pseudo-displacement of the inverted vacuum $\exp\left[\alpha\overline{A} - \alpha^* A\right]|0\rangle^r$,

(3) as states whose averages follow the classical trajectories of X, P and H^r.

However, the coherent states for the inverted oscillator constitute ”minimum uncertainty” wave packets. Therefore, the time evolution of an initially coherent state can be regarded as the quantum analog of a classical trajectory.

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i. It is useful to note that the following simplifie transformations (27) has been introduced in Ref. [17] as footnote.

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On the quantum dynamics of a general time-dependent coupled oscillator

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By using the Lewis–Riesenfeld invariants theory, we investigate the quantum dynamics of a two-dimensional (2D) time-dependent coupled oscillator. We introduce a unitary transformation and show the conditions under which the invariant operator is uncoupled to describe two simple harmonic oscillators with time-independent frequencies and unit masses. The decouplement allows us to easily obtain the corresponding eigenstates. The generalization to three-dimensional (3D) time-dependent coupled oscillator is briefly discussed where a diagonalized invariant, which is exactly the sum of three simple harmonic oscillators, is obtained under specific conditions on the parameters.

Keywords: Time-dependent quantum system; invariant theory; coupled oscillator; canonical and unitary transformation.

1. Introduction

The harmonic oscillator is one of the most important models in quantum mechanics, and one of the few ones that has an exact analytical solution that made it applicable in the study of the dynamical properties of different physical systems: natural systems and technological devices. The two- and the three-dimensional harmonic oscillators are particularly among the most representative and important

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harmonic oscillator models in quantum mechanics and therefore establishing the dynamical properties of coupled oscillatory systems is extremely demanded though their complex motion especially when the parameters are time-dependent and/or the dimension of the system is more than two.

The time-dependent coupled oscillator that has been a point of interest in the research field for a few years now,[1–6] model various physical systems[7–17] and helped in explaining numerous physical interacting systems including trapped atoms,[18] nanooptomechanical resonances,[19,20] electromagnetically induced transparency,[21] stimulated Raman effects,[22] time-dependent Josephson phenomena,[23] and systems of three isotropically coupled spins 1/2.[24] Coupled oscillators are fundamental for quantum technologies such as quantum computing and quantum cryptography.[25–27] The problem of a system of n-coupled harmonic oscillators with damping and driving forces was considered in the framework of Lie symmetries of differential equations.[28]

However, the study of time-dependent Hamiltonian systems has attracted much attention over the years leading to the development of variety of techniques, to cite but a few: adiabatic approximation,[29,30] sudden approximation, the perturbation theory and the Lewis–Reisenfeld invariants theory.[31] These cited techniques are devoted to Hermitian Hamiltonian systems and for the non-Hermitian ones an extension of the invariants theory[31] has been developed to treat explicitly time-dependent pseudo-Hermitian Hamiltonians which made the establishment of the solutions to the considered system in terms of the eigenstates of a pseudo-Hermitian invariant operator possible.[32–34]

In this paper, we investigate quantum dynamical properties of a two-dimensional (2D) time-dependent coupled oscillator. Our study is based on the theory of a two-dimensional (2D) coupled dynamical invariant. We show the conditions under which the invariant operator is uncoupled.

In Sec. 2, we evaluate the study of the time-dependent 2D coupled system and approach the whole subject in a scientifically coherent manner to show that the invariant operator of the system can be uncoupled under specific conditions. Then, we recapitulate the basic errors made in Ref. 35 while studying the same system. Hassoul *et al.*[35] claim that introducing two pairs of annihilation and creation operators uncouples the original invariant operator so that it becomes the one that describes two independent subsystems. We show also that the linear canonical transformation defined in Ref. 35 cannot be adapted to the considered problem without a constraint on the mass and to solve the problem in question (see Appendix A). In order to solve the problem in a clearer way we adopt, in the quantum mechanics framework the method used in classical case in Ref. 36; that the quantum mechanical unitary transformations are used instead of the classical canonical transformations, we introduce more adequate canonical transformations. Later, we highlight, with some simple mathematical calculation, all the flawed results found in Ref. 37 where a generalization to the 3D case of the investigation introduced in Ref. 35 using the same layout is considered.

