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RESOLUTION OF SOME OPTIMIZATION PROBLEMS VIA KERNEL FUNCTIONS

Author
SAFA GUERDOUH

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EXAMINING COMMITTEE MEMBERSHIP

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Wided CHIKOUCHE	Professor	University of Jijel	Supervisor
Imene TOUIL	Assoc.Prof	University of Jijel	Co-supervisor
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Abstract

Resolution of some optimization problems via kernel functions

by Safa GUERDOUH

This thesis topic falls within the realm of linear optimization and semidefinite optimization. The objective is to study primal-dual interior-point methods for solving linear optimization problems. These methods are based on introducing new hyperbolic kernel functions to determine new class of search directions. The analysis of the complexity will be established, and an extension to the semidefinite optimization case will be addressed.

In particular, we investigate the concept of feasible and infeasible interior-point methods that rely on kernel functions to define the search directions.

We first deal with feasible primal-dual interior-point methods for solving linear optimization problems. These methods are based on new hyperbolic kernel functions. We study the primal-dual interior-point method based on each kernel function and we derive the complexity bounds for each method. The proposed methods are then implemented in Matlab and compared with several existing kernel functions in the literature on various linear optimization examples.

Then, we extend primal-dual feasible interior-point methods to solve semidefinite optimization problems. We provide a concise summary on the basic of primal-dual interior-point methods for semidefinite optimization followed by the complexity analysis of primal-dual interior-point algorithms based on a class of kernel functions.

After that, we present a full-Newton step infeasible interior-point method for solving linear optimization problems based on a hyperbolic kernel function. Unlike the feasible interior-point methods, this method doesn't require a feasible starting point. Under some appropriate conditions, we guarantee that this method will converge to an optimal solution. In addition, we showcase the practical efficiency of the method by performing some numerical experiments.

Keywords: Linear programming, Semidefinite programming, Feasible interior-point methods, Infeasible interior-point methods, Kernel functions, full-Newton step, Complexity analysis, Large and small-update methods.

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”

Safa Guerdouh

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List of Abbreviations

LO	Linear optimization
SDO	Semidefinite optimization
SOCO	Second order conic optimization
IPM	Interior-point method
IIPM	Infeasible interior-point method
IPA	Interior-point algorithm
IIPA	Infeasible interior-point algorithm
IPC	Interior-point condition
KKT	Karush Khun Tucker
KF	Kernel fuction
EF	Example with fixed size
EV	Example with variable size
s.t.	such that
e.g.	for example
i.e.	that is

List of notations

Spaces

$\mathbb{R}^n$	the real n-dimensional space
$\mathbb{R}^n_+$	the nonnegative orthant of $\mathbb{R}^n$
$\mathbb{R}^n_{++}$	the positive orthant of $\mathbb{R}^n$
$\mathbb{R}^{m\times n}$	the space of $m \times n$ real matrices
$\mathbb{S}^n$	the space of all symmetric $n \times n$ real matrices
$\mathbb{S}^n_+$	the space of positive semidefinite symmetric matrices of order n
$\mathbb{S}^n_{++}$	the space of positive definite symmetric matrices of order n

Vectors

x, s	vectors of $\mathbb{R}^n$
$(x_i)_{1\le i\le n}$	components of the vector x
$x \ge 0$	means all the components of x are nonnegative
$x > 0$	means all the components of x are positive
x^T	the transpose of x
X	$=$ diag(x) the diagonal matrix that has components of the vector x on the diagonal and zeros everywhere else
$\|x\|$	$= \sqrt{\sum_{i=1}^{n} x_i^2}$ the Euclidean norm of $x \in \mathbb{R}^n$
$\|x\|_1$	$= \sum_{i=1}^{n} \|x_i\|$ the 1-norm of x
$\|x\|_\infty$	$= \max_{1\le i\le n}^{n} \|x_i\|$ the infinity norm of x
$x^{-1}, \sqrt{x}, xs, \frac{x}{s}$	component wise operations (inverse $(x_i \ne 0, 1 \le i \le n)$, square root $(x \ge 0)$, product, division $(s_i \ne 0, 1 \le i \le n)$)
e	$= (1, \ldots, 1)^T$ denotes a vector of all ones

Matrices

M	a matrix of $\mathbb{R}^{m\times n}$
$(M_{i,j})_{\substack{1\le i\le m \\ 1\le j\le n}}$	components of the matrix $M \in \mathbb{R}^{m\times n}$
M^T	the transpose of the matrix M
$\lambda_i(M)$	the ith eigenvalue of $M \in \mathbb{R}^{n\times n}$
$\lambda_{min}(M)$	the smallest eigenvalue of $M \in \mathbb{R}^{n\times n}$
$\lambda_{max}(M)$	the largest eigenvalue of $M \in \mathbb{R}^{n\times n}$
$\eta_i(M)$	the ith singular value of $M \in \mathbb{R}^{m\times n}$

$\mathrm{tr}(M)$	$= \sum_{i=1}^{n} M_{ii}$ the trace of $M \in \mathbb{R}^{n \times n}$
	$= \sum_{i=1}^{n} \lambda_i(M)$ if $M \in \mathbb{S}^n$
$M \bullet X$	$= \mathrm{tr}(X^T M)$ the inner-product between $M \in \mathbb{R}^{m \times n}$ and $X \in \mathbb{R}^{m \times n}$
I	the identity matrix
$M^{\frac{1}{2}}$ (or $\sqrt{M}$)	the symmetric square root of $M \in \mathbb{S}^n_+$
$\|M\|_F$	$= \sqrt{M \bullet M}$ the Frobenius matrix norm of $M \in \mathbb{R}^{n \times n}$
$\mathrm{vec(M)}$	$= [M_{1,1}, M_{2,1}, \ldots, M_{m,1}, M_{1,2}, M_{2,2}, \ldots, M_{m,n}]^T$ for $M \in \mathbb{R}^{m \times n}$
$M \succeq 0$ (resp. $M \succ 0$)	means that M is positive semidefinite (resp. definite)
$M \preceq 0$ (resp. $M \prec 0$)	means that M is negative semidefinite (resp. definite)

Functions

$f, g : \mathbb{R}^n \mapsto \mathbb{R}$	real valued functions with n variables
$\nabla f(x_1, \ldots, x_n)$	$= (\frac{\partial f}{\partial x_1}(x_1, \ldots, x_n), \ldots, \frac{\partial f}{\partial x_n}(x_1, \ldots, x_n))^T$ gradient vector of a differentiable function f at the point $(x_1, \ldots, x_n)$
$\nabla^2 f(x_1, \ldots, x_n)$	Hessian matrix of a twice differentiable function f at the point $(x_1, \ldots, x_n)$ whose components are $(\nabla^2 f(x_1, \ldots, x_n))_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1, \ldots, x_n)$, $i, j = 1, \ldots, n$
$f(x) = \mathcal{O}(g(x))$	means that $f(x) \leq Cg(x)$ for some positive constant C
$f(x) = \Theta(g(x))$	means that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ for two positive constants C_1 and C_2

This work is dedicated to my parents, my sister, my brothers, my friends and everyone who believed in me

Introduction

Mathematical programming, also known as mathematical optimization, is a field of mathematics and computer science that deals with finding the best solution from a set of feasible solutions to a given problem. The goal is to optimize an objective function, subject to a set of constraints. Mathematical programming has applications in various fields, including operations research, engineering, economics, finance, and data science.

The key concepts and components of mathematical programming are

- Objective Function: This is the function that needs to be either maximized or minimized. It represents the quantity you want to optimize, such as profit, cost, time, or efficiency.
- Decision Variables: These are the variables that you can adjust or "decide" on to achieve the optimal solution. The values of these variables impact the objective function.
- Constraints: They are conditions or limitations that must be satisfied. They can represent restrictions on the decision variables and are essential in modeling real-world problems accurately.
- Feasible Region: It is the set of all possible combinations of values for the decision variables that satisfy the given constraints.
- Optimal Solution: It is the combination of values for the decision variables that results in the best (maximum or minimum) value of the objective function while satisfying all constraints.
- Optimization Algorithms: They are computational procedures used to find the optimal solution among a set of feasible solutions to a particular problem. Various algorithms are used to solve optimization problems, including simplex method, interior-point methods and many more.

Mathematical programming provides a systematic and efficient way to make decisions and allocate resources in a wide range of applications, making it a powerful tool for addressing complex real-world problems.

Depending on the characteristics of the feasible set and of the objective function, some properties can be attributed to the problem, impacting significantly the scope of the problem and the method for its resolution. These properties are generally non-exclusive and are given above by order of importance. Indeed the key property of an optimization problem lies in its convexity, since it guarantees that the problem is polynomially solvable.

Convex optimization is an important class of mathematical optimization that deals with problems where both the objective function and the constraints are convex. Such problems are of significant importance in optimization since convexity is generally considered as the true discriminant between "easy" and "hard" problems

in optimization.
Convex optimization has been studied heavily not only for its very powerful and elegant theory, but also because of its widespread applications in many different fields of engineering and science such as data analysis, control theory, signal processing, relaxation and randomization, and robust optimization. In addition to strong and elegant theories, the potential for creating efficient and robust software has made convex optimization very popular.

Conic optimization deals with a class of problems that is essentially equivalent to the class of convex problems, i.e. minimization of a convex function over a convex set. However, formulating a convex problem in a conic way has the advantage of providing a very symmetric form for the dual problem, which often gives a new insight about its structure, especially dealing with duality. An example of a conic convex problem is as follows: for a proper cone $\mathcal{K}$

$$(P_c) \begin{cases} \inf c^T x \\ \text{s.t. } Ax = b \\ x \in \mathcal{K}, \end{cases}$$

for any $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

The feasible set of a conic optimization problem (P_c) is the intersection of the cone $\mathcal{K}$ with the polyhedron

$$\{x \in \mathbb{R}^n : Ax = b\}.$$

In particular, if

- $\mathcal{K} = \mathbb{R}^n_+$ (nonnegative orthant), then (P_c) is a linear optimization (LO) problem.
- $\mathcal{K} = \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^n : ||x|| \leq x_0\}$ (second-order cone), then (P_c) is a second-order conic optimization (SOCO) problem.
- $\mathcal{K} = \mathbb{S}^n_+$ (cone of positive semidefinite matrices), then (P_c) is a semidefinite optimization (SDO) problem.

These optimization areas are listed in such a way that each area includes the previous one i.e., any SOCO problem can be put under the form of a SDO, any LO problem can be put under the form of a SOCO and any LO problem can be put under the form of a SDO.

In conclusion, if the cone is the nonnegative orthant, the second order cone or the cone of positive semidefinite symmetric matrices, then we have respectively a LO problem, a SOCO problem or a semidefinite optimization problem. These three cones are the most relevant for the optimization field, and they were classified as belonging to the set of self- dual and homogenous cones, also called symmetric cones.

Linear programming

A linear programming or a LO problem involves the optimization (minimization or maximization) of a linear objective function under a finite set of linear constraints. These constraints may take the form of either equality or inequality. When the constraints are incompatible and do not permit any feasible solution, the problem is deemed infeasible; otherwise, it is considered feasible. In the feasible case, where

the feasible set is not empty, two possibilities arise: the objective function is either unbounded or bounded within the domain. In the former case, the problem is labeled as unbounded, while in the latter case, it is termed as bounded.

For any LO problem, a corresponding dual problem can be formulated. The dual problem is closely connected to the primal problem, and this relationship is succinctly expressed through the optimal sets of both problems. If the optimal set of one problem is nonempty, the optimal set of the other problem is also nonempty. Furthermore, the optimal values of the objective functions for both problems are equal. These significant outcomes constitute the fundamental principles of the duality theory in LO.

The initial duality results in LO were obtained through a nonconstructive approach. These results can be deduced from various forms of Farkas' lemma [25] or more general separation theorems for convex sets, as detailed in works of Osborne in [81] and Saigal in [93]. An alternative method relies on direct inductive proofs of theorems by Farkas, Weyl, and Minkowski, with the duality results for LO emerging as a corollary of these theorems, as discussed by Gale [28].

Methods for solving a linear programming problem

Simplex method

The simplex method, introduced by Dantzig in 1947, starts from a vertex within the feasible region, which is essentially a polyhedron. It proceeds by traversing along an edge to a vertex with non-increasing values of the objective function (for a minimization problem). This process is iteratively repeated until an optimal vertex is obtained. Despite its generally favorable practical performance, there exists an example, as provided by Klee and Minty [60], having $2n$ inequality constraints and n variables that necessitates 2^n iterations for the simplex method. This indicates that the simplex method might not exhibit a polynomial worst-case iteration bound. In fact, for numerous variants of the simplex method employing different pivoting rules, instances with exponential running times have been identified.

Ellipsoid method

The ellipsoid method was introduced by Khachiyan [55] in 1979. The ellipsoid method constructs a sequence of ellipsoids that enclose an optimal solution, if one exists, and the volumes of these ellipsoids uniformly decrease at each step. The iteration bound of the ellipsoid method is $\mathcal{O}(n^2L)$, where L represents the length of input data bits. However, despite its theoretical advantages, the ellipsoid method has proven to be impractical for real-world applications (see for instance [13, 35]), and the simplex method has remained the preferred choice in practice.

Interior-point method

The first "interior-point" methods (IPMs) and their polynomial complexity emerged in the mid-fifties. These methods were primarily generated by a barrier method proposed by [27] to solve nonlinear problems. In 1967, P. Huard introduced the central path method to solve problems with nonlinear constraints [50]. The term "interior-point" comes from the fact that unlike traditional methods like the simplex algorithm that move along the boundary of the feasible region, IPMs navigate through the interior, maintaining feasibility and approaching the optimal solution.

Although, IPMs have been known since 1960, they received renewed attention after Karmarkar's result [53] in 1984. He presented a polynomial-time method with an iteration bound that is better than that of Khachiyan's ellipsoid method by a factor of $\mathcal{O}(n)$. Later, Renegar [89] improved Karmarkar's iteration bound, namely (nL), by a factor of $\sqrt{n}$.

A short historical account

First steps of linear optimization.

- 1930-1940. First appearance of LO formulations.
- 1939-1945. Second World War: operations research makes its debuts with military applications.
- 1947. G. B. Dantzig publishes the first article about the simplex method for LO [21].
- 1970. V. Klee and G. Minty prove that the simplex method has exponential worst-case complexity [60].

First steps of interior-point methods

- 1955. K. R. Frisch proposes a barrier method to solve nonlinear programs [27].
- 1967. P. Huard introduces the method of centers to solve problems with nonlinear constraints [50].
- 1968. A. V. Fiacco and G. P. McCormick develop barrier methods for convex nonlinear optimization [26].

It's noteworthy that these barrier methods were originally designed for addressing nonlinear optimization problems. While they can be extended to LO, it's crucial to highlight that the developers do not regard them as practical alternatives to the simplex method.

The interior-point methods revolution

- 1984. N. Karmarkar discovers a polynomial IPM that is practically more efficient than the ellipsoid method. He also claims superior performance compared to the simplex method [53].
- 1994. Y. Nesterov and A. Nemirovski publish a monograph on polynomial IPMs for convex optimization [77].
- 1995. A. Alizadeh [1] elegantly applied IPMs to solve SDO problems arising from combinatorics.
- 2000. Since Karmarkar's first breakthrough, more than 3000 articles have been published on the topic of IPMs. A few textbooks have been also published (see e.g. [92, 105, 106]).

Primal-dual interior-point methods

Primal-dual IPMs offer a robust and efficient approach for solving linear and semidefinite programming problems, particularly for large-scale applications, thanks to their polynomial-time complexity and ability to handle a wide range of constraints. They operate by following a trajectory called the central path, which lies in the interior of the feasible region. This path connects feasible solutions that simultaneously satisfy the primal and dual optimality conditions. In fact, they employ a path-following strategy to trace the central path, maintaining a balance between primal feasibility, dual feasibility, and optimality. This strategy ensures that the solution remains in the interior of the feasible region. At each iteration, primal-dual IPMs compute a search direction that moves along the central path towards the optimal solution. This is typically done by solving a system of linear equations derived from the Karush-Kuhn-Tucker (KKT) conditions. In addition, they have polynomial-time complexity, making them attractive for large-scale linear and convex quadratic programming problems. The polynomial-time complexity is an improvement over the exponential worst-case behaviour of the simplex method.

Primal-dual IPMs are divided into feasible IPMs and infeasible IPMs (IIPMs). Feasible IPMs start from a primal-dual strictly feasible triple and generate a sequence of strictly feasible triples converging to an optimal solution of the primal and dual pair of problems. In contrast, in IIPMs the initial iterates are not feasible, and apart from reaching optimality one needs to strive for feasibility.

Primal-dual IPMs based on kernel functions (KFs) form an intriguing subset within the realm of IPMs. This approach seamlessly combines two fundamental concepts: primal-dual IPMs and KFs. The selection of the KF holds significance not only for theoretical analysis but also for the algorithm's performance. Specifically, the central path followed by IPMs to solve a LO problem is derived by solving a parametric system, using a barrier function defined in terms of a KF with appropriate barrier parameters.

Feasible primal-dual interior-point methods based on kernel functions

Feasible primal-dual IPMs represent the class of primal-dual IPMs designed for solving constrained optimization problems, where the initial starting point satisfies the constraints.

Feasible primal-dual IPMs based on KFs represent a fascinating subclass of feasible IPMs. This approach combines two fundamental concepts: primal-dual IPMs and KFs. The choice of the KF plays an important role not only for the IPM analysis but also for the performance of the corresponding interior-point algorithm (IPA). Specifically, the KF is used to define an equivalent form of the central path of the IPM, to define the proximity measure, and to obtain search directions. In fact, the central path followed by IPMs is obtained by solving a parametric system that is characterized by a barrier function defined in terms of a KF. In addition, the gradient of the KF serves to define a measure of the distance between the iterates and the central path. Both the KF and the proximity measure effect the iteration bounds since some of their properties play an important role in the complexity analysis. That is why the complexity rate depends on the proposed KF.

Roos et al. [92] introduced the first primal-dual IPM based on the classical logarithmic barrier function. Subsequently, Peng et al. [83] presented primal-dual IPMs for LO based on the so-called self-regular (SR) barrier functions, significantly

improving the theoretical complexity achieved with the classical logarithmic KF. They obtained the currently best iteration bound for large-update IPMs, namely $\mathcal{O}(\sqrt{n}\log n \log \frac{n}{\epsilon})$, where n represents the number of variables, and ϵ is the desired accuracy in terms of the objective value. This success motivated the exploration of alternative KFs in lieu of the classical logarithmic KF.

In 2002, Bai et al. [11] presented a primal-dual IPA for LO problems based on a simple non SR KF. They established that the complexity bound for large- and small-update methods are $\mathcal{O}\left(qn \log \frac{n}{\epsilon}\right)$ and $\mathcal{O}(q^2\sqrt{n}\log \frac{n}{\epsilon})$ respectively.

In 2004, Bai et al. [8] introduced the first KF with a trigonometric barrier term. The evaluation of this function has been done furthermore by El Ghami et al. [30] in 2012. They established the worst case iteration complexity as $\mathcal{O}(n^{\frac{3}{4}}\log \frac{n}{\epsilon})$.

Since then, a number of various KFs with a trigonometric barrier term has been proposed and analyzed. For these, we refer the reader for example to Li et al. [66], Bouafia et al. [15] and Peyghami et al. [45]. The authors in [15] are the first to reach the best known complexity bound for large-update methods based on trigonometric KFs for LO. In 2014, Peyghami et al. [88] and Cai et al. [19] proposed new KFs with trigonometric-logarithmic barrier terms for LO. In the same year, Peyghami et al. [86] presented an other KF with an exponential-trigonometric barrier term which has $\mathcal{O}(\sqrt{n}\log^2 n \log \frac{n}{\epsilon})$ complexity bounds for large-update methods. The complexity bound derived in [88], namely $\mathcal{O}(n^{\frac{2}{3}}\log \frac{n}{\epsilon})$, improves the complexity bounds obtained in [30, 19, 86]. Note that the KF proposed in [19] yields the same complexity derived in [88].

