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Master

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Theme

Nonsmooth dynamical system gouverned by the normal cone

Presented by:

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Nadjla

Contents

Notations

a.e.	almost everywhere.
i.e.	it mean.
$:=$	equal by definition.
$\equiv$	identically equal.
H	real Hilbert space.
$\Vert \cdot \Vert$	the norm of H.
$\langle \cdot, \cdot \rangle$	the scalar product of H.
$\overline{C}$	the closure of C.
$int(C)$	the interior of C.
$\mathbb{B}$	the closed unit ball.
$B(x, \delta)$	the open ball centered at x of radius δ.
$B[x, \delta]$	the closed ball centered at x of radius δ.
$C([T_0, T], H)$	The Banach space of all continuous functions defined from $[T_0, T]$ to H equipped with the norm of the uniform convergence $\Vert f(\cdot)\Vert_\infty = \sup_{t \in [T_0, T]} \Vert f(t) \Vert$.
$L^p([T_0, T], H)$	The space of measurable p-integrable mapping ($1 \leq p < \infty$) defined on $[T_0, T]$ with values in H equipped with the norm $\Vert f(\cdot)\Vert_{L^p} = \left(\int_{T_0}^{T} \vert f(t)\vert^p \, dt \right)^{\frac{1}{p}}$.
$L^\infty([T_0, T], H)$	The space of essentially bounded functions defined on $[T_0, T]$ with values in H equipped with the norm $\Vert f(\cdot)\Vert_{L^\infty} = \inf \left\{ c \geq 0 : \Vert f(t) \Vert \leq c \text{ a.e. on } [T_0, T] \right\}$.

Introduction

The sweeping process, also known as the Moreau's sweeping process, is a special case of a class of differential inclusion introduced and extensively studied by J.J. Moreau in a series of influential papers. The concept of the sweeping process arises from the need to model systems with unilateral constraints, such as frictional and frictionless contact problems or mechanical systems with impacts.

In Moreau's work, the sweeping process is formulated as a differential inclusion, which is a generalization of ordinary differential equations. While ordinary differential equations describe the evolution of a single valued function, differential inclusions describe the possible evolution of a set-valued function. This set-valued function represents the set of possible states that the system can attain at any given time.

The sweeping process is often represented as follows:

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) \quad \text{a.e. in } [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $N_{C(t)}(\cdot)$ denotes the usual normal cone of the closed convex set $C(t)$ in the sense of convex analysis, x_0 is a given initial condition.

Nowdays sweeping process becomes a famous mathematical model to analyze and solve problems involving unilateral constraints and impacts. It has applications in various fields, including mechanics, control theory, and optimization.

The aim of this master's thesis is to study the existence and uniquness of solution (following the lines in [14], [4] and [12]) for a new variant of the sweeping process, which can be described by the following differential inclusion:

$$(DP) \begin{cases} -\dot{x}(t) \in N_{C(t)}(Ax(t)) + \int\limits_{T_0}^{t} f(t, s, x(s))\, ds \qquad \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0, \quad Ax_0 \in C(T_0), \end{cases}$$

where A is a linear bounded operator and $f : [T_0, T] \times [T_0, T] \times H \longrightarrow H$ is a Lipschitz single-valued function.

It worth mentioning that the nonlinear problem (DP) without any perturbations (i.e. $f \equiv 0$) has been introduced and studied for the first time by Kunze and Marques in [15] where the moving set $C(t)$ varies in a Hausdorff continuous way. After, Kecies et al. in [14] extended the above results for the absolutely continuous moving set $C(t)$ by adding a Lipschitz perturbation. Recently in [4] and [5] the authors proved the well posedness of (DP) with $A = id$ (identical operator). The general degenerate sweeping process perturbed by the sum of a Caratheodory mapping and an integral forcing term has been studied in [13]. Therefore, (DP) is the goal of our master's thesis.

The outline of our work is divided into three chapters as follows:

An opening chapter serves as a reminder and presentation of the fundamental results and basic concepts that will play a crucial role in the subsequent chapters. It covers key notions in convex analysis, accompanied by illustrative examples. Additionally, it highlights classical results in functional analysis and discusses Gronwall's inequalities.

In the second chapter we state our main result where we prove the well posdness of the degenerate perturbed integro-differential sweeping process (DP).

The third chapter concludes our master's thesis by showcasing the application of our main result to the field of differential complementarity problems. This application demonstrates how our findings can be effectively utilized in this specific area of study.

Chapter 1

Preliminaries and auxiliary results

In this chapter, we recall all definitions and auxiliary results that will be used in the sequel of this master's thesis. For more details we refer the reader to [16] [8] [9] [19] [3] [7] [6] [10] [2] [11] [15] [18] [1].

1.1 Convex sets

Definition 1.1.1. *Let x and y be two points of H, x and y are called the endpoints of the line-segment denoted by $[x, y]$ where*

$$[x, y] := \{\lambda x + (1 - \lambda)y,\ 0 \leqslant \lambda \leqslant 1\}.$$

Definition 1.1.2 (Convex set). *A subset C of H is convex if it includes for every pair of points the line segment that joins them, or in other words, if for every choice of $x \in C$ and $y \in C$ one has $[x, y] \subset C$:*

$$\lambda x + (1 - \lambda)y \ \in C \quad \textit{for all } \lambda \in [0, 1].$$

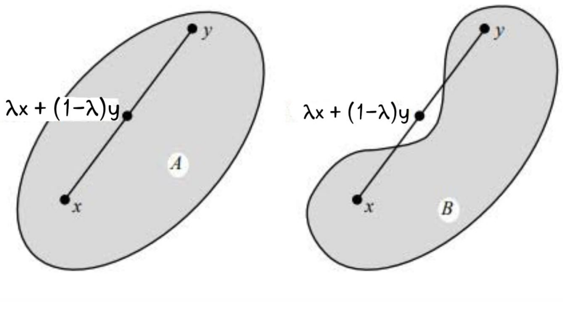


Figure 1.1: A convex, B not convex.

Example 1.1.3.

- *The empty set and H are convex;*
- *The affine subspaces are convex;*
- *In $H = \mathbb{R}$, the convex sets are exactly the intervals;*
- *The open or closed ball is convex.*

Proposition 1.1.4.

1. *If C is convex, then $\overline{C}$ and $int(C)$ are also convex.*
2. *Given an index set I, let $(C_i)_{i\in I}$ be a family of convex sets in H. Then the set $C := \bigcap\limits_{i\in I} C_i$ is also convex.*
3. *C convex and $\lambda \in \mathbb{R}$ implies λC convex.*
4. *If C_1 and C_2 are convex, then $C_1 + C_2$ is also convex.*
5. *If C is convex, then $\lambda_1 C + \lambda_2 C = (\lambda_1 + \lambda_2)C$ for every $\lambda_1, \lambda_2 \geqslant 0$. Consequently $\lambda_1 C + \lambda_2 C$ is also convex.*
6. *A set C is convex if and only if C contains all convex combinations of its elements.*
7. *The union of convex sets is not convex in general.*
   Indeed, if we take singletons $C_1 = \{x\}$ and $C_2 = \{y\}$ which are convex but $C_1 \cup C_2 = \{x, y\}$ it isn't.

Definition 1.1.5 (Convex combination). *A convex combination of the points $(x_i)_{1\leq i\leq k}$ of H is a point of the form*

$$x = \sum_{i=1}^{k} \lambda_i x_i, \text{ with } \lambda_i \geq 0, \ \forall i = 1, 2, ..., k \text{ and } \sum_{i=1}^{k} \lambda_i = 1,$$

i.e. x is an affine combination of $x_1, ..., x_k$, with all the scalars $\lambda_i \geq 0$ for $i = 1, 2, ..., k$.

Definition 1.1.6 (Convex hull). *Let $S \subset H$. The convex hull of S is the intersection of all the convex subsets of H containing S, i.e. the smallest convex subset of H containing S. It is denoted by $co(S)$.*

$$co(S) := \cap\{C \text{ is convex and } S \subset C\}.$$

Proposition 1.1.7. *Let $S \subset H$. The convex hull of S, $co(S)$, is the set of all convex combinations of the points in S, i.e.*

$$co(S) = \left\{ \sum_{i=1}^{n} \lambda_i x_i, (\lambda_i)_{1 \leqslant i \leqslant n} \geq 0, \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N}^*, (x_i)_{1 \leqslant i \leqslant n} \in S \right\}.$$

Remark 1.1.8. *Convex hulls satisfy the following identity*

$$co(A + B) = co(A) + co(B).$$

Definition 1.1.9 (Closed convex hull). *The closed convex hull of a subset $S \subset H$, denoted by $\overline{co}(S)$, is the intersection of all closed convex sets containing S. We note that $\overline{co}(S) = \overline{co(S)}$.*

$$\overline{co}(S) := \cap \{C \, is \, a \, closed \, convex \, and \; S \subset C\}.$$

Remark 1.1.10. *If $S \subset H$ is closed, then $co(S)$ is not necessarily closed.*

1.2 Convex functions

Definition 1.2.1. *Let $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function.*

The effective domain of f is defined by

$$dom f := \{x \in H : f(x) < +\infty\}.$$

The function f is said to be proper if $dom f \neq \emptyset$ and $f(x) \neq -\infty$, $\forall x \in H$.

Definition 1.2.2. *A proper function $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be convex if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

for every $\lambda \in [0, 1]$ and every $x, y \in dom f$.

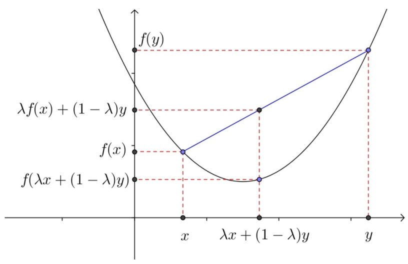


Figure 1.2: Illustration of the definition.

Definition 1.2.3. *The epigraph of a function $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by*

$$epi f := \{(x, r) \in H \times \mathbb{R} : f(x) \leq r\}.$$

Proposition 1.2.4. *Let $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function. Then*

$$f \, is \; convex \Longleftrightarrow epi(f) \, is \; convex \; in \, H \times \mathbb{R}.$$

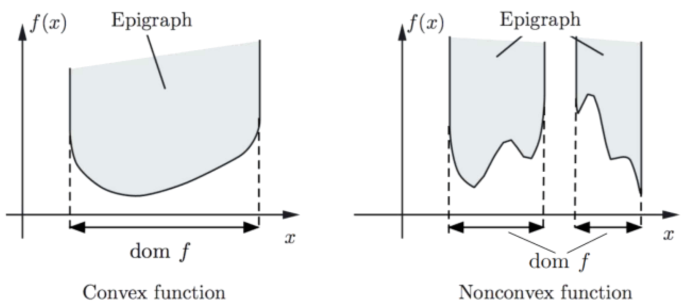


Figure 1.3: Epigraphs of different functions.

Proposition 1.2.5 (Jensen's inequality). *A function $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ is convex if and only if*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i),$$

for all finite families $(\lambda_i)_{1 \leqslant i \leqslant n}$ in $[0, 1]$ such that $\sum\limits_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N}^*$ and $(x_i)_{1 \leqslant i \leqslant n} \in dom(f)$.

1.3 Distance function

Definition 1.3.1. *Let C be a nonempty subset of H and $x \in H$. The distance of x to C, denoted by $d_C(x)$ or $d(x, C)$, is defined by*

$$d_C(x) := \inf_{y \in C} \|x - y\|.$$

Proposition 1.3.2. *Let C be a nonempty subset of H,*

1. $d_C(x) = 0 \Leftrightarrow x \in \overline{C}$.
2. $\mid d_C(x) - d_C(y) \mid \leq \|x - y\|, \quad \forall\, x, y \in H,$
   i.e. $d_C(.)$ *is* $1 -$ *Lipschitz.*
3. *When C is closed we have*
   $d_C(.)$ *is convex if and only if C is convex.*

Proof.

1. We suppose that $d_C(x) = 0 = \inf_{y\in C}\|x-y\|$.

$$\forall\, \varepsilon > 0,\ \exists\, c \in C;\ d_C(x) \leq \|x-c\| < d_C(x)+\varepsilon$$

   Taking $\varepsilon = \frac{1}{n}$ we obtain

$$\forall n \in \mathbb{N}^*,\ \exists\, c_n \in C;\ \|x-c\| \leq \frac{1}{n},$$

   i.e. $c_n \longrightarrow x$, that ensure $x \in \overline{C}$.

   Conversely, we suppose $x \in \overline{C}$ i.e. $\exists\,(c_n)_n \subset C$ such that $c_n \longrightarrow x$.

   We have

$$0 \leq d_C(x) = \inf_{c\in C}\|x-c\| \leq \|x-c\| \quad \forall c \in C.$$

   Or $(c_n)_n \subset C$, then $0 \leq d_C(x) \leq \|x-c_n\| \quad \forall n \in \mathbb{N}^*$,

   taking $n \longrightarrow +\infty$ we deduce $d_C(x) = 0$.

2. We have $d_C(x) = \inf_{c\in C}\|x-c\|$, hence

$$\begin{aligned} d_C(x) &\leq \|x-c\| \quad ;\forall c \in C \\ &\leq \|x-y\| + \|y-c\| \quad ;\forall c \in C,\ y \in H. \end{aligned}$$

   Taking the infimum over C on the above inequality, we obtain

$$\begin{aligned} d_C(x) &\leq \|x-y\| + \inf_{c\in C}\|y-c\| \\ &\leq \|x-y\| + d_C(y), \end{aligned}$$

   then $d_C(x) - d_C(y) \leq \|x-y\|$.

   Exchanging the roles between x and y we get

$$d_C(y) - d_C(x) \leq \|x-y\|,$$

   which gives the desired inequality.

3. We suppose that $d_C(\cdot)$ is convex.

   Let $x, y \in C$, $\alpha \in [0,1]$ we have

$$d_C(\cdot)\ convex \Leftrightarrow d_C(\alpha x + (1-\alpha)y) \leq \alpha\, d_C(x) + (1-\alpha)\, d_C(y),$$

   then $0 \leq d_C(\alpha x + (1-\alpha)y) \leq 0$ (C is closed),

   hence $d_C(\alpha x + (1-\alpha)y) = 0$, then $\alpha x + (1-\alpha)y \in C$, therefore C is convex.

Conversely, let $x, y \in C$, $\alpha \in [0,1]$. We have

$$d_C(x) = \inf_{c_1 \in C} \|x - c_1\| \Rightarrow \forall\, \varepsilon > 0,\ \exists\, c_1 \in C;\ \|x - c_1\| < d_C(x) + \varepsilon,$$

and

$$d_C(y) = \inf_{c_2 \in C} \|y - c_2\| \Rightarrow \forall\, \varepsilon > 0,\ \exists\, c_2 \in C;\ \|y - c_2\| < d_C(y) + \varepsilon,$$

then

$$\alpha\, \|x - c_1\| + (1-\alpha)\, \|y - c_2\| < \alpha\, d_C(x) + (1-\alpha)\, d_C(y) + \varepsilon.$$

On the other hand,

$$d_C(\alpha\, x + (1-\alpha)\, y) = \inf_{c \in C} \|\alpha\, x + (1-\alpha)\, y - c\| \leq \|\alpha\, x + (1-\alpha)\, y - c\| \quad , \forall c \in C.$$

Taking $c := \alpha\, c_1 + (1-\alpha)\, c_2 \in C \quad (c_1, c_2 \in C\, and\, C\, is\, convex)$,

therefore,

$$\begin{aligned} d_C(\alpha\, x + (1-\alpha)\, y) &\leq \|\alpha\, x + (1-\alpha)\, y - \alpha\, c_1 + (1-\alpha)\, c_2\| \\ &\leq \alpha\, \|x - c_1\| + (1-\alpha)\, \|y - c_2\| \\ &\leq \alpha\, d_C(x) + (1-\alpha)\, d_C(y) + \varepsilon, \end{aligned}$$

taking $\varepsilon \longrightarrow 0$, then

$$d_C(\alpha\, x + (1-\alpha)\, y) \leq \alpha\, d_C(x) + (1-\alpha)\, d_C(y).$$

∎

1.4 Normal cone

Definition 1.4.1. *A nonempty subset C of H is called a cone if for each $x \in C$ and each $\lambda \geq 0$ we have $\lambda x \in C$.*
Also if we have C is a convex set, then we say C is a convex cone.

Definition 1.4.2. *The normal cone to a convex subset C of H at a point $x \in C$ is defined as*

$$N_C(x) := \{z \in H : \langle z, y - x \rangle \leq 0, \quad \forall y \in C\}.$$

If $x \notin C$, then $N_C(x) = \emptyset$ (by convention).

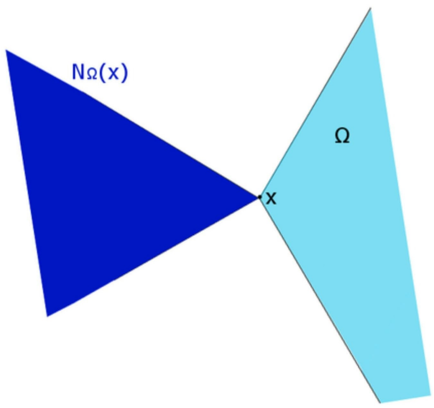


Figure 1.4: Normal cone to the convex set Ω.

