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Pair creation in graphene via path integral approach in non commutative phase space

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Dedications

In the name of Allah, the Most Gracious, Most Merciful, All the Praise is to Allah alone, the Sustainer of all the world.

I dedicate this work to:

My lovely father **Abdelmalek** *who is always beside me and encourage me to fulfill all my dreams. Today I do declare in front of everyone here that I love you with all my heart, thank you so much.*

My lovely ***mother****, I would like to say it is impossible to thank you because it is not enough for you and for all sacrifices that you made for me. Today I realize my dream which is yours of being a doctor of theoretical physics and it will be a fact not a dream soon Inchaa-Allah, darling.*

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My ***grandfathers*** *may God have mercy on them and my* ***grandmothers****.*

My **aunts** *and* **uncles***.*

My ***friends*** *and* ***friend work****.*

To all who support me in my education as well as for every one respect and love me.

To all my relatives and those who know me.

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Abstract

In our thesis, we have presented an exact analytical solution of the massless Dirac-Graphene equation in the presence of two plane wave fields using Volkov's ansatz. We have also adapted the supersymmetric path integral formalism to construct the corresponding Dirac-graphene propagator. Finally, the wave functions are deduced. On the other hand, we have studied the problem of graphene's quasiparticle-hole pair creation from the vacuum under the action of two gauges different of an electromagnetic field and in NC phase space coordinates by using Schwinger's method. Also, we applied the same formalism to analyze the pair creation process of both scalar and spinorial relativistic particles. For each case, the effective action is calculated by the supersymmetric path integral formalism. As an application, all special cases of (θ, η, E, B) have been studied and discussed. In addition, the influence of one and two orthogonal plane wave fields on the pair creation process in graphene is examined. The essential result of this work is that noncommutativity has an influence on the process of pair creation from the vacuum, and the plane wave does not contributes to this process. Furthermore, we have solved the Dirac-graphene equation for quasiparticles in interaction with the combination of a plane wave and a parallel magnetic field, following two different techniques. The first one is by using the Redmond method, and the second is by using the delta functional method. We have also studied the pair creation of graphene's quasiparticle-hole from the vacuum by this configuration of the field.

Key words: Graphene, quasi-particles, path integral, Fradkin and Gitmann formalism, Green's function, propagator, Dirac equation, Grassmann variables, Non-commutative geometry, Schwinger effect, plane wave, electromagnetic field, Redmond field.

Contents

Chapter 1

General introduction

The award of the Nobel Prize in Physics in 2010 went to Sir Konstantin Sergeevich Novoselov and Andre Geim, Dutch and Russian-British scientists, for the importance of their revolutionary work on graphene [1]. graphene is a two-dimensional material made of carbon atoms arranged in a honeycomb lattice, having excellent optical, mechanical, and electrical properties [2]. Nowadays, researchers' attention is increasingly focused on researching the extraordinary features of graphene [3, 4]. For example, the graphene quasiparticles have a linear relationship between energy and momentum [5, 6] and behave as massless relativistic fermions [7, 8, 9] satisfying the two-dimensional Dirac equation with an effective speed of light of $10^6\ m.s^{-1}$ so-called "the Fermi velocity υ_F"[10, 11, 5]. This property resulted in a number of new phenomena, particularly the Hall effect and the Klein tunneling effect [5]. This makes graphene a unique model system that gives the ability to examine the effects of quantum electrodynamics in strong fields [12, 13]. Additionally, it introduces a promising new direction of research about gravity-like phenomena (also known as "analogue gravity") on graphene [14]; the Hawking effect, [15] and the Schwinger effect, which is the fundamental topic of this work.

It is well known that the fundamental sources of the strong field are modern optical lasers such as the HERCULES laser [16] and the Extreme Light Infrastructure (ELI) [17] can produce intense electromagnetic fields, which are of the order $E = 10^{15}\ V/m$ corresponding to field strengths $\mathcal{E} = 10^{-3}\mathcal{E}_{cr}$ where ($\mathcal{E}_{cr} = 1.32 \times 10^{18}\ V/m$) [18]. In addition, the European X-ray Free-Electron Laser (XFEL), where the field strength is of the order $\mathcal{E} = 10^{17}\ V/m$ [19].

Astrophysical objects are another source for strong electromagnetic fields. Among these sources, we find black holes, active galactic nuclei, gamma-ray bursts, which generate superstrong electromagnetic fields [20], and pulsars [21], like rotating neutron stars, which emit electromagnetic radiation periodically. There are many other sources, like linear colliders and relativistic heavy ion collisions [18].

The use of a laser as a strong electromagnetic radiation source revealed the new characteristics of elementary particles, and in the near future, the system of two lasers can form a new scientific revolution. There are also a series of studies concerning the Compton scattering and the Schwinger effect in the presence of a strong field, particularly in graphene material.

Before World War II, British physicist Paul Dirac made many contributions to quantum mechanics. In 1926, Dirac in his thesis demonstrated the equivalence of the two recent formalisms of quantum physics, Heisenberg's matrix mechanics and Schrödinger's wave mechanics. The fundamental contributions are: the magnetic monopole [27, 28, 29], which makes it possible to explain the quantification of the electric charge and which has not yet been detected by experiments; the quantum statistics of fermions, called Fermi-Dirac; and the first mathematical formalism of quantum field theory. What attracted attention was how Dirac could imagine the existence of antimatter (the proton in that period) in 1930 in his paper "***A Theory of the Electron and the Proton***" [30].

Two years before, exactly in 1928, [31] Dirac wrote an equation that combined quantum theory and special relativity to describe the behavior of spin 1/2 quantum particles (electrons, for example) moving at a speed close to that of light [32].

There remained the problem of the negative energy solutions [holes] of these equations (in the classical case, one can purely and simply reject the negative solutions, but in the quantum case, transitions can take place between these states). Dirac then proposes a solution to this problem by assuming that the universe consists of both negative and positive energy states. After having imagined protons, he indicated in 1931 that these states of negative energy known as the Dirac sea could be occupied by "a new kind of particle, unknown to experimental physics, having the same mass and opposite charge to those of the electron". Since the positron was not observed in that period, it was the proton that was first interpreted as the electron's antiparticle. As early as 1932, Anderson confirmed Dirac's theory by discovering the positive electron, or positron, in cosmic rays and in β^+ radioactivity [33, 32].

After the discovery of the Dirac theory and with the discovery of the famous paradox of Klein [34], the most important non-perturbative quantum effect in quantum field theory is the pair production phenomenon, first predicted in the presence of a strong and static external electric field by Julian Schwinger, one of the founders of quantum field theory.

Despite the fact that the theory was originally put forward in 1930 by a team of researchers consisting of Fritz Sauter, Werner Heisenberg, and Hans Euler, they computed the leading quantum correction to the Maxwell Lagrangian in their paper [35, 36]. Also, they found that the effective action in quantum electrodynamics (QED) of a particle interacting with a classical electric field has an imaginary part, which means that the vacuum is unstable in an electric field, which prevents the creation of particle-antiparticle pairs. But Schwinger took his best effort to determine exactly the conditions under which this effect should appear and gave a full theoretical description in 1951 [37].

As a result, it takes his name "the Schwinger effect". He showed that the vacuum-vacuum transition amplitude has an intimate bond with the effective action [37] as follows

$$A\left(vac-vac\right)=\exp\left(\imath S_{eff}\right), \tag{1.1}$$

and the pair creation probability can be extracted from the imaginary part of this action,

$$\mathcal{P}_{Creat.}=1-\left|A\left(vac-vac\right)\right|^{2}\simeq 2\,\mathrm{Im}\,S_{eff}. \tag{1.2}$$

On the other hand, according to Shwinger, it is well known that a strong electric field creates scalar particles, while a magnetic field and a plane wave do not create pairs of particles.

In the presence of a constant and strong electric field $\mathcal{E}$, the famous formula concerning the probability of pair creation per unit of volume and time, referred to as the Schwinger formula, is given by [37, 38, 39]

$$\mathcal{P}_{Creat.}=\frac{e^{2}\mathcal{E}^{2}}{8\pi^{3}}\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n^{2}}\exp\left[-n\pi\frac{m^{2}}{e\mathcal{E}}\right]. \tag{1.3}$$

In the presence of a constant and strong electromagnetic field $(\mathcal{E},\mathcal{B})$, the probability of pair creation per unit of volume and time is modified to [39]

$$\mathcal{P}_{Creat.}=\frac{e^{2}\mathcal{B}}{8\pi^{2}}\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n}\csc\left(n\pi\frac{\mathcal{B}}{\mathcal{E}}\right)\exp\left[-n\pi\frac{m^{2}}{e\mathcal{E}}\right]. \tag{1.4}$$

Following [126], in lower dimension (2+1)D, for graphene, the previous equations were reduced to

$$\mathcal{P}_{Creat.}=\frac{\left(e\mathcal{E}\right)^{3/2}}{4\pi^{2}\upsilon_{F}^{1/2}}\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n+1}}{n^{3/2}}\quad for\ m=0, \tag{1.5}$$

and

$$\mathcal{P}_{Creat.} = \frac{\left(\sqrt{(e\mathcal{E})^2 - (\upsilon_F e\mathcal{B})^2}\right)^{3/2}}{4\pi^2\upsilon_F^{1/2}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{3/2}} \quad for\ m = 0. \tag{1.6}$$

In theoretical side, the study of pair production processes driven by a strong external force from the vacuum requires theoretical ideas that can effectively describe nonperturbative physics. We can use various techniques, such as, for example, the semi-classical method WKB [40, 41, 42], the wordline path integral formalism [43, 44, 45], Parker's adiabatic approach [46], the diagonalization of the Hamiltonian [47, 48, 49, 50, 51, 52, 53], and the Bogoliubov transformation linking the "in" states with the "out" states [54, 55, 56, 57, 58, 59, 60, 61, 62] and the instanton method [63, 64, 65, 66, 67, 68] and also the technique of Schwinger via the computation of the imaginary part of the effective action [37, 69, 70, 71, 72, 73].

The pair production process attracts the attention of many researchers, and its interest started when researchers started studying the Big Bang theory, which led to the creation of the universe as we know it, despite this effect can not be solved the problem of the Big Bang theory (In the first few minutes after the big bang, there is an increase in the number of matter over the number of antimatter "where did this increase in the number of particles come from?").

There are numerous applications for the Schwinger effect in modern physics. This effect may also be applicable in other contexts, such as cosmological pair creation [74, 75, 76, 77, 78, 79], Hawking radiation [80, 81, 82, 83, 84, 85, 86], black hole creation [87] and heavy nuclei to black holes [88]. In modern cosmology, the production of particles can have an important effect on the inflation phase problem and have an impact on how our universe evolves. Additionally, in the context of quantum field theory, gravitational and electromagnetic fields can both produce particle-antiparticle pairs from vacuum in curved space [89].

Furthermore, the event horizon of a black hole casually separates pairs into the interior and the exterior under the influence of gravitational and electromagnetic fields, and the black hole emits all species of particles [89].

On the other hand, the production of charged particle-antiparticle pairs in an external electric field was studied for $(2+1)$ −dimensional theories, and examining the Schwinger effect in semiconductors and Dirac materials like graphene has advanced significantly in recent years.

Nowaday, a series of studies concern the Schwinger effect in graphene in the presence of a strong electric field [90, 91], because of the possibility of observing it experimentally in graphene

[92, 93]. Following [94], the rate and the probability of pair production for a constant electric field are calculated using the semi-classical approach for multilayer graphene and via an exact solution of the Schrödinger equation for the case of monolayer graphene. Furthermore, in the framework of non-commutativity, the issue of pair creation is also treated [95], and the pair production probability is deduced for both scalar and spinor relativistic particles in the presence of an electromagnetic field in a non-commutative space considering Schwinger's method.

On the experimental side, the theoretical physicist attempts to incorporate the Schwinger effect in the laboratory, but this is very difficult to achieve experimentally. because it requires an electric field of the order of the critical value $\mathcal{E}_{cr} = \dfrac{m^2}{e} = 1.32 \times 10^{18} V/m$ for the electrons, which exceeds the current technological capacities [96].

Actually, the pair production mechanism has never been tested experimentally. The prospects for observing the Schwinger mechanism in future laser installations of X-rays are studied in [97].

However, the fact that the effective masses of the electron and hole in graphene are zero will open a window for an experimental investigation of QED phenomena in strong fields, in particular the Schwinger effect, because it might be observed experimentally in graphene [92, 93]. In January 2022, a team of researchers from Manchester University announced a surprising discovery: they confirmed the Schwinger effect and showed that the theory first proposed 70 years ago was correct when the scientists were able to really create the matter and antimatter from the vacuum under the influence of a strong electric field with a simple laboratory in monolayer graphene.

Researchers accelerated electrons in a vacuum simulator to the highest speed permitted by graphene's vacuum (1/300 c) [98]. Electrons appeared to become superluminous at this point, producing an electric current greater than what is permitted by the fundamental laws of quantum condensed matter physics. This effect's cause was attributed to the spontaneous production of extra charge carriers, or holes. Thus, as anticipated, the researchers' wish to scientifically confirm Schwinger's phenomenon in graphene material in the lab came true.

On the other hand, path integral formalism is one of the most important techniques for describing graphene's electronic structure and solving problems of its conductivity [22, 23].

Norbert Wiener first proposed the idea of a path integral in 1920 as a technique for solving problems in the theory of diffusion and Brownian motion, involving integrals in infinite dimension but unrelated to the quantum domain and its phenomena [99]. The propagator, which expresses the physical system's transition amplitude between its initial and final states, is the heart of this formulation. This latter is also known as Green's function.

Feynman's path integral is a pleasant subject of discussion between mathematicians and physicists. It allows physicists to present, thanks to a mathematical idea, a simple and rich formalism that describes physical phenomena, especially quantum phenomena. Originally, it has been used in the study of quantum mechanics and quantum field theory, where it can be the starting point of the covariant formalism or of the canonical formalism.

On the other hand, the path integral does not contain the operators, and due to its compatibility with calculation techniques, it has succeeded in solving several problems in non-relativistic quantum mechanics. Also, it is the origin of the Feynman diagrams. Despite the success of this formulation and the interest it has received from physicists in different branches of quantum mechanics, such as theoretical physics, statistical physics, and other branches, this formulation is not ideal because it is difficult to suggest a continuous path for the spin because of its discrete nature.

There have been numerous attempts to include spin in path integral formulation. In 1966, Fradkin developed the calculations for relativistic particles interacting with an electromagnetic wave [100, 101], to see him again with Gitman in 1991, where finally they succeeded to describe correctly the spinning particles by means of the formalism of path integrals and formulate the propagator of Dirac particles by using fermionic variables (Grassmann variables). This particular model is known as the "Fradkin-Gitman Model", this model was re-examined by Berezin and Marinov [102, 103].

In the case of the Dirac equation, the fundamental idea of this formalism is to write the causal Green's function like the inverse of an operator, and then we used a generalized proper time having two parts, one bosonic and the other fermionic. This formalism is used for solving many problems. For example, the problem of relativistic spinning particles in interaction with an electromagnetic plane wave field was treated via path integrals [100, 104] and the path-integral representations for the propagators of scalar and spinorial relativistic particles in an external electromagnetic field were derived [105].

Additionally, for graphene, the exact Green's function is constructed in uniform electric and magnetic fields [24], and the exact solutions for Green's function for quasiparticles in the field of slow-light pulse was constructed using the Fock-Schwinger proper time method [25, 26].

Furthermore, in the last few years, deformed-commutative geometry algebra played an important role in the field of physics due to the continuity of its applications in all branches of the subject. For example, the non-commutative space is necessary when studying the low energy yield of the **D-brane** with $\mathcal{B}$-field background. The effects of non-commutativity may also appear on a very small scale or in very high energy conditions. Furthermore, one of the strong motivations of non-commutative geometry is to obtain a coherent mathematical framework in which it would be possible to study quantum gravity. Particularly, it's interesting to test noncommutativity on graphene.

Modern physics is based on quantum mechanics, which clarifies the three main forces in the micro-world (the electromagnetic, weak and strong forces), and applies the theory of operator algebras acting on a Hilbert space (Von Neumann algebras) and on general relativity (GR), which explains the force of gravity in the macro-world. It mainly uses Riemannian geometry as a mathematical formalism.

Nowadays, scientists aspire to unify quantum field theory and gravity into one theory known as "the Grand Unified Theory". Several solutions have been proposed to solve this issue, but they are contradictory. We cite the widely used standard model that has been very successful as an example for the unification of the three main forces (strong, electromagnetic, and weak). There were other models for the unification of the four forces into one model, including the supersymmetry model, extra-dimensional model, string theory, and M. theory.

All of these attempts led to the appearance of a new concept of non-commutative geometry. This concept is based on generalizing ordinary legal commutation relations and Heisenberg's uncertainty principle, which leads to a new theory known as "non-commutative algebra" [106].

Geometry is based on the principle of describing spaces, which are sets of points equipped with an additional structure. Nowadays, physically, a point is an elusive theoretical concept, and, moreover, even though we describe the world as a space (a collection of points), effectively, we always use coordinates, which are functions on the space. Therefore, the notion of a function (in particular, a continuous function) appears to be more fundamental than that of a point. In physics, this is clearly visible when we take into account quantum effects, in particular the Heisenberg uncertainty principle. Then, it is impossible (both in theory and in practice) to observe a point or to fix the coordinates with an arbitrarily small accuracy.

Noncommutative geometry is an extensive theory that has numerous intriguing applications in both mathematics and physics, such as the quantum hall effect, quantum computing, the standard model, quantum field theory, and the list goes on.

The origin of non-commutative geometry was first related to the idea of non-commutative space-time that was suggested by Heisenberg in 1930 and presented in 1947 by Snyder [107], which is strongly driven largely by the foundations of quantum mechanics within the framework of canonical quantification, and that firstly related to the idea of non-commutative space-time and to the need to regularize the divergence of quantum field theory.

In 1985, the term noncommutative geometry was introduced by Alain Connes and others in a program aiming to generalize the different concepts of ordinary geometry into equivalent concepts for noncommutative algebras [108].

In recent years, N. Seiberg and E. Witten published their famous article [109] which was the most cited article at that time. This aroused and encouraged great interest in non-commutative geometry, which has become extremely interesting for the study of many physical problems, and it became clear that there is an intimate connection between these concepts and string theory. Studies of this geometric type and its implication largely contribute to bring out various fields of physics, in particular relativistic and non-relativistic quantum mechanics [110, 111, 112, 113, 114, 115, 116, 117, 118] and in the description of the theories of quantum gravity.

On the other hand, the non-commutative theory replaces the non-commutativity of operators linked to space-time coordinates by a deformation of the algebra of defined functions in space-time and replaces the ordinary theory by a non-commutative theory, including replacing ordinary fields with non-commutative fields and ordinary products with Moyal-Weyl products. Taking into account the fact that the notions of non-commutativity in phase space are based mainly on the Seiberg-Witten maps, the star product of Moyal-Weyl, and the linear transformation of the Bopp shift.

Several authors have solved many related problems, for example: Klein Gordon oscillators [119], central potential [120] an Aharonov-Bhom effect [121]....etc.

The deformation of space due to non-commutativity in field theory can be expressed by the commutation relations of the Hermitian operator [122]

$$\left[\hat{X}_i, \hat{P}_j\right] = i\hbar^{eff.}\delta_{ij}, \quad \left[\hat{X}_i, \hat{X}_j\right] = i\theta_{ij}, \quad \left[\hat{P}_i, \hat{P}_j\right] = i\eta_{ij}, \tag{1.7}$$

$$\text{with } \theta, \eta \lll \quad i, j = 1, 2, 3, \tag{1.8}$$

where the effective Planck constant $\hbar^{eff.}$ can be written as [123]

$$\hbar^{eff.} = \hbar\left(1 + \frac{\theta\eta}{4\hbar}\right). \tag{1.9}$$

In the same way as the previous deformation, a formulation of path integrals was constructed in the context of non-commutative, and various attempts were presented. As a case study, the problem of a charged particle with spin 1/2 moving in any electromagnetic field was treated in this relativistic case [124]

Furthermore, we take into account a noncommutative graphene description. For massless Dirac fermions, this description consists of a Dirac equation plus noncommutative corrections that are handled in the presence of an external magnetic field. We contend that since graphene is a two-dimensional Dirac system, it is especially intriguing to investigate noncommutativity in this material. We discover that whereas momentum noncommutativity has an impact on graphene's energy levels, Hall conductivity is unaffected [125].

The principal motivation of the present thesis is to study some relativistic problems of physics via the supersymmetric path integrals formalism in graphene material. One of these problems is the pair production process from the vacuum by an external field without and within the non-commutative geometry, due to its importance and advantages, mainly in QED and QFT, as well as the study of the behavior of quasiparticles in graphene.

This thesis is consistes of eight chapters organized in the following way: The first chapter is a general introduction; the second chapter gives a brief overview of graphene, its structure, its properties, and their applications in theoretical physics.

Whereas in the third chapter, we give an exact analytical solution of the massless Dirac equation for graphene in the presence of two plane wave fields using the Volkov ansatz and deduce the corresponding wave functions.

In addition, in the fourth chapter, we aim to address the problem of graphene quasiparticles in interaction with a single and two orthogonal electromagnetic plane wave fields. In our calculations, we rely on the supersymmetric path integral proposed by Fradkin and Gitman [105]. This formalism gives results identical to the results obtained via the exact solution for both cases [149]. The solution of the Green function for these two special configurations of plane waves is determined, and the wave functions are exactly deduced.

In the five chapter, we use the original paper [126] as a basis and principal part of this work, we study the problem of pair production from the vacuum in monolayer graphene, subjected to two different gauges of a constant electromagnetic field in the framework of NC phase space coordinates using Schwinger's method. We calculate the effective action and the corresponding pair creation probability. The special cases of the results are also studied and discussed.

Also in the sixth chapter, we study the problem of pair creation of both scalar and spinorial relativistic particles from the vacuum by a constant electromagnetic field in the framework of non-commutative phase space coordinates using Schwinger's method, and we discuss the special cases of pair production probability in per unit volume and time and compare them with those of the literature [126] by taking the limit $\upsilon_F \to c, m \to 0$.

In the seventh chapter, we solve the Dirac-graphene equation for quasiparticles in interaction with the combination of a Volkov's plane wave and a constant magnetic field parallel to the direction of propagation of the electromagnetic wave using two methods: the first is the Redmond method, and the second is the Delta functional method. Then, we examine particular cases of our result by taking a limit on the fields and comparing them with those in the literature. On the other hand, we study the influence of this configuration of fields on the process of pair creation. The last chapter is the general conclusion, which presents a summary of our main results.

The thesis concludes with one appendix, in which we present the details of diverse calculations of the inverse of the matrix elements $\left(\mathcal{M}^{-1}\right)^{\mu\nu}$ for determining the Polyakov spin factor.

Chapter 2

Fundamental properties of graphene

The main goal of this chapter is to provide a brief overview of graphene, its structure, properties, and applications in theoretical physics.

2.1 Forms of Carbon

Carbon is different from other elements due to its special capacity to hybridize, which also enables it to form $0D$, $1D$, $2D$, and $3D$ structures. Diamond and graphite are two materials constructed from carbon in three dimensions ($3D$).

The lower-dimensional forms of carbon $(0-2)D$ are graphene with a $2D$ structure, carbon nanotubes with a $1D$ structure (the sheet of graphene is rolled into a cylindrical tube with a diameter of around $1nm$ to form carbon nanotubes), and fullerene with a $0D$ structure, which are composed of derivatives of the two-dimensional carbon (graphene) and the one-dimensional carbon nanotubes [127, 128].

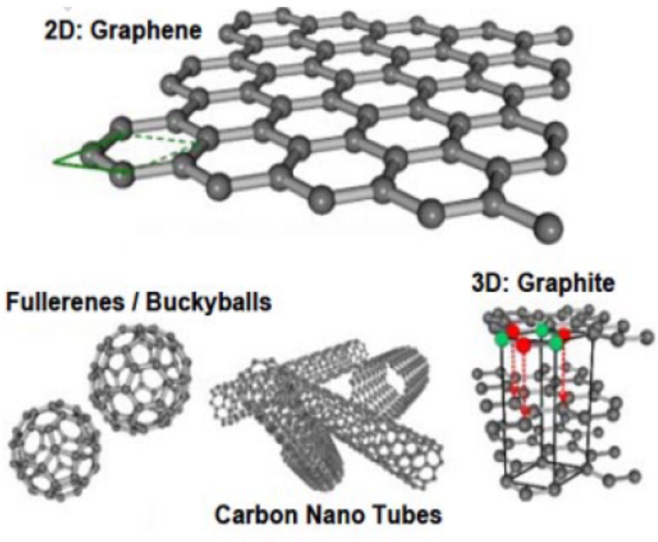


Figure1 : Forms of carbon materials: graphite, graphene, carbon nanotube, and fullerene. Image taken from [129].

2.2 The discovery of graphene material and its fundamental properties

Recently, the attention of researchers has turned to a two-dimensional material called graphene, which is the building block of graphite and consists of a monolayer of carbon atoms arranged in hexagonal cells that are only one atom thick.

The first study on graphene was investigated in 1947 by Wallace, who derived its band structure [128] and it was isolated experimentally for the first time in 2004 by Andre Geim and Kostaya Novoselov using the micromechanical cleavage technique; for that, they were awarded the Nobel Prize in Physics in 2010 [130, 131, 132, 133].

Due to its amazing and extraordinary properties, such as zero band gap when its electronic structure is characterized by conical valence and conduction band dynamics, graphene is one of the most crucial topics in condensed matter research.

As a result, it is the best thermal and electrical conductor at room temperature, and its vast surface area makes it an extremely chemically inert nanoparticle [134] that is completely impermeable to even the smallest atom (Helium).

Among other characteristics, one can notice the high electron mobility (more than $200,000$

$cm^2.V^{-1}.s^{-1}$) and the ballistic transport [135]. As a result of these characteristics, the electron wave packet can travel over great distances without scattering. Due to this, considerable research has been attempted to incorporate graphene into electronic devices. It has been employed in a number of applications, such as supercapacitors, lithium-ion batteries, and conducting electrodes [151, 137].

Currently, it is the strongest known material and one of the thinnest objects conceivable, which is also 200 times stronger than materials with Young's modulus of $1TPa$ [138, 139]. Additionally, it is one of the most elastic and flexible materials and is approximately completely transparent (97.3%).

2.3 Crystallographic structure of graphene

Graphene is a two-dimensional material composed of carbon atoms arranged in a honeycomb lattice. The carbon-carbon bond length is $a_0 = 1.42\mathring{A}$. The carbon atom is the sixth element in the periodic table; it has four covalent electrons with the configuration $1S^2$ $2S^2$ $2P_x^1$ $2P_y^1$ $2P_z^0$ see Figure 2(b).

There is no electron in the energy level of $2P_z$, therefore, it is equivalent to the energy levels $2P_x^1$ and $2P_y^1$ .Figure 2(a).

The nucleus of a carbon atom is surrounded by six electrons, among them four valence electrons, which are three types of hybridization (SP^1, $SP^2$and SP^3). Figure2(c).The carbon atom shares SP^2 electron with three neighboring carbon atoms to form a lattice. Thus forming a monolayer graphene. Figure 2(d)

The hybrizidation SP^2 in graphene forms two bands σ and π. Figure 2(e) [128].

The monolayer graphene honeycomb lattice consists of two atoms in the unit cell, which form the two triangular sublattices A and B. the two primitive lattice vectors are written as

$$\vec{a}_1 = \frac{a}{2}\begin{pmatrix} 1, & \sqrt{3} \end{pmatrix}, \quad \vec{a}_2 = \frac{a}{2}\begin{pmatrix} 1, & -\sqrt{3} \end{pmatrix}, \quad a = \sqrt{3}a_0 \approx \sqrt{3} \times 1.42 = 2.64\mathring{A}, \tag{2.1}$$

where a is the lattice constant and a_0 is the inter-atom distance. (see Fig. 2).

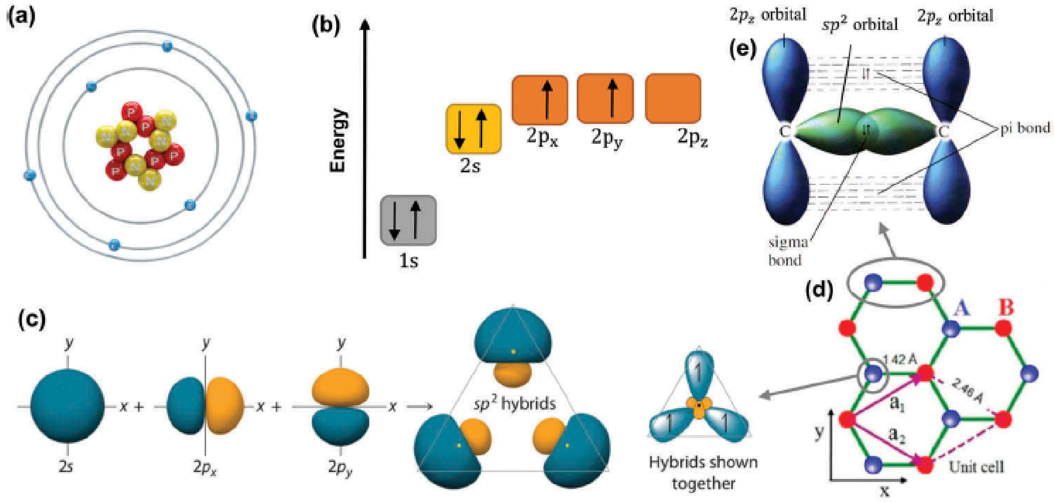
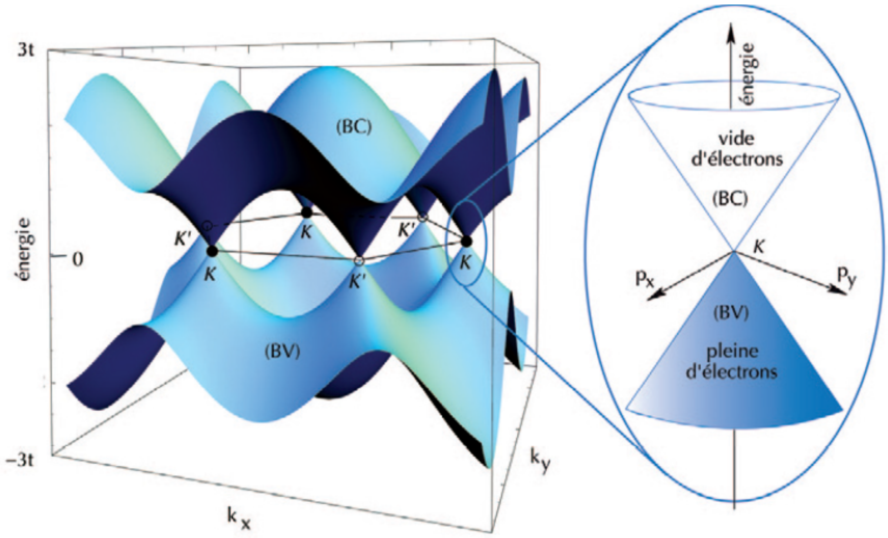


Figure 2 : The basics of graphene structure. Image taken from [128].

2.4 Band structure of graphene

Graphene is a model of a gapless semiconductor in general band theory. In graphene, the valence band is completely filled, the conduction band is empty, and there is no band gap in between. The Fermi energy of this material corresponds to the energy at conical points. On the other hand, Sn (grey tin) and HgTe are three-dimensional crystals known as gapless semiconductors.

The unusual, chiral nature of the electron states, as well as the high degree of electron-hole symmetry, are what distinguish graphene from other materials. Rather than the gapless state itself [140].



Fugure 3 : Fermi energy at the Dirac point. Image taken from [141].

2.5 Synthesis of graphene

Novoselov and Geim received the Nobel Prize in Physics six years later "for pioneering experiments regarding the two-dimensional material graphene." During this time, several techniques for producing monolayer graphene have been created. Depending on the physical or chemical procedure used to create the sheet of graphene, among these techniques we have:

2.5.1 Mechanical exfoliation

Everything starts with graphite. As in the pencils, if we zoom in, we would see a bunch of graphene layers stuck together.

We just need to exfoliate them in each layer. The atoms are very tightly bonded, but the layers are only weakly bonded among them, so we can just exfoliate them which a normal scotch tape. We can just tear some of these layers apart. Using this method, the scientists managed to get the first two-dimensional material ever, but the samples appear randomly distributed, with uneven shapes and sizes. This is not an industrial way to make graphene [142].

2.5.2 The chemical method

The Chemical Vapor Deposition (CVD) method is a process used to fabricate graphene material. It begins with a copper wafer that is then submerged in methane ($CH4$) and heated to extremely high temperatures.

When methane molecules ($CH4$) hit copper, the carbon atom gets trapped while the other hydrogen atoms continue to move around. This effectively creates a single layer of carbon atoms that can be several centimeters long, but the carbon is too bonded to the copper, making it difficult to do anything with it.

The copper wafer is deposited on acid, which dissolves the copper but has no effect on the graphene, to transport the graphene around. First, the graphene is coated with an organic polymer ($PMMA$) that serves as a protective layer.

The graphene ($+PMMA$) is then removed from the water using another wafer, typically composed of (Si/SiO_2) to remove any remaining acid residues.

Acetone or other solvents are used to remove the ($PMMA$) when the graphene is ready to be employed in a device or experiment. and in doing so, we can create a two-dimensional material [143].

2.6 The relation between the physics of graphene and relativistic quantum mechanics

The main characteristic of graphene that has attracted more attention of researchers is its electrons known as graphene quasiparticles, which behave as massless relativistic fermions and are described by the $(2+1)$-dimensional Dirac equation with a fermi velocity of $v_F = c/300$, where c is the speed of light [1, 2, 4].

In graphene, the dispersion energy of electrons and holes is linear, similar to that of photons, $E_{\pm} = \pm\hbar v_F\,|\mathbf{k}|$. Its bipartite crystal structure and the particular, regular arrangement of atoms are the root causes of all these properties. It is the best material for investigating such single electron physics. As a result, quantum field theory approaches are extremely useful in graphene physics. Additionally, it makes it possible to investigate the effects of strong fields on quantum electrodynamics [12, 13].

The study of the interactions between charge carriers in graphene and similar systems is interesting because of their application to superconductivity. Also, it is very useful in energy storage systems, electronics, chemical sensors, optoelectronics, and nanocomposites.

To move from Dirac physics to graphene physics, we make the following replacements [18]:

$$m \longrightarrow 0, \tag{2.2}$$

$$c \longrightarrow \upsilon_f, \tag{2.3}$$

$$E_\pm \longrightarrow \pm\hbar\upsilon_F\,|\mathbf{k}|\,, \hbar = 1 \tag{2.4}$$

2.6.1 Dirac-graphene equation

The dynamics of graphene quasiparticles in an external field is decribed by the Dirac equation for massless femions.

Which has the form of two equations: the first is for electron wave function, and the second is for hole wave function.

$$i\frac{\partial\psi}{\partial t} = \upsilon_F\sigma\left(k - eA\left(x,y,t\right)\right)\psi, \tag{2.5}$$

where ψ is the two-component wave function $\psi = \binom{\psi_A}{\psi_B}$

2.6.2 Dirac-graphene Hamiltonian

To describe electron and hole states in graphene material, one needs to define the effective Hamiltonian (Dirac-graphene Hamiltonian) around the Dirac points K and K' which is analogous to the Dirac Hamiltonian for massless fermions [140].

In the free case, the Dirac-graphene Hamiltonian near the points K and K' is defined by

$$H_K = \upsilon_F \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix}, \tag{2.6}$$

and

$$H_{K'} = \upsilon_F \begin{pmatrix} 0 & p_x + ip_y \\ p_x - ip_y & 0 \end{pmatrix}. \tag{2.7}$$

The Dirac Hamiltonian of graphene quasiparticles in interaction with an external field $A_\mu\left(x\right) \equiv A_\mu\left(x,y,t\right)$ around one of the special points K and K' is given as

$$H_{K,K'} = \upsilon_F \begin{pmatrix} 0 & (p_x - eA_x(x)) \mp i(p_y - eA_y(x)) \\ (p_x - eA_x(x)) \pm i(p_y - eA_y(x)) & 0 \end{pmatrix}. \quad (2.8)$$

For multilayer graphene, the Hamiltonian is defined by

$$H_{K,K'} = \upsilon_F \begin{pmatrix} 0 & ((p_x - eA_x(x)) \mp i(p_y - eA_y(x)))^J \\ ((p_x - eA_x(x)) \pm i(p_y - eA_y(x)))^J & 0 \end{pmatrix}, \quad (2.9)$$

where J is the chirality index or the number of layers.

2.6.3 Slowed light in graphene

The researchers have used the light to observe the nature of quantum electronic material. In graphene, light was captured and slowed to the speed of the material's electrons.

The research involved confining the plasmons vertically down to $5nm$ using heterostructures made of high-quality graphene, hexagonal Boron Nitride (h-BN), and adjacent metals [144], due to the capacity to modify its plasmon phase velocity to low values, close to its Fermi velocity of $\upsilon_F \approx c/300$, where c is the speed of light in vacuum, as a result, slow down the propagation velocity.

2.6.4 Quantum imaging of graphene's current flow

The movement of electron currents in devices composed of ultra-thin materials has attracted the attention of researchers. The first image presenting the motion of electrons in graphene was finally captured by a research team from the University of Melbourne. When scientists shine a green laser light on diamonds and then watch and analyze the intensity of the red light arising from the magnetic field created by the electric current, they are able to image the flow of electric currents in graphene. The team led by Hollenberg used a special quantum probe based on an atomic-sized "color center" found only in diamonds to image the flow of electric currents in graphene [145].

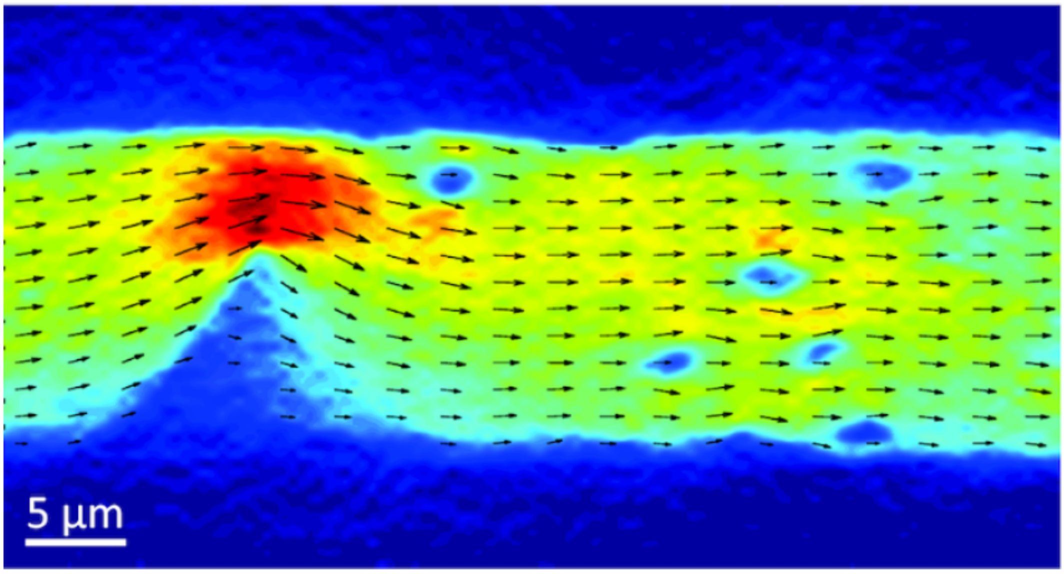


Figure 4 : An image of the current flow in graphene was obtained using a diamond quantum. Image taken from [145]

Chapter 3

Volkov-type solution of the Dirac-graphene equation in the presence of two orthogonal plane waves

3.1 Introduction

The exact solution of the Klein Gordon and Dirac equations in the presence of an electromagnetic plane wave field is very important in relativistic quantum mechanics due to it's widespread use in laser beam applications.

The Volkov solution, which describes how Klein Gordon and Dirac particles behave in an external electromagnetic plane wave field, was treated for the first time by Volkov [146] in 1935.

In the presence of two electromagnetic plane waves, Volkov's solution for an electron was reviewed in Refs [147, 148]. These publications actually provided the exact solutions of the Dirac equation for two orthogonal electromagnetic plane wave fields. This makes it possible to calculate the modified Compton formula for the scattering of two photons onto an electron with accuracy.

In the same context, Volkov's solution of Dirac-graphene equation that describes graphene quasiparticles in the presence of an electromagnetic plane wave was derived in Refs. [149].

The principal motivation of the present chapter is to give an exact analytical solution of the massless Dirac equation for graphene quasiparticles in the presence of two plane wave fields by using the Volkov ansatz. The identification of the properties of the interaction between relativistic particles and laser light is the most interesting aspect of this solution, wich has also been applied to the treatment of nonlinear Compton scattering.

3.2 Solution of the Dirac-graphene equation without a field

The Dirac-graphene equation without a field is defined as

$$i\frac{\partial\psi}{\partial t} = \upsilon_F\left(\sigma.p\right)\psi, \tag{3.1}$$

where $\upsilon_F \sim 10^6\ m/s$ is the velocity of quasiparticles in graphene named the Fermi velocity [10, 11, 5],

and ψ is the two-component of the wave function $\psi = \binom{\psi_A}{\psi_B}$ that corresponds to electron and hole states.

Setting the natural units $c = \hbar = 1$ then the momentum $p \equiv (p_\tau, p_x, p_y)$ is the wave number $k \equiv (k_\tau, k_x, k_y)$ and $\hat{k}_\mu = -i\partial_\mu$ and the Minkowski tensor has signature $g_{\mu\nu} = diag(-1, +1, +1)$; $\mu, \nu = 0, 1, 2$.

The last equation has the following simple solution [149]

$$\psi_k = u_k e^{i(\mathbf{kq}-E\tau)}, u_k = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi_k} \\ \pm e^{i\phi_k}\end{pmatrix}, \tag{3.2}$$

where $\tau = \upsilon_F t$, $\mathbf{q} = (x, y)$and $\mathbf{k} = (k_x, k_y)$. Whereas $\phi_k = \arctan(\frac{k_y}{k_x})$ is the polar angle and the energies (electrons-holes) are given as

$$E_\pm = \pm k_\tau = \pm\hbar\upsilon_F\left|\mathbf{k}\right|. \tag{3.3}$$

3.3 Volkov's solution for the Dirac-graphene equation in the presence of a single plane wave field

In this section, we study the interaction between graphene electrons and a single electromagnetic plane wave field. The dynamics of graphene quasiparticles in an external plane wave field is described by the Dirac equation for massless fermions, which is given by [151, 152]

$$i\frac{\partial\psi}{\partial t} = \upsilon_F\sigma.\left(k - eA\left(x, y, t\right)\right)\psi, \tag{3.4}$$

where k is the momentum of a quasiparticle in the natural units defined as

$$k = -i\left(e_x \partial_x + e_y \partial_y\right), \ \sigma = e_x \sigma_x + e_y \sigma_y, \ \tau = \upsilon_F t, \tag{3.5}$$

with (e_x, e_y) are vectors direction of plane (x, y). Whereas the four-potential A^μ is a function of the variable ξ, it is given as

$$A^\mu = A^\mu(\xi)\,; \qquad \xi = nx. \tag{3.6}$$

We consider that the field of a plane electromagnetic wave is linearly polarized along the graphene surface chosen as

$$A^\mu(x, y, t) = A(\xi) = \left(0, \vec{A}(\xi)\right), \tag{3.7}$$

$$\vec{A}(\xi) = \vec{e}_x A(\xi)\,. \tag{3.8}$$

Moreover, we have the following properties

$$n^\mu = (1, \vec{n}) = (1, 0, 1) \ \text{ with } n^2 = 0. \tag{3.9}$$

These allow us to write the variable ξ as $\xi = \alpha y - \tau$, which will play an important role in the next calculation.

We put the parameter value $\alpha = 1$ which means that velocities of electromagnetic wave and quasiparticles in graphene coincide [149].

To solve this problem, let us introduce the following notation

$$L^+ = \hat{h} + i\partial_\tau, \quad L^- = \hat{h} - i\partial_\tau, \tag{3.10}$$

where

$$\hat{h} = \sigma_x.\left(k_x - eA(\xi)\right) + \sigma_y.k_y, \tag{3.11}$$

by taking the condition $\left(\vec{A}(\xi)\ \text{ is parallel to the vector } \vec{e}_x\right)$ since ξ dependent on (y, τ) [149].Then the Hamiltonian expression is defined as

$$\begin{aligned} H &= L^+ L^- = \partial_\tau^2 - \partial_x^2 - \partial_y^2 + 2ieA(\xi)\,\partial_x \\ &\quad + e^2 A^2(\xi) + ie\partial_y A(\xi)\,\sigma_z - ie\partial_\tau A(\xi)\,\sigma_x. \end{aligned} \tag{3.12}$$

We impose that the four-potential $A^{\mu}(\xi)$ satisfies the Lorentz gauge condition

$$\begin{aligned} \partial_{\mu}A^{\mu}(\xi) &= n^{\mu}.\left(A_{\mu}(\xi)\right)' = 0, & (3.13) \\ &\Rightarrow e^{\mu}n_{\mu} = 0 \Rightarrow n^{\mu}A_{\mu} = 0, & (3.14) \end{aligned}$$

we define then the equation $H\psi = 0$ as follows

$$\left(\partial_{\tau}^{2} - \partial_{x}^{2} - \partial_{y}^{2} + 2ieA(\xi)\partial_{x} + e^{2}A^{2}(\xi) + e\partial_{y}A(\xi)\sigma_{z} - ie\partial_{\tau}A(\xi)\sigma_{x}\right) = 0. \quad (3.15)$$

According to the Volkov ansatz, the solution is of the form

$$_{k} = e^{i\chi}F(\xi)u_{k}, \quad (3.16)$$

with $\chi = \mathbf{kq} - E\tau$, $\xi = y - \tau$.