Recently, Touil and Chikouche [100] introduced the first IPM based on a hyperbolic-logarithmic KF. They demonstrated that the corresponding IPA attains $\mathcal{O}(n^{\frac{2}{3}}\log \frac{n}{\epsilon})$ iterations as the worst-case complexity bound for the large-update method. In [98], they presented an IPM based on a pure hyperbolic barrier term. The complexity analysis for large-update IPMs using this KF resulted in an $\mathcal{O}(n^{\frac{3}{4}}\log \frac{n}{\epsilon})$ iteration bound.

We end this overview by mentioning the works of Bai et al. [9, 10], Amini et al. [4, 5, 3] and Bouafia et al. [16] where the authors proposed KFs with exponential barrier terms.

In view of the precedent, most of KFs used in IPMs can be categorized into five main types: logarithmic, simple algebraic, exponential, trigonometric and hyperbolic. The remaining KFs are just compositions or binary combinations of these types, see e.g. [17, 46] for recent proposed KFs.

Infeasible primal-dual interior-point methods based on kernel functions

IIPMs provide a framework for solving optimization problems where the starting point doesn't satisfy the constraints. These methods iteratively navigate through the interior of the feasible region, restoring feasibility and approaching an optimal solution.

Lustig [70] introduced the first infeasible-start IPM. His approach underwent further refinement in Mehrotra's predictor-corrector algorithm [74]. Subsequently, Roos [90] introduced a novel primal-dual infeasible IPA (IIPA) utilizing only full-Newton steps. Some extensions on LO were explored by Liu and Sun [67], Liu et al. [68], and Mansouri and Roos [71].

Salahi et al. [94] presented a new primal-dual IIPM for LO, based on a specific SR KF. Recently, Kheirfam and Haghighi [58, 57] and Moslemi and Kheirfam [76]

investigated the complexity analysis of trigonometric proximity-based IIPMs for LO and SDO problems.

Semidefinite programming

The study of semidefinite programming also known as SDO is currently one of the most active research areas in optimization. The Interest in SDO has grown even more in recent years as numerous applications have been identified in various fields such as control, statistics, finance, localization, robust optimization, and engineering.

SDO is a subfield of convex optimization concerned with the optimization of a linear objective function over the intersection of the cone of positive semidefinite matrices with an affine space.

IPMs were first developed by Karmarkar [53] for LO problems. After that, using the fact that LO is a special case of SDO, many primal-dual IPMs were extended to solve SDO problems including even primal-dual IPMs based on KFs which was a significant contribution initiated by Nesterov and Todd [78].

Scope of the thesis

In this thesis, we deal with the complexity analysis and numerical implementation of IPMs for LO and SDO problems. In particular, we investigate the concept of feasible and IIPMs that rely on KFs to define the search directions.

In Chapter 1, we deal with feasible primal-dual IPMs based on KFs. We first give a summary on the basics of primal-dual IPMs for LO. In Section 2, we present the main steps to obtain the complexity of primal-dual IPAs based on a specific class of KFs. This section is hugely inspired by the work of Bai et al. [8]. After that, we apply the procedure used in Section 2 on three new hyperbolic KFs: an exponential hyperbolic KF [39], a parameterized hyperbolic logarithmic KF which can be considered as a generalization of the KF proposed in [100] and a hyperbolic KF [41] which is a generalization, up to a multiplicative constant, of the KF introduced in [98]. The primal-dual IPA based on each of these functions is studied and the complexity bounds for large- and small-update methods are derived. In addition to the theoretical study, we showcase the practical performance of each algorithm by comparing it with other existing IPAs based on KFs.

In Chapter 2, we extend our exploration of primal-dual feasible IPMs based on KFs, as discussed in Chapter 1, into the realm of SDO. Hence, we deal with feasible primal-dual IPMs based on KFs for solving SDO problems. We first give a concise summary on the basic of primal-dual IPMs for SDO. Then, we present the main steps to obtain the complexity of primal-dual IPAs based on the same class of KFs studied for LO in Chapter 1. As an application, we study an IPM based on a new twice parametric KF which is a combination of the prototype SR KF and the hyperbolic KF introduced in [83] and [41] respectively. We end this chapter by presenting some numerical experiments to showcase the practical performance of the IPA based on the twice parametrized KF in solving SDO problems.

In Chapter 3, we study a full-Newton step IIPA for solving LO problems based on a new hyperbolic KF. Unlike the feasible IPAs previously studied in Chapter 1, this algorithm doesn't require a feasible starting point. In addition, the algorithm avoids a big-M or a two-phase approach. Each main iteration of this algorithm involves a sequence of actions, including a feasibility step, and a series of centrality steps.

The feasibility search directions are computed using the hyperbolic KF; however, the centering search directions are obtained using the classical KF. Furthermore, under general conditions we guarantee that our algorithm will converge to an optimal solution. Using some mild properties, the complexity analysis for the primal-dual IIPM based on the corresponding proximity function indicates that the iteration bound of the algorithm matches the currently best iteration bound for IIPMs. We consolidate these theoretical results by performing some numerical experiments. These experiments were split into two segments. In the initial part, we conducted a comparison between the IIPM based on the considered KF and other established IIPMs using a set of problems from the Netlib repository. In the subsequent part, we evaluated our algorithm against the well-known SeDuMi solver.

Finally, we end this thesis with some conclusions and recommendations for potential avenues of future works.

Chapter 1

Feasible primal-dual IPMs based on kernel functions for linear optimization

In this chapter, we deal with feasible primal-dual interior-point methods (IPMs) based on kernel functions (KFs) for linear optimization (LO). We first give a summary on primal-dual IPMs. In Section 1.2, we present the main steps to obtain the complexity of primal-dual interior-point algorithms (IPAs) based on a specific class of KFs. The presented procedure is then applied on three new hyperbolic KFs.

1.1 Preliminaries

In this section, we outline some concepts and basic tools required in feasible IPMs based on KFs such as central path, search direction, proximity measure, step size, proximity function, complexity analysis, etc.

1.1.1 Description of the generic primal-dual interior-point algorithm based on kernel functions

In this subsection, we only showcase the main steps of feasible primal-dual IPMs based on KFs. At the end, we provide a formal description of the primal-dual IPA based on KFs. For more details and informations on the theory of feasible primal-dual IPMs, we refer the readers to the monograph of Peng et al. [84] as well as the references provided therein.

Recall that in this chapter, we are concerned about solving LO problems. To be more specific, we deal with a LO problem (P) which is formulated in the following standard form: Given the vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$, the matrix $A \in \mathbb{R}^{m\times n}$, find a vector $x \in \mathbb{R}^n$, such that

$$(P)\begin{cases} \min c^T x \\ Ax = b, \\ x \geq 0, \end{cases}$$

x is called the vector of variables. The set of feasible solutions for (P) is defined as follows

$$\mathcal{F}_P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}.$$

$\mathcal{F}_P$ is called the primal feasible region.

The corresponding dual problem of (P) is given by

$$(D)\begin{cases}\max b^T y\\ A^T y + s = c,\\ s \geq 0,\end{cases}$$

where $y \in \mathbb{R}^m$ is called the vector of variables and $s \in \mathbb{R}^n$ is called the vector of dual slack variables. The set of feasible solutions for (D) is given by

$$\mathcal{F}_D = \left\{(y,s) \in \mathbb{R}^m \times \mathbb{R}^n : A^T y + s = c,\ s \geq 0\right\}.$$

$\mathcal{F}_D$ is called the dual feasible region. The relative interiors of $\mathcal{F}_P$ and $\mathcal{F}_D$ are defined respectively as follows

$$\mathring{\mathcal{F}}_P = \{x \in \mathbb{R}^n : Ax = b,\ x > 0\},$$

$$\mathring{\mathcal{F}}_D = \left\{(y,s) \in \mathbb{R}^m \times \mathbb{R}^n : A^T y + s = c,\ s > 0\right\}.$$

Throughout this chapter, we assume that the matrix A has full rank i.e., $\text{rank}(A) = m < n$. This assumption implies that for a given dual feasible vector s, the vector y is uniquely defined. This means that we can determine a feasible solution of (D) only by s.

The connections and relationships between the primal problem (P) and the dual problem (D) have been studied by many authors. We refer the readers to the duality theory presented in [92]. We only recall the following main results. The first result is the Duality Theorem (due to J. von Neumann, 1947, [79]), and the second result will be referred to as the Goldman–Tucker Theorem (Goldman and Tucker, 1956, [36]).

Theorem 1.1.1. *([79, Duality Theorem]) If (P) and (D) are feasible then both problems have optimal solutions. Then, if $x \in \mathcal{F}_P$ and $(y,s) \in \mathcal{F}_D$, these are optimal solutions if and only if $x^T s = 0$. Otherwise neither of the two problems has optimal solutions: either both (P) and (D) are infeasible or one of the two problems is infeasible and the other one is unbounded.*

Theorem 1.1.2. *([36, Goldman-Tucker Theorem]) If (P) and (D) are feasible then there exists a strictly complementary pair of optimal solutions, that is an optimal solution pair $(\overline{x}, \overline{s})$ satisfying*

$$\overline{x} + \overline{s} > 0.$$

Central path

In what follows, we suppose that problems (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists (x^0, y^0, s^0) such that

$$Ax^0 = b,\ \ x^0 > 0,\ \ A^T y^0 + s^0 = c,\ \ s^0 > 0,$$

which is equivalent to

$$\mathring{\mathcal{F}}_P \times \mathring{\mathcal{F}}_D \neq \emptyset.$$

Under the IPC, it follows from Theorem 1.1.1 that a primal-dual feasible triple (x,y,s) is optimal if and only if $x^T s = 0$. This is called the complementarity condition for (P) and (D). Because the vectors x and s are nonnegative, the complementarity condition is equivalent to $xs = 0$. Hence, any primal-dual optimal solution $(\bar{x}, \bar{y}, \bar{s})$ of

(P) and (D) satisfies the following conditions:

$$\begin{cases} Ax = b, x \geq 0, \\ A^T y + s = c, s \geq 0, \\ xs = 0. \end{cases} \tag{1.1}$$

System (1.1) is called The Karuch-Kuhn-Tucker (KKT) optimality conditions for LO. The first and second equations represent primal and dual feasibility. Whereas, the last equation signify the complementarity condition for (P) and (D).

The theory of IPMs suggests that the third equation in (1.1) has to be perturbed. Hence, replacing the last equation in (1.1) by the parameterized equation

$$xs = \mu e, \mu > 0,$$

leads to the following system

$$\begin{cases} Ax = b, x > 0, \\ A^T y + s = c, s > 0, \\ xs = \mu e. \end{cases} \tag{1.2}$$

Theorem 1.1.3. *([84, Theorem 1.2.4]) If the IPC holds, then for each $\mu > 0$, the parameterized system (1.2) has a unique solution.*

Observing the last equation in system (1.2), any solution of this system verifies that

$$x > 0 \text{ and } s > 0.$$

Hence, a solution exists if and only if the IPC is verified. Since the IPC holds, the previous theorem indicates that for each $\mu > 0$, system (1.2), has a unique solution denoted by (x_μ, y_μ, s_μ). The vector x_μ is called the μ- center of (P) and (y_μ, s_μ) the μ-center of (D). The set of unique solutions $\{(x_\mu, y_\mu, s_\mu) : \mu > 0\}$ creates a homotopy trajectory in the interior of the feasible region labeled as the central path of problems (P) and (D). The behaviour of the central path as μ approaches zero has been a prominent focus. McLinden delved into this aspect in [72], particularly exploring the limiting behaviour of the path for monotone complementarity problems. After that, Megiddo [73] established that as μ tends to zero, the central path of (P) and (D) converges to a primal-dual optimal solution.

However, the last equation in system (1.2) makes it difficult to obtain the μ centers since it is nonlinear. A remedy to this issue is to use a numerical iterative procedure based on the popular Newton-Raphson's method. The details of this procedure will be shown in what follows.

Search directions

Introducing a parameter $\tau > 0$ as a threshold value and fixing $\mu > 0$, let us defined the τ-neighbourhood $\mathcal{N}(\tau, \mu)$ as follows

$$\mathcal{N}(\tau, \mu) = \{(x, y, s) \in \mathring{\mathcal{F}}_P \times \mathring{\mathcal{F}}_D,\ \Phi(x, s, \mu) \leq \tau\},$$

with τ the radius of the neighbourhood and Φ a so-called proximity measure. Φ is used to measure the distance from the point (x, s) to (x_μ, s_μ) and will be defined later in terms of a KF.

Starting with a point in the neighborhood $\mathcal{N}(\tau, \mu)$, we solve the following system of linear equations

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = \mu e - xs, \end{cases} \tag{1.3}$$

to obtain the search direction $(\Delta x, \Delta y, \Delta s)$. Since A has full row rank, system (1.3) uniquely defines $(\Delta x, \Delta y, \Delta s)$ for any $x > 0$ and $s > 0$.

Therefore, by taking a step along the search direction, one constructs a new iterate point

$$x_+ := x + \alpha\Delta x, \quad y_+ := y + \alpha\Delta y, \quad s_+ := s + \alpha\Delta s. \tag{1.4}$$

The step size $\alpha \in]0, 1]$ has to be chosen approximately so that the new iterates satisfies the strict positivity condition i.e.,

$$(x_+, s_+) > 0.$$

The concept of IPMs is visually represented in the figure provided below.

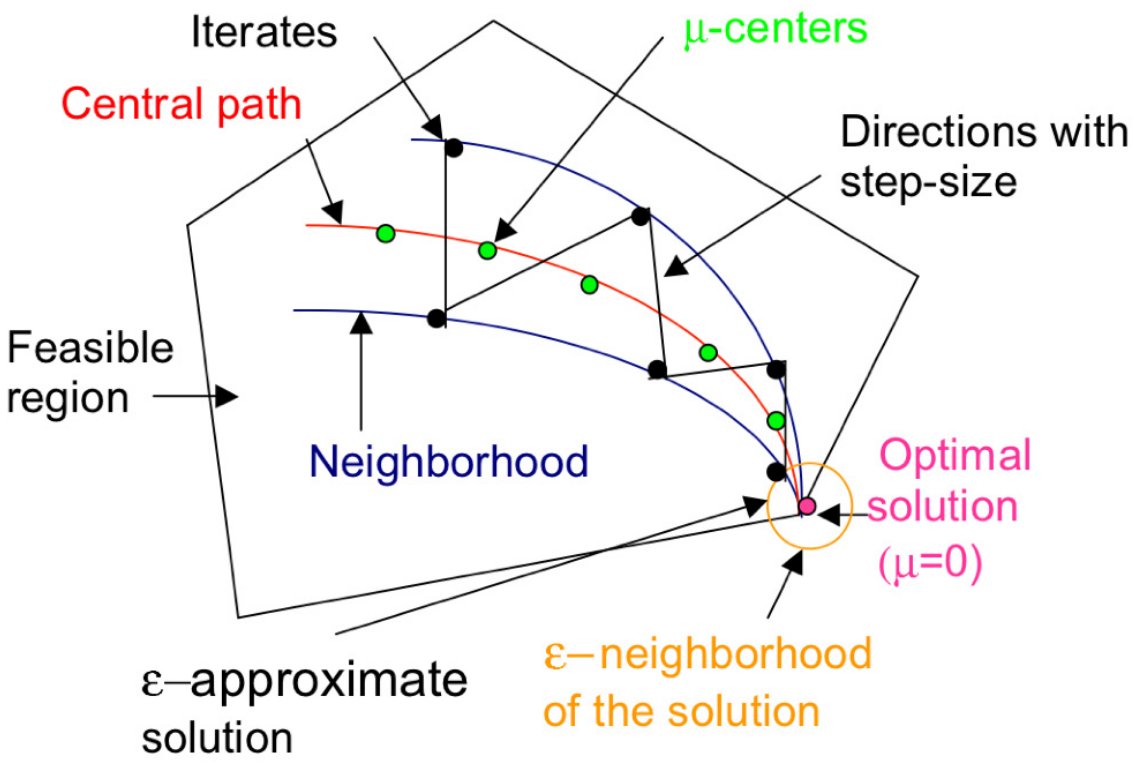

FIGURE 1.1: Graphical Interpretation of IPMs

Now, we define the scaled vector v and the scaled search directions d_x and d_s as follows

$$v = \sqrt{\frac{xs}{\mu}}, \quad d_x = \frac{v\Delta x}{x}, \quad d_s = \frac{v\Delta s}{s}. \tag{1.5}$$

Utilizing these notations, system (1.3) is restated in the subsequent way

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T\Delta y + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{1.6}$$

where $\overline{A} = \frac{1}{\mu}AV^{-1}X$, $V = \text{diag}(v)$, $X = \text{diag}(x)$.

A significant observation is that the right-hand side in the last equation of (1.6) is equal to minus gradient of the following function

$$\Psi_c(v) = \sum_{i=1}^{n} \psi_c(v_i), \tag{1.7}$$

with

$$\psi_c(t) = \frac{t^2 - 1}{2} - \log t.$$

Thus the third equation in (1.6), often called the scaled centering equation, can be rewritten as follows

$$v^{-1} - v = -\nabla\Psi_c(v).$$

This indicates that Ψ_c essentially determines the search direction. In addition, it's easy to verify that

$$\nabla^2\Psi_c(v) = \text{diag}(e + v^{-2}),$$

and that the matrix $\text{diag}(e + v^{-2})$ is positive definite. This implies that the Hessian $\nabla^2\Psi_c(v)$ is positive definite which indicates that Ψ_c is strictly convex. Moreover,

$$\nabla\Psi_c(e) = 0.$$

It follows that $\Psi_c(v)$ attains its minimal value at $v = e$, with $\Psi_c(e) = 0$. This means that $\Psi_c(v)$ is nonnegative everywhere and vanishes if and only if $v = e$, that is, if and only if $x = x_\mu$ and $s = s_\mu$. Therefore, the μ- center (x_μ, s_μ) can be identified as the minimizer of the function Ψ_c. For this reason, Ψ_c serves mainly as a "proximity" measure of closeness for (x, s) with respect to the μ-center.

These observations concerning Ψ_c led to the creation of the concept of primal-dual IPMs based on KFs. The basic idea in these methods is to replace ψ_c by any strictly convex function $\psi :]0, +\infty[\to [0, +\infty[$ which is minimal at $t = 1$ with $\psi(1) = 0$. The corresponding proximity function Ψ is then obtained by replacing ψ_c by ψ in (1.7). This explains the reason for calling ψ the KF of the barrier function Ψ. Besides, Ψ is minimal at $v = e$ and $\Psi(e) = 0$ i.e.,

$$\Psi(v) = 0 \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow v = e.$$

Thus, Ψ still serves as a proximity measure for closeness with respect to the μ-center (x_μ, s_μ) and the inequality

$$\Psi(v) \leq \tau,$$

defines a $\tau-$neighbourhood of the μ-center. Hence we can define $\Phi(x, s; \mu)$ as follows

$$\Phi(x, s; \mu) = \Psi(v).$$

In the sequel, we will use the norm-based proximity measure $\sigma(v)$ defined by

$$\sigma(v) := \sigma(x, s; \mu) = \frac{1}{2}\|d_x + d_s\| = \frac{1}{2}\|\nabla\Psi(v)\|. \tag{1.8}$$

One can easily verify that $\sigma(v) = 0$ if and only if $v = e$. This means that σ vanishes only at the μ-center. In other words,

$$\sigma(v) = 0 \Leftrightarrow v = e \Leftrightarrow xs = \mu e.$$

In this chapter, we will consider three alternatives for ψ_c:

1. the exponential hyperbolic KF [39]

$$\psi(t) = \frac{t^2-1}{2} + \sinh^2(1)\left(e^{\coth(t)-\coth(1)} - 1\right), \ \forall t > 0.$$

2. the parameterized hyperbolic logarithmic KF [42] which can be considered as a generalization of the KF proposed in [100]

$$\psi(t) = \frac{t^2-1}{2} + \frac{\sinh^2(1)}{\sinh^2(1) + p\coth^{p-1}(1)}\Big(\coth^p(t) - \log t - \coth^p(1)\Big), \ \forall t > 0,$$

with $p \geq 1$.