Example 1.4.3.

- $N_H(x) = \{0\}$.
- $N_{\{x\}}(x) = H$.
- $C = [0, 1]$

$$N_C(x) = \begin{cases} \mathbb{R}_- & if\, x = 0 \\ \mathbb{R}_+ & if\, x = 1 \\ \{0\} & if\, x \in]0,1[\\ \emptyset & otherwise. \end{cases}$$

Theorem 1.4.4. *Let $C \subset H$ be a convex subset where $int(C) \neq \emptyset$. If $x \in int(C)$ then $N_C(x) = \{0\}$.*

Proof.
By the definition of the normal cone we have $0 \in N_C(x)$ i.e. $\{0\} \subset N_C(x)$. Then, it remains to prove that $N_C(x) \subset \{0\}$.
Let $v \in N_C(x)$.
Since $x \in int(C)$; $\exists \delta > 0$ such that $x + \delta B(0, 1) \subset C$.

$$\begin{aligned} \langle v, x + \delta t - x\rangle &\leq 0, \quad \forall t \in B(0,1), \\ \langle v, \delta t\rangle &\leq 0, \quad \forall t \in B(0,1), \\ \langle v, t\rangle &\leq 0, \quad \forall t \in B(0,1). \end{aligned}$$

Let $r > 0$ small enouph such that $rv \in B(0, 1)$, then, $\langle v, rv\rangle \leq 0$, or $r\|v\|^2 \leq 0$.
So $v = 0$. ■

Proposition 1.4.5. *Let $C \subset H$ be a convex subset and $x \in C$. Then $N_C(x)$ is a closed convex cone.*

Proof.

• $N_C(x)$ is a cone:

Let $v \in N_C(x)$ and $\lambda \geq 0$,

$$\begin{aligned} v \in N_C(x) &\Leftrightarrow \langle v, y - x\rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow \langle \lambda v, y - x\rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow \lambda v \in N_C(x). \end{aligned}$$

•• $N_C(x)$ is convex:

Let $v_1, v_2 \in N_C(x)$ and $\lambda \in]0,1[$

$\lambda\, v_1 + (1-\lambda)\, v_2 \in N_C(x)$?

Let $y \in C$

$$\langle \lambda\, v_1 + (1-\lambda)\, v_2, y - x\rangle = \lambda\, \langle v_1, y - x\rangle + (1-\lambda)\, \langle v_2, y - x\rangle \leq 0.$$

• • • $N_C(x)$ is closed:

Let $(v_n)_n \in N_C(x)$ such that $v_n \longrightarrow v$,

$$\begin{aligned} v_n \in N_C(x), \forall n &\Rightarrow \langle v_n, y - x\rangle \leq 0, \quad \forall y \in C,\ \forall n \\ &\Rightarrow \lim_{n\to+\infty} \langle v_n, y - x\rangle \leq 0, \quad \forall y \in C \\ &\Rightarrow \langle v, y - x\rangle \leq 0, \quad \forall y \in C \\ &\Rightarrow v \in N_C(x). \end{aligned}$$

■

Proposition 1.4.6. *Let C, C_1, C_2 be three subsets of H:*

1. $N_C(-x) = -N_{-C}(x) \quad \forall x \in -C$.

2. $N_{C+a}(x+a) = N_C(x)$ *for all* $x \in C$, $a \in H$.

3. $N_{C_1\times C_2}(x_1, x_2) = N_{C_1}(x_1) \times N_{C_2}(x_2)$ *for all* $x_1 \in C_1$, $x_2 \in C_2$.

Definition 1.4.7. *Let K be a nonempty closed convex subset of H. The dual cone of K is definded by*

$$K^* := \{p \in H,\ \langle p, v\rangle \geq 0,\ \textit{for all}\, v \in K\}.$$

1.5 The subdifferential of convex functions

Definition 1.5.1. *Let $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper and convex function. We say that $\xi \in H$ is a subgradient of f at a point $x_0 \in dom(f)$ if*

$$\langle \xi, x - x_0 \rangle \leq f(x) - f(x_0) \quad for\, all\, x \in H. \tag{1.1}$$

The set of all $\xi \in H$ satisfying (1.1), denoted by $\partial f(x_0)$, is called the subdifferential of f at x_0 and definded by

$$\partial f(x_0) := \{\xi \in H : \langle \xi, x - x_0 \rangle \leq f(x) - f(x_0) \quad \forall x \in H\}.$$

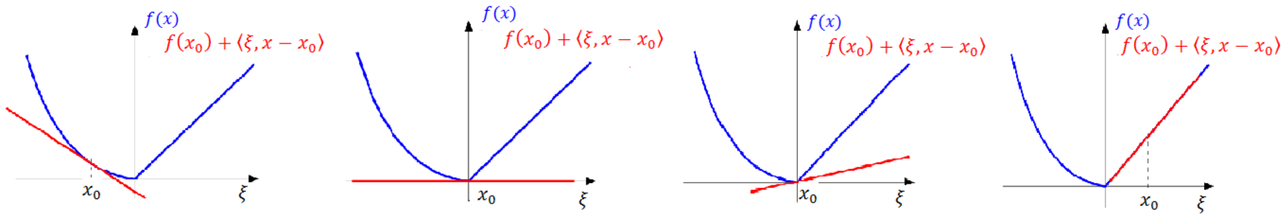


Figure 1.5: The subdifferential at different points.

Remark 1.5.2. *By convention we set $\partial f(x_0) = \emptyset$ if $x_0 \notin dom(f)$.*

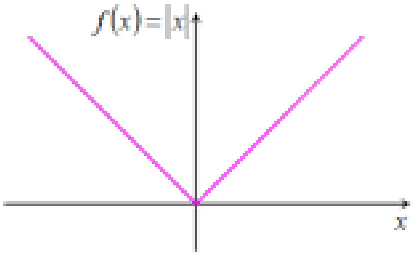
Example 1.5.3. *Let $f : \mathbb{R} \to \mathbb{R}, x \mapsto f(x) = |x|$. Then $\partial f(0) = [-1, 1]$, indeed,*

$$\begin{aligned}
\partial f(0) &= \{\xi \in \mathbb{R} : \langle \xi, x \rangle \leq f(x) - f(0) \quad \forall x \in \mathbb{R}\} \\
&= \{\xi \in \mathbb{R} : x \cdot \xi \leq |x|, \quad \forall x \in \mathbb{R}\} \\
&= \{\xi \in \mathbb{R} : x \cdot \xi \leq x, \ \forall\, x > 0\} \cap \{\xi \in \mathbb{R} : x \cdot \xi \leq -x, \ \forall\, x < 0\} \cap \mathbb{R} \\
&= \{\xi \in \mathbb{R} : \xi \leq 1\} \cap \{\xi \in \mathbb{R} : \xi \geq -1\} \cap \mathbb{R} \\
&= [-1, 1].
\end{aligned}$$

In the next proposition we give a geometrical caractirisation of the normal cone.

Proposition 1.5.4. *Let $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function. If $x_0 \in dom(f)$ then*

$$\partial f(x_0) = \left\{\xi \in H;\ (\xi, -1) \in N_{epi(f)}(x_0, f(x_0))\right\}.$$



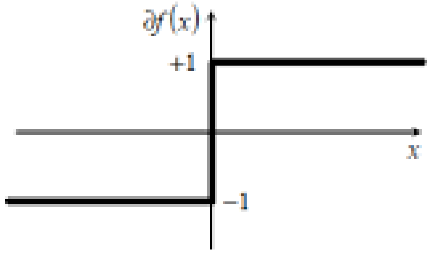


Figure 1.6: The graphs of the abslute function and its subdifferential.

Proof.

We start by the opposite incusion " $\supset$ " :

Let $(\xi, -1) \in N_{epi(f)}(x_0, f(x_0))$ this implies by definition

$$\big\langle (\xi, -1), (x, \lambda) - (x_0, f(x_0)) \big\rangle_{H\times\mathbb{R}} \leq 0 \quad \forall (x, \lambda) \in epi(f).$$

Or,

$$\big\langle \xi, x - x_0 \big\rangle_H + \big\langle -1, \lambda - f(x_0) \big\rangle_{\mathbb{R}} \leq 0 \quad \forall (x, \lambda) \in epi(f),$$

$$\big\langle \xi, x - x_0 \big\rangle_H \leq \lambda - f(x_0) \quad \forall (x, \lambda) \in epi(f).$$

Two cases eraise:

- If $x \notin dom(f) : f(x) = +\infty$, then we obtain $\big\langle \xi, x - x_0 \big\rangle \leq f(x) - f(x_0) \quad \forall x \in H$, i.e. $\xi \in \partial f(x_0)$.
- If $x \in dom(f) : f(x) < +\infty$, then $(x, f(x)) \in epi(f) \implies \big\langle \xi, x - x_0 \big\rangle \leq f(x) - f(x_0)$, hence $\xi \in \partial f(x_0)$.

Now, we prove the direct inclusion " $\subset$ ":

Let $\xi \in \partial f(x_0)$, we have $\big\langle \xi, x - x_0 \big\rangle \leq f(x) - f(x_0) \quad , \forall x \in H$.

$$\begin{aligned} \big\langle (\xi, -1), (x, \lambda) - (x_0, f(x_0)) \big\rangle_{H\times\mathbb{R}} &= \big\langle \xi, x - x_0 \big\rangle_H + \big\langle -1, \lambda - f(x_0) \big\rangle_{\mathbb{R}} \\ &= \big\langle \xi, x - x_0 \big\rangle - \lambda + f(x_0) \\ &\leq \big\langle \xi, x - x_0 \big\rangle - f(x) + f(x_0) \leq 0. \end{aligned}$$

Therefore, $(\xi, -1) \in N_{epi(f)}(x_0, f(x_0))$. ■

Example 1.5.5. *The subdifferential of the abslute function at* $(0, 0)$*:*

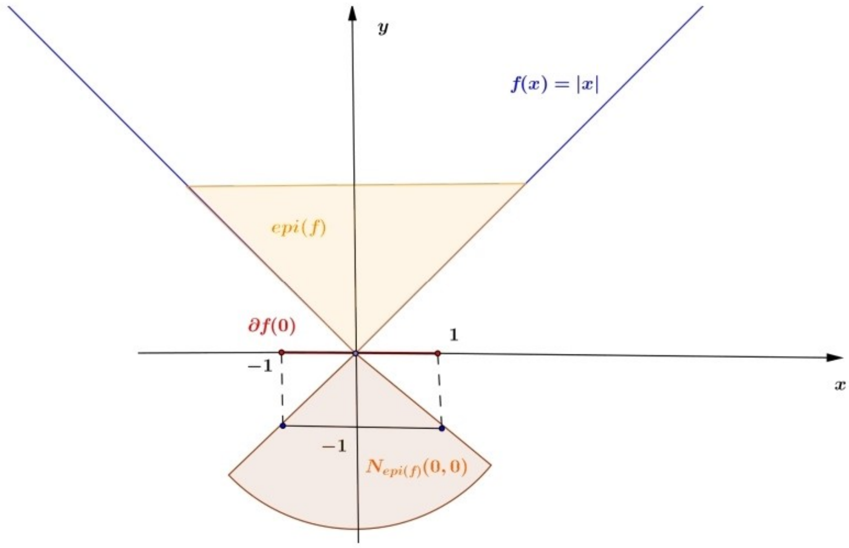


Figure 1.7: The subdifferential of the abslute function at $(0, 0)$.

1.6 Indicator function

Definition 1.6.1. *Let S be a non empty subset of H, the indicator function at S is definded by*

$$\begin{aligned} \delta_S(\cdot) : H &\longrightarrow \mathbb{R} \cup \{+\infty\} \\ x &\longmapsto \delta_S(x) = \begin{cases} 0 & \text{if} \quad x \in S, \\ +\infty & \text{if} \quad x \notin S. \end{cases} \end{aligned}$$

Remark 1.6.2.

- $dom(\delta_S) = \{x \in H : \delta_S(x) < +\infty\} = S.$
- $epi(\delta_S) = \{(x, r) \in H \times \mathbb{R} : \delta_S(x) \leq r\} = S \times [0, +\infty[.$
- *The function $\delta_S(\cdot)$ is convex if and only if S convex.*

Proposition 1.6.3. *Let S be a non empty convex subset of H, and $x_0 \in S$, then the subdifferential of the indicator function is*

$$\partial \delta_S(x_0) = N_S(x_0).$$

Proof.
Let $\xi \in \partial \delta_S(x_0)$, then

$$\langle \xi, x - x_0 \rangle \leq \delta_S(x) - \delta_S(x_0), \quad \forall x \in H.$$

In particular for $x \in S$ we have

$$\langle \xi, x - x_0 \rangle \leq 0, \quad \forall x \in S.$$

Therefore $\xi \in N_S(x_0)$.
Conversely, if $\xi \in N_S(x_0)$ then

$$\langle \xi, x - x_0 \rangle \leq 0 \quad, \forall x \in S.$$

Or $\langle \xi, x - x_0 \rangle \leq \delta_S(x) - \delta_S(x_0), \quad \forall x \in S$.
More $\langle \xi, x - x_0 \rangle \leq \delta_S(x) - \delta_S(x_0), \quad \forall x \in H \backslash S$ (because $\delta_S(x) = +\infty$),
therefore $\xi \in \partial \delta_S(x_0)$. ∎

1.7 The support function

Definition 1.7.1. *The support function of a nonempty subset S of H is the function $\sigma(S, \cdot) : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by*

$$\sigma(S, x) := \sup_{\xi \in S} \langle x, \xi \rangle.$$

The support function allows us to express the inclusion $\xi \in S$ in an analytical form.

Theorem 1.7.2. *[19] Let S be a convex closed set. Then $\xi \in S$ if and only if*

$$\langle x, \xi \rangle \leq \sigma(S, x) \quad \textit{for all } x \in H.$$

1.8 Weak topology

Let E be a Banach space and E' its topological dual i.e., $E' = \{f : E \longrightarrow \mathbb{R},\ f \textit{ continuous linear}\}$ such that for all $f \in E'$

$$\|f\|_{E'} = \sup_{\substack{x \in E \\ \|x\| \leq 1}} |\langle f, x \rangle|.$$

We denote by $\varphi_f : E \longrightarrow \mathbb{R}$ the linear functional $\varphi_f(x) = \langle f, x \rangle$. As f runs through E' we obtain a collection $(\varphi_f)_{f \in E'}$ of maps from E into $\mathbb{R}$.

Definition 1.8.1. *The weak topology $\sigma(E, E')$ on E is the weakest topology on E making all maps $(\varphi_f)_{f \in E'}$ continuous.*

Definition 1.8.2. *We say that a sequence* (x_n) *in* E *converges to* x *in the weak topology* $\sigma(E, E')$ *and we note* $x_n \rightharpoonup x$ *if and only if* $\langle f, x_n \rangle \longrightarrow \langle f, x \rangle \quad, \forall f \in E'$.

Remark 1.8.3. *If* E *is an Hilbert space then we can identify* E' *by* E*, then* $x_n \rightharpoonup x \Leftrightarrow \langle y, x_n \rangle \longrightarrow \langle y, x \rangle \quad, \forall y \in E$*, where* $\langle \cdot, \cdot \rangle$ *is the scalar product of* E.

Proposition 1.8.4. *Let* (x_n) *be a sequence in* E*, we have if* $x_n \longrightarrow x$ *strongly, then* $x_n \rightharpoonup x$ *weakly in* $\sigma(E, E')$.

Proposition 1.8.5. *Let* C *be a closed convex subset of* E*, then* C *is closed in the weak topology* $\sigma(E, E')$.

1.9 Semicontinuity

We start by recalling well-known definitions.

Definition 1.9.1. *We say that a real function* $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ *is continuous at* $x \in H$ *provided that for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $|f(x) - f(y)| < \varepsilon$ *whenever* $|x - y| < \delta$*. If this property holds for every* $x \in H$ *we say that* f *is continuous on* H.

Definition 1.9.2. *Let* $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be a proper function and let* $x_0 \in H$*. We say that* f *is lower semicontinuous (lsc), respectively upper semicontinuous (usc), at* x_0*, provided that* $f(x_0) \leq \liminf_{n \to +\infty} f(x_n)$*, respectively* $f(x_0) \geq \liminf_{n \to +\infty} f(x_n)$*, for every sequence* $(x_n)_{n \in \mathbb{N}} \subset H$ *such that* $x_n \to x_0$ *as* $n \to +\infty$.
If the property holds for every point $x_0 \in H$ *we say that* f *is lsc, usc respectively, on* H.

Remark 1.9.3.