Incorporating in (3.15) with (3.16), we check for $F(\xi)$ the following equation

$$2i(k_{\tau} - k_{y})\dot{F} + \left(-2eA(\xi)k_{x} + e^{2}A^{2}(\xi) + e\left(\frac{\partial\xi}{\partial y}A'(\xi)\sigma_{z} - i\frac{\partial\xi}{\partial\tau}A'(\xi)\sigma_{x}\right)\right)F = 0. \quad (3.17)$$

Its solution will be given as

$$F = \exp\left(\frac{i}{2}\int_{0}^{y-\tau} d\xi \frac{-2eA(\xi)k_{x} + e^{2}A^{2}(\xi) + e\left(A'(\xi)\sigma_{z} + iA'(\xi)\sigma_{x}\right)}{(k_{\tau} - k_{y})}\right). \quad (3.18)$$

Consequently, the solution of the Dirac-Graphene equation in the presence of an electromagnetic plane wave is given as

$$\begin{aligned} \psi_{k} &= \exp\left(i(\mathbf{kq} - E\tau) + \frac{i}{2}\int_{0}^{y-\tau} d\xi \frac{-2eAk_{x} + e^{2}A^{2} + e\left(A'(\xi)\sigma_{z} + iA'(\xi)\sigma_{x}\right)}{(k_{\tau} - k_{y})}\right)u_{k}, \\ &= \left[1 + e\frac{A'(\xi)\sigma_{z} + iA'(\xi)\sigma_{x}}{2(k_{\tau} - k_{y})}\right]u_{k}e^{i\chi}e^{iS}, & (3.19) \end{aligned}$$

where

$$S = -\int \frac{e^{2}A^{2}(\xi) - 2eA(\xi)k_{x}}{2(k_{\tau} - k_{y})}d\xi. \quad (3.20)$$

3.4 Volkov's solution for the Dirac-Graphene equation in the presence of two plane wave fields

In this section, we study the interaction between the graphene electrons and two orthogonal electromagnetic plane wave fields linearly polarized along the graphene surface that is chosen as

$$A^{\mu}(x,y,t) = A_1^{\mu}(\xi_1) + A_2^{\mu}(\xi_2), \tag{3.21}$$

where

$$A_1^{\mu}(\xi_1) = \left(0, \vec{A}_1(\xi_1)\right), \quad A_2^{\mu}(\xi_2) = \left(0, \vec{A}_2(\xi_2)\right), \tag{3.22}$$

and

$$\Rightarrow A^{\mu}(x,y,t) = \vec{e}_1 A_1(\xi_1) + \vec{e}_2 A_2(\xi_2). \tag{3.23}$$

Whereas

$$\xi_1 = n_1^{\mu} x_{\mu}, \qquad \xi_2 = n_2^{\mu} x_{\mu}. \tag{3.24}$$

Also, we have the following properties

$$n_1^{\mu} = (1, \vec{n}_1), n_2^{\mu} = (1, \vec{n}_2) \Rightarrow n_1^{\mu} n_{1\mu} = n_2^{\mu} n_{2\mu} = 0, \tag{3.25}$$

$$n_1^{\mu} n_{2\mu} = 1 - \vec{n}_1.\vec{n}_2 = 0 \Leftrightarrow \vec{n}_1 // \vec{n}_2. \tag{3.26}$$

By using the same procedure as in the previous section and by taking into account the condition $\left(\vec{A}_1(\xi_1) + \vec{A}_2(\xi_2)\right.$ is parallel to the vector $\left.\vec{e}_x\right)$ since ξ_1 and ξ_2 dependent on (y, τ), the Hamiltonian expression is defined as

$$\begin{aligned} H &= L^{+}L^{-} = \partial_{\tau}^2 - \partial_x^2 - \partial_y^2 + 2ie\left(A_1(\xi_1) + A_2(\xi_2)\right)\partial_x \\ &+ e^2\left((A_1(\xi_1))^2 + (A_2(\xi_2))^2 + 2A_1(\xi_1)A_2(\xi_2)\vec{e}_1.\vec{e}_2\right) \\ &+ e\partial_y\left(A_1(\xi_1) + A_2(\xi_2)\right)\sigma_z - ie\partial_{\tau}(A_1(\xi_1) + A_2(\xi_2))\sigma_x. \end{aligned} \tag{3.27}$$

We also consider that the four potentials $A_1^{\mu}(\xi_1)$ and $A_2^{\mu}(\xi_2)$ satisfy the Lorentz gauge conditions

$$\begin{cases} \partial_{\mu} A_1^{\mu}(\xi_1) = n_1^{\mu}.\left(A_{1\mu}(\xi_1)\right)' = 0, \\ \partial_{\mu} A_2^{\mu}(\xi_2) = n_2^{\mu}.\left(A_{2\mu}(\xi_2)\right)' = 0, \end{cases} \tag{3.28}$$

and we consider that the two waves are orthogonal, then we can write

$$e_1^{\mu}n_{1\mu}=0,\qquad e_2^{\mu}n_{2\mu}=0\Rightarrow n_1^{\mu}A_{1\mu}=0,n_2^{\mu}A_{2\mu}=0,$$
$$\text{and}\qquad A_{1\mu}\left(\xi_1\right)A_2^{\mu}\left(\xi_2\right)=0. \tag{3.29}$$

Now, towards the solution of the equation $H\psi=0$ and by using the Volkov ansatz, the solution is of the form [147]

$$\psi_k=e^{-ikq}F\left(\xi_1,\xi_2\right)u_k. \tag{3.30}$$

Then, we write the equation for $F\left(\xi_1,\xi_2\right)$ as follows

$$\begin{aligned}&2ik_{\mu}n_1^{\mu}\frac{\partial\xi_1}{\partial y}F_{\xi_1}+2ik_{\mu}n_2^{\mu}\frac{\partial\xi_2}{\partial y}F_{\xi_2}+e^2\left(\left(A_1\left(\xi_1\right)\right)^2+\left(A_2\left(\xi_2\right)\right)^2\right)F\left(\xi_1,\xi_2\right)\\&-2e\left(A_1\left(\xi_1\right)+A_2\left(\xi_2\right)\right)k_xF\left(\xi_1,\xi_2\right)+ie\partial_y\left(A_1\left(\xi_1\right)+A_2\left(\xi_2\right)\right)\sigma_zF\left(\xi_1,\xi_2\right)\\&-ie\partial_{\tau}(A_1\left(\xi_1\right)+A_2\left(\xi_2\right))\sigma_xF\left(\xi_1,\xi_2\right)=0.\end{aligned} \tag{3.31}$$

At this stage, we are looking for the solution in the most simple form

$$F\left(\xi_1,\xi_2\right)=X\left(\xi_1\right)Y\left(\xi_2\right). \tag{3.32}$$

After insertion of Eq. (3.32) into Eq. (3.31) and division of the new equation by XY, we get the terms depending only on ξ_1, and on ξ_2. Then we get

$$\begin{aligned}&\left(2ik_{\mu}n_1^{\mu}\frac{\partial\xi_1}{\partial x^{\mu}}\frac{X'}{X}+e^2A_1^2\left(\xi_1\right)-2eA_1\left(\xi_1\right)k_x+e\partial_yA_1\left(\xi_1\right)\sigma_z-ie\partial_{\tau}(A_1\left(\xi_1\right)\sigma_x\right)\\&+\left(2ik_{\mu}n_2^{\mu}\frac{\partial\xi_2}{\partial x^{\mu}}\frac{Y'}{Y}+e^2\left(A_2^2\left(\xi_2\right)\right)^2-2eA_2\left(\xi_2\right)k_x+e\partial_yA_2\left(\xi_2\right)\sigma_z-ie\partial_{\tau}A_2\left(\xi_2\right))\sigma_x\right)=0.\end{aligned} \tag{3.33}$$

Next, the solution of Eq. (3.33) is reduced to the solution of two equations only; after the simplification, we obtain

$$X\left(\xi_1\right)=\left[1+e\frac{\frac{\partial\xi_1}{\partial y}A_1'(\xi_1)\sigma_z-i\frac{\partial\xi_1}{\partial\tau}A_1'(\xi_1)\sigma_x}{2(k_{\tau}-k_y)}\right]\exp\left[-i\int\frac{e^2(A_1(\xi_1))^2-2eA_1(\xi_1)k_x}{2(k_{\tau}-k_y)}d\xi_1\right], \tag{3.34}$$

and

$$Y\left(\xi_2\right)=\left[1+e\frac{\frac{\partial\xi_2}{\partial y}A_2'(\xi_2)\sigma_z-i\frac{\partial\xi_2}{\partial\tau}A_2'(\xi_2)\sigma_x}{2(k_{\tau}-k_y)}\right]\exp\left[-i\int\frac{e^2(A_2(\xi_2))^2-2eA_2(\xi_2)k_x}{2(k_{\tau}-k_y)}d\xi_2\right]. \tag{3.35}$$

With $\xi_1 = y_1 - \tau_1, \xi_2 = y_2 - \tau_2$. The results can be written as

$$X(\xi_1) = \left[1 + e\frac{A_1'(\xi_1)\sigma_z + iA_1'(\xi_1)\sigma_x}{2(k_\tau - k_y)}\right] \exp\left[-i\int \frac{e^2(A_1(\xi_1))^2 - 2eA_1(\xi_1)k_x}{2(k_\tau - k_y)} d\xi_1\right], \tag{3.36}$$

and

$$Y(\xi_2) = \left[1 + e\frac{A_2'(\xi_2)\sigma_z + iA_2'(\xi_2)\sigma_x}{2(k_\tau - k_y)}\right] \exp\left[-i\int \frac{e^2\left(A_2'(\xi_2)\right)^2 - 2eA_2'(\xi_2)k_x}{2(k_\tau - k_y)} d\xi_2\right]. \tag{3.37}$$

The total solution then takes the following form

$$\psi_k = \left[1 + e\frac{A_1'(\xi_1)\sigma_z + iA_1'(\xi_1)\sigma_x}{2(k_\tau - k_y)}\right]\left[1 + e\frac{A_2'(\xi_2)\sigma_z + iA_2'(\xi_2)\sigma_x}{2(k_\tau - k_y)}\right] u_k e^{i\chi} e^{iS_1(A_1) + iS_2(A_2)}, \tag{3.38}$$

where

$$S_1 = -\int \frac{e^2(A_1(\xi_1))^2 + 2eA_1(\xi_1)k_x}{2(k_\tau - k_y)} d\xi_1, \tag{3.39}$$

$$S_2 = -\int \frac{e^2(A_2(\xi_2))^2 + 2eA_2(\xi_2)k_x}{2(k_\tau - k_y)} d\xi_2. \tag{3.40}$$

3.5 Conclusion

In this chapter, we have presented an exact analytical Volkov solution of the massless Dirac equation for graphene in the presence of a single and two plane wave fields using the Volkov ansatz. We have derived the partial differential equation (Volkov equation) and deduced the wave function for graphene quasiparticles.

Chapter 4

Path integral formulation for graphene quasiparticles in interaction with two plane wave fields

4.1 Introduction

The path integral formalism was able to successfully resolve several issues in NRQM because of its compatibility with calculation techniques and the absence of operators. Although, the success of this formulation did not make it the perfect one due to the problem that occurred in the spin because of its discrete nature, it has not been easy to introduce it as a suggestion for a continuous path.

Many various attempts to integrate spin into a path integral formulation can be classified into two categories. The first one was suggested by Feynmann, in which he described the spin by using the bosonic variables. The second one was suggested by Fradkin in 1965 [100, 101] in which he described the spin by using the fermionic variables (Grassmann variables). Then by Berezin and Marinov [102, 103]. In the last decade, exactly in 1991, Fradkin and Gitmann [153] have returned to this model and succeeded in establishing a rigorous formulation of path integral representation with effective classical actions [105] following the standard Feynman form $\sum_{paths} exp[iS(path)]$, where $S(path)$ is a supersymmetric action describing both the external motions of the particle by bosonic-type variables and the internal dynamics relating to the spin of the particle by Grassmann variables.

The fundamental idea of this formalism is to write the causal Green's function as the inverse of an operator. Then multiplying this inverse by an adequate conjugate operator in order to reduce the problem to a quadratic operator of Klein-Gordon type plus a spin-orbit type coupling term. After that, we write it in integral representation by using a generalized proper time that has two parts, one bosonic and the other fermionic. Then the Dirac matrices are replaced by Grassmann variables. After doing all of this, we apply the functional integration method to give the explicit result of Green's function.

The aim of this chapter is to adapt the formalism of the path integral for graphene quasiparticles. We construct the causal Green's function via the supersymmetric formalism proposed by Fradkin and Gitman [105] for graphene quasiparticles in the free case, in the presence of a single plane wave field and two orthogonal plane wave fields. Before that, we determine the Polyakov spin factor by using the Grassmann functional integration technique. Finally, we deduce the corresponding wave functions for each case.

4.2 The causal Green function for graphene quasiparticles in the presence of an external electromagnetic field

We consider a graphene's quasiparticle in the presence of an external electromagnetic field described by a vector potential $A_\mu(x) \equiv A^\mu(x, y, t)$. For a $(2+1)-$dimensional space-time, the corresponding causal Green's function $S^c(x_b^\mu, x_a^\mu)$ satisfies the $2D$ Dirac-graphene equation

$$\mathcal{O}^{gr} S^c(x_b^\mu, x_a^\mu) = \delta^3(x_b - x_a), \tag{4.1}$$

where the Dirac-graphene operator $\mathcal{O}^{gr}$ is defined as

$$\mathcal{O}^{gr} = \hat{h} - i\partial_\tau, \tag{4.2}$$

and $\hat{h}$ represents the Hamiltonian of Dirac massless particles, defined as

$$\hat{h} = \left(\hat{p}_x - e\hat{A}(x, y, \tau)\right)\sigma_x + \hat{p}_y\sigma_y. \tag{4.3}$$

Here $x = x^\mu, \mu = 0, 1, 2$ and $\tau = \upsilon_F t$ where the characteristic $\upsilon_F = (1.12 \pm 0.02) \times 10^6 m/s$ is the Fermi velocity in graphene, which replaces the speed of light in the Dirac theory.

$(\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices; they are given as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.4}$$

Setting the natural units $c = \hbar = 1$, then the momentum $p \equiv (p_\tau, p_x, p_y)$ is the wave number $k \equiv (k_\tau, k_x, k_y)$ and $\hat{k}_\mu = -i\partial_\mu$ and the Minkowski tensor has signature $g_{\mu\nu} = diag(-1, +1, +1); \mu, \nu = 0, 1, 2$.

In this work, we perform our study in (2 + 1)-dimensional, which allows us to express the Dirac gamma matrices γ^μ as a function of the Pauli matrices in the following way

$$\gamma^0 = \sigma_3,\ \gamma^1 = i\sigma_2,\ \gamma^2 = -i\sigma_1. \tag{4.5}$$

Eq. (4.2) can be written as

$$\mathcal{O}^{gr} = \gamma^0 i\partial_\tau - \gamma^1 \left(i\partial_x + eA\left(x, y, \tau\right)\right) - \gamma^2 i\partial_y. \tag{4.6}$$

According to the Schwinger proper time method [37], the propagator $S^c\left(x_b, x_a\right)$ can be written as a matrix element of the operator $\hat{S}^c$ in coordinate space

$$S^c\left(x_b^\mu, x_a^\mu\right) = \left\langle x_b\right| \hat{S}^c \left|x_a\right\rangle, \tag{4.7}$$

where $\langle x_b|$, $|x_a\rangle$ are eigenvectors of the self-adjoint operator $\hat{x}^\mu$ and form a complete orthonormal system with

$$\hat{x}^\mu \left|x_a\right\rangle = x^\mu \left|x_a\right\rangle, \left\langle x_b \right| x_a\rangle = \delta^3\left(x_b - x_a\right), \tag{4.8}$$

by using Eqs. (4.1) and (4.7), we get

$$\left\langle x_b\right| \mathcal{O}^{gr} \hat{S}^c \left|x_a\right\rangle = \delta^3\left(x_b - x_a\right), \tag{4.9}$$

or

$$\mathcal{O}^{gr} \hat{S}^c = I. \tag{4.10}$$

So, the operator $\hat{S}^c$ is the inverse of the graphene operator $\mathcal{O}^{gr}$, defined as

$$\hat{S}^c = \left[\mathcal{O}^{gr}\right]^{-1} = \left[\mathcal{O}^{gr}\right]\left[\mathcal{O}^{gr}\right]^{-2}. \tag{4.11}$$

According to the habitual construction procedure of the path integral (the Schwinger trick), we have

$$\begin{aligned} -i \int_0^\infty d\lambda \exp\left[i\lambda\left[\left[\mathcal{O}^{gr}\right]^2 + i\varepsilon\right]\right] &= -\left[\mathcal{O}^{gr}\right]^{-2}\left[\exp\left(i\infty\right) - \exp\left(0\right)\right] \\ &= \left[\mathcal{O}^{gr}\right]^{-2} + i\epsilon, \end{aligned} \tag{4.12}$$

and

$$i\int d\chi \exp\left[i\chi\mathcal{O}^{gr}\right] = -\left[\mathcal{O}^{gr}\right], \tag{4.13}$$

the operator $\hat{S}^c$ takes then the following form

$$\hat{S}^c = \int_0^\infty d\lambda \int d\chi \exp\left[i\left[\lambda\left[\mathcal{O}^{gr}\right]^2 + i\epsilon\right] + i\chi\left[\mathcal{O}^{gr}\right]\right], \tag{4.14}$$

where λ is an even variable and χ is an odd (Grassmann) variable [154]. Notice that χ anticommuting with the $\mathcal{O}^{gr}$-operator and verifying the following properties

$$\chi^2 = 1; \quad \int d\chi = 0; \quad \int \chi d\chi = 1. \tag{4.15}$$

We omit the infinitesimal quantity ϵ. We express the operator $\hat{S}^c$ as follows

$$\hat{S}^c = \int_0^\infty d\lambda \int \exp\left(-i\hat{H}\right) d\chi. \tag{4.16}$$

Here $\hat{H}$ is the Hamiltonian that governs the movement of the graphene quasiparticle expressed by the following equation

$$H\left(\lambda, \chi\right) = -\lambda\left[\mathcal{O}^{gr}\right]^2 + \left[\mathcal{O}^{gr}\right]\chi, \tag{4.17}$$

with

$$\mathcal{O}^{gr} = \gamma^0 i\partial_\tau - \gamma^1\left(i\partial_x + eA\left(x, y, \tau\right)\right) - \gamma^2 i\partial_y, \tag{4.18}$$

otherwise

$$\begin{aligned}\hat{H} &= \lambda\left[\partial_\tau^2 - \partial_x^2 - \partial_y^2 + 2ieA\left(x,y,\tau\right)\partial_x + e^2A^2\left(x,y,\tau\right)\right.\\ &\left.-ie\left(\frac{A\left(x,y,\tau\right)}{\partial\tau}\gamma^0\gamma^1 - \frac{A\left(x,y,\tau\right)}{\partial y}\gamma^1\gamma^2\right)\right] + \left[\mathcal{O}^{gr}\right]\chi.\\ &= \lambda\left[\partial_\tau^2 - \partial_x^2 - \partial_y^2 + 2ieA\left(x,y,\tau\right)\partial_x + e^2A^2\left(x,y,\tau\right)\right.\\ &\left.-\frac{ie}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu\right] + \left[\mathcal{O}^{gr}\right]\chi.\end{aligned} \tag{4.19}$$

Where $F_{\mu\nu}$ is the electromagnetic field tensor antisymmetric defined as a derivable of a potential

$$\begin{aligned}F_{\mu\nu}\gamma^\mu\gamma^\nu &= \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\gamma^\mu\gamma^\nu,\\ &= 2\left(\frac{A\left(x,y,\tau\right)}{\partial\tau}\gamma^0\gamma^1 - \frac{A\left(x,y,\tau\right)}{\partial y}\gamma^1\gamma^2\right).\end{aligned} \tag{4.20}$$

Now, the Green function is written in coordinate representation as

$$S^c\left(x_b^\mu, x_a^\mu\right) = \int_0^\infty d\lambda \int \langle x_b| \exp\left(-i\hat{H}\right)|x_a\rangle\, d\chi. \tag{4.21}$$

To pass to the path integral representation for the Green function $S^c\left(x_b^\mu, x_a^\mu\right)$, we follow the standard discretization method given in [153]. Subdividing the interval $[x_a; x_b]$ into N parts that we will assume equal for simplicity, we write $\exp\left(-i\hat{H}\right)$ by $\left[\exp\left[-i\hat{H}/\left(N+1\right)\right]\right]^{(N+1)}$.Then introducing between them N completeness relation of the space-time eigen-states $1 = \int d^3x\, |x\rangle\langle x|$ each pair of infinitesimal operators $\exp(-i\varepsilon\hat{H})$, where $\varepsilon = 1/(N+1)$.

We follow the standard discretization method for the kernel of Eq. (4.21). As it is known, usually we write

$$\exp\left(-i\lambda\hat{H}\right) = \left[\exp\left(-i\lambda\hat{H}/\left(N+1\right)\right)\right]^{N+1}. \tag{4.22}$$

After all of this, the Green function then takes the following discrete form

$$\begin{aligned} S^c\left(x_b^\mu, x_a^\mu\right) &= \int_0^\infty d\lambda \int d\chi \int_{-\infty}^{+\infty} dx_1 ... dx_N \langle x_{N+1}| \exp\left(-i\frac{\hat{H}\left(\hat{x},\hat{k}\right)}{N+1}\lambda\right)|x_N\rangle \\ &\quad \times ... \langle x_j| \exp\left(-i\frac{\hat{H}\left(\hat{x},\hat{k}\right)}{N+1}\lambda\right)|x_{j-1}\rangle ... \langle x_1| \exp\left(-i\frac{\hat{H}\left(\hat{x},\hat{k}\right)}{N+1}\lambda\right)|x_0\rangle \\ &= \int_0^\infty d\lambda \int d\chi \lim_{\varepsilon\to 0} \Pi_{j=1}^N \int_0^\infty d^3x_j \Pi_{j=1}^N \langle x_j| \exp\left[i\varepsilon\lambda\hat{H}\right]|x_{j-1}\rangle, \varepsilon = \frac{1}{N+1}. \end{aligned}$$

Then we insert $(N+1)$ times the identities of this formula

$$1 = \int d^3k\, |k\rangle\langle k|, \tag{4.23}$$

where the momentum k verifies the following relations

$$\hat{k}^\mu |k\rangle = k^\mu |k\rangle \qquad ; \langle k\,|k'\rangle = \delta^3\left(k - k'\right) \qquad ; \langle x\,|k\rangle = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{ikx}.$$

At the end, we take the limit $N \to \infty$, which transforms the expression of $S^c\left(x_b^\mu, x_a^\mu\right)$ into the following path integral

$$S^c\left(x_b^\mu, x_a^\mu\right) = \int_0^\infty d\lambda \int d\chi \lim_{\varepsilon\to 0} \Pi_{j=1}^{N-1} \int_0^\infty d^3x_j \Pi_{j=1}^N \int_0^\infty d^3k_j \langle x_j| \exp\left[i\varepsilon\lambda\hat{H}\right]|k_j\rangle\langle k_j \mid x_{j-1}\rangle. \tag{4.24}$$

To eliminate the derivation operators, we inject the following relations

$$\langle x_j \mid x_{j-1}\rangle = \frac{\exp\left(ix_j x_{j-1}\right)}{(2\pi)^3}. \tag{4.25}$$

Then, the Green function (4.24) takes the following form

$$\begin{aligned} S^c\left(x_b^\mu, x_a^\mu\right) &= \int_0^\infty d\lambda \int d\chi \int \mathcal{D}x \int \mathcal{D}k \\ \times \mathbb{T}\exp &\left\{ i\sum_j \left[k_j \Delta x_j + \lambda\varepsilon\left(k_j^2 - 2ieA\left(x_j, y_j, \tau_j\right) k_{x_j} + e^2 A^2\left(x_j, y_j, \tau_j\right)\right)\right.\right. \\ &\left.\left.-\frac{ie}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu\right) + \left(\gamma^0 k_{\tau_j} + \gamma^1\left(k_{x_j} - eA\left(x_j, y_j, \tau_j\right)\right) + \gamma^2 k_{y_j}\right)\chi\right]\right\}, \end{aligned} \tag{4.26}$$

where, $x_j^\mu \equiv \left[x_j^0 = \tau_j = \tau(s_j), x_j^1 = x_j = x(s_j), x_j^2 = y_j = y(s_j)\right]$, $x_b^\mu = x^\mu(s_b)$, $x_a^\mu = x^\mu(s_a)$, and $\varepsilon = s_j - s_{j-1} = 1/\left(N+1\right)$.

Whereas $\mathbb{T}$ is the time ordering operator, also called the chronological product operator of Dyson (Dyson time ordering symbol), it affects only on the phase relative to the coupling term that ordered the x, k and $\gamma-$matrices, which are formally supposed to depend on the time parameter s.

A more explicit expression for $S^c\left(x_b^\mu, x_a^\mu\right)$ is easily obtained. Taking at the end the limit $N \to \infty$ or $\left(\varepsilon = \frac{1}{N+1} \to 0\right)$, the continuous form of the Green function $S^c\left(x_b^\mu, x_a^\mu\right)$ becomes as

$$\begin{aligned} S^c\left(x_b^\mu, x_a^\mu\right) &= \int_0^\infty d\lambda \int d\chi \int \mathcal{D}x \int \mathcal{D}k \\ \times \mathbb{T}\exp &\left\{ i\int_0^1 ds\left[k\dot{x} + \lambda\left(-k_\tau^2 + k_x^2 + k_y^2 - 2eA\left(x, y, \tau\right) k_x + e^2 A^2\left(x, y, \tau\right)\right.\right.\right. \\ &\left.\left.\left.-\frac{ie}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu\right) + \left(\gamma^0 k_\tau + \gamma^1\left(k_x - eA\left(x, y, \tau\right)\right) + \gamma^2 k_y\right)\chi\right]\right\}, \end{aligned} \tag{4.27}$$

where the integration over trajectories $x\left(s\right)$ is parametrized by parameter $s \in [0, 1]$, giving the boundary conditions

$$x\left(0\right) = x_a,\ x\left(1\right) = x_b.$$

We note that $x \equiv (\tau, x, y)$ and $k \equiv (k_\tau, k_x, k_y)$ respectively represent the quadratic vector coordinate momentum.

Let us integrate over the $x-$component. First, we integrate the term $\int_0^1 \dot{x}k_x ds$ by part

$$\int_0^1 \dot{x}k_x ds = k_{x_b}x_b - k_{x_a}x_a - \int_0^1 x\dot{k}_x ds, \tag{4.28}$$

then Green's function $S^c\left(x_b^\mu, x_a^\mu\right)$ rewrites as follows

$$S^c\left(x_b^\mu, x_a^\mu\right) = \int_0^\infty d\lambda \int d\chi \int \mathcal{D}x \int \mathcal{D}k$$
$$\times \mathbb{T}\exp\left\{i\int_0^1 ds\left[k_{x_b}x_b - k_{x_a}x_a - x\dot{k} + k\dot{q} + \lambda\left(-k_\tau^2 + k_x^2 + k_y^2 - 2eA\left(x,y,\tau\right)k_x + e^2A^2\left(x,y,\tau\right)\right.\right.\right.$$
$$\left.\left.\left.-\frac{ie}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu\right) + \left(\gamma^0 k_\tau + \gamma^1\left(k_x - eA\left(x,y,\tau\right)\right) + \gamma^2 k_y\right)\chi\right]\right\}. \tag{4.29}$$

The integrations over the $x-$component give the Dirac functions $\delta\left(\dot{k}\right)$, which are defined as

$$\int \mathcal{D}x \exp\left(-i\int_0^1 x\dot{k}ds\right) = \delta\left(\dot{k}\right), \tag{4.30}$$

which leads to the conservation of k_x-momentum during the motion

$$k_{x_1} = k_{x_2} = ...k_{x_N} = k_x, \tag{4.31}$$

this gives for the Green function $S^c\left(x_b^\mu, x_a^\mu\right)$ the following result

$$S^c\left(x_b^\mu, x_a^\mu\right) = \int_0^\infty d\lambda \int d\chi \int \mathcal{D}x \int \mathcal{D}k$$
$$\times \mathbb{T}\exp\left\{i\int_0^1 ds\left[k_x\left(x_b - x_a\right) + k\dot{q} + \lambda\left(-k_\tau^2 + k_x^2 + k_y^2 - 2eA\left(x,y,\tau\right)k_x + e^2A^2\left(x,y,\tau\right)\right.\right.\right.$$
$$\left.\left.\left.-\frac{ie}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu\right) + \left(\gamma^0 k_\tau + \gamma^1\left(k_x - eA\left(x,y,\tau\right)\right) + \gamma^2 k_y\right)\chi\right]\right\}. \tag{4.32}$$

Otherwise

$$S^c\left(x_b^\mu, x_a^\mu\right) = \int \frac{dk_x}{2\pi} e^{ik_x\left(x_b - x_a\right)} S^c\left(q_b^\mu, q_a^\mu\right), \tag{4.33}$$

where $S^c\left(q_b^\mu, q_a^\mu\right)$ is written as

$$S^c\left(q_b^\mu, q_a^\mu\right) = \int_0^\infty d\lambda \int d\chi \int \mathcal{D}q \int \mathcal{D}\bar{k}$$
$$\times \mathbb{T}\exp\left\{i\int_0^1 ds\left[\bar{k}\dot{q} + \lambda\left(-k_\tau^2 + k_x^2 + k_y^2 - 2eA\left(x,y,\tau\right)k_x + e^2A^2\left(x,y,\tau\right)\right.\right.\right.$$
$$\left.\left.\left.-\frac{ie}{2}F_{\mu\nu}\left(x\right)\gamma^\mu\gamma^\nu\right) + \left(\gamma^0 k_\tau + \gamma^1\left(k_x - eA\left(x,y,\tau\right)\right) + \gamma^2 k_y\right)\chi\right]\right\}. \tag{4.34}$$

We note that $q \equiv (q_1, q_2) = (\tau, y)$ and $\bar{k} \equiv (k_1, k_2) = (k_\tau, k_y)$, and the functional measure symbols $\mathcal{D}q$ and $\mathcal{D}\bar{k}$ are defined as

$$\mathcal{D}q = \prod_{n=1}^{N} d\tau dy, \quad \mathcal{D}\bar{k} = \prod_{n=1}^{N+1} \frac{dk_\tau}{2\pi}\frac{dk_y}{2\pi}. \tag{4.35}$$

Next, one must obligatorily eliminate the ordering operator $\mathbb{T}$ by using the source technique of Fradkin [105], associated to $\gamma^\mu(s)$ the odd sources $\rho^\mu(s)$ anticommuting with $\gamma-$matrices by the definition such that γ dependent on time.

$$\mathbb{T}\exp\left\{F\left(\gamma^\mu(s)\right)\right\} = \exp\left\{F\left(\frac{\delta_l}{\delta\rho_\mu}\right)\right\}\mathbb{T}\exp\left\{\int_0^1 \rho_\mu(s)\gamma^\mu ds\right\}|_{\rho=0}\ , \tag{4.36}$$

where $\frac{\delta_l}{\delta\rho_\mu}$ presents the derivation with respect to the Grassmann source $\rho^\mu(s)$.
Then $S^c(q_b^\mu, q_a^\mu)$ can be transformed as follows

$$\begin{aligned} S^c(q_b^\mu, q_a^\mu) &= \int_0^\infty d\lambda \int d\chi \int \mathcal{D}q \int \mathcal{D}\bar{k} \\ \times \exp\Bigg\{i\int_0^1 ds \Bigg[\bar{k}\dot{q} &+ \lambda\left(k^2 - 2eA(x,y,\tau)k_x + e^2A^2(x,y,\tau) - \frac{ie}{2}F_{\mu\nu}\frac{\delta_l}{\delta\rho_\mu}\frac{\delta_l}{\delta\rho_\nu}\right) \\ + \left(k_\tau\frac{\delta_l}{\delta\rho_0} + (k_x - eA(x,y,\tau))\frac{\delta_l}{\delta\rho_1} + k_y\frac{\delta_l}{\delta\rho_2}\right)\chi\Bigg]\Bigg\} &\mathbb{T}\exp\left\{\int_0^1 \rho_\mu(s)\gamma^\mu ds\right\}\Bigg|_{\rho=0}. \end{aligned} \tag{4.37}$$

In the next step, we write the quantity $\mathbb{T}\exp\left\{\int_0^1 \rho_\mu(s)\gamma^\mu ds\right\}$ via a Grassmannian path integral [105] as follows

$$\begin{aligned} \mathbb{T}\exp\left\{\int_0^1 \rho_\mu(s)\gamma^\mu ds\right\}\Bigg|_{\rho=0} &= \exp\left(i\gamma^\mu\frac{\delta_l}{\delta\theta^\mu}\right)\int_{\psi_\mu(0)+\psi_\mu(1)=\theta_\mu} \exp\left[\int_0^1 \left({}_\mu\dot{\psi}^\mu - 2i\rho_\mu\psi^\mu\right) ds \right. \\ &\left. +\psi_\mu(1)\psi^\mu(0)\right]\mathcal{D}\psi\ |_{\theta=0}^{\rho=0}, \end{aligned} \tag{4.38}$$

where $D\psi$ is defined by the following expression

$$\mathcal{D}\psi = D\psi\left[\int_{\psi_\mu(0)+\psi_\mu(1)=0} D\psi\exp\left(\int_0^1 \psi_\mu\dot{\psi}^\mu ds\right)\right]^{-1}. \tag{4.39}$$

Here q, λ are even variables, χ, θ^μ and $\psi(s)$ are (odd)-Grassmann variables, anticommuting with the $\gamma-$matrices, and satisfy the following boundary conditions

$$x(0) = x_a \quad , x(1) = x_a \quad , \lambda(0) = \lambda_0 \quad , \chi(0) = \chi_0, \tag{4.40}$$

and

$$\psi_{\mu}(0)+\psi_{\mu}(1)=\theta_{\mu}. \tag{4.41}$$

By inserting the identity (4.38) in Eq. (4.37), the Green function will take the following form

$$S^{c}\left(q_{b}^{\mu},q_{a}^{\mu}\right)=\exp\left(i\gamma^{\mu}\frac{\delta_{l}}{\delta\theta^{\mu}}\right)\int_{0}^{\infty}d\lambda\int d\chi\int\mathcal{D}q\int\mathcal{D}\bar{k}\int\mathcal{D}$$
$$\exp\left\{i\int_{0}^{1}ds\left[\bar{k}\dot{q}+\lambda\left(k^{2}-2eA\left(x,y,\tau\right)k_{x}+e^{2}A^{2}\left(x,y,\tau\right)\right)+2ie\lambda F_{\mu\nu}\psi^{\mu}\ \ ^{\nu}\right.\right.$$
$$\left.\left.-2i\left(k_{\tau}\psi^{0}+\left(k_{x}-eA\left(x,y,\tau\right)\right)\psi^{1}+k_{y}\psi^{2}\right)\chi-i\psi_{\mu}\dot{\psi}^{\mu}\right]+\psi_{\mu}\left(1\right)\psi^{\mu}\left(0\right)\right\}_{\theta=0}. \tag{4.42}$$

4.3 The causal Green's function for free graphene's quasiparticles

The causal Green's function for the quasiparticles graphene in interaction with an electromagnetic field $S^{c}\left(x_{b}^{\mu},x_{a}^{\mu}\right)$ is given as

$$S^{c}\left(q_{b}^{\mu},q_{a}^{\mu}\right)=\exp\left(i\gamma^{\mu}\frac{\delta_{l}}{\delta\theta^{\mu}}\right)\int_{0}^{\infty}d\lambda\int d\chi\int\mathcal{D}q\int\mathcal{D}\bar{k}\int\mathcal{D}$$
$$\exp\left\{i\int_{0}^{1}ds\left[\bar{k}\dot{q}+\lambda\left(k^{2}-2eA\left(x,y,\tau\right)k_{x}+e^{2}A^{2}\left(x,y,\tau\right)\right)+2ie\lambda F_{\mu\nu}\psi^{\mu}\ \ ^{\nu}\right.\right.$$
$$\left.\left.-2i\left(k_{\tau}\psi^{0}+\left(k_{x}-eA\left(x,y,\tau\right)\right)\psi^{1}+k_{y}\psi^{2}\right)\chi-i\psi_{\mu}\dot{\psi}^{\mu}\right]+\psi_{\mu}\left(1\right)\psi^{\mu}\left(0\right)\right\}_{\theta=0}. \tag{4.43}$$

For the simplest and fundamental case of a free graphene quasiparticles, we put the electromagnetic field $A\left(x,y,\tau\right)=0$, and the corresponding causal Green's function $S^{c}\left(x_{b}^{\mu},x_{a}^{\mu}\right)$ is given as [155]

$$S^{c}\left(x_{b}^{\mu},x_{a}^{\mu}\right)=\exp\left(i\gamma^{\mu}\frac{\delta_{l}}{\delta\theta^{\mu}}\right)\int_{0}^{\infty}d\lambda\int d\chi\int\mathcal{D}x\int\mathcal{D}k\exp\left\{i\int_{0}^{1}ds\left[k\dot{x}+\lambda k^{2}\right]\right\}$$
$$\int\mathcal{D}\psi\exp\left\{i\int_{0}^{1}ds\left[-2ik_{\mu}\psi^{\mu}\chi-i\psi_{\mu}\dot{\psi}^{\mu}\right]+\psi_{\mu}\left(1\right)\psi^{\mu}\left(0\right)\right\}_{\theta=0}. \tag{4.44}$$

4.3.1 The evaluation of Green's function

In order to evaluate the Green function in the free case, we integrate x first and perform the functional integral over the paths $x\left(s\right)$, which implies that the momentum k is conserved

$$k=cst. \tag{4.45}$$

Therefore, the Green function becomes as

$$S^{c}\left(x_{b}^{\mu},x_{a}^{\mu}\right)=\exp\left(i\gamma^{\mu}\frac{\delta_{l}}{\delta\theta^{\mu}}\right)\int_{0}^{\infty}d\lambda\int d\chi\int\mathcal{D}k\exp\left\{i\int_{0}^{1}ds\left[k\left(x_{b}-x_{a}\right)+\lambda k^{2}\right]\right\}$$
$$\int\mathcal{D}\psi\exp\left\{i\int_{0}^{1}ds\left[-2ik_{\mu}\psi^{\mu}\chi-i\psi_{\mu}\dot{\psi}^{\mu}\right]+\psi_{\mu}\left(1\right)\psi^{\mu}\left(0\right)\right\}_{\theta=0}. \qquad (4.46)$$

Now, we calculate the functional integrations over Grassmaniann variables $\psi\left(s\right)$, and we replace the integration variables ψ by the velocity $\omega(s)$ (there is no restriction on $\omega(s)$) as follows

$$\psi\left(s\right)=\frac{1}{2}\int_{0}^{1}\varepsilon\left(s-s'\right)\omega\left(s'\right)ds'+\frac{\theta}{2}, \qquad (4.47)$$

where $\varepsilon\left(s-s'\right)$ is the sign function, and it is defined as

$$\varepsilon\left(s-s'\right)=sign\left(s-s'\right)=\begin{cases}-1 & \text{for } s'\succ s\\ 0 & \text{for } s'=s\\ 1 & \text{for } s\succ s'\end{cases}, \qquad (4.48)$$

and the velocity (odd Grassmaian variable) $\omega(s)$ by using the relation $\frac{d}{ds}\varepsilon\left(s-s'\right)=2\delta\left(s-s'\right)$, is the derivative of ψ with respect to s as

$$\dot{\psi}\left(s\right)=\omega\left(s\right), \qquad (4.49)$$

which gives

$$\psi_{\mu}\dot{\psi}^{\mu}=-\frac{1}{2}\int ds'\omega\left(s\right)\varepsilon\left(s-s'\right)\omega\left(s'\right)+\frac{\theta}{2}\omega\left(s\right), \qquad (4.50)$$

and

$$\begin{aligned}\psi^{\mu}\left(1\right)\psi_{\mu}\left(0\right) &= \left[\frac{1}{2}\int_{0}^{1}\varepsilon\left(1-s\right)\omega^{\mu}\left(s\right)ds+\frac{\theta^{\mu}}{2}\right]\left[\frac{1}{2}\int_{0}^{1}\varepsilon\left(-s\right)\omega_{\mu}\left(s\right)ds+\frac{\theta_{\mu}}{2}\right]\\ &= -\frac{\theta_{\mu}}{2}\int_{0}^{1}\omega^{\mu}\left(s\right)ds,\end{aligned} \qquad (4.51)$$

on the other hand

$$\omega\left(1\right)+\omega\left(0\right)=\frac{d}{ds}\psi|_{s=1}+\frac{d}{ds}\psi|_{s=0}=\frac{d}{ds}\theta=0, \qquad (4.52)$$

whereas the measure $\mathcal{D}\psi$ has the following definition

$$\mathcal{D}\psi = D\psi\left[\int D\psi \exp\left(\int_0^1 \psi_\mu \dot{\psi}^\mu ds\right)\right]^{-1}. \tag{4.53}$$

Finally, Green's function can be transformed as follows

$$\begin{aligned}S^c\left(x_b^\mu, x_a^\mu\right) = \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right)\int_0^\infty d\lambda \int d\chi \int \mathcal{D}k \exp\left\{i\left[k\left(x_b - x_a\right) + \lambda k^2\right]\right\} \\ \int \mathcal{D}\omega \exp\left\{i\int_0^1 ds\left(-2ik_\mu\left[\frac{1}{2}\int_0^1 \varepsilon\left(s-s'\right)\omega^\mu\left(s'\right)ds' + \frac{\theta^\mu}{2}\right]\chi\right.\right. \\ \left.\left.\left[-i\left(\frac{1}{2}\int_0^1\left(\varepsilon\left(s-s'\right)\omega^\mu\left(s'\right)ds' + \frac{\theta^\mu}{2}\right)\right)\omega_\mu\left(s\right)\right]ds - \frac{\theta^\mu}{2}\int_0^1 \omega_\mu\left(s\right)ds\right\}\right|_{\theta=0}.\end{aligned} \tag{4.54}$$

By using the following convolution notation

$$\omega_\mu \varepsilon \omega^\mu = \int_0^1 \int_0^1 \omega_\mu\left(s\right)\varepsilon\left(s-s'\right)\omega^\mu\left(s\right)dsds'. \tag{4.55}$$

The causal Green function becomes as

$$\begin{aligned}S^c\left(x_b^\mu, x_a^\mu\right) = \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right)\int_0^\infty d\lambda \int d\chi \int \mathcal{D}k \exp\left\{i\left[k\left(x_b - x_a\right) + \lambda k^2\right]\right\} \\ \int \mathcal{D}\omega \exp\left\{i\int_0^1 ds\left(-ik_\mu\left(\varepsilon\omega^\mu + \theta^\mu\right)\chi + \frac{i}{2}\omega_\mu\varepsilon\omega^\mu\right)\right\}\right|_{\theta=0}.\end{aligned} \tag{4.56}$$

At this stage, we integrate over the $\chi-$Grassmann proper time, and we get

$$\begin{aligned}S^c\left(x_b^\mu, x_a^\mu\right) = \exp\left(i\gamma^\mu \tfrac{\delta_l}{\delta\theta^\mu}\right)\textstyle\int_0^\infty d\lambda \int \mathcal{D}k \exp\left\{i\left[k\left(x_b - x_a\right) + \lambda k^2\right]\right\} \\ \times \textstyle\int \mathcal{D}\omega\left[\int_0^1\left[k_\mu\left(\varepsilon\omega^\mu + \theta^\mu\right)\right]ds\right]\exp\left\{-\frac{1}{2}\int_0^1 \omega_n\varepsilon\omega^n d\tau\right\}\Big|_{\theta=0},\end{aligned} \tag{4.57}$$

by inserting the odd source $\rho\left(\tau\right)$, the Green function transforms to

$$\begin{aligned}S^c\left(x_b, x_a\right) = \exp\left(i\tilde{\gamma}^n \tfrac{\partial_l}{\partial\theta^n}\right)\textstyle\int_0^\infty d\lambda \int \mathcal{D}k \exp\left\{i\left[k\left(x_b - x_a\right) + \lambda k^2\right]\right\} \\ \times \textstyle\int \mathcal{D}\omega\left[\int_0^1\left[k_\tau\left(\varepsilon\frac{\delta}{\delta\rho^0} + \theta^0\right) + k_x\left(\varepsilon\frac{\delta}{\delta\rho^1} + \theta^1\right) + k_y\left(\varepsilon\frac{\delta}{\delta\rho^2} + \theta^2\right)\right]ds\right] I\left(\rho\right)\Big|_{\theta=0}^{\rho=0},\end{aligned} \tag{4.58}$$

where the Gaussian integration $I\left(\rho\right)$ over ω gives

$$\begin{aligned}I\left(\rho\right) &= \int \mathcal{D}\omega \exp\left\{\int_0^1\left[-\frac{1}{2}\omega_\mu\varepsilon\omega^\mu + \rho_\mu\omega^\mu\right]ds\right\} \\ &= \exp\left\{-\frac{1}{2}\int_0^1\left[\rho_\mu\varepsilon^{-1}\left(s-s'\right)\rho^\mu\right]ds\right\},\end{aligned} \tag{4.59}$$

by using the following property $\varepsilon^{-1}(s-s')=\frac{1}{2}\frac{d}{ds}\delta(s-s')$, the equation (4.59) takes the following form

$$\begin{aligned} I(\rho) &= \exp\left\{-\frac{1}{4}\int_0^1\left[\rho_\mu\frac{d}{ds}\delta(s-s')\rho^\mu\right]ds\right\} \\ &= \exp\left\{\frac{1}{4}\int_0^1\left[\dot{\rho}_\mu\rho^\mu\right]ds\right\}. \end{aligned} \tag{4.60}$$

The expression of the Green function becomes as

$$\begin{gathered} S^c(x_b,x_a)=\exp\left(i\tilde{\gamma}^n\frac{\partial_l}{\partial\theta^n}\right)\int_0^\infty d\lambda\int\mathcal{D}k\exp\left\{i\left[k(x_b-x_a)+\lambda k^2\right]\right\} \\ \times\left[k_\tau\left(\varepsilon\frac{\delta}{\delta\rho^0}+\theta^0\right)+k_x\left(\varepsilon\frac{\delta}{\delta\rho^1}+\theta^1\right)+k_x\left(\varepsilon\frac{\delta}{\delta\rho^2}+\theta^2\right)\right]\exp\left\{\frac{1}{4}\int_0^1\left[\dot{\rho}_\mu\rho^\mu\right]ds\right\}\Big|_{\theta=0}^{\rho=0}. \end{gathered} \tag{4.61}$$

At this level, we perform the differentiation with respect to ρ, after that, we use these identities

$$\exp\left(i\gamma^\mu\frac{\delta_l}{\delta\theta^\mu}\right)f(\theta)\Big|_{\theta=0}=f\left(\frac{\partial_l}{\partial\theta}\right)\exp\left(i\gamma^\mu\theta_\mu\right)\Big|_{\theta=0}, \tag{4.62}$$

and expanding $\exp(i\gamma^\mu\theta_\mu)$ to the first order

$$\exp\left(i\gamma^\mu\theta_\mu\right)=1+i\gamma^\mu\theta_\mu, \tag{4.63}$$

the Green function takes the following form

$$\begin{gathered} S^c(x_b,x_a)=\exp\left(i\tilde{\gamma}^n\frac{\partial_l}{\partial\theta^n}\right)\int_0^\infty d\lambda\int\mathcal{D}k\exp\left\{i\left[k(x_b-x_a)+\lambda k^2\right]\right\} \\ \times\left[k_\tau\left(\frac{\delta}{\delta\theta^0}\right)+k_x\left(\frac{\delta}{\delta\theta^1}\right)+k_y\left(\frac{\delta}{\delta\theta^2}\right)\right]\left(1+i\gamma^\mu\theta_\mu\right)\big|_{\theta=0}, \end{gathered} \tag{4.64}$$

where

$$\frac{\delta}{\delta\rho}\exp\left\{\frac{1}{4}\int_0^1\left[\dot{\rho}_\mu\rho^\mu\right]ds\right\}\Big|_{\rho=0}=0. \tag{4.65}$$

Finally, by integrating over the even variable λ, we get

$$\begin{aligned} S^c(x_b,x_a) &= i\int\mathcal{D}k\left(k_\tau\gamma^0+k_x\gamma^1+k_y\gamma^2\right)\exp\left\{i\left[k(x_b-x_a)\right]\right\}\int_0^\infty d\lambda\exp\left\{i\lambda k^2\right\} \\ &= \int\mathcal{D}k\hat{k}\exp\left\{i\left[k(x_b-x_a)\right]\right\}\int_0^\infty d\lambda\exp\left\{i\lambda k^2\right\} \\ &= -\int\mathcal{D}k\frac{\hat{k}}{k^2}\exp\left\{i\left[k(x_b-x_a)\right]\right\}. \end{aligned} \tag{4.66}$$

Where $\hat{k}=k_\mu\gamma^\mu$ and

$$\mathcal{D}k=\prod_{n=1}^{N+1}\frac{dk_\tau}{2\pi}\frac{dk_x}{2\pi}\frac{dk_y}{2\pi}. \tag{4.67}$$

4.3.2 The derivation of the wave function from Green's function

The suitably normalized wave functions that describe the motion of the free graphene quasiparticles are given as

$$\psi_{s,k}^{(+)}(x) = \exp\{-ikx\}\, u(k,s), \tag{4.68}$$

and

$$\psi_{s,k}^{(-)}(x) = \exp\{ikx\}\, \upsilon(k,s). \tag{4.69}$$

The sets $\left\{\psi_{s,k}^{(+)}\right\}$ and $\left\{\psi_{s,k}^{(-)}\right\}$ are each orthonormalized relative to the usual scalar product

$$(\psi_{s,k}^{(+)}, \psi_{s',k'}^{(+)}) = (\psi_{s,k}^{(-)}, \psi_{s',k'}^{(-)}) = \delta(k-k')\,\delta_{ss'}, \tag{4.70}$$

and

$$(\psi_{s,k}^{(+)}, \psi_{s',k'}^{(-)}) = 0. \tag{4.71}$$

Eqs. (4.68) and (4.69) satisfy the $2D$ massless Dirac equation

$$i\frac{\partial\psi}{\partial t} = \upsilon_F(\boldsymbol{\sigma}.\mathbf{k})\,\psi, \tag{4.72}$$

whilst $u(k,s) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi_k} \\ +e^{i\phi_k}\end{pmatrix}$ and $\upsilon(k,s) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi_k} \\ -e^{i\phi_k}\end{pmatrix}$, are the solutions of the free Dirac-graphene equation (the spinors), such as $\bar{u}(k,s)\,u(k,s) = 1$ and $\bar{\upsilon}(k,s)\,\upsilon(k,s) = -1$, and they verify the below equations

$$\begin{cases} \Lambda_+ = \sum_{\pm s} u(k,s)\, u^+(k,s) = \dfrac{\hat{k}}{k} \\ \Lambda_- = \sum_{\pm s} \upsilon(k,s)\, \upsilon^+(k,s) = \dfrac{-\hat{k}}{k} \end{cases}. \tag{4.73}$$

4.4 The causal Green's function for graphene quasiparticles in interaction with a single plane wave field

To construct the causal Green's function for a Dirac massless particles in the presence of an electromagnetic plane wave field, let us consider the vector potential $A(x)$ chosen as

$$A^\mu(x,y,t) = A^\mu(\xi) = \left(0, \vec{A}(\xi)\right), \tag{4.74}$$

$$\vec{A}(\xi) = \vec{e}_x A(\xi), \tag{4.75}$$

where $A^{\mu}(\xi)$ is an arbitrary function of the single variable

$$\xi = n^{\mu}x_{\mu} \equiv \alpha y - \tau,$$

whereas $\tau = \upsilon_F t$ and $x^{\mu} = (\tau, x, y)$.