3. the hyperbolic KF [41] which is a generalization, up to a multiplicative constant, of the KF introduced in [98]

$$\psi(t) = \frac{t^2-1}{2} + \frac{\sinh^2(1)}{p\coth^{p-1}(1)}\coth^p(t) - \frac{\sinh^2(1)}{p}\coth(1), \ \forall t > 0,$$

with $p \geq 2$.

Coming back to system (1.6), we can convert it to

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T\Delta y + d_s = 0, \\ d_x + d_s = -\nabla\Psi(v). \end{cases} \tag{1.9}$$

Since A has full row rank, system (1.9) has a unique solution. Furthermore, the vectors d_x and d_s are orthogonal and thus

$$d_x = d_s = 0 \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow v = e \Leftrightarrow \Psi(v) = 0 \Leftrightarrow x = x_\mu \text{ and } s = s_\mu.$$

Solving system (1.9) and using notations (1.5), we get the new search direction $(\Delta x, \Delta y, \Delta s)$.

Another way to obtain these search direction is to solve the following system

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = -\mu v\nabla\Psi(v). \end{cases}$$

This system can be condensed to the following form

$$M\Delta y = r,$$

where

$$\begin{aligned} M &= AS^{-1}XA^T \\ r &= \mu AS^{-1}v\nabla\Psi(v). \end{aligned}$$

Once Δy is found, Δs and Δx are determined through back substitutions

$$\begin{aligned}\Delta s &= -A^T \Delta y \\ \Delta x &= -s^{-1}\left(x\Delta s - \mu v \nabla \Psi(v)\right).\end{aligned}$$

We end this section by providing a brief description of the algorithm corresponding to the primal-dual IPM based on KFs summarized in Algorithm 1. Given a strictly feasible point (x^0, y^0, s^0) situated in a τ-neighbourhood of the given μ-center, we reduce μ to $\mu_+ := (1-\theta)\mu$ for some fixed $0 < \theta < 1$, and then solve the system (1.9) to obtain the search direction $(d_x, \Delta y, d_s)$; Then, we use notations (1.5) to obtain $(\Delta x, \Delta s)$. The new iterates are then computed according to (1.4). The positivity condition of a new iterate is ensured by choosing an appropriate step size α. This procedure is repeated until we find a new iterate (x_+, y_+, s_+) that again belongs to the τ-neighbourhood of the current μ-center, that is, until $\Psi(v) \leq \tau$. Then, we update the parameter μ to μ_+ and we let $(x, y, s) = (x_+, y_+, s_+)$. This procedure is repeated until we find an iterate (x_+, y_+, s_+) such that $x_+^T s_+ < \epsilon$. In this case, an ϵ-approximate optimal solution of problems (P) and (D) is found.

Remark 1.1.4.

- *According to [92], we call the step where the present iterate is in a certain neighbourhood of the current μ-center an outer iteration, and the procedure to get a primal-dual pair (x, y, s) in the neighborhood of this μ-center an inner iteration. In the algorithm, we use the proximity $\Phi(x, y, s)$ to control the iterates.*

- *The choice of the parameter θ is an important ingredient of IPMs. Generally, when θ is a constant independent of n, for example $\theta = \frac{1}{2}$, the algorithm is called a large-update (or long-step) method. On the other hand, if θ depends on the problem dimension n, for example $\theta = \frac{1}{2\sqrt{n}}$, then the algorithm is referred to as a small-update (or short-step) method. It's noteworthy that there exists a gap between the practical efficiency and the theoretical worst-case complexity within these two categories of IPMs in the literature. Specifically, large-update methods tend to exhibit higher practical efficiency, albeit with a more unfavorable theoretical complexity bound when compared to small-update methods, which possess a complexity of $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$.*

Algorithm 1 : Generic Feasible Primal-Dual Interior-Point Algorithm for Linear Optimization

Input data

a threshold parameter $\tau \geq 1$;

an accuracy parameter $\epsilon > 0$;

a fixed barrier update parameter $\theta \in]0,1[$;

(x^0, y^0, s^0) satisfy the IPC and $\mu^0 = 1$ such that

$\Phi\left(x^0, s^0; \mu^0\right) := \Psi(v^0) \leq \tau$.

begin

$x := x^0;\ y := y^0;\ s := s^0;$

$\mu := x^0 s^0;$

while $x^T s \geq \epsilon$ **do**

begin (outer iteration)

$\mu := (1-\theta)\mu;$

$v := \sqrt{\frac{xs}{\mu}};$

while $\Phi(x, s; \mu) := \Psi(v) > \tau$ **do**

begin (inner iteration)

Solve system (1.9) to get $(d_x, \Delta y, d_s)$;

Use (1.5) to obtain $(\Delta x, \Delta s)$;

Choose a suitable step size α;

$x := x + \alpha \Delta x;\ y := y + \alpha \Delta y;\ s := s + \alpha \Delta s;$

$v := \sqrt{\frac{xs}{\mu}};$

end while (inner iteration)

end while (outer iteration)

end

1.1.2 Definitions and technical lemmas

Let $\mathcal{C}^2(\mathbb{R}_{++}, \mathbb{R}_+)$ be the set of twice differentiable functions defined from $\mathbb{R}_{++}$ to $\mathbb{R}_+$.

Definition 1.1.5. *Let $\psi \in \mathcal{C}^2(\mathbb{R}_{++}, \mathbb{R}_+)$. ψ is a KF if it satisfies the following conditions*

- ***KF1*** $\psi(1) = \psi'(1) = 0$.
- ***KF2*** $\psi''(t) > 0,\ \forall t \in \mathbb{R}_{++}$ *(strict convexity).*
- ***KF3*** ψ *is a coercive KF ,i.e.,*

$$\lim_{t \to 0^+} \psi(t) = \lim_{t \to +\infty} \psi(t) = +\infty.$$

The corresponding proximity function (barrier function) is defined as follows

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i),\ \forall v > 0. \tag{1.10}$$

Remark 1.1.6. *If*

$$\lim_{t \to 0^+} \psi(t) = c < +\infty,$$

then, ψ has a finite barrier term. An example is the following KF introduced in [9]

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{1}{b}(e^{b(1-t)} - 1),\ \forall t > 0 \text{ with b} > 0.$$

Proposition 1.1.7. *Let ψ be as defined in Definition 1.1.5, then*

(i) ψ *is nonnegative and vanishes at its global minimal point $t = 1$.*

(ii) $\psi'(t) \geq 0$ *if* $t \geq 1$ *and* $\psi'(t) < 0$ *if* $t < 1$.

(iii) ψ *is completely defined by its second derivative, i.e.,*

$$\psi(t) = \int_1^t \int_1^{\xi} \psi''(z)\, dz\, d\xi\ \forall t > 0.$$

(iv) the derivative function ψ' is monotonically increasing on $\mathbb{R}_{++}$.

Remark 1.1.8. *Let ψ be a KF. Then, the restriction functions*

$$\psi : [1, +\infty[\to [0, +\infty[,$$

and

$$\psi' :]0, 1] \to]-\infty, 0],$$

are both monotonically increasing.

This remark allows us to introduce the following definition

Definition 1.1.9. *Let ϱ and ρ be the inverse functions:*

- $\varrho : [0, +\infty[\longrightarrow [1, +\infty[$ *the monotonically increasing inverse function of the restriction of ψ to $[1, +\infty[$.*

- $\rho : [0, +\infty[\longrightarrow]0, 1]$ *the monotonically decreasing inverse function of the restriction of* $-\frac{1}{2}\psi'$ *to* $]0, 1]$.

The inverse functions ϱ and ρ, thus defined, play a centric role in the complexity analysis of feasible primal-dual IPAs based on KFs.

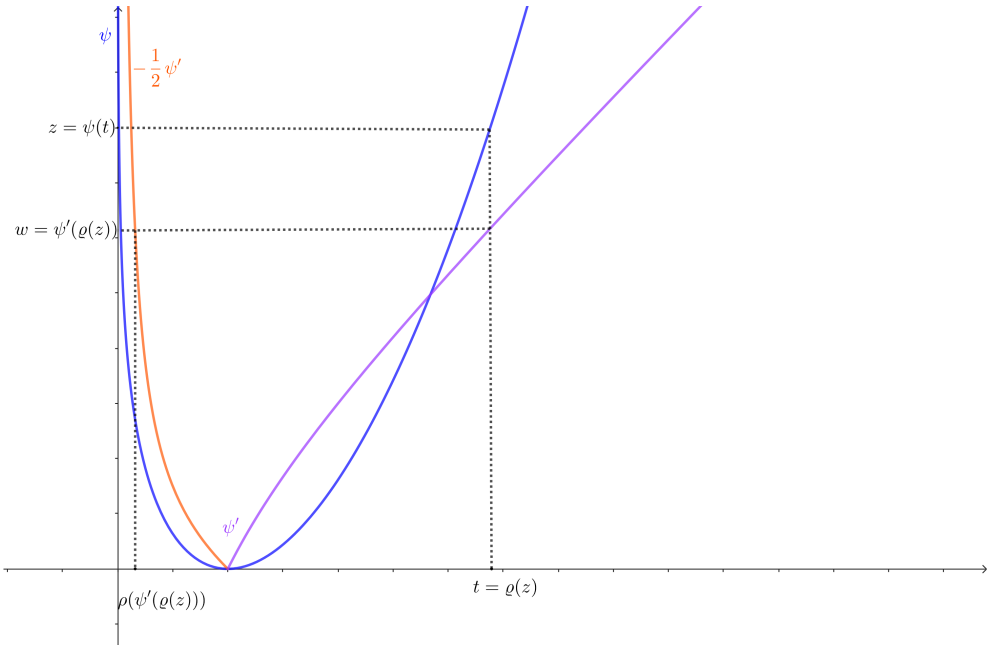

FIGURE 1.2: Graphical Interpretation of the inverse functions ϱ and ρ

1.2 Analysis of the primal-dual interior-point algorithm for a class of kernel functions

1.2.1 A specific class of kernel functions

In this section, we are inspired by the work of Bai et al. [8]. In particular, we study a class of KFs which satisfies all the eligibility conditions introduced in [8] except the following condition

$$2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0, \forall t < 1.$$

Let ψ be a KF. In this section, we require that ψ is three times continuous differentiable and satisfies hypothesis

- **H1** For all $t \in \mathbb{R}_{++}$, $\psi'''(t) < 0$.
- **H2** For all $t > 1$, $t\psi''(t) - \psi'(t) > 0$, ($\psi$ is sqrt-convex).
- **H3** For all $t < 1$, $t\psi''(t) + \psi'(t) > 0$, ($\psi$ is e-convex).

We also assume, as in [8], that the KF is written in the following form

$$\psi(t) = \frac{t^2 - 1}{2} + \psi_b(t), \ \forall t > 0,$$

with $\psi_b'(t) < 0$ and $\psi_b''(t) \geq 0$ for all $t > 0$.
ψ_b is called the barrier term and $\frac{t^2-1}{2}$ is called the growth term. The growth term

dominates the behavior of $\psi(t)$ when t goes to infinity, whereas the barrier term dominates its behavior when t approaches zero.

These conditions will facilitate the complexity analysis of the IPM based on the KF ψ.

Remark 1.2.1. *We call the property described in **H3** the exponential convexity, or shortly e-convexity (Lemma A.0.9). This property is not satisfied for a non-coercive KF. An example for such KF is the function introduced in [9].*

Proposition 1.2.2. *One has the following properties*

(i) $\psi^{''}(t) \geq 1, \ \forall t > 0.$

(ii) $\psi_b(1) = 0,$ *and* $\psi_b^{'}(1) = -1.$

(iii) $\psi_b^{'''}(t) < 0, \ \forall t > 0.$

Lemma 1.2.3. *([8, Lemma 2.4]) We have*

$$\psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t) > 0, \ \forall t > 1, \ \forall \beta > 1.$$

Proof. Fixing $t > 1$, we define the following function

$$g_t(\beta) = \psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t), \ \forall \beta \geq 1.$$

Differentiating g_t with respect to β, we get for all $\beta \geq 1$

$$\begin{aligned} g_t^{'}(\beta) &= t\psi^{''}(t)\psi^{''}(\beta t) - \beta t\psi^{'}(t)\psi^{'''}(\beta t) - \psi^{'}(t)\psi^{''}(\beta t), \\ &= \psi^{''}(\beta t)\left(t\psi^{''}(t) - \psi^{'}(t)\right) - \beta t\psi^{'}(t)\psi^{'''}(\beta t) > 0, \end{aligned}$$

since ψ is a KF and due to **H1** and **H2**. It follows that the function g_t is strictly increasing on $[1, +\infty[$. Hence

$$g_t(\beta) > g_t(1) = 0, \ \forall \beta > 1.$$

The desired inequality is then obtained. □

Lemma 1.2.4. *([8, Lemma 2.5]) One has*

$$\psi(t) < \frac{1}{2}(t-1)\psi'(t) \text{ and } \psi^{'}(t) > (t-1)\psi^{''}(t), \ \forall t < 1, \tag{1.11}$$

while

$$\psi(t) > \frac{1}{2}(t-1)\psi'(t) \text{ and } \psi^{'}(t) > (t-1)\psi^{''}(t), \ \forall t > 1. \tag{1.12}$$

Proof. Let's define the function l as follows

$$l(t) = 2\psi(t) - (t-1)\psi^{'}(t), \ \forall t > 0.$$

Deriving l twice, we get

$$\begin{aligned} l'(t) &= \psi^{'}(t) - (t-1)\psi^{''}(t), \\ l^{''}(t) &= -(t-1)\psi^{'''}(t). \end{aligned}$$

Obviously, by **KF1**

$$l(1) = l^{'}(1) = 0.$$

Furthermore, using **H1**, we obtain

$$l^{''}(t) > 0, \ \forall t > 1,$$
$$l^{''}(t) < 0, \ \forall t < 1.$$

This implies that the function $t \mapsto l^{'}(t)$ in monotonically increasing for $t > 1$, and monotonically decreasing for $t < 1$. Therefore,

$$l^{'}(t) > l^{'}(1) = 0, \ \forall t > 1,$$
$$l^{'}(t) > l^{'}(1) = 0, \ \forall t < 1.$$

This gives the second parts of (1.11) and (1.12). In addition, it implies that the function $t \mapsto l(t)$ is increasing on $\mathbb{R}_{++}$. Consequently,

$$l(t) > l(1) = 0, \ \forall t > 1,$$
$$l(t) < l(1) = 0, \ \forall t < 1.$$

This concludes the proof. □

Lemma 1.2.5. *([8, Lemma 2.6]) We have*

$$\frac{1}{2}\psi^{''}(1)(t-1)^2 < \psi(t) < \frac{1}{2}\psi^{''}(t)(t-1)^2, \ \forall t < 1,$$

and

$$\frac{1}{2}\psi^{''}(t)(t-1)^2 < \psi(t) < \frac{1}{2}\psi^{''}(1)(t-1)^2, \ \forall t > 1.$$

Proof. Since ψ is thrice continuous differentiable and $\psi(1) = \psi^{'}(1) = 0$, the Taylor expansion of ψ implies that

$$\psi(t) = \frac{1}{2}\psi^{''}(1)(t-1)^2 + \frac{1}{6}\psi^{'''}(\xi)(\xi-1)^3,$$

with $t < \xi < 1$ for $t < 1$ and $1 < \xi < t$ for $t > 1$. Using **H1**, we get

$$\psi(t) > \frac{1}{2}\psi^{''}(1)(t-1)^2, \forall t < 1,$$
$$\psi(t) < \frac{1}{2}\psi^{''}(1)(t-1)^2, \forall t > 1.$$

In addition, using the previous lemma, the other remaining inequalities are achieved. □

Lemma 1.2.6. *([8, Lemma 3.1]) Let $\beta, t_1, t_2 \in \mathbb{R}_{++}$ such that $\beta \geq 1$ and $t_1 \leq 1 \leq t_2$ with $\psi(t_1) = \psi(t_2)$. Then,*

$$\psi(\beta t_2) \leq \psi(\beta t_1).$$

The equality holds if and only if $t_1 = t_2 = 1$ or $\beta = 1$.

Proof. We define the function h as follows

$$h(\beta) = \psi(\beta t_1) - \psi(\beta t_2), \ \forall \beta \geq 1.$$

Differentiating h with respect to β, we get

$$h'(\beta) = t_1\psi'(\beta t_1) - t_2\psi'(\beta t_2).$$

Obviously $h(1) = 0$. Using **KF2**, it follows that

$$\psi'(\beta t_1) \leq \psi'(\beta t_2) \text{ and } \psi'(\beta t_2) \geq \psi'(1) = 0.$$

Therefore,

$$h'(\beta) \leq (t_1 - t_2)\psi'(\beta t_2) \leq 0,$$

i.e., h is decreasing on $[1, +\infty[$. This proves the inequality since $h(1) = 0$.

For the second part of the lemma, the equality is obviously satisfied for $\beta = 1$. On the other hand, for $\beta > 1$ and $h(\beta) = 0$, applying the mean value theorem on h yields that there exists $\xi \in]1, \beta[$ such that

$$h'(\xi) = \frac{h(\beta) - h(1)}{\beta - 1}.$$

Or $h(\beta) = h(1) = 0$. This implies that

$$h'(\xi) = 0,$$

and

$$t_1\psi'(\xi t_1) = t_2\psi'(\xi t_2).$$

Utilizing the strict monotonicity of ψ' and the fact that $t_1 \leq 1 \leq t_2$, we get $\xi t_1 = \xi t_2$ and therefore $t_1 = t_2 = 1$. □

Lemma 1.2.7. *([8, Lemma 6.2]) Let ϱ be the function defined by Definition 1.1.9. Then for all $w \in [0, +\infty[$, one has*

$$\sqrt{1 + 2w} \leq \varrho(w) \leq 1 + \sqrt{2w}.$$

Proof. Let $w = \psi(t)$, for $t \in [1, +\infty[$. We start with the right hand side inequality. Using (iii) of Proposition 1.1.7 and (i) of Proposition 1.2.2, we have

$$w = \psi(t) = \int_1^t \int_1^\zeta \psi''(\xi)d\xi d\zeta \geq \int_1^t \int_1^\zeta d\xi d\zeta = \frac{1}{2}(t-1)^2.$$

Therefore, substituting by $t = \varrho(w)$ we get the first inequality.

For the second inequality, using the fact that ψ_b is decreasing on $]0, +\infty[$ and the second item of Proposition 1.2.2 we obtain

$$w = \psi(t) \leq \frac{t^2 - 1}{2}, \ \forall t \geq 1. \tag{1.13}$$

Hence putting $t = \varrho(w)$, the desired inequality is achieved. □

1.2.2 Properties of proximity functions and proximity measures

In this section, we provide upper bounds of the proximity function Ψ defined in (1.10) and the proximity measure σ defined in (1.8) after the μ- update. These bounds will intervene in the complexity analysis. In accordance with Algorithm 1, at the beginning of an outer iteration, we have $\Psi(v) \leq \tau$ before the update of μ with the factor $(1 - \theta)$. After updating μ in an outer iteration, the value of $\Psi(v)$ increases

since v is divided by the factor $\sqrt{1-\theta}$. Then during the inner iteration, the value of $\Psi(v)$ decreases until it passes the threshold τ.