- f *is continuous (at* $x_0 \in H$*) if it is both lower and upper semicontinuous (at* x_0*),*
- f *is lsc if and only if* $-f$ *is usc,*

Proposition 1.9.4. *Let* $f, g : H \longrightarrow \mathbb{R} \cup \{+\infty\}$*,* $\lambda > 0$*. We have*

1. *If* f*,* g *are lsc, respectively usc, then* $f + g$ *is lsc, respectively usc;*
2. *If* f *is lsc, respectively usc, then* λf *is also lsc, respectively usc;*

3. f is lower semicontinuous if and only if epi(f) is closed in $H \times \mathbb{R}$.

Definition 1.9.5. *A function $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ is weakly lower semicontinuous at x_0 if, for every sequence $(x_n)_{n\in\mathbb{N}}$ in H such that $x_n \rightharpoonup x_0$ as $n \to +\infty$, we have*

$$\liminf_{n\to+\infty} f(x_n) \geq f(x_0).$$

1.10 Maximally monotone operators

Definition 1.10.1. *Let X and Y be two sets. $F : X \rightrightarrows Y$ is a set-valued mapping (multifunction) defined from X to $\mathscr{P}(Y)$ (subsets of Y).*

The set $dom(F) = \{x \in X : F(x) \neq \phi\}$ is called the domain of F.

The graph of F is denoted $gr(F)$ and is defined by

$$gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

The image of F is defined as

$$im(F) = \{y \in Y / \exists\, x \in X : y \in F(x)\}.$$

Definition 1.10.2. *The operator $F : X \rightrightarrows Y$ is monotone if*

$$\forall x_1, x_2 \in dom(F)\ \ \forall y_1 \in F(x_1)\ \ \forall y_2 \in F(x_2),\ \ \langle y_1 - y_2, x_1 - x_2\rangle \geq 0.$$

Example 1.10.3. *Let $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be convex and proper. Then the subdifferential of f (∂f) is monotone.*

Proof. Take $x_1, x_2 \in dom(\partial f)$ and $y_1 \in \partial f(x_1)$, $y_2 \in \partial f(x_2)$. Then

$$y_1 \in \partial f(x_1) \Leftrightarrow f(x) \geq f(x_1) + \langle y_1, x - x_1\rangle\,,\ \forall\, x \in H, \tag{1.2}$$

$$y_2 \in \partial f(x_2) \Leftrightarrow f(x) \geq f(x_2) + \langle y_2, x - x_2\rangle\,,\ \forall\, x \in H, \tag{1.3}$$

in particular for $x = x_2$ in (1.2) and for $x = x_1$ in (1.3) we have, $f(x_2) \geq f(x_1) + \langle y_1, x_2 - x_1\rangle$ and $f(x_1) \geq f(x_2) + \langle y_2, x_1 - x_2\rangle$. Adding these inequalities, we conclude that $\langle y_1 - y_2, x_1 - x_2\rangle \geq 0$. ■

Remark 1.10.4. *If F and G are monotone, then $F + G$ is monotone.*

Definition 1.10.5. *Let $F : X \rightrightarrows Y$ be monotone. Then F is maximally monotone (or maximal monotone) if there exists no monotone operator $G : X \rightrightarrows Y$ such that $gr(F) \subset gr(G)$.*

Remark 1.10.6. *Let $\lambda > 0$ and $F : X \rightrightarrows Y$ be a maximally monotone operator then λF is maximal monotone.*

Corollary 1.10.7. *[3] Let $F : H \to H$ be monotone and continuous. Then F is maximally monotone.*

1.11 Absolutely continuous functions

Definition 1.11.1. *A function $u : [T_0, T] \to H$, $t \mapsto u(t)$, is called absolutely continuous on $[T_0, T] \subset \mathbb{R}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that: $\sum\limits_{i=1}^{n} \|u(b_i) - u(a_i)\| < \varepsilon$, for any n and any disjoint collection of intervals $[a_i, b_i] \in [T_0, T]$ satisfying $\sum\limits_{i=1}^{n} (b_i - a_i) < \delta$.*

Theorem 1.11.2. *A function $u : [T_0, T] \to H$ is absolutely continuous if and only if it is the integral of its derivative, i.e.*

$$u(t) - u(T_0) = \int_{T_0}^{t} \dot{u}(s) ds, \quad \forall\, t \in]T_0, T[.$$

Remark 1.11.3.

- *An absolutely continuous function is continuous but the inverse is false;*
- *A Lipschitz function is absolutely continuous;*
- *An absolutely continuous function is derivable a.e.*

Lemma 1.11.4. *Let $x : [T_0, T] \longrightarrow H$ be an absolutely continuous function, and let $A : H \longrightarrow H$ be a symmetric bounded linear operator, then*

$$\frac{d}{dt}\big\langle x(t), Ax(t) \big\rangle = 2\big\langle \dot{x}(t), Ax(t) \big\rangle, \quad a.e.\ t \in [T_0, T].$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\big\langle x(t), Ax(t)\big\rangle &= \lim_{h\to 0}\frac{1}{h}\Big[\big\langle x(t+h), Ax(t+h)\big\rangle - \big\langle x(t), Ax(t)\big\rangle\Big] \\
&= \lim_{h\to 0}\frac{1}{h}\Big[\big\langle x(t+h), A(x(t+h)+x(t)-x(t))\big\rangle - \big\langle x(t), Ax(t)\big\rangle\Big] \\
&= \lim_{h\to 0}\frac{1}{h}\Big[\big\langle x(t+h), A(x(t+h)-x(t))\big\rangle + \big\langle x(t+h), Ax(t)\big\rangle - \big\langle x(t), Ax(t)\big\rangle\Big] \\
&= \lim_{h\to 0}\frac{1}{h}\Big[\big\langle x(t+h), A(x(t+h)-x(t))\big\rangle + \big\langle x(t+h)-x(t), Ax(t)\big\rangle\Big] \\
&= \lim_{h\to 0}\frac{1}{h}\big\langle x(t+h), A(x(t+h)-x(t))\big\rangle + \lim_{h\to 0}\frac{1}{h}\big\langle x(t+h)-x(t), Ax(t)\big\rangle \\
&= \big\langle x(t), A\dot{x}(t)\big\rangle + \big\langle \dot{x}(t), Ax(t)\big\rangle \\
&= 2\big\langle \dot{x}(t), Ax(t)\big\rangle.
\end{aligned}$$

■

1.12 Some convergence results

Lemma 1.12.1 (Mazur's lemma). *Let V be a Banach space and $(u_n)_{n\in\mathbb{N}} \subset V$ a sequence converging weakly to u in V. Then*

1. *$u \in \overline{co}\{u_k, k \geq n\}$, for any $n \in \mathbb{N}$.*

2. *There exists $(v_n)_n \subset V$ such that $v_n \in co\{u_k; k \geq n\}$ for any $n \in \mathbb{N}$ and $v_n \longrightarrow u$ strongly in V.*

Theorem 1.12.2 (Lebesgue's dominated convergence theorem). *Let (f_n) be a sequence of functions in $L^p([T_0,T],H)$ $(1 \leq p < +\infty)$ that satisfy*

1. *$f_n \longrightarrow f$ a.e. on $[T_0,T]$.*

2. *There is a function $g(\cdot) \in L^p([T_0,T],H)$ such that for all $n \in \mathbb{N}$,*

$$\|f_n(t)\| \leq g(t) \;\; a.e.\ t \in [T_0,T].$$

Then $f(\cdot) \in L^p([T_0,T],H)$ and $\|f_n - f\|_p \longrightarrow 0$.

Theorem 1.12.3. *Let (f_n) be a sequence in $L^p([T_0,T],H)$ and let $f \in L^p([T_0,T],H)$ be such that $\|f_n - f\|_p \longrightarrow 0$. Then, there exists a subsequence $(f_{n_k})_k$ converges to f a.e. on $[T_0,T]$.i.e.,*

$$f_{n_k}(x) \longrightarrow f(x)\ a.e.\ on\ [T_0,T].$$

1.13 Gronwall's inequality

Lemma 1.13.1 (Gronwall's inequality). *[4] Let $T > T_0$ be given reals and $a(\cdot)$, $b(\cdot) \in L^1([T_0,T],\mathbb{R})$ with $b(t) \geq 0$ for almost all $t \in [T_0,T]$. Let the absolutely continuous function $w : [T_0,T] \longrightarrow \mathbb{R}_+$ satisfy:*

$$(1-\alpha)w'(t) \leq a(t)w(t) + b(t)w^{\alpha}(t), \quad a.e.\ t \in [T_0,T],$$

where $0 \leq \alpha < 1$. Then for all $t \in [T_0,T]$, one has

$$w^{1-\alpha}(t) \leq w^{1-\alpha}(T_0)\exp\left(\int_{T_0}^{t} a(\tau)d\tau\right) + \int_{T_0}^{t} \exp\left(\int_{s}^{t} a(\tau)d\tau\right)b(s)ds.$$

Lemma 1.13.2. *Let $\rho : [T_0,T] \longrightarrow \mathbb{R}$ be a non-negative absolutely continuous function and let a, $b : [T_0,T] \longrightarrow \mathbb{R}_+$ be non-negative Lebesgue integrable functions. Assume that*

$$\dot{\rho}(t) \leq a(t) + b(t)\int_{T_0}^{t} \rho(s)\,ds, \quad a.e.\ t \in [T_0,T]. \tag{1.4}$$

Then for all $t \in [T_0,T]$, one has

$$\rho(t) \leq \rho(T_0)\exp\left(\int_{T_0}^{t} B(\tau)d\tau\right) + \int_{T_0}^{t} a(s)\exp\left(\int_{s}^{t} B(\tau)d\tau\right)ds, \quad a.e.\ t \in [T_0,T],$$

where $\qquad B(t) := \max\{1, b(t)\}, \quad a.e.\ t \in [T_0,T].$

Proof. Setting $z(t) = \int_{T_0}^{t} \rho(s)\,ds \implies \dot{z}(t) = \rho(t),\ \ddot{z}(t) = \dot{\rho}(t)$. Then from (1.4) we see that

$$\ddot{z}(t) \leq a(t) + b(t)z(t), \quad \text{a.e.}\, t \in [T_0,T].$$

Setting $w(t) = \dot{z}(t) + z(t)$, for all $t \in [T_0,T]$. Hence, for a.e. $t \in [T_0,T]$

$$\dot{w}(t) = \ddot{z}(t) + \dot{z}(t) \leq a(t) + b(t)z(t) + \dot{z}(t),$$

$$\begin{aligned} &\Rightarrow \dot{w}(t) \leq a(t) + \max\{b(t),1\}(\dot{z}(t) + z(t)), \\ &\Leftrightarrow \dot{w}(t) \leq a(t) + B(t)w(t), \quad \text{a.e. } t \in [T_0,T]. \end{aligned}$$

Applying the Gronwall inequalety 1.13.1 with w, we obtain for all $t \in [T_0,T]$

$$w(t) \leq w(T_0)\exp\left(\int_{T_0}^{t} B(\tau)\,d\tau\right) + \int_{T_0}^{t} a(s)\,\exp\left(\int_{s}^{t} B(\tau)\,d\tau\right)ds,$$

which gives

$$\rho(t) \le \dot{z}(t) + z(t) = w(t) \le \rho(T_0)\, \exp\left(\int_{T_0}^{t} B(\tau)\, d\tau\right) + \int_{T_0}^{t} a(s)\, \exp\left(\int_{s}^{t} B(\tau)\, d\tau\right) ds.$$

■

Lemma 1.13.3. *Let $\rho : [T_0, T] \longrightarrow \mathbb{R}$ be a non-negative absolutely continuous function and let $K, \varepsilon : [T_0, T] \longrightarrow \mathbb{R}_+$ be non-negative Lebesgue integrable functions. Suppose for some $\epsilon > 0$*

$$\dot{\rho}(t) \le \varepsilon(t) + \epsilon + K(t)\sqrt{\rho(t)} \int_{T_0}^{t} \sqrt{\rho(s)}\, ds, \quad a.e.\ t \in [T_0, T]. \tag{1.5}$$

Then for all $t \in [T_0, T]$, one has

$$\begin{aligned} \sqrt{\rho(t)} \le \sqrt{\rho(T_0) + \epsilon}\, \exp\left(\int_{T_0}^{t} H(s)\, ds\right) + \frac{\sqrt{\epsilon}}{2} \int_{T_0}^{t} \exp\left(\int_{s}^{t} H(\tau)\, d\tau\right) ds \\ + 2\left(\sqrt{\int_{T_0}^{t} \varepsilon(s)\, ds + \epsilon} - \sqrt{\epsilon}\, \exp\left(\int_{T_0}^{t} H(\tau)\, d\tau\right)\right) \\ + 2\int_{T_0}^{t} H(s) \exp\left(\int_{s}^{t} H(\tau)\, d\tau\right) \sqrt{\int_{T_0}^{s} \varepsilon(\tau)\, d\tau + \epsilon}\; ds, \end{aligned} \tag{1.6}$$

where $H(t) := \max\left\{1, \dfrac{K(t)}{2}\right\}$, *a.e.* $t \in [T_0, T]$.

Proof.

Let $\lambda(t) = \sqrt{\int_{T_0}^{t} \varepsilon(s)\, ds + \epsilon}$ and $z_\varepsilon(t) = \sqrt{\rho(t) + \lambda^2(t)}$ for all $t \in [T_0, T]$.

From (1.5) we have for a.e. $t \in [T_0, T]$

$$\dot{\rho}(t) \le \varepsilon(t) + \epsilon + K(t)\sqrt{\rho(t) + \lambda^2(t)} \int_{T_0}^{t} \sqrt{\rho(s) + \lambda^2(s)}\, ds, \tag{1.7}$$

and

$$\dot{z}_\varepsilon(t) = \frac{\dot{\rho}(t) + 2\dot{\lambda}(t)\lambda(t)}{2\sqrt{\rho(t) + \lambda^2(t)}} = \frac{\dot{\rho}(t) + \varepsilon(t)}{2 z_\varepsilon(t)},$$

or equivalently

$$\dot{\rho}(t) = 2 z_\varepsilon(t) \dot{z}_\varepsilon(t) - \varepsilon(t).$$

Hence from (1.7)

$$2z_\varepsilon(t)\dot{z}_\varepsilon(t) \le 2\varepsilon(t) + \epsilon + K(t)z_\varepsilon(t)\int_{T_0}^{t} z_\varepsilon(s)\,ds.$$

Therefore

$$\dot{z}_\varepsilon(t) \le \frac{\varepsilon(t)}{z_\varepsilon(t)} + \frac{\epsilon}{2z_\varepsilon(t)} + \frac{K(t)}{2}\int_{T_0}^{t} z_\varepsilon(s)\,ds. \tag{1.8}$$

On the other hand, we note that

$$\lambda(t) = \sqrt{\int_{T_0}^{t}\varepsilon(s)\,ds + \epsilon} \le \sqrt{\rho(t) + \int_{T_0}^{t}\varepsilon(s)\,ds + \epsilon} = \sqrt{\rho(t)+\lambda^2(t)} = z_\varepsilon(t),$$

then,

$$\frac{1}{z_\varepsilon(t)} \le \frac{1}{\lambda(t)} \quad\Longleftrightarrow\quad \frac{\varepsilon(t)}{z_\varepsilon(t)} \le \frac{\varepsilon(t)}{\lambda(t)}. \tag{1.9}$$

Also we have $\dot{\lambda}(t) = \dfrac{\varepsilon(t)}{2\lambda(t)}$. Then $\dfrac{\varepsilon(t)}{z_\varepsilon(t)} \le 2\dot{\lambda}(t)$, and

$\sqrt{\epsilon} \le \sqrt{\epsilon + \int_{T_0}^{t}\varepsilon(s)\,ds} = \lambda(t) \le z_\varepsilon(t)$, hence

$$\frac{\epsilon}{2z_\varepsilon(t)} \le \frac{\sqrt{\epsilon}}{2}. \tag{1.10}$$

The relations (1.8), (1.9), (1.10) give the inequality

$$\dot{z}_\varepsilon(t) \le 2\dot{\lambda}(t) + \frac{\sqrt{\epsilon}}{2} + \frac{K(t)}{2}\int_{T_0}^{t} z_\varepsilon(s)\,ds. \tag{1.11}$$

Letting $H(t) := \max\left\{1, \dfrac{K(t)}{2}\right\}$ and applying the Gronwall lemma 1.13.2 with z_ε, one obtains for all $t \in [T_0, T]$

$$\begin{aligned} z_\varepsilon(t) &\le z_\varepsilon(T_0)\exp\Big(\int_{T_0}^{t} H(s)\,ds\Big) + \int_{T_0}^{t}(2\dot{\lambda}(s) + \frac{\sqrt{\epsilon}}{2})\exp\Big(\int_{s}^{t} H(\tau)\,d\tau\Big)\,ds \\ &= \sqrt{\rho(T_0)+\epsilon}\exp\Big(\int_{T_0}^{t} H(s)\,ds\Big) + \int_{T_0}^{t}(2\dot{\lambda}(s) + \frac{\sqrt{\epsilon}}{2})\exp\Big(\int_{s}^{t} H(\tau)\,d\tau\Big)\,ds, \end{aligned}$$

or equivalently

$$\begin{aligned} z_\varepsilon(t) \le \sqrt{\rho(T_0)+\epsilon}\exp\Big(\int_{T_0}^{t} H(s)\,ds\Big) + 2\int_{T_0}^{t}\dot{\lambda}(s)\exp\Big(\int_{s}^{t} H(\tau)\,d\tau\Big)\,ds \\ + \frac{\sqrt{\epsilon}}{2}\int_{T_0}^{t}\exp\Big(\int_{s}^{t} H(\tau)\,d\tau\Big)\,ds. \end{aligned}$$