Here $\alpha = 1$, indicating that the velocities of electromagnetic waves and quasiparticles in graphene are the same and equal to $\upsilon_F = (1.12 \pm 0.02) \times 10^6 m/s$ [144, 149].

The vector of propagation (the wave vector) n has the following components

$$n = (1, 0, 1), \tag{4.76}$$

and verifying the condition of

$$n^2 = n^{\mu}n_{\mu} = 0. \tag{4.77}$$

Where Minkowski tensor has the signature $g_{\mu\nu} = diag(-1, +1, +1)$.

In addition, we impose that the electromagnetic plane wave satisfy the Lorentz gauge condition

$$\partial_{\mu}A^{\mu} = n_{\mu}(A^{\mu})' = (n_{\mu}A^{\mu})' = 0, \tag{4.78}$$

where the prime denotes the derivative functions of $A(\xi)$ with regard to ξ.

The electromagnetic field tensor $F_{\mu\nu}$ is then defined as

$$F_{\mu\nu}\gamma^{\mu}\gamma^{\nu} = 2A'(\xi)\left(\frac{\partial\xi}{\partial\tau}\gamma^0\gamma^1 - \frac{\partial\xi}{\partial y}\gamma^1\gamma^2\right) \tag{4.79}$$

$$= 2A'(\xi)\left(\dot{\xi}_{\tau}\gamma^0\gamma^1 - \xi'_y\gamma^1\gamma^2\right). \tag{4.80}$$

Where $\dot{\xi}_{\tau} = \frac{\partial\xi}{\partial\tau}$ and $\xi'_y = \frac{\partial\xi}{\partial y}$.

For all of this, the causal Green's function for a Dirac massless particles in the presence of an electromagnetic plane wave field is given as

$$S^c(q_b^{\mu}, q_a^{\mu}) = \exp\left(i\gamma^{\mu}\frac{\delta_l}{\delta\theta^{\mu}}\right)\int_0^{\infty} d\lambda \int d\chi \int \mathcal{D}q \int \mathcal{D}\bar{k} \int \mathcal{D}$$
$$\exp\left\{i\int_0^1 ds\left[\bar{k}\dot{q} + \lambda\left(k^2 - 2eA(y,\tau)k_x + e^2A^2(y,\tau)\right) + 2ie\lambda F_{\mu\nu}\psi^{\mu} \ {}^{\nu}\right.\right.$$
$$\left.\left.-2i\left(k_{\tau}\psi^0 + (k_x - eA(y,\tau))\psi^1 + k_y\psi^2\right)\chi - i\psi_{\mu}\dot{\psi}^{\mu}\right] + \psi_{\mu}(1)\psi^{\mu}(0)\right\}_{\theta=0}. \tag{4.81}$$

4.4.1 The evaluation of Green's function

The integrations over $q(s) = (\tau(s), y(s))$ seem to be difficult due to the dependence of $A(n^\mu q_\mu = \xi)$. We propose to introduce a new variable ξ that considers the plane wave variable $\xi = nq$ as independent from the quadriposition q via the following easily proved identity [100, 104]

$$\int d\xi_b d\xi_a \delta\left(\xi_a - n^\mu q_{\mu_a}\right) \delta\left(\xi_b - \xi_a - n^\mu (q_{\mu_b} - q_{\mu_a})\right) = 1. \tag{4.82}$$

Or rather its generalization which lets all time intervals $[n-1, n]$ occur

$$\int d\xi_b d\xi_a \delta\left(\xi_a - n^\mu q_{\mu_a}\right) \int \prod_{n=1}^{N} d\xi_n \prod_{n=1}^{N+1} \delta\left(\Delta\xi_n - n^\mu \Delta q_{\mu_n}\right) = 1, \tag{4.83}$$

where

$$\delta\left(\Delta\xi_n - n^\mu q_{\mu_n}\right) = \int \frac{dk_{\xi_n}}{2\pi} \exp\left[ik_{\xi n}\left(\Delta\xi_n - n^\mu \Delta q_{\mu_n}\right)\right], \tag{4.84}$$

with $\xi = n^\mu q_\mu = y - \tau$ and $\Delta\xi_n = \xi_n - \xi_{n-1}$. By inserting this in equation (4.81), the Green function takes the following form

$$\begin{aligned} S^c\left(q_b^\mu, q_a^\mu\right) &= \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right) \int_0^\infty d\lambda \int d\chi \\ &\times \int d\xi_b d\xi_a \delta\left(\xi_a - (y_a - \tau_a)\right) \int \mathcal{D}\xi \int \mathcal{D}k_\xi \int \mathcal{D}q \int \mathcal{D}\bar{k} \int \mathcal{D}\psi \\ &\times \exp\left\{i \int_0^1 ds \left[\left(\bar{k} - nk_\xi\right)\dot{q} + k_\xi\dot{\xi} + \lambda\left(k^2 - 2eA(\xi)k_x + e^2A^2(\xi)\right) + 2ie\lambda F_{\mu\nu}(\xi)\psi^\mu\psi^\nu \right.\right. \\ &\left.\left. -2i\left(k_\tau\psi^0 + (k_x - eA(y,\tau))\psi^1 + k_y\psi^2\right)\chi - i\psi_\mu\dot{\psi}^\mu\right] + \psi_\mu(1)\psi^\mu(0)\right\}_{\theta=0}, \end{aligned} \tag{4.85}$$

where

$$\mathcal{D}\xi = \prod_{n=1}^{N} d\xi_n, \quad \mathcal{D}k_\xi = \prod_{n=1}^{N+1} \frac{dk_{\xi_n}}{2\pi}. \tag{4.86}$$

Then shifting the momentum from $\bar{k} + nk_\xi$ into $\bar{k}$ by taking into account the equations $n^\mu A_\mu = 0$ and $n^\mu n_\mu = 0$, at the limit continuous $N \to \infty$, we get for the Green function the following expression

$$
\begin{aligned}
S^c\left(q_b^\mu, q_a^\mu\right) &= \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right) \int_0^\infty d\lambda \int d\chi \\
&\times \int d\xi_b d\xi_a \delta\left(\xi_a - \left(y_a - \tau_a\right)\right) \int \mathcal{D}\xi \int \mathcal{D}k_\xi \int \mathcal{D}q \int \mathcal{D}\bar{k} \int \mathcal{D}\psi \\
&\times \exp\left\{ i \int_0^1 ds \left[\bar{k}\dot{q} + k_\xi \dot{\xi} + \lambda\left(k^2 + 2k_\xi nk - 2eA(\xi)k_x + e^2A^2(\xi)\right) + 2ie\lambda F_{\mu\nu}(\xi)\psi^\mu\psi^\nu \right.\right. \\
&\left.\left. -2i\left(k_\tau\psi^0 + \left(k_x - eA(y,\tau)\right)\psi^1 + k_y\psi^2 + k_\xi n_\mu\psi^\mu\right)\chi - i\psi_\mu\dot{\psi}^\mu \right] + \psi_\mu(1)\psi^\mu(0) \right\}_{\theta=0}. \quad (4.87)
\end{aligned}
$$

Now it becomes possible to integrate over q_j and $\bar{k}_j$. Let us perform the functional integration on q_j-variables; this latter gives N Dirac functions $\delta\left(\bar{k}_j - \bar{k}_{j-1}\right)$ defined as

$$
\int \mathcal{D}q e^{-i\int_0^1 q\dot{\bar{k}}ds} = \delta\left(\dot{\bar{k}}\right), \quad (4.88)
$$

which leads to the conservation of $(k_\tau, k_y)-$momentum of the quasiparticles in graphene during the motion

$$
k_{\tau_1} = k_{\tau_2}...k_{\tau_{N+1}} = k_\tau = cst, \qquad k_{y1} = k_{y2}...k_{y_{N+1}} = k_y = cst. \quad (4.89)
$$

For this conservation, the Green function $S^c\left(x_b^\mu, x_a^\mu\right)$ takes the following form

$$
\begin{aligned}
S^c\left(x_b^\mu, x_a^\mu\right) &= \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right) \int_0^\infty d\lambda \int d\chi \int \frac{d^3k}{(2\pi)^3} e^{ik_\mu\left(x_b^\mu - x_a^\mu\right) + i\lambda k^2} \\
&\times \int d\xi_b d\xi_a \delta\left(\xi_a - \left(y_a - \tau_a\right)\right) \int \mathcal{D}\xi \int \mathcal{D}k_\xi \int \mathcal{D}\psi \\
&\times \exp\left\{ i \int_0^1 ds \left[k_\xi\left(\dot{\xi} + 2\lambda nk - 2in_\mu\psi^\mu\chi\right) + \lambda\left(-2eA(\xi)k_x + e^2A^2(\xi)\right) + 2ie\lambda F_{\mu\nu}(\xi)\psi^\mu\psi^\nu \right.\right. \\
&\left.\left. -2i\left(k_\tau\psi^0 + \left(k_x - eA(y,\tau)\right)\psi^1 + k_y\psi^2\right)\chi - i\psi_\mu\dot{\psi}^\mu \right] + \psi_\mu(1)\psi^\mu(0) \right\}_{\theta=0}. \quad (4.90)
\end{aligned}
$$

For performing the functional integrations over Grassmannian odd variables, we write the spin-field couplage $F_{\mu\nu}(\xi)\psi^\mu\psi^\nu$ as follows

$$
F_{\mu\nu}(\xi)\psi^\mu\psi^\nu = 2(n\psi)\left(A'(\xi)\psi\right). \quad (4.91)
$$

This writing inspires us to introduce a new Grassmannian variable $\eta = n^\mu\psi_\mu$ as an independent variable from ψ via the following Grassmann functional identity [100]

$$
\int d\eta_b d\eta_a \delta\left(\eta_a - n^\mu\psi_{\mu_a}\right) \int D\eta\delta\left(\dot{\eta} - n^\mu\dot{\psi}_\mu\right) = 1, \quad (4.92)
$$

where

$$\delta\left(\dot{\eta}-n^{\mu}\dot{\psi}_{\mu}\right)=\int\frac{dk_{\eta}}{2\pi}\exp\left[ik_{\eta}\left(\dot{\eta}-n^{\mu}\dot{\psi}_{\mu}\right)\right], \tag{4.93}$$

with k_η is the momentum Grassmannian odd variable.

From Eqs. (4.92) and (4.93), the Green function becomes

$$\begin{aligned}S^{c}\left(x_b^{\mu},x_a^{\mu}\right)&=\exp\left(i\gamma^{\mu}\frac{\delta_l}{\delta\theta^{\mu}}\right)\int_0^{\infty}d\lambda\int d\chi\int\frac{d^3k}{(2\pi)^3}e^{ik_{\mu}\left(x_b^{\mu}-x_a^{\mu}\right)+i\lambda k^2}\\&\times\int d\xi_b d\xi_a\delta\left(\xi_a-\left(y_a-\tau_a\right)\right)\int D\xi\int Dk_{\xi}\int d\eta_b d\eta_a\delta\left(\eta_a-n^{\mu}\psi_{\mu_a}\right)\\&\times\int D\eta\int Dk_{\eta}\int\mathcal{D}\psi\exp\left\{i\int_0^1 ds\left[k_{\xi}\left(\dot{\xi}+2\lambda nk-2i\eta\chi\right)+k_{\eta}\left(\dot{\eta}-n^{\mu}\dot{\psi}_{\mu}\right)\right.\right.\\&+\lambda\left(-2eA\left(\xi\right)k_x+e^2A^2\left(\xi\right)\right)+4ie\lambda\eta A'\left(\xi\right)\psi\\&\left.\left.-2i\left(k_{\tau}\psi^0+\left(k_x-eA\left(y,\tau\right)\right)\psi^1+k_y\psi^2\right)\chi-i\psi_{\mu}\dot{\psi}^{\mu}\right]+\psi_{\mu}\left(1\right)\psi^{\mu}\left(0\right)\right\}_{\theta=0},\end{aligned} \tag{4.94}$$

where

$$D\eta=\prod_{n=1}^{N}d\eta_n,\quad \mathcal{D}k_{\eta}=\prod_{n=1}^{N+1}\frac{dk_{\eta_n}}{2\pi}. \tag{4.95}$$

In order to determine the functional integrations over Grassmannian variables $\psi(s)$, we note that these odd trajectories $\psi(s)$ obey the antiperiodic boundary condition $\psi_\mu(0)+\psi_\mu(1)=\theta_\mu$. This condition can be suitably absorbed by replacing the integration over odd trajectories $\psi(s)$ by one over odd velocities $\omega(s)$ is defined as

$$\psi\left(s\right)=\frac{1}{2}\int\varepsilon\left(s-s'\right)\omega\left(s'\right)ds'+\frac{\theta}{2}, \tag{4.96}$$

where

$$\varepsilon\left(s-s'\right)=sign\left(s-s'\right)=\begin{cases}-1 & \text{for } s'\succ s\\ 0 & \text{for } s'=s\\ 1 & \text{for } s\succ s'\end{cases},$$

and the measure $\mathcal{D}\psi$ has the following definition

$$\mathcal{D}\psi=D\psi\left[\int D\psi\exp\left(\int_0^1\psi_{\mu}\dot{\psi}^{\mu}ds\right)\right]^{-1}. \tag{4.97}$$

We use the following convolution notation [100].

$$f\varepsilon g=\int_0^1\int_0^1 f\left(s\right)\varepsilon\left(s-s'\right)g\left(s'\right)dsds'. \tag{4.98}$$

For the transformation of $\omega^n(s)$, the Green function takes the Gaussian form and is written as

$$
\begin{aligned}
S^c\left(x_b^\mu, x_a^\mu\right) &= \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right) \int_0^\infty d\lambda \int d\chi \int \frac{d^3k}{(2\pi)^3} e^{ik_\mu\left(x_b^\mu - x_a^\mu\right) + i\lambda k^2} \\
&\times \int d\xi_b d\xi_a \delta\left(\xi_a - (y_a - \tau_a)\right) \int D\xi \int Dk_\xi \int d\eta_b d\eta_a \delta\left(\eta_a + \frac{1}{2} n^\mu \left(\omega_\mu - \theta_\mu\right)\right) \\
&\times \int D\eta \int Dk_\eta \int \mathcal{D}\omega \exp\left\{ i \int_0^1 ds \left[k_\xi \left(\dot{\xi} + 2\lambda nk - 2i\eta\chi\right) + k_\eta\left(\dot{\eta} - n^\mu\omega_\mu\right) \right.\right. \\
&\quad + \lambda\left(-2eA(\xi)k_x + e^2A^2(\xi)\right) + 2ie\lambda\eta A'(\xi)\left(\varepsilon\omega + \theta\right) \\
&\quad \left.\left. - i\left(\left(k_x - eA(\xi)\right)\left(\varepsilon\omega^1 + \theta^1\right) + k_\mu\left(\varepsilon\omega^\mu + \theta^\mu\right)\right)\chi + \frac{i}{2}\omega_\mu \varepsilon\omega^\mu \right] \right\}_{\theta=0}. \qquad (4.99)
\end{aligned}
$$

In order to extract the classical equation of motion relative to spin, therefore, make a link between classical and quantum evolution, then perform the following translation shift of velocity, which facilitates the calculation over the spin variables

$$\omega^\mu(s) \rightarrow \omega^\mu(s) + in^\mu \int_0^1 \varepsilon^{-1}(s - s')\, k_\eta(s')ds'. \qquad (4.100)$$

At this step, by using the plane wave properties $n^2 = 0$ and $nA = 0$, the terms containing the velocities will transform as follows.

For the linear term

$$\omega_\mu \varepsilon\omega^\mu \rightarrow \omega_\mu \varepsilon\omega^\mu - 2ik_\eta n^\mu \omega_\mu. \qquad (4.101)$$

For the Delta bilinear term

$$\varepsilon\omega^\mu \rightarrow \varepsilon\omega^\mu + in^\mu k_\eta, \qquad (4.102)$$

the measure $\mathcal{D}\omega$ is unchanged. Then the Green function can be written as

$$S^c\left(x_b^\mu, x_a^\mu\right) = \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right) \int_0^\infty d\lambda \int d\chi \int \frac{d^3k}{(2\pi)^3} e^{ik_\mu\left(x_b^\mu - x_a^\mu\right) + i\lambda k^2}$$
$$\times \int d\xi_b d\xi_a \delta\left(\xi_a - \left(y_a - \tau_a\right)\right) \int D\xi \int Dk_\xi \int d\eta_b d\eta_a \delta\left(\eta_a + \frac{1}{2} n^\mu\left(\omega_\mu - \theta_\mu\right)\right)$$
$$\times \int D\eta \int Dk_\eta \int \mathcal{D}\omega \exp\left\{ i \int_0^1 ds \left[k_\xi \left(\dot{\xi} + 2\lambda nk - 2i\eta\chi\right) + k_\eta\left(\dot{\eta} - n^\mu\omega_\mu\right)\right.\right.$$
$$+\lambda\left(-2eA(\xi)k_x + e^2A^2(\xi)\right) + 2ie\lambda\eta A'(\xi)\left(\varepsilon\omega + ink_\eta + \theta\right)$$
$$\left.\left. -i\left(\left(k_x - eA(\xi)\right)\left(\varepsilon\omega^1 + ink_\eta + \theta^1\right) + k_\mu\left(\varepsilon\omega^\mu + ink_\eta + \theta^\mu\right)\right)\chi + \frac{i}{2}\left(\omega_\mu\varepsilon\omega^\mu - 2ik_\eta n\omega\right)\right]\right\}_{\theta=0}. \tag{4.103}$$

Now, we replace the delta functional $\delta\left(\eta_a + \frac{1}{2}n^\mu\left(\omega_\mu - \theta_\mu\right)\right)$ by the exponential form as

$$\delta\left(\eta_a + \frac{1}{2}n^\mu\left(\omega_\mu - \theta_\mu\right)\right) = \int dk_{\eta_a} \exp\left(ik_{\eta_a}\left(\eta_a + \frac{1}{2}n^\mu\left(\omega_\mu - \theta_\mu\right)\right)\right), \tag{4.104}$$

the Green function is then determined by the following expression

$$S^c\left(x_b^\mu, x_a^\mu\right) = \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right) \int_0^\infty d\lambda \int d\chi \int \frac{d^3k}{(2\pi)^3} e^{ik_\mu\left(x_b^\mu - x_a^\mu\right) + i\lambda k^2}$$
$$\times \int d\xi_b d\xi_a \delta\left(\xi_a - \left(y_a - \tau_a\right)\right) \int D\xi \int Dk_\xi \int d\eta_b d\eta_a \int dk_{\eta_a}$$
$$\times \int D\eta \int Dk_\eta \exp\left\{ i \int_0^1 ds \left[k_\xi\left(\dot{\xi} + 2\lambda k^\mu n_\mu - 2i\eta\chi\right) + k_\eta\left(\dot{\eta} + n^\mu k_\mu \chi\right)\right.\right.$$
$$+k_{\eta_a}\left(\eta_a - \frac{1}{2}n^\mu\theta_\mu\right) + \lambda\left(-2eA(\xi)k_x + e^2A^2(\xi)\right) + 2ie\lambda\eta A'(\xi)\theta$$
$$\left.\left. -i\left(\left(k_x - eA(\xi)\right)\theta^1 + k_\mu\theta^\mu\right)\chi\right]\right\} I(\xi,\eta)\,|_{\theta=0}, \tag{4.105}$$

where $I(\xi,\eta)$ gives the Gaussian integration over the odd $\omega-$velocities trajectories

$$I(\xi,\eta) = \int \mathcal{D}\omega \exp\left\{\int_0^1 \left[-\frac{1}{2}\omega^\mu \mathcal{M}_{\mu\nu}\omega^\nu + \mathcal{J}_\mu\omega^\mu\right] ds\right\}, \tag{4.106}$$

noting that $\mathcal{M}_{\mu\nu}(s,s') = g_{\mu\nu}\varepsilon(s-s')$ and $g_{\mu\nu}$ refers to the metric tensor of the Minkowski space. While the expressions of external current sources $\mathcal{J}_\mu$ are given as

$$\mathcal{J}_\mu = -\chi \int_0^1 \left(k_\mu - eA_\mu\left(\xi\left(s'\right)\right)\right)\varepsilon\left(s' - s\right) ds'$$
$$- 2e\lambda \int_0^1 \eta\left(s'\right) A'\left(\xi\left(s'\right)\right)\varepsilon\left(s'-s\right) ds' + \frac{i}{2}k_{\eta_a}n^\mu.$$
$$= -\chi\left(k_\mu - eA_\mu\xi\right)\varepsilon - 2e\lambda \int_0^1 \eta A'\varepsilon + \frac{i}{2}k_{\eta_a}n^\mu. \tag{4.107}$$

According to Refs. [100], we determine the spin factor $I(\xi,\eta)$, which takes the Gaussian form as

$$\begin{aligned} I(\xi,\eta) &= \int \mathcal{D}\omega \exp\left\{\int_0^1 \left[-\frac{1}{2}\omega^\mu \mathcal{M}_{\mu\nu}\omega^\nu + \mathcal{J}_\mu \omega^\mu\right] ds\right\}, \\ &= \sqrt{\frac{\det \mathcal{M}}{\det \varepsilon}} \exp\left\{-\frac{1}{2}\mathcal{J}_\mu \mathcal{M}^{-1}_{\mu\nu}\mathcal{J}^\nu\right\} \\ &= \exp\left\{-\frac{1}{2}\mathcal{J}_\mu \mathcal{M}^{-1}_{\mu\nu}\mathcal{J}^\nu\right\} \\ &= \exp\left\{-\frac{1}{2}\mathcal{J}_\mu \varepsilon^{-1}\mathcal{J}^\mu\right\}, \end{aligned} \tag{4.108}$$

where $\sqrt{\frac{\det \mathcal{M}}{\det \varepsilon}} = 1$ and $\mathcal{M}^{-1}_{\mu\nu}$ is the inverse of the matrix $\mathcal{M}_{\mu\nu}$.

For the factor $\mathcal{J}_\mu \varepsilon^{-1}\mathcal{J}^\mu$, it is easy to show that

$$\begin{aligned} \mathcal{J}_\mu \varepsilon^{-1}\mathcal{J}^\mu &= \int_0^1 \mathcal{J}_\mu(s)\,\varepsilon^{-1}(s-s')\,\mathcal{J}^\mu(s')\,ds ds' = \left[2e\lambda(k_\mu - eA_\mu)\,\varepsilon\eta(s_2)\,A' + ik_{\eta_a}k_\mu n^\mu\right]\chi \\ &\quad -4(e\lambda)^2\,\eta A'\varepsilon\eta A', \end{aligned} \tag{4.109}$$

which gives us the following form of spin factor $I(\xi,\eta)$

$$I(\xi,\eta) = \exp\left\{i\int_0^1 ds\left[\left[ie\lambda(k_\mu - eA_\mu)\,\varepsilon\eta A' - \frac{1}{2}k_\mu n^\mu k_{\eta_a}\right]\chi - 2ie^2\lambda^2\eta A'\varepsilon\eta A'\right]\right\}. \tag{4.110}$$

The Green function then takes the following form

$$\begin{aligned} S^c(x_b^\mu, x_a^\mu) &= \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right)\int_0^\infty d\lambda \int d\chi \int \frac{d^3k}{(2\pi)^3} e^{ik_\mu\left(x_b^\mu - x_a^\mu\right) + i\lambda k^2} \\ &\times \int d\xi_b d\xi_a \delta\left(\xi_a - (y_a - \tau_a)\right) \int D\xi \int Dk_\xi \int d\eta_b d\eta_a \int dk_{\eta_a} \\ &\times \int D\eta \int Dk_\eta \exp\left\{i\int_0^1 ds\left[k_\xi\left(\dot\xi + 2\lambda k^\mu n_\mu - 2i\eta\chi\right) + k_\eta\left(\dot\eta + n^\mu k_\mu \chi\right)\right.\right. \\ &+ k_{\eta_a}\left(\eta_a - \frac{1}{2}n^\mu\theta_\mu - \frac{k_\mu n^\mu}{2}\chi\right) + \lambda\left(-2eA(\xi)\,k_x + e^2A^2(\xi)\right) + 2ie\lambda\eta A'(\xi)\,\theta \\ &\left.\left. -i\left((k_x - eA(\xi))\left(\theta^1 - e\lambda\varepsilon\eta A'\right) + k_\mu\theta^\mu\right)\chi - 2ie^2\lambda^2\eta A'\varepsilon\eta A'\right]\right\}\Big|_{\theta=0}. \end{aligned} \tag{4.111}$$

After that, noticing that the integrations over k_η and k_ξ give the following delta functional

$$\delta(\dot{\eta} + nk\chi), \tag{4.112}$$

and

$$\delta\left(\dot{\xi} + 2\lambda\,(nk) - 2i\eta\chi\right). \tag{4.113}$$

By integrating the arguments of delta functionals $\delta(\dot{\eta} + nk\chi)$ and $\delta\left(\dot{\xi} + 2\lambda\,(nk) - 2i\eta\chi\right)$, we get the following results

$$\dot{\eta} + n^{\mu}k_{\mu}\chi = 0 \Rightarrow \eta\,(s) = \eta_a - (nk)\,\chi s, \tag{4.114}$$

and

$$\dot{\xi} + 2\lambda\,(nk) - 2i\eta\chi = 0 \Rightarrow \xi(s) = \xi_a - 2\lambda\,(nk)\,s + 2i\eta_a\chi s. \tag{4.115}$$

At this level, we insert the explicit solutions of the classical equations of motion (4.114) and (4.115) in the expression of the previous Green's function, and after straightforward and long computations, we obtain

$$\begin{aligned} S^{c}\left(x_b^{\mu}, x_a^{\mu}\right) = \exp\left(i\gamma^{\mu}\frac{\delta_l}{\delta\theta^{\mu}}\right)\int_0^{\infty} d\lambda \int d\chi \int \frac{d^3k}{(2\pi)^3} e^{ik_{\mu}\left(x_b^{\mu}-x_a^{\mu}\right)+i\lambda k^2} \\ \times \int d\xi_b d\xi_a \delta\left(\xi_a - (y_a - \tau_a)\right) \int d\eta_b d\eta_a \delta\left(\eta_b - \eta_a + n^{\mu}k_{\mu}\chi\right) \\ \times\delta\left(\xi_b - \xi_a + 2\lambda nk - 2i\eta_a\chi\right)\delta\left(\eta_a - \frac{1}{2}n^{\mu}\theta_{\mu} - \frac{n^{\mu}k_{\mu}}{2}\chi\right) \\ \times\exp\left\{i\int_0^1 ds\left[\lambda\left(-2eA\,(\xi)\,k_x + e^2A^2\,(\xi)\right) + 2ie\lambda\left(\eta_a - n^{\mu}k_{\mu}\chi s\right)A'\,(\xi)\,\theta\right.\right. \\ -i\left(\left(k_x - eA\,(\xi)\right)\left(\theta^1 - e\lambda\varepsilon\left(\eta_a - n^{\mu}k_{\mu}\chi s'\right)A'\right) + k_{\mu}\theta^{\mu}\right)\chi \\ \left.\left.-2ie^2\lambda^2\left(\eta_a - n^{\mu}k_{\mu}\chi s\right)A'\varepsilon\left(\eta_a - n^{\mu}k_{\mu}\chi s'\right)A'\right]\right\}|_{\theta=0}, \end{aligned} \tag{4.116}$$

we use the following simplifications

$$\left(\eta_a - n^{\mu}k_{\mu}\chi s\right)A'\varepsilon\left(\eta_a - n^{\mu}k_{\mu}\chi s'\right)A' = -2k_{\mu}n^{\mu}A'\varepsilon s'A'\eta_a\chi. \tag{4.117}$$

The Green function expression becomes as

$$
\begin{aligned}
S^{c}\left(x_{b}^{\mu}, x_{a}^{\mu}\right) = \exp\left(i\gamma^{\mu}\frac{\delta_{l}}{\delta\theta^{\mu}}\right)\int_{0}^{\infty} d\lambda \int d\chi \int \frac{d^{3}k}{(2\pi)^{3}} e^{ik_{\mu}\left(x_{b}^{\mu}-x_{a}^{\mu}\right)+i\lambda k^{2}} \\
\times \int d\xi_{b} d\xi_{a} \delta\left(\xi_{a}-\left(y_{a}-\tau_{a}\right)\right) \int d\eta_{b} d\eta_{a} \delta\left(\eta_{b}-\eta_{a}+n^{\mu}k_{\mu}\chi\right) \\
\times \delta\left(\xi_{b}-\xi_{a}+2\lambda nk\text{-}2i\eta_{a}\chi\right) \delta\left(\eta_{a}-\frac{1}{2}n^{\mu}\theta_{\mu}-\frac{n^{\mu}k_{\mu}}{2}\chi\right) \\
\times \exp\left\{i\int_{0}^{1} ds \left[\lambda\left(-2eA(\xi)k_{x}+e^{2}A^{2}(\xi)\right)+2ie\lambda\left(\eta_{a}-n^{\mu}k_{\mu}\chi s\right)A'(\xi)\theta\right.\right. \\
-i\left(\left(k_{x}-eA(\xi)\right)\left(\theta^{1}-e\lambda\varepsilon\left(\eta_{a}-n^{\mu}k_{\mu}\chi s'\right)A'\right)+k_{\mu}\theta^{\mu}\right)\chi \\
\left.\left.+4ie^{2}\lambda^{2}k_{\mu}n^{\mu}A'\varepsilon s'A'\eta_{a}\chi\right]\right\}|_{\theta=0}\,. \qquad (4.118)
\end{aligned}
$$

From the two delta functions shown in the above expression for the Green function, we can get the following equations

$$
\eta_{a} = \frac{1}{2}n^{\mu}\theta_{\mu} + \frac{1}{2}nk\chi, \qquad (4.119)
$$

and

$$
\eta_{b} - \eta_{a} = -nk\chi. \qquad (4.120)
$$

The summation of the previous equations (4.119) and (4.120) gives

$$
\eta_{a} + \eta_{b} = n^{\mu}\theta_{\mu}, \qquad (4.121)
$$

this boundary condition represents the conservation of spin for quasiparticles during motion. By summing the above two equations, we can find this equality $(\eta_{b}+\eta_{a}=n\theta)$.

Let us now integrate over η_{a} and over η_{b}. The propagator is therefore reduced to the following expression

$$
\begin{aligned}
S^{c}\left(x_{b}^{\mu}, x_{a}^{\mu}\right) = \exp\left(i\gamma^{\mu}\frac{\delta_{l}}{\delta\theta^{\mu}}\right)\int_{0}^{\infty} d\lambda \int d\chi \int \frac{d^{3}k}{(2\pi)^{3}} e^{ik_{\mu}\left(x_{b}^{\mu}-x_{a}^{\mu}\right)+i\lambda k^{2}} \\
\times \int d\xi_{b} d\xi_{a} \delta\left(\xi_{a}-\left(y_{a}-\tau_{a}\right)\right) \delta\left(\xi_{b}-\xi_{a}+2\lambda k^{\mu}n_{\mu}-in^{\mu}\theta_{\mu}\chi\right) \\
\exp\left\{i\int_{0}^{1} ds \left[\lambda\left(-2eA(\xi)k_{x}+e^{2}A^{2}(\xi)\right)+ie\lambda\left(-2n^{\mu}k_{\mu}\chi s+n^{\mu}\theta_{\mu}+n^{\mu}k_{\mu}\chi\right)A'(\xi)\theta\right.\right. \\
\left.\left.-i\left(\left(k_{x}-eA(\xi)\right)\left(\theta^{1}-e\lambda\frac{n^{\mu}\theta_{\mu}}{2}\varepsilon A'\right)+k_{\mu}\theta^{\mu}\right)\chi+2ie^{2}\lambda^{2}n^{\mu}k_{\mu}A'\varepsilon sA'n^{\mu}\theta_{\mu}\chi\right]\right\}|_{\theta=0}\,. \qquad (4.122)
\end{aligned}
$$

Now, we integrate over k_{ξ}; this integration gives us the delta function $\delta\left(\dot{\xi}+2\lambda nk-in\theta\chi\right)$

$$\dot{\xi} = -2\lambda nk + in\theta\chi, \tag{4.123}$$

$$\frac{d\xi}{ds} = -2\lambda nk + in\theta\chi. \tag{4.124}$$

Its inverse is given as

$$\frac{ds}{d\xi} = -\frac{1}{2\lambda nk}\left(1 + \frac{i}{2}\frac{n\theta}{\lambda nk}\chi\right). \tag{4.125}$$

Performing now the integration over the proper time s, and by simplification, the Green function becomes as

$$S^c\left(x_b^\mu, x_a^\mu\right) = \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right)\int_0^\infty d\lambda \int d\chi \int \frac{d^3k}{(2\pi)^3} e^{ik_\mu\left(x_b^\mu - x_a^\mu\right) + i\lambda k^2}$$
$$\times \int d\xi_b d\xi_a \delta\left(\xi_a - (y_a - \tau_a)\right)\delta\left(\xi_b - \xi_a + 2\lambda k^\mu n_\mu - in^\mu\theta_\mu\chi\right) \tag{4.126}$$

$$\exp\left\{\left[\frac{i}{2nk}\int_{\xi_a}^{\xi_b}\left(-2eA(\xi)k_x + e^2A^2(\xi)\right)d\xi + \frac{e}{2nk}n^\mu\theta_\mu\left(A_b - A_a\right)\theta\right.\right.$$
$$\left.\left. + \left[-\frac{e}{4}\left(A_b + A_a\right)\theta - \frac{e}{4nk}\left[k_x\left(A_b + A_a\right) + eA_aA_b\right]n^\mu\theta_\mu + k_\mu\theta^\mu\right]\chi\right\}|_{\theta=0}. \tag{4.127}$$

To eliminate the constraints $\xi_b = nx_b$, replace the delta function $\delta\left(\xi_b - \xi_a + 2\lambda k^\mu n_\mu - in^\mu\theta_\mu\chi\right)$ by its integral representation

$$\delta\left(\xi_b - \xi_a + 2\lambda k^\mu n_\mu - in^\mu\theta_\mu\chi\right) = \int \frac{dk_{\xi_b}}{2\pi}\exp\left[ik_{\xi_b}\left(\xi_b - \xi_a + 2\lambda nk - in^\mu\theta_\mu\chi\right)\right]. \tag{4.128}$$

Next, shifting the momentum k by $k - nk_{\xi_b}$, and according to integrate over the $\chi-$Grassmann proper time, one obtains

$$S^c\left(x_b^\mu, x_a^\mu\right) = \exp\left(i\gamma^\mu\frac{\delta_l}{\delta\theta^\mu}\right)\int_0^\infty d\lambda\int\frac{d^3k}{(2\pi)^3}$$
$$\times\left[k\theta\left[1 + \frac{e}{2nk}n^\mu\theta_\mu\left(A_b - A_a\right)\theta\right] - \frac{e}{4}\left(A_b + A_a\right)\theta\right.$$
$$\left.-\frac{e}{4nk}\left[k_x\left(A_b + A_a\right) + eA_aA_b\right]n^\mu\theta_\mu - \frac{e}{8nk}\left(A_b + A_a\right)\theta\left(n^\mu\theta_\mu\right)\left(A_b - A_a\right)\theta\right]\Bigg|_{\theta=0}$$
$$\exp\left\{ik\left(x_b - x_a\right) + i\lambda k^2 + \frac{i}{2nk}\int_{\xi_a}^{\xi_b}\left(-2eA(\xi)k_x + e^2A^2(\xi)\right)d\xi\right\}. \tag{4.129}$$

Then performing the derivation with respect to θ and using the relation

$$\exp\left(i\gamma^\mu\frac{\delta_l}{\delta\theta^\mu}\right)f(\theta)\Bigg|_{\theta=0} = f\left(\frac{\partial_l}{\partial\theta}\right)\exp\left(i\gamma^\mu\theta_\mu\right)\Bigg|_{\theta=0}. \tag{4.130}$$

Expanding $\exp(i\gamma^\mu\theta_\mu)$ to the second order as follows

$$\exp(i\gamma^\mu\theta_\mu) = 1 + i\gamma^\mu\theta_\mu - \frac{1}{2}\theta_\mu\theta_\nu\gamma^\mu\gamma^\nu. \tag{4.131}$$

Using the properties $n^2 = 0$, $nA = 0$, and $\hat{A}\hat{B}+\hat{B}\hat{A} = 2AB$, after this step, Green's function is transformed to

$$\begin{aligned} S^c(x_b^\mu, x_a^\mu) = \int_0^\infty d\lambda \int \frac{d^3k}{(2\pi)^3} \Big[\hat{k}\left[1 + \frac{e}{2nk}\hat{n}\left(\hat{A}_b - \hat{A}_a\right)\right] - e\hat{A}_b \\ + \frac{e}{2kn}\hat{n}(kA_b) - \frac{e^2}{2kn}\hat{n}(A_aA_b) + \frac{e^2}{2kn}\hat{n}\hat{A}_a\hat{A}_b\Big] \\ \exp\left\{ik(x_b - x_a) + i\lambda k^2 + \frac{i}{2nk}\int_{\xi_a}^{\xi_b}\left(-2eA(\xi)k_x + e^2A^2(\xi)\right)d\xi\right\}, \end{aligned} \tag{4.132}$$

where, $(\hat{n} = n_\mu\gamma^\mu, \hat{A} = A_\mu\gamma^\mu, \hat{k} = k_\mu\gamma^\mu)$.

By using the idea of the relationship

$$\begin{aligned} \hat{k}\left[1 + \frac{e}{2kn}\hat{n}\left(\hat{A}_b - \hat{A}_a\right)\right] &= \hat{k}\left[1 + \frac{e}{2kn}\hat{n}\hat{A}_b\right]\left[1 - \frac{e}{2kn}\hat{n}\hat{A}_a\right] \\ &= \left[1 + \frac{e}{2kn}\hat{n}\hat{A}_b\right]\hat{k}\left[1 - \frac{e}{2kn}\hat{n}\hat{A}_a\right] + e\hat{A}_b \\ &\quad - \frac{e}{2kn}\hat{n}(kA_b) + \frac{e^2}{2kn}\hat{n}(A_aA_b) - \frac{e^2}{2kn}\hat{n}\hat{A}_a\hat{A}_b. \end{aligned} \tag{4.133}$$

The expression of Green's function becomes as

$$\begin{aligned} S^c(x_b^\mu, x_a^\mu) = i\int_0^\infty d\lambda \int \frac{d^2k}{(2\pi)^2}\int\frac{dk_\tau}{2\pi}\left[1 + \frac{e}{2k^\mu n_\mu}\hat{n}\hat{A}_b\right]\hat{k}\left[1 - \frac{e}{2k^\mu n_\mu}\hat{n}\hat{A}_a\right] \\ \exp\left\{ik(x_b - x_a) + i\lambda k^2 + \frac{i}{2k^\mu n_\mu}\int_{\xi_a}^{\xi_b}\left(-2eA(\xi)k_x + e^2A^2(\xi)\right)d\xi\right\}. \end{aligned} \tag{4.134}$$

At the end, the integration over the bosonic proper time λ one obtains

$$\begin{aligned} S^c(x_b^\mu, x_a^\mu) = -\int \frac{d^2k}{(2\pi)^2}\int\frac{dk_\tau}{2\pi}\left[1 + \frac{e}{2k^\mu n_\mu}\hat{n}\hat{A}_b\right]\frac{\hat{k}}{k^2 + i\varepsilon}\left[1 - \frac{e}{2k^\mu n_\mu}\hat{n}\hat{A}_a\right] \\ \exp\left\{ik(x_b - x_a) + \frac{i}{2k^\mu n_\mu}\int_{\xi_a}^{\xi_b}\left(-2eA(\xi)k_x + e^2A^2(\xi)\right)d\xi\right\}. \end{aligned} \tag{4.135}$$

4.4.2 The derivation of the wave function from the Green function

To determine the wave functions, we integrate over the energy k_τ by applying the residue theorem. The poles of Green's function give us the positive and negative energies (the plus sign stands for graphene quasiparticles, and the minus sign

stands for holes), they are given by

$$k^2 = \left(k^0\right)^2 - k_x^2 - k_y^2 = 0 \Longrightarrow k_\pm^0 = \pm\sqrt{k_x^2 + k_y^2} = \pm E \mp i\varepsilon. \tag{4.136}$$

The application of the residue theorem gives

$$\begin{aligned} S^c\left(q_b^\mu, q_a^\mu\right) &= -i\Theta\left(t_b - t_a\right)\int \frac{d^2k}{\left(2\pi\right)^2}\left[1 + \frac{e}{2k^\mu n_\mu}\hat{n}\hat{A}_b\right]\frac{\hat{k}}{k^2}\left[1 - \frac{e}{2k^\mu n_\mu}\hat{n}\hat{A}_a\right] \\ &\exp\left\{-ik\left(x_b - x_a\right) + \frac{i}{2k^\mu n_\mu}\int_{\xi_a}^{\xi_b}\left(-2eA\left(\xi\right)k_x + e^2A^2\left(\xi\right)\right)d\xi\right\} \\ &-i\Theta\left(t_a - t_b\right)\int \frac{d^2k}{\left(2\pi\right)^2}\left[1 - \frac{e}{2k^\mu n_\mu}\hat{n}\hat{A}_b\right]\frac{-\hat{k}}{k^2}\left[1 + \frac{e}{2k^\mu n_\mu}\hat{n}\hat{A}_a\right] \\ &\exp\left\{ik\left(x_b - x_a\right) + \frac{i}{2k^\mu n_\mu}\int_{\xi_a}^{\xi_b}\left(-2eA\left(\xi\right)k_x + e^2A^2\left(\xi\right)\right)d\xi\right\}, \end{aligned} \tag{4.137}$$

we have a projection of the positive and negative energy states [156]

$$\begin{cases} \Lambda_+ = \sum_{\pm s} u\left(k,s\right)u^+\left(k,s\right) = \dfrac{\hat{k}}{k^2} \\ \Lambda_- = \sum_{\pm s} \upsilon\left(k,s\right)\upsilon^+\left(k,s\right) = \dfrac{-\hat{k}}{k^2} \end{cases}. \tag{4.138}$$

where $u\left(k,s\right)$ and $\upsilon\left(k,s\right)$ are the spinors, which are the solutions of the free graphene quasi-particles equation verifying $\bar{u}\left(k,s\right)u\left(k,s\right) = 1$ and $\bar{\upsilon}\left(k,s\right)\upsilon\left(k,s\right) = -1$,

The expansion of Green's function in terms of the complete basis of states for quasiparticles in an external field is given as

$$\begin{aligned} S^c\left(x_b^\mu, x_a^\mu\right) = -i\Theta\left(t_b - t_a\right)\int \frac{d^2k}{\left(2\pi\right)^2}\sum_{s=\pm1}\psi_{s,k}^{(+)}\left(x_b\right)\bar{\psi}_{s,k}^{(+)}\left(x_a\right) \\ + i\Theta\left(t_a - t_b\right)\int \frac{d^2k}{\left(2\pi\right)^2}\sum_{s=\pm1}\psi_{s,k}^{(-)}\left(x_b\right)\bar{\psi}_{s,k}^{(-)}\left(x_a\right), \end{aligned} \tag{4.139}$$

where

$$\psi_{s,k}^{(+)}(x) = \left[1 + \frac{e}{2k^{\mu}n_{\mu}}\hat{n}\hat{A}\right] u(k,s)$$
$$\times \exp\left\{-ikx + \frac{i}{2(k_y - k_\tau)}\int_0^{nq}\left(-2eA(\xi)k_x + e^2A^2(\xi)\right)d\xi\right\}, \quad (4.140)$$

and

$$\psi_{s,k}^{(-)}(x) = \left[1 - \frac{e}{2k^{\mu}n_{\mu}}\hat{n}\hat{A}\right] \upsilon(k,s)$$
$$\times \exp\left\{ikx + \frac{i}{2(k_y - k_\tau)}\int_0^{nq}\left(-2eA(\xi)k_x + e^2A^2(\xi)\right)d\xi\right\}. \quad (4.141)$$

The sets $\left\{\psi_{s,k}^{(+)}\right\}$ and $\left\{\psi_{s,k}^{(-)}\right\}$ are each orthonormalized relative to the usual scalar product

$$(\psi_{s,k}^{(+)}, \psi_{s',k'}^{(+)}) = (\psi_{s,k}^{(-)}, \psi_{s',k'}^{(-)}) = \delta(k-k')\delta_{ss'}; \quad (\psi_{s,k}^{(+)}, \psi_{s',k'}^{(-)}) = 0. \quad (4.142)$$

According to Ref. [156]. The solutions of a complete and orthonormal system relative can be rewritten as

$$\psi_{s,k}^{(+)}(x) = \exp\left\{-ikx + \frac{i}{2(k_y - k_\tau)}\int_0^{nq}\left(-2eA(\xi)k_x + e^2A^2(\xi) - ie\hat{n}\hat{A}'(\xi)\right)d\xi\right\}u(k,s), \quad (4.143)$$

and

$$\psi_{s,k}^{(-)}(x) = \exp\left\{ikx + \frac{i}{2(k_y - k_\tau)}\int_0^{nq}\left(-2eA(\xi)k_x + e^2A^2(\xi) + ie\hat{n}\hat{A}'(\xi)\right)d\xi\right\}\upsilon(k,s). \quad (4.144)$$

which satisfy the $2D$ massless Dirac equation

$$i\frac{\partial\psi}{\partial t} = \upsilon_F\sigma.(k - eA(x,y,t))\psi. \quad (4.145)$$

This result agrees exactly with that of Refs. [149] and with that obtained in the third chapter of this thesis.

4.5 The causal Green function for Dirac-graphene quasiparticles in interaction with two plane wave fields

The main goal of this section of the chapter is to calculate the relative causal Green's function for graphene quasiparicles in interaction with orthogonal two plane wave fields described by the

4-vector waves n_1^μ and n_2^μ with components $n_1^\mu = n_2^\mu = (1, 0, 1)$, via path integral formalism. For this choice, n_1^μ and n_2^μ verify the properties $n_1^\mu n_{1\mu} = n_2^\mu n_{2\mu} = 0$ and also $n_1^\mu n_{2\mu} = 0$. After that, we deduce the corresponding wave function.