Theorem 1.2.8. *([8, Theorem 3.2]) For any $v \in \mathbb{R}^n_{++}$ and $\beta \geq 1$, we have*

$$\Psi(\beta v) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right).$$

Proof. For the first case $\beta = 1$, the equality is immediate since ϱ is the inverse of the restriction of ψ to $[1, +\infty[$.

As for the proof of the case $\beta > 1$, we first consider the constrained maximization problem (MP) defined for any $z \in \mathbb{R}_+$ as follows

$$(MP)\begin{cases} \max\limits_{v} \Psi(\beta v) \\ \Psi(v) = z. \end{cases}$$

Applying the Lagrange multiplier theorem [69] implies that solving (MP) is equivalent to solving the following problem

$$\beta\psi'(\beta v_i) = \lambda\psi'(v_i),\ i = 1\ldots n, \tag{1.14}$$

with $\lambda \in \mathbb{R}$ called the Lagrange multiplier. Using **KF1** and the fact that ψ' is strictly increasing, we have for all $\beta > 1$,

$$\psi'(\beta) > \psi'(1) = 0,$$

which implies that

$$v_i \neq 1,\ i = 1\ldots n.$$

In addition, we assume that $v_i > 1$ for all i. In fact, let $(z_i)_{1\leq i\leq n}$ be a linear decomposition of z, i.e., $z = \sum\limits_{i=1}^{n} z_i$. Then, the equation $\Psi(v) = z$ can be reduced to the system of equations

$$\psi(v_i) = z_i,\ i = 1\ldots n.$$

There are two solutions for this equation: $v_i = v_i^{(1)} < 1$ and $v_i = v_i^{(2)} > 1$. Using Lemma 1.2.6 for $t_1 = v_i^{(1)}$ and $t_2 = v_i^{(2)}$, we have

$$\psi(\beta v_i^{(1)}) \leq \psi(\beta v_i^{(2)}),$$

and since we are maximizing $\Psi(\beta v)$, we may assume without lost of generality that

$$v_i = v_i^{(2)} > 1,\ i = 1\ldots n.$$

A direct consequence from this assumption and equation (1.14) is

$$\psi'(v_i) > 0 \text{ and } \psi'(\beta v_i) > 0,$$

hence $\lambda > 0$. Let us define for all $\beta > 1$ a new univariate function h as follows

$$h(t) = \frac{\psi'(t)}{\psi'(\beta t)},\ \forall t \geq 1.$$

Differentiating h, we get

$$h^{'}(t) = \frac{\psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t)}{(\psi^{'}(\beta t))^2} > 0, \ \forall t > 1.$$

The strict positivity of $h^{'}$ is due to Lemma 1.2.3. Thus, h is strictly monotonically increasing. Since we have from equation (1.14)

$$h(v_i) = \frac{\beta}{\lambda}, \ i = 1 \ldots n,$$

it follows that v_i are all mutually equal. Let $v_i = t > 1$ for all $i = 1 \ldots n$. From the constraint of problem (MP) and the definition of Ψ, we may conclude that

$$n\psi(t) = z.$$

As a consequence, Ψ attains its maximal value at v with $v_i = t = \varrho(\frac{z}{n})$. The maximal value that $\Psi(\beta v)$ can reach is then given by

$$\Psi(\beta te) = n\psi(\beta t) = n\psi\left(\beta\varrho\left(\frac{z}{n}\right)\right) = n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right).$$

□

Corollary 1.2.9. *Let $v_+ = \frac{v}{\sqrt{1-\theta}}$ with $0 \leq \theta < 1$. If we assume that $\Psi(v) \leq \tau$ just before the μ-update to $(1-\theta)\mu$, we have the following upper bound*

$$\Psi(v_+) \leq n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right).$$

Proof. Applying Theorem 1.2.8 with $\beta = \frac{1}{\sqrt{1-\theta}}$, and using the increasing of the functions ϱ and ψ on $[0, +\infty[$ and $[1, +\infty[$ respectively, we get the desired inequality. □

As a consequence, we have the following lemmas.

Lemma 1.2.10. *([30]) Let $v_+ = \frac{v}{\sqrt{1-\theta}}$ and $0 \leq \theta < 1$. If we assume that $\Psi(v) \leq \tau$ just before the μ-update to $(1-\theta)\mu$, we have the following upper bound*

$$\Psi(v_+) \leq \frac{\theta n + 2\tau + 2\sqrt{2\tau n}}{2(1-\theta)} := \Psi_0.$$

Ψ_0 is an upper bound for $\Psi(v_+)$ during the process of the algorithm.

Proof. From (1.13), we have

$$\psi(t) \leq \frac{t^2 - 1}{2}, \ \forall t \geq 1.$$

Hence putting $t = \frac{\varrho(\frac{\tau}{n})}{\sqrt{1-\theta}} \geq 1$ (from the definition of ϱ), and using Corollary 1.2.9 we get

$$\Psi(v_+) \leq \frac{n}{2}\left(\frac{\varrho\left(\frac{\tau}{n}\right)^2}{1-\theta} - 1\right).$$

Thanks to Lemma 1.2.7, we obtain

$$\begin{aligned}\Psi(v_+) &\leq \frac{n}{2}\left(\frac{\left(1+\sqrt{\frac{2\tau}{n}}\right)^2}{1-\theta} - 1\right)\\ &= \frac{\theta n + 2\tau + 2\sqrt{2\tau n}}{2(1-\theta)}.\end{aligned}$$

□

Lemma 1.2.11. *([8, Lemma 6.3]) Let $v_+ = \frac{v}{\sqrt{1-\theta}}$ with $0 \leq \theta < 1$. If we assume that $\Psi(v) \leq \tau$ just before the $\mu-$update to $(1-\theta)\mu$, we have the following upper bound*

$$\Psi(v_+) \leq \frac{\psi''(1)}{2}\frac{\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2}{1-\theta}.$$

Proof. Recall that from Corollary 1.2.9, $\Psi(v_+)$ is bounded by

$$\Psi(v_+) \leq n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right).$$

Hence, using Lemma 1.2.7 for $w = \frac{\tau}{n}$ we have

$$\Psi(v_+) \leq n\psi\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right).$$

But, from Lemma 1.2.5, we have

$$\psi(t) \leq \frac{1}{2}\psi''(1)(t-1)^2, \ \forall t \geq 1.$$

Using this inequality for $t = \dfrac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}} \geq 1$, we get

$$\begin{aligned}\Psi\left(v_{+}\right) &\leq \frac{n\psi^{''}(1)}{2}\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}-1\right)^{2}\\ &\leq \frac{n\psi^{''}(1)}{2}\left(\frac{\theta+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}}\right)^{2}\\ &= \frac{\psi''(1)}{2}\frac{\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^{2}}{1-\theta},\end{aligned}$$

where the last inequality is due to the fact that

$$1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta.$$

□

In the following lemma, we give a lower bound of the proximity measure $\sigma(v)$ defined by (1.8) in terms of the proximity function $\Psi\left(v\right)$ defined by (1.10).

Lemma 1.2.12. *For any $v > 0$, we have*

$$\sigma(v) \geq \sqrt{\frac{\Psi\left(v\right)}{2}}.$$

Proof. By item (iii) of Proposition 1.1.7, we can write

$$\psi(t) = \int_{1}^{t}\int_{1}^{\xi}\psi^{''}(z)dzd\xi,\ \forall t > 0.$$

Using the first item of Proposition 1.2.2, it follows that

$$\begin{aligned}\psi(t) &\leq \int_{1}^{t}\int_{1}^{\xi}\psi^{''}(\xi)\psi^{''}(z)dzd\xi\\ &= \int_{1}^{t}\psi^{''}(\xi)\psi^{'}(\xi)d\xi\\ &= \frac{1}{2}\psi^{'}(t)^{2},\end{aligned} \tag{1.15}$$

where the equalities are due to the fact that $\psi^{'}(1) = 0$. Therefore, using (1.10), (1.15) and (1.8), we obtain

$$\begin{aligned}\Psi(v) = \sum_{i=1}^{n}\psi\left(v_i\right) &\leq \sum_{i=1}^{n}\frac{1}{2}\psi'\left(v_i\right)^{2}\\ &= \frac{1}{2}\|\nabla\Psi(v)\|^{2}\\ &= 2\sigma(v)^{2}.\end{aligned}$$

□

Remark 1.2.13. *Through this chapter, we assume that* $\tau \geq 1$. *Using Lemma* 1.2.12 *and the assumption that* $\Psi(v) \geq \tau$, *we have*

$$\sigma(v) \geq \sqrt{\frac{1}{2}}.$$

1.2.3 Computation of the displacement step

The purpose of this subsection is to compute a default step size α such that (x_+, y_+, s_+) defined in (1.4) are strictly feasible and the proximity function decreases sufficiently. Due to (1.5) for fixed μ, we may write

$$x_+ = \frac{x}{v}(v + \alpha d_x), \quad s_+ = \frac{s}{v}(v + \alpha d_s).$$

This leads to

$$v_+ = \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s)}.$$

Now, we define the difference of proximities between a new iterate and a current iterate for a fixed μ as below

$$f(\alpha) = \Psi(v_+) - \Psi(v).$$

Recall that the step size α is chosen to satisfy

$$v + \alpha d_x > 0 \text{ and } v + \alpha d_s > 0.$$

Therefore using the e-convexity property of ψ (hypothesis **H3**), we get by Lemma A.0.9

$$\Psi(v_+) \leq \frac{1}{2}\left(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\right),$$

Thus, $f(\alpha) \leq f_1(\alpha)$ where

$$f_1(\alpha) = \frac{1}{2}\left(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\right) - \Psi(v). \tag{1.16}$$

Differentiating the function f_1 with respect to α, we get

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi'(v_i + \alpha d_{x_i})d_{x_i} + \psi'(v_i + \alpha d_{s_i})d_{s_i}\right).$$

and

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi''(v_i + \alpha d_{x_i})d_{x_i}^2 + \psi''(v_i + \alpha d_{s_i})d_{s_i}^2\right). \tag{1.17}$$

Hence, using (1.8) and the last equation of (1.9) we obtain

$$f_1'(0) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\sigma(v)^2.$$

To simplify the notation, we set $\sigma(v) := \sigma$, and $v_1 = \min(v)$.

Lemma 1.2.14. *([8, Lemma 4.1]) Let $f_1(\alpha)$ be as defined in (1.16). Then, we have*

$$f_1''(\alpha) \leq 2\sigma^2\psi''(v_1 - 2\alpha\sigma).$$

Proof. Recall that d_x and d_s are orthogonal and $d_x + d_s = -\nabla\Psi(v)$. Moreover, using (1.8) one obtains

$$\|d_x + d_s\|^2 = \|d_x\|^2 + \|d_s\|^2 = 4\sigma^2.$$

This implies that

$$\|d_x\| \leq 2\sigma \text{ and } \|d_s\| \leq 2\sigma.$$

Consequently, for all $i \in \{1, \ldots, n\}$ one has

$$v_i + \alpha d_{x_i} \geq v_1 - 2\alpha\sigma \text{ and } v_i + \alpha d_{s_i} \geq v_1 - 2\alpha\sigma. \tag{1.18}$$

In addition, using **H1**, we can easily see that ψ'' is monotonically decreasing on $\mathbb{R}_{++}$. Hence , using (1.17) and (1.18), it follows that

$$f_1''(\alpha) \leq \frac{1}{2}\psi''(v_1 - 2\alpha\sigma)\sum_{i=1}^{n}\left(d_{x_i}^2 + d_{s_i}^2\right) = \frac{1}{2}\psi''(v_1 - 2\alpha\sigma)\|d_x + d_s\|^2 = 2\sigma^2\psi''(v_1 - 2\alpha\sigma).$$

□

Lemma 1.2.15. *([8, Lemma 4.2]) If the step size α satisfies the inequality*

$$\psi'(v_1) - \psi'(v_1 - 2\alpha\sigma) \leq 2\sigma, \tag{1.19}$$

then

$$f_1'(\alpha) \leq 0.$$

Proof. Since f_1' is continuous and differentiable, we can rewrite $f_1'(\alpha)$ as follows

$$f_1'(\alpha) = f_1'(0) + \int_0^\alpha f_1''(\xi)d\xi.$$

Therefore, using Lemma 1.2.14 and the fact that $f_1'(0) = -2\sigma^2$, we obtain

$$\begin{aligned} f_1'(\alpha) &\leq -2\sigma^2 + 2\sigma^2\int_0^\alpha \psi''(v_1 - 2\sigma\xi)d\xi \\ &= -2\sigma^2 - \sigma\int_0^\alpha \psi''(v_1 - 2\sigma\xi)d(v_1 - 2\sigma\xi) \\ &= -2\sigma^2 - \sigma\left(\psi'(v_1 - 2\sigma\alpha) - \psi'(v_1)\right) \\ &= \sigma\left(-\psi'(v_1 - 2\sigma\alpha) + \psi'(v_1) - 2\sigma\right). \end{aligned}$$

As a result, the inequality

$$f_1'(\alpha) \leq 0,$$

will surely holds if α verifies

$$\psi'(v_1) - \psi'(v_1 - 2\sigma\alpha) \leq 2\sigma.$$

□

Lemma 1.2.16. *([8, Lemma 4.3]) Let ρ be the function defined in Definition 1.1.9. Then the largest possible value of the step size α^* satisfying (1.19) is given by*

$$\alpha^* = \frac{\rho(\sigma) - \rho(2\sigma)}{2\sigma}.$$

Proof.
We would like to find the largest value of α such that inequality (1.19) is satisfied. Differentiating the left hand side of (1.19) with respect to v_1, and using the monotonically decreasing behavior of ψ'', we obtain

$$\psi''(v_1) - \psi''(v_1 - 2\alpha\sigma) \leq 0.$$

Thus, by fixing the value of σ, the smaller v_1 is, the smaller α will be. In addition, by the Definition of σ, we have

$$\sigma(v) = \frac{1}{2}\|\nabla\Psi(v)\| \geq \frac{1}{2}|\psi'(v_1)| \geq -\frac{1}{2}\psi'(v_1)$$

The equality occurs if and only if v_1 is the sole coordinate in v that deviates from 1, and $v_1 < 1$ (in which case $\psi'(v_1) < 0$). Therefore, the worst-case scenario for the step size α happens when v_1 is taken such that

$$-\frac{1}{2}\psi'(v_1) = \sigma. \tag{1.20}$$

In addition, differentiating the left hand side of inequality (1.19) with respect to α, we get $2\sigma\psi''(v_1 - 2\sigma\alpha)$, which is obviously positive since ψ is convex. This means that the function $\alpha \mapsto \psi'(v_1) - \psi'(v_1 - 2\alpha\sigma)$ is monotonically increasing. Thus, the largest potential value of the step size α such that (1.19) is verified satisfies the following equality

$$-\frac{1}{2}\psi'(v_1 - 2\sigma\alpha) = 2\sigma. \tag{1.21}$$

Moreover, using the Definition 1.1.9, equalities (1.20) and (1.21) are rewritten in the following forms

$$v_1 = \rho(\sigma)$$

and

$$v_1 - 2\sigma\alpha = \rho(2\sigma).$$

Hence, we may conclude that

$$\alpha^* = \frac{\rho(\sigma) - \rho(2\sigma)}{2\sigma}.$$

□

Lemma 1.2.17. *([8, Lemma 4.4]) Let ρ be the function defined in Definition 1.1.9 and α^* be as defined in Lemma 1.2.16. Then, we have*

$$\alpha^* \geq \frac{1}{\psi''(\rho(2\sigma))}.$$

Proof.
Since ρ is the inverse of the restriction of $-\frac{1}{2}\psi'$ to $]0,1]$, we can write

$$-\psi'(\rho(\sigma)) = 2\sigma. \tag{1.22}$$

Differentiating the equality (1.22) with respect to σ, we get

$$-\psi''(\rho(\sigma))\rho'(\sigma) = 2,$$

and consequently

$$\rho'(\sigma) = \frac{-2}{\psi''(\rho(\sigma))}. \tag{1.23}$$

Furthermore, from Lemma 1.2.16 we have

$$\begin{aligned}\alpha^* &= \frac{\rho(\sigma) - \rho(2\sigma)}{2\sigma} \\ &= \frac{1}{2\sigma}\int_{2\sigma}^{\sigma} \rho'(\xi)d\xi \\ &= \frac{1}{\sigma}\int_{\sigma}^{2\sigma} \frac{1}{\psi''(\rho(\xi))}d\xi.\end{aligned} \tag{1.24}$$

The last equality is due to (1.23). Therefore, to obtain a lower bound of α^*, we need to change the argument of the integral in (1.24) with its minimal value. So we need to find the maximum of the function $\xi \to \psi''(\rho(\xi))$ for $\sigma \leq \xi \leq 2\sigma$. Recall that since ψ satisfies **H1**, ψ'' is monotonically decreasing on $\mathbb{R}_{++}$. This implies that for $\sigma \leq \xi \leq 2\sigma$, the function $\xi \to \psi''(\rho(\xi))$ is maximal when the function $\xi \to \rho(\xi)$ is minimal. Taking into account that ρ is monotonically decreasing, this happens when $\xi = 2\sigma$. Hence, from (1.24) it follows that

$$\begin{aligned}\alpha^* &= \frac{1}{\sigma}\int_{\sigma}^{2\sigma} \frac{1}{\psi''(\rho(\xi))}d\xi \\ &\geq \frac{1}{\sigma}\frac{1}{\psi''(\rho(2\sigma))}\int_{\sigma}^{2\sigma} d\xi \\ &= \frac{1}{\psi''(\rho(2\sigma))}.\end{aligned}$$

□

Theorem 1.2.18. *([8, Theorem 4.6]) Let us set* $\bar{\alpha} = \dfrac{1}{\psi''(\rho(2\sigma))}$, *as the default step size. Then*

$$f(\bar{\alpha}) \leq -\sigma^2\bar{\alpha} = -\frac{\sigma^2}{\psi''(\rho(2\sigma))}.$$

Proof. Let us define the univariate function g as follows

$$g(\alpha) = -2\alpha\sigma^2 + \alpha\sigma\psi'(v_1) + \frac{1}{2}\Big(\psi(v_1 - 2\alpha\sigma) - \psi(v_1)\Big).$$

Differentiating g twice we get

$$\begin{aligned}g'(\alpha) &= -2\sigma^2 - \sigma\Big(\psi'(v_1 - 2\alpha\sigma) - \psi'(v_1)\Big) \\ g''(\alpha) &= 2\sigma^2\psi''(v_1 - 2\alpha\sigma).\end{aligned}$$

We can easily see that

$$g(0) = f_1(0) = 0,\ g'(0) = f_1'(0) = -2\sigma^2 < 0.$$

On the other hand, from Lemma 1.2.14 we have

$$f_1^{''}(\alpha) \leq g^{''}(\alpha).$$

Therefore, $f_1^{'}(\alpha) \leq g^{'}(\alpha)$ and $f_1(\alpha) \leq g(\alpha)$. Taking $\alpha \in [0, \alpha^*]$ with α^* defined in Lemma 1.2.16, we get

$$g'(\alpha) = -2\sigma^2 - \sigma\Big(\psi'(v_1 - 2\alpha\sigma) - \psi'(v_1)\Big) \leq 0.$$

Due to the increase of $g^{''}$ in α, using Lemma A.0.11 we obtain

$$f(\alpha) \leq f_1(\alpha) \leq g(\alpha) \leq \frac{1}{2}\alpha g^{'}(0) = -\alpha\sigma^2.$$

The proof is then completed by replacing α by $\bar{\alpha}$. □

1.3 A primal-dual interior-point algorithm based on a kernel function with an exponential-hyperbolic barrier term

In this section, we study a path-following IPM for solving LO problems based on the following new KF

$$\psi(t) = \frac{t^2 - 1}{2} + \sinh^2(1)\left(e^{\coth(t) - \coth(1)} - 1\right),\ \forall t > 0. \tag{1.25}$$

This KF differs from other KFs in having an exponential-hyperbolic barrier term that belongs to the hyperbolic type, recently introduced by I. Touil and W. Chikouche [100, 98, 99]. We prove that the new KF belongs to the class defined in Section 1.2.1 and we estimate the decrease behaviour of the barrier function induced by the new KF. After that, we derive the iteration bounds of the new IPA for both large- and small-update methods. We back up the obtained theoretical results with some preliminary numerical tests which show that our algorithm outperformed other algorithms with better theoretical convergence complexity. The results of this section have been the subject of a publication [39].