2. Discussion

2.1. *Background on the subject*

Let us recall the method used to solve the quantum dynamics of a general time-dependent coupled oscillator, mainly the one adopted in Ref. 35. We should point out that the notations used in this paper are slightly different from those of Ref. 35. For instance, we used $\alpha_i = \rho_i^2$ while in Ref. 35, $\alpha_i = \frac{1}{m_i}$. The Hamiltonian of the time-dependent coupled oscillator that we consider is of the form

$$H(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\frac{P_i^2}{m_i(t)} + c_i(t)X_i^2\right] + \frac{1}{2}c_3(t)X_1X_2, \tag{1}$$

where $m_i(t)$, $c_i(t)$ and $c_3(t)$ are arbitrary functions of time. We choose a quantum invariant operator of the system of the form

$$I(t) = \frac{1}{2}\sum_{i=1}^{2}[\alpha_i(t)P_i^2 + \beta_i(t)(X_iP_i + P_iX_i) + \gamma_i(t)X_i^2] + \frac{1}{2}\eta(t)X_1X_2, \tag{2}$$

the parameters $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$ $(i = 1, 2)$ and $\eta(t)$ are real and differentiable functions of time. The substitution of (1) and (2) into the invariance condition

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H] = 0, \tag{3}$$

implies the auxiliary equations given as

$$\dot{\alpha}_i(t) = \frac{-2\beta_i(t)}{m_i(t)}, \tag{4}$$

$$\dot{\beta}_i(t) = c_i(t)\alpha_i(t) - \frac{\gamma_i(t)}{m_i(t)}, \tag{5}$$

$$\dot{\gamma}_i(t) = 2c_i(t)\beta_i(t), \tag{6}$$

$$\dot{\eta}(t) = c_3(t)[\beta_1(t) + \beta_2(t)], \tag{7}$$

in addition to the following important condition:

$$\frac{\eta(t)}{c_3(t)} = \alpha_1(t)m_1(t) = \alpha_2(t)m_2(t). \tag{8}$$

Now, by noting that

$$\alpha_i(t)\gamma_i(t) - \beta_i^2(t) = \delta_i, \tag{9}$$

with δ_i being a real constant, we set

$$\alpha_i(t) = \rho_i^2, \tag{10}$$

to get after simple calculation

$$\beta_i(t) = -m_i\rho_i\dot{\rho}_i, \tag{11}$$

$$\gamma_i(t) = c_i(t)m_i\rho_i^2 + m_i\dot{m}_i\rho_i\dot{\rho}_i + m_i^2\dot{\rho}_i^2 + m_i^2\rho_i\ddot{\rho}_i, \tag{12}$$

and the auxiliary equation for ρ_i

$$\ddot{\rho}_i + \frac{\dot{m}_i}{m_i}\dot{\rho}_i = \frac{\delta_i}{m_i^2\rho_i^3} - \frac{c_i(t)\rho_i}{m_i}, \tag{13}$$

and since

$$\rho_1^2 m_1 = \rho_2^2 m_2, \tag{14}$$

one can write

$$\eta(t) = -\int^t c_3(t')m_1\rho_1(\dot{\rho}_1 + \rho_1\frac{\dot{\rho}_2}{\rho_2})dt', \tag{15}$$

thus, the invariant (2) is written as

$$I(t) = \frac{1}{2}\sum_{i=1}^{2}\left[\rho_i^2 P_i^2 - m_i\rho_i\dot{\rho}_i\left(X_iP_i + P_iX_i\right) + \left(\frac{\delta_i}{\rho_i^2} + m_i^2\dot{\rho}_i^2\right)X_i^2\right]$$
$$-\frac{1}{2}\left[\int^t c_3(t')m_1\rho_1\left(\dot{\rho}_1 + \rho_1\frac{\dot{\rho}_2}{\rho_2}\right)dt'\right]X_1X_2. \tag{16}$$