4.5.1 The evaluation of Green's function

According to the previous section, the relative causal Green's function $S^c(q_b^\mu, q_a^\mu)$ (setting the natural unit $c = \hbar = 1$) has the following expression

$$S^c(q_b^\mu, q_a^\mu) = \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right) \int_0^\infty d\lambda \int d\chi$$
$$\times \int Dq \int D\bar{k} \int \mathcal{D}\psi \exp\left\{ i \int_0^1 ds \left[\bar{k}\dot{q} + \lambda\left(k^2 - 2e\left(A_1(\xi_1) + A_2(\xi_2)\right) k_x\right.\right.\right.$$
$$\left. + e^2 \left(A_1(\xi_1) + A_2(\xi_2)\right)^2 + 2ie\lambda\left(F_{1\mu\nu}(\xi_1) + F_{2\mu\nu}(\xi_2)\right) \quad ^{\mu} \quad ^{\nu}\right)$$
$$\left. + \left[k_\tau\psi^0 + \left(k_x - e\left(A_1(\xi_1) + A_2(\xi_2)\right)\right)\psi^1 + k_y\psi^2\right]\chi - i\psi_\mu\dot{\psi}^\mu\right] + \psi_\mu(1)\psi^\mu(0)\Big\}_{\theta=0}. \quad (4.146)$$

Where $\left(\bar{k} = (k_\tau, k_y)\right)$ and $(q = (\tau, y))$. and the electromagnetic field tensor $F_{\mu\nu}$ is then defined as

$$F_{1\mu\nu}(\xi_1)\gamma^\mu\gamma^\nu = 2A_1'(\xi)\left(\dot{\xi}_\tau\gamma^0\gamma^1 - \xi_y'\gamma^1\gamma^2\right), \quad (4.147)$$
$$F_{2\mu\nu}(\xi_1)\gamma^\mu\gamma^\nu = 2A_2'(\xi)\left(\dot{\xi}_\tau\gamma^0\gamma^1 - \xi_y'\gamma^1\gamma^2\right). \quad (4.148)$$

with $\dot{\xi}_\tau = \frac{\partial\xi}{\partial\tau}$ and $\xi_y' = \frac{\partial\xi}{\partial y}$.

At this stage, it is useful to introduce two new variables ξ_1 and ξ_2 that consider the plane wave variables $n_1 q$ and $n_2 q$ respectively, and are independent from the quadriposition q. So, we use the following identity

$$\prod_{i=1}^{2} \int d\xi_{ib} d\xi_{ia} \delta\left(\xi_{ia} - n_i q_{\mu_a}\right) \int D\xi_i \int Dk_{\xi_i} \exp\left[ik_{\xi_i}\left(\dot{\xi}_i - n_i\dot{q}\right)\right] = 1. \quad (4.149)$$

We suggest the introduction of the two new Grassmannian variables $\eta_1 = n_1\psi$ and $\eta_2 = n_2$ as independent variables from ψ via the following identities

$$\prod_{i=1}^{2} \int d\eta_{ib} d\eta_{ia} \delta\left(\eta_{ia} - n_i^\mu \psi_{\mu_a}\right) \int D\eta_i Dk_{\eta_i} \exp\left[ik_{\eta_i}\left(\dot{\eta}_i - n_i^\mu\dot{\psi}_\mu\right)\right] = 1. \quad (4.150)$$

Also, using this replacement, which represents the coupling term of the two electromagnetic plane wave fields with the spin variables, is defined as

$$\left(F_{1\mu\nu}\left(\xi_1\right)+F_{2\mu\nu}\left(\xi_2\right)\right)\psi^{\mu}\psi^{\nu}=2\left(\eta_i\right)\left(A_i'\left(\xi_i\right)\psi\right). \tag{4.151}$$

By inserting the Eqs. (4.149), (4.150) and (4.151) in the Eq (4.146) and shifting the momentum k by $k+n_i k_{\xi i}$ taking into account the properties $n_i A\left(\xi_i\right)=0$, $n_1^{\mu}n_{1\mu}=n_2^{\mu}n_{2\mu}=0$, and $n_1^{\mu}n_{2\mu}=0$, the causal Green function expression is then simplified to

$$\begin{aligned}
S^c\left(q_b^{\mu},q_a^{\mu}\right)&=\exp\left(i\gamma^{\mu}\frac{\delta_l}{\delta\theta^{\mu}}\right)\int_0^{\infty}d\lambda\int d\chi\\
&\times\prod_{i=1}^{2}\int d\xi_{ib}d\xi_{ia}\delta\left(\xi_{ia}-n_i^{\mu}q_{\mu_a}\right)\int D\xi_i\int Dk_{\xi_i}\int d\eta_{ib}d\eta_{ia}\delta\left(\eta_{i_a}-n_i\psi_a\right)\\
&\times\int Dq\int D\bar{k}\int D\eta_i\int Dk_{\eta_i}\int\mathcal{D}\psi\exp\Big\{i\int_0^1 ds\,\Big[\left(\bar{k}+n_ik_{\xi i}\right)\dot{q}+k_{\xi_i}\left(\dot{\xi}_i-n_i^{\mu}\dot{q}_{\mu}\right)\\
&+k_{\eta_i}\left(\dot{\eta}_i-n_{i\mu}\dot{\psi}^{\mu}\right)+\lambda\left(k^2+2k_{\xi_i}(nk)-2e\left(A_1\left(\xi_1\right)+A_2\left(\xi_2\right)\right)k_x\right.\\
&\left.+e^2\left(A_1\left(\xi_1\right)+A_2\left(\xi_2\right)\right)^2\right)+4ie\lambda\eta_i\left(A'\left(\xi_1\right)+A'\left(\xi_2\right)\right)\psi\\
&-2i\left(k_{\tau}\psi^0+\left(k_x-e\left(A_1\left(\xi_1\right)+A_2\left(\xi_2\right)\right)\right)\psi^1+k_y\psi^2+n_{i\mu}k_{\xi i}\psi^{\mu}\right)\chi\\
&-i\psi_n\dot{\psi}^n\Big]+\psi_n(1)\psi^n(0)\Big\}_{\theta=0}.
\end{aligned} \tag{4.152}$$

Performing the integration over the $q-$variables. This integration gives us the delta Dirac functions, which implies that the momentum (k_{τ},k_y)-momentum is conserved. After this, Eq. (6.26) the Green function will take the following form

$$\begin{aligned}
S^c\left(q_b^{\mu},q_a^{\mu}\right)&=\exp\left(i\gamma^{\mu}\frac{\delta_l}{\delta\theta^{\mu}}\right)\int_0^{\infty}d\lambda\int d\chi\int\frac{dk_{\tau}}{(2\pi)}\frac{dk_y}{(2\pi)}e^{i\bar{k}(q_b-q_a)+i\lambda k^2}\\
&\times\prod_{i=1}^{2}\int d\xi_{ib}d\xi_{ia}\delta\left(\xi_{ia}-n_i^{\mu}q_{\mu_a}\right)\int D\xi_i\int Dk_{\xi_i}\int d\eta_{ib}d\eta_{ia}\delta\left(\eta_{ia}-n_i^{\mu}\psi_{\mu_a}\right)\\
&\times\int D\eta_i\int Dk_{\eta_i}\int\mathcal{D}\psi\exp\Big\{i\int_0^1 ds\,\Big[k_{\xi_i}\left(\dot{\xi}_i+2\lambda nk-2i\eta_i\chi\right)+k_{\eta_i}\left(\dot{\eta}_i-n_i^{\mu}\dot{\psi}_{\mu}\right)\\
&+\lambda\left(-2e\left(A_1\left(\xi_1\right)+A_2\left(\xi_2\right)\right)k_x+e^2\left(A_1\left(\xi_1\right)+A_2\left(\xi_2\right)\right)^2\right)+4ie\lambda\eta_i\left(A'\left(\xi_1\right)+A'\left(\xi_2\right)\right)\psi\\
&-2i\left(k_{\tau}\psi^0+\left(k_x-e\left(A_1\left(\xi_1\right)+A_2\left(\xi_2\right)\right)\right)\psi^1+k_y\psi^2\right)\chi-i\psi_{\mu}\dot{\psi}^{\mu}\Big]+\psi_{\mu}(1)\psi^{\mu}(0)\Big\}_{\theta=0}.
\end{aligned} \tag{4.153}$$

Next, we perform the functional integration over Grassmannian odd variables $\psi(s)$, which requires the introduction of the variables $\omega^{\mu}-$velocity defined in Eq. (4.96). By making the following shift

$$\omega^{\mu}(s)\longrightarrow\omega^{\mu}(s)+in_i^{\mu}\int_0^1\varepsilon^{-1}\left(s-s'\right)k_{\eta_i}(s')ds'. \tag{4.154}$$

After some calculations, the causal Green function takes the following form

$$S^c\left(q_b^\mu, q_a^\mu\right) = \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right)\int_0^\infty d\lambda \int d\chi \int \frac{d^3k}{(2\pi)^3} e^{i\bar{k}(q_b-q_a)+i\lambda k^2}$$
$$\times \prod_{i=1}^{2}\int d\xi_{ib}d\xi_{ia}\delta\left(\xi_{ia}-n_iq_a\right)\int D\xi_i \int Dk_{\xi_i}\int d\eta_b d\eta_a \int dk_{\eta_a}\int D\eta_i \int Dk_{\eta_i}$$
$$\times \exp\left\{i\int_0^1 ds\left[\lambda\left(-2e\left(A_{\xi_1}+A_{\xi_2}\right)k_x + e^2\left(A_{\xi_1}+A_{\xi_2}\right)^2\right)\right.\right.$$
$$+k_{\xi_i}\left(\dot{\xi}_i + 2\lambda k^\mu n_{\mu i} - 2i\eta_i\chi\right) + k_{\eta_i}\left(\dot{\eta}_i + n_i^\mu k_\mu \chi\right) + k_{\eta_{ia}}\left(\eta_{ia} - \frac{1}{2}n_i^\mu \theta_\mu\right)$$
$$\left.\left.-i\left(\left(k_x - e\left(A_{\xi_1}+A_{\xi_2}\right)\right)\theta^1 + k_\mu\theta^\mu\right)\chi + 2ie\lambda\eta_i\left(A'_{\xi_1}+A'_{\xi_2}\right)\theta\right]\right\} I\left(\xi_i,\eta_i\right)|_{\theta=0}\,. \tag{4.155}$$

Where

$$I\left(\xi_i,\eta_i\right) = \int \mathcal{D}\omega \exp\left\{\int_0^1\left[-\frac{1}{2}\omega^\mu \mathcal{M}_{\mu\nu}\omega^\nu + \mathcal{J}_\mu\omega^\mu\right]ds\right\}, \tag{4.156}$$

with $\mathcal{M}_{\mu\nu}(s,s') = g_{\mu\nu}\varepsilon\left(s-s'\right)$ and the external current sources $\mathcal{J}_\mu$ has the following expression

$$\mathcal{J}_\mu = -\chi\int_0^1\left(k_\mu + eA_{i\mu}\left(\xi_i\left(\tau'\right)\right)\right)\varepsilon\left(\tau'-\tau\right)d\tau'$$
$$-2e\lambda\int_0^1 \eta_i\left(\tau'\right)A'_i\left(\xi_i\left(\tau'\right)\right)\varepsilon\left(\tau'-\tau\right)d\tau' + \frac{i}{2}k_{\eta_{ia}}n_i^\mu. \tag{4.157}$$

The integration of the spin term can now be easily applied thanks to its Gaussian form, which is simplified to

$$I\left(\xi_i,\eta_i\right) = \exp\left\{\int_0^1 ds\left[-e\lambda\left(k_\mu + e\left(A_{1\mu}+A_{2\mu}\right)\right)\varepsilon\eta\left(A'_1+A'_2\right) - \frac{i}{2}k_\mu n_i^\mu k_{\eta_{ia}}\chi\right.\right.$$
$$\left.\left.+2e^2\lambda^2\eta_i\left(A'_1+A'_2\right)\varepsilon\eta_i\left(A'_1+A'_2\right)\right]\right\}. \tag{4.158}$$

Let us now perform the integration over k_{η_1} and k_{η_2} and over k_{ξ_1} and k_{ξ_2}. These integrations are easy and directly given the following products of the delta functionals

$$\prod_{i=1}^{2}\delta\left(\dot{\eta}_i + n_i^\mu k_\mu\chi\right) \rightarrow \eta_i(s) = \left(\eta_{i_a} - n_i^\mu k_\mu \chi s\right), \tag{4.159}$$

and

$$\prod_{i=1}^{2}\delta\left(\dot{\xi}_i + 2\lambda k^\mu n_{\mu i} - 2i\eta_i\chi\right) \Rightarrow \dot{\xi}_i + 2\lambda k^\mu n_{\mu i} - 2i\eta_i\chi = 0. \tag{4.160}$$

The argument of the delta functional in Eqs. (4.159) and (4.160) gives us the explicit solutions of the classical equations of motion

$$\eta_i(s) = \eta_{ia} - n_i k \chi s, \tag{4.161}$$

and

$$\xi_i(s) = \xi_{ia} - 2\lambda (n_i k) s + 2i\eta_{ia}\chi s. \tag{4.162}$$

By substituting all the previous results in Eqs. (4.155) and after straightforward and long computations. We immediately obtain the expression of Green's function, which is determined by

$$\begin{aligned}
S^c(q_b^\mu, q_a^\mu) = {} & \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right) \int_0^\infty d\lambda \int d\chi \int \frac{d^3k}{(2\pi)^3} e^{i\bar{k}(q_b - q_a) + i\lambda k^2} \\
& \times \prod_{i=1}^{2} \int d\xi_{ib} d\xi_{ia} \delta\left(\xi_{ia} - n_i^\mu q_{\mu_a}\right) \int d\eta_{ib} d\eta_{ia} \\
& \times \delta\left(\eta_{ib} - \eta_{ia} + n_i^\mu k_\mu \chi\right) \delta\left(\xi_{i_b} - \xi_{i_a} + 2\lambda k^\mu n_{i\mu} - 2i\eta_{ia}\chi\right) \\
& \times \int dk_{\eta_{ia}} \exp\left\{ i \int_0^1 ds \left[\lambda\left(-2e\left(A_{\xi_1} + A_{\xi_2}\right) k_x + e^2\left(A_{\xi_1} + A_{\xi_2}\right)^2\right) \right.\right. \\
& + k_{\eta_{ia}}\left(\eta_{ia} - \frac{1}{2} n_i^\mu \theta_\mu - \frac{1}{2} k_\mu n_i^\mu k_{\eta_{ia}} \chi\right) + 2ie\lambda\left(\eta_{ia} - n_i^\mu k_\mu \chi s\right)\left(A'_{\xi_1} + A'_{\xi_2}\right)\theta \\
& - i\left(\left(k_x - e\left(A_{\xi_1} + A_{\xi_2}\right)^2\right)\left(\theta^1 - e\lambda\varepsilon\eta_{ia}\left(A'_{\xi_1} + A'_{\xi_2}\right)\right) + k_\mu \theta^\mu\right)\chi \\
& \left.\left. - 4e^2\lambda^2 n_i^\mu k_\mu \left(A'(\xi_1) + A'(\xi_2)\right)\varepsilon s \eta_{ia}\chi\left(A'(\xi_1) + A'(\xi_2)\right)\right]\right\} |_{\theta=0} .
\end{aligned} \tag{4.163}$$

Let us now integrate over $k_{\eta_{ia}}$, which gives the delta functional $\delta\left(\eta_{ia} - \frac{1}{2} n_i^\mu \theta_\mu - \frac{1}{2} k_\mu n_i^\mu k_{\eta_{ia}} \chi\right)$. From this, we get the constraints

$$\eta_{ia} = \frac{1}{2} n_i^\mu \theta_\mu + \frac{1}{2} k_\mu n_i^\mu \chi, \tag{4.164}$$

and

$$\eta_{ib} = \eta_{ia} - n_i^\mu k_\mu \chi, \tag{4.165}$$

it is easy to show that the antiperiodic boundary condition on the spin variables $\eta_{ib} + \eta_{ia} = n_i^\mu \theta_\mu$ is satisfied and conserved.

$$\eta_{ib} + \eta_{ia} = n_i^\mu \theta_\mu. \tag{4.166}$$

Using the above equations in the Eqs. (4.163) and by performing the integration over η_a and η_b. After this, the propagator is reduced to the following form

$$S^c\left(x_b^\mu, x_a^\mu\right) = \exp\left(i\gamma^\mu \frac{\delta_l}{\delta\theta^\mu}\right)\int_0^\infty d\lambda \int d\chi \int \frac{d^3k}{(2\pi)^3} e^{ik(x_b - x_a)+i\lambda k^2}$$
$$\times \prod_{i=1}^{2}\int d\xi_{ib} d\xi_{ia}\delta\left(\xi_{ia} - n_i^\mu q_{\mu_a}\right)\delta\left(\xi_{i_b} - \xi_{i_a} + \lambda k^\mu n_{i\mu} - i\eta_{ia}\chi\right)$$
$$\exp\left\{i\int_0^1 ds\left[\lambda\left(-2e\left(A_1(\xi_1) + A_2(\xi_2)\right)k_x + e^2\left(A_1(\xi_1) + A_2(\xi_2)\right)^2\right)\right.\right.$$
$$+ie\lambda\left(-2n_i^\mu k_\mu \chi s + n_i^\mu \theta_\mu + k_\mu n_i^\mu \chi\right)\left(A'(\xi_1) + A'(\xi_2)\right)\theta$$
$$-i\left(\left(k_x - e\left(A_1(\xi_1) + A_2(\xi_2)\right)\right)\left(\theta^1 - e\lambda\frac{n_i^\mu\theta_\mu}{2}\varepsilon\left(A'(\xi_1) + A'(\xi_2)\right)\right) + k_\mu\theta^\mu\right)\chi$$
$$\left.\left. -2ie^2\lambda^2 n_i^\mu k_\mu\left(A'(\xi_1) + A'(\xi_2)\right)\varepsilon s\left(A'(\xi_1) + A'(\xi_2)\right)n_i^\mu\theta_\mu\chi\right]\right\}|_{\theta=0}\,. \tag{4.167}$$

Replacing the Delta functional $\delta\left(\xi_{ib} - \xi_{ia} + 2\lambda k^\mu n_{i\mu} - in_i^\mu\theta_\mu\chi\right)$ by its integral representation

$$\delta\left(\xi_{ib} - \xi_{ia} + 2\lambda k^\mu n_{i\mu} - in_i^\mu\theta_\mu\chi\right) = \int \frac{dk_{\xi_{ib}}}{2\pi}\exp\left\{ik_{\xi_{ib}}\left(\xi_{ib} - \xi_{ia} + 2\lambda k^\mu n_{i\mu} - in_i^\mu\theta_\mu\chi\right)\right\}, \tag{4.168}$$

and change the momentum k^μ by $k^\mu - n_i^\mu k_{\xi_{ib}}$ and after that, integrating over the Grassmann proper time χ, the result simplifies to

$$S^c\left(x_b^\mu, x_a^\mu\right) = \exp\left(i\gamma^\mu\frac{\delta_l}{\delta\theta^\mu}\right)\int_0^\infty d\lambda\int\frac{d^3k}{(2\pi)^3}$$
$$\times\left[k\theta\left[1 + \frac{e}{2k^\mu n_{i\mu}}n_i^\mu\theta_\mu\left(A_{ib} - A_{ia}\right)\theta\right] - \frac{e}{4}\left(A_{ib} + A_{ia}\right)\theta\right.$$
$$\left.\left. -\frac{e}{4k^\mu n_{i\mu}}\left(k_x\left(A_{ib} + A_{ia}\right) + eA_{ia}A_{ib}\right)n_i^\mu\theta_\mu - \frac{e}{8k^\mu n_{i\mu}}\left(A_{ib} + A_{ia}\right)\theta\left(n_i^\mu\theta_\mu\right)\left(A_{ib} - A_{ia}\right)\theta\right]\right|_{\theta=0}$$
$$\exp\left\{ik\left(x_b - x_a\right) + i\lambda k^2 + \frac{i}{2k^\mu n_{i\mu}}\int_{\xi_{ia}}^{\xi_{ib}}\left(-2eA_i(\xi_i)k_x + e^2\left(A_i(\xi_i)\right)^2\right)d\xi_i\right\}. \tag{4.169}$$

At this stage, we perform the differentiation with respect to θ by using the identities (4.130) and (4.131). The expression of the causal Green's function is rewritten as

$$S^c\left(x_b^\mu, x_a^\mu\right) = \int\frac{d^2k}{(2\pi)^2}\int_0^\infty d\lambda\prod_{i=1}^{2}\left\{\hat{k}\left[1 - \frac{e}{2k^\mu n_{i\mu}}\hat{n}_i\left(\hat{A}_{ib} - \hat{A}_{ia}\right)\right]\right.$$
$$\left. -e\hat{A}_{ib} + \frac{e^2}{2k^\mu n_\mu}\hat{n}_i\hat{A}_{ia}\hat{A}_{ib} + \frac{e}{k^\mu n_{i\mu}}\hat{n}_i k A_{ib} - \frac{e^2}{k^\mu n_{i\mu}}\hat{n}A_{ia}A_{ib}\right\}$$
$$\times\exp\left\{ik\left(x_b - x_a\right) + i\lambda k^2 + \frac{i}{2k^\mu n_{i\mu}}\int_{\xi_{ia}}^{\xi_{ib}}\left(-2e\left(A_i(\xi_i)\right)k_x + e^2\left(A_i(\xi_i)\right)^2\right)d\xi_i\right\}. \tag{4.170}$$

At the end, where we used the relation $\hat{A}\hat{B} + \hat{B}\hat{A} = 2AB$ with $\hat{A} = A_\mu \gamma^\mu$, taking into account the properties $n^2 = 0$ and $n_i A_i = 0$, and after performing the integration over the bosonic proper time λ, the Green function related to the graphene's quasiparticles in interaction with two electromagnetic plane wave fields is given as

$$S^c\left(x_b^\mu, x_a^\mu\right) = -\int \frac{d^3k}{(2\pi)^3} \prod_{i=1}^{2} \left[1 + \frac{e}{2k^\mu n_{i\mu}} \hat{n}_i \hat{A}_{ib}\right] \frac{\hat{k}}{k^2 + i\varepsilon} \prod_{i=1}^{2} \left[1 - \frac{e}{2k^\mu n_{i\mu}} \hat{n}_i \hat{A}_{ia}\right]$$
$$\times \prod_{i=1}^{2} \exp\left\{ik\left(x_b - x_a\right) + \frac{i}{2k^\mu n_{i\mu}} \int_{\xi_{ia}}^{\xi_{ib}} \left(-2eA_i\left(\xi_i\right) k_x + e^2 \left(A_i\left(\xi_i\right)\right)^2\right) d\xi_i\right\}. \tag{4.171}$$

4.5.2 The derivation of the wave function from Green's function

The suitably normalized wave functions that describe the motion of the graphene quasiparticles in interaction with two electromagnetic plane wave fields are given by

$$^{(+)}_{s,k}(x) = \prod_{i=1}^{2} \exp\left\{-ikx + \frac{i}{2\left(k_\tau - k_y\right)} \int_0^{n_i q} \left(-2e\left(A_i\left(\xi_i\right)\right) k_x + e^2 \left(A_i\left(\xi_i\right)\right)^2\right) d\xi_i\right\}$$
$$\times \left[1 + \frac{e}{2k^\mu n_{i\mu}} \hat{n}_i \hat{A}_i\right] u(k,s), \tag{4.172}$$

and

$$\psi^{(-)}_{s,k}(x) = \prod_{i=1}^{2} \exp\left\{ikx + \frac{i}{2\left(k_\tau - k_y\right)} \int_0^{n_i q} \left(-2e\left(A_i\left(\xi_i\right)\right) k_x + e^2 \left(A_i\left(\xi_i\right)\right)^2\right) d\xi_i\right\}$$
$$\times \left[1 - \frac{e}{2k^\mu n_{\mu}} \hat{n}_i \hat{A}_i\right] \upsilon(k,s). \tag{4.173}$$

The solutions of a complete and orthonormal system relative can be rewritten as

$$^{(+)}_{s,k}(x) = \exp\left[\sum_i \left\{-ikx + \frac{i}{2\left(k_\tau - k_y\right)} \int_0^{n_i q} \left(-2e\left(A_i\left(\xi_i\right)\right) k_x + e^2 \left(A_i\left(\xi_i\right)\right)^2 - ie\hat{n}_i \hat{A}_i'(\xi)\right) d\xi_i\right\}\right] u(k, \tag{4.174}$$

and

$$\psi^{(-)}_{s,k}(x) = \exp\left[\sum_i \left\{ikx + \frac{i}{2\left(k_\tau - k_y\right)} \int_0^{n_i q} \left(-2e\left(A_i\left(\xi_i\right)\right) k_x + e^2 \left(A_i\left(\xi_i\right)\right)^2 + ie\hat{n}_i \hat{A}_i'(\xi)\right) d\xi_i\right\}\right] \upsilon(k,s) \tag{4.175}$$

This result agrees exactly with that obtained in the third chapter by using Volkov's method.

4.6 Conclusion

In this chapter, we have calculated the causal Green's function of graphene quasiparticles via the supersymmetric path integral formalism. First, in the free case, then for the quasiparticles in interaction with one electromagnetic plane wave field and with two electromagnetic plane wave fields. Finally, the wave functions are exactly deduced and presented for the three considered cases. The results obtained from the path integral formalism are identical to the results obtained via Volkov's method in the previous section. These results give us all information about the behavior of graphene's quasiparticles which helps us study the electronic, magnetic, and nonlinear properties of graphene at high energy.

If the angle between the two waves is so small $\left(\hat{\theta} \ll 1\right)$ (see Ref. [157]), in the future it can help us solve many problems, such as, for example, the Schwinger effect and the Compton effect..... etc.

Chapter 5

Schwinger pair production in monolayer graphene under the action of a constant electromagnetic field and in non-commutative phase space coordinates

5.1 Introduction

At first, the Schwinger effect was studied for a constant electric field [37]. More generally, the issue of pair creation was studied for various configurations of fields, such as the electromagnetic field. For example, this problem was treated in detail in Ref. [158] in the presence of a constant electromagnetic field for both scalar and spinorial particles using the Bogoliubov transformation method.

Furthermore, in Ref. [39] the author used Schwinger's method for calculating the effective action and the pair production probability for both scalar and spinorial relativistic particles in the presence of a constant electromagnetic field plus a volkov plane wave. He shown that the results for scalar and spinning particles are different by the spin factor, and he deduced that the plane wave has no influence on the process of pair creation.

On the other hand, it is well known that the amazing properties of graphene material

[1, 2, 4, 161] allow to study the quantum electrodynamics effects of the strong fields [12, 13]. In recent years, a series of studies concerning the Schwinger effect have been conducted through a strong external field in graphene, because pair creation is essential for investigating this system. Moreover, the Schwinger effect was studied also for monolayer graphene under the action of a strong electric field [162, 163, 164] and in the presence of an electric current [165], for bilayer graphene in anisotropic QED [166] and for multilayer graphene by several researchers [94]. Also, the Schwinger effect was treated for the time-dependent Schwinger mechanism in Ref. [167]. In the same context, in Ref. [94] the probability of pair production for a constant electric field was calculated using the semi-classical approach for multilayer graphene also via an exact solution of the Schrödinger equation for the case of monolayer graphene was obtained.

There are some researches in the literature that study the influence of the pair creation problem on the NC space coordinates developed by Chikh Jabbari in Ref. [95], the author calculated the effective action and deduced the rate and the pair creation probability for both scalar and spinorial relativistic particles in the presence of an electromagnetic field in non-commutative space coordinate considering Schwinger's method.

In this chapter, we apply the Schwinger method in NC phase space coordinates for Dirac-graphene quasiparticles in interaction with a constant electromagnetic field via path integral formalism using the method of Fradkin and Gitman [105, 168].

We consider two gauges of quadri-vector potential in NC phase space. For calculating the effective action and the pair creation probability, we assume that the direction of the magnetic field $\vec{\mathcal{B}}$ is along the $z-$axis, $\vec{\mathcal{B}} = \mathcal{B}\vec{k}$ and electric field is along the $y-$axis, $\vec{\mathcal{E}} = \mathcal{E}\vec{j}$. Thus, our aim is to formulate the effective action under these two gauges in NC phase space coordinates. The first one is the Landau gauge, defined by

$$A_\mu = (0, -\frac{\mathcal{B}}{2}\mathcal{X}_2, \frac{\mathcal{B}}{2}\mathcal{X}_1 + \mathcal{E}t), \tag{5.1}$$

and the second is

$$A_\mu = (-\mathcal{E}\mathcal{X}_2, -\frac{\mathcal{B}}{2}\mathcal{X}_2, \frac{\mathcal{B}}{2}\mathcal{X}_1), \tag{5.2}$$

where $\mathcal{X}_1 = x - \frac{\theta}{2}p_y$ and $\mathcal{X}_2 = y + \frac{\theta}{2}p_x$.

We made the corresponding Lagrangian function to calculate Green's function, and after that, we formulate the corresponding effective action for calculating the pair production probability. At the end, we discuss the results for special cases. On the other hand, we study the influence of the plane wave on the process of particle-antiparticle pair creation.

5.2 The formal method of effective action

5.2.1 Dirac-Graphene equation for quasiparticles in interaction with an electromagnetic field in non-commutative phase-space coordinates

The massless Dirac equation on this NC phase space can be written as

$$\imath\gamma^{\mu}\left(p_{\mu}-eA_{\mu}(x)\right)\star\psi\left(x\right)=0. \tag{5.3}$$

Where the $\star$ represents the star product, it has been used to incorporate the non-commutativity between the coordinates and between the momentum operators.

Following Refs. [169, 170], the Moyal star product is defined as

$$\begin{aligned}\left(f\star g\right)(x,p) &= e^{\frac{\imath}{2}\theta_{ij}\partial_i^x\partial_j^x+\frac{\imath}{2}\eta_{ij}\partial_i^p\partial_j^p}f(x,p)g(x,p)\\ &= f(x,p)g(x,p)+\frac{\imath}{2}\theta_{ij}\partial_i^x f\partial_j^x g\mid_{x_i=x_j}+\frac{\imath}{2}\eta_{ij}\partial_i^p f\partial_j^p g\mid_{p_i=p_j}+\mathcal{O}\left(\Theta^2\right),\end{aligned} \tag{5.4}$$

where $f(x,p)$ and $g(x,p)$ are two arbitrary functions and $\mathcal{O}\left(\Theta^2\right)$ indicates the higher order terms of $(\theta,\,\eta)$, then by using this formula (5.4), we return to the usual product.

As a result, the corresponding massless Dirac equation for monolayer graphene on NC phase space coordinates (5.3) will be simplified as

$$\hat{O}^{\star}_{Graph}\psi(x)=0, \tag{5.5}$$

where the Dirac-graphene operator of the monolayer graphene on NC phase space is defined as

$$\hat{O}^{\star}_{Graph}=\imath\gamma^{\mu}\mathcal{D}_{\mu}=\imath\gamma^{\mu}\left(\hat{\mathcal{P}}_{\mu}-eA_{\mu}(\hat{\mathcal{X}})\right),\quad \mu=0,1,2. \tag{5.6}$$

While $A_{\mu}(\hat{\mathcal{X}})$ represents the quadri-vector potential of a constant electromagnetic field and γ^{μ} are Dirac matrices, in $(2+1)$-dimensions are represented by the Pauli matrices as follows

$$\gamma^0=\sigma_3,\;\gamma^1=\imath\sigma_2,\;\gamma^2=-\imath\sigma_1. \tag{5.7}$$

By applying the Bopp shift transformation [171, 172, 173, 169, 170, 174, 175], the non-commuting coordinates $\left(\hat{\mathcal{X}}^{\mu},\hat{\mathcal{P}}^{\mu}\right)$ can be expressed in terms of the commuting coordinates $(\hat{x}_i,\hat{p}_i)$. Where the time component rests unchanged in the following way [122]

$$\hat{\mathcal{X}}_0 = x_0 \equiv t,\ \hat{\mathcal{P}}_0 = \partial_0 = \imath\partial/\partial\tau,\ \hat{\mathcal{X}}_i = \hat{x}_i - \frac{\theta_{ij}}{2}\hat{p}_j,\ \hat{\mathcal{P}}_i = \hat{p}_i + \frac{\eta_{ij}}{2}\hat{x}_j, i = 1, 2. \tag{5.8}$$

Where $\tau = \upsilon_F t$, $\upsilon_F = (1.12 \pm 0.02) \times 10^6\ m/s$ is the Fermi velocity in graphene and θ_{ij}, η_{ij} are the results of non-commutativity in plane (x, y) are defined as

$$\theta_{ij} = \theta\varepsilon_{ij}\ ,\ \eta_{ij} = \eta\varepsilon_{ij}. \tag{5.9}$$

Here the parameter ε_{ij} is the Levi-Civita symbol [176], and θ, η are the parameters of the deformation.

The operators $\left(\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}^\mu\right)$ satisfy the below commutation relations

$$\left[\hat{\mathcal{P}}_i, \hat{\mathcal{P}}_j\right] = \imath\eta_{ij},\ \ \left[\hat{\mathcal{X}}_i, \hat{\mathcal{X}}_j\right] = \imath\theta_{ij},\ \ \left[\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_j\right] = \imath\delta_{ij},\ \ i, j = 1, 2. \tag{5.10}$$

While the operators $(\hat{x}_i, \hat{p}_i)$ are new variables that satisfy the usual canonical commutation relations,

$$[\hat{x}_i, \hat{x}_j] = 0,\ \ [\hat{p}_i, \hat{p}_j] = 0,\ \ [\hat{p}_i, \hat{x}_j] = i\delta_{ij}. \tag{5.11}$$

5.2.2 The vacuum-vacuum transition amplitude $\mathcal{A}\,(vac - vac)$

In quantum field theory (QFT), the vacuum-vacuum transition amplitude for spinning particles that existed at the point (x_i, t_i) to be found at the point (x_f, t_f) is written as a functional integral over all Grassmann field configurations $\psi(x)$ and $\bar{\psi}(x)$ which is clarified in the papers [37, 177], given by

$$\begin{aligned} \mathcal{A} &= \langle 0_{out} | 0_{in} \rangle, & (5.12)\\ &= \int D\psi D\bar{\psi} \exp\left[\imath \int d^3x \mathcal{L}_{Graph}\right] & (5.13)\\ &= \exp\left[\imath S^{(\mathrm{NC})}_{eff.}\right], & (5.14)\end{aligned}$$

where $\mathcal{L}_{Graph}$ is the Lagrangian density.

In NC phase space coordinates, the Lagrangian density is defined as

$$\mathcal{L}_{Graph} = \bar{\psi}\hat{O}^{\star}_{Graph}\psi, \tag{5.15}$$

with $\hat{O}^{\star}_{Graph}$ representing the massless Dirac-graphene electron operator in NC phace space coordinates, that is defined as

$$\hat{O}^{\star}_{Graph} = \imath\gamma^\mu\left(\hat{\mathcal{P}}_\mu - eA_\mu(\hat{\mathcal{X}}_\mu)\right),\ \ \mu = 0, 1, 2. \tag{5.16}$$

The functional integral (5.13) is Gaussian, then the vacuum-vacuum transition amplitude has the following form

$$A_{spin.} = \det\left(\hat{O}^{\star}_{Graph}\right). \tag{5.17}$$

In order to evaluate the determinant $\det\left(\hat{O}^{\star}_{Graph}\right)$, we use the following formula

$$\det \hat{A} = \det \hat{A}^{\dagger} = \Pi_i \lambda_i = \det\left[\hat{A}\hat{A}^{\dagger}\right]^{1/2}, \tag{5.18}$$

where $\hat{A}^{\dagger}$ is the conjugate of the operator $\hat{A}^{\dagger}$. Notice that the eigenvalues of $\hat{A}$ and $\hat{A}^{\dagger}$ are conjugate. Then we can write

$$\begin{aligned} A_{spin.} &\approx \det\left[\hat{O}^{\star}_{Graph}\hat{O}^{\star\dagger}_{Graph}\right]^{1/2}, & (5.19)\\ &\approx \det\left[\hat{O}^{\star}_{KG} - \frac{e}{2}\sigma^{\mu\nu}\mathcal{F}^{\star}_{\mu\nu}\right]^{1/2}. & (5.20)\end{aligned}$$

Here $\hat{O}^{\star}_{KG}$ represents the Klein-Gordon operator, which is defined by

$$\hat{O}^{\star}_{KG} = \left(\hat{\mathcal{P}}_{\mu} - eA_{\mu}(\hat{\mathcal{X}})\right)\left(\hat{\mathcal{P}}^{\mu} - eA^{\mu}(\hat{\mathcal{X}})\right), \tag{5.21}$$

and $\sigma^{\mu\nu} = \frac{\imath}{2}\gamma^{\mu}\gamma^{\nu}$ is the spin tensor and $\mathcal{F}^{\star}_{\mu\nu}$ is the strength antisymmetric tensor, of a gauge field related to noncommutative geometry given by

$$\mathcal{F}^{\star}_{\mu\nu} = \partial_{\mu}\mathcal{A}_{\nu} - \partial_{\nu}\mathcal{A}_{\mu} + \imath e\left[\mathcal{A}_{\mu}, \mathcal{A}_{\nu}\right]_{\star}. \tag{5.22}$$

Form [177] we have the formula

$$\begin{aligned} \det\left[\hat{O}^{\star}_{KG} - \frac{e}{2}\sigma^{\mu\nu}\mathcal{F}^{\star}_{\mu\nu}\right] &= \exp\left[\ln\left[\hat{O}^{\star}_{KG} - \frac{e}{2}\sigma^{\mu\nu}\mathcal{F}^{\star}_{\mu\nu}\right]\right] & (5.23)\\ &= \exp\left[\frac{1}{2}Tr\left(\ln\left[\hat{O}^{\star}_{KG} - \frac{e}{2}\sigma^{\mu\nu}\mathcal{F}^{\star}_{\mu\nu}\right]\right)\right]. & (5.24)\end{aligned}$$

Using the representation

$$\ln \hat{A} = Cst - \int_0^{\infty}\frac{d\lambda}{\lambda}\exp\left[-\imath\lambda\hat{A}\right], \tag{5.25}$$

we obtain

$$\ln\left[\hat{O}^{\star}_{KG} - \frac{e}{2}\sigma^{\mu\nu}\mathcal{F}^{\star}_{\mu\nu}\right] = Cst - \int_0^{\infty}\frac{d\lambda}{\lambda}\exp\left[-\imath\lambda\left[\hat{O}^{\star}_{KG} - \frac{e}{2}\sigma^{\mu\nu}\mathcal{F}^{\star}_{\mu\nu}\right]\right]. \tag{5.26}$$

5.2.3 The Schwinger effective action

Formally, the effective action expression is given as [158]

$$S_{eff.}^{(\mathrm{NC})} = \int_0^\infty \frac{d\lambda}{\lambda} Tr\ \mathcal{G}^{(\mathrm{NC})}\left(x_b^\mu, x_a^\mu, \lambda\right), \tag{5.27}$$

Here "Tr" indicates the complete diagonal summation over the continuous space-time coordinates and $\gamma-$matrices ($Tr = Tr\gamma Trx$).

It is obvious that the determination of $S_{eff.}^{(\mathrm{NC})}$ requires the knowledge of the kernel propagator defined by

$$\mathcal{G}^{(\mathrm{NC})}\left(x_b^\mu, x_a^\mu, \lambda\right) = -\imath\mathbb{T}\left\langle x_b\right| \exp\left\{-\imath\lambda\left[\hat{\mathcal{H}}\left(\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}^\mu\right)\right]\right\}\left|x_a\right\rangle, \tag{5.28}$$

where

$$\hat{\mathcal{H}}\left(\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}^\mu\right) = \hat{O}_{KG}^\star\left(\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}^\mu\right) - \frac{e}{2}\sigma^{\mu\nu}\mathcal{F}_{\mu\nu}^\star\left(\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}^\mu\right). \tag{5.29}$$

Whilst "$\mathbb{T}$" represents the time ordering operator that affects on the phase relative to the coupling term, which ordered the x, p and $\gamma-$matrices.

We understand from Eq. (5.27), that the kernel propagator calculation helps us in determining the effective action as well as the probability of particle creation. To do this, we follow the standard discretization method for the kernel (5.28) and write as usual $\exp\left(-\frac{\imath}{h}\lambda_0\hat{\mathcal{H}}\right) = \left[\exp\left(-\frac{\imath}{h}\varepsilon\hat{\mathcal{H}}\right)\right]^{N+1}$, with $\varepsilon = \lambda_0/(N+1)$, and then insert N identities $\int\left|x_\mu\right\rangle\left\langle x_\mu\right| dx_\mu = 1$ and $(N+1)$ times the identities $\int\left|p_\mu\right\rangle\left\langle p_\mu\right| dp_\mu = 1$ between all the infinitesimal operators $\exp\left(-\frac{\imath}{h}\varepsilon\hat{\mathcal{H}}\right)$. Therefore, the expression of $\mathcal{G}^{(\mathrm{NC})}\left(x_b, x_a, \lambda\right)$ will be taken as the following Hamiltonian path-integral representation

$$\begin{aligned}\mathcal{G}^{(\mathrm{NC})}\left(x_b^\mu, x_a^\mu, \lambda\right) = \imath \lim_{N\to\infty}\int_0^\infty d\lambda_0 \prod_{k=1}^{N}\int d^3x_{\mu_k}\prod_{k=1}^{N+1}\int\frac{d^3p_{\mu_k}}{2\pi}\exp\left\{i\sum_{k=1}^{N+1}\left[p_{\mu_k}\Delta x_k^\mu\right.\right.\\ \left.\left.+\left[\varepsilon\left[\frac{1}{v_F^2}\left(p_{0k} - eA_0(x_k^0, \mathcal{X}_k^i)\right)^2 - \left(\mathcal{P}_k^i - eA^i(x_k^0, \mathcal{X}_k^i)\right)^2 - \frac{\imath e}{2}\gamma^\mu\gamma^\nu\mathcal{F}_{\mu\nu}^\star\right]\right\}\right],\end{aligned} \tag{5.30}$$

Following the method of Fradkin and Gitman [168], we can write the effective action as follows

$$\begin{aligned}S_{eff.}^{(\mathrm{NC})} = -\imath\int\frac{d\lambda}{\lambda}\int dt_b dx_b dy_b\\ \times\int DtDxDy\int\mathcal{D}\psi\exp\left\{\imath\int_0^\lambda ds\left[L\left(x, \dot{x}, s\right) - \imath e\mathcal{F}_{\mu\nu}^\star\psi^\mu\psi^\nu + \imath\psi\dot{\psi}\right]\right\}_{x_b = x_a},\end{aligned} \tag{5.31}$$

where $L(x,\dot{x},s)$ represents the Lagrangian function associated with a uniform electromagnetic field in NC phase space coordinates, and as usual, we define (Dx, Dy and Dt) as,

$$Dx^{\mu}=\lim_{N\rightarrow\infty}\sqrt{\frac{m_{x^{\mu}}}{2\imath\pi\lambda}}\prod_{k=1}^{N}\left(dx_{k}^{\mu}\sqrt{\frac{m_{x^{\mu}}}{2\imath\pi\lambda}}\right), \tag{5.32}$$

and the measure $\mathcal{D}\psi$ has the following definition

$$\mathcal{D}\psi=D\psi\left[\int D\psi\exp\left\{-i\int_{0}^{\lambda}ds\psi_{\mu}(s)\dot{\psi}^{\mu}(s)\right\}\right]^{-1}, \tag{5.33}$$

where $\psi_{\mu}(s)$ are odd Grassmann variables that obey the boundary condition $\psi(0)+\psi(\lambda)=0$. We note that the difference between the effective action and the kernel propagator is the factor $(1/\lambda)$, with the boundary condition $x^{\mu}(0)=x^{\mu}(\lambda)$.

5.2.4 The pair production probability

The probability of transition vacuum-vacuum amplitude $A_{spin.}$ is defined by

$$\begin{aligned}\mathcal{P}_{vac-vac} &= \left|A_{spin.}(vac-vac)\right|^{2}\\ &= \left|\langle 0_{out}\,|0_{in}\rangle\right|^{2}\\ &= \exp\left(-\int d^{3}x2\,\mathrm{Im}\,\mathcal{L}_{eff}\right)\\ &= \exp\left(-2\,\mathrm{Im}\,S_{eff}\right).\end{aligned} \tag{5.34}$$

Schwinger [37] has shown that the probability of pair creation is the imaginary part of the effective action S_{eff} defined as

$$\begin{aligned}\mathcal{P}_{Creat} &= 1-\mathcal{P}_{vac-vac}\\ &= 1-\exp\left(-\int d^{3}x2\,\mathrm{Im}\,\mathcal{L}_{eff}\right)\\ &\simeq 2\,\mathrm{Im}\,S_{eff}.\end{aligned} \tag{5.35}$$

5.2.5 Gauge invariance

It is well known that, in the case of the commutative phase space (i.e., $\eta=0$ and $\theta=0$), the previous quadri-potentials ($A_{\mu}=(0,-\frac{\mathcal{B}}{2}\mathcal{X}_{2},\frac{\mathcal{B}}{2}\mathcal{X}_{1}+\mathcal{E}t)$ and $A_{\mu}=(-\mathcal{E}\mathcal{X}_{2},-\frac{\mathcal{B}}{2}\mathcal{X}_{2},\frac{\mathcal{B}}{2}\mathcal{X}_{1})$) are connected by this gauge transformation $A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$, where $\Lambda(x)$ denotes the gauge function are physically equivalent (also known as gauge invariance).

However, in the noncommutative case (i.e., $\eta \neq 0$ and $\theta \neq 0$) they are no longer equivalent according to this usual gauge transformation , and as we shall observe in our application concerning the production of pairs. Otherwise, the equation (5.5) is not invariant uder this usual gauge transformation because of the NC phase space. We may deduce, without making any claims, that one should introduce adequate gauge transformations containing the noncommutativity parameters (η, θ) to satisfy a corresponding gauge invariance of the equation (5.5). These gauge transformations would also ensure the invariance of the Maxwell equations corresponding to the fields in this NC phase space.

5.3 The construction of effective action for the first gauge of field

5.3.1 The evaluation of propagator $\mathcal{G}^{(\mathrm{NC})}(x_b, x_a, \lambda)$

In this section, we study the creation of particles from the vacuum in monolayer graphene, under the action of a uniform electromagnetic fields $A_\mu = (0, -\frac{\mathcal{B}}{2}\mathcal{X}_2, \frac{\mathcal{B}}{2}\mathcal{X}_1 + \mathcal{E}t)$, in a non-commutative space coordinate considering Schwinger's method.