1.3.1 The new kernel function and its properties

For conveniency, we give the first three derivatives of ψ for all $t > 0$

$$\psi'(t) = t - \frac{\sinh^2(1)}{\sinh^2(t)} e^{\coth(t) - \coth(1)}, \tag{1.26}$$

$$\psi''(t) = 1 + \sinh^2(1) e^{\coth(t) - \coth(1)} \left(2\frac{\coth(t)}{\sinh^2(t)} + \frac{1}{\sinh^4(t)}\right) \geq 1, \tag{1.27}$$

and

$$\psi'''(t) = -\sinh^2(1)e^{\coth(t)-\coth(1)}\left(4\frac{\coth^2(t)}{\sinh^2(t)} + \frac{6\coth(t)+2}{\sinh^4(t)} + \frac{1}{\sinh^6(t)}\right) < 0.$$

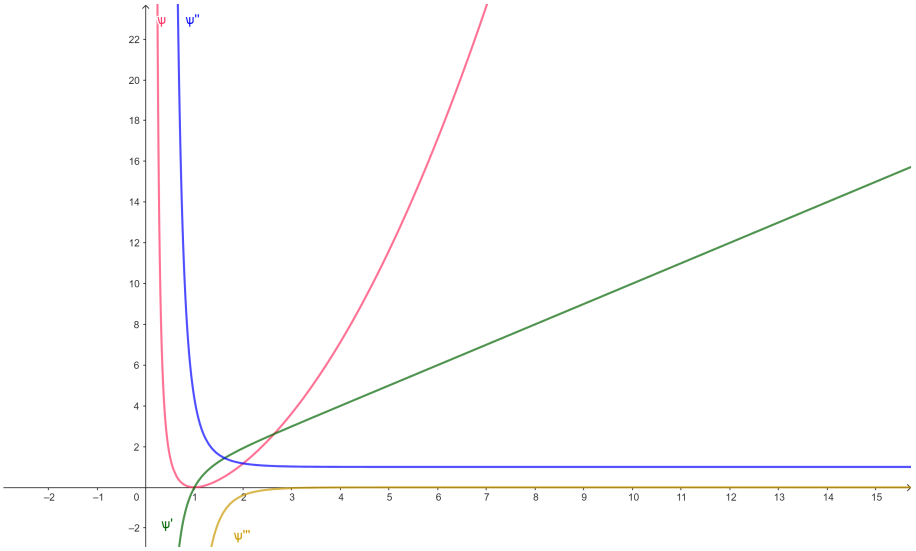

FIGURE 1.3: Graphs of ψ, ψ' ψ'' and ψ'''.

We clearly see that

$$\psi(1) = 0 \text{ and } \psi'(1) = 0.$$

Hence, ψ verifies both conditions **KF1** and **KF2**. Moreover, since

$$\lim_{t\to 0^+} \coth(t) = +\infty \text{ and } \lim_{t\to+\infty} \coth(t) = 1,$$

we get

$$\lim_{t\to 0^+} \psi(t) = \lim_{t\to+\infty} \psi(t) = +\infty.$$

This means that ψ is a KF satisfying **H1**.

Furthermore, we can write

$$\psi(t) = \frac{t^2-1}{2} + \psi_b(t),$$

with

$$\psi_b(t) = \sinh^2(1)\left(e^{\coth(t)-\coth(1)} - 1\right).$$

From (1.26) and (1.27), it follows that

$$\psi_b'(t) < 0 \text{ and } \psi_b''(t) > 0, \forall t > 0.$$

The following lemma provides an important feature of the hyperbolic cotangent function.

Lemma 1.3.1. *One has for all* $t > 0$

(i)

$$t\coth(t) - 1 > 0, \tag{1.28}$$

(ii)

$$2t\coth(t) - 1 > 0. \tag{1.29}$$

Proof. For the first item, let's define the function l as follows:

$$l(t) = t\coth(t) - 1,\ \forall t > 0.$$

Deriving l, we obtain

$$\begin{aligned} l'(t) &= \coth(t) - \frac{t}{\sinh^2(t)} \\ &= \frac{\cosh(t)\sinh(t) - t}{\sinh^2(t)}. \end{aligned}$$

Recall that

$$\sinh(2t) = 2\cosh(t)\sinh(t),\ \forall t \in \mathbb{R}.$$

Using this property of hyperbolic functions, we can rewrite l' as follows:

$$l'(t) = \frac{\sinh(2t) - 2t}{2\sinh^2(t)}.$$

Moreover, the Taylor expansion of the hyperbolic sine function implies that

$$\sinh(t) \geq t,\ \forall t \geq 0.$$

It follows that the function l is strictly increasing on the interval $]0, +\infty[$. Since

$$\lim_{t\to 0^+} l(t) = 0 \text{ and } \lim_{t\to+\infty} l(t) = +\infty,$$

we conclude that $t\coth(t) - 1 > 0,\ \forall t > 0$. The second item is a direct consequence of the first item. □

The next lemma reveals that ψ defined in (1.25) satisfies conditions **H2** and **H3**.

Lemma 1.3.2. *Let ψ be as defined in* (1.25)*. Then,*

(i) $t\psi''(t) - \psi'(t) > 0,\ \forall t > 0.$

(ii) $t\psi''(t) + \psi'(t) > 0,\ \forall t > 0.$

Proof. Using (1.26) and (1.27), we have for all $t > 0$

$$t\psi''(t) - \psi'(t) = \frac{\sinh^2(1)}{\sinh^2(t)} e^{\coth(t)-\coth(1)} \left(2t\coth(t) + 1 + \frac{t}{\sinh^2(t)}\right) > 0,$$

and

$$t\psi''(t) + \psi'(t) = 2t + \frac{\sinh^2(1)}{\sinh^2(t)} e^{\coth(t)-\coth(1)} \left(2t\coth(t) - 1 + \frac{t}{\sinh^2(t)}\right) > 0,$$

by taking into account (1.29) of Lemma 1.3.1. □

As a result ψ belongs to the class of KFs introduced in Section 1.2. Thus, we can take advantage of all the results presented there without proofs.

Let ϱ and ρ be as defined in Definition 1.1.9. We have the following lemma.

Lemma 1.3.3. *For all* $(z,t) \in [0,+\infty[\times]0,1]$ *such that* $z = -\frac{1}{2}\psi'(t)$, *one has*

$$\coth(t) \leq \log\left(e^{\coth(1)}(2z+1)\right).$$

Proof. Let $z \geq 0$ and $0 < t \leq 1$ such that $z = -\frac{1}{2}\psi'(t)$, then $\rho(z) = t$. Using (1.26), we have

$$\begin{aligned} 2z &= -\psi'(t) \\ &= -t + \frac{\sinh^2(1)}{\sinh^2(t)} e^{\coth(t)-\coth(1)}. \end{aligned}$$

Since sinh is a monotonically increasing function, we obtain

$$e^{\coth(t)-\coth(1)} \leq (2z+1),$$

which implies that

$$\coth(t) \leq \log(e^{\coth(1)}(2z+1)).$$

□

1.3.2 Decrease of the proximity during a (damped) Newton step

We first study the effect of updating the barrier parameter μ on the value of the proximity function $\Psi(v)$, defined in (1.10) corresponding to ψ defined in (1.25). Since

$$\begin{aligned} \psi''(1) &= 1 + 2\coth(1) + \frac{1}{\sinh^2(1)} \\ &\leq 2(3\coth(1)+1), \end{aligned}$$

we deduce directly from Lemma 1.2.11 the following corollary.

Corollary 1.3.4. *Let* θ *be such that* $0 < \theta < 1$. *If* $\Psi(v) \leq \tau$, *then*

$$\Psi(v_+) \leq \frac{3\coth(1)+1}{(1-\theta)}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2 := \Psi_0.$$

Ψ_0 *is an upper bound for* $\Psi(v_+)$ *during the process of the algorithm.*

Now, we would like to compute a default step size $\bar{\alpha}$ such that (x_+, y_+, s_+) defined in (1.4) are strictly feasible and the proximity function Ψ decreases sufficiently. Recall that during an inner iteration the parameter μ is fixed and from (1.5) we have

$$x_+ = \frac{x}{v}(v + \bar{\alpha}d_x),\ s_+ = \frac{s}{v}(v + \bar{\alpha}d_s),\ v_+ = \sqrt{\frac{x_+ s_+}{\mu}}.$$

We provide an upper bound for the decreasing value of the proximity function in an inner iteration by the next theorem.

Theorem 1.3.5. *If* $\bar{\alpha}$ *is the default step size, then*

$$\Psi(v_+) - \Psi(v) \leq -\frac{\sqrt{\Psi(v)}}{80\log^2\left(e^{\coth(1)}(\sqrt{\Psi(v)}+1)\right)}. \tag{1.30}$$

Proof. From (1.27) , we have

$$\begin{aligned}\psi''(t) =& 1+\left(t-\psi'(t)\right)\left(2\coth(t)+\frac{1}{\sinh^2(t)}\right)\\ &\leq 1+3\left(1-\psi'(t)\right)\coth^2(t),\ \forall t>0.\end{aligned}$$

Putting $t=\rho\left(2\sigma\right)$, we get

$$\psi''\left(\rho\left(2\sigma\right)\right)\leq 1+3\left(1+4\sigma\right)\coth^2\left(\rho\left(2\sigma\right)\right).$$

Thus, applying Lemma 1.3.3 for $t=\rho(2\sigma)$ and $z=2\sigma$ we get

$$\begin{aligned}\psi''\left(\rho\left(2\sigma\right)\right)&\leq 1+3\left(1+4\sigma\right)\log^2\left(e^{\coth(1)}(4\sigma+1)\right)\\ &\leq 4(1+4\sigma)\log^2\left(e^{\coth(1)}(4\sigma+1)\right)\\ &\leq 4(\sigma+4\sigma)\log^2\left(e^{\coth(1)}(4\sigma+1)\right)\\ &= 20\sigma\log^2\left(e^{\coth(1)}(4\sigma+1)\right),\end{aligned}$$

where the second inequality is obtained since

$$(1+4\sigma)\log^2\left(e^{\coth(1)}(4\sigma+1)\right)\geq\log^2\left(e^{\coth(1)}\right)>1,$$

while the last one is obtained using Remark 1.2.13. Hence, from Theorem 1.2.18 it follows that

$$\Psi(v_+)-\Psi(v)=f(\bar{x})\leq-\frac{\sigma}{20\log^2\left(e^{\coth(1)}(4\sigma+1)\right)}.$$

Let's define the function

$$t\mapsto g(t):=-\frac{t}{\log^2\left(e^{\coth(1)}(4t+1)\right)},\ t>0.$$

Differentiating g, we get

$$g'(t)=-\frac{(4t+1)\log(e^{\coth(1)}(4t+1))-8t}{(4t+1)\log^3(e^{\coth(1)}(4t+1))}.$$

To study the sign of $g'(t)$, let's define another function

$$t\mapsto h(t):=(4t+1)\log(e^{\coth(1)}(4t+1))-8t,\ t\geq 0.$$

Differentiating h, we have

$$\begin{aligned}h'(t)=&4\log(e^{\coth(1)}(4t+1))-4.\\ &=4\log(4t+1)+4(\coth(1)-1).\end{aligned}$$

Since $\coth(1)>1$, h' is strictly positive on $\mathbb{R}_+$ which implies that h is monotonically increasing. Hence, for all $t>0$

$$h(t)>h(0)=\coth(1)>0.$$

It follows that g' is is strictly negative on $]0,+\infty[$ and therefore g is monotonically decreasing. Thanks to Lemma 1.2.12, we have $\sigma \geq \frac{\sqrt{\Psi(v)}}{4}$, which gives

$$\Psi(v_+) - \Psi(v) \leq -\frac{\sqrt{\Psi(v)}}{80\log^2\left(e^{\coth(1)}(\sqrt{\Psi(v)}+1)\right)}.$$

□

1.3.3 Iteration complexity

Now, we compute how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$ after μ-update. Let us define the value of $\Psi(v)$ after μ-update as Ψ_0, and the subsequent values in the same outer iteration as $\Psi_i, i = 1,\ldots,K$, where K stands for the total number of inner iterations in the outer iteration. The decrease on each inner iteration is given by (1.30), that is,

$$\Psi_{i+1} \leq \Psi_i - \frac{1}{80\log^2\left(e^{\coth(1)}(\sqrt{\Psi_0}+1)\right)}\Psi_i^{\frac{1}{2}},\ i = 0,1\ldots,K-1.$$

A direct application of Lemma A.0.13 for $t_k = \Psi_k$, $\beta = \frac{1}{80\log^2\left(e^{\coth(1)}(\sqrt{\Psi_0}+1)\right)}$ and $\gamma = \frac{1}{2}$, produces the following lemma.

Lemma 1.3.6. *One has*

$$K \leq 160\Psi_0^{\frac{1}{2}}\log^2(e^{\coth(1)}(\sqrt{\Psi_0}+1)).$$

We recall the following important result.

Lemma 1.3.7. *([92, Lemma II.17]) If the barrier parameter μ has an initial value μ^0 and is repeatedly multiplied by $1-\theta$ with $0<\theta<1$, then after at most*

$$\frac{1}{\theta}\log\frac{n}{\epsilon},$$

iterations we have

$$n\mu \leq \epsilon.$$

Corollary 1.3.8. *An upper bound for the total number of iterations is obtained by multiplying the upper bound K by the number of barrier parameter updates, which is bounded above by* $\frac{1}{\theta}\log\frac{n}{\epsilon}$.

Using Lemma 1.3.6, Corollary 1.3.8 and the fact that

$$\sqrt{\Psi_0} \leq \Psi_0,$$

we arrive at the final result of this subsection which summarizes the complexity bound.

Theorem 1.3.9. *Let Ψ_0 be an upper bound for $\Psi(v_+)$ and $\tau \geq 1$. Then, the total number of iterations to obtain an approximate solution with $n\mu \leq \epsilon$ is bounded by*

$$\mathcal{O}\left(\log^2(\Psi_0)\Psi_0^{\frac{1}{2}}\frac{\log\frac{n}{\epsilon}}{\theta}\right).$$

For small-update methods with $\tau = \mathcal{O}(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$, Corollary 1.3.4 implies that $\Psi_0 = \mathcal{O}(1)$. Hence, the complexity of the primal-dual IPA for linear programming problem based on the new KF is $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ iterations complexity.

As for large-update methods i.e., $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, Corollary 1.3.4 implies that $\Psi_0 = \mathcal{O}(n)$. Thus, we obtain $\mathcal{O}\left(\sqrt{n}\log^2 n\log\frac{n}{\epsilon}\right)$ iterations complexity.

1.3.4 Numerical tests

In this section, we carried out through numerical experiments to show the computational performance of the proposed algorithm comparing it with other algorithms based on the KFs provided in Table 1.1. Our experiments are implemented in MATLAB R2012b using a Supermicro dual-2.80 GHz Intel Core i5 server with 4.00 Go RAM. We have taken $\epsilon = 10^{-8}$, $\tau = n$, and $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9, 0.99\}$.

TABLE 1.1: Considered kernel functions.

Kernel functions	Complexity	Ref.
$\psi_c(t) = \frac{t^2-1}{2} - \log t$	$\mathcal{O}\left(n\log\frac{n}{\epsilon}\right)$	[92]
$\psi_1(t) = \frac{t^2-1}{2} + \frac{e^{p(\frac{1}{t}-1)}-1}{p},\ p=2$	$\mathcal{O}\left(\sqrt{n}\log^2 n\log\frac{n}{\epsilon}\right)$	[3]
$\psi_2(t) = \frac{t^2-1}{2} - \int_1^t e^{p(\frac{1}{x}-1)}dx,\ p=\log(1+n)$	$\mathcal{O}\left(\sqrt{n}\log n\log\frac{n}{\epsilon}\right)$	[10]
$\psi_3(t) = \frac{t^2-1}{2} - \log t + \frac{1}{8}\tan^2(\pi\frac{1-t}{4t+2})$	$\mathcal{O}\left(n^{\frac{2}{3}}\log\frac{n}{\epsilon}\right)$	[88]
$\psi_4(t) = \frac{1+2\coth(1)}{2\sinh^2(1)}(t^2-1) + \coth^2(t) - \coth^2(1) - \log t$	$\mathcal{O}\left(n^{\frac{2}{3}}\log\frac{n}{\epsilon}\right)$	[100]
$\psi_{\text{new}}(t) = \frac{t^2-1}{2} + \sinh^2(1)\left(e^{\coth(t)-\coth(1)}-1\right)$	$\mathcal{O}\left(\sqrt{n}\log^2 n\log\frac{n}{\epsilon}\right)$	New

To analyze the computational performance fairly, we choose a practical step size α as in [54] i.e., $\alpha = \min(\alpha_x, \alpha_s)$, with

$$\alpha_x = \min_{i=1,\dots,n}\begin{cases} -\dfrac{x_i}{\Delta x_i} & \text{if } \Delta x_i < 0, \\ 1 & \text{elsewhere,}\end{cases} \quad \text{and } \alpha_s = \min_{i=1,\dots,n}\begin{cases} -\dfrac{s_i}{\Delta s_i} & \text{if } \Delta s_i < 0, \\ 1 & \text{elsewhere.}\end{cases}$$

This choice of α guarantees the strict positivity of the new point. Moreover, we increase the step size by a fixed factor $0 < \beta < 1$ (in our case we choose $\beta = 0.9$). We conducted comparative numerical tests between the KFs provided in Table 1.1 on four fixed size test problems and a variable size test problem taken from Table B.2 and Table B.3 respectively. The summary of results is given in the following table. From Table 1.2, it becomes clear that smaller values of the parameter θ influence the iteration count negatively. Thus, for the variable size test problem EV3 from Table B.3, we only choose $\theta \in \{0.9, 0.99\}$ for seven different sizes $n = 2m$ where $m \in \{5, 25, 50, 100, 200, 400, 1000\}$.

For each example, we used **bold** font to highlight the best, i.e., the smallest, iteration number.

TABLE 1.2: Number of inner iterations for fixed size examples.