According to the Lewis–Riesenfeld theory,[31] the invariant operator $I(t)$ has time-independent eigenvalues $\lambda_{n,m}$

$$I(t)|\varphi_{n_1,n_2}\rangle = \lambda_{n_1,n_2}|\varphi_{n_1,n_2}\rangle, \tag{17}$$

and his eigenfunctions $|\varphi_{n_1,n_2}\rangle$ are time-dependent whose multiplication by suitable phases $\exp[i\mu_{n_1,n_2}(t)]$, with $\mu_{n_1,n_2}(t)$ verifying

$$\dot{\mu}_{n,m}(t) = \left\langle\varphi_{n_1,n_2}\middle|\left(i\hbar\frac{\partial}{\partial t} - H(t)\right)\middle|\varphi_{n_1,n_2}\right\rangle, \tag{18}$$

is a solution of the time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{19}$$

2.2. *Unitary transformations: Results and discussion*

In order to solve the eigenvalues equation (17), we introduce the unitary transformation U such that

$$|\varphi'_{n_1,n_2}\rangle = U|\varphi_{n_1,n_2}\rangle = U_1U_2|\varphi_{n_1,n_2}\rangle, \tag{20}$$

where

$$U_1 = \prod_{i=1}^{2}\exp\left[\frac{i}{2\hbar}(X_iP_i + P_iX_i)\ln\sqrt{\alpha_i}\right], \tag{21}$$

$$U_2 = \sum_{i=1}^{2}\exp\left[\frac{i}{2\hbar}\frac{\beta_i}{\alpha_i}X_i^2\right]. \tag{22}$$

under which X_i and P_i transform into

$$U_1 X_i U_1^+ = \sqrt{\alpha_i} X_i, \qquad U_1 P_i U_1^+ = \frac{1}{\sqrt{\alpha_i}} P_i, \tag{23}$$

and

$$U_2 X_i U_2^+ = X_i, \tag{24}$$

$$U_2 P_i U_2^+ = P_i - \frac{\beta_i}{\alpha_i} X_i. \tag{25}$$

Finally, we get

$$U X_i U^+ = \sqrt{\alpha_i} X_i, \tag{26}$$

$$U P_i U^+ = \frac{1}{\sqrt{\alpha_i}}, \quad P_i - \frac{\beta_i}{\sqrt{\alpha_i}} X_i \tag{27}$$

and consequently the invariant (2) becomes

$$I' = \frac{1}{2}\sum_{i=1}^{2}[P_i^2 + (\gamma_i\alpha_i - \beta_i^2)X_i^2] + \frac{1}{2}\eta\sqrt{\alpha_1\alpha_2}X_1X_2. \tag{28}$$

We can further simplify the invariant operator I' by introducing the unitary operator U_3

$$U_3 = \exp\left[\frac{i}{\hbar}\frac{\theta}{2}(P_2X_1 - P_1X_2)\right], \tag{29}$$

under which X_i and P_i transform into

$$U_3 X_1 U_3^+ = \cos\left(\frac{\theta}{2}\right) X_1 - \sin\left(\frac{\theta}{2}\right) X_2, \quad U_3 X_2 U_3^+ = \cos\left(\frac{\theta}{2}\right) X_2 + \sin\left(\frac{\theta}{2}\right) X_1,$$
$$U_3 P_1 U_3^+ = \cos\left(\frac{\theta}{2}\right) P_1 - \sin\left(\frac{\theta}{2}\right) P_2, \quad U_3 P_2 U_3^+ = \cos\left(\frac{\theta}{2}\right) P_2 + \sin\left(\frac{\theta}{2}\right) P_1, \tag{30}$$

the above Eqs. (30) are similar to the canonical classical variables (12–15) in Ref. 36.