For constructing the effective action, we must calculate the Green function via the supersymmetric path integral formalism. It is convenient to write the Hamiltonian $\hat{O}^{\star}_{Graph}\left(\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}^\mu\right)$ associated with this gauge in $(2+1)$-dimensions as follows

$$\hat{\mathcal{H}}\left(\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}^\mu\right) = \hat{O}^{\star}_{Graph}\left(\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}^\mu\right) = -(p_0)^2 + \left(p_x\left(1 + \frac{e\mathcal{B}\theta}{4}\right) + \frac{e\mathcal{B}}{2}\left(1 + \frac{\eta}{e\mathcal{B}}\right)y\right)^2 + \left(p_y\left(1 + \frac{e\mathcal{B}\theta}{4}\right) - \frac{e\mathcal{B}}{2}\left(1 + \frac{\eta}{e\mathcal{B}}\right)x - e\mathcal{E}t\right)^2 - \frac{\imath e}{2}\gamma^\mu\gamma^\nu\mathcal{F}^{\star}_{\mu\nu}. \tag{5.36}$$

By making the shift $p_\mu \to p_\mu + p^0_\mu$ and after a long and straightforward calculation, we obtain the classical Lagrangian function corresponding to this gauge of field

$$L_{G1}\left(x, \dot{x}, \psi, \dot{\psi}, s\right) = -\frac{m_t}{2}\dot{t}^2 + \frac{m_x}{2}\dot{x}^2 + \frac{m_y}{2}\dot{y}^2 + \omega_1 x\dot{y} - \omega_2\dot{x}y + \omega_3 t\dot{y} - ie\mathcal{F}^{\star}_{\mu\nu}\psi^\mu\psi^\nu + i \quad \dot{\psi}, \tag{5.37}$$

Where m_t, m_x and m_y are the mass of particles in NC phase space along the t, x, y directions, respectively, and ω_1, ω_2 and ω_3 are the frequencies along the x and y directions.

$$m_t = \frac{\upsilon_F^2}{2}, \quad m_x = m_y = \frac{1}{2\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2}, \quad \omega_1 = \omega_2 = \frac{e\mathcal{B}}{2}\frac{\left(1+\frac{\eta}{e\mathcal{B}}\right)}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)},$$
$$\omega_3 = \frac{e\mathcal{E}}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)}, \tag{5.38}$$

and

$$\mathcal{F}_{02}^{\star} = \frac{\mathcal{E}}{\upsilon_F}, \quad \mathcal{F}_{12}^{\star} = \mathcal{B}(1+\tfrac{\theta e\mathcal{B}}{4}+\tfrac{\eta}{e\mathcal{B}}). \tag{5.39}$$

Note that the mass of graphene quasiparticles, m, is posed as $m = 0$. By applying the path integral formalism using the method of Fradkin and Gitman [105], we obtain

$$\mathcal{G}^{(\mathrm{NC})}\left(x_b^{\mu}, x_a^{\mu}; \lambda\right) = \int DtDxDy \int \mathcal{D}\psi \exp\left\{i\int_0^{\lambda} dt\left[-\frac{m_t}{2}\dot{t}^2 + \frac{m_x}{2}\dot{x}^2 + \frac{m_y}{2}\dot{y}^2 \right.\right.$$
$$\left.\left. +\omega_1 x\dot{y} - \omega_2\dot{x}y + \omega_3 t\dot{y} + ie\mathcal{F}_{\mu\nu}^{\star}\psi^{\mu}\psi^{\nu} - i\psi\dot{\psi}\right]\right\}, \tag{5.40}$$

where Dx, Dy and Dt are defined as

$$Dx^{\mu} = \lim_{N\to\infty}\sqrt{\frac{m_{x^{\mu}}}{2\imath\pi\lambda}}\prod_{k=1}^{N}\left(dx_k^{\mu}\sqrt{\frac{m_{x^{\mu}}}{2\imath\pi\lambda}}\right), \tag{5.41}$$

where the measure $\mathcal{D}\psi$ is given by

$$\mathcal{D}\psi = D\psi\left[\int D\psi \exp\left\{-\int_0^{\lambda} dt\psi\dot{\psi}\right\}\right]^{-1}. \tag{5.42}$$

The standard technique of the Gaussian integration over the Grassmannian variables gives us $\sqrt{\frac{\det\mathcal{M}(e)}{\det\mathcal{M}(e=0)}}$, the propagator then takes the following form

$$\mathcal{G}^{(\mathrm{NC})}\left(x_b^{\mu}, x_a^{\mu}; \lambda\right) = \sqrt{\frac{\det\mathcal{M}\left(e\right)}{\det\mathcal{M}\left(e=0\right)}}$$
$$\int DtDxDy \exp\left\{i\int_0^{\lambda} dt\left[-\frac{m_t}{2}\dot{t}^2 + \frac{m_x}{2}\dot{x}^2 + \frac{m_y}{2}\dot{y}^2 + \omega_1 x\dot{y} - \omega_2\dot{x}y + \omega_3 t\dot{y}\right]\right\}, \tag{5.43}$$

with

$$\mathcal{M}_{\mu\nu}\left(e,\tau,\tau'\right) = \left[\eta_{\mu\nu}\delta'(\tau-\tau') - e\mathcal{F}_{\mu\nu}^{\star}\left(\tau\right)\delta(\tau-\tau')\right]. \tag{5.44}$$

It is easy to show that the spin factor (SF) is written as

$$\sqrt{\frac{\det\mathcal{M}\left(e\right)}{\det\mathcal{M}\left(0\right)}} = \exp\left\{-\frac{\lambda}{2}\int_0^{e} de' Tr\int d\tau\left(\mathcal{M}^{-1}\right)^{\mu\nu}\left(e',\tau,\tau\right)\mathcal{F}^{\star}{}_{\mu\nu}\left(\tau\right)\right\}, \tag{5.45}$$

where $\left(\mathcal{M}^{-1}\right)^{\mu\nu}(\tau,\tau')$ is the inverse of the matrix elements $\mathcal{M}^{\mu\nu}(\tau,\tau')$ satisfying the following relation

$$\int_0^1 \mathcal{M}_{\mu\nu}(e,\tau,s)\left(\mathcal{M}^{\nu\beta}\right)^{-1}(e,s,\tau') = \delta_\mu^\beta \delta(\tau-\tau'). \tag{5.46}$$

Following Ref. [168] and after a long and simple calculation, which is presented in the appendix, we obtain this result

$$\sqrt{\frac{\det \mathcal{M}(e)}{\det \mathcal{M}(e=0)}} = \cosh(e\Upsilon_1\lambda), \tag{5.47}$$

with $\Upsilon_1 = \sqrt{\left(\frac{\mathcal{E}}{v_F}\right)^2 - (\mathcal{B})^2\left(1+\frac{\theta e\mathcal{B}}{4}+\frac{\eta}{e\mathcal{B}}\right)^2}$.

Following [178], the kernel propagator $\mathcal{G}^{(\mathrm{NC})}(x_b^\mu, x_a^\mu;\lambda)$ can be written as

$$\mathcal{G}^{(\mathrm{NC})}(x_b^\mu, x_a^\mu;\lambda) = \exp(\imath\omega_2(x_a y_a - x_b y_b))\,\mathcal{K}(x_b^\mu, x_a^\mu;\lambda)\cosh(e\Upsilon_1\lambda), \tag{5.48}$$

where $\mathcal{K}(x_b,x_a,\lambda)$ is defined as

$$\mathcal{K}(x_b^\mu, x_a^\mu, \lambda) = \int Dy \exp\left\{\imath\int_0^\lambda \left[\frac{m_y}{2}\dot{y}^2\right]dt\right\}\mathbb{K}[y(t)], \tag{5.49}$$

$\mathbb{K}[y(t)]$ is the propagator of a free particle in a time-dependent external force $(2\omega_1\dot{y})$ and $\omega_3\dot{y}$, respectively, on the axes (Ox) and (Ot) defined by

$$\begin{aligned}\mathbb{K}[y(t)] &= \mathbb{K}_x[y(t)]\times\mathbb{K}_t[y(t)]\\ &= \int Dx\exp\left\{\imath\int_0^\lambda\left[\tfrac{m_x}{2}\dot{x}^2+(2\omega_1)\dot{y}x\right]ds\right\}\\ &\quad\times\int Dt\exp\left\{\imath\int_0^\lambda\left[-\frac{m_t}{2}\dot{t}^2+\omega_3\dot{y}t\right]ds\right\}.\end{aligned} \tag{5.50}$$

Using the well known result of time-dependent forcing of a free particle [179], the propagators $\mathbb{K}_x[y(t)]$ and $\mathbb{K}_t[y(t)]$ respectively have the following forms

$$\begin{aligned}\mathbb{K}_x[y(t)] &= \sqrt{\frac{m_x}{2\pi\imath\lambda}}\exp\left\{\imath\tfrac{m_x}{2\lambda}(x_b-x_a)^2\right\}\exp\left\{\imath(2\omega_1)(x_b y_b - x_a y_a)\right\}\\ &\times\exp\left\{\imath\int_0^\lambda\left[\frac{2\omega_1}{\lambda}(x_a-x_b)y-\frac{4\omega_1^2}{2m_x}y^2+\frac{4\omega_1^2}{m_x\lambda}y(s)\int_0^s y(s')ds'\right]ds\right\}.\end{aligned} \tag{5.51}$$

and

$$\begin{aligned}\mathbb{K}_t[y(t)] &= \sqrt{\frac{-m_t}{2\pi\imath\lambda}}\exp\left\{-\imath\tfrac{m_t}{2\lambda}(t_b-t_a)^2\right\}\exp\left\{-\imath\omega_3(t_b y_b - t_a y_a)\right\}\\ &\times\exp\left\{\imath\int_0^\lambda\left[-\frac{\omega_3}{\lambda}(t_a-t_b)y+\frac{\omega_3^2}{2m_t}y^2-\frac{\omega_3^2}{m_t\lambda}y(s)\int_0^s y(s')ds'\right]ds\right\}.\end{aligned} \tag{5.52}$$

Consequently, we can write propagator $\mathbb{K}[y(t)]$ as follows

$$\mathbb{K}[y(t)] = \sqrt{\frac{-m_t}{2\pi\imath\lambda}}\sqrt{\frac{m_x}{2\pi\imath\lambda}}\exp\left\{-\imath\tfrac{m_t}{2\lambda}(t_b-t_a)^2\right\}\exp\left\{-\imath\omega_3(t_by_b-t_ay_a)\right\}$$
$$\times\exp\left\{\imath\tfrac{m_x}{2\lambda}(x_b-x_a)^2\right\}\exp\left\{2\imath\omega_1(x_by_b-x_ay_a)\right\}$$
$$\times\exp\left\{\imath\int_0^\lambda\left[\left[\frac{2\omega_1}{\lambda}(x_a-x_b)-\frac{\omega_3}{\lambda}(t_a-t_b)\right]y-\left[\frac{4\omega_1^2}{2m_x}-\frac{\omega_3^2}{2m_t}\right]y^2\right.\right.$$
$$\left.\left.+\left[\frac{4\omega_1^2}{m_x\lambda}-\frac{\omega_3^2}{m_\tau\lambda}\right]\left[y(s)\int_0^s y(s')\,ds'\right]\right]ds\right\}. \tag{5.53}$$

Performing the integration by parts of the term ($\int_0^\lambda dt\left[y(t)\int_0^t y(s)\,ds\right]$), the kernel propagator $\mathcal{K}(x_b^\mu,x_a^\mu;\lambda)$ then will express as follows

$$\mathcal{K}(x_b^\mu,x_a^\mu;\lambda) = \int Dy\exp\left\{\imath\frac{m_y}{2}\int_0^\lambda\left[\dot{y}^2+\left[\frac{2\omega_3}{\lambda m_y}(t_a-t_b)+\frac{4\omega_1}{\lambda m_y}(x_a-x_b)\right]y\right.\right.$$
$$\left.\left.-\left[\frac{4\omega_1^2}{m_xm_y}-\frac{\omega_3^2}{m_tm_y}\right]y^2\right]ds\right\}\exp\left\{\imath\frac{m_y}{2\lambda}\left[\frac{4\omega_1^2}{m_xm_y}-\frac{\omega_3^2}{m_tm_y}\right]\left[\int_0^\lambda y(s)\,ds\right]^2\right\}. \tag{5.54}$$

By using the following formula,

$$\sqrt{\frac{a}{\pi}}\int d\chi\exp\left[-a\chi^2+b\chi\right] = \exp\left[\left(\frac{b}{2\sqrt{a}}\right)^2\right]. \tag{5.55}$$

The Gaussian function $\exp\left\{\imath\frac{m_y}{2\lambda}\left[\frac{4\omega_1^2}{m_xm_y}-\frac{\omega_3^2}{m_tm_y}\right]\left[\int_0^\lambda y(s)\,ds\right]^2\right\}$ can be written as follows

$$\exp\left\{\imath\frac{m_y}{2\lambda}\left[\frac{4\omega_1^2}{m_xm_y\lambda}-\frac{\omega_3^2}{m_tm_y\lambda}\right]\left[\int_0^\lambda y(s)\,ds\right]^2\right\} = \sqrt{\frac{\imath m_y}{2\pi\lambda}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{\imath m_y}{2\lambda}\chi^2\right.$$
$$\left.+\imath\frac{m_y}{\lambda}\sqrt{\frac{4\omega_1^2}{m_xm_y}-\frac{\omega_3^2}{m_tm_y}}\chi\int_0^\lambda y(s)\,ds\right\}d\chi. \tag{5.56}$$

Incorporating Eq. (5.56) into Eq. (5.54), this gives for the kernel propagator $\mathcal{K}(x_b^\mu,x_a^\mu;\lambda)$ the following result

$$\mathcal{K}(x_b^\mu,x_a^\mu;\lambda) = \sqrt{\frac{\imath m_y}{4\pi\lambda}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{\imath m_y}{4\lambda}\chi^2\right\}\mathbf{K}(x_b^\mu,x_a^\mu,\lambda,\chi)\,d\chi, \tag{5.57}$$

where $\mathbf{K}(x_b^\mu,x_a^\mu,\lambda,\chi)$ is given by

$$\mathbf{K}(x_b^\mu,x_a^\mu,\lambda,\chi) = \int Dy\exp\left\{\imath\int_0^\lambda\left[\tfrac{m_y}{2}\left(\dot{y}^2-\Omega_1^2y^2\right)\right.\right.$$
$$\left.\left.+\frac{1}{\lambda}\left[2\omega_1(x_a-x_b)-\omega_3(t_a-t_b)+m_y\Omega_1\chi\right]y\right]ds\right\}. \tag{5.58}$$

This propagator represents a one-dimensional forced harmonic oscillator with frequency Ω_1 defined as

$$\Omega_1 = \sqrt{\frac{4\omega_1^2}{m_x m_y} - \frac{\omega_3^2}{m_t m_y}} \tag{5.59}$$

$$= 2\sqrt{(e\mathcal{B})^2\left(1+\frac{\eta}{e\mathcal{B}}\right)^2\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2 - \frac{(e\mathcal{E})^2}{\upsilon_F^2}}, \tag{5.60}$$

and a time-independent external force defined as

$$F_y = \frac{1}{\lambda}\left[2\omega_1\left(x_a - x_b\right) - \omega_3\left(t_a - t_b\right) + m_y \Omega \chi\right]. \tag{5.61}$$

Following Ref. [179], the propagator $\mathbf{K}\left(x_b^\mu, x_a^\mu, \lambda, \chi\right)$ transformed to

$$\begin{aligned}\mathbf{K}\left(x_b^\mu, x_a^\mu, \lambda, \chi\right) &= \sqrt{\tfrac{m_y\Omega_1}{2\pi\imath\sin(\Omega_1\lambda)}}\exp\left\{\imath\tfrac{m_y\Omega_1}{2\sin(\Omega_1\lambda)}\left(\left(y_a^2+y_b^2\right)\cos\left(\Omega_1\lambda\right) - 2y_b y_a\right)\right\} \\ &\times\exp\left\{\imath\left[-\tfrac{\frac{1}{\lambda}(2\omega_1(x_a-x_b)-\omega_3(t_a-t_b)+m_y\Omega_1\chi)}{\Omega_1\sin(\Omega_1\lambda)}\left(y_b+y_a\right)\left(\cos\left(\Omega_1\lambda\right)-1\right)\right]\right\} \\ &\times\exp\left\{\imath\tfrac{\frac{1}{\lambda^2}(2\omega_1(x_a-x_b)-\omega_3(t_a-t_b)+m_y\Omega_1\chi)^2}{m_y\Omega_1^3\sin(\Omega_1\lambda)}\left(\cos\left(\Omega_1\lambda\right)-1\right)\right\} \\ &\times\exp\left[\imath\tfrac{\lambda\left(\frac{1}{\lambda}(2\omega_1(x_a-x_b)-\omega_3(t_a-t_b)+m_y\Omega_1\chi)\right)^2}{2m_y\Omega_1^2}\right].\end{aligned} \tag{5.62}$$

At this stage, by substituting Eq. (5.62) into Eq. (5.57) and by performing the integration over χ from $-\infty$ to $+\infty$, then with some simplifications [178], we get for the expression of the Green function the following result

$$\begin{aligned}\mathcal{G}^{(\mathrm{NC})}\left(x_b^\mu, x_a^\mu;\lambda\right) &= K_0\left(t_b,t_a,\lambda\right)K_0\left(x_b,x_a,\lambda\right)e^{-\imath\omega_3(t_b y_b - t_a y_a)}e^{\imath\omega_1(x_b y_b - x_a y_a)} \\ &\times\sqrt{\tfrac{m_y\Omega_1}{2\pi\imath\sin(\Omega_1\lambda)}}\sqrt{\tfrac{\imath m_y}{2\pi\lambda}}\sqrt{\tfrac{\pi}{\imath\frac{m_y}{\lambda^2\Omega_1}\tan(\Omega_1\lambda/2)}}\exp\left\{\imath\tfrac{m_y\Omega_1}{4}\left[\left(y_a-y_b\right)^2\cot\left(\frac{\Omega_1\lambda}{2}\right)\right.\right. \\ &-\frac{\imath m_y\Omega_1}{2\lambda}\left(\frac{1}{m_y^2\Omega_1^3} - \tfrac{\lambda}{2m_y^2\Omega_1^2\tan(\Omega_1\lambda/2)}\right)\left(-4\omega_1\omega_3\left(x_a-x_b\right)\left(t_a-t_b\right)\right) \\ &\left.\left.-\frac{\imath m_y\Omega_1}{2\lambda}\left(\frac{1}{m_y^2\Omega_1^3} - \tfrac{\lambda}{2m_y^2\Omega_1^2\tan(\Omega_1\lambda/2)}\right)\left(4\omega_1^2\left(x_a-x_b\right)^2 - \omega_3^2\left(t_a-t_b\right)^2\right)\right]\right\}\cosh\left(e\Upsilon_1\lambda\right),\end{aligned} \tag{5.63}$$

where $K_0\left(t_b,t_a,\lambda\right)$ and $K_0\left(x_b,x_a,\lambda\right)$ are the propagators of free particles olong t and x direction respectively.

5.3.2 The effective action expression

As it is usually known, we write the effective action corresponding to the first gauge, which takes the following form

$$S_{eff.}^{(\mathrm{NC,G1})} = -\imath \int_0^\infty \frac{d\lambda}{\lambda} \int dt_b dx_b dy_b \cosh\left(e\Upsilon_1 \lambda\right)$$
$$\times K_0\left(t_b, t_a, \lambda\right) K_0\left(x_b, x_a, \lambda\right) e^{-\imath\omega_3(t_b y_b - t_a y_a)} e^{\imath\omega_1(x_b y_b - x_a y_a)} \tag{5.64}$$
$$\times \sqrt{\frac{m_y \Omega_1}{2\pi\imath \sin(\Omega_1\lambda)}} \sqrt{\frac{\imath m_y}{2\pi\lambda}} \sqrt{\frac{\pi}{\imath \frac{m_y}{\lambda^2\Omega_1} \tan(\Omega_1\lambda/2)}} \exp\left\{ \imath \frac{m_y \Omega_1}{4} \left[\left(y_a - y_b\right)^2 \cot\left(\frac{\Omega_1 \lambda}{2}\right) \right.\right.$$
$$- \frac{\imath m_y \Omega_1}{2\lambda} \left(\frac{1}{m_y^2 \Omega_1^3} - \frac{\lambda}{2m_y^2\Omega_1^2 \tan(\Omega_1\lambda/2)} \right) \left(-4\omega_1\omega_3 \left(x_a - x_b\right)\left(t_a - t_b\right)\right)$$
$$\left.\left.- \frac{\imath m_y \Omega_1}{2\lambda} \left(\frac{1}{m_y^2 \Omega_1^3} - \frac{\lambda}{2m_y^2\Omega_1^2 \tan(\Omega_1\lambda/2)} \right) \left(4\omega_1^2 \left(x_a - x_b\right)^2 - \omega_3^2 \left(t_a - t_b\right)^2\right) \right] \right\}\Bigg|_{x_a = x_b}. \tag{5.65}$$

Consequently, by some simplification, the effective action takes this simple form

$$S_{eff.}^{(\mathrm{NC,G1})} = -\imath \int_0^\infty \frac{d\lambda}{\lambda} \int dt_b dx_b dy_b \cosh\left(e\Upsilon_1 \lambda\right)$$
$$\times K_0\left(t_b, t_a, \lambda\right) K_0\left(x_b, x_a, \lambda\right) \sqrt{\frac{m_y \Omega_1}{2\pi\imath \sin(\Omega_1\lambda)}} \sqrt{\frac{\imath m_y}{2\pi\lambda}} \sqrt{\frac{\pi}{\imath \frac{m_y}{\lambda^2\Omega_1} \tan\left(\Omega_1\lambda/2\right)}}. \tag{5.66}$$

where

$$K_0\left(t_b, t_a, \lambda\right) = \sqrt{\frac{-m_t}{2\pi i\lambda}} \tag{5.67}$$
$$K_0\left(x_b, x_a, \lambda\right) = \sqrt{\frac{m_x}{2\pi i\lambda}} \tag{5.68}$$

Performing now the final space-time coordinates integral for all terms we obtain, the volume that is defined as

$$\int dt_b dx_b dy_b = TL^2, \tag{5.69}$$

the effective action then becomes as follows

$$S_{eff.}^{(\mathrm{NC,G1})} = TL^2 \frac{\imath^{1/2}\tilde{\Omega}_1 v_F}{8\pi^{3/2}\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \int_0^\infty \frac{d\lambda}{\lambda^{3/2}} \frac{\cosh\left(e\Upsilon_1\lambda\right)}{\sinh\left(\tilde{\Omega}_1\lambda\right)}. \tag{5.70}$$

Returning to real time via the replacements $(T \to \imath T)$ in the Eq. (5.70). Consequently, we get the expression of $S_{eff}^{(\mathrm{NC})}$ in the framework of the NC phase space coordinate for first gauge, which is defined as

$$S_{eff.}^{(\mathrm{NC,G1})} = TL^2 \tfrac{\iota^{3/2}\tilde{\Omega}_1 \upsilon_F}{8\pi^{3/2}\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \int_0^{\infty} \frac{d\lambda}{\lambda^{3/2}} \frac{\cosh\left(e\Upsilon_1\lambda\right)}{\sinh\left(\tilde{\Omega}_1\lambda\right)}, \tag{5.71}$$

with $\tilde{\Omega}_1 = \iota\Omega_1/2$ which has the following form

$$\tilde{\Omega}_1 = \frac{\iota}{2}\sqrt{\frac{4\omega_1^2}{m_x m_y} - \frac{\omega_3^2}{m_t m_y}}, \tag{5.72}$$

$$= \sqrt{\frac{(e\mathcal{E})^2}{\upsilon_F^2} - (e\mathcal{B})^2\left(1+\frac{\eta}{e\mathcal{B}}\right)^2\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2}. \tag{5.73}$$

In commutative phase space ($\theta = \eta \to 0$) and $\mathcal{B} = 0$, Eq. (5.71) agrees exactly with that of Ref. [162, 94, 163].

5.3.3 The pair production probability

The pair production probability per unit volume and per unit time in the framework of the NC phase space coordinate for the first gauge is defined as

$$\mathcal{P}^{(\mathrm{NC,G1})}(\mathrm{pair}) = 2\,\mathrm{Im}\, S_{eff.}^{(\mathrm{NC})}. \tag{5.74}$$

Where the imaginary part of $S_{eff.}^{(\mathrm{NC})}$ can be written as follows

$$2\,\mathrm{Im}\, S_{eff.}^{(\mathrm{NC})} = \frac{1}{\imath}\left(S_{eff.}^{(\mathrm{NC})} - S_{eff.}^{*(\mathrm{NC})}\right), \tag{5.75}$$

$$= \frac{1}{\imath} TL^2 \tfrac{\iota^{3/2}\tilde{\Omega}_1 \upsilon_F}{8\pi^{3/2}\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda^{3/2}} \frac{\cosh\left(e\Upsilon_1\lambda\right)}{\sinh\left(\tilde{\Omega}_1\lambda\right)}, \tag{5.76}$$

this result (5.76) is verified when $\tilde{\Omega}_1^2 > 0$, while it equals zero in the opposite case.

Let us now achieve the integration over λ by using the residue theorem. We close the integration contour at infinity with a semicircle in the upper half-plane.

The function $\sinh(\tilde{\Omega}\lambda)$ has these poles $\tilde{\Omega}\lambda_k = ik\pi \Longrightarrow \lambda_k = \frac{k\pi}{\tilde{\Omega}}$ along the integration contour, then we can write

$$\int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda^{3/2}} \tfrac{\cosh(e\Upsilon_1\lambda)}{\sinh(\tilde{\Omega}_1\lambda)} = 2\pi\iota \sum_{k=1}^{\infty} \mathrm{Res}\left(\tfrac{\cosh(e\Upsilon_1\lambda)}{\lambda^{3/2}\sinh(\tilde{\Omega}_1\lambda)}, \frac{\iota k\pi}{\tilde{\Omega}_1}\right) \tag{5.77}$$

$$= 2\pi\iota \sum_{k=1}^{\infty} \frac{(-1)^k}{\left(\iota k\pi/\tilde{\Omega}_1\right)^{3/2}} \frac{\cos\left(\frac{e\Upsilon_1}{\tilde{\Omega}_1}k\pi\right)}{\tilde{\Omega}_1}. \tag{5.78}$$

Finally, the expression of pair production probability per unit volume per unit time in the framework of a NC phase space coordinate for the first gauge is simplified in the following manner

$$\mathcal{P}^{(\mathrm{NC,G1})}(\text{pair}) = TL^2 \frac{\tilde{\Omega}_1^{3/2}\upsilon_F}{4\pi^2\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{3/2}} \cos\left(\frac{e\Upsilon_1}{\tilde{\Omega}_1}k\pi\right), \tag{5.79}$$

As a result, when there are no noncommutative coefficients (i.e., $\theta = \eta \to 0$) in the case when $\mathcal{E} \neq 0$, $\mathcal{B} = 0$, we have ($e\Upsilon_1 = \tilde{\Omega}_1 = \frac{\mathcal{E}}{\upsilon_F}$), and the pair production probability per unit volume per unit time becomes as

$$\mathcal{P}^{(\mathrm{NC,G1})}(\text{pair}) = \frac{\upsilon_F TL^2}{4\pi^2}\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^{3/2} \sum_{k=1}^{\infty} \frac{(-1)^{2k}}{k^{3/2}} = \frac{TL^2 (e\mathcal{E})^{3/2}}{4\pi^2\upsilon_F^{1/2}}\zeta(3/2), \tag{5.80}$$

with ζ is the Riemann zeta function and $\zeta(3/2) = \sum_{k=1}^{\infty} k^{-3/2} = 2.61238$.

The Eq. (5.80) confirms the Schwinger results in graphene, which agrees exactly with those of Refs. [162, 94].

Moreover, following Eq. (5.76) in the case of large magnetic fields, the problem of pair production is non-existent. The same result exists in the absence of electromagnetic fields.

5.4 The construction of effective action for the second gauge of field

5.4.1 The evaluation of propagator $\mathcal{G}^{(\mathrm{NC,G2})}(x_b, x_a, \lambda)$

In this section, we study the creation of particle-anti particle pairs from the vacuum in monolayer graphene under the action of the uniform electromagnetic field and in a non-commutative space-phase coordinates considering Schwinger's method.

We consider the vector potential $A_\mu(\mathcal{X})$ that is given by

$$A_\mu = (-\mathcal{E}\mathcal{X}_2, -\frac{\mathcal{B}}{2}\mathcal{X}_2, \frac{\mathcal{B}}{2}\mathcal{X}_1), \tag{5.81}$$

where $\mathcal{X}_1 = x - \frac{\theta}{2}p_y$ and $\mathcal{X}_2 = y + \frac{\theta}{2}p_x$.

The Hamiltonian $\hat{O}^{\star}_{Graph}\left(\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}^\mu\right)$ associated with this gauge is defined as

$$\hat{O}^{\star}_{Graph}\left(\hat{\mathcal{X}}^{\mu},\hat{\mathcal{P}}^{\mu}\right)=(\hat{p}_0+e\mathcal{E}(\hat{y}+\frac{\theta}{2}\hat{p}_x))^2-(\hat{p}_x+\frac{\eta}{2}\hat{y}+\frac{e\mathcal{B}}{2}(\hat{y}+\frac{\theta}{2}\hat{p}_x))^2$$
$$-(\hat{p}_y-\frac{\eta}{2}\hat{x}-\tfrac{e\mathcal{B}}{2}(\hat{x}-\frac{\theta}{2}\hat{p}_y))^2-\frac{\imath e}{2}\gamma^{\mu}\gamma^{\nu}\mathcal{F}^{\star}_{\mu\nu}, \tag{5.82}$$

and the corresponding Lagrangian function $L_{G2}\left(x,\dot{x},\psi,\dot{\psi},s\right)$ for this gauge can be written as

$$L_{G_2}\left(x,\dot{x},\psi,\dot{\psi},s\right)=\tfrac{m_x}{2}\dot{x}^2+\tfrac{m_Y}{2}\dot{Y}^2+\omega_1 x\dot{Y}-\omega_2 Y\dot{x}-\omega_Y^2Y^2$$
$$+\frac{p_0}{\upsilon_F}\delta\left(\theta,\eta\right)\left(x_b-x_a\right)-\imath e\mathcal{F}^{\star}_{\mu\nu}\psi^{\mu}\psi^{\nu}+\imath\psi\dot{\psi}, \tag{5.83}$$

with

$$Y=y+\frac{\left(1+\frac{e\mathcal{B}\theta}{4}\right)}{e\mathcal{E}\left[\left(1+\frac{e\mathcal{B}\theta}{2}\right)-\frac{e\mathcal{B}\theta}{4}\left(1+\frac{\eta}{2e\mathcal{B}}\right)\right]}p_0, \tag{5.84}$$

and

$$m_x=\frac{1}{2\left[\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2-\left(\frac{e\theta\mathcal{E}}{2\upsilon_F}\right)^2\right]},\ m_Y=\frac{1}{2\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2}$$
$$\omega_1=\frac{e\mathcal{B}}{2}\frac{\left(1+\frac{\eta}{e\mathcal{B}}\right)}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)},\ \omega_2=\frac{\frac{eB}{2}\left(1+\frac{e\mathcal{B}\theta}{4}\right)\left(1+\frac{\eta}{e\mathcal{B}}\right)-\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2\frac{\theta}{2}}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2-\left(\frac{e\theta\mathcal{E}}{2\upsilon_F}\right)^2},$$
$$\omega_Y^2=-\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2\frac{\left[\left(1+\frac{e\mathcal{B}\theta}{4}\right)-\frac{e\mathcal{B}\theta}{4}\left(1+\frac{\eta}{e\mathcal{B}}\right)\right]^2}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2-\left(\frac{e\theta\mathcal{E}}{2\upsilon_F}\right)^2}\text{ and }\delta\left(\theta,\eta\right)=\frac{\frac{e\mathcal{B}}{2}\left(1+\frac{\eta}{e\mathcal{B}}\right)}{\frac{e\mathcal{E}}{\upsilon_F}\left[\left(1+\frac{e\mathcal{B}\theta}{4}\right)-\frac{e\mathcal{B}\theta}{4}\left(1+\frac{\eta}{e\mathcal{B}}\right)\right]}, \tag{5.85}$$

also

$$\mathcal{F}^{\star}_{02}=-\frac{\mathcal{E}}{\upsilon_F}(1+\tfrac{e\mathcal{B}\theta}{2}),\quad \mathcal{F}^{\star}_{12}=-\mathcal{B}(1+\tfrac{\theta e\mathcal{B}}{4}+\tfrac{\eta}{e\mathcal{B}}). \tag{5.86}$$

Whereas m_x and m_Y are the masses of quasiparticles in a NC phase space along the x and Y directions, respectively, and ω_Y is the frequency along the Y. Note that in the ordinary case, it is well known that the effective masses and the frequencies are respectively given by $m_x=m_Y=1/2$ and $\omega_1=\omega_2=\frac{e\mathcal{B}}{2}$.

By applying the path integral formalism using the method of Fradkin and Gitman [105], the corresponding Green's function is defined as

$$\mathcal{G}^{(\mathrm{NC;G2})}\left(x_b^{\mu},x_a^{\mu},\lambda\right)=\int_{-\infty}^{+\infty}\frac{dp_0}{2\pi}e^{-\imath p_0(t_b-t_a)}$$
$$\times\int DxDy\int\mathcal{D}\psi\exp\left\{\imath\int_0^{\lambda}d\tau\left[\tfrac{m_x}{2}\dot{x}^2+\tfrac{m_Y}{2}\dot{Y}^2+\omega_1x\dot{Y}-\omega_2Y\dot{x}\right.\right.$$
$$\left.\left.-\omega_Y^2Y^2+\frac{p_0}{\upsilon_F}\left(x_b-x_a\right)\delta\left(\theta,\eta\right)-\imath e\mathcal{F}^{\star}_{\mu\nu}\psi^{\mu}\psi^{\nu}+\imath\psi\dot{\psi}\right]\right\}. \tag{5.87}$$

The integration over spin is given as (i.e., see the Appendix),

$$\sqrt{\frac{\det \mathcal{M}(e)}{\det \mathcal{M}(e=0)}} = \cosh\left(e\Upsilon_2\lambda\right), \tag{5.88}$$

with $\Upsilon_2 = \sqrt{\frac{\mathcal{E}^2}{v_F^2}(1+\frac{e\mathcal{B}\theta}{2})^2 - \mathcal{B}^2(1+\frac{\theta e\mathcal{B}}{4}+\frac{\eta}{e\mathcal{B}})^2}$.

By using the same methodology that was presented in the previous section, we perform all the path integrations on the bosonic trajectories. Therefore, the kernel propagator $\mathcal{G}^{(\mathrm{NC;G2})}\left(x_b^\mu, x_a^\mu, \lambda\right)$ can be expressed as,

$$\mathcal{G}^{(\mathrm{NC;G2})}\left(x_b^\mu, x_a^\mu, \lambda\right) = \exp\left(\imath\omega_2\left(x_a Y_a - x_b Y_b\right)\right)\mathcal{K}\left(x_b^\mu, x_a^\mu, \lambda\right), \tag{5.89}$$

where the kernel propagator $\mathcal{K}\left(x_b^\mu, x_a^\mu, \lambda\right)$ can be written as,

$$\mathcal{K}\left(x_b^\mu, x_a^\mu, \lambda\right) = \int DY \exp\left\{\imath\int_0^\lambda \left[\tfrac{m_Y\dot{Y}^2}{2} - \omega_Y^2 Y^2\right] ds\right\}\mathbb{K}\left[Y(t)\right]. \tag{5.90}$$

We note that $\mathbb{K}\left[Y(t)\right]$ is the free particle propagator in a time-dependent external force $\left((\omega_1+\omega_2)\dot{Y}\right)$ defined as

$$\mathbb{K}\left[Y(t)\right] = \int Dx \exp\left\{\imath\int_0^\lambda \left[\tfrac{m_x}{2}\dot{x}^2 + (\omega_1+\omega_2)\dot{Y}x\right] ds\right\}. \tag{5.91}$$

Following [179], the propagator $\mathbb{K}\left[Y(t)\right]$ is transformed to

$$\begin{aligned}\mathbb{K}\left[Y(t)\right] &= \sqrt{\frac{m_x}{2\pi\imath\lambda}}\exp\left\{\imath\tfrac{m_x}{2\lambda}\left(x_b-x_a\right)^2\right\}\exp\left\{\imath\left(\omega_1+\omega_2\right)\left(x_bY_b - x_aY_a\right)\right\}\\ &\times\exp\left\{\imath\int_0^\lambda\left[\frac{(\omega_1+\omega_2)}{\lambda}\left(x_a-x_b\right)Y - \frac{(\omega_1+\omega_2)^2}{2m_x}Y^2 + \frac{(\omega_1+\omega_2)^2}{m_x\lambda}Y\int_0^s Y(s')\,ds'\right]ds\right\}.\end{aligned} \tag{5.92}$$

After the integration by parts of the term $\left(\int_0^\lambda ds\left[Y(s)\int_0^s Y(s')\,ds'\right]\right)$, we find

$$\begin{aligned}\mathcal{K}\left(x_b^\mu, x_a^\mu, \lambda\right) &= \int DY\exp\left\{i\tfrac{m_Y}{2}\int_0^\lambda\left[\dot{Y}^2 + \tfrac{2(\omega_1+\omega_2)}{\lambda m_Y}\left(x_a-x_b\right)Y - \left(\tfrac{(\omega_1+\omega_2)^2}{m_xm_Y} + 2\tfrac{\omega_Y^2}{m_Y}\right)Y^2\right]ds\right\}\\ &\times\exp\left\{i\tfrac{m_Y}{2}\frac{(\omega_1+\omega_2)^2}{m_xm_Y\lambda}\left[\int_0^\lambda Y(s)\,ds\right]^2\right\}.\end{aligned} \tag{5.93}$$

Then, we write the final term in Eq. (5.93) as follows

$$\begin{aligned}\exp\left\{i\tfrac{m_Y}{2}\frac{(\omega_1+\omega_2)^2}{m_xm_Y\lambda}\left[\int_0^\lambda Y(s)\,ds\right]^2\right\} &= \sqrt{\frac{im_Y}{2\pi\lambda}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{im_Y}{2\pi\lambda}\chi^2\right.\\ &\left.+im_Y\frac{(\omega_1+\omega_2)}{\sqrt{m_xm_Y}\lambda}\chi\int_0^\lambda Y(s)\,ds\right\}d\chi.\end{aligned} \tag{5.94}$$

Then, the Eq. (5.93) reduces to

$$\mathcal{K}\left(x_b^\mu, x_a^\mu, \lambda\right) = \sqrt{\frac{im_Y}{2\pi\lambda}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{im_Y}{2\lambda}\chi^2\right\} \mathcal{K}\left(x_b^\mu, x_a^\mu, \lambda, \chi\right) d\chi, \tag{5.95}$$

where $\mathcal{K}\left(x_b^\mu, x_a^\mu, \lambda, \chi\right)$ is given by

$$\begin{aligned}\mathcal{K}\left(x_b^\mu, x_a^\mu, \lambda, \chi\right) &= \int DY \exp\left\{i \int_0^\lambda \left[\tfrac{m_Y}{2}\left(\dot{Y}^2 - \omega^2 Y^2\right)\right.\right. \\ &\quad \left.\left. + \left(\tfrac{(\omega_1+\omega_2)}{\lambda}\left((x_a - x_b) + \sqrt{\frac{m_Y}{m_x}}\chi\right)\right) Y\right] ds\right\},\end{aligned} \tag{5.96}$$

This propagator represents a one-dimensional forced harmonic oscillator with a time-independent external force that is defined as

$$F_Y = \frac{(\omega_1 + \omega_2)}{\lambda}\left((x_a - x_b) + \sqrt{\frac{m_Y}{m_x}}\chi\right). \tag{5.97}$$

Following [179], in this gauge, the expression of the kernel (5.96) takes the following form

$$\begin{aligned}\mathcal{K}\left(x_b^\mu, x_a^\mu, \lambda, \chi\right) &= \sqrt{\tfrac{m_Y \Omega_2}{2\pi\imath \sin(\Omega_2\lambda)}} \exp\left\{\imath \tfrac{m_Y\Omega_2}{2\sin(\Omega_2\lambda)}\left(\left(Y_a^2 + Y_b^2\right)\cos\left(\Omega_2\lambda\right) - 2Y_bY_a\right)\right\} \\ &\times \exp\left\{\imath\left[-\frac{\frac{(\omega_1+\omega_2)}{\lambda}\left((x_a-x_b)+\sqrt{\frac{m_Y}{m_x}}\chi\right)}{\Omega_2\sin(\Omega_2\lambda)}\left(Y_b + Y_a\right)\left(\cos\left(\Omega_2\lambda\right) - 1\right)\right]\right\} \\ \times \exp&\left\{\imath\frac{\left(\frac{(\omega_1+\omega_2)}{\lambda}\left((x_a-x_b)+\sqrt{\frac{m_Y}{m_x}}\chi\right)\right)^2}{m_Y\Omega_2^3\sin(\Omega_2\lambda)}\left(\cos\left(\Omega_2\lambda\right) - 1\right)\right\} \exp\left[\imath\frac{\lambda\left(\frac{(\omega_1+\omega_2)}{\lambda}\left((x_a-x_b)+\sqrt{\frac{m_Y}{m_x}}\chi\right)\right)^2}{2m_Y\Omega_2^2}\right],\end{aligned} \tag{5.98}$$

with

$$\Omega_2 = \sqrt{\frac{(\omega_1 + \omega_2)^2}{m_x m_Y} + 2\frac{\omega_Y^2}{m_Y}}. \tag{5.99}$$

Finally, after substituting Eq. (5.98) into Eq. (5.95) and performing the integration over χ from $(-\infty$ to $+\infty)$, by combining Eq. (5.95) into Eq. (5.89), the Green function will be expressed as follows

$$\begin{aligned}\mathcal{G}^{(\mathrm{NC;G2})}\left(x_b^\mu, x_a^\mu, \lambda\right) &= \int \frac{dp_0}{2\pi}\cosh\left(e\Upsilon_2\lambda\right) \\ &\times e^{-\imath p_0(t_b - t_a)}\left[A\left(\Omega_2\lambda\right)\right]^{-1/2} K_0\left(x_b, x_a, \lambda\right) K_{\Omega_2}\left(Y_b, Y_a, \lambda\right) \\ &\times \exp\left\{\imath\omega_1\left(x_bY_b - x_aY_a\right) + \imath\frac{p_0}{\upsilon_F}\left(x_b - x_a\right)\delta\left(\theta, \eta\right)\right\} \\ &\times \exp\left[D_x\left(\lambda\right)\left(x_a - x_b\right)^2 + D_{xY}\left(\lambda\right)\left(x_a - x_b\right)\left(Y_b + Y_a\right) + D_Y\left(\lambda\right)\left(Y_b + Y_a\right)^2\right],\end{aligned} \tag{5.100}$$

where, $K_0\left(x_b, x_a, \lambda\right)$ is the propagator of free particles along the x direction.

$K_{\Omega_2}(Y_b, Y_a, \lambda)$ is the propagator of a one-dimensional harmonic oscillator with frequency Ω_2.

Whereas $D_x(\lambda)$, $D_{xY}(\lambda)$, $D_Y(\lambda)$ are given as

$$D_x(\lambda) = \imath \frac{m_x\left[(\omega_1+\omega_2)^2/m_x m_Y\right]}{2\lambda^2\Omega_2^3} \frac{(\Omega_2\lambda - 2\tan(\Omega_2\lambda/2))}{A(\Omega_2\lambda)},$$

$$D_{xY}(\lambda) = \imath \frac{\left[(\omega_1+\omega_2)/\sqrt{m_x m_Y}\right]}{\lambda\Omega_2} \frac{\sqrt{m_x m_Y}\tan(\Omega_2\lambda/2)}{A(\Omega_2\lambda)} \text{ and } D_Y(\lambda) = \imath \frac{m_Y\left[(\omega_1+\omega_2)^2/m_x m_Y\right]}{2\lambda\Omega_2^2} \frac{\tan^2(\Omega_2\lambda/2)}{A(\Omega_2\lambda)}, \tag{5.101}$$

with

$$A(\Omega_2\lambda) = \left[1 + \frac{(\omega_1+\omega_2)^2}{\lambda m_Y m_x \Omega_2^3}\left[2\tan(\Omega_2\lambda/2) - \Omega_2\lambda\right]\right]. \tag{5.102}$$

5.4.2 The effective action expression

Through the definition, the effective action for the second gauge of quasiparticles of graphene in the NC phase space coordinates takes the following form

$$\begin{aligned} S_{eff.}^{(\mathrm{NC})} &= -\imath \int_0^\infty \frac{d\lambda}{\lambda} \int \frac{dp_0}{2\pi} \int dt_b dx_b dy_b \cosh(e\Upsilon_2\lambda) \\ &\times e^{-\imath p_0(t_b - t_a)} \left[A(\Omega_2\lambda)\right]^{-1/2} K_0(x_b, x_a, \lambda) K_{\Omega_2}(Y_b, Y_a, \lambda) \\ &\times \exp\left\{\imath\omega_1(x_b Y_b - x_a Y_a) + \imath\frac{p_0}{\upsilon_F}(x_b - x_a)\delta(\theta, \eta)\right\} \\ &\times \exp\left[D_x(\lambda)(x_a - x_b)^2 + D_{xY}(\lambda)(x_a - x_b)(Y_b + Y_a) + D_Y(\lambda)(Y_b + Y_a)^2\right]|_{x_b = x_a}. \end{aligned} \tag{5.103}$$

By performing the integration for all terms within the expression of $S_{eff}^{(\mathrm{NC})}$, we get

$$\begin{aligned} S_{eff.}^{(\mathrm{NC})} &= -\imath T L^2 \int_0^\infty \frac{d\lambda}{\lambda} \int \frac{dp_0}{2\pi} \cosh(e\Upsilon_2\lambda) \\ &\times \left[A(\Omega\lambda)\right]^{-1/2} \sqrt{\frac{m_x}{2\pi\imath\lambda}} \sqrt{\frac{m_Y\Omega_2}{2\pi\imath\sin(\Omega_2\lambda)}} \\ &\times \int dY \exp\left\{\imath m_Y\Omega_2\left[\frac{\cos(\Omega_2\lambda)-1}{\sin(\Omega_2\lambda)} + \frac{2\left[(\omega_1+\omega_2)^2/m_x m_Y\right]}{\lambda\Omega_2^2}\frac{\tan^2(\Omega_2\lambda/2)}{A(\Omega_2\lambda)}\right]Y^2\right\}. \end{aligned} \tag{5.104}$$

For our study, we know that $y = Y - \frac{p_0}{e\mathcal{E}} \frac{\left(1+\frac{e\mathcal{B}\theta}{4}\right)}{\left[\left(1+\frac{e\mathcal{B}\theta}{2}\right) - \frac{e\mathcal{B}\theta}{4}\left(1+\frac{\eta}{2e\mathcal{B}}\right)\right]}$, for this reason p_0 must be constrained to be in the range $0 < p_0 < e\mathcal{E}L((1+\frac{e\mathcal{B}\theta}{2}) - \frac{e\mathcal{B}\theta}{4}(1+\frac{\eta}{2e\mathcal{B}}))/(1+\frac{e\mathcal{B}\theta}{4})$, in order that the entire range of time is included as p_0 is varied. After simplifying this integral and then integrating over Y, the Eq. (6.34) gives us the following result

$$S_{eff.}^{(\text{NC})} = -\imath TL \int_0^\infty \frac{d\lambda}{\lambda} \int_0^{\frac{e\mathcal{E}L\left[\left(1+\frac{e\mathcal{B}\theta}{2}\right)-\frac{e\mathcal{B}\theta}{4}\left(1+\frac{\eta}{2e\mathcal{B}}\right)\right]}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)}} \frac{dp_0}{2\pi}$$
$$\times\sqrt{\frac{m_x}{2\pi\imath\lambda}} \sqrt{\frac{\pi\Omega_2}{-2\imath\omega_Y^2 \tan\left(\Omega_2\lambda/2\right)}}^{\frac{m_Y\Omega_2}{2\pi\imath\sin(\Omega_2\lambda)}} \cosh\left(e\Upsilon_2\lambda\right). \tag{5.105}$$

Integrating now over p_0 and returning to real time via the replacements ($T \to \imath T$) [37], we obtain

$$S_{eff.}^{(\text{NC})} = \imath^{3/2} \frac{TL^2\tilde{\Omega}_2 \upsilon_F}{8\pi^{3/2}\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \int_0^\infty \frac{d\lambda}{\lambda^{3/2}} \frac{\cosh\left(e\Upsilon_2\lambda\right)}{\sinh\left(\tilde{\Omega}_2\lambda/2\right)}, \tag{5.106}$$

where $\Omega_2 = 2\imath\tilde{\Omega}_2$ and $\tilde{\Omega}_2^2$ is given by

$$\begin{aligned}\tilde{\Omega}_2^2 \;=\; & \frac{\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2\left[\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2-\frac{e\mathcal{B}\theta}{4}\left(1+\frac{\eta}{e\mathcal{B}}\right)\left(1+\frac{e\mathcal{B}\theta}{4}\right)\right]^2}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2-\left(\frac{e\theta\mathcal{E}}{2\upsilon_F}\right)^2} \\ & -\frac{\left(e\mathcal{B}\left(1+\frac{\eta}{e\mathcal{B}}\right)\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2-\frac{e\mathcal{B}}{2}\left(1+\frac{\eta}{e\mathcal{B}}\right)\left(\frac{e\theta\mathcal{E}}{2\upsilon_F}\right)^2-\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2\frac{\theta}{2}\left(1+\frac{e\mathcal{B}\theta}{4}\right)\right)^2}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2-\left(\frac{e\theta\mathcal{E}}{2\upsilon_F}\right)^2}.\end{aligned} \tag{5.107}$$

In commutative phase space ($\theta = \eta \to 0$) and $\mathcal{B} = 0$, Eq. (5.106) agrees exactly with that of Ref. [162, 94, 163].

5.4.3 The pair production probability

The same steps that we made in the previous section to calculate the term ($2\,\text{Im}\, S_{eff.}^{(\text{NC})}$), which will give the pair production probability per unit volume per unit time in the framework of the NC phase space coordinates for second gauge

$$\mathcal{P}^{(\text{NC,G2})}\left(\text{pair}\right) = TL^2 \frac{\tilde{\Omega}_2^{3/2}\upsilon_F}{4\pi^2\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k)^{3/2}} \cos\left(\frac{e\Upsilon_2}{\tilde{\Omega}_2}k\pi\right), \tag{5.108}$$

when $\theta \to 0$ and $\eta \to 0$, the above result agrees with Refs. [162, 94].