Examples	θ	ψ_c	ψ_1	ψ_2	ψ_3	ψ_4	ψ_{new}
EF1	0.1	**188**	**188**	**188**	**188**	200	**188**
	0.3	**56**	**56**	**56**	**56**	70	**56**
	0.5	**29**	**29**	**29**	**29**	44	**29**
	0.7	**17**	**17**	**17**	**17**	23	**17**
	0.9	**11**	13	**11**	**11**	14	**11**
EF2	0.1	**191**	**191**	**191**	**191**	215	**191**
	0.3	**57**	**57**	**57**	**57**	75	**57**
	0.5	**29**	**29**	**29**	**29**	39	**29**
	0.7	**17**	**17**	**17**	**17**	21	**17**
	0.9	11	**9**	10	11	**9**	**9**
EF3	0.1	**192**	**192**	**192**	**192**	204	**192**
	0.3	**57**	60	**57**	**57**	66	**57**
	0.5	**30**	33	**30**	**30**	33	**30**
	0.7	**17**	20	20	**17**	20	18
	0.9	33	21	**19**	39	25	**19**
EF4	0.1	**196**	**196**	**196**	**196**	212	**196**
	0.3	**58**	**58**	**58**	**58**	78	**58**
	0.5	31	**30**	**30**	31	42	**30**
	0.7	28	24	24	**22**	24	24
	0.9	24	**16**	17	23	20	23

TABLE 1.3: Number of inner iterations for Example EV3

θ	m	ψ_c	ψ_1	ψ_2	ψ_3	ψ_4	ψ_{new}
$\theta = 0.9$	5	11	**9**	10	11	**9**	**9**
	25	12	**10**	**10**	12	**10**	**10**
	50	12	**10**	**10**	12	**10**	**10**
	100	13	**11**	**11**	13	**11**	**11**
	200	13	**11**	**11**	13	**11**	**11**
	400	13	**11**	**11**	13	**11**	**11**
	1000	15	**12**	**12**	15	**12**	**12**
$\theta = 0.99$	5	11	**10**	11	11	**10**	**10**
	25	11	**10**	**10**	11	**10**	**10**
	50	13	**12**	**12**	13	**12**	**12**
	100	13	**12**	**12**	13	**12**	**12**
	200	13	**12**	**12**	13	**12**	**12**
	400	13	**12**	**12**	13	**12**	**12**
	1000	13	**12**	**12**	13	**12**	**12**

Although, three out of five considered KFs in Table 1.1 have better theoretical convergence complexity, numerical results show that by using our new KF, with exponential-hyperbolic barrier term, the best iteration complexity was achieved in 91% of the realized experiments.

1.4 Complexity of a primal-dual interior-point algorithm based on a class of hyperbolic-logarithmic kernel functions

In this section, we first study a primal-dual IPA for solving LO problems, based on the new KF with hyperbolic-logarithmic barrier term

$$\psi(t) = \frac{t^2-1}{2} + \tanh^2(1)\big(\coth(t) - \log t\big) - \tanh(1), \ \forall t > 0. \tag{1.31}$$

To improve the iteration bound, we generalize the barrier term of ψ by applying a positive parameter p to obtain the following parametric KF

$$\psi_p(t) = \frac{t^2-1}{2} + \frac{\sinh^2(1)}{\sinh^2(1) + p\coth^{p-1}(1)}\big(\coth^p(t) - \log t - \coth^p(1)\big), \ \ p \geq 2. \tag{1.32}$$

The latter contains the first hyperbolic-logarithmic KF, proposed recently by Touil and Chikouche [100], as a special case up to a multiplicative constant and improves significantly its theoretical complexity. In fact, the complexity analysis proves that the new method enjoys the currently best iterations bounds for both large- and small-update methods namely, $\mathcal{O}(\sqrt{n}\log n \log \frac{n}{\epsilon})$ and $\mathcal{O}(\sqrt{n}\log \frac{n}{\epsilon})$. To illustrate the effectiveness of the proposed KFs, we conducted numerical experiments comparing with all existing KFs with $\log t$ in their barrier term. The results of this section were the subject of a published paper [42].

1.4.1 The analysis of the interior-point method based on the non-parametric kernel function

Some technical results

In the analysis of the algorithm based on ψ defined in (1.31), we need its first three derivatives with respect to t which are given for all $t > 0$ by

$$\psi'(t) = t - \tanh^2(1)\left(\frac{1}{\sinh^2(t)} + \frac{1}{t}\right), \tag{1.33}$$

$$\psi''(t) = 1 + \tanh^2(1)\left(2\frac{\coth(t)}{\sinh^2(t)} + \frac{1}{t^2}\right) \geq 1, \tag{1.34}$$

and

$$\psi'''(t) = -\tanh^2(1)\left(\frac{2}{\sinh^4(t)} + 4\frac{\coth^2(t)}{\sinh^2(t)} + \frac{2}{t^3}\right) < 0.$$

Clearly, $\psi'(1) = \psi(1) = 0$. Moreover, since

$$\lim_{t\to 0^+}\coth(t) = +\infty \text{ and } \lim_{t\to+\infty}\coth(t) = 1,$$

we get

$$\lim_{t\to 0^+}\psi(t)=\lim_{t\to+\infty}\psi(t)=+\infty,$$

which imply that ψ is a KF satisfying **H1**.

Furthermore, we can write

$$\psi(t)=\frac{t^2-1}{2}+\psi_b(t),$$

with

$$\psi_b(t)=\tanh^2(1)\big(\coth(t)-\log t\big)-\tanh(1).$$

We can easily see from (1.33) and (1.34) that

$$\psi_b'(t)<0 \text{ and } \psi_b''(t)>0, \forall t>0.$$

Lemma 1.4.1. *Let ψ be as defined in (1.31). Then,*

(i) $t\psi''(t)-\psi'(t)>0,\ \forall t>0.$

(ii) $t\psi''(t)+\psi'(t)>0,\ \forall t>0.$

Proof. From (1.33) and (1.34), we have

$$t\psi''(t)-\psi'(t)=\tanh^2(1)\left(\frac{2t\coth(t)+1}{\sinh^2(t)}+\frac{2}{t}\right)>0,$$

and

$$t\psi''(t)+\psi'(t)=2t+\tanh^2(1)\left(\frac{2t\coth(t)-1}{\sinh^2(t)}\right)>0,$$

by taking into account (1.29) of Lemma 1.3.1. □

The previous lemma reveals that ψ satisfies conditions **H2** and **H3**. As a consequence, ψ belongs to the class of KFs introduced in Section 1.2.
Let ρ and ϱ be the functions defined in Definition 1.1.9. We give an implicit lower bound for ρ in the following lemma.

Lemma 1.4.2. *For all $(z,t)\in[0,+\infty[\times]0,1]$ such that $z=-\frac{1}{2}\psi'(t)$, one has*

$$\coth(t)\leq\coth(1)\,(2z+2)^{\frac{1}{2}}.$$

Proof. Let $z\geq 0$ and $t\in\,]0,1]$ such that $z=-\frac{1}{2}\psi'(t)$, then $\rho(z)=t$.
Since $\dfrac{1}{\sinh^2(t)}=\coth^2(t)-1$, using (1.33), we have

$$\begin{aligned}2z&=-\psi'(t)\\&=-t+\tanh^2(1)\left(\coth^2(t)-1+\frac{1}{t}\right),\end{aligned}$$

which implies that

$$\begin{aligned}\coth^2(t)&=\coth^2(1)(2z+t)+1-\frac{1}{t}\\&\leq\coth^2(1)(2z+2).\end{aligned}$$

□

Iteration complexity

Before updating μ in the generic IPA, we have $\Psi(v) \leq \tau$. After updating μ in an outer iteration, the vector v is divided by the factor $(1-\theta)$, which generally leads to an increase of the value of $\Psi(v)$. Thus, during the inner iterations, the value of $\Psi(v)$ decreases until it passes the threshold τ. We proceed by studying the effect of updating the barrier parameter μ on the value of $\Psi(v)$.

Corollary 1.4.3. *Let θ be such that $0 < \theta < 1$. If $\Psi(v) \leq \tau$, then*

$$\Psi(v_+) \leq \frac{3\coth(1)+1}{(1-\theta)}\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2 := \Psi_0.$$

Ψ_0 is an upper bound for $\Psi(v_+)$ during the process of the algorithm.

Proof. Recall from Lemma 1.2.11 that

$$\Psi(v_+) \leq \frac{\psi''(1)}{2}\frac{\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2}{1-\theta}.$$

In addition, using (1.34)

$$\begin{aligned}\psi''(1) = 1+\tanh^2(1)\left(2\frac{\coth(1)}{\sinh^2(1)}+1\right) &\leq 1+\left(2\frac{\coth(1)}{\sinh^2(1)}+1\right)\\ &\leq 6\coth(1)+2.\end{aligned}$$

Therefore the desired inequality follows □

Now, we would like to have a default step size α such that (x_+, y_+, s_+) defined in Algorithm 1 are strictly feasible and the proximity function (1.10) decreases sufficiently. We recall that during an inner iteration the parameter μ is fixed and from (1.5) we have

$$x_+ = \frac{x}{v}(v+\bar{\alpha}d_x), s_+ = \frac{s}{v}(v+\bar{\alpha}d_s), v_+ = \sqrt{\frac{x_+s_+}{\mu}}.$$

We then present an upper bound for the decreasing value of the proximity in the inner iteration in the following theorem:

Theorem 1.4.4. *If $\bar{\alpha}$ is the default step size and $\sigma \geq 1$, then we have*

$$\Psi(v_+) - \Psi(v) \leq -\frac{\Psi(v)^{\frac{1}{4}}}{273}. \tag{1.35}$$

Proof. Using (1.34) and (1.29), we have for all $t > 0$

$$\begin{aligned}\psi''(t) &\leq 1+\tanh^2(1)\left(2\coth(t)(\coth^2(t)-1)+4\coth^2(t)\right)\\ &\leq 1+\tanh^2(1)\left(2\coth^3(t)+4\coth^2(t)\right)\\ &\leq 1+6\tanh^2(1)\coth^3(t).\end{aligned}$$

Using Lemma 1.4.2 for $t = \rho(2\sigma) < 1$ and $z = 2\sigma > 0$, we obtain

$$\begin{aligned}\psi''\left(\rho\left(2\sigma\right)\right) &\leq \left(1+6\tanh^2(1)\right)\coth^3(\rho\left(2\sigma\right)) \\ &\leq \coth^3(1)\left(1+6\tanh^2(1)\right)\left(4\sigma+2\right)^{\frac{3}{2}} \\ &\leq \coth^3(1)\left(1+6\tanh^2(1)\right)\left(4\sigma+2(2\sigma)\right)^{\frac{3}{2}}.\end{aligned}$$

Moreover, from Theorem 1.2.18 we have

$$\Psi(v_+) - \Psi(v) \leq -\frac{\sigma^2}{\psi''\left(\rho\left(2\sigma\right)\right)}.$$

This implies that

$$\begin{aligned}\Psi(v_+) - \Psi(v) &\leq -\frac{\sigma^2}{\coth^3(1)\left(1+6\tanh^2(1)\right)\left(8\sigma\right)^{\frac{3}{2}}} \\ &= -\frac{\sqrt{\sigma}}{8^{\frac{3}{2}}\coth^3(1)\left(1+6\tanh^2(1)\right)}.\end{aligned}$$

Using Lemma 1.2.12, we get

$$\begin{aligned}\Psi(v_+) - \Psi(v) &\leq -\frac{\Psi(v)^{\frac{1}{4}}}{2^{\frac{1}{4}}8^{\frac{3}{2}}\coth^3(1)\left(1+6\tanh^2(1)\right)} \\ &\leq -\frac{\Psi(v)^{\frac{1}{4}}}{273}.\end{aligned}$$

□

In the rest of this subsection, we need to compute how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$ after μ-update. Let us define the value of $\Psi(v)$ after μ-update as Ψ_0, and the subsequent values in the same outer iteration as $\Psi_k, k = 1, ..., K$, where K stands for the total number of inner iterations in the outer iteration. By the definition of $f(\alpha)$ and according to (1.35), for $k = 1, ..., K-1$, we obtain

$$\Psi_{k+1} \leq \Psi_k - \frac{\Psi_k^{\frac{1}{4}}}{273}.$$

As a consequence of Lemma A.0.13, by taking $t_k = \Psi_k$, $\beta = \frac{1}{273}$ and $\gamma = \frac{3}{4}$, we get the following lemma.

Lemma 1.4.5. *Let K be the total number of inner iterations in the outer iteration. Then, we have*

$$K \leq 364\,\Psi_0^{\frac{3}{4}},$$

where Ψ_0 is the value of $\Psi(v)$ after the μ-update in an outer iteration.

Using Lemma 1.4.5 and Corollary 1.3.8, we derive an upper bound for the total number of iterations.

Theorem 1.4.6. *The total number of iterations to obtain an approximate solution with* $n\mu \leq \epsilon$ *is bounded by*

$$\left(364\,\Psi_0^{\frac{3}{4}}\right)\left(\frac{\log\frac{n}{\epsilon}}{\theta}\right).$$

For large-update IPMs with $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, we have $\mathcal{O}\left(n^{\frac{3}{4}}\log\frac{n}{\epsilon}\right)$ iteration complexity for LO problems.

For small-update IPMs with $\tau = \mathcal{O}(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$, we get the currently best known iteration bound, namely $\mathcal{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ iterations.

Numerical tests

In this section, we show that the generic IPA based on the hyperbolic-logarithmic KF (1.31) can be very efficient in solving LO problems. As in Section 1.3.4, we conducted comparative numerical tests between the KFs provided in Table 1.4. The latter contains all KFs with barrier terms constructed by $\log t$ combined with another type of functions (to our knowledge) and the complexity results for the corresponding algorithms, starting with the classical logarithmic KF ψ_c in [92] and its generalized version $\psi_{1,p}$ proposed by El Ghami et al. in [32]. Each KF was tested with the same parameters used in Section 1.3.4 on the eight test problems defined in Tables B.2 and B.3 with different sizes, ranging from very small to big size problems. The problems with fixed size EF1, EF2, EF3 and EF4 were tested for multiple values of θ, $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. Table 1.5 That corresponds to fixed size problems shows that, in most cases, larger θ gives better iteration numbers. Thus, for variable size problems EV1, EV2, EV3 and EV4, we only choose $\theta \in \{0.7, 0.9, 0.99\}$. This reduces the number of experiments since we perform Algorithm 1 with all considered KFs on four test problems for seven different sizes $n = 2m$ where $m \in \{5, 25, 50, 100, 200, 400, 1000\}$. This left us with 104 experiments for each KF. The summary of results is given in tables below.

TABLE 1.4: Existing kernel functions with $\log t$ in their barrier terms.

Type	Kernel functions	Ref. & Type of problem	Complexity
Simple-logarithmic	$\psi_c(t) = \frac{t^2-1}{2} - \log t$ $\psi_{1,p}(t) = \frac{t^{p+1}-1}{p+1} - \log t, \ \ 0 < p \leq 1$ $\psi_2(t) = 8t^2 - 11t + \frac{2}{\sqrt{t}} - 4\log t$ $\psi_{3,p}(t) = \frac{t^{2}-1-\log t}{2} + \frac{t^{1-p}-1}{2(p-1)}, \ \ p \geq 2$	[92, LO] [32, LO] [33, LO] [14, LO]	$\mathcal{O}\left(n \log \frac{n}{\epsilon}\right)$ $\mathcal{O}\left(n \log \frac{n}{\epsilon}\right)$ $\mathcal{O}\left(n^{\frac{5}{6}} \log \frac{n}{\epsilon}\right)$ $\mathcal{O}\left(pn^{\frac{p+1}{2p}} \log \frac{n}{\epsilon}\right)$
Exponential-logarithmic	$\psi_{4,p}(t) = \frac{t^2-1-\log t}{2} + \frac{e^{\frac{1}{p}t^p-1}}{2p}, \ \ p \geq 1$ $\psi_{5,p}(t) = t^2 - 1 - \log t + \frac{e^{p\left(\frac{1}{t}-1\right)}-1}{p}, \ \ p \geq 1$	[24, LO] [12, LO]	$\mathcal{O}\left(p\sqrt{n}(\log n)^{\frac{p+1}{p}} \log \frac{n}{\epsilon}\right)$ $\mathcal{O}\left(p^{-1}\sqrt{n}(\log n)^2 \log \frac{n}{\epsilon}\right)$
Trigonometric-logarithmic	$\psi_6(t) = \frac{t^2-1}{2} - \log t + \frac{1}{8}\tan^2(\pi \frac{1-t}{4t+2})$ $\psi_{6,\lambda}(t) = \frac{t^2-1}{2} - \log t + \lambda \tan^2(\pi \frac{1-t}{3t+2}), \ \ 0 < \lambda \leq \frac{8}{25\pi}$ $\psi_{7,p}(t) = \frac{t^2-1}{2} - \log t - \int_1^t \frac{1}{2p(2+4x)^2} \tan^{2p}(\pi \frac{1-x}{4x+2}) dx, \ \ p \in \mathbb{N}$	[88, LO] [19, LO] [43, LCP]	$\mathcal{O}\left(n^{\frac{2}{3}} \log \frac{n}{\epsilon}\right)$ $\mathcal{O}\left(n^{\frac{2}{3}} \log \frac{n}{\epsilon}\right)$ $\mathcal{O}\left((1+2k)n^{\frac{2p+1}{4p}} p^{\frac{2p+1}{2p}} \log \frac{n}{\epsilon}\right)$
Hyperbolic-logarithmic	$\psi_8(t) = \frac{1+2\coth(1)}{2\sinh^2(1)}(t^2-1) + \coth^2(t) - \coth^2(1) - \log t$ $\psi_{\text{new}}(t) = \frac{t^2-1}{2} + \tanh^2(1)\Big(\coth(t) - \log t\Big) - \tanh(1)$	[100, SDO] [new, LO]	$\mathcal{O}\left(n^{\frac{2}{3}} \log \frac{n}{\epsilon}\right)$ $\mathcal{O}\left(n^{\frac{3}{4}} \log \frac{n}{\epsilon}\right)$

TABLE 1.5: Number of inner iterations for fixed size examples.

Example	θ	ψ_c	$\psi_{1,\frac{1}{2}}$	ψ_2	$\psi_{3,2}$	$\psi_{3,\frac{\log n}{2}}$	$\psi_{4,1}$	$\psi_{4,2}$	$\psi_{5,2}$	$\psi_{5,\log n}$	ψ_6	$\psi_{6,0.05}$	$\psi_{7,2}$	$\psi_{7,\lceil\log n\rceil}$	ψ_8	ψ_{new}
EF1	0.1	**188**	**188**	–	**188**	**188**	**188**	**188**	190	**188**	**188**	**188**	**188**	**188**	200	**188**
	0.3	**56**	**56**	104	**56**	**56**	**56**	**56**	60	**56**	**56**	**56**	**56**	**56**	70	**56**
	0.5	**29**	38	47	**29**	**29**	**29**	32	31	**29**	**29**	**29**	**29**	**29**	44	**29**
	0.7	**17**	41	36	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	23	**17**
	0.9	**11**	57	18	**11**	33	**11**	13	13	12	**11**	**11**	**11**	**11**	14	**11**
EF2	0.1	**191**	**191**	243	**191**	**191**	**191**	**191**	215	**191**	**191**	**191**	**191**	**191**	215	**191**
	0.3	**57**	**57**	63	**57**	**57**	**57**	**57**	75	**57**	**57**	**57**	**57**	**57**	75	**57**
	0.5	**29**	36	31	**29**	**29**	**29**	**29**	35	33	**29**	**29**	**29**	**29**	39	**29**
	0.7	**17**	36	19	**17**	**17**	**17**	**17**	25	21	**17**	**17**	**17**	**17**	21	**17**
	0.9	11	45	12	10	11	10	10	**9**	**9**	11	11	11	11	**9**	10
EF3	0.1	**192**	**192**	–	**192**	**192**	197	197	206	195	**192**	**192**	**192**	**192**	204	**192**
	0.3	**57**	**57**	61	**57**	**57**	**57**	62	69	62	**57**	**57**	**57**	**57**	66	**57**
	0.5	**30**	41	32	**30**	**30**	**30**	**30**	37	**30**	**30**	**30**	**30**	**30**	33	**30**
	0.7	**17**	44	**17**	18	**17**	20	20	23	21	**17**	**17**	**17**	**17**	20	18
	0.9	33	73	34	28	33	**19**	21	22	22	39	26	42	–	25	21
EF4	0.1	**196**	**196**	240	**196**	**196**	**196**	**196**	220	214	**196**	**196**	**196**	**196**	212	**196**
	0.3	**58**	**58**	112	**58**	**58**	**58**	**58**	66	74	**58**	**58**	**58**	**58**	78	**58**
	0.5	31	38	34	**30**	31	33	**30**	36	36	31	31	31	31	42	33
	0.7	28	45	22	**21**	28	**21**	24	28	28	22	22	22	22	24	**21**
	0.9	24	67	41	21	24	24	24	22	23	23	23	23	23	**20**	21

TABLE 1.6: Number of inner iterations for EV1 with different sizes $n = 2m$.