Therefore, the invariant (28) is written as

$$\begin{aligned} I'' &= U_3 I' U_3^+ \\ &= \frac{P_1^2}{2} + \frac{P_2^2}{2} + \frac{1}{2}\left[\delta_1\cos^2\left(\frac{\theta}{2}\right) + \delta_2\sin^2\left(\frac{\theta}{2}\right) + \frac{\eta\sqrt{\alpha_1\alpha_2}}{2}\sin\theta\right]X_1^2 \\ &\quad + \frac{1}{2}\left[\delta_1\sin^2\left(\frac{\theta}{2}\right) + \delta_2\cos^2\left(\frac{\theta}{2}\right) - \frac{\eta\sqrt{\alpha_1\alpha_2}}{2}\sin\theta\right]X_2^2 \\ &\quad + \frac{1}{2}[\eta\sqrt{\alpha_1\alpha_2}\cos\theta - (\delta_1 - \delta_2)\sin\theta]X_1X_2, \end{aligned} \tag{31}$$

We observe from Eq. (31) that the separation of variables is complete for

$$\eta\sqrt{\alpha_1\alpha_2}\cos\theta - [\delta_1 - \delta_2]\sin\theta = 0, \tag{32}$$

and this leads to the following transformed invariant:

$$I'' = \frac{1}{2}\sum_{i=0}^{2}(P_i^2 + \widetilde{\Omega}_i^2 X_i^2), \tag{33}$$

where

$$\widetilde{\Omega}_1^2 = \delta_1\cos^2\left(\frac{\theta}{2}\right) + \delta_2\sin^2\left(\frac{\theta}{2}\right) + \frac{\eta\sqrt{\alpha_1\alpha_2}}{2}\sin\theta, \tag{34}$$

$$\widetilde{\Omega}_2^2 = \delta_1\sin^2\left(\frac{\theta}{2}\right) + \delta_2\cos^2\left(\frac{\theta}{2}\right) - \frac{\eta\sqrt{\alpha_1\alpha_2}}{2}\sin\theta, \tag{35}$$

are constants. From Eq. (32), we deduce

$$\tan(\theta) = \frac{\eta\sqrt{\alpha_1\alpha_2}}{[\delta_1 - \delta_2]}, \tag{36}$$

where

$$\theta = \arctan(\eta\sqrt{\alpha_1\alpha_2}.[\delta_1 - \delta_2]^{-1}), \tag{37}$$

is time-independent, to be convinced it is enough to follow the calculation method of $\frac{\partial\theta}{\partial t}$ developed in Ref. 38. We confirm that the frequencies $\widetilde{\Omega}_i$ are time-independent.

After decouplement, the invariant (33) becomes the sum of two invariants describing simple harmonic oscillators with time-independent frequencies $\widetilde{\Omega}_i$ and unit masses whose eigenstates are well known and expressed as

$$|\varphi_{n_1,n_2}\rangle = \prod_{i=1}^{2}\left(\frac{\sqrt{\widetilde{\Omega}_i}}{(\pi\hbar)^{1/2}n_i!2^{n_i}}\right)^{1/2} H_{n_i}\left(\sqrt{\frac{\widetilde{\Omega}_i}{\hbar}}X_i\right)\exp\left[-\frac{i\widetilde{\Omega}_i}{2\hbar}X_i^2\right], \tag{38}$$

where H_{n_i} are the Hermite polynomials. Finally, let us note that it is easy to express the invariant operator in terms of the well known annihilation and creation operators associated to the usual harmonic oscillator.

3. Short Discussion on the Generalization to 3D Coupled Oscillator

A generalization of the 2D coupled oscillator to a 3D one has been incorrectly made by Hassoul *et al.* in Ref. 37 where the authors study general time-dependent three coupled nanooptomechanical oscillators. They start to obtain the formulae of time-dependent parameters of the time-dependent invariant operator by deriving six differential equations (Eqs. (8)–(13) in[37]) with their possible solutions (Eqs. (14)–(19). In fact, they failed again in mentioning other conditions just like before, since the invariance condition implies nine equations given as