On the other hand, from integration by parts, we note that

$$\begin{aligned}
&\int_{T_0}^{t} \dot{\lambda}(s) \exp\big(\int_{s}^{t} H(\tau)\, d\tau\big)\, ds \\
&= \Big[\exp\big(\int_{s}^{t} H(\tau)\, d\tau\big)\lambda(s)\Big]_{s=T_0}^{s=t} + \int_{T_0}^{t} H(s) \exp\big(\int_{s}^{t} H(\tau)\, d\tau\big)\lambda(s)\, ds \\
&= \lambda(t) - \exp\big(\int_{T_0}^{t} H(\tau)\, d\tau\big)\sqrt{\epsilon} + \int_{T_0}^{t} H(s) \exp\big(\int_{s}^{t} H(\tau)\, d\tau\big)\lambda(s)\, ds,
\end{aligned}$$

which combined with what precedes gives

$$\begin{aligned}
z_\varepsilon(t) \le &\sqrt{\rho(T_0)+\epsilon}\, \exp\big(\int_{T_0}^{t} H(s)\, ds\big) + \frac{\sqrt{\epsilon}}{2}\int_{T_0}^{t} \exp\big(\int_{s}^{t} H(\tau)\, d\tau\big)\, ds \\
&+ 2\lambda(t) - 2\exp\big(\int_{T_0}^{t} H(\tau)\, d\tau\big)\sqrt{\epsilon} + 2\int_{T_0}^{t} H(s) \exp\big(\int_{s}^{t} H(\tau)\, d\tau\big)\lambda(s)\, ds.
\end{aligned}$$

Consequently, observing that $\sqrt{\rho(t)} \le \sqrt{\rho(t)+\lambda^2(t)} = z_\varepsilon(t)$ we obtain

$$\begin{aligned}
\sqrt{\rho(t)} \le &\sqrt{\rho(T_0)+\epsilon}\, \exp\big(\int_{T_0}^{t} H(s)\, ds\big) + \frac{\sqrt{\epsilon}}{2}\int_{T_0}^{t} \exp\big(\int_{s}^{t} H(\tau)\, d\tau\big)\, ds + 2\lambda(t) \\
&- 2\exp\big(\int_{T_0}^{t} H(\tau)\, d\tau\big)\sqrt{\epsilon} + 2\int_{T_0}^{t} H(s) \exp\big(\int_{s}^{t} H(\tau)\, d\tau\big)\lambda(s)\, ds.
\end{aligned}$$

Since $\lambda(t) = \sqrt{\int_{T_0}^{t} \varepsilon(s)\, ds + \epsilon}$, we obtain (1.6),

which completes the proof of the lemma . ■

In [14] the authors proved the following existence and uniquness result for the degenerate sweeping process where the moving set moved in an absolutely continuous way.

Theorem 1. *[14] Let H be a real Hilbert space with $\langle \cdot, \cdot \rangle$ as its scalar product. Let $T > 0$ be a positive real number and $C : [T_0, T] \rightrightarrows H$ a multi-valued mapping. Suppose that the following hypotheses are satisfied:*

($\mathcal{H}_1$) *For each* $t \in [T_0, T]$*, the set* $C(t)$ *is a nonempty closed and convex subset of* H*.*

($\mathcal{H}_2$) *There exists a non-negative absolutely continuous function* $\upsilon(.) : [0, T] \longrightarrow \mathbb{R}_+$ *with* $\upsilon(0) = 0$ *such that*

$$\forall\, x \in H,\ \forall s, t \in [0, T]: \quad |d_{C(t)}(x) - d_{C(s)}(x)| \leq |\upsilon(t) - \upsilon(s)|.$$

($\mathcal{H}_3$) $A : H \longrightarrow H$ *is a bounded linear operator which is symmetric and* β*-coercive, that is there exists* $\beta > 0$ *such that*

$$\langle Ax, x\rangle = \langle x, Ax\rangle \geq \beta \|x\|^2 \quad , \forall x \in H.$$

Then for all $Au_0 \in C(0)$ *there exists one and unique absolutely continuous solution for the following differential inclusion:*

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(Au(t)) \quad a.e.\ t \in [0, T]\,, \\ u(0) = u_0, \quad Au_0 \in C(0). \end{cases}$$

Moreover, the solution verifies the following inequality:

$$\|\dot{u}(t)\| \leq \frac{1}{\beta} |\dot{\upsilon}(t)| \quad a.e.\ \ t \in [0, T]\,.$$

Chapter 2

The well-posedness of the degenerate perturbed integro-differential sweeping process

2.1 Standing hypotheses

Let H be a real Hilbert space with $\langle \cdot, \cdot \rangle$ as its scalar product. Let $T > 0$ be a positive real number and $C : [T_0, T] \rightrightarrows H$ a multi-valued mapping with nonempty closed and convex values. Let $f : [T_0, T] \times [T_0, T] \times H \longrightarrow H$ be a single-valued mapping. Given the following hypotheses:

($\mathcal{H}_1$) For each $t \in [T_0, T]$, the set $C(t)$ is a nonempty closed and convex subset of H and moves in an absolutely continuous way, that is there exists an absolutely continuous function $\upsilon : [T_0, T] \longrightarrow \mathbb{R}$ such that

$$C(t) \subset C(s) + |\upsilon(t) - \upsilon(s)|\mathbb{B}, \quad \forall\, t, s \in [T_0, T],$$

equivalent to

$$|d_{C(t)}(y) - d_{C(s)}(y)| \leq |\upsilon(t) - \upsilon(s)|, \quad \forall\, y \in H,\ \forall t, s \in [T_0, T].$$

($\mathcal{H}_2$) $A : H \longrightarrow H$ is a bounded linear operator which is symmetric and γ-coercive, that is there exists $\gamma > 0$ such that:

$$\langle Ax, x \rangle = \langle x, Ax \rangle \geq \gamma \|x\|^2 \quad , \forall x \in H.$$

$(\mathcal{H}_3)$ $f : Q \times H \longrightarrow H$ is a measurable mapping such that

$(\mathcal{H}_{3,1})$ f satisfies the linear growth condition that is there exists a non-negative function $\beta(\cdot,\cdot) \in L^1(Q, \mathbb{R}_+)$ such that

$$\|f(t,s,x)\| \leq \beta(t,s)(1+\|x\|), \text{ for all } (t,s) \in Q \text{ and for any } x \in \bigcup_{t\in[T_0,T]} C(t),$$

$(\mathcal{H}_{3,2})$ f is locally Lipschitz integrable that is for each real $\eta > 0$ there exists a non-negative function $k_\eta(\cdot) \in L^1([T_0,T], \mathbb{R}_+)$ such that for all $(t,s) \in Q$ and for any $x, y \in B[0,\eta]$:

$$\|f(t,s,x) - f(t,s,y)\| \leq k_\eta(t)\|x - y\|,$$

where

$$Q := \{(t,s) \in [T_0,T] \times [T_0,T] : s \leq t\}.$$

This chapter is devoted to study the following new nonsmooth dynamical system involving two operators, the first one is the normal cone to a convex moving set $C(t)$ where as the second one is the linear bounded operator A. Also, this dynamic is perturbed by an integral forcing term:

$$(DP)\begin{cases} -\dot{x}(t) \in N_{C(t)}(Ax(t)) + \displaystyle\int_{T_0}^{t} f(t,s,x(s))\,ds & \text{a.e. } t \in [T_0,T], \\ x(T_0) = x_0, \quad Ax_0 \in C(T_0). \end{cases}$$

2.2 Well posedness result

Before stating our main result we recall and we detail the following auxiliary existence and uniqueness result proved in [14].

Proposition 2.2.1. *Let H be a real Hilbert space, suppose that $C(\cdot)$ satisfies $(\mathcal{H}_1)$ and that the operator A satisfies $(\mathcal{H}_2)$. Let $h : [T_0,T] \longrightarrow H$ be a single-valued mapping in $L^1([T_0,T], H)$. Then for all $Ax_0 \in C(T_0)$ there exists one and unique absolutely continuous solution $x(\cdot)$ for the following differential inclusion:*

$$(I)\begin{cases} -\dot{x}(t) \in N_{C(t)}(Ax(t)) + h(t) & a.e.\ t \in [T_0,T], \\ x(T_0) = x_0, \quad Ax_0 \in C(T_0). \end{cases}$$

Moreover $x(\cdot)$ satisfies the following inequality:

$$\|\dot{x}(t) + h(t)\| \leq \frac{\|A\|\,\|h(t)\| + |\dot{v}(t)|}{\gamma} \quad a.e.\ t \in [T_0,T]. \tag{2.1}$$

Proof. Let us define

$$D(t) := C(t) + A\Psi(t) \quad and \quad y(t) = x(t) + \Psi(t),$$

where $\Psi(t) = \int_{T_0}^{t} h(s)\,ds$. We have

$$N_{C(t)}(Ax(t)) = N_{D(t)}(Ax(t) + A\Psi(t)) = N_{D(t)}(A(x(t) + \Psi(t))) = N_{D(t)}(Ay(t)).$$

Moreover the set-valued map $D(\cdot)$ satisfies also $(\mathcal{H}_1)$ with the absolutely continuous function $V(\cdot)$, where for all $t \in [T_0, T]$

$$V(t) = \int_{T_0}^{t} (\|A\|\,\|h(r)\| + |\dot{v}(r)|)\,dr.$$

Therefore (I) is equivalent to the following degenerate differential inclusion:

$$\begin{cases} -\dot{y}(t) \in N_{D(t)}(Ay(t)) \quad \text{a.e. } t \in [T_0, T]\,, \\ y(T_0) = x_0, \quad Ax_0 \in D(T_0). \end{cases}$$

Therefore by Theorem 1, (I) has one and unique absolutely continuous solution $y(t)$ satisfying:

$$\|\dot{y}(t)\| \le \frac{1}{\gamma}\,|\dot{V}(t)|, \quad \text{a.e. } in\,[T_0, T]$$

i.e:

$$\begin{aligned} \|\dot{x}(t) + h(t)\| &\le \frac{1}{\gamma}\,|\|A\|\,\|h(r)\| + |\dot{v}(r)||\\ &= \frac{\|A\|\,\|h(r)\| + |\dot{v}(r)|}{\gamma}. \end{aligned}$$

■

Let us now state and prove our main result, which is an existence and uniqueness result for the perturbed degenerate integro-differential sweeping process.

Theorem 2.2.2. *Let H be a real Hilbert space, assume that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, and $(\mathcal{H}_3)$ are satisfied. Then for all $Ax_0 \in C(T_0)$ there exists an unique absolutely continuous solution $x : [T_0, T] \longrightarrow H$ for the following differential inclusion:*

$$(DP)\begin{cases} -\dot{x}(t) \in N_{C(t)}(Ax(t)) + \displaystyle\int_{T_0}^{t} f(t, s, x(s))\,ds \qquad a.e.\ t \in [T_0, T], \\ x(T_0) = x_0, \quad Ax_0 \in C(T_0). \end{cases}$$

Moreover this solution satisfies:

1. *For a.e.* $(t,s) \in Q$

$$\|\dot{x}(t) + \int_{T_0}^{t} f(t,s,x(s))\,ds\| \le \frac{\|A\|}{\gamma} \int_{T_0}^{t} \|f(t,s,x(s))\|\,ds + \frac{1}{\gamma}\,|\dot{v}(t)|.$$

2. *Assume the following strengthened form of assumption* $(\mathcal{H}_{3,1})$ *on the function* f *holds:*

   $(\mathcal{H}'_{3,1})$ *: There exist non-negative functions* $\lambda(\cdot) \in L^1([T_0,T],\mathbb{R}_+)$ *and* $g(\cdot) \in L^1(Q,\mathbb{R}_+)$ *such that*

$$\|f(t,s,x)\| \le g(t,s) + \lambda(t)\|x\|, \text{ for any } (t,s) \in Q \text{ and any } x \in \bigcup_{t\in[T_0,T]} C(t).$$

   Then we have

$$\|f(t,s,x(s))\| \le g(t,s) + \lambda(t)\,l, \quad a.e.\ (t,s) \in Q,$$

   and for almost all $t \in [T_0,T]$

$$\|\dot{x}(t) + \int_{T_0}^{t} f(t,s,x(s))\,ds\| \le \frac{1}{\gamma}\,|\dot{v}(t)| + \frac{\|A\|}{\gamma} \int_{T_0}^{t} g(t,s)\,ds + (T-T_0)\,\lambda(t)\,l,$$

   where

$$l := \|x_0\| \exp\left(\int_{T_0}^{T} B(\tau)\,d\tau\right) + \int_{T_0}^{T} \left[\frac{1}{\gamma}\,|\dot{v}(s)| + \left(1+\frac{\|A\|}{\gamma}\right) \int_{T_0}^{T} g(s,\tau)\,d\tau\right] \exp\left(\int_{T_0}^{T} B(\tau)\,d\tau\right) ds.$$

   and

$$B(t) := \max\{1,\lambda(t)\} \text{ for all } t \in [T_0,T].$$

Proof.

Step 1. Discretization of the interval $I := [T_0,T]$.

For each $n \in \mathbb{N}^*$, divide the interval I into n intervals of length $h = \frac{T-T_0}{n}$ and define, for all $i \in \{0,\cdots,n-1\}$

$$\begin{cases} t_{i+1}^n := t_i^n + h = T_0 + ih, \\ I_i^n := \left[t_i^n, t_{i+1}^n\right], \end{cases}$$

so that

$$T_0 = t_0^n < t_1^n < \cdots < t_i^n < t_{i+1}^n < \cdots < t_n^n = T.$$

Step 2. Construction of the sequence of the approximate solution $(x_n(\cdot))_n$**.**

We construct a sequence of functions $(x_n(\cdot))_{n\in\mathbb{N}}$ in $\mathcal{C}(I,H)$ that converges uniformly to a solution $x(\cdot)$ of (DP).

Our method consists in establish a sequence of discrete solutions $(x_k^n(\cdot))_{n\in\mathbb{N}}$ in each interval $I_k^n := [t_k^n, t_{k+1}^n]$ $(0 \le k \le n-1)$ by using Proposition 2.2.1. Indeed, we proceed as follows:

Given the following problem

$$(P_0)\begin{cases} -\dot{x}(t) \in N_{C(t)}(Ax(t)) + \displaystyle\int_{T_0}^{t} f(t,s,x_0)\,ds \quad \text{a.e.} \quad t \in [T_0, t_1^n]\,, \\ x(T_0) = x_0, \quad Ax_0 \in C(T_0). \end{cases}$$

Then (P_0) is a perturbed degenerate sweeping process with the perturbation depending only on time.

Let $h_0 : [T_0, t_1^n] \to H$ defined by $h_0(t) := \displaystyle\int_{T_0}^{t} f(t,s,x_0)\,ds$ for all $t \in [T_0, t_1^n]$.

By definition we have $h_0(\cdot)$ is a measurable function.

Moreover we have

$$\begin{aligned} \|h_0(t)\| &= \|\int_{T_0}^{t} f(t,s,x_0)\,ds\| \\ &\le \int_{T_0}^{t} \|f(t,s,x_0)\|\,ds, \end{aligned}$$

we use the condition $(\mathcal{H}_{3,1})$ we get

$$\begin{aligned} \|h_0(t)\| &\le \int_{T_0}^{t} (1+\|x_0\|)\beta(t,s)ds \\ &= (1+\|x_0\|)\int_{T_0}^{t} \beta(t,s)ds. \end{aligned}$$

We apply the integral for $t \in [T_0, t_1^n]$, we obtain

$$\begin{aligned} \int_{T_0}^{t_1^n} \|h_0(t)\|dt &\le \int_{T_0}^{T} \|h_0(t)\|dt \\ &\le (1+\|x_0\|)\int_{T_0}^{T}\int_{T_0}^{t} \beta(t,s)ds\,dt, \end{aligned}$$

and since $\beta(\cdot,\cdot) \in L^1(Q,\mathbb{R}_+)$, then $h_0(\cdot)$ is an integrable function. So $h_0(\cdot) \in L^1([T_0,t_1^n],H)$. Therefore, by Proposition 2.2.1 the differential inclusion (P_0) has an unique absolutely continuous solution denoted by

$$x_0^n(\cdot) : [T_0, t_1^n] \longrightarrow H,$$

satisfying the following inequality

$$\left\| \dot{x}_0^n(t) + \int_{T_0}^{t} f(t,s,x_0)\,ds \right\| \leq \frac{\|A\| \left\| \int_{T_0}^{t} f_2(t,s,x_0)\,ds \right\| + |\dot{v}(t)|}{\gamma} \qquad \text{a.e. } t \in [T_0, t_1^n]\,.$$

Next, let us consider the following problem

$$(P_1) \begin{cases} -\dot{x}(t) \in N_{C(t)}(Ax(t)) \int_{T_0}^{t_1^n} f(t,s,x_0)\,ds + \int_{t_1^n}^{t} f(t,s,x_0^n(t_1^n))\,ds \;\; \text{a.e. } t \in [t_1^n, t_2^n]\,, \\ x(t_1^n) = x_0^n(t_1^n), \;\; A(x_0^n(t_1^n)) \in C(t_1^n). \end{cases}$$

(P_1) is a perturbed degenerate sweeping process with the perturbation depending only on time.