5.5 Special cases and discussion

Table 1 represents a list of special effective actions for graphene quasiparticles in NC phase space with electric and magnetic fields different from or equal to zero.

In order to discuss all the special cases of two gauges and compare the results between them, we will show the results of the effective actions for four cases ($\mathcal{E} \neq 0, \mathcal{B} \neq 0$, $\mathcal{E} \neq 0, \mathcal{B} = 0$, $\mathcal{E} = 0, \mathcal{B} \neq 0$, and $\mathcal{E} = 0, \mathcal{B} = 0$) of both gauges.

TABLE I. Effective actions of special uniform electromagnetic fields within the NC phase space.

$\forall\,(\eta,\theta)$	$\mathcal{E}\neq 0$ gauge 1	$\mathcal{E}\neq 0$ gauge 2	$\mathcal{E}=0$ gauge 1	$\mathcal{E}=0$ gauge 2
$\mathcal{B}\neq 0$	Eq. (48)	Eq. (77)	$\imath^{3/2}\frac{\upsilon_F T L^2}{8\pi^{3/2}}\frac{e\mathcal{B}\left(1+\frac{\eta}{e\mathcal{B}}\right)}{1+\frac{e\mathcal{B}\theta}{4}}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\frac{\cos\left(e\mathcal{B}(1+\frac{\theta e\mathcal{B}}{4}+\frac{\eta}{e\mathcal{B}})\lambda\right)}{\sin\left(e\mathcal{B}\left(1+\frac{\eta}{e\mathcal{B}}\right)\left(1+\frac{e\mathcal{B}\theta}{4}\right)\lambda\right)}$	$\imath^{3/2}\frac{\upsilon_F T L^2}{8\pi^{3/2}}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\frac{e\mathcal{B}(1+\frac{\theta e\mathcal{B}}{4}+\frac{\eta}{e\mathcal{B}})}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2}\cot\left(e\mathcal{B}(1+\frac{\theta e\mathcal{B}}{4}+\frac{\eta}{e\mathcal{B}})\lambda\right).$
$\mathcal{B}=0$	$\imath^{3/2}\frac{\upsilon_F T L^2}{8\pi^{3/2}}\sqrt{\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2-\eta^2}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\coth\left(\sqrt{\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2-\eta^2}\lambda\right)$	$\imath^{3/2}\frac{\upsilon_F T L^2}{8\pi^{3/2}}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\omega\left(1-\left(\frac{e\theta\mathcal{E}}{2\upsilon_F}\right)^2\right)^{-1/2}\frac{\cosh\sqrt{\frac{(e\mathcal{E})^2}{\upsilon_F^2}-\eta^2}}{\sinh\left[\omega\left(1-\left(\frac{e\theta\mathcal{E}}{2\upsilon_F}\right)^2\right)^{-1/2}\lambda\right]}.$ with $\omega=\sqrt{\frac{e^2\mathcal{E}^2}{\upsilon_F^2}\left(1-\frac{\eta\theta}{4}\right)^2-\left(\eta-\frac{\theta}{2}\frac{e^2\mathcal{E}^2}{\upsilon_F^2}\left(1+\frac{\eta\theta}{4}\right)\right)^2}$	$\imath^{3/2}\frac{\upsilon_F T L^2}{8\pi^{3/2}}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\eta\cot(\eta\lambda)$	$\imath^{3/2}\frac{\upsilon_F T L^2}{8\pi^{3/2}}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\eta\cot(\eta\lambda)$

Let us start with the case of $\mathcal{E} \neq 0, \mathcal{B} \neq 0$, when $\theta \neq 0$ and $\eta \neq 0$. From the table, the effective actions are defined in the equations (Eq. (5.71) and Eq. (5.106)) as follows

$$S_{eff.}^{(\text{NC,G1})} = TL^2 \frac{\imath^{3/2}\tilde{\Omega}_1 \upsilon_F}{8\pi^{3/2}\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \int_0^\infty \frac{d\lambda}{\lambda^{3/2}} \frac{\cosh\left(e\Upsilon_1\lambda\right)}{\sinh\left(\tilde{\Omega}_1\lambda\right)}, \tag{5.109}$$

and

$$S_{eff.}^{(\text{NC,G2})} = \imath^{3/2} \frac{TL^2\tilde{\Omega}_2\upsilon_F}{8\pi^{3/2}\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \int_0^\infty \frac{d\lambda}{\lambda^{3/2}} \frac{\cosh\left(e\Upsilon_2\lambda\right)}{\sinh\left(\tilde{\Omega}_2\lambda/2\right)}. \tag{5.110}$$

As a result, in commutative phase space coordinates (i.e., $\theta = \eta \to 0$), the effective actions for two gauges are the same and become

$$S_{eff.}^{(\text{NC,G1,G2})} = \frac{\upsilon_F TL^2}{8\pi^{3/2}} \imath^{3/2} \sqrt{\frac{(e\mathcal{E})^2}{\upsilon_F^2} - (e\mathcal{B})^2} \int_0^\infty \frac{d\lambda}{\lambda^{3/2}} \coth\left(\sqrt{\frac{(e\mathcal{E})^2}{\upsilon_F^2} - (e\mathcal{B})^2}\lambda\right), \tag{5.111}$$

Consequently, when $\theta = 0$ and $\eta \neq 0$, the $S_{eff.}^{(\text{NC})}$ expression is the same for both gauges, and it is given as

$$\begin{aligned} S_{eff.}^{(\text{NC,G2,G1})} &= \imath^{3/2} \frac{\upsilon_F TL^2}{8\pi^{3/2}} \int_0^\infty \frac{d\lambda}{\lambda^{3/2}} \sqrt{\frac{(e\mathcal{E})^2}{\upsilon_F^2} - (e\mathcal{B})^2 \left(1+\frac{\eta}{e\mathcal{B}}\right)^2} \\ &\times \coth\left(\sqrt{\frac{(e\mathcal{E})^2}{\upsilon_F^2} - (e\mathcal{B})^2 (1+\frac{\eta}{e\mathcal{B}})^2}\lambda\right). \end{aligned} \tag{5.112}$$

Also, when $\theta \neq 0$ and $\eta = 0$, the expression of $S_{eff.}^{(\text{NC})}$ for two gauges, respectively, become as

$$\begin{aligned} S_{eff.}^{(\text{NC,G1})} &= \imath^{3/2} \frac{\upsilon_F TL^2}{8\pi^{3/2}} \int_0^\infty \frac{d\lambda}{\lambda^{3/2}} \frac{\sqrt{\frac{(e\mathcal{E})^2}{\upsilon_F^2} - (e\mathcal{B})^2\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2}}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \\ &\times \coth\left(\sqrt{\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2 - (e\mathcal{B})^2 (1+\frac{\theta e\mathcal{B}}{4})^2}\lambda\right), \end{aligned} \tag{5.113}$$

and

$$S_{eff.}^{(\text{NC,G2})} = \imath^{3/2} \frac{\upsilon_F TL^2}{8\pi^{3/2}} \int_0^\infty \frac{d\lambda}{\lambda^{3/2}} \frac{\tilde{\Omega}_{2,\eta=0}}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \frac{\cosh\left(\Upsilon_{2,\eta=0}\lambda\right)}{\sinh\left(\tilde{\Omega}_{2,\eta=0}\lambda\right)}. \tag{5.114}$$

Where

$$\tilde{\Omega}_{2,\eta=0}^2 = \frac{\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2\left[\left(1+\frac{e\mathcal{B}\theta}{4}\right)-\frac{e\mathcal{B}\theta}{4}\right]^2 - \left[e\mathcal{B}\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2 - \frac{e\mathcal{B}}{2}\left(\frac{e\theta\mathcal{E}}{2\upsilon_F}\right)^2 - \left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2\frac{\theta}{2}\left(1+\frac{e\mathcal{B}\theta}{4}\right)\right]^2}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2 - \left(\frac{e\theta\mathcal{E}}{2\upsilon_F}\right)^2}, \tag{5.115}$$

and $\Upsilon_{2,\eta=0}=(1+\frac{e\mathcal{B}\theta}{2})\sqrt{(e\mathcal{E})^2/\upsilon_F^2-(e\mathcal{B})^2}$.

From the above Eqs. (5.113) and (5.114), for each gauge, the effective action is different. This indicates that in this case, the pair production problem becomes essential because if $\theta\neq 0$ and $\eta=0$, we conclude that the creation of pairs for the two gauges gives different results. On the other hand, in Ref. [95], the author "Sheikh-Jabbari" has studied the problem of pair creation by an external electromagnetic field in NC space given in a gauge configuration that differs from ours; his study gives necessarily different results, and this is due to the properties of non-commutativity.

Then, for the second case ($\mathcal{E}\neq 0$, $\mathcal{B}=0$), the effective actions are defined in the **Table1** . In particular, when $\theta=0$ and $\eta\neq 0$, we find the same expression of the $S_{eff.}^{(\mathrm{NC})}$ for both gauges that is given by

$$S_{eff.}^{(\mathrm{NC,G1,G2})}=\imath^{3/2}\frac{\upsilon_F TL^2}{8\pi^{3/2}}\sqrt{\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2-\eta^2}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\coth\left(\sqrt{\left(\frac{e\mathcal{E}}{\upsilon_F}\right)^2-\eta^2}\lambda\right), \tag{5.116}$$

while in the case when $\theta\neq 0$ and $\eta=0$, the $S_{eff.}^{(\mathrm{NC})}$ expression for both gauges is reduced to

$$S_{eff.}^{(\mathrm{NC,G1,G2})}=\imath^{3/2}\frac{\upsilon_F TL^2}{8\pi^{3/2}}\frac{e\mathcal{E}}{\upsilon_F}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\coth\left(\frac{e\mathcal{E}}{\upsilon_F}\lambda\right), \tag{5.117}$$

which gives the same expression in an ordinary uniform electric field [162, 94].

Then, in the third case ($\mathcal{E}=0,\mathcal{B}\neq 0$), the expression of effective actions is defined in **Table1**. Especially if $\theta=0$ and $\eta\neq 0$, the result of $S_{eff.}^{(\mathrm{NC})}$ is the same for two gauges and is written as

$$S_{eff.}^{(\mathrm{NC,G1,G2})}=\imath^{3/2}\frac{\upsilon_F TL^2}{8\pi^{3/2}}e\mathcal{B}\left(1+\frac{\eta}{e\mathcal{B}}\right)\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\cot\left(e\mathcal{B}(1+\tfrac{\eta}{e\mathcal{B}})\lambda\right). \tag{5.118}$$

As well as when $\theta\neq 0$ and $\eta=0$, the $S_{eff.}^{(\mathrm{NC})}$ expression for two gauges is given by

$$S_{eff.}^{(\mathrm{NC,G1,G2})}=\imath^{3/2}\frac{\upsilon_F TL^2}{8\pi^{3/2}}\frac{e\mathcal{B}}{1+\frac{e\mathcal{B}\theta}{4}}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\cot\left(e\mathcal{B}(1+\tfrac{e\mathcal{B}\theta}{4})\lambda\right). \tag{5.119}$$

In the free case ($\mathcal{B}=\mathcal{E}=0$), from Eq. (5.71) and Eq. (5.106), the effective action in these NC phase space coordinates for both gauges is the same result specified in the following equation

$$S_{eff-free}^{(\mathrm{NC,G1,G2})}=\imath^{3/2}TL^2\frac{\upsilon_F}{8\pi^{3/2}}\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\eta\cot(\eta\lambda), \tag{5.120}$$

which is the same result of the effective action of a charged massless particle in a constant magnetic field (η/e).

5.6 Schwinger's pair production in monolayer graphene under the action of electromagnetic plane wave fields

In this section, we suggest to study the influence of a single plane wave field on the process of particle-antiparticle pair creation from the vacuum in ordinary space phase coordinates in monolayer graphene following the Green function that exists in the fourth chapter.

5.6.1 The effective action expression for a single-plane wave field

As it is well known, Schwinger's effective action is defined as

$$S_{eff.}=\int_0^{\infty}\frac{d\lambda}{\lambda}Tr\ \mathcal{G}\left(x_b^{\mu},x_a^{\mu},\lambda\right), \tag{5.121}$$

where the corresponding kernel propagator $\mathcal{G}\left(x_b^{\mu},x_a^{\mu},\lambda\right)$ can be presented in the form

$$\mathcal{G}\left(x_b^{\mu},x_a^{\mu},\lambda\right)=-\imath\mathbb{T}\left\langle x_b\right|\exp\left\{-\imath\lambda\left[\mathcal{H}\left(x,k\right)\right]\right\}\left|x_a\right\rangle . \tag{5.122}$$

The Green function for a (2 + 1)-dimensional relativistic Dirac massless particles in the presence of an electromagnetic plane wave field is defined as [100]

$$\begin{aligned}\mathcal{G}\left(x_b^{\mu},x_a^{\mu},\lambda\right)=&\int_0^{\infty}d\lambda\int\frac{d^3k}{\left(2\pi\right)^3}\left[\hat{k}\left[1+\frac{e}{2nk}\hat{n}\left(\hat{A}_b-\hat{A}_a\right)\right]-e\hat{A}_b\right.\\&\left.+\frac{e}{2kn}\hat{n}\left(kA_b\right)-\frac{e^2}{2kn}\hat{n}\left(A_aA_b\right)+\frac{e^2}{2kn}\hat{n}\hat{A}_a\hat{A}_b\right]\\&\exp\left\{ik\left(x_b-x_a\right)+i\lambda k^2+\frac{i}{2nk}\int_{\xi_a}^{\xi_b}\left(-2eA\left(\xi\right)k_x+e^2A^2\left(\xi\right)\right)d\xi\right\},\end{aligned} \tag{5.123}$$

where, $\left(\hat{n}=n_{\mu}\gamma^{\mu},\hat{A}=A_{\mu}\gamma^{\mu},\hat{k}=k_{\mu}\gamma^{\mu}\right)$.

The effective action then takes the following form

$$\begin{aligned}S_{eff.}\ =\ &-i\int\frac{d\lambda}{\lambda}\int_{x_b=x_a}dx_bTr\left[\int e^{i\lambda k^2}\frac{d^3k}{\left(2\pi\right)^3}\left\{\hat{k}\left[1+\frac{e}{2k^{\mu}n_{\mu}}\hat{n}\left(\hat{A}_b-\hat{A}_a\right)\right]\right.\right.\\&\left.-e\hat{A}_b+\frac{e}{2kn}\hat{n}\left(kA_b\right)-\frac{e^2}{2kn}\hat{n}\left(A_aA_b\right)+\frac{e^2}{2kn}\hat{n}\hat{A}_a\hat{A}_b\right\}\\&\left.\times\exp\left\{ik\left(x_b-x_a\right)+\frac{i}{2nk}\int_{\xi_a}^{\xi_b}\left(-2eA\left(\xi\right)k_x+e^2A^2\left(\xi\right)\right)d\xi\right\}\right],\end{aligned} \tag{5.124}$$

we have $\xi=n^{\mu}x_{\mu}=y-\tau\rightarrow\xi_a=y_a-\tau_a,\xi_b=y_b-\tau_b$ and $\hat{A}_a=A_{a\mu}\gamma^{\mu},\hat{A}_b=A_{b\mu}\gamma^{\mu},\hat{n}=n_{\mu}\gamma^{\mu}$, also $A_{b\nu}=A_{\nu}\left(nx_b\right),A_{a\nu}=A_{\nu}\left(nx_a\right).$

We can write the effective action expression with the component in the following form

$$\begin{aligned}
S_{eff.} &= -i\int\frac{d\lambda}{\lambda}\int_{x_b=x_a}dx_b\int\frac{d^3k}{(2\pi)^3}e^{i\lambda k^2}\\
&Tr\left[\left\{k_\tau\gamma^0\left[1+\frac{e}{2k^\mu n_\mu}\left[\left(n_\tau\gamma^0+n_y\gamma^2\right)\left(A_x\left(nx_b\right)-A_x\left(nx_a\right)\right)\gamma^1\right]\right]\right.\right.\\
&+k_x\gamma^1\left[1+\frac{e}{2k^\mu n_\mu}\left[\left(n_\tau\gamma^0+n_y\gamma^2\right)\left(A_x\left(nx_b\right)-A_x\left(nx_a\right)\right)\gamma^1\right]\right]\\
&+k_y\gamma^2\left[1+\frac{e}{2k^\mu n_\mu}\left[\left(n_\tau\gamma^0+n_y\gamma^2\right)\left(A_x\left(nx_b\right)-A_x\left(nx_a\right)\right)\gamma^1\right]\right]\\
&-eA_x\left(nx_b\right)\gamma^1+\frac{e^2}{2k^\mu n_\mu}\left(n_\tau\gamma^0+n_y\gamma^2\right)\left(A_x\left(nx_a\right)\gamma^1\right)\left(A_x\left(nx_b\right)\gamma^1\right)\\
&\left.+\frac{e}{k^\mu n_\mu}\left(n_\tau\gamma^0+n_y\gamma^2\right)k^\nu A_\nu\left(nx_b\right)-\frac{e^2}{k^\mu n_\mu}\left(n_\tau\gamma^0+n_y\gamma^2\right)A_{a\nu}A^\nu\left(nx_b\right)\right\}\\
&\left.\times\exp\left\{ik\left(x_b-x_a\right)+\frac{i}{2k^\mu n_\mu}\int_{nx_a}^{nx_b}\left(-2eA\left(nx\right)k_x+e^2A^2\left(nx\right)\right)n^\mu dx_\mu\right\}\right] \qquad (5.125)
\end{aligned}$$

By using the fundamental properties of trace of γ matrices, which are represented by the Pauli matrices in two dimensions, they are defined as

$$\gamma^0=\sigma_3,\ \gamma^1=i\sigma_2,\ \gamma^2=-i\sigma_1. \qquad (5.126)$$

it is easy to obtain the traces

$$\begin{aligned}
Tr_\gamma Tr_x\quad &\mathcal{G}\left(x_b^\mu,x_a^\mu,\lambda\right)=Tr_\gamma Tr_x\left[\left\{\hat{k}\left[1+\frac{e}{2k^\mu n_\mu}\hat{n}\left(\hat{A}_b-\hat{A}_a\right)\right]\right.\right.\\
&\left.-e\hat{A}_b+\frac{e}{2kn}\hat{n}\left(kA_b\right)-\frac{e^2}{2kn}\hat{n}\left(A_aA_b\right)+\frac{e^2}{2kn}\hat{n}\hat{A}_a\hat{A}_b\right\}\\
&\left.\exp\left\{ik\left(x_b-x_a\right)+i\lambda k^2-\frac{i}{2nk}\int_{\xi_a}^{\xi_b}\left(2eA\left(\xi\right)k_x+e^2A^2\left(\xi\right)\right)d\xi\right\}\right]=0. \qquad (5.127)
\end{aligned}$$

The effective action in this case thus vanishes.

$$S_{eff.}=0. \qquad (5.128)$$

Consequently, the pair production probability of quasiparticles-holes in graphene by an electromagnetic plane wave is null.

$$\mathcal{P}_{Creat}=0. \qquad (5.129)$$

5.6.2 The effective action expression for two plane wave fields

In this section, we suggest to study the influence of the two orthogonal plane wave fields on the creation of quasiparticles-holes) in monolayer graphene. The Green function for a $(2+1)$-dimensional relativistic Dirac massless particles in this configuration of the field is defined as

$$\mathcal{G}\left(x_b^\mu, x_a^\mu, \lambda\right) = \int \frac{d^2k}{(2\pi)^2} \int_0^\infty d\lambda \prod_{i=1}^{2} \left\{ \hat{k}\left[1 - \frac{e}{2k^\mu n_{i\mu}} \hat{n}_i \left(\hat{A}_{ib} - \hat{A}_{ia}\right)\right] - e\hat{A}_{ib} + \frac{e^2}{2k^\mu n_\mu} \hat{n}_i \hat{A}_{ia}\hat{A}_{ib} + \frac{e}{k^\mu n_{i\mu}} \hat{n}_i k A_{ib} - \frac{e^2}{k^\mu n_{i\mu}} \hat{n} A_{ia} A_{ib} \right\} \times \exp\left\{ ik\left(x_b - x_a\right) + i\lambda k^2 + \frac{i}{2k^\mu n_{i\mu}} \int_{\xi_{ia}}^{\xi_{ib}} \left(-2e\left(A_i\left(\xi_i\right)\right) k_x + e^2 \left(A_i\left(\xi_i\right)\right)^2\right) d\xi_i \right\}. \tag{5.130}$$

It is easy to show that the traces of $\mathcal{G}\left(x_b^\mu, x_a^\mu, \lambda\right)$ are null.

$$Tr_\gamma Tr_x \mathcal{G}\left(x_b^\mu, x_a^\mu, \lambda\right) = Tr_\gamma Tr_x \prod_{i=1}^{2} \left\{ \hat{k}\left[1 - \frac{e}{2k^\mu n_{i\mu}} \hat{n}_i \left(\hat{A}_{ib} - \hat{A}_{ia}\right)\right] - e\hat{A}_{ib} + \frac{e^2}{2k^\mu n_\mu} \hat{n}_i \hat{A}_{ia}\hat{A}_{ib} + \frac{e}{k^\mu n_{i\mu}} \hat{n}_i k A_{ib} - \frac{e^2}{k^\mu n_{i\mu}} \hat{n} A_{ia} A_{ib} \right\} \times \exp\left\{ ik\left(x_b - x_a\right) + i\lambda k^2 + \frac{i}{2k^\mu n_{i\mu}} \int_{\xi_{ia}}^{\xi_{ib}} \left(-2e\left(A_i\left(\xi_i\right)\right) k_x + e^2 \left(A_i\left(\xi_i\right)\right)^2\right) d\xi_i \right\} = 0, \tag{5.131}$$

Then the corresponding effective action is also null.

$$S_{eff.} = 0. \tag{5.132}$$

Therefore, the probability of pair production for graphene quasiparticles under the action of two orthogonal electromagnetic plane wave fields is null.

$$\mathcal{P}_{Creat} = 0. \tag{5.133}$$

5.7 Conclusion

In this chapter, we have studied the problem of particle-antiparticle pair creation from the vacuum in monolayer graphene by a constant electromagnetic field in the framework of non-commutative phase space coordinates using Schwinger's method.

we have made the corresponding quadratic Lagrangian, and then we have calculated the effective action using the supersymetric path integral formalism for the first and second gauges that are defined in the Eqs. (5.1) and (5.2). Also, all special cases of $(\theta, \eta, \mathcal{E}, \mathcal{B})$ for each gauge are discussed. It is shown that the results are identical to the Schwinger result in $(2+1)$ QED.

When we put the limits $\theta \longrightarrow 0, \eta \longrightarrow 0$ and $\mathcal{B}=0$, we get the same results for the probability that are obtained in the Refs. [94]. Also, we have obtained an important result, and we show that when $\mathcal{E} = \mathcal{B} = 0$, the corresponding effective action is equivalent to the effective action in a constant magnetic field $(\mathcal{B} \equiv \frac{\eta}{e})$.

On the other hand, the configuration of the field consists of one plane wave field, and two orthogonal plane wave fields do not contribute to the effective action. Consequently, there is no influence of the plane wave fields on the process of pair creation.

Chapter 6

Schwinger pair production of scalar and spinorial particles by a constant electromagnetic field and in non-commutative phase space coordinates

6.1 Introduction

The main goal of this chapter is the study of the creation of relativistic scalar and spinorial particles from the vacuum by an electromagnetic field in noncommutative (NC) phase space coordinates, considering the Schwinger method. We calculate the effective action and the probability of the creation of both scalar and spinorial particles. Also, we discuss all special cases of pair creation probability for each case of $(\theta, \eta, \mathcal{E}, \mathcal{B})$.

6.2 Scalar particles

For determine the pair production probability of scalar particles in the framework of the NC phase space coordinates of quantum field theory, we must use the vacuum-vacuum transition amplitude $A^{(\mathrm{NC})}_{scalar.}$ as a functional integral over all scalar field configurations $\varphi(x)$ and $\varphi^*(x)$, which considered independents and given as [177, 39]

$$A^{(\text{NC})}_{scalar.}(vac - vac) = \int D\varphi D\varphi^* \exp\left[\imath \int d^4x \mathcal{L}^{(\text{NC})}_{KG}\left(\varphi, \varphi^*, \partial_\mu\varphi, \partial_\mu\varphi^*\right)\right], \mu = 0, 1, 2, 3, \quad (6.1)$$

where $\mathcal{L}_{KG}$ is the scalar Lagrangian density defined by

$$\mathcal{L}^{(\text{NC})}_{KG} = \varphi^*(x)\hat{O}_{KG} \star \varphi(x) = \varphi^*(x)\left[\left(\hat{\mathcal{P}}^\mu - eA_\mu\left(\hat{\mathcal{X}}\right)\right)^2 - m^2\right] \star \varphi(x). \quad (6.2)$$

While the operators $\left(\hat{\mathcal{X}}^\mu, \hat{\mathcal{P}}^\mu\right)$ satisfy the canonical commutation relations defined as

$$\left[\hat{\mathcal{P}}_i, \hat{\mathcal{P}}_j\right] = \imath\eta_{ij}, \quad \left[\hat{\mathcal{X}}_i, \hat{\mathcal{X}}_j\right] = \imath\theta_{ij}, \quad \left[\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_j\right] = \imath\delta_{ij}, \quad i, j = 1, 2, \quad (6.3)$$

and the operators $(\hat{x}_i, \hat{p}_i)$ are new variables that satisfy the usual canonical commutation relations given as

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{p}_i, \hat{x}_j] = i\delta_{ij}. \quad (6.4)$$

For solving the previous functional integral, we consider this equation

$$\left[\left(\hat{\mathcal{P}}_\mu - eA_\mu\left(\hat{\mathcal{X}}\right)\right)^2 - m^2\right] \star \phi_n(x) = \lambda_n\phi_n(x), \quad (6.5)$$

on the other hand, we have

$$\int d^4x\phi_n(x)\phi^*_m(x) = \delta_{nm}. \quad (6.6)$$

If we expand $\varphi(x)$ on the basis $\{\phi_n(x)\}$as

$$\varphi(x) = \sum_n a_n\phi_n(x). \quad (6.7)$$

we find

$$A^{(\text{NC})}_{scalar.} = N\int \prod_{i,j} da_i da_j \exp\left[\imath\sum_n |a_n|^2 \lambda_n\right] = \mathcal{N}\left[\det\left(\hat{O}^\star_{KG}\right)\right]^{-1}. \quad (6.8)$$

We can write the term $\frac{1}{\det(\hat{O}^\star_{KG})}$ as follows

$$\frac{1}{\det(\hat{O}^\star_{KG})} = \exp\left[-Tr\left(\ln\hat{O}^\star_{KG}\right)\right], \quad (6.9)$$

by using the following representation

$$\ln\hat{O}^\star_{KG} = Const - \int_0^\infty \frac{d\lambda}{\lambda}\int Dx \exp\left[-\imath\lambda\hat{O}^\star_{KG}\right]. \quad (6.10)$$

From Eqs. (6.8), (6.9) and (6.10) we obtain the famous expression of the effective action

$$S^{(\text{NC})}_{eff.} = -\imath\int_0^\infty \frac{d\lambda}{\lambda} Tr\langle x_b|\,\mathbb{T}.\exp\left\{-\imath\lambda\hat{O}^\star_{KG}\left(\hat{\mathcal{X}}, \hat{\mathcal{P}}\right)\right\}|x_a\rangle. \quad (6.11)$$

Where "Tr" indicates the complete diagonal summation over the continuous space-time coordinates.

6.2.1 The evaluation of propagator $\mathcal{G}^{(\text{NC})}(x_b, x_a, \lambda)$

We consider the vector potential $A_\mu(\mathcal{X})$ which is the same as the second gauge in previous chapter, defined as

$$A_\mu = (-\mathcal{E}\mathcal{X}_2, -\frac{\mathcal{B}}{2}\mathcal{X}_2, \frac{\mathcal{B}}{2}\mathcal{X}_1, 0), \tag{6.12}$$

The corresponding Hamiltonian $\hat{O}^{\star}_{KG}$ subjected to the action of this uniform electromagnetic field is defined as follows

$$\hat{O}^{\star}_{KG}\left(\hat{\mathcal{X}}, \hat{\mathcal{P}}\right) = (\hat{E} + e\mathcal{E}(\hat{y} + \frac{\theta}{2}\hat{p}_x))^2 - (\hat{p}_x + \frac{\eta}{2}\hat{y} + \frac{e\mathcal{B}}{2}(\hat{y} + \frac{\theta}{2}\hat{p}_x))^2$$
$$-(\hat{p}_y - \frac{\eta}{2}\hat{x} - \tfrac{e\mathcal{B}}{2}(\hat{x} - \frac{\theta}{2}\hat{p}_y))^2 - m^2. \tag{6.13}$$

where

$$\hat{\mathcal{X}}_i = \hat{x}_i - \frac{\theta_{ij}}{2}\hat{p}_j,\ \hat{\mathcal{P}}_i = \hat{p}_i + \frac{\eta_{ij}}{2}\hat{x}_j, i = 1, 2. \tag{6.14}$$

It is simple to demonstrate that the action of a scalar propagator of a charged particle in a constant external electromagnetic field in NC phase space coordinates is identical to that of a propagator in a constant external magnetic field in the z direction and with a quadratic potential in standard quantum field theory.

By using the Lagrangian representation, we find

$$S^{(\text{NC})}_{eff.} = -\imath \int_0^\infty \frac{d\lambda}{\lambda} e^{-\imath\lambda m^2} Tr\left[\int \frac{dE}{2\pi} e^{-\imath E(t_b - t_a)} \mathcal{G}^{(\text{NC})}(x_b, x_a)\right]. \tag{6.15}$$

The propagator $\mathcal{G}^{(\text{NC})}(x_b, x_a)$ is distinguished from the effective action $S^{(\text{NC})}_{eff.}$ by the parameter $1/\lambda$ and the boundary condition $\vec{x}(0) = \vec{x}(\lambda)$.Then, to calculate the effective action ($S^{(\text{NC})}_{eff.}$), we must calculate the propagator $\mathcal{G}^{(\text{NC})}(x_b, x_a)$ as follows

$$\mathcal{G}^{(\text{NC})}(x_b, x_a) = \int DxDyDz \exp\left[\imath \int_0^\lambda L(x, \dot{x}, t)\, dt\right]. \tag{6.16}$$

Where the Lagrangian function is given as

$$L(x, \dot{x}, t) = \tfrac{m_x}{2}\dot{x}^2 + \tfrac{m_Y}{2}\dot{Y}^2 + \frac{m_z}{2}\dot{z}^2 + \omega_1 x\dot{Y} - \omega_2 Y\dot{x} - \omega_Y^2 Y^2$$
$$+E(x_b - x_a)\,\delta(\theta, \eta). \tag{6.17}$$

Whereas the variable Y is dependent on term energy defined as ($Y = y + \frac{E}{e\mathcal{E}} \frac{\left(1 + \frac{e\mathcal{B}\theta}{4}\right)}{\left[\left(1 + \frac{e\mathcal{B}\theta}{2}\right) - \frac{e\mathcal{B}\theta}{4}\left(1 + \frac{\eta}{2e\mathcal{B}}\right)\right]}$) and m_x, m_Y and m_z are the effective masses along the x, Y and z directions, respectively, while

ω_Y is the frequency along the Y and ω_1, ω_2 play the role of the components of the effective magnetic field. These masses and frequencies depend on the values of the external electromagnetic field $(\mathcal{E}, \mathcal{B})$ and the parameters of deformation (θ, η) and are given by

$$m_x = \frac{1}{2\left[\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2 - \left(\frac{e\theta\mathcal{E}}{2}\right)^2\right]},\ m_Y = \frac{1}{2\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2},\ m_z = \frac{1}{2},$$
$$\omega_1 = \frac{e\mathcal{B}}{2}\frac{\left(1+\frac{\eta}{e\mathcal{B}}\right)}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)},\ \omega_2 = \frac{\frac{eB}{2}\left(1+\frac{e\mathcal{B}\theta}{4}\right)\left(1+\frac{\eta}{e\mathcal{B}}\right) - (e\mathcal{E})^2\frac{\theta}{2}}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2 - \left(\frac{e\theta\mathcal{E}}{2}\right)^2},$$
$$\omega_Y^2 = -\left(e\mathcal{E}\right)^2 \frac{\left[\left(1+\frac{e\mathcal{B}\theta}{4}\right) - \frac{e\mathcal{B}\theta}{4}\left(1+\frac{\eta}{e\mathcal{B}}\right)\right]^2}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2 - \left(\frac{e\theta\mathcal{E}}{2}\right)^2} \text{ and } \delta\left(\theta,\eta\right) = \frac{\frac{e\mathcal{B}}{2}\left(1+\frac{\eta}{e\mathcal{B}}\right)}{e\mathcal{E}\left[\left(1+\frac{e\mathcal{B}\theta}{4}\right) - \frac{e\mathcal{B}\theta}{4}\left(1+\frac{\eta}{e\mathcal{B}}\right)\right]}. \tag{6.18}$$

Knowing that the coordinate z is free, then the propagator is given as

$$\mathcal{G}^{(\text{NC})}\left(x_b, x_a\right) = \sqrt{\frac{m_z}{2\pi\imath\lambda}} \exp\left\{\imath \tfrac{m_z}{2\lambda}\left(z_b - z_a\right)^2\right\} \exp\left(\imath\omega_2\left(x_a Y_a - x_b Y_b\right)\right) \mathcal{K}\left(x_b, x_a, \lambda\right), \tag{6.19}$$

where the kernel propagator $\mathcal{K}\left(x_b^\mu, x_a^\mu, \lambda\right)$ can be written as

$$\mathcal{K}\left(x_b, x_a, \lambda\right) = \int DY \exp\left\{\imath \int_0^\lambda \left[\tfrac{m_Y \dot{Y}^2}{2} - \omega_Y^2 Y^2\right] dt\right\} \mathbb{K}\left[Y\left(t\right)\right], \tag{6.20}$$

We note that $\mathbb{K}\left[Y\left(t\right)\right]$ is the free particle propagator in a time-dependent external force $\left(\left(\omega_1 + \omega_2\right)\dot{y}\right)$ defined by

$$\mathbb{K}\left[Y\left(t\right)\right] = \int Dx \exp\left\{\imath \int_0^\lambda \left[\tfrac{m_x}{2}\dot{x}^2 + \left(\omega_1 + \omega_2\right)\dot{Y}x\right] dt\right\}. \tag{6.21}$$

Following [179], the propagator $\mathbb{K}\left[Y\left(t\right)\right]$ is transformed to

$$\mathbb{K}\left[Y\left(t\right)\right] = \sqrt{\frac{m_x}{2\pi\imath\lambda}} \exp\left\{\imath\tfrac{m_x}{2\lambda}\left(x_b - x_a\right)^2\right\} \exp\left\{\imath\left(\omega_1+\omega_2\right)\left(x_b Y_b - x_a Y_a\right)\right\}$$
$$\times \exp\left\{\imath \int_0^\lambda \left[\frac{\left(\omega_1+\omega_2\right)}{\lambda}\left(x_a - x_b\right)Y - \frac{\left(\omega_1+\omega_2\right)^2}{2m_x}Y^2 + \frac{\left(\omega_1+\omega_2\right)^2}{m_x\lambda}Y\int_0^t Y\left(s\right)ds\right] dt\right\}. \tag{6.22}$$

By substituting Eq. (6.22) into Eq. (6.20) we obtain

$$\mathcal{G}^{(\text{NC})}\left(x_b, x_a, \lambda\right) = \sqrt{\frac{m_z}{2\pi\imath\lambda}} e^{\imath\frac{m_z}{2\lambda}\left(z_b - z_a\right)^2} \sqrt{\frac{m_x}{2\pi\imath\lambda}} e^{\imath\frac{m_x}{2\lambda}\left(x_b - x_a\right)^2} e^{\imath\omega_1\left(x_b Y_b - x_a Y_a\right)}$$
$$\times \int DY \mathcal{K}\left(x_b, x_a, \lambda\right). \tag{6.23}$$

After integration by part of the term $\left(\int_0^\lambda dt \left[Y(t)\int_0^t Y(s)\,ds\right]\right)$, we find

$$\mathcal{K}(x_b,x_a,\lambda) = \int DY \exp\left\{\imath\tfrac{m_Y}{2}\int_0^\lambda \left[\dot{Y}^2 + \tfrac{2(\omega_1+\omega_2)}{\lambda m_Y}(x_a-x_b)Y - \left(\tfrac{(\omega_1+\omega_2)^2}{m_x m_Y}+\tfrac{2\omega_Y^2}{m_Y}\right)Y^2\right]dt\right\}$$
$$\times \exp\left\{\imath\tfrac{m_Y}{2}\frac{(\omega_1+\omega_2)^2}{m_x m_Y \lambda}\left[\int_0^\lambda Y(s)\,ds\right]^2\right\}. \tag{6.24}$$

Then, we can write the final term in Eq. (6.24) as follows [178]

$$\exp\left\{\imath\tfrac{m_Y}{2}\frac{(\omega_1+\omega_2)^2}{m_x m_Y \lambda}\left[\int_0^T Y(s)\,ds\right]^2\right\} = \sqrt{\frac{\imath m_Y}{2\pi\lambda}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{\imath m_Y}{2\lambda}\chi^2 + \imath m_Y\frac{(\omega_1+\omega_2)}{\sqrt{m_x m_Y}\lambda}\chi\int_0^\lambda Y(t)\,dt\right\}d\chi. \tag{6.25}$$

Therefore, Eq. (6.24) becomes as

$$\mathcal{K}(x_b,x_a,\lambda) = \sqrt{\frac{\imath m_Y}{2\pi\lambda}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{\imath m_Y}{2\lambda}\chi^2\right\}\mathbf{K}(x_b,x_a,\lambda,\chi)\,d\chi, \tag{6.26}$$

where $\mathbf{K}(x_b,x_a,\lambda,\chi)$ is given by

$$\mathbf{K}(x_b,x_a,\lambda,\chi) = \int DY \exp\left\{\imath\int_0^\lambda\left[\tfrac{m_Y}{2}\left(\dot{Y}^2-\Omega^2Y^2\right) + \left(\tfrac{(\omega_1+\omega_2)}{\lambda}\left((x_a-x_b)+\sqrt{\frac{m_Y}{m_x}}\chi\right)\right)Y\right]dt\right\}. \tag{6.27}$$

This propagator represents a one-dimensional forced harmonic oscillator with a time-independent external force defined as

$$F_Y = \frac{(\omega_1+\omega_2)}{\lambda}\left((x_a-x_b)+\sqrt{\frac{m_Y}{m_x}}\chi\right). \tag{6.28}$$

Following [179], in this gauge, the expression of the kernel (6.27) takes the following form

$$\mathbf{K}(x_b,x_a,\lambda,\chi) = \sqrt{\tfrac{m_Y\Omega}{2\pi\imath\sin(\Omega\lambda)}}\exp\left\{\imath\tfrac{m_Y\Omega}{2\sin(\Omega\lambda)}\left(\left(Y_a^2+Y_b^2\right)\cos(\Omega\lambda)-2Y_bY_a\right)\right\}$$
$$\times\exp\left\{\imath\left[-\frac{\tfrac{(\omega_1+\omega_2)}{\lambda}\left((x_a-x_b)+\sqrt{\tfrac{m_Y}{m_x}}\chi\right)}{\Omega\sin(\Omega\lambda)}(Y_b+Y_a)(\cos(\Omega\lambda)-1)\right]\right\}$$
$$\times\exp\left\{\imath\frac{\left(\tfrac{(\omega_1+\omega_2)}{\lambda}\left((x_a-x_b)+\sqrt{\tfrac{m_Y}{m_x}}\chi\right)\right)^2}{m_Y\Omega^3\sin(\Omega\lambda)}(\cos(\Omega\lambda)-1)\right\}\exp\left[\imath\frac{\lambda\left(\tfrac{(\omega_1+\omega_2)}{\lambda}\left((x_a-x_b)+\sqrt{\tfrac{m_Y}{m_x}}\chi\right)\right)^2}{2m_Y\Omega^2}\right], \tag{6.29}$$

with

$$\Omega^2 = \left(\frac{(\omega_1+\omega_2)^2}{m_x m_Y}+2\frac{\omega_Y^2}{m_Y}\right). \tag{6.30}$$

Finally, after substituting Eq.(5.98) into Eq. (5.95) and performing the integration over χ from $(-\infty$ to $+\infty)$, then by combining (5.95) into (5.89), the Green function will express as follows

$$\mathcal{G}^{(\text{NC})}(x_b,x_a,\lambda) = -\imath\int_0^\infty \frac{d\lambda}{\lambda}e^{-\imath\lambda m^2}\int\frac{dE}{2\pi}\int dt_b dx_b dy_b dz_b$$
$$\times e^{-\imath E(t_b-t_a)}\left[A(\Omega\lambda)\right]^{-1/2}K_0(z_b,z_a,\lambda)\,K_0(x_b,x_a,\lambda)\,K_\Omega(Y_b,Y_a,\lambda)$$
$$\times\exp\left\{\imath\omega_1(x_bY_b-x_aY_a)+\imath E(x_b-x_a)\,\delta(\theta,\eta)\right\}$$
$$\times\exp\left[D_x(\lambda)(x_a-x_b)^2+D_{xY}(\lambda)(x_a-x_b)(Y_b+Y_a)+D_Y(\lambda)(Y_b+Y_a)^2\right]|_{x_b=x_a}\,. \tag{6.31}$$

where $K_0(x_b,x_a,\lambda)$ and $K_0(z_b,z_a,\lambda)$ are the propagators of free particles along the axes x and z, respectively.

$K_\Omega(Y_b,Y_a,\lambda)$ is the propagator of a one-dimensional harmonic oscillator with frequency Ω_2.

Whereas $D_x(\lambda)$, $D_{xY}(\lambda)$, $D_Y(\lambda)$ are given as,

$$D_x(\lambda)=\imath\frac{m_x\left[(\omega_1+\omega_2)^2/m_xm_Y\right]}{2\lambda^2\Omega^3}\frac{(\Omega\lambda-2\tan(\Omega\lambda/2))}{A(\Omega\lambda)},$$
$$D_{xY}(\lambda)=\imath\frac{\left[(\omega_1+\omega_2)/\sqrt{m_xm_Y}\right]}{\lambda\Omega}\frac{\sqrt{m_xm_Y}\tan(\Omega\lambda/2)}{A(\Omega\lambda)}\text{ and }D_Y(\lambda)=\imath\frac{m_Y\left[(\omega_1+\omega_2)^2/m_xm_Y\right]}{2\lambda\Omega^2}\frac{\tan^2(\Omega\lambda/2)}{A(\Omega\lambda)}, \tag{6.32}$$

with

$$A(\Omega\lambda)=\left[1+\frac{(\omega_1+\omega_2)^2}{\lambda m_Y m_x\Omega^3}\left[2\tan(\Omega\lambda/2)-\Omega\lambda\right]\right]. \tag{6.33}$$

6.2.2 The effective action expression $S_{eff}^{(\text{NC})}$

We calculate the final space-time coordinates integral for each term in the $S_{eff}^{(\text{NC})}$ expression, we obtain

$$S_{eff.}^{(\text{NC})}=-\imath TL^2\int_0^\infty\frac{d\lambda}{\lambda}e^{-\imath\lambda m^2}\int\frac{dE}{2\pi}$$
$$\times\left[A(\Omega\lambda)\right]^{-1/2}\sqrt{\frac{m_z}{2\pi\imath\lambda}}\sqrt{\frac{m_x}{2\pi\imath\lambda}}\sqrt{\frac{m_Y\Omega}{2\pi\imath\sin(\Omega\lambda)}}$$
$$\times\int dY\exp\left\{\imath m_Y\Omega\left[\frac{\cos(\Omega\lambda)-1}{\sin(\Omega\lambda)}+\frac{2\left[(\omega_1+\omega_2)^2/m_xm_Y\right]}{\lambda\Omega^2}\frac{\tan^2(\Omega\lambda/2)}{A(\Omega\lambda)}\right]Y^2\right\}. \tag{6.34}$$

In our study, we have $y=Y-\frac{E}{e\mathcal{E}}\frac{\left(1+\frac{e\mathcal{B}\theta}{4}\right)}{\left[\left(1+\frac{e\mathcal{B}\theta}{2}\right)-\frac{e\mathcal{B}\theta}{4}\left(1+\frac{\eta}{2e\mathcal{B}}\right)\right]}$, for this E must be constrained in the range $0<E<e\mathcal{E}L((1+\frac{e\mathcal{B}\theta}{2})-\frac{e\mathcal{B}\theta}{4}(1+\frac{\eta}{2e\mathcal{B}}))/(1+\frac{e\mathcal{B}\theta}{4})$ in order that the entire range of time is

included as E varies. By simplification and after integrating over the variable Y, the effective action takes the following form

$$S_{eff.}^{(\mathrm{NC})}=-\imath TL^{2}\int_{0}^{\infty}\frac{d\lambda}{\lambda}e^{-\imath\lambda m^{2}}\int_{0}^{\frac{e\mathcal{E}L\left[\left(1+\frac{e\mathcal{B}\theta}{2}\right)-\frac{e\mathcal{B}\theta}{4}\left(1+\frac{\eta}{2e\mathcal{B}}\right)\right]}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)}}\frac{dE}{2\pi}$$
$$\times\sqrt{\frac{m_{z}}{2\pi\imath\lambda}}\sqrt{\frac{m_{x}}{2\pi\imath\lambda}}\sqrt{\frac{\pi\Omega}{2\imath\omega_{Y}^{2}\tan\left(\Omega\lambda/2\right)}\frac{m_{Y}\Omega}{2\pi\imath\sin(\Omega\lambda)}}. \tag{6.35}$$

Returning to real time via the replacements ($T\rightarrow\imath T$) in the Eq. 6.35. Consequently, we get the final expression of $S_{eff}^{(\mathrm{NC})}$in the presence of a constant electromagnetic $\left(\vec{\mathcal{E}}=\mathcal{E}\vec{j},\ \mathcal{B}=\mathcal{B}\vec{k}\right)$ field in the framework of a NC phase space coordinate for the scalar case defined as [37]

$$S_{eff.}^{(\mathrm{NC})}=TL^{3}\frac{\tilde{\Omega}/(1+\frac{e\mathcal{B}\theta}{4})^{2}}{16\pi^{2}}\int_{0}^{\infty}\frac{d\lambda}{\lambda^{2}}\frac{e^{-\imath\lambda m^{2}}}{\sin\left(\tilde{\Omega}\lambda\right)}. \tag{6.36}$$

where $\tilde{\Omega}^{2}=-\Omega^{2}/4$.

In the free case ($\mathcal{B}=\mathcal{E}=0$), the effective action expression in this NC phase space becomes as follows

$$S_{eff-free.}^{(\mathrm{NC})}=\imath TL^{3}\frac{\eta}{16\pi^{2}}\int_{0}^{\infty}\frac{d\lambda}{\lambda^{2}}\frac{e^{-\imath\lambda m^{2}}}{\sinh(\eta\lambda)}, \tag{6.37}$$

this formula agrees exactly with that of Ref. [37] for a charged particle in a constant magnetic field ($\mathcal{B}=\eta/e$).

6.2.3 The pair production probability

The pair production probability per unit volume and per unit time in the framework of a NC phase space coordinates is defined a

$$\mathcal{P}^{(\mathrm{NC,G1})}\left(\mathrm{pair}\right)=2\,\mathrm{Im}\,S_{eff.}^{(\mathrm{NC})}. \tag{6.38}$$

Where the imaginary part of $S_{eff.}^{(\mathrm{NC})}$ can be written as follows

$$2\,\mathrm{Im}\,S_{eff.}^{(\mathrm{NC})}=\frac{1}{\imath}\left(S_{eff.}^{(\mathrm{NC})}-S_{eff.}^{*(\mathrm{NC})}\right)=\frac{1}{\imath}TL^{3}\frac{\tilde{\Omega}/(1+\frac{e\mathcal{B}\theta}{4})^{2}}{16\pi^{2}}\int_{-\infty}^{+\infty}\frac{d\lambda}{\lambda^{2}}\frac{e^{-\imath\lambda m^{2}}}{\sinh\left(\tilde{\Omega}\lambda\right)}. \tag{6.39}$$

This result is verified when $\tilde{\Omega}^{2}>0$, whereas it equals zero in the opposite case.