$$\dot{A}_i(t) = -\frac{2B_i(t)}{m_i(t)}, \tag{39}$$

$$\dot{B}_i(t) = -\frac{C_i(t)}{m_i(t)} + m_i(t)\omega_i^2(t)A_i(t), \tag{40}$$

$$\dot{C}_i(t) = 2m_i(t)\omega_i^2(t)B_i(t), \tag{41}$$

$$\dot{D}_{12}(t) = \frac{k_{12}(t)}{2}[B_1(t) + B_2(t)], \tag{42}$$

$$\dot{D}_{13}(t) = \frac{k_{13}(t)}{2}[B_1(t) + B_3(t)], \tag{43}$$

$$\dot{D}_{23}(t) = \frac{k_{23}(t)}{2}[B_2(t) + B_3(t)], \tag{44}$$

$$\frac{D_{13}(t)}{D_{12}(t)} = \frac{k_{13}(t)}{k_{12}(t)}, \tag{45}$$

$$\frac{D_{12}(t)}{D_{23}(t)} = \frac{k_{12}(t)}{k_{23}(t)}, \tag{46}$$

$$\frac{D_{23}(t)}{D_{13}(t)} = \frac{k_{23}(t)}{k_{13}(t)}. \tag{47}$$

Similarly to the 2D case, the authors of Ref. 37 pretend to obtain the solution of Eqs. (39)–(44), omitting to mention an important detail: the following constraint equation that the mass must obey when considering $m_i(t) = 1/\alpha_i(t)$

$$\ddot{m}_i(t) - \frac{1}{2}\frac{\dot{m}_i^2(t)}{m_i(t)} + 2(\delta_i - \omega_i^2(t))m_i(t) = 0, \tag{48}$$

which is difficult to solve. This last Eq. (48) is obtained from the auxiliary Eqs. (39)–(44) by noting that

$$A_i(t)C_i(t) - B_i^2(t) = \delta_i, \tag{49}$$

with δ_i being a real constant. Condition (49) is not mentioned in Ref. 37. Note that putting $\frac{1}{\rho_i^2} = m_i(t)$ gives the famous auxiliary equation of ρ_i.[31,39–41]

Thus, the given solutions (Eq. (14)–(19) in Ref. 37) impose a constraint on the system that the authors did not pay attention to. The system cannot be resolved for any given mass.

The authors proceed, with the aim to have a diagonalized invariant operator $\mathcal{O}(t)$, to diagonalize the matrix $\mathbb{k}$ (formula (30) in Ref. 37) using the invertible matrix $\mathbb{R}$ (formula (46) in Ref. 37). Note that substituting the expressions (49–51) of x_i[37] in the diagonalized invariant operator (48) $\mathcal{O}(t)$ does not lead to expression (28) and the formula (35) of Ref. 37 cannot be obtained unless the parameters of $\mathbb{k}$ obey

$$K_{12} = K_{13} = K_{23} \quad \text{and} \quad \varpi_1^2 = \varpi_2^2 = \varpi_3^2, \tag{50}$$

which implies that $\Omega^2 = 0$ and the eigenvalues Ω_i^2 read

$$\Omega_1^2 = \varpi_1^2 + K_{12}, \tag{51}$$

$$\Omega_2^2 = \varpi_1^2 - \frac{K_{12}}{2}, \tag{52}$$

$$\Omega_3^2 = \varpi_1^2 - \frac{K_{12}}{2}. \tag{53}$$

Furthermore, as mentioned before, it is crucial in the Lewis and Riesenfeld theory[31] for the invariant operator to have time-independent eigenvalues whereas in Ref. 37 the eigenvalues Ω_i^2 are time-dependent. To see this, we calculate the time derivative of Ω_i^2 as

$$\frac{d\Omega_1^2}{dt} = \frac{d}{dt}\left(\frac{D_{12}}{\sqrt{m_1 m_2}}\right) + \frac{d}{dt}\left(\frac{D_{13}}{\sqrt{m_1 m_3}}\right), \tag{54}$$

$$\frac{d\Omega_2^2}{dt} = -\frac{d}{dt}\left(\frac{D_{23}}{\sqrt{m_2 m_3}}\right) + \frac{d\Omega^2}{dt}, \tag{55}$$