Let $h_1 : [t_1^n, t_2^n] \to H$ defined by

$$h_1(t) := \int_{T_0}^{t_1^n} f(t,s,x_0)\,ds + \int_{t_1^n}^{t} (t,s,x_0^n(t_1^n))\,ds \quad \text{for all } t \in [t_1^n, t_2^n].$$

We have $h_1(\cdot)$ is a measurable function by definition.

Moreover we have

$$\begin{aligned} \|h_1(t)\| &= \| \int_{T_0}^{t_1^n} f(t,s,x_0)\,ds + \int_{t_1^n}^{t} f(t,s,x_0^n(t_1^n))\,ds \| \\ &\leq \int_{T_0}^{t_1^n} \|f(t,s,x_0)\| ds + \int_{t_1^n}^{t} \|f(t,s,x_0^n(t_1^n))\| ds, \end{aligned}$$

we use the conditon $(\mathcal{H}_{3,1})$, we get

$$\|h_1(t)\| \leq (1 + \|x_0\|) \int_{T_0}^{t_1^n} \beta(t,s)\,ds + (1 + \|x_0^n(t_1^n)\|) \int_{t_1^n}^{t} \beta(t,s) ds.$$

Integring on $[t_1^n, t_2^n]$, we get

$$\int_{t_1^n}^{t_2^n} \|h_1(t)\|dt \le \int_{T_0}^{T} \|h_1(t)\|dt \le (1+\|x_0\|)\int_{T_0}^{T}\int_{T_0}^{t_1^n} \beta(t,s)ds\,dt + (1+\|x_0^n(t_1^n)\|)\int_{T_0}^{T}\int_{t_1^n}^{t} \beta(t,s)ds\,dt$$

$$\le \int_{T_0}^{T}\left(\int_{T_0}^{t_1^n}\beta(t,s)ds + \int_{t_1^n}^{t}\beta(t,s)ds\right)dt$$

$$+\max\{\|x_0\|,\|x_0^n(t_1^n)\|\}\int_{T_0}^{T}\left(\int_{T_0}^{t_1^n}\beta(t,s)ds + \int_{t_1^n}^{t}\beta(t,s)ds\right)dt$$

$$\le \int_{T_0}^{T}\int_{T_0}^{t}\beta(t,s)ds\,dt + \max\{\|x_0\|,\|x_0^n(t_1^n)\|\}\left(\int_{T_0}^{T}\int_{T_0}^{t}\beta(t,s)ds\,dt\right)$$

$$\le (1+\max\{\|x_0\|,\|x_0^n(t_1^n)\|\})\left(\int_{T_0}^{T}\int_{T_0}^{t}\beta(t,s)ds\,dt\right).$$

We know from the above problem (P_0) that the mapping $x_0^n(\cdot)$ is absolutely continuous, then in particular bounded on $[T_0,T]$. Further, since $\beta(\cdot,\cdot)\in L^1(Q,\mathbb{R}_+)$, then $h_1(\cdot)$ is an integrable mapping, so $h_1(\cdot)\in L^1([t_1^n,t_2^n],H)$. The same arguments as above show that (P_1) has an unique absolutely continuous solution denoted by

$$x_1^n(\cdot):[t_1^n,t_2^n]\longrightarrow H,$$

and this solution satisfies the following inequality

$$\left\|\dot{x}_1^n(t)+\int_{T_0}^{t_1^n} f(t,s,x_0)\,ds + \int_{t_1^n}^{t}(t,s,x_0^n(t_1^n))\,ds\right\|$$

$$\le \frac{\|A\|\left\|\int_{T_0}^{t_1^n} f(t,s,x_0)\,ds + \int_{t_1^n}^{t} f(t,s,x_0^n(t_1^n))\,ds\right\| + |\dot{v}(t)|}{\gamma} \quad \text{a.e. } t\in[t_1^n,t_2^n].$$

Successively, for each n, we have a family of absolutely continuous mappings $(x_k^n(\cdot))_{0\le k\le n-1}$ definded by

$$x_k^n(\cdot):\left[t_k^n,t_{k+1}^n\right]\longrightarrow H,$$

such that for each $k\in\{0,\cdots,n-1\}$ we have,

$$(P_k)\begin{cases} -\dot{x}_k^n(t)\in N_{C(t)}(Ax_k^n(t)+\sum_{j=0}^{k-1}\int_{t_j^n}^{t_{j+1}^n} f(t,s,x_{j-1}^n(t_j^n))\,ds + \int_{t_k^n}^{t} f(t,s,x_{k-1}^n(t_k^n))\,ds \quad \text{a.e. } t\in\left[t_k^n,t_{k+1}^n\right], \\ x_k^n(t_k^n)=x_{k-1}^n(t_k^n),\quad A(x_{k-1}^n(t_k^n))\in C(t_k^n), \end{cases} \tag{2.2}$$

where for $k=0$ we put $x^n_{-1}(T_0) := x_0$. Moreover for almost every where $t \in [t^n_k, t^n_{k+1}]$, we have

$$\left\| \dot{x}^n_k(t) + \sum_{j=0}^{k-1} \int_{t^n_j}^{t^n_{j+1}} f(t,s,x^n_{j-1}(t^n_j))\,ds + \int_{t^n_k}^{t} f(t,s,x^n_{k-1}(t^n_k))\,ds \right\|$$

$$\leq \frac{1}{\gamma}\left(\|A\| \left\| \sum_{j=0}^{k-1} \int_{t^n_j}^{t^n_{j+1}} f(t,s,x^n_{j-1}(t^n_j))\,ds + \int_{t^n_k}^{t} f(t,s,x^n_{k-1}(t^n_k))\,ds \right\| + |\dot{v}(t)| \right). \quad (2.3)$$

Defining for each $k \in \{0,1,\cdots,n-1\}$ the mapping $h_k : [t^n_k, t^n_{k+1}] \to H$ by

$$h_k(t) := \sum_{j=0}^{k-1} \int_{t^n_j}^{t^n_{j+1}} f(t,s,x^n_{j-1}(t^n_j))\,ds + \int_{t^n_k}^{t} f(t,s,x^n_{k-1}(t^n_k))\,ds,$$

for all $t \in \left[t^n_k, t^n_{k+1}\right]$. Using the integral linearity and the condition $(\mathcal{H}_{3,1})$, we obtain

$$\int_{t^n_k}^{t^n_{k+1}} \|h_k(t)\| dt \leq \int_{T_0}^{T} \|h_k(t)\| dt$$

$$\leq \sum_{j=0}^{k-1} \int_{T_0}^{T} \int_{t^n_j}^{t^n_{j+1}} \|f(t,s,x^n_{j-1}(t^n_j))\|\,ds\,dt + \int_{T_0}^{T} \int_{t^n_k}^{t} \|f(t,s,x^n_{k-1}(t^n_k))\|\,ds\,dt$$

$$\leq \sum_{j=0}^{k-1} (1 + \|x^n_{j-1}(t^n_j)\|) \int_{T_0}^{T} \int_{t^n_j}^{t^n_{j+1}} \beta(t,s) ds\,dt + (1 + \|x^n_{k-1}(t^n_k)\|) \int_{T_0}^{T} \int_{t^n_k}^{t} \beta(t,s) ds\,dt$$

$$\leq \sum_{j=0}^{k-1} (1 + \max_{0 \leq j \leq k} \|x^n_{j-1}(t^n_j)\|) \int_{T_0}^{T} \int_{t^n_j}^{t^n_{j+1}} \beta(t,s) ds\,dt$$

$$+ (1 + \max_{0 \leq j \leq k} \|x^n_{k-1}(t^n_k)\|) \int_{T_0}^{T} \int_{t^n_k}^{t} \beta(t,s) ds\,dt$$

$$\leq (1 + \max_{0 \leq j \leq k} \left\|x^n_{j-1}(t^n_j)\right\|) \int_{T_0}^{T} \int_{T_0}^{t} \beta(t,s) ds\,dt.$$

We know from the above problem $(P_j)_{0 \leq j \leq k-1}$ that the mapping $x^n_{k-1}(\cdot)$ is absolutely continuous, then in particular bounded on $[T_0, T]$. Further, since $\beta(\cdot,\cdot) \in L^1(Q, \mathbb{R}_+)$, the mapping $h_k(\cdot)$ is integrable on $\left[t^n_k, t^n_{k+1}\right]$, so $h_k(\cdot) \in L^1([t^n_k, t^n_{k+1}], H)$.

Now, we define the sequence $(x_n(\cdot))_n$ from the discrete sequences $(x^n_k(\cdot))$ as follows for all $n \in \mathbb{N}$, let $x_n(\cdot) : [T_0, T] \longrightarrow H$, such that

$$x_n(t) := x^n_k(t), \quad \text{if } t \in \left[t^n_k, t^n_{k+1}\right], k \in \{0,1,\cdots,n-1\}. \quad (2.4)$$

It is clear from this definition that $x_n(\cdot)$ is absolutely continuous.

Let the function $\theta_n(\cdot) : [T_0, T] \longrightarrow [T_0, T]$ defined by

$$\begin{cases} \theta_n(T) := T, \\ \theta_n(t) := t_k^n, \ \text{ if } t \in \left[t_k^n, t_{k+1}^n\right[. \end{cases} \tag{2.5}$$

We obtain from (2.2), (2.3), (2.4) and (2.5) that

$$\begin{cases} -\dot{x}_n(t) \in N_{C(t)}(Ax_n(t)) + \int\limits_{T_0}^{t} f(t, s, x_n(\theta_n(s)))\, ds \ \text{ a.e. } t \in [T_0, T], \\ x_n(T_0) = x_0, \quad Ax_0 \in C(T_0). \end{cases} \tag{2.6}$$

and a.e. $t \in [T_0, T]$ we have

$$\left\| \dot{x}_n(t) + \int\limits_{T_0}^{t} f(t, s, x_n(\theta_n(s)))\, ds \right\| \le \frac{\|A\| \left\| \int\limits_{T_0}^{t} f(t, s, x_n(\theta_n(s)))\, ds \right\| + |\dot{v}(t)|}{\gamma}. \tag{2.7}$$

<u>Step 3.</u> The upper bound of the sequence $(\dot{x}_n(\cdot))_n$.

Since $\beta(\cdot,\cdot) \in L^1(Q, \mathbb{R}_+)$, assume without loss of generality that

$$\int\limits_{T_0}^{T} \int\limits_{T_0}^{\tau} \beta(\tau, s)\, ds\, d\tau < \frac{\gamma}{\gamma + \|A\|}.$$

By construction we have for each $i \in \{0, \cdots, n-1\}$ and for a.e. $t \in [t_i^n, t_{i+1}^n]$

$$\begin{aligned} &\left\| \dot{x}_n(t) + \sum_{j=0}^{i-1} \int\limits_{t_j^n}^{t_{j+1}^n} f(t, s, x_n(t_j^n))\, ds + \int\limits_{t_i^n}^{t} f(t, s, x_n(t_i^n))\, ds \right\| \\ &\le \frac{1}{\gamma} \left(\|A\| \left\| \sum_{j=0}^{i-1} \int\limits_{t_j^n}^{t_{j+1}^n} f(t, s, x_n(t_j^n))\, ds + \int\limits_{t_i^n}^{t} f(t, s, x_n(t_i^n))\, ds \right\| + |\dot{v}(t)| \right). \end{aligned} \tag{2.8}$$

On the other hand, we have

$$\begin{aligned}\|\dot{x}_n(t)\| &= \Bigg\|\dot{x}_n(t) + \sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} f(t,s,x_n(t_j^n))\,ds + \int_{t_i^n}^{t} f(t,s,x_n(t_i^n))\,ds \\ &\quad - \sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} f(t,s,x_n(t_j^n))\,ds - \int_{t_i^n}^{t} f(t,s,x_n(t_i^n))\,ds\Bigg\| \\ &\le \Bigg\|\dot{x}_n(t) + \sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} f(t,s,x_n(t_j^n))\,ds + \int_{t_i^n}^{t} f(t,s,x_n(t_i^n))\,ds\Bigg\| \\ &\quad + \Bigg\|\sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} f(t,s,x_n(t_j^n))\,ds + \int_{t_i^n}^{t} f(t,s,x_n(t_i^n))\,ds\Bigg\|.\end{aligned}$$

Hence from (2.8), we have for a.e. $t \in [t_i^n, t_{i+1}^n]$

$$\begin{aligned}\|\dot{x}_n(t)\| &\le \left(1+\frac{\|A\|}{\gamma}\right)\Bigg\|\sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n} f(t,s,x_n(t_j^n))\,ds + \int_{t_i^n}^{t}(t,s,x_n(t_i^n))\,ds\Bigg\| + \frac{1}{\gamma}|\dot{v}(t)| \\ &\le \left(1+\frac{\|A\|}{\gamma}\right)\sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n}\|f(t,s,x_n(t_j^n))\|\,ds + \left(1+\frac{\|A\|}{\gamma}\right)\int_{t_i^n}^{t}\|f(t,s,x_n(t_i^n))\|\,ds + \frac{1}{\gamma}|\dot{v}(t)|.\end{aligned} \tag{2.9}$$

Moreover we observe that

$$\int_{t_0^n}^{t_{i+1}^n}\|\dot{x}_n(\tau)\|\,d\tau = \int_{t_0^n}^{t_1^n}\|\dot{x}_n(\tau)\|\,d\tau + \int_{t_1^n}^{t_2^n}\|\dot{x}_n(\tau)\|\,d\tau + \cdots + \int_{t_{i-1}^n}^{t_i^n}\|\dot{x}_n(\tau)\|\,d\tau + \int_{t_i^n}^{t_{i+1}^n}\|\dot{x}_n(\tau)\|\,d\tau.$$

So (2.9) implies that

$$\begin{aligned}\int_{t_0^n}^{t_{i+1}^n}\|\dot{x}_n(\tau)\|\,d\tau &\le \sum_{k=0}^{i}\int_{t_k^n}^{t_{k+1}^n}\Bigg[\left(1+\frac{\|A\|}{\gamma}\right)\sum_{j=0}^{k-1}\int_{t_j^n}^{t_{j+1}^n}\|f(\tau,s,x_n(t_j^n))\|\,ds \\ &\quad + \left(1+\frac{\|A\|}{\gamma}\right)\int_{t_k^n}^{\tau}\|f(\tau,s,x_n(t_k^n))\|\,ds + \frac{1}{\gamma}|\dot{v}(\tau)|\Bigg]\,d\tau \\ &\le \frac{1}{\gamma}\int_{t_0^n}^{t_{i+1}^n}|\dot{v}(\tau)|\,d\tau + \left(1+\frac{\|A\|}{\gamma}\right)\sum_{k=0}^{i}\sum_{j=0}^{k-1}\int_{t_k^n}^{t_{k+1}^n}\int_{t_j^n}^{t_{j+1}^n}\|f(\tau,s,x_n(t_j^n))\|\,ds\,d\tau \\ &\quad + \left(1+\frac{\|A\|}{\gamma}\right)\sum_{k=0}^{i}\int_{t_k^n}^{t_{k+1}^n}\int_{t_k^n}^{\tau}\|f(\tau,s,x_n(t_k^n))\|\,ds\,d\tau.\end{aligned}$$

Since $C(t_k^n) \in \bigcup_{t\in[T_0,T]} C(t)$, using the growth condition $(\mathcal{H}_{3,1})$, we have the following: for all $i \in \{0, \cdots, n-1\}$

- $$\sum_{k=0}^{i} \int_{t_k^n}^{t_{k+1}^n} \int_{t_k^n}^{\tau} \|f(\tau, s, x_n(t_k^n))\| \, ds \, d\tau \leq \sum_{k=0}^{i} \int_{t_k^n}^{t_{k+1}^n} \int_{t_k^n}^{\tau} \big(1 + \|x_n(t_k^n)\|\big)\beta(\tau, s) \, ds \, d\tau$$
$$\leq (1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|) \int_{t_0^n}^{t_{i+1}^n} \int_{t_i^n}^{\tau} \beta(\tau, s) \, ds \, d\tau.$$