θ	m	ψ_c	$\psi_{1,\frac{1}{2}}$	ψ_2	$\psi_{3,2}$	$\psi_{3,\frac{\log n}{2}}$	$\psi_{4,1}$	$\psi_{4,2}$	$\psi_{5,2}$	$\psi_{5,\log n}$	ψ_6	$\psi_{6,0.05}$	$\psi_{7,2}$	$\psi_{7,\lceil \log n \rceil}$	ψ_8	ψ_{new}
$\theta = 0.7$	5	**18**	44	–	**18**	**18**	**18**	21	26	32	**18**	**18**	**18**	**18**	35	**18**
	25	**19**	46	–	**19**	**19**	**19**	22	27	36	**19**	**19**	**19**	**19**	36	**19**
	50	**20**	49	–	**20**	**20**	**20**	23	28	39	**20**	**20**	**20**	**20**	37	**20**
	100	**20**	49	–	**20**	**20**	**20**	23	28	46	**20**	**20**	**20**	**20**	37	**20**
	200	**21**	52	–	**21**	**21**	**21**	24	31	62	**21**	**21**	**21**	**21**	40	**21**
	400	**21**	52	–	**21**	**21**	**21**	24	31	49	**21**	**21**	**21**	**21**	40	**21**
	1000	**22**	54	–	**22**	**22**	**22**	25	34	77	**22**	**22**	**22**	**22**	41	**22**
$\theta = 0.9$	5	21	58	–	14	25	13	23	31	26	16	16	15	15	**12**	14
	25	23	63	–	15	15	14	24	32	–	17	17	17	20	**13**	15
	50	23	63	–	15	15	14	25	32	–	17	17	17	22	**13**	15
	100	24	69	–	16	16	16	25	33	–	19	18	18	17	**14**	17
	200	24	69	–	16	15	16	25	33	–	19	18	18	17	**14**	17
	400	24	69	–	16	16	16	25	33	–	19	18	18	18	**14**	17
	1000	25	69	–	18	17	17	26	34	–	20	20	19	20	**15**	18
$\theta = 0.99$	5	21	89	20	29	21	39	40	**15**	32	28	30	37	37	21	28
	25	21	89	20	29	30	39	40	**15**	41	28	30	37	45	21	28
	50	24	104	22	31	29	41	42	**17**	42	31	33	40	57	23	30
	100	24	104	22	31	27	41	42	**17**	42	31	33	40	57	23	30
	200	24	104	22	31	30	41	42	**17**	46	31	33	40	57	23	30
	400	24	104	22	31	24	41	42	**17**	–	31	33	40	55	23	30
	1000	24	104	22	31	31	41	42	**17**	–	31	33	40	54	23	30

TABLE 1.7: Number of inner iterations for EV2 with different sizes $n = 2m$.

θ	m	ψ_c	$\psi_{1,\frac{1}{2}}$	ψ_2	$\psi_{3,2}$	$\psi_{3,\frac{\log n}{2}}$	$\psi_{4,1}$	$\psi_{4,2}$	$\psi_{5,2}$	$\psi_{5,\log n}$	ψ_6	$\psi_{6,0.05}$	$\psi_{7,2}$	$\psi_{7,\lceil\log n\rceil}$	ψ_8	ψ_{new}
	5	**18**	39	54	**18**	**18**	**18**	**18**	34	30	**18**	**18**	**18**	**18**	28	**18**
	25	**19**	41	–	**19**	**19**	**19**	**19**	37	28	**19**	**19**	**19**	**19**	29	**19**
	50	**20**	44	–	**20**	**20**	**20**	**20**	40	45	**20**	**20**	**20**	**20**	32	**20**
$\theta = 0.7$	100	**20**	44	–	**20**	**20**	**20**	**20**	40	32	**20**	**20**	**20**	**20**	32	**20**
	200	**21**	46	–	**21**	**21**	**21**	**21**	41	–	**21**	**21**	**21**	**21**	33	**21**
	400	**21**	46	–	**21**	**21**	**21**	**21**	41	–	**21**	**21**	**21**	**21**	33	**21**
	1000	**22**	48	–	**22**	**22**	**22**	**22**	42	–	**22**	**22**	**22**	**22**	34	**22**
	5	11	40	**9**	10	11	10	10	**9**	**9**	11	11	11	11	**9**	10
	25	12	45	**10**	11	11	11	11	**10**	**10**	12	12	12	12	**10**	11
	50	12	45	**10**	11	11	11	11	**10**	**10**	12	12	12	12	**10**	11
$\theta = 0.9$	100	13	51	**11**	12	12	12	12	**11**	**11**	13	13	13	13	**11**	12
	200	13	51	**11**	12	12	12	12	**11**	**11**	13	13	13	13	**11**	12
	400	13	51	**11**	12	12	12	12	**11**	**11**	13	13	13	13	**11**	12
	1000	15	55	**12**	14	13	13	13	**12**	**12**	14	14	15	15	**12**	14
	5	11	60	**10**	**10**	11	**10**	**10**	**10**	**10**	11	11	11	11	**10**	**10**
	25	11	60	**10**	**10**	**10**	**10**	**10**	**10**	**10**	11	11	11	11	**10**	**10**
	50	13	73	**12**	13	13	13	13	**12**	**12**	13	13	13	13	**12**	13
$\theta = 0.99$	100	13	73	**12**	13	13	13	13	**12**	**12**	13	13	13	13	**12**	13
	200	13	73	**12**	13	13	13	13	**12**	**12**	13	13	13	13	**12**	13
	400	13	73	**12**	13	13	13	13	**12**	**12**	13	13	13	13	**12**	13
	1000	13	73	**12**	13	13	13	13	**12**	**12**	13	13	13	13	**12**	13

TABLE 1.8: Number of inner iterations for EV3 with different sizes $n = 2m$.

θ	m	ψ_c	$\psi_{1,\frac{1}{2}}$	ψ_2	$\psi_{3,2}$	$\psi_{3,\frac{\log n}{2}}$	$\psi_{4,1}$	$\psi_{4,2}$	$\psi_{5,2}$	$\psi_{5,\log n}$	ψ_6	$\psi_{6,0.05}$	$\psi_{7,2}$	$\psi_{7,\lceil \log n \rceil}$	ψ_8	ψ_{new}
$\theta = 0.7$	5	**18**	42	–	**18**	**18**	**18**	**18**	–	–	**18**	**18**	**18**	**18**	–	**18**
	25	**19**	44	–	**19**	**19**	**19**	**19**	–	–	**19**	**19**	**19**	**19**	–	**19**
	50	**20**	47	–	**20**	**20**	**20**	**20**	–	–	**20**	**20**	**20**	**20**	–	**20**
	100	**20**	47	–	**20**	**20**	**20**	**20**	–	–	**20**	**20**	**20**	**20**	–	**20**
	200	**21**	49	–	**21**	**21**	**21**	**21**	–	–	**21**	**21**	**21**	**21**	–	**21**
	400	**21**	49	–	**21**	**21**	**21**	**21**	–	–	**21**	**21**	**21**	**21**	–	**21**
	1000	**22**	52	–	**22**	**22**	**22**	**22**	–	–	**22**	**22**	**22**	**22**	–	**22**
$\theta = 0.9$	5	11	44	**9**	10	11	10	10	**9**	**9**	11	11	11	11	**9**	10
	25	12	49	**10**	11	11	11	11	**10**	**10**	12	12	12	12	**10**	11
	50	12	49	**10**	11	11	11	11	**10**	**10**	12	12	12	12	**10**	11
	100	13	54	**11**	12	12	12	12	**11**	**11**	13	13	13	13	**11**	12
	200	13	54	**11**	12	12	12	12	**11**	**11**	13	13	13	13	**11**	12
	400	13	54	**11**	12	12	12	12	**11**	**11**	13	13	13	13	**11**	12
	1000	15	60	**12**	14	13	13	13	**12**	14	15	15	15	15	**12**	14
$\theta = 0.99$	5	11	64	**10**	**10**	11	**10**	**10**	**10**	**10**	11	11	11	11	**10**	**10**
	25	11	64	**10**	**10**	**10**	**10**	**10**	**10**	**10**	11	11	11	11	**10**	**10**
	50	13	80	**12**	13	13	13	13	**12**	**12**	13	13	13	13	**12**	13
	100	13	80	**12**	13	13	13	13	**12**	**12**	13	13	13	13	**12**	13
	200	13	80	**12**	13	13	13	13	**12**	**12**	13	13	13	13	**12**	13
	400	13	80	**12**	13	13	13	13	**12**	14	13	13	13	13	**12**	13
	1000	13	80	**12**	13	13	13	13	**12**	14	13	13	13	13	**12**	13

TABLE 1.9: Number of inner iterations for EV4 with different sizes $n = 2m$.

θ	m	ψ_c	$\psi_{1,\frac{1}{2}}$	ψ_2	$\psi_{3,2}$	$\psi_{3,\frac{\log n}{2}}$	$\psi_{4,1}$	$\psi_{4,2}$	$\psi_{5,2}$	$\psi_{5,\log n}$	ψ_6	$\psi_{6,0.05}$	$\psi_{7,2}$	$\psi_{7,\lceil\log n\rceil}$	ψ_8	ψ_{new}
$\theta = 0.7$	5	**18**	42	50	**18**	**18**	**18**	**18**	27	29	**18**	**18**	**18**	**18**	28	**18**
	25	**19**	45	53	**19**	**19**	**19**	**19**	28	31	**19**	**19**	**19**	**19**	31	**19**
	50	**20**	47	56	**20**	**20**	**20**	**20**	29	52	**20**	**20**	**20**	**20**	32	**20**
	100	**20**	47	56	**20**	**20**	**20**	**20**	29	59	**20**	**20**	**20**	**20**	32	**20**
	200	**21**	50	59	**21**	**21**	**21**	**21**	30	–	**21**	**21**	**21**	**21**	33	**21**
	400	**21**	50	59	**21**	**21**	**21**	**21**	30	–	**21**	**21**	**21**	**21**	33	**21**
	1000	**22**	52	62	**22**	**22**	**22**	**22**	31	–	**22**	**22**	**22**	**22**	36	**22**
$\theta = 0.9$	5	**12**	56	32	**12**	**12**	13	16	16	16	**12**	**12**	**12**	**12**	**12**	**12**
	25	14	61	33	**13**	**13**	14	17	17	49	14	14	14	14	**13**	**13**
	50	14	61	33	**13**	**13**	14	17	17	25	14	14	14	15	**13**	**13**
	100	15	67	34	**14**	**14**	15	19	18	50	15	15	15	16	**14**	**14**
	200	15	67	34	**14**	**14**	15	19	18	92	15	15	15	16	**14**	**14**
	400	15	67	34	**14**	**14**	15	19	18	32	15	15	15	16	**14**	**14**
	1000	16	72	35	**15**	16	16	20	19	29	16	16	16	18	**15**	**15**
$\theta = 0.99$	5	21	82	23	26	22	28	36	28	33	24	24	27	27	**19**	23
	25	21	82	23	26	26	28	36	28	31	24	24	27	32	**19**	23
	50	23	96	25	28	26	30	38	30	36	26	26	29	40	**21**	26
	100	23	96	25	28	22	30	38	30	100	26	26	29	40	**21**	26
	200	23	96	25	28	23	30	38	30	38	26	26	29	40	**21**	26
	400	23	96	25	28	24	30	38	30	45	26	26	29	33	**21**	26
	1000	23	96	25	28	28	30	38	30	43	26	26	29	34	**21**	26

Comments

Recall that the numerical results were obtained by performing Algorithm 1 with the KFs defined in Table 1.4 (except $\psi_{\text{new,p}}$) on eight test problems.
When there are parameters $p \geq 1$ involved in the definition of a KF, we used two values of these parameters: the parameter that gives the best theoretical iteration bound and a common value $p = 2$, except for $\psi_{1,p}$ and $\psi_{6,\lambda}$ where we chose $p = \frac{1}{2}$ and $\lambda = 0.05$ since the parameters p and λ take their values in the intervals $]0, 1]$ and $]0, \frac{8}{25\pi}]$ respectively. This left us with 15 different KFs.

For each example, we used **bold** font to highlight the best, i.e., the smallest, iteration number.

From Tables 1.5-1.9, we may draw a few conclusions:

- For $\psi_{1,p}$ (the classical logarithmic barrier function ψ_c occurs if $p = 1$), although the theoretical iteration bound of the algorithm is independent of the parameter p, numerical tests show that p influences the iteration count. In both types of problems (fixed or variable size), $p = 1$ gives better results which is in accordance with the analysis carried out by [32].

- The function $\psi_{1,\frac{1}{2}}$ never gives the smallest iteration number in examples with variable size, even for examples with fixed size it gives the smallest iteration number only for the values 0.1 and 0.3 of θ.

- For KFs ψ_2, $\psi_{5,2}$, $\psi_{5,\log n}$, $\psi_{7,[\log n]}$ and ψ_8, the dashes in the corresponding columns of Tables 1.5-1.9 indicate that the algorithms require more than 10^4 iterations to obtain an optimal solution. Despite this, ψ_2, $\psi_{5,2}$, $\psi_{5,\log n}$ and ψ_8, (and exclusively ψ_8) are the only ones to give the smallest iteration number for $\theta = 0.9$ in EV2 and EV3 (in EV1), while for $\theta = 0.99$ in EV1 (resp. in EV4), $\psi_{5,2}$ (resp. ψ_8) is the only function to achieve the best iteration number.

- The iteration numbers of the algorithm based on our KF ψ_{new} depend on the values of the parameter θ. In fact, the value 0.9 of θ gives better iteration numbers in general.

- In all examples with variable size for $\theta = 0.7$, the KF ψ_{new} has the smallest iterations number with the KFs ψ_c, $\psi_{3,p}$, $\psi_{4,p}$, ψ_6, $\psi_{6,0.05}$, $\psi_{7,2}$, $\psi_{7,[\log n]}$ except $\psi_{4,2}$ in Example 5.

- For our KF, the obtained iteration numbers coincided with, or at worst was close to the best ones with a slackness of at most 4 iterations, except the case $\theta = 0.99$ in EV1, where the slackness attain thirty.

To confirm the superiority of our algorithm in terms of the total number of iterations, we compute, for each KF the percentage of cases where the KF gives the best iteration number. As an illustration, we plot a histogram that we use as a statistical tool to compare the performance of the algorithms.

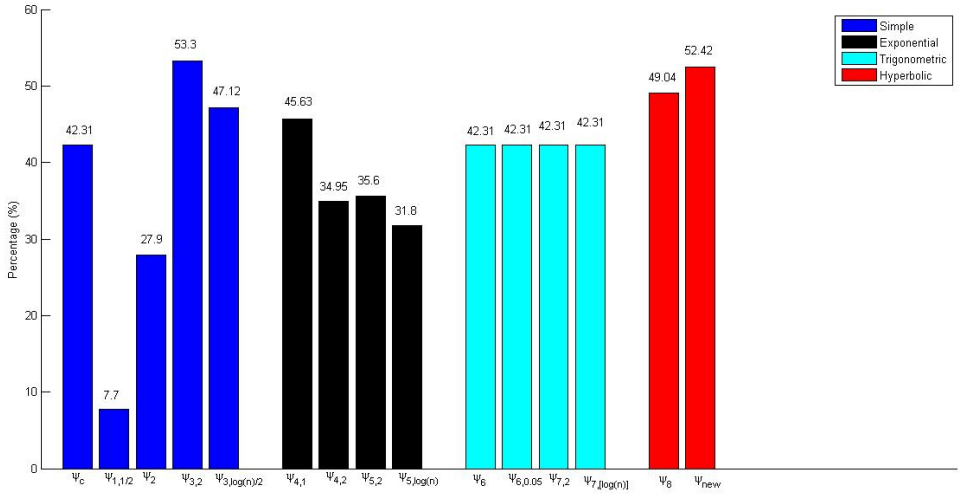

FIGURE 1.4: Performance comparison between KFs in Table 1.4

Supported by the performance bar graph, We may conclude some remarks:

- For $\psi_{4,p}$, the parameter which gives the best theoretical complexity bound has better percentage than $p = 2$. But the concordance between the theoretical and the numerical results is not always satisfied, as we can see in the case of $\psi_{3,p}$, $\psi_{5,p}$ and $\psi_{7,p}$.
- We can easily see that the algorithms based on hyperbolic KFs attain the most wins, on average, among the considered algorithms based on other types.
- Although the theoretical complexity obtained for ψ_8 is better than the one of ψ_{new}, but numerical tests reveal that ψ_{new} has better percentage. In fact, a thorough analysis shows that in the examples with fixed size, it's ψ_{new} that gives the smallest iteration number with a slackness which can amount up to 24 iterations. As for examples with variable size for $\theta = 0.7$, ψ_{new} performs better than ψ_8 in EV1, EV2 and EV4 while in EV3, ψ_8 doesn't have the ability to complete the run successfully. In contrast, for the values 0.9 and 0.99 of θ, ψ_8 meets or exceeds ψ_{new} with a slackness of no more than 7 iterations.

The numerical effectiveness of ψ_{new} comparing with all KFs with logarithmic barrier term motivates us to propose a generalization of this function in order to achieve a better theoretical complexity.

1.4.2 The analysis of the interior-point method based on the parametric kernel function

In the previous section, we performed a well-detailed complexity analysis for ψ_{new}. Thus, we discuss more briefly the analysis for the generalized KF $\psi_{\text{new},p}$.

$$\psi_{\text{new},p}(t) = \frac{t^2-1}{2} + \psi_{b,p}(t),\ t > 0,\ p \geq 2,$$

with $\psi_{b,p}(t) = a_p\Big(\coth^p(t) - \log t - \coth^p(1)\Big)$ and $a_p = \dfrac{\sinh^2(1)}{\sinh^2(1) + p\coth^{p-1}(1)}$.