$$\frac{d\Omega_3^2}{dt} = -\frac{d}{dt}\left(\frac{D_{23}}{\sqrt{m_2 m_3}}\right) - \frac{d\Omega^2}{dt}, \tag{56}$$

one can see that even if $\Omega^2 = 0$, the parameters D_{12}, D_{13}, D_{23} and the masses m_i are defined as time-dependent. Therefore, the eigenvalues Ω_i^2 are not time-independent as they are supposed to be. This is once again a fundamental error that contradicts with the Lewis–Riesenfeld theory.

Finally, we suspect that the authors have calculated the phases (56) in Ref. 35 ((76) in Ref. 37) by taking the invariant operator instead of the Hamiltonian operator in which the term X_1X_2 has been omitted knowing that the dynamics of the system is ruled by the Hamiltonian operator and not by the invariant operator. It seems that the authors take the results of Refs. 31 and 39–41 and simply set $\frac{1}{\rho^2} = m(t)$ as if the invariant operator is the generator of the dynamics. Apparently, they present a study of coupled systems from a quantum point of view (this has an analog in the classical theory) and claim to prove that the solution to the time-dependent Schrödinger equation with the mixed term X_1X_2 in the Hamiltonian can be reduced to the solution of a time-independent Schrödinger equation involving the quantum invariant. We believe that is incorrect because the coupled terms X_1X_2 in the Hamiltonian have a contribution and cannot be omitted.

4. Conclusion

This paper is an opportunity to draw the reader's attention to the Refs. 14 and 15 cited in Ref. 35 which contain errors in dealing with time-dependent systems without taking into account the generating function of the canonical transformation.[42,43]

It is clear from the analysis above that Hassoul *et al.*'s analytical expressions (23), (26)–(32), (41) and (42) and (56) in Ref. 35 and expressions (28), (38)–(45) and (76)–(80) in Ref. 37 cannot be correct. Consequently, all the physical conclusions derived from such equations, are based on a wrong analytical layout.

Appendix A. Reminder on the Method of Ref. 35 by Pointing Out the Inconsistencies

Recently, Hassoul *et al.*[35] have studied the same system (1) considering the time-dependent invariant operator $I(t)$ (2). They claim that introducing two pairs of annihilation and creation operators uncouples the original invariant operator (2) so that it becomes the one that describes two independent subsystems. In fact, the authors of Ref. 35 pretend to obtain the solution of the Eqs. (4)–(7), omitting to mention that the so-called solution $\alpha_i(t) = 1/m_i(t)$ must obey the following constraint equation:

$$\ddot{m}_i(t) - \frac{1}{2}\frac{\dot{m}_i^2(t)}{m_i(t)} + 2(\delta_i m_i(t) - c_i(t)) = 0, \tag{A.1}$$

which is difficult to solve. They omit to mention condition (8) as well. Later, they claim that the invariant operator (23) in Ref. 35 can be decoupled into (35) simply using the canonical transformations (41) and (42) which cannot be valid if and only if $m_1(t) = m_2(t)$ and $\omega_1^2(t) = \omega_2^2(t)$.

Moreover, to our knowledge, the invariant operator in the Lewis and Riesenfeld theory[31] has time-independent eigenvalues whereas the frequencies ω_i^2 are time-dependent which does not imply time-independent eigenvalues of the invariant as claimed in Ref. 35 and consequently it is not easy to obtain the phases.We show that the frequencies (30) and (31) presented in Ref. 35 are time-dependent.