- $$\sum_{k=0}^{i} \sum_{j=0}^{k-1} \int_{t_k^n}^{t_{k+1}^n} \int_{t_j^n}^{t_{j+1}^n} \|f(\tau, s, x_n(t_j^n))\| \, ds \, d\tau \leq \sum_{k=0}^{i} \sum_{j=0}^{k-1} \int_{t_k^n}^{t_{k+1}^n} \int_{t_j^n}^{t_{j+1}^n} \big(1 + \|x_n(t_k^n)\|\big)\beta(\tau, s) \, ds \, d\tau$$
$$\leq (1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|) \sum_{j=0}^{i-1} \int_{t_0^n}^{t_{i+1}^n} \int_{t_j^n}^{t_{j+1}^n} \beta(\tau, s) \, ds \, d\tau.$$

Therefore, noticing that $t_0^n = T_0$, we have

$$\int_{T_0}^{t_{i+1}^n} \|\dot{x}_n(\tau)\| \, d\tau \leq \frac{1}{\gamma} \int_{T_0}^{t_{i+1}^n} |\dot{v}(\tau)| \, d\tau + \left(1 + \frac{\|A\|}{\gamma}\right)(1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|) \sum_{j=0}^{i-1} \int_{T_0}^{t_{i+1}^n} \int_{t_j^n}^{t_{j+1}^n} \beta(\tau, s) \, ds \, d\tau$$
$$+ \left(1 + \frac{\|A\|}{\gamma}\right)(1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|) \int_{T_0}^{t_{i+1}^n} \int_{t_i^n}^{\tau} \beta(\tau, s) \, ds \, d\tau.$$
$$\leq \frac{1}{\gamma} \int_{T_0}^{t_{i+1}^n} |\dot{v}(\tau)| \, d\tau + \left(1 + \frac{\|A\|}{\gamma}\right)(1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|)$$
$$\left[\int_{T_0}^{t_{i+1}^n} \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} \beta(\tau, s) \, ds \, d\tau + \int_{T_0}^{t_{i+1}^n} \int_{t_i^n}^{\tau} \beta(\tau, s) \, ds \, d\tau \right]$$
$$= \frac{1}{\gamma} \int_{T_0}^{t_{i+1}^n} |\dot{v}(\tau)| \, d\tau + \left(1 + \frac{\|A\|}{\gamma}\right)(1 + \max_{0\leq k\leq n} \|x_n(t_k^n)\|) \int_{T_0}^{t_{i+1}^n} \int_{T_0}^{\tau} \beta(\tau, s) \, ds \, d\tau. \tag{2.10}$$

Now, using the fact that $x_n(\cdot)$ is absolutely continuous, we have

$$x_n(t_{i+1}^n) - x_n(t_0^n) = \int_{t_0^n}^{t_{i+1}^n} \dot{x}_n(\tau) d\tau.$$

This yields to the following inequality:

$$\|x_n(t_{i+1}^n)\| \leq \|x_0\| + \int_{t_0^n}^{t_{i+1}^n} \|\dot{x}_n(\tau)\| d\tau.$$

Consequently, the inequality (2.10) give

$$\|x_n(t_{i+1}^n)\| \leq \|x_0\| + \frac{1}{\gamma}\int_{T_0}^{t_{i+1}^n} |\dot{v}(\tau)|\, d\tau + \left(1+\frac{\|A\|}{\gamma}\right)(1+\max_{0\leq k\leq n}\|x_n(t_k^n)\|)\int_{T_0}^{t_{i+1}^n}\int_{T_0}^{\tau} \beta(\tau,s)\, ds\, d\tau. \tag{2.11}$$

The relation (2.11) holds for all $i \in \{0,\cdots,n-1\}$, we have the following

$$\begin{aligned}
\max_{0\leq k\leq n}\|x_n(t_k^n)\| &\leq \|x_0\| + \frac{1}{\gamma}\int_{T_0}^{T} |\dot{v}(\tau)|\, d\tau + \left(1+\frac{\|A\|}{\gamma}\right)(1+\max_{0\leq k\leq n}\|x_n(t_k^n)\|)\int_{T_0}^{T}\int_{T_0}^{\tau} \beta(\tau,s)\, ds\, d\tau \\
&\leq \|x_0\| + \frac{1}{\gamma}\int_{T_0}^{T} |\dot{v}(\tau)|\, d\tau + \left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{T}\int_{T_0}^{\tau} \beta(\tau,s)\, ds\, d\tau \\
&\quad + \left(1+\frac{\|A\|}{\gamma}\right)\max_{0\leq k\leq n}\|x_n(t_k^n)\|\int_{T_0}^{T}\int_{T_0}^{\tau} \beta(\tau,s)\, ds\, d\tau,
\end{aligned}$$

then

$$\begin{aligned}
&\left[1-\left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{T}\int_{T_0}^{\tau} \beta(\tau,s)\, ds\, d\tau\right]\max_{0\leq k\leq n}\|x_n(t_k^n)\| \\
&\leq \|x_0\| + \frac{1}{\gamma}\int_{T_0}^{T} |\dot{v}(\tau)|\, d\tau + \left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{T}\int_{T_0}^{\tau} \beta(\tau,s)\, ds\, d\tau,
\end{aligned}$$

hence

$$\begin{aligned}
\max_{0\leq k\leq n}\|x_n(t_k^n)\| \leq &\frac{1}{1-\left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{T}\int_{T_0}^{\tau} \beta(\tau,s)\, ds\, d\tau} \\
&\left[\|x_0\| + \frac{1}{\gamma}\int_{T_0}^{T} |\dot{v}(\tau)|\, d\tau + \left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{T}\int_{T_0}^{\tau} \beta(\tau,s)\, ds\, d\tau\right].
\end{aligned}$$

Or equivalently

$$\max_{0\leq k\leq n}\|x_n(t_k^n)\| \leq M,$$

where

$$M := \frac{1}{1-\left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{T}\int_{T_0}^{\tau} \beta(\tau,s)\, ds\, d\tau}\left[\|x_0\| + \frac{1}{\gamma}\int_{T_0}^{T} |\dot{v}(\tau)|\, d\tau + \left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{T}\int_{T_0}^{\tau} \beta(\tau,s)\, ds\, d\tau\right]. \tag{2.12}$$

On one hand, from the growth condition of f ($\mathcal{H}_{3,1}$) and (2.12) we have for almost all t and for all n,

$$\|f(t,s,x_n(\theta_n(s)))\| \leq \beta(t,s)(1+\|x_n(\theta_n(s))\|) \leq (1+M)\beta(t,s), \;\; \forall\, (t,s) \in Q. \quad (2.13)$$

(2.7) and (2.13) imply for almost all t and for all n, the following

$$\begin{aligned}\Big\|\dot{x}_n(t) + \int_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds\Big\| &\leq \frac{\|A\|}{\gamma}\int_{T_0}^{t} \big\|f(t,s,x_n(\theta_n(s)))\big\|\,ds + \frac{1}{\gamma}|\dot{v}(t)| \\ &\leq \frac{\|A\|}{\gamma}(1+M)\int_{T_0}^{t}\beta(t,s)\,ds + \frac{1}{\gamma}|\dot{v}(t)|. \qquad (2.14)\end{aligned}$$

Further,

$$\begin{aligned}\|\dot{x}_n(t)\| &= \left\|\dot{x}_n(t) + \int_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds - \int_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds\right\| \\ &\leq \left\|\dot{x}_n(t) + \int_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds\right\| + \Big\|\int_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds\Big\| \\ &\leq \frac{\|A\|}{\gamma}(1+M)\int_{T_0}^{t}\beta(t,s)\,ds + \frac{1}{\gamma}|\dot{v}(t)| + \int_{T_0}^{t}\|f(t,s,x_n(\theta_n(s)))\|\,ds \\ &\leq \frac{\|A\|}{\gamma}(1+M)\int_{T_0}^{t}\beta(t,s)\,ds + \frac{1}{\gamma}|\dot{v}(t)| + (1+M)\int_{T_0}^{t}\beta(t,s)\,ds \\ &= \left(1+\frac{\|A\|}{\gamma}\right)(1+M)\int_{T_0}^{t}\beta(t,s)\,ds + \frac{1}{\gamma}|\dot{v}(t)|,\end{aligned}$$

which gives for almost all t and for all n

$$\|\dot{x}_n(t)\| \leq \gamma(t), \qquad (2.15)$$

where

$$\gamma(t) := (1+M)\left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{t}\beta(t,s)\,ds + \frac{1}{\gamma}|\dot{v}(t)|.$$

Step 4. The compactness of the sequence $(x_n(\cdot))_n$**.**

As $(\mathcal{C}(I,H), \|\cdot\|_\infty)$ is a Banach space with uniforme convergence norm, it remains to prove that $(x_n(\cdot))_n$ is a Cauchy sequence i.e.

$$\lim_{n,m\to+\infty}\|x_n(\cdot) - x_m(\cdot)\|_\infty = 0,$$

such as

$$\|x_n(\cdot) - x_m(\cdot)\|_\infty := \sup_{t\in[T_0,T]} \|x_n(\cdot) - x_m(\cdot)\|.$$

Let $m, n \in \mathbb{N}$, for almost all $t \in [T_0, T]$, we have

$$\begin{cases} -\dot{x}_n(t) - \int\limits_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds \in N_{C(t)}(Ax_n(t)), \\ -\dot{x}_m(t) - \int\limits_{T_0}^{t} f(t,s,x_m(\theta_m(s)))\,ds \in N_{C(t)}(Ax_m(t)). \end{cases}$$

Using the fact that the normal cone is monotone, we get the following:

$$\left\langle -\dot{x}_n(t) - \int\limits_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds + \dot{x}_m(t) + \int\limits_{T_0}^{t} f(t,s,x_m(\theta_m(s)))\,ds, Ax_n(t) - Ax_m(t) \right\rangle \geq 0.$$

This implies that

$$\begin{aligned} &\left\langle \dot{x}_n(t) + \int\limits_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds - \dot{x}_m(t) - \int\limits_{T_0}^{t} f(t,s,x_m(\theta_m(s)))\,ds, A(x_n(t) - x_m(t)) \right\rangle \\ &= \langle \dot{x}_n(t) - \dot{x}_m(t), A(x_n(t) - x_m(t)) \rangle \\ &+ \left\langle \int\limits_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds - \int\limits_{T_0}^{t} f(t,s,x_m(\theta_m(s)))\,ds, A(x_n(t) - x_m(t)) \right\rangle \\ &\leq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\langle \dot{x}_n(t) - \dot{x}_m(t), A(x_n(t) - x_m(t)) \rangle \\ &\leq \left\langle \int\limits_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds - \int\limits_{T_0}^{t} f(t,s,x_m(\theta_m(s)))\,ds, A(x_m(t) - x_n(t)) \right\rangle. \end{aligned}$$

From Lemma 1.11.4 we have

$$\frac{1}{2}\frac{d}{dt}\langle x_n(t) - x_m(t), A(x_n(t) - x_m(t)) \rangle = \langle \dot{x}_n(t) - \dot{x}_m(t), A(x_n(t) - x_m(t)) \rangle, \quad \text{a.e. } t \in [T_0, T].$$

Then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\langle x_n(t) - x_m(t), A(x_n(t) - x_m(t)) \rangle \\ &\leq \left\langle \int\limits_{T_0}^{t} f(t,s,x_n(\theta_n(s)))\,ds - \int\limits_{T_0}^{t} f(t,s,x_m(\theta_m(s))), A(x_m(t) - x_n(t)) \right\rangle \\ &\leq \|A\|\,\|x_n(t) - x_m(t)\| \int\limits_{T_0}^{t} \|f(t,s,x_n(\theta_n(s)))\,ds - f(t,s,x_m(\theta_m(s)))\|\,ds. \end{aligned} \tag{2.16}$$

On the other hand, we have

$$\|\dot{x}_n(t)\| \leq \gamma(t) \qquad a.e.\ t \in [T_0, T]. \tag{2.17}$$

$(x_n(\cdot))_n$ is absolutely countiuous, then we can write

$$\|x_n(t)\| - \|x_n(T_0)\| \leq \|x_n(t) - x_n(T_0)\| = \|\int_{T_0}^{t} \dot{x}_n(s)\,ds\| \leq \int_{T_0}^{t} \|\dot{x}_n(s)\|\,ds \leq \int_{T_0}^{t} \gamma(s)\,ds$$
$$\leq \int_{T_0}^{T} \gamma(s)\,ds.$$

Thus

$$\|x_n(t)\| \leq \|x_n(T_0)\| + \int_{T_0}^{T} \gamma(s)\,ds.$$

Therefore

$$\|x_n(t)\| \leq \eta \qquad a.e.\ t \in [T_0, T], \tag{2.18}$$

where

$$\eta := \|x_0\| + \int_{T_0}^{T} \gamma(s)\,ds.$$

This gives for almost all t and for all $n \in \mathbb{N}$

$$x_n(t) \in B[0, \eta].$$

Consequently

$$x_m(t)\,,\ x_n(\theta_n(t))\,,\ x_m(\theta_m(t)) \in B[0, \eta].$$

Applying the Lipschitz continuity of f with Lipschitz radius $k_\eta(\cdot) \in L^1(I, \mathbb{R}_+)$ on the bounded subset $B[0, \eta]$ and using (2.16) it follows that

$$\frac{1}{2}\frac{d}{dt}\big\langle x_n(t) - x_m(t), A(x_n(t) - x_m(t))\big\rangle$$
$$\leq \|A\|\,\|x_n(t) - x_m(t)\| \int_{T_0}^{t} k_\eta(t)\,\|x_n(\theta_n(s)) - x_m(\theta_m(s))\|\,ds$$
$$= k_\eta(t)\,\|A\|\,\|x_n(t) - x_m(t)\| \int_{T_0}^{t} \|x_n(\theta_n(s)) - x_n(s) + x_n(s) - x_m(s) + x_m(s) - x_m(\theta_m(s))\|\,ds$$
$$\leq k_\eta(t)\,\|A\|\,\|x_n(t) - x_m(t)\| \Bigg(\int_{T_0}^{t} \|x_n(\theta_n(s)) - x_n(s)\|\,ds + \int_{T_0}^{t} \|x_n(s) - x_m(s)\|\,ds$$
$$+ \int_{T_0}^{t} \|x_m(t) - x_m(\theta_m(t))\|\,ds \Bigg).$$

$(x_n(\cdot))_n$ is absolutely continuous, by (2.17) we have for each $n \in \mathbb{N}$ and for all t,

$$\|x_n(t) - x_n(\theta_n(t))\| = \left\| \int_{\theta_n(t)}^{t} \dot{x}_n(s)\, ds \right\| \leq \int_{\theta_n(t)}^{t} \|\dot{x}_n(s)\|\, ds \leq \int_{\theta_n(t)}^{t} \gamma(s)\, ds.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\langle x_n(t) - x_m(t), A(x_n(t) - x_m(t))\big\rangle \\
&\leq k_\eta(t)\, \|A\|\, \|x_n(t) - x_m(t)\| \Big(\int_{T_0}^{t} \int_{\theta_n(s)}^{s} \gamma(\tau)\, d\tau ds + \int_{T_0}^{t} \int_{\theta_m(s)}^{s} \gamma(\tau)\, d\tau ds \Big) \\
&+ k_\eta(t)\, \|A\|\, \|x_n(t) - x_m(t)\| \int_{T_0}^{t} \|x_n(s) - x_m(s)\|\, ds.
\end{aligned}$$

Moreover, by (2.18) , we have

$$\|x_n(t) - x_m(t)\| \leq \|x_n(t)\| + \|x_m(t)\| \leq 2\eta,$$

this implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\langle x_n(t) - x_m(t), A(x_n(t) - x_m(t))\big\rangle \\
&\leq k_\eta(t)\, \|A\|\, 2\eta \Big(\int_{T_0}^{t} \Big[\int_{\theta_n(s)}^{s} \gamma(\tau)\, d\tau + \int_{\theta_m(s)}^{s} \gamma(\tau)\, d\tau \Big] ds \Big) \\
&+ k_\eta(t)\, \|A\|\, \|x_n(t) - x_m(t)\| \int_{T_0}^{t} \|x_n(s) - x_m(s)\|\, ds.
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{d}{dt}\big\langle x_n(t) - x_m(t), A(x_n(t) - x_m(t))\big\rangle &\leq 4\eta\ k_\eta(t)\, \|A\| \int_{T_0}^{t} G_{n,m}(s)\, ds \\
&+ 2\ k_\eta(t)\, \|A\|\, \|x_n(t) - x_m(t)\| \int_{T_0}^{t} \|x_n(s) - x_m(s)\|\, ds,
\end{aligned} \tag{2.19}$$

where

$$G_{n,m}(s) := \int_{\theta_n(s)}^{s} \gamma(\tau)\ d\tau + \int_{\theta_m(s)}^{s} \gamma(\tau)\ d\tau.$$