Finally, the expression of pair production probability per unit volume per unit time in the framework of NC phase space coordinates for scalar particles is simplified in the following manner

$$\mathcal{P}^{(\mathrm{NC})}(\text{pair}) = TL^3 \frac{\tilde{\Omega}^2}{8\pi^3\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} e^{-\frac{k\pi}{\tilde{\Omega}}m^2}. \tag{6.40}$$

Moreover, in the case when $\mathcal{E} \neq 0$, $\mathcal{B} = 0$, $\theta = 0$, and $\eta = 0$, the pair production probability per unit volume per unit time becomes as

$$\mathcal{P}^{(\mathrm{NC})}(\text{pair}) = \frac{TL^3}{8\pi^3} (e\mathcal{E})^2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} e^{-\frac{k\pi}{e\mathcal{E}}m^2}. \tag{6.41}$$

Following the Eq. (6.39), for the large magnetic field $\mathcal{B} \gg$, the pair production problem is non-existent, which is the same result in the case of the absence of the electromagnetic field.

In the absence of the parameters of deformation (i.e., $\eta \to 0$ and $\theta \to 0$), we obtained the Schwinger results for scalar relativistic particles (see, Refs [37, 35]).

6.3 Spinorial case

6.3.1 The evaluation of propagator $\mathcal{G}^{(\mathrm{NC})}(x_b, x_a, \lambda)$

In this case of the Dirac particle, the vacuum-vacuum transition amplitude $A^{(\mathrm{NC})}_{spin.}$ is defined as a functional integral over all Grassmann field configurations $\psi(x)$ and $\bar{\psi}(x)$ defined by

$$A^{(\mathrm{NC})}_{spin.}(vac - vac) = \int D\psi D\bar{\psi} \exp\left[\imath \int d^4x \bar{\psi} \hat{O}_{Dirac} \psi\right], \tag{6.42}$$

$$\approx \det\left[\hat{O}^{\star}_{KG} - \frac{e}{2}\sigma^{\mu\nu}\mathcal{F}^{\star}_{\mu\nu}\right]^{1/2}. \tag{6.43}$$

where $\hat{O}^{\star}_{KG}$ represents the Hamiltonian of the Klein-Gordon equation in NC phase space coordinates defined in the previous section, and $\mathcal{F}^{\star}_{\mu\nu}$ is the strength tensor of a gauge field related to the non-commutativity of phase space given as

$$\tilde{\mathcal{F}}^{\star}_{02} = \mathcal{E}(1 + \tfrac{e\mathcal{B}\theta}{2}),\ \tilde{\mathcal{F}}^{\star}_{12} = -\mathcal{B}(1 + \tfrac{\theta e\mathcal{B}}{4} + \tfrac{\eta}{e\mathcal{B}}). \tag{6.44}$$

6.3.2 The effective action expression

To determine the effective action expression for relativistic spinorial particles, we introduce two paths of integral representations, including spinor indices represented by Grassmann variables

[105, 168] and the variables relative to space-time coordinates. Consequently, we can write

$$S^{(\text{NC})}_{eff.} = -\imath \int \frac{d\lambda}{\lambda} \exp\left(-\imath m^2 \lambda\right) \int dt_b dx_b dy_b dz_b \int \frac{dE}{2\pi} e^{-\imath E(t_b - t_a)}$$
$$\times \int DxDyDz \int \mathcal{D}\psi \exp\left\{\imath \int_0^\lambda dt \left[\tfrac{m_x}{2}\dot{x}^2 + \tfrac{m_Y}{2}\dot{Y}^2 + \frac{m_z}{2}\dot{z}^2 + \omega_1 x\dot{Y} - \omega_2 Y\dot{x}\right.\right.$$
$$\left.\left. -\omega_Y^2 Y^2 + E\left(x_b - x_a\right)\delta\left(\theta,\eta\right) - \imath e\mathcal{F}^{\star}_{\mu\nu}\psi^\mu\psi^\nu + \imath \ \dot{\psi}\right]\right\}_{x_b = x_a}, \tag{6.45}$$

where the measure $\mathcal{D}\psi$ is given by

$$\mathcal{D}\psi = D \quad \left[\int D\psi \exp\left\{-\int_0^\lambda dt\psi\dot{\psi}\right\}\right]^{-1}. \tag{6.46}$$

Directly, the integration over Grassmann variables gives the following result

$$S^{(\text{NC})}_{eff.} = -\imath \int \frac{d\lambda}{\lambda} \exp\left(-\imath m^2 \lambda\right) \int dt_b dx_b dy_b dz_b \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \sqrt{\frac{\det \mathcal{M}\left(e\right)}{\det \mathcal{M}\left(e=0\right)}}$$
$$\times \int DxDyDz \exp\left\{\imath \int_0^\lambda dt \left[\tfrac{m_x}{2}\dot{x}^2 + \tfrac{m_Y}{2}\dot{Y}^2 + \frac{m_z}{2}\dot{z}^2 + \omega_1 x\dot{Y} - \omega_2 Y\dot{x}\right.\right.$$
$$\left.\left. -\omega_Y^2 Y^2 + E\left(x_b - x_a\right)\delta\left(\theta,\eta\right)\right]\right\}_{x_b = x_a}, \tag{6.47}$$

with

$$\mathcal{M}_{\mu\nu}\left(e,\tau,\tau'\right) = \left[\eta_{\mu\nu}\delta'(\tau-\tau') - e\mathcal{F}^{\star}_{\mu\nu}\left(\tau\right)\delta(\tau-\tau')\right]. \tag{6.48}$$

Returning to real time via the replacement $(T \rightarrow \imath T)$ in the Eq. 6.35. Consequently, the final form of $S^{(\text{NC})}_{eff}$ in the presence of a constant electromagnetic $\left(\vec{\mathcal{E}} = \mathcal{E}\vec{j},\ \mathcal{B} = \mathcal{B}\vec{k}\right)$ field in the framework of NC phase space coordinates for relativistic spinorial particles becomes as [37]

$$S^{(\text{NC})}_{eff.} = -\imath \frac{L^3 T}{16\pi^2} \int_0^\infty \frac{d\lambda}{\lambda^2} \frac{\tilde{\Omega}}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \frac{\cosh(e\Upsilon\lambda)}{\sinh\left(\tilde{\Omega}\lambda\right)} e^{-\imath\lambda m^2}. \tag{6.49}$$

In the free case $(\mathcal{B} = \mathcal{E} = 0)$, the effective action in this NC phase space becomes as follows

$$S^{(\text{NC})}_{eff-free.} = -\imath T L^3 \frac{\eta}{16\pi^2} \int_0^\infty \frac{d\lambda}{\lambda^2} \coth\left(\eta\lambda\right) e^{-\imath\lambda m^2}, \tag{6.50}$$

6.3.3 The pair production probability

In this case, the pair production probability per unit volume and per unit time in the framework of the NC phase space coordinates for spinorial relativistic particles takes the following form

$$\mathcal{P}^{(\text{NC})}\left(\text{pair}\right) = \frac{1}{8\pi^3} \frac{\tilde{\Omega}^2}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2} \sum_{n=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \cos\left(\frac{e\Upsilon}{\tilde{\Omega}} k\pi\right) e^{-\frac{k\pi m^2}{\tilde{\Omega}}}, \tag{6.51}$$

In the absence of the parameters of deformation (i.e., $\eta \to 0$ and $\theta \to 0$), we obtained the same results as Schwinger for $1/2-$spin particles [39, 37]. In the free case ($\mathcal{B} = \mathcal{E} = 0$), the pair production problem is non-existent.

6.4 Special cases

Table 2 represents a list of pair production probabilities of bosonic and spinorial particles in NC phase space coordinates with electric and magnetic fields different from or equal to zero.

TABLE II. The pair production probability of special uniform electromagnetic fields within the NC phase space.

$\forall\,(\eta,\theta)$	$\theta\neq0,\ \eta\neq0$	$\theta\neq0,\ \eta=0$	$\theta=0,\ \eta\neq0$	$\theta=0,\ \eta=0$
$\mathcal{B}\neq0,\ \mathcal{E}\neq0$	$\mathrm{P}^{(\mathrm{NC})}\left(\mathrm{pair}\right)\simeq\frac{1}{8\pi^3}\frac{\tilde{\Omega}^2}{\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2}\cos\left(\epsilon\frac{e\Upsilon}{\tilde{\Omega}}\pi\right)e^{-\frac{\pi m^2}{\tilde{\Omega}}}$	$\mathrm{P}^{(\mathrm{NC})}\left(\mathrm{pair}\right)\simeq\frac{1}{8\pi^3}\tilde{\Omega}^2e^{-\frac{\pi m^2}{\tilde{\Omega}}}\cos\left(\epsilon\frac{e\sqrt{\mathcal{E}^2(1+\frac{e\mathcal{B}\theta}{2})^2-\mathcal{B}^2(1+\frac{e\mathcal{B}\theta}{4})^2}}{\tilde{\Omega}}\pi\right)$ with $\tilde{\Omega}=\sqrt{\frac{(e\mathcal{E})^2\left[\left(1+\frac{e\mathcal{B}\theta}{4}\right)-\frac{e\mathcal{B}\theta}{4}\right]^2-\left(e\mathcal{B}\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2-e\mathcal{B}\left(\frac{e\theta\mathcal{E}}{2}\right)^2-(e\mathcal{E})^2\frac{\theta}{2}\right)^2}{\left(\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2-\left(\frac{e\theta\mathcal{E}}{2}\right)^2\right)}}$	$\mathrm{P}^{(\mathrm{NC})}\left(\mathrm{pair}\right)\simeq\frac{1}{8\pi^3}e\left(\mathcal{E}^2-\mathcal{B}^2\left(1+\frac{\eta}{e\mathcal{B}}\right)^2\right)e^{-\frac{\pi m^2}{e\sqrt{\mathcal{E}^2-\mathcal{B}^2\left(1+\frac{\eta}{e\mathcal{B}}\right)^2}}}\cos\left(\epsilon\pi\right)$	$\mathrm{P}^{(\mathrm{NC})}\left(\mathrm{pair}\right)\simeq\frac{1}{8\pi^3}\left((e\mathcal{E})^2-(e\mathcal{B})^2\right)e^{-\frac{\pi m^2}{\sqrt{(e\mathcal{E})^2-(e\mathcal{B})^2}}}\cos\left(\epsilon\pi\right)$
$\mathcal{B}=0,\ \mathcal{E}\neq0$	$\mathrm{P}^{(\mathrm{NC})}\left(\mathrm{pair}\right)\simeq\frac{1}{8\pi^3}\frac{\tilde{\Omega}^2}{1-\frac{(e\mathcal{E}\theta)^2}{4}}e^{-\frac{\pi m^2\sqrt{1-\frac{(e\mathcal{E}\theta)^2}{4}}}{\tilde{\Omega}}}\cos\left(\epsilon\frac{\pi\sqrt{(e\mathcal{E})^2-\eta^2}\sqrt{1-\frac{(e\mathcal{E}\theta)^2}{4}}}{\tilde{\Omega}}\right)$ With $\tilde{\Omega}=\sqrt{(e\mathcal{E})^2\left(1-\frac{\eta\theta}{4}\right)^2-\left(\eta-\frac{\theta}{2}(e\mathcal{E})^2\left(1+\frac{\eta\theta}{4}\right)\right)^2}$	$\mathrm{P}^{(\mathrm{NC})}\left(\mathrm{pair}\right)\simeq\frac{(e\mathcal{E})^2}{8\pi^3}e^{-\frac{\pi m^2}{e\mathcal{E}}}$	$\mathcal{P}^{(\mathrm{NC})}\left(\mathrm{pair}\right)=\frac{1}{8\pi^3}\left((e\mathcal{E})^2-\eta^2\right)e^{-\frac{\pi m^2}{\sqrt{(e\mathcal{E})^2-\eta^2}}}$	$\mathcal{P}^{(\mathrm{NC})}\left(\mathrm{pair}\right)\simeq\frac{(e\mathcal{E})^2}{8\pi^3}e^{-\frac{\pi m^2}{e\mathcal{E}}}$

From the effective action expressions of both bosonic and fermionic particles , we will see that if ($\mathcal{B} \neq 0$, $\mathcal{E} = 0$, $\forall(\theta, \eta)$) the imaginary part of this latter is zero, this leads to the corresponding pair production probability also being zero, which indicates that there is no pair creation.

$$\mathcal{P}^{(\text{NC})}(\text{pair}) = 0. \tag{6.52}$$

But when ($\mathcal{B} \neq 0$, $\mathcal{E} \neq 0$, $\theta \neq 0$, and $\eta \neq 0$), the pair production probability $\mathcal{P}^{(\text{NC})}(\text{pair})$ is defined in Eqs. 5.79 and 6.38 given as

$$\mathcal{P}^{(\text{NC})}(\text{pair}) \simeq \frac{1}{8\pi^3} \frac{\tilde{\Omega}^2}{\left(1 + \frac{e\mathcal{B}\theta}{4}\right)^2} \cos\left(\epsilon \frac{e\Upsilon}{\tilde{\Omega}} \pi\right) e^{-\frac{\pi m^2}{\tilde{\Omega}}}, \tag{6.53}$$

with $\epsilon = (0, 1)$ representing the bosonic and fermionic particles, respectively.

While when ($\mathcal{B} \neq 0$, $\mathcal{E} \neq 0$, $\theta \neq 0$, and $\eta = 0$) the pair production probability becomes as

$$\mathcal{P}^{(\text{NC})}_{\eta=0}(\text{pair}) = \frac{1}{8\pi^3} \tilde{\Omega}^2 e^{-\frac{\pi m^2}{\tilde{\Omega}}} \cos\left(\epsilon \frac{e\sqrt{\mathcal{E}^2(1+\frac{e\mathcal{B}\theta}{2})^2 - \mathcal{B}^2(1+\frac{e\mathcal{B}\theta}{4})^2}}{\tilde{\Omega}} \pi\right), \tag{6.54}$$

with $\tilde{\Omega} = \sqrt{\frac{(e\mathcal{E})^2\left[\left(1+\frac{e\mathcal{B}\theta}{4}\right) - \frac{e\mathcal{B}\theta}{4}\right]^2 - \left(e\mathcal{B}\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2 - e\mathcal{B}\left(\frac{e\theta\mathcal{E}}{2}\right)^2 - (e\mathcal{E})^2\frac{\theta}{2}\right)^2}{\left(\left(1+\frac{e\mathcal{B}\theta}{4}\right)^2 - \left(\frac{e\theta\mathcal{E}}{2}\right)^2\right)}}$.

In addition, we conclude that if ($\mathcal{B} \neq 0$, $\mathcal{E} \neq 0$, $\theta = 0$, and $\eta \neq 0$) the pair production probability will end up with the following result

$$\mathcal{P}^{(\text{NC})}_{\theta=0}(\text{pair}) \simeq \frac{1}{8\pi^3} e\left(\mathcal{E}^2 - \mathcal{B}^2\left(1 + \frac{\eta}{e\mathcal{B}}\right)^2\right) e^{-\frac{\pi m^2}{e\sqrt{\mathcal{E}^2 - \mathcal{B}^2\left(1+\frac{\eta}{e\mathcal{B}}\right)^2}}} \cos(\epsilon\pi). \tag{6.55}$$

Likewise, when the effects of the deformation are absent ($\mathcal{B} \neq 0$, $\mathcal{E} \neq 0$, $\theta = 0$, and $\eta = 0$), we obtain the following result

$$\mathcal{P}^{(\text{NC})}_{\theta=\eta=0}(\text{pair}) \simeq \frac{1}{8\pi^3} \left((e\mathcal{E})^2 - (e\mathcal{B})^2\right) e^{-\frac{\pi m^2}{\sqrt{(e\mathcal{E})^2 - (e\mathcal{B})^2}}} \cos(\epsilon\pi). \tag{6.56}$$

In addition, in the absence of the magnetic field ($\mathcal{E} \neq 0$, $\mathcal{B} = 0$) and in the presence of NC phase space ($\theta \neq 0$, $\eta \neq 0$), the probability of this type of creation pair becomes as

$$\mathcal{P}^{(\text{NC})}_{\mathcal{B}=0}(\text{pair}) \simeq \frac{1}{8\pi^3} \frac{\tilde{\Omega}^2}{1 - \frac{(e\mathcal{E}\theta)^2}{4}} e^{-\frac{\pi m^2\sqrt{1-\frac{(e\mathcal{E}\theta)^2}{4}}}{\tilde{\Omega}}} \cos\left(\epsilon \frac{\pi\sqrt{(e\mathcal{E})^2 - \eta^2}\sqrt{1 - \frac{(e\mathcal{E}\theta)^2}{4}}}{\tilde{\Omega}}\right). \tag{6.57}$$

With $\tilde{\Omega} = \sqrt{(e\mathcal{E})^2\left(1 - \frac{\eta\theta}{4}\right)^2 - \left(\eta - \frac{\theta}{2}(e\mathcal{E})^2\left(1 + \frac{\eta\theta}{4}\right)\right)^2}$.

Also, the result of Eq. (6.53) if $\theta \to 0$ and $\eta \neq 0$ agree with the following result,

$$\mathcal{P}^{(\text{NC})}_{\theta=0}\,(\text{pair}) \simeq \frac{1}{8\pi^3}\left((e\mathcal{E})^2 - \eta^2\right) e^{-\frac{\pi m^2}{\sqrt{(e\mathcal{E})^2-\eta^2}}}. \tag{6.58}$$

But if $\eta \to 0$ and $\theta \neq 0$, the Eq. (6.53) leads to the standard result

$$\mathcal{P}^{(\text{NC})}_{\eta=0}\,(\text{pair}) \simeq \frac{(e\mathcal{E})^2}{8\pi^3} e^{-\frac{\pi m^2}{e\mathcal{E}}}. \tag{6.59}$$

Also, when the effects of the deformation are absent ($\mathcal{B}=0$, $\mathcal{E}\neq 0$, $\theta=0$, and $\eta=0$), will recover the standard result of the uniform electric field for the bosonic pair production probability in Ref. [37] given as

$$\mathcal{P}^{(\text{NC})}_{\theta=\eta=0}\,(\text{pair}) \simeq \frac{(e\mathcal{E})^2}{8\pi^3} e^{-\frac{\pi m^2}{e\mathcal{E}}}. \tag{6.60}$$

If we take the limit $m \longrightarrow 0, c \longrightarrow v_F$ in $(2+1)$ QED, we obtain the same result in Ref. [126] in the case of graphene.

6.5 Conclusion

In this chapter, we have studied the problem of pair creation of scalar and spinorial relativistic particles from a vacuum by a constant electromagnetic field in the framework of NC phase space coordinates using Schwinger's method.

we have made the corresponding quadratic Lagrangian, and then we have calculated the effective action using the supersymetric path integral formalism for two cases of relativistic particles defined by Eqs. (6.36) and (6.49). Also, all particular cases of $(\theta, \eta, \mathcal{E}, \mathcal{B})$ for each case are discussed. It is shown that in the absence of the parameter of deformations (i.e., $\theta \longrightarrow 0, \eta \longrightarrow 0$), we recover that the results agree with the Schwinger results. Also, it is shown that when $\mathcal{E} = \mathcal{B} =0$, the effective action is equivalent to the effective action in a constant magnetic field ($\mathcal{B} \equiv \eta/e$).

Chapter 7

Dirac-graphene quasiparticles in the combination of a Volkov plane wave and a parallel constant magnetic field

7.1 Introduction

In recent years, the Redmond solution for charged particles moving in an arbitrary electromagnetic plane wave and a uniform static magnetic field was studied in Refs [180]. Furthermore, the same problem was solved via the path integral formalism, using the delta-functionals method [181] and via the Schwinger action principle method by solving the Heisenberg equations [182]. Also using a supersymmetric action [183, 184].

As a special case, the Green functions for charged particles of spin zero and spin 1/2, which are in interaction with an electromagnetic plane wave, were constructed according to the path integral formalism through different methods, the direct method and the semi-classiclal approach [104], and by using a supersymmetric action proposed by Fradkin and Gitman [100].

On the other hand, the Volkov solution of the massless Dirac equation for graphene in the field of slow-light pulse with arbitrary time dependence has also been discussed [185]. Furthermore, its generalization for two orthogonal electromagnetic plane wave fields via path integral formalism using the method of Fradkin and Gitman is analyzed in the fourth chapter of the thesis.

In this chapter, we solve the Dirac equation for graphene quasiparticles in interaction with the combination of a plane wave and a parallel magnetic field using the Redmond method. Also, we solve in the framework of Feynman's path integrals the Redmond problem using the method of delta-functionals, where the integrations of the spin factor are done by using the $\mathbb{T}$-product technique [186]. We construct the corresponding Green's function via Feynman's technique.

At the end, we examine the issue of the creation of quasiparticle-hole pairs in monolayer graphene under the action of this configuration of field.

7.2 Formulation of problem

In this section of the chapter, we present a short review of our notation and conventions. We consider massless graphene quasiparticles of charge $e \prec 0$ in interaction with an electromagnetic plane wave field plus a static magnetic field parallel to the direction of propagation of the plane wave. We write the vector potential as a sum of two terms as follows [180, 181, 182]

$$\mathcal{A}_{\mu}^{Comb}(x) = A_{\mu}(\xi) + A_{\mu}\left(x^{T}\right), \tag{7.1}$$

where

$$\begin{aligned} A^{\mu}(\xi) &= \vec{e}_{x} A(\xi), & (7.2)\\ A^{\mu}\left(x^{T}\right) &= i\frac{\mathcal{B}}{2}\left(\epsilon^{\mu}\epsilon^{*}\cdot x + \epsilon^{\mu *}\epsilon\cdot x\right). & (7.3)\end{aligned}$$

The first term generates the electromagnetic plane wave field with the vector of propagation n which has a real components $n^{\mu} = (1, \vec{n}) = (1, 0, 1)$ satisfy $\epsilon_{\mu} n^{\mu} = \epsilon_{\mu}^{*} n^{\mu} = n_{\mu} n^{\mu} = 0$ and $\xi = nx \equiv n_{\mu} x^{\mu} = \alpha y - \tau$.

Here $\alpha = 1$, indicating that the velocities of electromagnetic waves and quasiparticles in graphene coincide. While the second term generates the constant magnetic field.

We introduce the vectors ϵ and ϵ^{*} which are complex conjugates of each other, with components $\epsilon_{\mu} = \frac{1}{\sqrt{2}}(0, 1, i)$ and $\epsilon_{\mu}^{*} = \frac{1}{\sqrt{2}}(0, 1, -i)$ are satisfying the orthogonality conditions $\epsilon\epsilon = \epsilon^{*}\epsilon^{*} = 0$ and $\epsilon\epsilon^{*} = 1$.

We note that in the $(2+1)$-dimension, the Minkowski tensor has the signature $g_{\mu\nu} = diag(-1, +1, +1)$ and units are chosen such that $\hbar = c = 1$. Also, we consider that the graphene quasiparticles with a magnetic field is moving in the z direction, which allows us to introduce the quantities

$$x \pm iy. \tag{7.4}$$

We suppose that the potential $A^{\mu}(\xi)$ satisfies the Lorentz gauge condition

$$\partial_{\mu} A^{\mu}(\xi) = n_{\mu}(A^{\mu})' = 0, \tag{7.5}$$

where the prime denotes the derivative with respect to ξ, which is equivalent to

$$n_{\mu} A^{\mu}(\xi) = 0. \tag{7.6}$$

7.3 Solution of the Dirac-graphene equation in the combination of a Volkov plane wave and a constant magnetic field

The dynamics of graphene quasiparticles in the combination of a Volkov plane wave and a parallel magnetic field is described by the Dirac equation for massless fermions, defined as

$$\left[\hat{k}^2 - 2e\left(\hat{A}\left(x^T\right) + \hat{A}(\xi)\right)\hat{k}_x + e^2\left(\hat{A}\left(x^T\right) + \hat{A}(\xi)\right)^2 - \frac{ie}{2}\mathcal{F}^{Comb}_{\mu\nu}\gamma^{\mu}\gamma^{\nu}\right] \quad (x) = 0, \tag{7.7}$$

where $\hat{k}^{\mu} = -i\partial_{\mu} \equiv \left(\frac{E}{\upsilon_F}, \vec{k}\right) - i\left(\frac{\partial}{\upsilon_F \partial t}, \vec{\nabla}\right)$ is the momentum operator in the natural units, with υ_F representing the Fermi velocity equals $\upsilon_F = (1.12 \pm 0.02) \times 10^6 m/s$. Since the vector potential $\mathcal{A}^{Comb}_{\mu}$ and $\psi(x)$ depend only on x^T and ξ.

and γ^{μ} are Dirac matrices, in two dimensions, are represented by the Pauli matrices in the following manner

$$\gamma^0 = \sigma_3,\ \gamma^1 = \imath\sigma_2,\ \gamma^2 = -\imath\sigma_1. \tag{7.8}$$

With

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.9}$$

According to the known Volkov ansatz, we seek a solution of the form

$$(x) = e^{i\chi}\psi_k\left(x^T, \xi\right)u_k, \tag{7.10}$$

where

$$\chi = \mathbf{kq} - E\tau,\ \mathbf{q} = (q_x, q_y)\ ,\ \xi = y - \tau. \tag{7.11}$$

By substituting (7.10) into Eq. (7.7), we obtain the equation for $\psi_k\left(x^T, \xi\right)$

$$\Big[k^2 + 2ik\partial - \partial^2 - 2ie\left(\hat{A}\left(x^T\right) + \hat{A}\left(\xi\right)\right)k_x + e^2\left(\hat{A}\left(x^T\right) + \hat{A}\left(\xi\right)\right)^2$$

$$-\frac{ie}{2}\mathcal{F}^{Comb}_{\mu\nu}\gamma^\mu\gamma^\nu\Big]\psi_k\left(x^T, \xi\right) = 0, \tag{7.12}$$

where

$$\mathcal{F}^{Comb}_{\mu\nu}\gamma^\mu\gamma^\nu = \left(F_{\mu\nu}\left(\mathcal{B}\right) + f_{\mu\nu}\left(\xi\right)\right)\gamma^\mu\gamma^\nu \tag{7.13}$$

$$= 2A'\left(\xi\right)\left(\frac{\partial\xi}{\partial\tau}\gamma^0\gamma^1 - \frac{\partial\xi}{\partial y}\gamma^1\gamma^2\right) + 2\left(\frac{\partial A\left(x^T\right)}{\partial\tau}\gamma^0\gamma^1 - \frac{\partial A\left(x^T\right)}{\partial y}\gamma^1\gamma^2\right) \tag{7.14}$$

$$= 2\left(\gamma^\mu n_\mu\right)\left(\gamma^\nu \dot{A}_\nu\left(\alpha s\right)\right) + i\mathcal{B}\left(\gamma\epsilon\gamma\epsilon^* - \gamma\epsilon^*\gamma\epsilon\right). \tag{7.15}$$

This may be rewritten in the following form

$$\Big[2i\left(k_\tau - k_y\right)\frac{\partial}{\partial\xi} + k^2 - \partial^2 - 2ie\left(\hat{A}\left(x^T\right) + \hat{A}\left(\xi\right)\right)k_x + e^2\left(\hat{A}\left(x^T\right) + \hat{A}\left(y, \tau\right)\right)^2$$

$$+\frac{e\mathcal{B}}{2}\left(\gamma\epsilon\gamma\epsilon^* - \gamma\epsilon^*\gamma\epsilon\right) - ie\left(\gamma^\mu n_\mu\right)\left(\gamma^\nu \dot{A}_\nu\left(\alpha s\right)\right)\Big]\psi_k\left(x^T, \xi\right) = 0, \tag{7.16}$$

At this stage, to complete the solution of equation (7.16), we need the analogy to the classical situation. From Ref. [180], we have the following classical equations of motion

$$m\ddot{x}_\mu = -e\mathcal{F}^{Comb}_{\mu\nu}\dot{x}^\nu, \tag{7.17}$$

The equation (7.17) is transformed to

$$m\ddot{x}_\mu + ie\mathcal{B}\epsilon_\mu\epsilon^*_\nu\dot{x}^\nu - ie\mathcal{B}\epsilon_\nu\epsilon^*_\mu\dot{x}^\nu + en_\mu\frac{dA_\nu}{d\xi}\dot{x}^\nu - e\dot{A}_\mu = 0, \tag{7.18}$$

The mass parameter m must be equal to $1/2$. It also seems that the classical equations of motion (7.17) lead to the following relation after multiplying this latter by n. Consequently, we can write

$$n\ddot{x} = 0 \Rightarrow nx = \beta s, \tag{7.19}$$

which represents the ξ-variable is defined by $\xi = \beta s$ and β is a constant. We make these transformations

$$\begin{cases} e^{iK.\pi}\pi_\mu^T e^{-iK.\pi} = \pi_\mu^T - eF_{\mu\nu}(\mathcal{B})K^\nu \\ e^{iK.\pi}\left(-2in.k\frac{\partial}{\partial\xi}\right)e^{-iK.\pi} = -2in.k\frac{\partial}{\partial\xi} + \pi.\dot{K} - \frac{e}{2}\dot{K}^\mu F_{\mu\nu}(\mathcal{B})K^\nu \end{cases}. \tag{7.20}$$

where $K(s)$ is the transverse vector and

$$\pi^T = k^T - eA\left(x^T\right), \tag{7.21}$$

The combination $e^{iK.\pi}\psi_k$ then satisfies the equation

$$\left[-2in.k\frac{\partial}{\partial\xi} + \pi.\dot{K} - \frac{e}{2}\dot{K}^\mu F_{\mu\nu}(\mathcal{B})K^\nu - \left(k^2\right)^T + \left(\pi^T - eF_{\mu\nu}(\mathcal{B})K^\nu + eA(\xi)\right)^2\right.$$
$$\left. + \frac{e\mathcal{B}}{2}(\gamma\epsilon\gamma\epsilon^* - \gamma\epsilon^*\gamma\epsilon) - ie(\gamma^\mu n_\mu)\left(\gamma^\nu \dot{A}_\nu(\beta s)\right)\right]e^{iK.\pi}\psi_k = 0, \tag{7.22}$$

where $(k^2)^T = -k^2$. We determine $K(s)$ using the condition $K(-\infty) = 0$, the equation (7.22) separates to

$$\pi.\dot{K} + 2\pi^T(-eF_{\mu\nu}(\mathcal{B})K^\nu + eA(\beta s)) = 0 \longrightarrow \frac{1}{2}\dot{K} = -eF_{\mu\nu}(\mathcal{B})K^\nu + eA(\beta s), \tag{7.23}$$

which is the classical equation with $m = 1/2$ and

$$\left[-2in.k\frac{\partial}{\partial\xi} - \dot{J}(s)\right]e^{iK.\pi}\psi_k = 0, \tag{7.24}$$

where

$$J(s) = \int_{-\infty}^{s}\left[\frac{e}{2}\dot{K}^\mu F_{\mu\nu}(\mathcal{B})K^\nu + (-eF_{\mu\nu}(\mathcal{B})K^\nu + eA(\xi))^2\right]. \tag{7.25}$$

The solution of Eq. (7.24) is given as

$$\begin{aligned} \left[-2in.k\frac{\partial}{\partial\xi} - \dot{J}(\xi)\right]e^{iK.\hat{\Pi}}\psi_k &= 0, \\ -2in.k\frac{\partial}{\partial\xi}\psi_k &= \dot{J}(\xi)\psi_k \\ &\rightarrow \psi_k = Ce^{\frac{i}{2n.k}J(\xi)}, C = cst \end{aligned} \tag{7.26}$$

and

$$\psi_k\left(x^T, \xi\right) = e^{iK\cdot\pi}\psi_k, \tag{7.27}$$

with the terms linear in π^T dropping out, the equation becomes as follows

$$\Big(2i\,(k_\tau - k_y)\,\frac{\partial}{\partial\xi}\psi_k - \left(k^2\right)^T + \Big(\left(\pi^T\right)^2 + \frac{e\mathcal{B}}{2}\,(\gamma\epsilon\gamma\epsilon^* - \gamma\epsilon^*\gamma\epsilon)$$

$$-ie\,(\gamma^\mu n_\mu)\left(\gamma^\nu \dot{A}_\nu(\xi)\right)\Big)\Big)\, e^{iJ} e^{iK\cdot\pi}\psi_k\left(x^T,\xi\right) = 0, \tag{7.28}$$

The separation constant was determined just as in the case of the Klein Gordon equation. Then we can write

$$\left[2i\,(k_\tau - k_y)\,\frac{\partial}{\partial\xi} + e\mathcal{B}\,(S \mp 1) - ie\,(\gamma^\mu n_\mu)\left(\gamma^\nu \dot{A}_\nu(\xi)\right)\right] e^{iJ} e^{iK\cdot\pi}\psi_k^\pm = 0, \tag{7.29}$$

and

$$\left[\left(k^2\right)^T - \left(\left(\pi^2\right)^T \pm e\mathcal{B}\right)\right] e^{iJ} e^{iK\cdot\pi}\psi_k^\pm = 0, \tag{7.30}$$

with $S = \frac{1}{2}\,(\gamma\epsilon\gamma\epsilon^* - \gamma\epsilon^*\gamma\epsilon)$.

The equation (7.30) describes the particles in an uniform static magnetic field. This latter analogy to the equation of harmonic oscilator. We have this commutation relation

$$[\pi_\mu, \pi_\nu] = -ieF_{\mu\nu}(\mathcal{B}) \tag{7.31}$$

multyplying the right hand side and the left hand side of the equation by this term $\epsilon^{*\mu}\epsilon^\nu$ we get the following result

$$[\epsilon^{*\mu}\pi_\mu, \epsilon^\nu\pi_\nu] = e\mathcal{B} \Longrightarrow \left[\frac{\epsilon^{*\mu}\pi_\mu}{\sqrt{e\mathcal{B}}}, \frac{\epsilon^\nu\pi_\nu}{\sqrt{e\mathcal{B}}}\right] = 1. \tag{7.32}$$

This is the commutaion relation for annihilation and creation operators. If $e \succ 0$ then the following term $\epsilon^* \cdot \pi$ is the annihilation operator and the ground states determined by

$$\epsilon^* \cdot \pi\psi_0\left(x^T\right) = 0 \tag{7.33}$$

This is a first-order partial differential equation which separates if $A\left(x^T\right)$ is a linear function of x^T, then other eigenfunctions (the excited states) is obtained by repeatedly applying the creation operator $\epsilon \cdot \pi$ to ground state given as

$$\Psi_n\left(x^T\right) = (n)^{-\frac{1}{2}}\,(eH)^{-\frac{n}{2}}\,(\epsilon\cdot\pi)_0^n\,\Psi\left(x^T\right) \tag{7.34}$$

Since

$$\left(\pi^T\right)^2 = (\epsilon^*\cdot\pi)(\epsilon\cdot\pi) + (\epsilon\cdot\pi)(\epsilon^*\cdot\pi) \tag{7.35}$$

The allowed values of $(k^2)^T$ give us the following expression

$$\left(k^2\right)^T = 2ne\mathcal{B}. \tag{7.36}$$

Consequently, the complete set of solutions to the Dirac-graphene equation has the following form

$$(k,n,x) = e^{ipx-iJ-iK\cdot\pi}\left\{\Psi_{n-1}\left(x^T\right)f_+(\xi) + \Psi_n\left(x^T\right)f_-(\xi)\right\}, \tag{7.37}$$

where $\Psi_{n-1}\left(x^T\right)$ and $\Psi_n\left(x^T\right)$ are the solutions of the oscillator equations respectively of order $n-1$ and n.

whereas $f_\pm$ are spinors determined by Eq. (6.26) .The equation for $f_+(s)$ is given in Eq.

$$2i\left(k_\tau - k_y\right)\dot{f}_+ = \left[e\mathcal{B}\left(\gamma\epsilon^*\right)\left(\gamma\epsilon\right) - ie\left(\gamma^\mu n_\mu\right)\left(\gamma^\nu \dot{A}_\nu\left(y-\tau\right)\right)\right]f_+ = 0. \tag{7.38}$$

By solving the equation (7.38) by iteration starting with f_+ on the right hand side represented by a constant spinor u_+ satisfying $Su_+ = u_+$ which implies $\gamma\epsilon u_+ = 0$.

We get for $f_+(\xi)$ the following form

$$f_+(\xi) = \left[1 + \gamma n\gamma\epsilon^* R(\xi)\right]u_+, \tag{7.39}$$

where $R(\xi)$ is a scalar function of ξ.

Incorporating Eq. (7.39) into Eq. (7.38) and after some manipulation of the γ matrices, the equation takes the following form

$$\left[\dot{R}(\xi) + i\omega_0 R(\xi) - \frac{e}{2\left(k_\tau - k_y\right)}\epsilon\dot{A}(\xi)\right]\left(\gamma n\right)\left(\gamma\epsilon^*\right)u_+ = 0, \tag{7.40}$$

the solution of the above equation is given as follows

$$R(\xi) = \frac{ie}{2\left(k_\tau - k_y\right)}e^{-i\omega_0\xi}\int_{-\infty}^{\xi}\epsilon\dot{A}(\xi)\,e^{i\omega_0\xi'}d\xi', \tag{7.41}$$

By applying for f_- the same procedure as f_+, we obtain the folowing solution

$$f_-(\xi) = \left[1 + R^*(\xi)\,\gamma n\gamma\epsilon\right]u_-. \tag{7.42}$$

7.4 Path integral formulation for Dirac-graphene quasiparticles in the combination of a Volkov plane wave and a parallel constant magnetic field

7.4.1 Construction of the causal Green's function $S^c(x_b, x_a)$

The Green function for graphene quasiparticles in interaction with the combination of a plane wave and a constant magnetic field parallel to the direction of propagation of the electromagnetic wave is defined as

$$\mathcal{O}^{Graph} S^c(x_b, x_a) = \delta(x_b - x_a), \tag{7.43}$$

where

$$\mathcal{O}^{Graph} = \gamma^0 (i\partial_\tau) - \gamma^1 \left(i\partial_x - e\mathcal{A}^{Comb}(x)\right) - \gamma^2 (i\partial_y), \tag{7.44}$$

with $\tau = \upsilon_F t$ and

$$\mathcal{A}^{Comb}_\mu (x) = A_\mu \left(x^T\right) + A_\mu (\xi), \quad \mu = 0, 1, 2. \tag{7.45}$$

In lower dimensions, the Dirac gamma matrices γ^μ are represented by the Pauli matrices in the following way

$$\gamma^0 = \sigma_3, \ \gamma^1 = i\sigma_2, \ \gamma^2 = -i\sigma_1. \tag{7.46}$$

We can write $S^c(x_b, x_a)$ as a matrix element of the operator S^c as follows

$$S^c(x_b, x_a) = \langle x_a | \hat{S}^c | x_b \rangle, \tag{7.47}$$

by using the Schwinger proper time method, we get

$$\hat{S}^c = \int_0^\infty d\lambda \exp\left(-i\lambda \hat{H}\right), \tag{7.48}$$

where $\hat{H}$ is the Hamiltonian operator, defined as

$$\begin{aligned} \hat{H}\left(\hat{x}, \hat{k}\right) &= \left[\mathcal{O}^{Graph}\right]^2, \\ &= \hat{O}_{KG} - \frac{ie}{2}\gamma^\mu\gamma^\nu \mathcal{F}^{Comb}_{\mu\nu}, \end{aligned} \tag{7.49}$$

and $\hat{O}_{KG}$ being the Klein Gordon operator, given by

$$\hat{O}_{KG} = \partial_\tau^2 - \partial_x^2 - \partial_y^2 + 2ie\mathcal{A}^{Comb}(x)\,\partial_x + e^2\left[\mathcal{A}^{Comb}(x)\right]^2, \tag{7.50}$$

where $\mathcal{F}_{\mu\nu}^{Comb}$ is the electromagnetic field tensor defined as a derivable of a potential $\mathcal{A}^{Comb}(x)$ can then be written as

$$\begin{aligned}\mathcal{F}_{\mu\nu}^{Comb} &= \partial_\mu \mathcal{A}_\nu^{Comb} - \partial_\nu \mathcal{A}_\mu^{Comb}, && (7.51)\\ &= F_{\mu\nu}(\mathcal{B}) + f_{\mu\nu}(\xi), && (7.52)\\ &= i\mathcal{B}\left(\epsilon_\mu \epsilon_\nu^* - \epsilon_\nu \epsilon_\mu^*\right) + n_\mu \frac{dA_\nu}{d\xi} - n_\nu \frac{dA_\mu}{d\xi}, && (7.53)\end{aligned}$$

whereas $\mathcal{B}$ is the static magnetic field.

Formally, knowing that $S^c(x_b, x_a) = \langle x_a| \exp\left(-i\hat{H}\right)|x_b\rangle$, we easily write for the Green function the following result

$$S^c(x_b, x_a) = \langle x_a| \exp\left(-i\hat{H}\right)|x_b\rangle. \quad (7.54)$$

Let us now derive the path integral representation for the Green function $S^c(x_b, x_a)$

$$\begin{aligned}S^c\left(x_b^\mu, x_a^\mu\right) &= \int \mathcal{D}x \int \mathcal{D}k \\ \times \mathbb{T}\exp\Bigg\{ i\int_0^1 ds \Big[k\dot{x} &+ \lambda\Big(-k_\tau^2 + k_x^2 + k_y^2 - 2e\mathcal{A}^{Comb}(x)\, k_x \\ &+ e^2\left[\mathcal{A}^{Comb}(x)\right]^2 - \frac{ie}{2}\gamma^\mu\gamma^\nu \mathcal{F}_{\mu\nu}^{Comb}\Big)\Big]\Bigg\}, \quad (7.55)\end{aligned}$$

Where $\mathbb{T}$ is the time ordering operator, effects only on the phase relative to the coupling term.