From the expression of ω_1^2 Eq. (30) in Ref. 35, the time-dependent derivative of ω_1^2 is

$$\begin{aligned}\frac{d\omega_1^2}{dt} &= \frac{\sin\theta}{2}\left(\left[\frac{\int_0^t [c_2 \frac{\dot{m}_2}{2m_2}]dt}{m_2} - \frac{\int_0^t [c_1 \frac{\dot{m}_1}{2m_1}]dt}{m_1}\right] - \left[\frac{\dot{m}_1^2}{4m_1^2} - \frac{\dot{m}_2^2}{4m_2^2}\right]\right)\dot{\theta} \\ &\quad + \cos(\theta)\left(\frac{\int_0^t c_3[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}]dt}{2\sqrt{m_1 m_2}}\right)\dot{\theta} \\ &\quad + \sin(\theta)\left[\frac{c_3[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}]}{2\sqrt{m_1 m_2}} - \frac{1}{4\sqrt[3]{m_1 m_2}}\left(\frac{\dot{m}_1}{m_1} + \frac{\dot{m}_2}{m_2}\right)\right. \\ &\quad \left.\times \int_0^t c_3\left[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}\right]dt\right].\end{aligned} \tag{A.2}$$

A short calculation of $\dot{\theta}(t)$ is

$$\begin{aligned}\dot{\theta} &= \cos^2\left(\frac{\theta(t)}{2}\right)\left[\frac{-1}{2\sqrt{m_1 m_2}}\left(\frac{\dot{m}_1}{m_1} + \frac{\dot{m}_2}{m_2}\right)\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}\right]dt\right. \\ &\quad \left. + \frac{\int_0^t c_3[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}]}{\sqrt{m_1 m_2}}\right] \times \left[\left(\frac{1}{m_1}\int_0^t \left[c_1 \frac{\dot{m}_1}{m_1}\right]dt\right.\right. \\ &\quad \left.\left. - \frac{\dot{m}_1^2}{4m_1^2}\right) - \left(\frac{1}{m_2}\int_0^t \left[c_2\frac{\dot{m}_2}{m_2}\right]dt - \frac{\dot{m}_2^2}{4m_2^2}\right)\right] \neq 0,\end{aligned} \tag{A.3}$$

which shows that $\theta(t)$ is time-dependent. When this last expression is inserted in Eq. (A.2), we obtain the following result:

$$\frac{d\omega_1^2}{dt} = \cos^2\left(\frac{\theta}{2}\right)\left[\left(\frac{1}{m_1}\int_0^t\left[c_1\frac{\dot{m}_1}{m_1}\right]dt - \frac{\dot{m}_1^2}{4m_1^2}\right.\right.$$
$$\left.\left. - \left(\frac{1}{m_2}\int_0^t\left[c_2\frac{\dot{m}_2}{m_2}\right]dt - \frac{\dot{m}_2^2}{4m_2^2}\right)\right]$$
$$\times\left[\frac{-1}{2\sqrt{m_1m_2}}\left(\frac{\dot{m}_1}{m_1} + \frac{\dot{m}_2}{m_2}\right)\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}\right]dt\right.$$
$$\left. + \frac{\int_0^t c_3[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}]}{\sqrt{m_1m_2}}\right]$$
$$\times\left[\frac{\sin\theta}{2}\left(\left[\frac{\int_0^t[c_2\frac{\dot{m}_2}{m_2}]dt}{m_2} - \frac{\int_0^t[c_1\frac{\dot{m}_1}{m_1}]dt}{m_1}\right] - \left[\frac{\dot{m}_1^2}{4m_1^2} - \frac{\dot{m}_2^2}{4m_2^2}\right]\right)\right.$$
$$\left. + \cos(\theta)\left(\frac{\int_0^t c_3[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}]dt}{2\sqrt{m_1m_2}}\right)\right] + \sin(\theta)\left[\frac{c_3[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}]}{2\sqrt{m_1m_2}}\right.$$
$$\left. - \frac{1}{4\sqrt[3]{m_1m_2}}\left(\frac{\dot{m}_1}{m_1} + \frac{\dot{m}_2}{m_2}\right)\int_0^t c_3\left[\frac{\dot{m}_1}{2m_1} + \frac{\dot{m}_2}{2m_2}\right]dt\right] \neq 0. \tag{A.4}$$

The above equation vanishes only if the conditions ($c_1 = c_2$, $c_3 = 0$ and $m_1 = m_2$) are satisfied.

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