On the other hand, we have A is $\gamma - coercive$, then

$$\big\langle x_n(t) - x_m(t), A(x_n(t) - x_m(t))\big\rangle \geq \gamma\, \|x_n(t) - x_m(t)\|^2,$$

this implies

$$\frac{1}{\gamma}\frac{d}{dt}\big\langle x_n(t)-x_m(t), A(x_n(t)-x_m(t))\big\rangle \geq \frac{d}{dt}\left\|x_n(t)-x_m(t)\right\|^2,$$

this and (2.19) give

$$\frac{d}{dt}\left\|x_n(t)-x_m(t)\right\|^2 \leq \frac{1}{\gamma}\frac{d}{dt}\big\langle x_n(t)-x_m(t), A(x_n(t)-x_m(t))\big\rangle \leq \frac{4}{\gamma}\eta\, k_\eta(t)\,\|A\|\int_{T_0}^{t} G_{n,m}(s)\,ds$$
$$+\frac{2}{\gamma}\,k_\eta(t)\,\|A\|\,\|x_n(t)-x_m(t)\|\int_{T_0}^{t}\|x_n(s)-x_m(s)\|\,ds.$$

Now, applying the Lemma 1.13.3 to the above differential inequality by setting

$$\rho(t)=\left\|x_n(t)-x_m(t)\right\|^2, \qquad K(t)=\frac{2}{\gamma}\,k_\eta(t)\,\|A\|,$$

$$\varepsilon(t)=\frac{4}{\gamma}\eta\, k_\eta(t)\,\|A\|\int_{T_0}^{T} G_{n,m}(s)\,ds, \qquad \epsilon>0,$$

we obtain

$$\begin{aligned}\|x_n(t)-x_m(t)\| \leq& \sqrt{\|x_n(T_0)-x_m(T_0)\|^2+\epsilon}\,\exp\left(\int_{T_0}^{t} H(s)\,ds\right)\\ &+\frac{\sqrt{\epsilon}}{2}\int_{T_0}^{t}\exp\left(\int_{s}^{t}H(\tau)\,d\tau\right)ds\\ &+2\left(\sqrt{\int_{T_0}^{t}\varepsilon(s)\,ds+\epsilon}-\sqrt{\epsilon}\exp\left(\int_{T_0}^{t}H(\tau)\,d\tau\right)\right)\\ &+2\int_{T_0}^{t}H(s)\exp\left(\int_{s}^{t}H(\tau)\,d\tau\right)\sqrt{\int_{T_0}^{s}\varepsilon(\tau)\,d\tau+\epsilon}\;ds,\end{aligned} \tag{2.20}$$

where

$$H(t):=\max\left\{1, k_\eta(t)\frac{\|A\|}{\gamma}\right\} \text{ for all } t\in[T_0,T].$$

Since $\gamma(\cdot)\in L^1(I,\mathbb{R}_+)$ and for each $t\in I$, we have

$$\theta_n(t),\theta_m(t)\longrightarrow t,$$

then the Lebesgue dominated theorem ensures that

$$\lim_{n,m\to+\infty} G_{n,m}(t)=0, \quad \text{a.e. } in\, I. \tag{2.21}$$

On the other hand, we have for each $n \in \mathbb{N}$,

$$\int_{\theta_n(t)}^{t} \gamma(s)\, ds \leq \int_{T_0}^{T} \gamma(s)\, ds, \qquad \int_{\theta_m(t)}^{t} \gamma(s)\, ds \leq \int_{T_0}^{T} \gamma(s)\, ds,$$

this implies

$$|G_{n,m}(t)| \leq 2 \int_{T_0}^{T} \gamma(s)\, ds.$$

Therefore, for all $t \in [T_0, T]$ by (2.21) and the dominated convergence theorem, we have

$$\lim_{n,m\to+\infty} \int_{T_0}^{T} G_{n,m}(t)\, dt = \int_{T_0}^{T} \lim_{n,m\to+\infty} G_{n,m}(t)\, dt = 0. \tag{2.22}$$

Hence by (2.20) , we get $\lim\limits_{n,m\to+\infty} \|x_n(\cdot) - x_m(\cdot)\| \leq 0$, because $x_n(T_0) = x_m(T_0) = x_0$, i.e. $\|x_n(T_0) - x_m(T_0)\| = 0$ and $\lim\limits_{n,m\to+\infty} \int_{T_0}^{t} \varepsilon(s)\, ds = 0$ and taking $\epsilon \to 0$.

Consequently

$$\lim_{n,m\to+\infty} \|x_n(t) - x_m(t)\| = 0.$$

The last equality verified for all $t \in [T_0, T]$ then

$$\lim_{n,m\to+\infty} \sup_{t\in[T_0,T]} \|x_n(t) - x_m(t)\| = 0,$$

Or

$$\lim_{n,m\to+\infty} \|x_n(t) - x_m(t)\|_\infty = 0.$$

Therefore, $(x_n(\cdot))_n$ is a Cauchy sequence in $(\mathcal{C}([T_0, T], H), \|\cdot\|_\infty)$ and so converges uniformly to a function $x(\cdot) \in \mathcal{C}([T_0, T], H)$.

Step 5. The regularity of the limit function $x(\cdot)$.

We have for almost all $t \in I$ and for any $n \in \mathbb{N}$,

$$\|\dot{x}_n(t)\| \leq (1 + M)\left(1 + \frac{\|A\|}{\gamma}\right) \int_{T_0}^{t} \beta(t, s)\, ds + \frac{1}{\gamma} |\dot{v}(t)| := \gamma(t).$$

So we can extract a subsequence of $(\dot{x}_n(\cdot))$ and without loss of generality, we suppose that this subsequence is denoted again by $(\dot{x}_n(\cdot))$ and converges weakly in $L^1(I, H)$ to a function $g(\cdot) \in L^1(I, H)$. That is equivalent to the following:

$$\int_{T_0}^{T} \langle \dot{x}_n(s), h(s) \rangle\, ds \longrightarrow \int_{T_0}^{T} \langle g(s), h(s) \rangle\, ds, \quad \forall\, h \in L^\infty(I, H).$$

Now observe that for all $z \in H$

$$\int_{T_0}^{T} \langle \dot{x}_n(s), z \cdot \mathbf{1}_{[T_0,t]}(s) \rangle \, ds = \int_{T_0}^{t} \langle \dot{x}_n(s), z \rangle \, ds = \langle \int_{T_0}^{t} \dot{x}_n(s) ds, z \rangle,$$

and

$$\int_{T_0}^{T} \langle g(s), z \cdot \mathbf{1}_{[T_0,t]}(s) \rangle \, ds = \int_{T_0}^{t} \langle g(s), z \rangle \, ds = \langle \int_{T_0}^{t} g(s) \, ds, z \rangle.$$

So from the weak convergence of $(\dot{x}_n(\cdot))$, we deduce that

$$\int_{T_0}^{t} \dot{x}_n(s) \, ds \longrightarrow \int_{T_0}^{t} g(s) \, ds \quad \text{weakly in } H.$$

This implies that

$$x_n(T_0) + \int_{T_0}^{t} \dot{x}_n(s) \, ds \longrightarrow x(T_0) + \int_{T_0}^{t} g(s) \, ds \quad \text{weakly in } H.$$

But $(x_n(\cdot))$ is absolutely continuous, so

$$x_n(t) = x_n(T_0) + \int_{T_0}^{t} \dot{x}_n(s) \, ds \longrightarrow x(T_0) + \int_{T_0}^{t} g(s) \, ds \quad \text{weakly in } H.$$

On the other hand, we have for all $t \in [T_0, T]$

$$x_n(t) \longrightarrow x(t) \quad \text{strongly in } H,$$

this implies that

$$x_n(t) \longrightarrow x(t) \quad \text{weakly in } H.$$

From the uniqueness of the limit, we have

$$x(t) = x(T_0) + \int_{T_0}^{t} g(s) \, ds.$$

Therefore $x(\cdot)$ is absolutely continuous, and moreover, it is derivable almost everywhere in $[T_0, T]$ with its derivative $\dot{x}(t) = g(t)$ a.e. $t \in [T_0, T]$.
This implies

$$\|x(t)\| - \|x(T_0)\| \leq \|x(t) - x(T_0)\| = \| \int_{T_0}^{t} \dot{x}(s) \, ds\| \leq \int_{T_0}^{t} \|\dot{x}(s)\| \, ds \leq \int_{T_0}^{t} \|g(s)\| \, ds,$$

therefore

$$\|x(t)\| - \|x(T_0)\| \leq \int_{T_0}^{T} \|g(s)\|\, ds.$$

Then for all $t \in [T_0, T]$

$$\|x(t)\| \leq \widetilde{\eta}, \tag{2.23}$$

where

$$\widetilde{\eta} := \|x_0\| + \int_{T_0}^{T} \|g(s)\|\, ds.$$

Step 6. $x(\cdot)$ is a solution of (DP).

Since $\theta_n(t) \longrightarrow t$ for all $t \in I$ and $x_n(\cdot)$ converges uniformly to $x(\cdot)$, we have $x_n(\theta_n(t)) \longrightarrow x(t)$ for each $t \in I$. On the other hand, the continuity of $f(t, s, \cdot)$ on $B[0, \eta]$ ensures that, for all $t, s \in I$,

$$f(t, s, x_n(\theta_n(t))) \longrightarrow f(t, s, x(t)) \quad \text{in } H.$$

Let us set for each $t \in I$

$$\begin{cases} y_n(t) := \displaystyle\int_{T_0}^{t} f(t, s, x_n(\theta_n(s)))ds, \\ y(t) := \displaystyle\int_{T_0}^{t} f(t, s, x(s))ds. \end{cases}$$

We have shown in the above step that $(\dot{x}_n(\cdot))_n$ converges weakly to $\dot{x}(\cdot)$ in $L^1(I, H)$. Moreover, by (2.18) and (2.23) we can choose some real $c > 0$ such that, for each n, $\|x_n(\theta_n(t))\| \leq c$ and $\|x(t)\| \leq c$ for all $t \in [T_0, T]$. Therefore, by assumption $(\mathcal{H}_{3,2})$, there exists $k_c(\cdot) \in L^1([T_0, T], \mathbb{R}_+)$ such that f is $k_c(t) - \textit{Lipschitz}$ on $B[0, c]$. It follows that

$$\int_{T_0}^{T} \|y_n(t) - y(t)\|\, dt \leq \int_{T_0}^{T} k_c(t) \int_{T_0}^{t} \|x_n(\theta_n(s)) - x(s)\|\, ds\, dt. \tag{2.24}$$

Note that for every $(t, s) \in Q$

$$\begin{aligned} k_c(t) \int_{T_0}^{t} \|x_n(\theta_n(s)) - x(s)\| ds &\leq 2\, c\, k_c(t) \int_{T_0}^{t} ds \leq 2\, c\, k_c(t) \int_{T_0}^{T} ds, \\ &= 2\, c\, (T - T_0)\, k_c(t). \end{aligned}$$

Then by (2.24) and by the Lebesgue dominated convergence theorem

$$y_n(\cdot) \longrightarrow y(\cdot) \text{ strongly in } L^1(I, H),$$

this implies that

$$y_n(\cdot) \longrightarrow y(\cdot) \text{ weakly in } L^1(I, H).$$

Consequently

$$\zeta_n(\cdot) := \dot{x}_n(\cdot) + y_n(\cdot) \longrightarrow \zeta(\cdot) := \dot{x}(\cdot) + y(\cdot) \text{ weakly in } L^1(I, H).$$

Due to Mazur's Lemma, we extract a subsequence of $\zeta_n(\cdot)$ and we denote it again by $\zeta_n(\cdot)$ such that: For almost all $t \in I$,

$$\zeta_n(t) \in co\{\dot{x}_k(t) + y_k(t), k \geqslant n\} \tag{2.25}$$

The subsequence $\zeta_n(\cdot)$ converges strongly to $\dot{x}(\cdot) + y(\cdot)$ in $L^1(I, H)$ and for almost all $t \in I$,

$$\zeta(t) = \dot{x}(\cdot) + y(\cdot) \in \bigcap_n \overline{co}\{\dot{x}_k(t) + y_k(t), k \geqslant n\} \tag{2.26}$$

By (2.6), we have for almost all $t \in I$

$$-\dot{x}_n(t) - y_n(t) \in N_{C(t)}(Ax_n(t)), \tag{2.27}$$

and by (2.14), we have

$$-\dot{x}_n(t) - y_n(t) \in \alpha(t)\, B[0, 1], \tag{2.28}$$

where

$$\alpha(t) := \frac{\|A\|}{\gamma}\,(1 + M) \int_{T_0}^{t} \beta(t, s)\, ds + \frac{1}{\gamma}\,|\dot{v}(t)|.$$

The relations (2.27), (2.28) imply that

$$-\frac{\dot{x}_n(t) + y_n(t)}{\alpha(t)} \in N_{C(t)}(Ax_n(t)) \cap B[0, 1].$$

Therefore

$$-\frac{\dot{x}_n(t) + y_n(t)}{\alpha(t)} \in \partial d_{C(t)}(Ax_n(t)).$$

It follows by applying Theorem 1.7.2 that for almost all $t \in I$, for all $\xi \in H$,

$$\left\langle \xi, \frac{\dot{x}_n(t) + y_n(t)}{\alpha(t)} \right\rangle \leq \sigma(-\partial d_{C(t)}(Ax_n(t)), \xi),$$

where $\sigma(-\partial d_{C(t)}(Ax_n(t)),.)$ is the support function associated to the closed convex set $-\partial d_{C(t)}(Ax_n(t))$. Then, we obtain

$$\big\langle \xi, \dot{x}_n(t) + y_n(t)\big\rangle \leq \alpha(t)\,\sigma(-\partial d_{C(t)}(Ax_n(t)),\xi). \tag{2.29}$$

On the other hand, by (2.25) and (2.29), we have for all $n \in \mathbb{N}$, for almost all $t \in I$, and for all $\xi \in H$,

$$\big\langle \xi, \zeta_n(t)\big\rangle \leq \sup_{k\geqslant n}\big\langle \xi, \zeta_k(t)\big\rangle \leq \alpha(t) \sup_{k\geqslant n} \sigma(-\partial d_{C(t)}(Ax_k(t)),\xi).$$

From (2.26), we have

$$\big\langle \xi, \zeta(t)\big\rangle \leq \alpha(t) \inf_{n} \sup_{k\geqslant n} \sigma(-\partial d_{C(t)}(Ax_k(t)),\xi).$$

That is

$$\big\langle \xi, \zeta(t)\big\rangle \leq \alpha(t) \limsup_{n} \sigma(-\partial d_{C(t)}(Ax_n(t)),\xi).$$

Since for all $t \in I$, $\sigma(-\partial d_{C(t)}(\cdot),\xi)$ is upper semi-continuous on H ([8]), we have, for almost all $t \in I$ and for all $\xi \in H$,

$$\big\langle \xi, \dot{x}(t) + y(t)\big\rangle \leq \alpha(t)\,\sigma(-\partial d_{C(t)}(Ax(t)),\xi),$$

this implies

$$\big\langle \xi, \dot{x}(t) + y(t)\big\rangle - \alpha(t)\,\sigma(-\partial d_{C(t)}(Ax(t)),\xi) \leq 0.$$

That is

$$\big\langle \xi, \dot{x}(t) + y(t)\big\rangle - \sigma(-\alpha(t)\,\partial d_{C(t)}(Ax(t)),\xi) \leq 0.$$

Since ξ is chosen arbitrarily, we have

$$\sup_{\xi\in H}\Big[\big\langle \xi, \dot{x}(t) + y(t)\big\rangle - \sigma(-\alpha(t)\,\partial d_{C(t)}(Ax(t)),\xi)\Big] \leq 0. \tag{2.30}$$

Therefore since $\partial d_{C(t)}(Ax(t))$ is a closed convex subset, we have

$$\begin{aligned} d(\dot{x}(t) + y(t), -\alpha(t)\,\partial d_{C(t)}(Ax(t)) &= \sup_{\xi\in\mathbb{B}}\Big[\big\langle \xi, \dot{x}(t) + y(t)\big\rangle - \sigma(-\alpha(t)\,\partial d_{C(t)}(Ax(t)),\xi)\Big] \\ &\leq \sup_{\xi\in H}\Big[\big\langle \xi, \dot{x}(t) + y(t)\big\rangle - \sigma(-\alpha(t)\,\partial d_{C(t)}(Ax(t)),\xi)\Big], \end{aligned}$$

this and inequality (2.30) give

$$d(\dot{x}(t) + y(t), -\alpha(t)\,\partial d_{C(t)}(Ax(t))) = 0.$$

Therefore

$$-\dot{x}(t) - y(t) \in \alpha(t)\,\partial d_{C(t)}(Ax(t)) \subset N_{C(t)}(Ax(t)) \qquad \text{a.e. } t \in I,$$

this is

$$-\dot{x}(t) - \int_{T_0}^{t} f(t,s,x(s))\,ds \in N_{C(t)}(Ax(t)) \qquad \text{a.e. } t \in I,$$

and thus

$$-\dot{x}(t) \in N_{C(t)}(Ax(t)) + \int_{T_0}^{t} f(t,s,x(s))\,ds \qquad \text{a.e. } t \in [T_0,T].$$

Moreover, $x(T_0) = \lim_{n\to\infty} x_n(T_0) = x_0$. Therefore, the function $x(\cdot)$ is a solution of (DP).