Here $x \equiv (\tau, x, y)$ represents the quadratic vector coordinate, and $k \equiv (k_\tau, k_x, k_y)$ is the quadratic vector momentum. Whereas $A'(\xi)$ is the abbreviated derivative functions of $A(\xi)$ with respect to ξ. The same goes for other derivatives: $\dot{\xi}_\tau = \frac{\partial \xi}{\partial \tau}$ and $\xi'_y = \frac{\partial \xi}{\partial y}$.

we can write the causal Green function $S^c(x_b, x_a)$ as follows

$$S^c(x_b, x_a) = \mathcal{K}_{KG}(x_b, x_a; \lambda)\, \mathcal{S}(x_b, x_a; \lambda). \quad (7.56)$$

where

$$\mathcal{S}(x_b, x_a; \lambda) = \mathbb{T}\exp\left\{ i\int_0^\lambda ds \left[-\frac{ie}{2}\gamma^\mu\gamma^\nu \mathcal{F}_{\mu\nu}^{Comb}\right]\right\}, \quad (7.57)$$

and

$$\mathcal{K}_{KG}(x_b, x_a; \lambda) = \int \mathcal{D}x \exp\left[i\int_0^\lambda \left[k\dot{x} - \left(k - e\mathcal{A}^{Comb}\right)^2\right] ds\right], \quad (7.58)$$

7.4.2 The evaluation of the kernel propagator $\mathcal{K}_{KG}(x_b, x_a; \lambda)$

Explicitly, the propagator $\mathcal{K}_{KG}(x_b, x_a; \lambda)$ is defined as

$$\begin{aligned}\mathcal{K}_{KG}(x_b, x_a; \lambda) &= \lim_{N\to\infty} \int \Pi_{j=1}^{N-1} d^4x_j \Pi_{j=1}^{N} \frac{d^3k_j}{(2\pi)^4} \times \Pi_{j=1}^{N} \\ &\times \exp\left[i\left[k_j \Delta x_j - \varepsilon\left(k_j - e\mathcal{A}_j^{Comb}\right)^2\right]\right],\end{aligned} \tag{7.59a}$$

where

$$\begin{aligned}x_j &= x(s_j), \quad x_a = x(0), \ \Delta x_j = x_j - x_{j-1}, \\ \mathcal{A}_j^{Comb} &= \mathcal{A}^{Comb}(s_j) \text{ and } \varepsilon = s_j - s_{j-1} = \frac{\lambda}{N}.\end{aligned} \tag{7.60}$$

Let us consider ξ as a variable independent of nx. The integration over $x(s)$ seems to be difficult due the dependence $A(nx = \xi)$. We insert this identity

$$\int d\xi_b d\xi_a \delta(\xi_a - kx_a) \delta[\xi_b - \xi_a - n(x_b - x_a)] = 1, \tag{7.61}$$

into the expression (7.59a) or rather its generalization, which lets all time intervals $[j-1, j]$ occur

$$\int d\xi_b d\xi_a \delta(\xi_a - nx_a) \int \Pi_{j=1}^{N-1} d\xi_j \Pi_{j=1}^{N} \delta\left(\Delta\xi_j - n\Delta x_j\right) = 1, \tag{7.62}$$

where $\Delta\xi_j = \xi_b - \xi_a$ and

$$\delta\left(\Delta\xi_j - n\Delta x_j\right) = \frac{1}{2\pi} \int dk_{\xi_j} \exp\left[ik_{\xi_j}\left(\Delta\xi_j - n\Delta x_j\right)\right]. \tag{7.63}$$

The propagator (7.59a) then becomes as

$$\mathcal{K}_{KG}(x_b, x_a; \lambda) = \int d\xi_b d\xi_a \delta(\xi_a - nx_a) \bar{\mathcal{K}}(\xi_b, x_b, \xi_a, x_a; \lambda), \tag{7.64}$$

where

$$\begin{aligned}\bar{\mathcal{K}}(\xi_b, x_b, \xi_a, x_a; \lambda) &= \int DxDkD\xi Dk_\xi \exp\left[i\int_0^\lambda \left[k\dot{x} - k^2 - 2ek_x\mathcal{A}^{Comb}(x)\right.\right. \\ &\left.\left. -e^2\left[\mathcal{A}^{Comb}(x)\right]^2 + k_\xi\left(\dot{\xi} - n\dot{x}\right)\right] ds\right].\end{aligned} \tag{7.65}$$

where

$$Dx = \Pi_{j=1}^{N-1} dx_j, \tag{7.66}$$

$$Dk = \Pi_{j=1}^{N-1} \frac{dk_j}{(2\pi)^3}, \tag{7.67}$$

$$D\xi = \Pi_{j=1}^{N-1} d\xi_j, \tag{7.68}$$

$$Dk_\xi = \Pi_{j=1}^{N-1} \frac{dk_{\xi j}}{2\pi}, \tag{7.69}$$

Let us use the transverse and longitudinal components of the vectors (x, k) defined as

$$x = \begin{pmatrix} x^T \\ x^L \end{pmatrix}, k = \begin{pmatrix} k^T \\ k^L \end{pmatrix}. \tag{7.70}$$

By changing $k^L - nk_\xi$ into k^L and taking $n^2 = 0$ and $nA(\xi) = 0$, By simplification, we obtain the following expression for the propagator $\bar{\mathcal{K}}(\xi_b, x_b, \xi_a, x_a; \lambda)$

$$\begin{aligned} \bar{\mathcal{K}}(\xi_b, x_b, \xi_a, x_a; \lambda) &= \int Dx^L Dk^L Dx^T Dk^T D\xi Dk_\xi \\ &\times \exp\left\{ i \int_0^\lambda \left[k^L \dot{x}^L + k^T \dot{x}^T - \left(k^L\right)^2 - \left(k^T\right)^2 \right.\right. \\ &-2e\left(k_x^L + k_x^T\right)\mathcal{A}^{Comb}(x) - e^2\left[\mathcal{A}^{Comb}(x)\right]^2 \\ &\left.\left. +k_\xi \dot{\xi} - \left(k^L n\right) k_\xi \right] ds \right\}. \end{aligned} \tag{7.71}$$

In discretized form

$$\sum_1^N k_j^L \Delta x_j^L = k_N^L x_N^L - k_1^L x_0^L + \sum_1^{N-1} \left(k_j^L - k_{j+1}^L\right) x_j^L. \tag{7.72}$$

The integrations over x_j^L give us the Dirac distributions $\delta(k_{j+1}^L - k_j^L)$ which lead to the conservation of the longitudinal component of the momentum of graphene quasiparticles during the motion

$$k_1^L = k_2^L = ... = k_N^L = k^L. \tag{7.73}$$

then the propagator $\bar{\mathcal{K}}(\xi_b, x_b, \xi_a, x_a; \lambda)$ becomes as

$$\begin{aligned} \bar{\mathcal{K}}(\xi_b, x_b, \xi_a, x_a; \lambda) &= \int Dk^L Dx^T Dk^T D\xi Dk_\xi \exp\left[i \int_0^\lambda \left[k^L \dot{x}^L - \left(k^L\right)^2 \right] ds \right] \\ &\times \exp\left[i \int_0^\lambda \left[k^T \dot{x}^T - \left(k^T\right)^2 - e k_x^T \mathcal{A}^{Comb}(x) \right.\right. \\ &\left.\left. -e^2\left[\mathcal{A}^{Comb}(x)\right]^2 + k_\xi\left(\dot{\xi} - \left(k^L n\right)\right)\right] ds \right]. \end{aligned} \tag{7.74}$$

Let us integrate over the variables k_{ξ_j}. The path integral $\int Dk_\xi$ over the variables k_{ξ_j} gives

$$\int Dk_\xi \exp\left[i \int_0^\lambda k_\xi \left(\dot{\xi}_j - \left(k^L n\right)\right) ds \right] = \Pi_{j=1}^{N-1} \int \frac{dk_{\xi_j}}{2\pi} \exp\left[i \int_0^\lambda k_{\xi_j} \left(\dot{\xi}_j - \left(k^L n\right)\right) ds \right] \tag{7,75}$$

$$= \Pi_{j=1}^{N-1} \delta\left(\xi_{jb} - \xi_{ja} - \left(k^L n\right)\right), \tag{7.76}$$

which imposes on the paths N constraints as follows

$$\Delta\xi_j = (k^L n)\varepsilon \ , \quad j \in [1, N]. \tag{7.77}$$

These constraints give us the equation

$$\frac{d\xi_j}{ds} = (k^L n) \text{ or else } \frac{\xi(s) - \xi(0)}{s} = \frac{\xi_b - \xi_a}{\lambda} = (k^L n). \tag{7.78}$$

The propagator $\bar{\mathcal{K}}(\xi_b, x_b, \xi_a, x_a; \lambda)$ is then transformed to

$$\begin{aligned}\bar{\mathcal{K}}(\xi_b, x_b, \xi_a, x_a; \lambda) &= \int Dx^L Dk^L Dx^T Dk^T D\xi \delta\left(\xi_b - \xi_a - \left(k^L n\right)\right) \\ &\times \exp\left[i \int_0^\lambda \left[k^L \dot{x}^L - \left(k^L\right)^2\right] ds\right] \\ &\times \exp\left[i \int_0^\lambda \left[k^T \dot{x}^T - \left(k^T - e\mathcal{A}^{Comb}(x)\right)^2\right] ds\right], \end{aligned}\tag{7.79}$$

By substituting (7.79) in (7.64), the transverse propagator $\mathcal{K}_{KG}(x_b, x_a; \lambda)$ then becomes

$$\mathcal{K}_{KG}(x_b, x_a; \lambda) = \int d\xi_b d\xi_a \delta(\xi_a - kx_a) \bar{K}(\xi_b, x_b, \xi_a, x_a; \lambda), \tag{7.80}$$

$$\begin{aligned}&= \int d\xi_b d\xi_a \delta(\xi_a - kx_a) \delta\left(\xi_b - \xi_a - \left(k^L n\lambda\right)\right) Dk^L \\ &\times \exp\left[i\left[k^L\left(x_b^L - x_a^L\right) - \left(k^L\right)^2 \lambda\right]\right] \mathcal{K}_{KG}^T\left(x_b^T, x_a^T; \lambda^T\right). \end{aligned}\tag{7.81}$$

Where

$$\mathcal{K}_{KG}^T\left(x_b^T, x_a^T; \lambda^T\right) = \int Dx^T Dk^T \exp\left[i \int_0^\lambda \left[k^T \dot{x}^T - \left(k^T - e\mathcal{A}^{Comb}(x)\right)^2\right] ds\right]. \tag{7.82}$$

By changing $k^T - e\mathcal{A}^{Comb}$ into k^T, we get

$$\begin{aligned}\mathcal{K}_{KG}^T\left(x_b^T, x_a^T; \lambda^T\right) &= \int Dx^T Dk^T \exp\left[i \int_0^\lambda \left[k^T \dot{x}^T - \left(k^T - e\mathcal{A}^{Comb}(x)\right)^2\right] ds\right] \\ &= \int Dx^T \frac{d^2 k^T}{(2\pi)^2} \exp\left[i \int_0^\lambda \left[-\left(k^T\right)^2 + k^T \dot{x}^T\right.\right. \\ &\left.\left. + ex^T F(H) \dot{x}^T + eA(\xi) \dot{x}^T\right] ds\right], \end{aligned}\tag{7.83}$$

where

$$A_\mu\left(x^T\right) \dot{x}^{\mu T} = x^\mu F_{\mu\nu}(H) \dot{x}^{\mu T}, \tag{7.84}$$

and by integrating on the variables k^T, we get for the transverse propagator $\mathcal{K}_{KG}^T(x_b, x_a; \lambda)$ the following expression

$$\mathcal{K}_{KG}^T\left(x_b^T, x_a^T; \lambda^T\right) = \frac{1}{4}\int Dx^T \exp\left[i\int_0^\lambda \left[\left(\dot{x}^T\right)^2 + ex^T F(H)\dot{x}^T + eA(\xi)\dot{x}^T\right] ds\right]. \tag{7.85}$$

To eliminate the term $A(\xi)\dot{x}^T$, we take the following transformation

$$X^T = x^T - \mathbb{Q}^T, \tag{7.86}$$

$$\dot{X}^T = \dot{x}^T - \dot{\mathbb{Q}}^T, \tag{7.87}$$

where $\mathbb{Q}^T$ satisfy this equation

$$\dot{\mathbb{Q}}^T = eF(H)\mathbb{Q}^T - eA(\xi),\ A(-\infty) = 0. \tag{7.88}$$

After lengthy manipulations, the propagator $\mathcal{K}_{KG}^T\left(x_b^T, x_a^T; \lambda^T\right)$ then becomes as

$$\begin{aligned}\mathcal{K}_{KG}^T\left(x_b^T, x_a^T; \lambda^T\right) &= \frac{1}{4}\int DX^T \exp\left[i\int_0^\lambda \left[\left(\dot{X}^T\right)^2 + eX^T F(H)\dot{X}^T\right.\right.\\ &\quad \left.\left.+eA(\xi)\dot{\mathbb{Q}}^T + e\frac{d}{ds}\left(x^T F(H)\mathbb{Q}^T\right)\right] ds\right]\\ &= \exp\left[-i\left(J(\xi) - e\left(x^T F(H)\mathbb{Q}^T\right)\right)\Big|_{\xi_a}^{\xi_b}\right]\mathcal{K}_{KG_0}^T\left(X_b^T, X_a^T; \lambda^T\right),\end{aligned} \tag{7.89}$$

where

$$J(\xi) = \int_{-\infty}^{\xi} d\xi A(\xi)\frac{d\mathbb{Q}^T}{d\xi}, \tag{7.90}$$

and $\mathcal{K}_{KG_0}^T\left(X_b^T, X_a^T; \lambda^T\right)$ is defined as

$$\mathcal{K}_{KG_0}^T\left(X_b^T, X_a^T; \lambda^T\right) = \int DX^T \exp\left[i\int_0^\lambda \left[\left(\dot{X}^T\right)^2 + eX^T F(H)\dot{X}^T\right] ds\right] \tag{7.91}$$

$$= \int dudv \exp\left[i\int_0^\lambda \left[\left(\dot{u}^2 + \dot{v}^2\right) - \omega_0\left(u\dot{v} - v\dot{u}\right)\right] ds\right], \tag{7.92}$$

with $X^T = \begin{pmatrix} u \\ v \end{pmatrix}$, and the cyclotron frequency ω_0 is defined as $\omega_0 = e\mathcal{B}$.

Now, to determine the kernal propagator $\mathcal{K}_{KG_0}\left(X_b^T, X_a^T; \lambda^T\right)$, we decouple the coordinates u and v by introducing a rotation of coordinate axes as follows

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\omega_0}{2}s\right) & -\sin\left(\frac{\omega_0}{2}s\right) \\ \sin\left(\frac{\omega_0}{2}s\right) & \cos\left(\frac{\omega_0}{2}s\right) \end{pmatrix}\begin{pmatrix} \zeta \\ \eta \end{pmatrix}.$$

Finally, the kernal propagator $\mathcal{K}_{KG_0}\left(X_b^T, X_a^T; \lambda^T\right)$ is reduced to

$$\begin{aligned}\mathcal{K}_{KG_0}\left(X_b^T, X_a^T; \lambda^T\right) &= \frac{\omega_0}{4i\pi\sin\left(\frac{\omega_0}{2}\lambda\right)}\exp\left[i\frac{\omega_0}{4}\left[\left(u_b^2-u_a^2\right)+\left(v_b^2-v_a^2\right)\right]\cot\left(\frac{\omega_0}{2}\lambda\right)\right. \\ &\left. +2\left(u_b v_a - v_b u_a\right)\right]. \qquad (7.93)\end{aligned}$$

After lengthy manipulations, the propagator $\mathcal{K}_{KG}\left(x_b, x_a; \lambda\right)$ takes the symmetrical form

$$\begin{aligned}\mathcal{K}_{KG}\left(x_b, x_a; \lambda\right) &= \frac{\omega_0}{16i\pi\sin\left(\frac{\omega_0}{2}\lambda\right)}\int d\xi_b d\xi_a \int \mathcal{D}k^L \delta\left(\xi_a - kx_a\right)\delta\left(\xi_b - \xi_a - \left(k^L n\lambda\right)\right) \\ &\times\exp\left[-i\left(J\left(\xi\right)+e\left(x^T F\left(H\right)\mathbb{Q}^T\right)\right)\Big|_{\xi_a}^{\xi_b}\right] \\ &\times\exp\left[i\left[k^L\left(x_b^L - x_a^L\right)-\left(k^L\right)^2\lambda\right]\right] \\ &\times\exp\left[i\frac{\omega_0}{4}\left[\left(u_b^2-u_a^2\right)+\left(v_b^2-v_a^2\right)\right]\cot\left(\frac{\omega_0}{2}\lambda\right)+2\left(u_b v_a - v_b u_a\right)\right] \qquad (7.94)\end{aligned}$$

with

$$\mathcal{D}k^L = \prod_{n=1}^{N+1}\frac{dk^L}{2\pi}, \qquad (7.95)$$

Now, we replace the distribution $\delta\left(\xi_b - \xi_a - \left(k^L n\lambda\right)\right)$ by its integral representation

$$\delta\left(\xi_b - \xi_a - \left(k^L n\lambda\right)\right) = \frac{1}{2\pi}\int dk_{\xi_b}\exp\left[ik_{\xi_b}\left(\xi_b - \xi_a - \left(k^L n\lambda\right)\right)\right], \qquad (7.96)$$

and taking the replacement $k^L \to k^L - nk_{\xi_b}$ with $n^2 = 0$, which indicates that $\xi_b = nx_b, \xi_a = nx_a$.

By performing the integration over ξ_a, ξ_b, the propagator $\mathcal{K}_{KG}\left(x_b, x_a; \lambda\right)$ takes then the following form

$$\begin{aligned}\mathcal{K}_{KG}\left(x_b, x_a; \lambda\right) &= \frac{\omega_0}{16i\pi\sin\left(\frac{\omega_0}{2}\lambda\right)}\int \frac{dk^L}{2\pi}\exp\left[i\left[k^L\left(x_b^L - x_a^L\right)-\left(k^L\right)^2\lambda\right]\right] \\ &\times\exp\left[-i\left(J\left(\xi\right)+e\left(x^T F\left(H\right)\mathbb{Q}^T\right)\right)\Big|_{\xi_a}^{\xi_b}\right] \\ &\times\exp\left[i\frac{\omega_0}{4}\left[\left(u_b^2-u_a^2\right)+\left(v_b^2-v_a^2\right)\right]\cot\left(\frac{\omega_0}{2}\lambda\right)+2\left(u_b v_a - v_b u_a\right)\right] \qquad (7.97)\end{aligned}$$

Finally, the integration over k^L give us the following result

$$\begin{aligned}\mathcal{K}_{KG}\left(x_b, x_a; \lambda\right) &= i^{1/2}\frac{\omega_0}{32\pi^{3/2}\lambda^{1/2}\sin\left(\frac{\omega_0}{2}\lambda\right)}\exp\left(i\frac{\left(x_b^L - x_a^L\right)^2}{4\lambda}\right) \\ &\times\exp\left[-i\left(J\left(\xi\right)+e\left(x^T F\left(H\right)\mathbb{Q}^T\right)\right)\Big|_{\xi_a}^{\xi_b}\right] \\ &\times\exp\left[i\frac{\omega_0}{4}\left[\left(u_b^2-u_a^2\right)+\left(v_b^2-v_a^2\right)\right]\cot\left(\frac{\omega_0}{2}\lambda\right)+2\left(u_b v_a - v_b u_a\right)\right] \qquad (7.98)\end{aligned}$$

with

$$J(\xi) = \int_{-\infty}^{\xi} d\xi A(\xi) \frac{d\mathbb{Q}^T}{d\xi}, \mathbb{Q}^T = x^T - X^T \text{ and } X^T = \begin{pmatrix} u \\ v \end{pmatrix}, \ \omega_0 = e\mathcal{B}. \tag{7.99}$$

7.4.3 The evaluation of spin factor $\mathcal{S}(x_b, x_a; \lambda)$

Let us evaluate the spin factor $\mathcal{S}(x_b, x_a; \lambda)$ defined by

$$\mathcal{S}(x_b, x_a; \lambda) = \mathbb{T}\exp\left\{i\int_0^{\lambda} ds \left[-\frac{ie}{2}\gamma^{\mu}\gamma^{\nu}\mathcal{F}_{\mu\nu}^{Comb}\right]\right\}. \tag{7.100}$$

To eliminate the $\mathbb{T}$ product, we use Schulman's formula, which is defined as [153]

$$\begin{aligned}\mathbb{T}\exp\left[-i\int_a^b (H_1(s) + H_2(s))\, ds\right] &= \mathbb{T}\exp\left[-i\int_a^b H_1(s)\, ds\right] \\ &\times \left\{\mathbb{T}\exp\left[-i\int_a^b \left[\left[\mathbb{T}\exp\left(-i\int_a^s H_1(s')\, ds'\right)\right]^{-1}\right.\right.\right. \\ &\left.\left.\left. H_2(s)\left[\mathbb{T}\exp\left(-i\int_a^s H_1(s')\, ds'\right)\right] ds\right]\right]\right\}. \end{aligned} \tag{7.101}$$

Let us write down the quantity $\mathbb{T}\exp\left\{i\int_0^{\lambda} ds\left[-\frac{ie}{2}\gamma^{\mu}\gamma^{\nu}\mathcal{F}_{\mu\nu}^{Comb}\right]\right\}$ as follows

$$\mathbb{T}\exp\left\{i\int_0^{\lambda} ds\left[-\frac{ie}{2}\gamma^{\mu}\gamma^{\nu}\mathcal{F}_{\mu\nu}^{Comb}\right]\right\} = \mathbb{T}\exp\left[-i\int_0^{\lambda}(H_1 + H_2)\, ds\right] \tag{7.102}$$

where

$$H_1 = -(i\omega_0/2)\gamma^1\gamma^2, \tag{7.103}$$

and

$$H_2 = -(ie/2)\hat{k}\frac{d\hat{A}}{d\xi} \text{ with } \hat{A} = A_{\mu}\gamma^{\mu}. \tag{7.104}$$

At first, we calculate the term $\mathbb{T}\exp\left[-i\int_0^{\lambda} H_1(s)\, ds\right]$. After the simplification, we obtain

$$\mathbb{T}\exp\left[-i\int_0^{\lambda} H_1(s)\, ds\right] = \exp\left[-i\frac{\omega_0}{2}\lambda\right]\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right) + \exp\left[i\frac{\omega_0}{2}\lambda\right]\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right). \tag{7.105}$$

In the second step, we calculate the following term

$$\left\{\mathbb{T}\exp\left[-i\int_0^{\lambda}\left[\left[\mathbb{T}\exp\left(-i\int_a^s H_1(s')\, ds'\right)\right]^{-1} H_2(s)\left[\mathbb{T}\exp\left(-i\int_a^s H_1(s')\, ds'\right)\right]\right] ds\right]\right\}, \tag{7.106}$$

where H_1 is hermitian

$$\left(\left(\gamma^i\right)^\dagger = -\gamma^i, (AB)^\dagger = B^\dagger A^\dagger\right) \longrightarrow H_1 = -\left(i\omega_0/2\right)\gamma^1\gamma^2 \rightarrow H_1^\dagger = -\left(i\omega_0/2\right)\gamma^1\gamma^2. \quad (7.107)$$

Then we can write

$$\left[\mathbb{T}\exp\left(-i\int_a^s H_1\left(s'\right)ds'\right)\right]^{-1} = \mathbb{T}\exp\left(i\int_a^s H_1^\dagger\left(s'\right)ds'\right). \quad (7.108)$$

Or

$$\left[\mathbb{T}\exp\left(-i\int_0^s H_1\left(s\right)ds\right)\right]^{-1} = \exp\left[i\frac{\omega_0}{2}s\right]\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right) + \exp\left[-i\frac{\omega_0}{2}s\right]\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right). \quad (7.109)$$

Whereas

$$\left[\mathbb{T}\exp\left(-i\int_0^s H_1\left(s'\right)ds'\right)\right]\left[\mathbb{T}\exp\left(-i\int_0^s H_1\left(s'\right)ds'\right)\right]^{-1} = 1. \quad (7.110)$$

This allows us to write

$$\begin{aligned}
&\left[\mathbb{T}\exp\left(-i\int_0^s H_1\left(s'\right)ds'\right)\right]^{-1} H_2\left(s\right)\left[\mathbb{T}\exp\left(-i\int_0^s H_1\left(s'\right)ds'\right)\right] \\
= \quad & -\left(ie/2\right)\left[\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right)\left(\hat{k}\frac{d\hat{A}}{d\xi}\right)\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right) + \left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right)\left(\hat{k}\frac{d\hat{A}}{d\xi}\right)\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right)\right] \\
& -\left(ie/2\right)\exp\left[-i\omega_0 s\right]\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right)\left(\hat{k}\frac{d\hat{A}}{d\xi}\right)\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right) \\
& -\left(ie/2\right)\exp\left[i\omega_0 s\right]\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right)\left(\hat{k}\frac{d\hat{A}}{d\xi}\right)\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right),
\end{aligned} \quad (7.111)$$

Now, it is easy to show that

$$\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right)\left(\hat{k}\frac{d\hat{A}}{d\xi}\right)\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right) + \left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right)\left(\hat{k}\frac{d\hat{A}}{d\xi}\right)\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right) = 0. \quad (7.112)$$

and

$$\begin{cases}
\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right)\left(\hat{k}\frac{d\hat{A}}{d\xi}\right)\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right) = \hat{k}\hat{\epsilon}^*\left(\epsilon\frac{dA}{d\xi}\right) \\
\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right)\left(\hat{k}\frac{d\hat{A}}{d\xi}\right)\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right) = \hat{k}\hat{\epsilon}\left(\epsilon^*\frac{dA}{d\xi}\right)
\end{cases}. \quad (7.113)$$

Finally, we can write

$$\begin{aligned}&\left[\mathbb{T}\exp\left(-i\int_0^s H_1(s')\,ds'\right)\right]^{-1} H_2(s)\left[\mathbb{T}\exp\left(-i\int_0^s H_1(s')\,ds'\right)\right]\\ =&\ -(ie/2)\exp\left[-i\omega_0 s\right]\hat{k}\hat{\epsilon}^*\left(\epsilon\frac{dA}{d\xi}\right)-(ie/2)\exp\left[i\omega_0 s\right]\hat{k}\hat{\epsilon}\left(\epsilon^*\frac{dA}{d\xi}\right),\end{aligned}\tag{7.114}$$

therefore, the equation (7.106) is reduced to

$$\begin{aligned}&\mathbb{T}\exp\left[-i\int_0^\lambda\left[\left[\mathbb{T}\exp\left(-i\int_0^s H_1(s')\,ds'\right)\right]^{-1} H_2(s)\left[\mathbb{T}\exp\left(-i\int_0^s H_1(s')\,ds'\right)\right]\right]\right]ds\\ =&\ 1-e/2\hat{k}\hat{\epsilon}^*\int_0^\lambda ds\exp\left[-i\omega_0 s\right]\left(\epsilon\frac{dA}{d\xi}\right)-e/2\hat{k}\hat{\epsilon}\int_0^\lambda ds\exp\left[i\omega_0 s\right]\left(\epsilon^*\frac{dA}{d\xi}\right),\end{aligned}\tag{7.115}$$

we have $\dfrac{d\xi}{ds}=k^L n\rightarrow d\xi=dsk^L n\rightarrow ds=\dfrac{1}{k^L n}d\xi$, then we can write

$$\begin{aligned}&\left[1-\frac{e}{2}\hat{k}\hat{\epsilon}^*\int_0^\lambda ds\exp\left[-i\omega_0 s\right]\left(\epsilon\frac{dA}{d\xi}\right)-\frac{e}{2}\hat{k}\hat{\epsilon}\int_0^\lambda ds\exp\left[i\omega_0 s\right]\left(\epsilon^*\frac{dA}{d\xi}\right)\right]\\ =&\ 1-\frac{e}{2k^L n}\hat{k}\hat{\epsilon}^*\int_0^\lambda d\xi'\exp\left[-i\omega_0\frac{\xi'}{k^L n}\right]\left(\epsilon\frac{dA}{d\xi'}\right)\\ &-\frac{e}{2k^L n}\hat{k}\hat{\epsilon}\int_0^\lambda d\xi'\exp\left[i\omega_0\frac{\xi'}{k^L n}\right]\left(\epsilon^*\frac{dA}{d\xi'}\right),\end{aligned}\tag{7.116}$$

Combining (7.105) and (7.115), the result of the calculation for the term $\mathbb{T}\exp\left\{i\int_0^\lambda ds\left[-\frac{ie}{2}\gamma^\mu\gamma^\nu\mathcal{F}_{\mu\nu}^{Com}\right.\right.$ is abbreviated to

$$\begin{aligned}\mathcal{S}(x_b,x_a;\lambda)\ =&\ \mathbb{T}\exp\left\{i\int_0^\lambda ds\left[-\frac{ie}{2}\gamma^\mu\gamma^\nu\mathcal{F}_{\mu\nu}^{Comb}\right]\right\}\\ =&\ \left[\exp\left[-i\frac{\omega_0}{2}\lambda\right]\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right)+\exp\left[i\frac{\omega_0}{2}\lambda\right]\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right)\right]\\ &\times\left[1-\frac{e}{2}\hat{k}\hat{\epsilon}^*\int_0^\lambda ds\exp\left[-i\omega_0 s\right]\left(\epsilon\frac{dA}{d\xi}\right)\right.\\ &\left.-\frac{e}{2}\hat{k}\hat{\epsilon}\int_0^\lambda ds\exp\left[i\omega_0 s\right]\left(\epsilon^*\frac{dA}{d\xi}\right)\right].\end{aligned}\tag{7.117}$$

Finally, by integrating over k^L, the Green function expression becomes as

$$
\begin{aligned}
S^{c}(x_b, x_a) &= \mathcal{S}(x_b, x_a; \lambda)\,\mathcal{K}_{KG}(x_b, x_a; \lambda) \\
&= i^{1/2}\frac{\omega_0}{32\pi^{3/2}\lambda^{1/2}\sin\left(\frac{\omega_0}{2}\lambda\right)}\exp\left(i\frac{\left(x_b^L - x_a^L\right)^2}{4\lambda}\right) \\
&\quad\times\exp\left[-i\left(J(\xi) + \frac{e}{2}\left(x^T F(H)\,\mathbb{Q}^T\right)\right)\Big|_{\xi_a}^{\xi_b}\right] \\
&\quad\times\exp\left[i\frac{\omega_0}{4}\left[\left(u_b^2 - u_a^2\right) + \left(\upsilon_b^2 - \upsilon_a^2\right)\right]\cot\left(\frac{\omega_0}{2}\lambda\right) + 2\left(u_b\upsilon_a - \upsilon_b u_a\right)\right] \\
&\quad\times\left[\exp\left[-i\frac{\omega_0}{2}\lambda\right]\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right) + \exp\left[i\frac{\omega_0}{2}\lambda\right]\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right)\right] \\
&\quad\times\left[1 - \frac{e}{2}\hat{k}\hat{\epsilon}^*\int_0^\lambda ds\exp\left[-i\omega_0 s\right]\left(\epsilon\frac{dA}{d\xi}\right)\right. \\
&\quad\left.-\frac{e}{2}\hat{k}\hat{\epsilon}\int_0^\lambda ds\exp\left[i\omega_0 s\right]\left(\epsilon^*\frac{dA}{d\xi}\right)\right], \qquad (7.118)
\end{aligned}
$$

with

$$
J(\xi) = \int_{-\infty}^{\xi} d\xi A(\xi)\frac{d\mathbb{Q}^T}{d\xi}, \qquad (7.119)
$$

7.5 Schwinger's pair production in monolayer graphene in the combination of an electromagnetic plane wave and a parallel magnetic field

7.5.1 Shwinger's effective action and the pair production probability

In this section, we will recall how to construct the same system [37] but for graphene quasiparticles in the combination of a Volkov plane wave and a parallel magnetic field. Firstly, we will calculate the effective action that is defined as

$$
\begin{aligned}
S_{eff.} &= \int_0^{\infty} \frac{d\lambda}{\lambda} Tr_{\gamma} Tr_{x} \mathcal{S}\left(x_b, x_a; \lambda\right) \mathcal{K}_{KG}\left(x_b, x_a; \lambda\right) \\
&= \int_0^{\infty} \frac{d\lambda}{\lambda} Tr_{x} Tr_{\gamma} \left[i^{1/2} \frac{\omega_0}{32\pi^{3/2}\lambda^{1/2} \sin\left(\frac{\omega_0}{2}\lambda\right)} \exp\left(i \frac{\left(x_b^L - x_a^L\right)^2}{4\lambda} \right) \right. \\
&\times \exp\left[-i \left(J\left(\xi\right) + \frac{e}{2}\left(x^T F\left(H\right) \mathbb{Q}^T\right) \right) \Big|_{\xi_a}^{\xi_b} \right] \\
&\times \exp\left[i \frac{\omega_0}{4} \left[\left(u_b^2 - u_a^2\right) + \left(v_b^2 - v_a^2\right) \right] \cot\left(\frac{\omega_0}{2}\lambda\right) + 2\left(u_b v_a - v_b u_a\right) \right] \\
&\times \left[\left[\exp\left[-i \frac{\omega_0}{2}\lambda \right] \left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2} \right) + \exp\left[i \frac{\omega_0}{2}\lambda \right] \left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2} \right) \right] \right. \\
&\times \left[1 - \frac{e}{2} \hat{k}\hat{\epsilon}^* \int_0^{\lambda} ds \exp\left[-i\omega_0 s \right] \left(\epsilon \frac{dA}{d\xi} \right) \right. \\
&\left. \left. \left. - \frac{e}{2} \hat{k}\hat{\epsilon} \int_0^{\lambda} ds \exp\left[i\omega_0 s \right] \left(\epsilon^* \frac{dA}{d\xi} \right) \right] \right] \right] .
\end{aligned}
\tag{7.120}
$$

In lower-dimensional space-time, the Dirac gamma matrices γ^{μ} as a function of the Pauli matrices are defined in the following way

$$
\gamma^0 = \sigma_3, \ \gamma^1 = i\sigma_2, \ \gamma^2 = -i\sigma_1. \tag{7.121}
$$

then we can write

$$
\begin{aligned}
Tr_{\gamma} \frac{\hat{\epsilon}\hat{\epsilon}^*}{2} &= Tr_{\gamma} \left(\frac{1}{2} I - \frac{i}{2} \left(\gamma^1 \gamma^2\right) \right) \\
&= Tr_{\gamma} \left(\frac{1}{2} I - \frac{i}{2} \left(\left(i\sigma_2\right) \left(-i\sigma_1\right) \right) \right) \\
&= Tr_{\gamma} \left(\frac{1}{2} I - \frac{1}{2} \sigma_3 \right) \\
&= 1,
\end{aligned}
\tag{7.122}
$$

and

$$
\begin{aligned}
Tr \frac{\hat{\epsilon}^*\hat{\epsilon}}{2} &= Tr_{\gamma} \left(\frac{1}{2} I + \frac{i}{2} \left(\gamma^1 \gamma^2\right) \right) \\
&= Tr_{\gamma} \left(\frac{1}{2} I + \frac{i}{2} \left(\left(i\sigma_2\right) \left(-i\sigma_1\right) \right) \right) \\
&= Tr_{\gamma} \left(\frac{1}{2} I + \frac{1}{2} \sigma_3 \right) \\
&= 1.
\end{aligned}
\tag{7.123}
$$

On the other hand, we have

$$Tr_\gamma\left[\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right)\hat{k}\hat{\epsilon}^*\right] = Tr_\gamma\left[\left(\frac{1}{2}I+\frac{i}{2}\left(\gamma^1\gamma^2\right)\right)\left(-\left(k_0\gamma^0\right)+\left(k_3\gamma^3\right)\right)\right]=0, \tag{7.124}$$

$$Tr_\gamma\left[\left(\frac{\hat{\epsilon}\hat{\epsilon}^*}{2}\right)\hat{k}\hat{\epsilon}\right] = Tr_\gamma\left[\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right)\hat{k}\hat{\epsilon}^*\right]=Tr_\gamma\left[\left(\frac{\hat{\epsilon}^*\hat{\epsilon}}{2}\right)\hat{k}\hat{\epsilon}\right]=0 \tag{7.125}$$

Also, we have $\xi = n^\mu x_\mu = y-\tau \longrightarrow \xi_a = y_a-\tau_a, \xi_b = y_b - \tau_b$ and $\hat{\epsilon}=\epsilon_\mu\gamma^\mu, \hat{\epsilon}^*=\epsilon^*_\mu\gamma^\mu, \hat{n}= n_\mu\gamma^\mu$. When we put $\xi_a=\xi_b$ and by simplification, we get the effective action expression given by

$$S_{eff.} = i^{1/2}\frac{\omega_0}{32\pi^{3/2}}\int dxdyd\tau\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\frac{1}{\sin\left(\frac{\omega_0}{2}\lambda\right)}\left[\exp\left[i\frac{\omega_0}{2}\lambda\right]+\exp\left[-i\frac{\omega_0}{2}\lambda\right]\right], \tag{7.126}$$

$$= i^{1/2}\frac{\left(\frac{\omega_0}{2}\right)}{8\pi^{3/2}}\int dxdyd\tau\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\frac{\cos\left(\frac{\omega_0}{2}\lambda\right)}{\sin\left(\frac{\omega_0}{2}\lambda\right)}, \tag{7.127}$$

We perform the final space-time coordinates integral for all terms, and returning to real time via the replacements ($T \longrightarrow iT$) on the expression $S_{eff.}$ we get

$$S_{eff.} = i^{3/2}\frac{\left(\frac{e\mathcal{B}}{2}\right)\upsilon_F}{8\pi^{3/2}}TL^2\int_0^\infty\frac{d\lambda}{\lambda^{3/2}}\cot\left(\frac{e\mathcal{B}}{2}\lambda\right), \tag{7.128}$$

this latter is the effetive action for charged massless particle in a constant magnetic field. Consequently, the pair production probability of quasiparticles-holes in graphene by the Redmond field is null

$$\mathcal{P}\,(\text{pair}) = 0, \tag{7.129}$$

7.6 Conclusion

In this chapter, we have solved the Dirac-graphene equation in the presence of an electromagnetic plane wave plus a uniform static magnetic field parallel to the direction of the propagation of the electromagnetic plane wave using the Redmond method and via path integral formalism using the delta-functionals approach.

For the path integral formalism, we have constructed the propagator and integrated the spin factor by using the $\mathbb{T}$-product technique. Furthermore, we have calculated the probability of pair production using Schwinger's method. As it is well known, the essential result is that the magnetic field does not create pairs; the pair production probability does not depend on the plane wave and its combination cannot creat pairs in monolayer graphene. This result agrees with Ref. [39] .

Chapter 8

General conclusion

In this thesis, we have studied some problems of relativistic quantum mechanics in monolayer graphene. We have treated the behavior of graphene quasiparticles in the presence of some configuration of fields consisting of a single plane wave, two plane waves, and the combination of the plane wave plus a uniform magnetic field. In addition, we analyzed the problem of pair creation from a vacuum under the action of an electromagnetic field in the framework of a non-commutative geometry.

In the third chapter, we find the solutions of the massless Dirac-graphene equation in the presence of two orthogonal electromagnetic plane waves via the ansatz proposed by Volkov.

Furthermore, in the fourth chapter, we have constructed the causal Green's function of graphene quasiparticles in interaction with two electromagnetic plane waves using a general representation for the propagator via bosonic and fermionic path integrals formalism. The wave functions have been deduced, and the results are agree with those obtained via Volkov's method.

In the fifth chapter, we have studied the issue of pair creation from the vacuum in monolayer graphene, under the action of a static external electromagnetic field in the framework of non-commutative phase space coordinates by using Schwinger's method. We have calculated the effective action for two different gauges by using the supersymetric path integral. It is shown that the results are identical to the Schwinger result when we put the limits $\theta \longrightarrow 0, \eta \longrightarrow 0$ and $\mathcal{B} \longrightarrow 0$. For the same limit, we recover the same results of the probability obtained in the literature [94] via the semi-classical approach for multilayer graphene with the number of layers $J = 1$, and the same results obtained via the exact solution of the Schrödinger equation for the case of monolayer graphene. Additionally, we have studied the influence of a plane wave and two orthogonal plane wave fields on the process of pair creation in monolayer graphene. In this case, we have deduced that a single plane wave and the two orthogonal plane waves don't create pairs.

Also in the sixth chapter, we have studied the problem of pair creation of both scalar and spinorial relativistic particles from the vacuum by a constant electromagnetic field in the framework of non-commutative phase space coordinates using the same method in the previous chapter, and we have discussed the special cases of the pair production probability and compare them with those of the literature [126] by taking the limit $v_F \to c, m \to 0$.

In the last chapter, we have presented the solution of Redmond problem in graphene using different methods, we solved the Dirac-graphene equation in the presence of an electromagnetic plane wave plus a uniform static magnetic field parallel to the direction of the propagation of the electromagnetic plane wave using the Redmond method and via path integral formalism using the delta-functional method.

For the path integral formalism, we have constructed the propagator and corresponding Green's function. For this role, we integrate the spin factor by using the $\mathbb{T}$-product technique. Furthermore, the pair creation problem is examined, and the null pair production probability is deduced.

Furthermore, we aspire from this research to generalize it to two waves propagating in different directions, which has been discussed in the special case when the angle between the directions of propagation of the waves is very small $\left(\hat{\theta} \ll 1\right)$ used in the application of laser beams to produce strong electromagnetic radiation.

Through this work, we can say that the path integral formalism is a successful technique that can be used in solving problems related to plane wave field, and the calculation of the pair production probability and its results are essential because the Schwinger mechanism can be realized experimentally in condensed matter, especially in graphene.

Appendix A: Inverse matrix $\left(\mathcal{M}^{-1}\right)^{\mu\nu}$:

In order to determine $\sqrt{\frac{\det \mathcal{M}(e)}{\det \mathcal{M}(e=0)}}$, we are going to calculate the inverse matrix $\left(\mathcal{M}^{-1}\right)^{\mu\nu}$, where

$$\mathcal{M}_{\mu\nu}\left(e,\tau,\tau'\right) = \left[\eta_{\mu\nu}\delta'(\tau-\tau') - e\mathcal{F}^{\star}_{\mu\nu}\left(\tau\right)\delta(\tau-\tau')\right]. \tag{1}$$

As we know the relation between $\mathcal{M}_{\mu\upsilon}(\tau,\tau')$ and its inverse matrix $(\mathcal{M}^{-1})^{\mu\nu}(\tau,\tau')$ satisfying :

$$\int_0^1 \mathcal{M}_{\mu\nu}(e,\tau,s)\left(\mathcal{M}^{\nu\beta}\right)^{-1}(e,s,\tau') = \delta_\mu^\beta \delta(\tau-\tau'). \tag{2}$$

Substituting (1) into (2) we obtain

$$\int_0^1 ds\left[\eta_{\mu\nu}\delta'(\tau-s) - e\mathcal{F}^\star_{\mu\nu}\delta(\tau-s)\right]\left(\mathcal{M}^{\nu\beta}\right)^{-1}(e,s,\tau') = \delta_\mu^\beta\delta(\tau-\tau'). \tag{3}$$

This latter is equivalent to the differential equation

$$\frac{d}{d\tau}\mathcal{M}^{-1}{}_{\mu\nu}(e,\tau,\tau') - e\mathcal{F}^{\star}_{\mu}{}^{\lambda}\mathcal{M}^{-1}{}_{\lambda\nu}(e,\tau,\tau') = \eta_{\mu\nu}\delta(\tau-\tau'). \tag{4}$$

Knowing that in our case $\mathcal{F}^\star$ is a constant, let us insert the following general solution [168]

$$\mathcal{M}^{-1}(e,\tau,\tau') = \exp(e\tau\mathcal{F}^\star)\,\tilde{c}(\tau,\tau'). \tag{5}$$

Into Eq. (4), we get

$$\tilde{c}(\tau,\tau') = \Theta(\tau-\tau')\exp(-e\tau'\mathcal{F}^\star) + \tilde{c}(\tau'). \tag{6}$$

Therefore, from (6) and (5) we obtain,

$$\mathcal{M}^{-1}(e,\tau,\tau') = \exp(e\tau\mathcal{F}^\star)\left[\Theta(\tau-\tau')\exp(-e\tau'\mathcal{F}^\star) + \tilde{c}(\tau')\right]. \tag{7}$$

In this step we can determine the condition $\mathcal{M}^{-1}_{\alpha\beta}(0,\tau') = \tilde{c}(\tau')$, by inserting the integral over τ on the equation (1), we find

$$\mathcal{M}^{-1}_{\mu\nu}(1,\tau') - \mathcal{M}^{-1}_{\mu\nu}(0,\tau') = e\mathcal{F}^\star{}_\mu{}^\lambda\int_0^1 \mathcal{M}^{-1}_{\lambda\nu}(e,\tau_1,\tau')\,d\tau_1 + \eta_{\mu\nu}\int_0^1 \delta(\tau_1-\tau')\,d\tau_1, \tag{8}$$

with $\mathcal{M}^{-1}_{\mu\nu}(1,\tau') = -\mathcal{M}^{-1}_{\mu\nu}(0,\tau')$, Eq. (8) becomes as

$$-2\tilde{c}(\tau')\,\eta_{\mu\nu} = e\mathcal{F}^\star{}_\mu{}^\lambda\int_0^1 \mathcal{M}^{-1}_{\lambda\nu}(e,\tau_1,\tau')\,d\tau_1 + \eta_{\mu\nu}. \tag{9}$$

After that, we insert the expression of $\mathcal{M}^{-1}_{\lambda\nu}$ in Eq. (9) and we can find $\tilde{c}(\tau')$ after a long and straightforward calculations as follows:

$$\tilde{c}(\tau') = -e^{-e\tau'\mathcal{F}^\star}e^{e\lambda\mathcal{F}^\star}\left[1+e^{e\lambda\mathcal{F}^\star}\right]^{-1}. \tag{10}$$

Substituting (10) into (7), we get

$$\mathcal{M}^{-1}(e,\tau,\tau') = \frac{1}{2}e^{e(\tau-\tau')\mathcal{F}^\star}\left[\varepsilon(\tau-\tau') - \tanh\left(\frac{e\lambda}{2}\mathcal{F}^\star\right)\right], \tag{11}$$

where $\Theta(\tau-\tau') = [1+\varepsilon(\tau-\tau')]/2$.

Further, with a simple calculation, we can conclude

$$\sqrt{\frac{\det \mathcal{M}(e)}{\det \mathcal{M}(0)}} = \exp\left\{\frac{1}{2}\int_0^e de' \int_0^\lambda d\tau \left(\mathcal{M}^{-1}\right)^{\mu\nu}(e',\tau,\tau)\,\mathcal{F}^\star{}_{\mu\nu}\right\} \tag{12}$$

$$= \exp\left\{\frac{\lambda}{4}\int_0^e de' \left[\mathcal{F}^\star{}_{\mu\nu}\tanh\left(\frac{e'\lambda}{2}\mathcal{F}^\star\right)^{\mu\nu}\right]\right\}. \tag{13}$$

To evaluate S_{eff} we start by finding $\sqrt{\det\cosh\left(\frac{\lambda\mathcal{F}^\star}{2}\right)}$. Since $\mathcal{F}^\star$ is antisymmetric it can be use the series of $\tanh(x)$ and the matrix characteristics of $\mathcal{F}^\star(\tau)$, (i.e., $(\mathcal{F}^\star)^3 \propto \mathcal{F}^\star$).

For example, for the first gauge, we have

$$\mathcal{F}^\star_{02} = \frac{\mathcal{E}}{\upsilon_F}, \quad \mathcal{F}^\star_{12} = -\mathcal{B}(1+\tfrac{\theta e\mathcal{B}}{4}+\tfrac{\eta}{e\mathcal{B}}), \tag{14}$$

and for the second gauge, we have

$$\mathcal{F}^\star_{02} = \frac{\mathcal{E}}{\upsilon_F}(1+\tfrac{e\mathcal{B}\theta}{2}), \quad \mathcal{F}^\star_{12} = -\mathcal{B}(1+\tfrac{\theta e\mathcal{B}}{4}+\tfrac{\eta}{e\mathcal{B}}). \tag{15}$$

From these, we can prove the following equality for two gauges respectively

$$\left[\tanh\left(\frac{e'\lambda}{2}\mathcal{F}^\star\right)\right]_{\mu\nu} = \frac{\tanh\left(\frac{e'\lambda}{2}\sqrt{\left(\frac{\mathcal{E}}{\upsilon_F}\right)^2 - \mathcal{B}^2(1+\frac{\theta e\mathcal{B}}{4}+\frac{\eta}{e\mathcal{B}})^2}\right)}{\sqrt{\mathcal{E}^2-\mathcal{B}^2(1+\frac{\theta e\mathcal{B}}{4}+\frac{\eta}{e\mathcal{B}})^2}}\mathcal{F}^\star_{\mu\nu}, \tag{16}$$

and

$$\left[\tanh\left(\frac{e'\lambda}{2}\mathcal{F}^\star\right)\right]_{\mu\nu} = \frac{\tanh\left(\frac{e'\lambda}{2}\sqrt{\left(\frac{\mathcal{E}}{\upsilon_F}\right)^2(1+\frac{e\mathcal{B}\theta}{2})^2 - \mathcal{B}^2(1+\frac{\theta e\mathcal{B}}{4}+\frac{\eta}{e\mathcal{B}})^2}\right)}{\sqrt{\mathcal{E}^2-\mathcal{B}^2(1+\frac{\theta e\mathcal{B}}{4}+\frac{\eta}{e\mathcal{B}})^2}}\mathcal{F}^\star_{\mu\nu}. \tag{17}$$

Finally, we get

$$\sqrt{\frac{\det \mathcal{M}(e)}{\det \mathcal{M}(e=0)}} = \exp\left\{\frac{\lambda}{4}\int_0^e de' \left[\mathcal{F}^\star_{\mu\nu}\mathcal{F}^{\star\mu\nu}\frac{\tanh(\frac{e'\Upsilon_{1,2}}{2}\lambda)}{\Upsilon_{1,2}}\right]\right\}, \tag{18}$$

with $\Upsilon_{1,2}$ are defined in above sections. While $\mathcal{F}^\star_{\mu\nu}\mathcal{F}^{\star\mu\nu}$ is given by

$$Tr\mathcal{F}^{\star 2} = \mathcal{F}^\star_{\mu\nu}\mathcal{F}^{\star\mu\nu} = 2\Upsilon^2_{1,2}. \tag{19}$$

Which we can easily rewrite (18) as follows

$$\sqrt{\frac{\det \mathcal{M}(e)}{\det \mathcal{M}(e=0)}} = \exp\left\{\int_0^e de' \left[\frac{\Upsilon_{1,2}\lambda}{2}\tanh\left(\frac{e'\Upsilon_{1,2}}{2}\lambda\right)\right]\right\} = \cosh\left(e\Upsilon_{1,2}\lambda\right). \tag{20}$$

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ملخص

في هاته الأطروحة قمنا بحل معادلة ديراك-جرافين ثنائية البعد في وجود موجتين كهرو مغناطيسيتين متعامدتين باستعمال حل فولكوف. كما طبقنا طريقة تكاملات المسار لبناء دالة غرين لأشباه جسيمات الجرافين التي تتفاعل مع نفس الحقل السابق باستخدام طريقة فرادكن و جيثمان و استنتجنا دالتي الموجة الموافقتين.

درسنا كذلك ظاهرة خلق الجسيمات من الفراغ في نظام الجرافين في فضاء لا تبادلي بوجود حالتين مختلفتين لحقل كهرومغناطيسي ثابت باستخدام طريقة شوينغر, حيث استنتجنا احتمال خلق أزواج الجسيمات باستخدام فعل شوينغر الفعال, و درسنا بعض الحالات الخاصة للنتائج التي تحصلنا عليها. من أهم النتائج التي تحصلنا عليها هي أن الفضاء الا تبادلي له دور في خلق أزواج الجسيمات . كذلك درسنا تأثير موجة كهرومغناطيسية واحدة ثم موجتسن كهرومغناطيسيتين متعامدتين على ظاهرة خلق أزواج الجسيمات والذي يكون معدوما.

بالإضافة إلى ذلك قمنا بحل معادلة ديراك الخاصة بأشباه جسيمات الجرافين في وجود تراكب حقلين مكون من موجة كهرومغناطيسية و حقل مغناطيسي ثابت يوازي اتجاه حركة الموجة الكهرومغناطيسية بطريقتين مختلفتين. الأولى باستعمال طريقة رادموند والثانية باستعمال طريقة تكامل المسارات لفاينمان. بعد الحصول على دالة غرين قمنا بحساب فعل شوينغر الفعال و استنتجنا احتمال خلق أزواج الجسيمات في الفراغ و كانت النتيجة أن هذا الحقل لا يمكنه خلق أزواج الجسيمات في مادة الجرافين أيضا.

الكلمات المفتاحية: الجرافين, أشباه الجسيمات, تكاملات المسار, دالة غرين, الناشر, معادل ديراك, متغيرات غراسمان, الهندسة اللا تبديلية, مفعول شوينغر, حقل مغناطيسي, موجة كهرومغناطيسية, حقل رادموند.

Résumé

Dans cette thèse, nous avons résolu l'équation de Dirac-Graphène en présence de deux ondes électromagnétique perpendiculaires en utilisant l'ansatz de Volkov, et nous avons finalement obtenu les fonctions d'onde correspondantes. Nous avons également accompli le même travail basé sur le formalisme des intégrales de chemins, où la fonction de Green a été construite en appliquant la méthode de Feynman en utilisant la technique de Fradkin et Gitman.

Aussi, nous avons voulu étudier le phénomène de création des particules à partir du vide dans le système du graphène dans un espace non commutatif par deux jauges différentes d'un champ électromagnétique constant en utilisant l'action effective de Schwinger suivant l'intégrale de chemin, d'où nous avons calculé la probabilité de création des particules. Comme application telle que nous l'avons déduite pour certains cas particuliers de (θ,η,E,B). De plus, nous avons également examiné l'influence d'un et deux champs d'ondes planes orthogonales sur le processus de création de paires dans le graphène qui est nul.

De plus, nous avons résolu l'équation de Dirac-Graphène pour des quasi-particules en interaction avec la combinaison d'une onde plane et d'un champ magnétique constant parallèle à la direction de propagation de l'onde électromagnétique, dans un premier temps en utilisant la méthode de Redmond, la deuxième est la méthode de Delta fonctionnelle. Enfin, la création de paires de quasiparticules de graphène à partir du vide par cette configuration de champ est analysée.

Mots-clés : Graphene, quasi-particules, integral de chemin, La fonction de Green, propagateur, L'equation de Dirac , Les variables de Grassmann, La geometrie non-commutative, L'effet de Shwinger, onde plane, Champ électromagnétique, Champ de Redmond.