Step 7. The a priori estimations.

Let $x(\cdot)$ be a solution of (DP).

1. We follow the same arguments as in [4], we have

$$\|\dot{x}(t) + \int_{T_0}^{t} f(t,s,x(s))\,ds\| \leq \frac{\|A\|}{\gamma}\int_{T_0}^{t} \|f(t,s,x(s))\|\,ds + \frac{1}{\gamma}\,|\dot{v}(t)| \;\text{ a.e. } (t,s)\in Q \tag{2.31}$$

2. If $\quad \|f(t,s,x)\| \leq g(t,s) + \lambda(t)\,\|x\|,$

   we have from (2.31)

$$\begin{aligned}
\|\dot{x}(t)\| &= \|\dot{x}(t) + \int_{T_0}^{t} f(t,s,x(s))\,ds - \int_{T_0}^{t} f(t,s,x(s))\,ds\| \\
&\leq \|\dot{x}(t) + \int_{T_0}^{t} f(t,s,x(s))\,ds\| + \|\int_{T_0}^{t} f(t,s,x(s))\,ds\| \\
&\leq \frac{1}{\gamma}\,|\dot{v}(t)| + \left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{t} \|f(t,s,x(s))\|\,ds \\
&\leq \frac{1}{\gamma}\,|\dot{v}(t)| + \left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{t} g(t,s)\,ds + \left(1+\frac{\|A\|}{\gamma}\right)\lambda(t)\int_{T_0}^{t} \|x(s)\|\,ds.
\end{aligned} \tag{2.32}$$

   The fact that $x(\cdot)$ is absolutely continuous implies

$$\|x(t)\| \leq \|x_0\| + \int_{T_0}^{t} \|\dot{x}(s)\|\,ds := \rho(t).$$

   The inequality (2.32) ensures that

$$\dot{\rho}(t) \leq \frac{1}{\gamma}\,|\dot{v}(t)| + \left(1+\frac{\|A\|}{\gamma}\right)\int_{T_0}^{t} g(t,s)\,ds + \left(1+\frac{\|A\|}{\gamma}\right)\lambda(t)\int_{T_0}^{t} \rho(s)\,ds.$$

Applying Gronwall's lemma (Lemma 1.13.2) with $\rho(\cdot)$, we obtain for a.e. $t \in [T_0, T]$

$$\|x(t)\| \leq \rho(t) \leq \|x_0\| \exp\left(\int_{T_0}^{t} B(\tau)\, d\tau\right) + \int_{T_0}^{t} \left[\frac{1}{\gamma}\,|\dot{v}(s)| + \left(1 + \frac{\|A\|}{\gamma}\right) \int_{T_0}^{s} g(s,\tau)\, d\tau\right] \exp\left(\int_{s}^{t} B(\tau)\, d\tau\right) ds,$$

where $B(\tau) := \max\{1, \lambda(\tau)\}$ for almost all $\tau \in [T_0, T]$.

The last inequality implies that for all $t \in [T_0, T]$

$$\|x(t)\| \leq l,$$

where

$$l := \|x_0\| \exp\left(\int_{T_0}^{T} B(\tau)\, d\tau\right) + \int_{T_0}^{T} \left[\frac{1}{\gamma}\,|\dot{v}(s)| + \left(1 + \frac{\|A\|}{\gamma}\right) \int_{T_0}^{T} g(s,\tau)\, d\tau\right] \exp\left(\int_{T_0}^{T} B(\tau)\, d\tau\right) ds.$$

Moreover for almost all $(t, s) \in Q$, we have

$$\|f(t,s,x)\| \leq g(t,s) + \lambda(t)\,\|x\|,$$

this entails

$$\|f(t,s,x)\| \leq g(t,s) + \lambda(t)\, l \qquad \text{a.e. } (t,s) \in Q. \tag{2.33}$$

By (2.31) and (2.33) we obtain

$$\|\dot{x}(t) + \int_{T_0}^{t} f(t,s,x(s))\, ds\| \leq \frac{1}{\gamma}\,|\dot{v}(t)| + \frac{\|A\|}{\gamma} \int_{T_0}^{t} g(t,s)\, ds + (T - T_0)\,\lambda(t)\, l.$$

Now consider the situation when

$$\int_{T_0}^{T} \int_{T_0}^{\tau} \beta(\tau, s)\, ds\, d\tau \geq \frac{\gamma}{\gamma + \|A\|}.$$

We fix a subdivision of $[T_0, T]$ given by $T_0, T_1, ..., T_k = T$ such that, for any $0 \leq i \leq k-1$,

$$\int_{T_i}^{T_{i+1}} \int_{T_0}^{\tau} \beta(\tau, s)\, ds\, d\tau < \frac{\gamma}{\gamma + \|A\|}.$$

Then, there exists an absolutely continuous map $x_0 : [T_0, T_1] \longrightarrow H$ such that $x_0(T_0) = x_0$, $Ax_0(t) \in C(t)$ for all $t \in [T_0, T_1]$, and

$$-\dot{x}_0(t) \in N_{C(t)}(Ax_0(t)) + \int_{T_0}^{t} f(t, s, x_0(s))\, ds, \quad \text{a.e. } t \in [T_0, T_1].$$

Similarly, there is an absolutely continuous map $x_1 : [T_1, T_2] \longrightarrow H$ such that $x_1(T_1) = x_0(T_1)$, $Ax_1(t) \in C(t)$ for all $t \in [T_1, T_2]$, and

$$-\dot{x}_1(t) \in N_{C(t)}(Ax_1(t)) + \int_{T_0}^{t} f(t, s, x_1(s))\, ds, \quad \text{a.e. } t \in [T_1, T_2].$$

By induction, we obtain for each $0 \leq i \leq k-1$ a finite sequence of absolutely continuous maps $x_i : [T_i, T_{i+1}] \longrightarrow H$ such that for each $0 \leq i \leq k-1$,
$x_i(T_i) = x_{i-1}(T_i)$ and $Ax_i(t) \in C(t)$ for all $t \in [T_i, T_{i+1}]$, and

$$-\dot{x}_i(t) \in N_{C(t)}(Ax_i(t)) + \int_{T_0}^{t} f(t, s, x_i(s))\, ds, \quad \text{a.e. } t \in [T_i, T_{i+1}].$$

We set $x_{-1}(0) = x_0$ and define the mapping $x : [T_0, T] \longrightarrow H$ given by

$$x(t) = x_i(t), \text{ if } t \in [T_i, T_{i+1}], \; 0 \leq i \leq k-1.$$

Obviously, $x(\cdot)$ is an absolutely continuous mapping satisfying $x(T_0) = x_0$,
$Ax(t) \in C(t)$ for all $t \in [T_0, T]$ and

$$-\dot{x}(t) \in N_{C(t)}(Ax(t)) + \int_{T_0}^{t} f(t, s, x(s))\, ds, \quad \text{a.e. } t \in [T_0, T].$$

Step 8. Uniqueness.

Let $x_1(\cdot), x_2(\cdot)$ be two solutions of the differential inclusion (DP), then

$$\begin{cases} -\dot{x}_1(t) \in N_{C(t)}(Ax_1(t)) + \displaystyle\int_{T_0}^{t} f(t, s, x_1(s))ds & \text{a.e. } t \in [T_0, T] \\ x_1(T_0) = x_0, \qquad A(x_0) \in C(T_0), \end{cases}$$

and

$$\begin{cases} -\dot{x}_2(t) \in N_{C(t)}(Ax_2(t)) + \displaystyle\int_{T_0}^{t} f(t, s, x_2(s))ds & \text{a.e. } t \in [T_0, T] \\ x_2(T_0) = x_0 \qquad A(x_0) \in C(T_0). \end{cases}$$

The fact that the normal cone is monotone, we have

$$\Big\langle -\dot{x}_1(t)-\int_{T_0}^{t} f(t,s,x_1(s))\,ds+\dot{x}_2(t)+\int_{T_0}^{t} f(t,s,x_2(s))\,ds, Ax_1(t)-Ax_2(t)\Big\rangle \geq 0,$$

this implies

$$\begin{aligned}\big\langle \dot{x}_1(t)-\dot{x}_2(t), A(x_1(t)-x_2(t))\big\rangle &\leq \Big\langle \int_{T_0}^{t} f(t,s,x_1(s))\,ds-\int_{T_0}^{t} f(t,s,x_2(s))\,ds, A(x_2(t)-x_1(t))\Big\rangle\\ &\leq \int_{T_0}^{t} \|f(t,s,x_1(s))\,ds-f(t,s,x_2(s))\|\,ds\,\|A(x_2(t)-x_1(t))\|.\end{aligned}$$

Since A is a bounded linear map and f Lipschitzian on bounded set ($B[0,\eta]$) we have

$$\big\langle \dot{x}_1(t)-\dot{x}_2(t), A(x_1(t)-x_2(t))\big\rangle \leq \|A\|\,\|x_1(t)-x_2(t)\|\,k_\eta(t)\int_{T_0}^{t}\|x_1(t)-x_2(t)\|\,ds,$$

this gives

$$\frac{1}{2}\frac{d}{dt}\big\langle x_1(t)-x_2(t), A(x_1(t)-x_2(t))\big\rangle \leq k_\eta(t)\,\|A\|\,\|x_1(t)-x_2(t)\|\int_{T_0}^{t}\|x_1(t)-x_2(t)\|\,ds.$$

The fact that A is $\gamma - \mathit{coercive}$ imply

$$\frac{d}{dt}\|x_1(t)-x_2(t)\|^2 \leq \frac{2}{\gamma}\,k_\eta(t)\,\|A\|\,\|x_1(t)-x_2(t)\|\int_{T_0}^{t}\|x_1(t)-x_2(t)\|\,ds.$$

Setting

$$\begin{cases}\rho(t) := \|x_1(t)-x_2(t)\|^2,\\ k(t) := \frac{2}{\gamma}\,\|A\|\,k_\eta(t).\end{cases}$$

Then

$$\dot{\rho}(t) \leq k(t)\sqrt{\rho(t)}\int_{T_0}^{t}\sqrt{\rho(s)}\,ds. \tag{2.34}$$

Applying Lemma 1.13.3 in (2.34) with $\varepsilon(\cdot), \epsilon > 0$ arbitrary, we obtain

$$\begin{aligned}\sqrt{\rho(t)} \leq{}& \sqrt{\rho(T_0)+\epsilon}\,\exp\Big(\int_{T_0}^{t} H(s)\,ds\Big)+\frac{\sqrt{\epsilon}}{2}\int_{T_0}^{t}\exp\Big(\int_{s}^{t} H(\tau)\,d\tau\Big)\,ds\\ &+2\Big(\sqrt{\int_{T_0}^{t}\varepsilon(s)\,ds+\epsilon}-\sqrt{\epsilon}\,\exp\Big(\int_{T_0}^{t} H(\tau)\,d\tau\Big)\Big)\\ &+2\int_{T_0}^{t}\Big[H(s)\exp\Big(\int_{s}^{t} H(\tau)\,d\tau\Big)\sqrt{\int_{T_0}^{s}\varepsilon(\tau)\,d\tau+\epsilon}\,\Big]\,ds,\end{aligned}$$

where $H(t) := \max\left\{1, \frac{\|A\|}{\gamma}\, k_\eta(t)\right\}$ for all $t \in [T_0, T]$.
Taking $\epsilon \longrightarrow 0$ and $\varepsilon(t) \longrightarrow 0$ then

$$\|x_1(t) - x_2(t)\| \leq 0,$$

this clearly implies that

$$\|x_1(t) - x_2(t)\| = 0.$$

So

$$x_1(t) = x_2(t),$$

which justifies the uniqueness.

■

Chapter 3

Application

The connection between degenerate sweeping processes and nonsmooth dynamical systems can be effectively illustrated through the framework of differential complementarity problems. These problems frequently arise in various fields such as mechanical and electrical engineering modeling, nonsmooth optimization, and other related areas.

In this chapter we give the following example as an application of Theorem 2.2.2 that include an existence and uniqueness result of a differential complementarity problem,

Let $K \subset H$ be a closed convex cone and $K^* = \{h \in H : \langle h, k \rangle \geq 0, \forall k \in K\}$ be its dual cone. Let $I = [T_0, T]$ and $A : H \longrightarrow H$ satisfying $(\mathcal{H}_2)$ and $g(\cdot) : [T_0, T] \times H \longrightarrow H$ be a single valued mapping that satisfies $(\mathcal{H}_{3,1})$ and $(\mathcal{H}_{3,2})$. Let $\alpha(\cdot) : I \longrightarrow H$ be an absolutely continuous function such that $\alpha(0) = 0$.

Consider the problem of finding an absolutely continuous mappings $x(\cdot) : I \longrightarrow H$ and an integrable function $u(\cdot) : I \longrightarrow H$ satisfying for almost every t in I the following differential complementarity problem

$$(DCP) \begin{cases} -\dot{x}(t) = \int_{T_0}^{t} g(t, s, x(s))ds + u(t) \qquad \text{a.e. } t \in I, \\ v(t) = Ax(t) + \alpha(t), \\ K^* \ni \; u(t) \perp v(t) \in K, \\ x(T_0) = x_0, \quad Ax_0 \in K. \end{cases}$$

Now, we recall and prove the following complementarity result which will be usefull to rewrite (DCP) in our abstruct frame:

Proposition 3.0.1.

A vector u is belong to $-N_K(v)$ *if and only if* $K^* \ni \; u \perp v \in K$.

Proof.

Fist we prove the ordinary implication which is

$$u \in -N_K(v) \Rightarrow K^* \ni \; u \perp v \in K.$$

Let $u \in -N_K(v)$ that is $-u \in N_K(v)$ i.e.

$$\langle -u, p - v \rangle \leq 0, \quad \forall p \in K, \tag{3.1}$$

and we prove the following properties:

(i) $v \in K$,

(ii) $u \in K^*$,

(iii) $\langle u, v \rangle = 0$.

The first property (i) is evident.
Let us proof the property (iii):
We recall that (3.1) is true for all $p \in K$ in particular:
•$p := 2v \; \in K$ (verified because K is a cone),

$$\langle -u, v \rangle \leq 0 \Rightarrow \langle u, v \rangle \geq 0.$$

•$p := 0 \; \in K$,

$$\langle -u, -v \rangle \leq 0 \Rightarrow \langle u, v \rangle \leq 0.$$

Combining the above inequalities we get $\langle u, v \rangle = 0$, that mean $u \perp v$.
Now it remains to prove (ii)
From (3.1) we have for all $p \in K$: $\langle -u, p \rangle + \langle -u, -v \rangle \leq 0$, then $\langle u, p \rangle \geq 0$, i.e. $u \in K^*$.
Therefore $K^* \ni \; u \perp v \in K$.

Conversely, we suppose that $K^* \ni \; u \perp v \in K$.
Let p be an arbitrary vector of K then

$$\begin{aligned} \langle -u, p - v \rangle &= \langle -u, p \rangle + \langle -u, -v \rangle \\ &= -\langle u, p \rangle + \langle u, v \rangle \leq 0 \end{aligned}$$

i.e. $-u \in N_K(v) \Leftrightarrow u \in -N_K(v)$.
which complete the proof. ■

From the above proposition (DCP) is equivalent to finding $x(\cdot) : I \longrightarrow H$ such that

$$\begin{cases} \dot{x}(t) \in -N_K(Ax(t) + \alpha(t)) + \int\limits_{T_0}^{t} g(t, s, x(s))ds & \text{a.e. } t \in I, \\ x(T_0) = x_0, \quad Ax_0 \in K. \end{cases}$$

We set $C(t) := K - \alpha(t)$ for all $t \in I$, and using the invariance of the normal cone by translation, so we see that the problem (DCP) is equivalent to finding $x(\cdot) : I \longrightarrow H$ satisfying

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(Ax(t)) + \int\limits_{T_0}^{t} g(t, s, x(s))ds & \text{a.e. } t \in I, \\ x(T_0) = x_0, \quad Ax_0 \in C(0). \end{cases}$$

Clearly, all the assumptions of Theorem 2.2.2 are satisfied and thus we obtain the existence and uniqueness of solution for (DCP).

Conclusion and future work

In this master's thesis we were proved the well posedness (in sence of existence and uniquness result) for the new variant of sweeping process which is degenerated by a linear bounded operator and involving an integral perturbation:

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(Ax(t)) + \int\limits_{T_0}^{t} f(t,s,x(s))ds \qquad \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0, \quad Ax_0 \in C(T_0), \end{cases}$$

where the moving set $C(t)$ moved in an absolutely continuous way.

To reach this goal, we have provided a semi discretization sheme and proved it convergence to an absolutely continuous solution by using a new type of Gronwall inequality.

In the last, we suggest as future research to study this variant of sweeping process in the case where the set $C(t)$ not convex (r-prox).

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