Remark 1.4.7. *This function can also be considered as a generalization, up to the multiplicative constant* $1/a$*, of the KF* ψ_8 *introduced in [100] (see Table 1.4).*

Using simple calculations, we can easily prove that $\psi_{\text{new},p}$ is indeed a KF and we have:

TABLE 1.10: The first three derivatives of $\psi_{\text{new},p}$ with $t\psi''_{\text{new},p} \pm \psi'_{\text{new},p}$

$\psi'_{\text{new},p}(t) = t - a_p\Big(\frac{p\coth^{p-1}(t)}{\sinh^2(t)} + \frac{1}{t}\Big)$
$\psi''_{\text{new},p}(t) = 1 + a_p\Big(\frac{p(p-1)\coth^{p-2}(t)}{\sinh^4(t)} + \frac{2p\coth^{p}(t)}{\sinh^2(t)} + \frac{1}{t^2}\Big) \geq 1$
$\psi'''_{\text{new},p}(t) = -a_p\Big(\frac{p(p-1)(p-2)\coth^{p-3}(t)}{\sinh^6(t)} + \frac{(6p^2-4p)\coth^{p-1}(t)}{\sinh^4(t)} + \frac{4p\coth^{p+1}(t)}{\sinh^2(t)} + \frac{2}{t^3}\Big) < 0$
$t\psi''_{\text{new},p} - \psi'_{\text{new},p} = a_p\Big(\frac{p(p-1)t\coth^{p-2}(t)}{\sinh^4(t)} + \frac{2pt\coth^{p}(t)}{\sinh^2(t)} + \frac{2}{t} + \frac{p\coth^{p-1}(t)}{\sinh^2(t)}\Big) > 0$
$t\psi''_{\text{new},p} + \psi'_{\text{new},p} = 2t + a_p\Big(\frac{p(p-1)t\coth^{p-2}(t)}{\sinh^4(t)} + \frac{p\coth^{p-1}(t)}{\sinh^2(t)}\big(2t\coth(t)-1\big)\Big) > 0$

Thus, $\psi_{\text{new},p}$ belongs to the class the defined in Section 1.2. Whereas, the equivalent of Lemma 1.4.2 for $\psi_{\text{new},p}$ is

Lemma 1.4.8. *For all* $(z,t) \in [0,+\infty) \times (0,1]$ *such that* $z = -\frac{1}{2}\psi'_{new,p}(t)$*, we have*

$$\coth(t) \leq \Big(\big(\sinh^2(1) + \coth^{p-1}(1)\big)\coth^2(1)\Big)^{\frac{1}{p+1}} (2z+1)^{\frac{1}{p+1}}.$$

Complexity analysis

The complexity analysis for the parameterized KF $\psi_{\text{new},p}$ proceeds in the same way as in the previous section for ψ_{new}. To avoid repetition, we do not present all the details of the computations. We only present the outcome of each step of our computational scheme in Table 1.11.

TABLE 1.11: Outcomes of the computational scheme.

$f(\bar{\alpha}) \leq -\dfrac{\sqrt{2}\Psi(v)^{\frac{p}{2(p+1)}}}{72\left(1+2\sinh^2(1)\coth^5(1)(p+5)\right)}$	$\Psi(v_+) \leq \dfrac{\coth(1)(p+2)}{(1-\theta)}\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2 := \Psi_0$
$\beta = \dfrac{\sqrt{2}}{72\left(1+2\sinh^2(1)\coth^5(1)(p+5)\right)}$ and $\gamma = \dfrac{p+2}{2(p+1)}$	$K \leq \left\lceil \dfrac{72\sqrt{2}(5+4p)(p+1)}{(p+2)} \right\rceil \Psi_0^{\frac{p+2}{2(p+1)}}$

For small-update methods with $\tau = \mathcal{O}(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$, the complexity of the primal-dual IPA for LO problems based on the new parametric KF is $\mathcal{O}(p^2\sqrt{n}\log\frac{n}{\epsilon})$ iterations complexity.

As for large-update methods i.e., $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, the substitution of these values into

$$K \leq \left\lceil \frac{72\sqrt{2}(5+4p)(p+1)}{(p+2)} \right\rceil \Psi_0^{\frac{p+2}{2(p+1)}},$$

does not give the best possible bound. A better bound is obtained using Ψ_0 defined in Lemma 1.2.10. As a consequence, we get $\mathcal{O}\left((p+1)n^{\frac{p+2}{2(p+1)}}\log\frac{n}{\epsilon}\right)$ iterations complexity for large-update methods. This expression is minimal at $p=\frac{\log n}{2}-1$ and then is equal to $\mathcal{O}\left(\sqrt{n}\log n\log\frac{n}{\epsilon}\right)$.

Numerical tests

Now, we would like to investigate the influence of parameterizing the KF (1.31) on the computational behavior of the generic primal-dual algorithm for LO presented in Algorithm 1. For this purpose, we present some numerical simulations for implementing the algorithm based on the new parametric KF (1.32) on the same set of problems and for different values of the parameter p. We also compare with the KF that gave the best performance in the previous section $\psi_{3,2}$ and with ψ_8 which is of the same hyperbolic-logarithmic type. The obtained results are listed in tables below.

TABLE 1.12: Number of inner iterations for fixed size examples.

Example	θ	ψ_{new}	$\psi_{\text{new},2}$	$\psi_{\text{new},3}$	$\psi_{\text{new},4}$	$\psi_{\text{new},\frac{\log n}{2}-1}$	$\psi_{3,2}$	ψ_8
EF1	0.1	**188**	**188**	**188**	**188**	**188**	**188**	200
	0.3	**56**	**56**	**56**	**56**	**56**	**56**	70
	0.5	**29**	**29**	**29**	**29**	**29**	**29**	44
	0.7	**17**	**17**	**17**	**17**	**17**	**17**	23
	0.9	11	11	**10**	12	24	11	14
EF2	0.1	**191**	**191**	**191**	**191**	**191**	**191**	215
	0.3	**57**	**57**	**57**	**57**	**57**	**57**	75
	0.5	**29**	**29**	**29**	**29**	**29**	**29**	39
	0.7	**17**	**17**	**17**	**17**	**17**	**17**	21
	0.9	10	**9**	**9**	**9**	11	10	**9**
EF3	0.1	**192**	**192**	**192**	**192**	**192**	**192**	204
	0.3	**57**	**57**	**57**	**57**	**57**	**57**	66
	0.5	**30**	**30**	**30**	**30**	**30**	**30**	33
	0.7	18	19	18	19	**17**	18	20
	0.9	21	**19**	**19**	22	33	28	25
EF4	0.1	**196**	**196**	**196**	**196**	**196**	**196**	212
	0.3	**58**	**58**	**58**	**58**	**58**	**58**	78
	0.5	33	**30**	**30**	**30**	31	**30**	42
	0.7	21	**20**	22	24	28	21	24
	0.9	21	**18**	22	20	24	21	20

TABLE 1.13: Number of inner iterations for EV1 and 2 with different sizes $n = 2m$.

θ	m	EV1							EV2						
		ψ_{new}	$\psi_{\text{new},2}$	$\psi_{\text{new},3}$	$\psi_{\text{new},4}$	$\psi_{\text{new},\frac{\log n}{2}-1}$	$\psi_{3,2}$	ψ_8	ψ_{new}	$\psi_{\text{new},2}$	$\psi_{\text{new},3}$	$\psi_{\text{new},4}$	$\psi_{\text{new},\frac{\log n}{2}-1}$	$\psi_{3,2}$	ψ_8
$\theta = 0.7$	5	**18**	**18**	21	22	**18**	**18**	35	**18**	**18**	**18**	**18**	**18**	**18**	28
	25	**19**	**19**	22	23	**19**	**19**	36	**19**	**19**	**19**	**19**	**19**	**19**	29
	50	**20**	**20**	23	24	**20**	**20**	37	**20**	**20**	**20**	**20**	**20**	**20**	32
	100	**20**	**20**	23	24	**20**	**20**	37	**20**	**20**	**20**	**20**	**20**	**20**	32
	200	**21**	**21**	24	25	**21**	**21**	40	**21**	**21**	**21**	**21**	**21**	**21**	33
	400	**21**	**21**	24	25	**21**	**21**	40	**21**	**21**	**21**	**21**	**21**	**21**	33
	1000	**22**	**22**	25	26	**22**	**22**	41	**22**	**22**	**22**	**22**	**22**	**22**	34
$\theta = 0.9$	5	14	14	16	16	22	14	**12**	10	**9**	**9**	**9**	11	10	**9**
	25	15	15	17	17	15	15	**13**	11	11	**10**	**10**	11	11	**10**
	50	15	16	17	17	16	15	**13**	11	11	**10**	**10**	11	11	**10**
	100	17	16	18	18	15	16	**14**	12	12	**11**	**11**	12	12	**11**
	200	17	16	18	18	16	16	**14**	12	12	**11**	**11**	12	12	**11**
	400	17	16	18	18	15	16	**14**	12	12	**11**	**11**	12	12	**11**
	1000	18	17	19	19	16	18	**15**	14	13	**12**	**12**	13	14	**12**
$\theta = 0.99$	5	28	24	29	30	**21**	29	**21**	**10**	**10**	**10**	**10**	11	**10**	**10**
	25	28	24	29	30	28	29	**21**	**10**	**10**	**10**	**10**	**10**	**10**	**10**
	50	30	27	31	32	38	31	**23**	13	**12**	**12**	**12**	**12**	13	**12**
	100	30	27	31	32	31	31	**23**	13	**12**	**12**	**12**	**12**	13	**12**
	200	30	27	31	32	27	31	**23**	13	**12**	**12**	**12**	**12**	13	**12**
	400	30	27	31	32	25	31	**23**	13	**12**	**12**	**12**	**12**	13	**12**
	1000	30	27	31	32	31	31	**23**	13	**12**	**12**	**12**	**12**	13	**12**

TABLE 1.14: Number of inner iterations for EV3 and 4 with different sizes $n = 2m$.

θ	m	EV3							EV4						
		ψ_{new}	$\psi_{\text{new},2}$	$\psi_{\text{new},3}$	$\psi_{\text{new},4}$	$\psi_{\text{new},\frac{\log n}{2}-1}$	$\psi_{3,2}$	ψ_8	ψ_{new}	$\psi_{\text{new},2}$	$\psi_{\text{new},3}$	$\psi_{\text{new},4}$	$\psi_{\text{new},\frac{\log n}{2}-1}$	$\psi_{3,2}$	ψ_8
$\theta = 0.7$	5	**18**	**18**	**18**	**18**	**18**	**18**	–	**18**	**18**	**18**	**18**	**18**	**18**	28
	25	**19**	**19**	**19**	**19**	**19**	**19**	–	**19**	**19**	**19**	**19**	**19**	**19**	31
	50	**20**	**20**	**20**	**20**	**20**	**20**	–	**20**	**20**	**20**	**20**	**20**	**20**	32
	100	**20**	**20**	**20**	**20**	**20**	**20**	–	**20**	**20**	**20**	**20**	**20**	**20**	32
	200	**21**	**21**	**21**	**21**	**21**	**21**	–	**21**	**21**	**21**	**21**	**21**	**21**	33
	400	**21**	**21**	**21**	**21**	**21**	**21**	–	**21**	**21**	**21**	**21**	**21**	**21**	33
	1000	**22**	**22**	**22**	**22**	**22**	**22**	–	**22**	**22**	**22**	**22**	**22**	**22**	36
$\theta = 0.9$	5	10	**9**	**9**	**9**	11	10	**9**	**12**	**12**	**12**	**12**	**12**	**12**	**12**
	25	11	11	**10**	**10**	11	11	**10**	**13**	**13**	**13**	**13**	**13**	**13**	**13**
	50	11	11	**10**	**10**	11	11	**10**	**13**	**13**	**13**	**13**	**13**	**13**	**13**
	100	12	12	**11**	**11**	12	12	**11**	**14**	**14**	**14**	**14**	**14**	**14**	**14**
	200	12	12	**11**	**11**	12	12	**11**	**14**	**14**	**14**	**14**	**14**	**14**	**14**
	400	12	12	**11**	**11**	12	12	**11**	**14**	**14**	15	15	15	**14**	**14**
	1000	14	13	**12**	**12**	13	14	**12**	**15**	**15**	16	16	16	**15**	**15**
$\theta = 0.99$	5	**10**	**10**	**10**	**10**	11	**10**	**10**	23	21	24	22	21	26	**19**
	25	**10**	**10**	**10**	**10**	**10**	**10**	**10**	23	21	24	22	26	26	**19**
	50	13	**12**	**12**	**12**	**12**	13	**12**	26	23	26	24	27	28	**21**
	100	13	**12**	**12**	**12**	**12**	13	**12**	26	23	26	24	25	28	**21**
	200	13	**12**	**12**	**12**	**12**	13	**12**	26	23	26	24	23	28	**21**
	400	13	**12**	**12**	**12**	**12**	13	**12**	26	23	26	24	24	28	**21**
	1000	13	**12**	**12**	**12**	**12**	13	**12**	26	23	26	24	82	28	**21**

TABLE 1.15: The percentage of cases where the KF gives the best iteration number.

ψ_i	ψ_{new}	$\psi_{\text{new},2}$	$\psi_{\text{new},3}$	$\psi_{\text{new},4}$	$\psi_{\text{new},\frac{\log n}{2}-1}$	$\psi_{3,2}$	ψ_8
%	50.48	66.99	68.93	66.99	58.25	51.45	55.33

Comments

By comparing the results in Tables 1.12-1.15, we notice that:

- The number of iterations clearly depends on the value of the parameter p as the gap between two different values of p can amount up to 59. It should also be noted that $\psi_{\text{new},3}$ significantly reduces the number of iterations although $\psi_{\text{new},\frac{\log n}{2}-1}$ has the best complexity bound theoretically.
- Comparing with ψ_{new}, the KFs $\psi_{\text{new},2}, \psi_{\text{new},3}, \psi_{\text{new},4}$ and $\psi_{\text{new},\frac{\log n}{2}-1}$ were able to produce even better iteration numbers especially in EV2 and EV3 for $\theta = 0.9$ and $\theta = 0.99$, while maintaining similar performance for $\theta = 0.7$ in Examples EV2, EV3 and EV4.
- $\psi_{\text{new},p}$, for all tested parameters, outperformed $\psi_{3,2}$ which had the best performance in the previous section. This confirms that the parametrization has effected the number of iterations of the algorithm positively.
- $\psi_{\text{new},2}$ far outperformed ψ_8 despite having the same complexity bounds and more than, the same expression up to a multiplicative constant (see Remark 1.4.7).

1.5 A primal dual interior-point algorithm based on a parameterized kernel function with a pure hyperbolic barrier term

In this section, we study a primal-dual IPA for solving linear programming problems based on the following parameterized hyperbolic KF

$$\psi(t) := \psi_p(t) = \frac{t^2-1}{2} + \frac{a_p}{p}\coth^p(t) - \frac{\sinh^2(1)}{p}\coth(1), \ \forall t > 0, \tag{1.36}$$

where $a_p = \dfrac{\sinh^2(1)}{\coth^{p-1}(1)}$ and $p \geq 2$. The parameter p is called the barrier degree. This function is a generalization, up to a multiplicative constant, of the newly introduced KF [98]

$$\phi(t) = \frac{t^2-1}{2\sinh^2(1)} + \coth(t) - \coth(1), \ \forall t > 0.$$

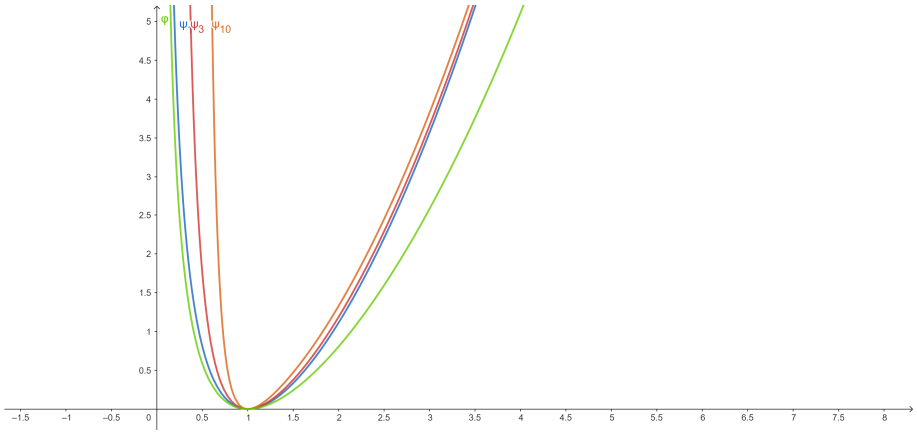

FIGURE 1.5: Graphs of $\psi_1, \psi_3, \psi_{10}$ and ϕ

We prove that the KF (1.36) belongs to the eligible class of KFs introduced by Bai et al. in [8] and hence it belongs to the class defined in Section 1.2. We derive the complexity analysis for algorithms with large-update primal-dual IPMs. We show that by choosing a specific parameter, the corresponding algorithm enjoys the currently best known iteration bound. Furthermore, we prove the practical efficiency of the new algorithm by presenting some numerical results. The results of this section are submitted for future publication. See [41] for a prepublication of these results.

1.5.1 Some technical lemmas

We first give the first three derivatives of ψ defined in (1.36) for all $t > 0$.

$$\psi'(t) = t - a_p \frac{\coth^{p-1}(t)}{\sinh^2(t)}, \tag{1.37}$$

$$\psi''(t) = 1 + a_p \left(2\frac{\coth^{p}(t)}{\sinh^2(t)} + (p-1)\frac{\coth^{p-2}(t)}{\sinh^4(t)} \right) > 1, \tag{1.38}$$

and

$$\psi'''(t) = -a_p \left(4\frac{\coth^{p+1}(t)}{\sinh^2(t)} + (6p-4)\frac{\coth^{p-1}(t)}{\sinh^4(t)} + (p-1)(p-2)\frac{\coth^{p-3}(t)}{\sinh^6(t)} \right) < 0. \tag{1.39}$$

We easily verify that ψ is definitely a KF as it satisfies the following conditions: $\psi'(1) = \psi(1) = 0$ and $\lim_{t\to 0^+} \psi(t) = \lim_{t\to+\infty} \psi(t) = +\infty$.

Furthermore, we can write $\psi(t)$ as

$$\psi(t) = \frac{t^2-1}{2} + \psi_b(t),$$

where

$$\psi_b(t) = \frac{a_p}{p}\coth^p(t) - \frac{\sinh^2(1)}{p}\coth(1),$$

with $\psi_b'(t) < 0$ and $\psi_b''(t) > 0$ for all $t > 0$.

The following lemma confirms the e-convexity and the eligibility of ψ.

Lemma 1.5.1. *Let ψ be as defined in (1.36). Then, we have*

(i) $t\psi''(t) - \psi'(t) > 0,\ \forall t > 0.$

(ii) $t\psi''(t) + \psi'(t) > 0,\ \forall t > 0.$

(iii) $2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0, \forall t < 1.$

Proof. From (1.37) and (1.38), it follows that

$$t\psi''(t) - \psi'(t) = a_p\left((p-1)t\frac{\coth^{p-2}(t)}{\sinh^4(t)} + \frac{\coth^{p-1}(t)}{\sinh^2(t)}(2t\coth(t)+1)\right),$$

and

$$t\psi''(t) + \psi'(t) = 2t + a_p\left((p-1)t\frac{\coth^{p-2}(t)}{\sinh^4(t)} + \frac{\coth^{p-1}(t)}{\sinh^2(t)}(2t\coth(t)-1)\right).$$

The first item is obviously strictly positive for all $t > 0$. As for the second item, it's obtained using the second item of Lemma 1.3.1.
Lastly, using (1.37), (1.38) and (1.39), simple calculation leads to

$$\begin{aligned}2\psi''(t)^2 - \psi'(t)\psi'''(t) = 2 + a_p t + 4a_p^2\frac{\coth^{2p}(t)}{\sinh^4(t)} + 8a_p\frac{\coth^{p}(t)}{\sinh^2(t)} + 4a_p(p-1)\frac{\coth^{p-2}(t)}{\sinh^4(t)}\\ + a_p^2 p(p-1)\frac{\coth^{2p-4}(t)}{\sinh^8(t)} + 2a_p^2(p-2)\frac{\coth^{2p-2}(t)}{\sinh^6(t)} > 0,\end{aligned}$$

since $a_p > 0$ and $p \geq 2$. □

Remark 1.5.2. *It is worth noting that any KF that satisfies **H1** and the three conditions of Lemma 2.4.1 is an eligible KF according to [8, p.109]. Thus, ψ is eligible.*

After that, we give in the following lemma an implicit lower bound for ρ defined in Definition 1.1.9.

Lemma 1.5.3. *For all $(z,t) \in [0,+\infty[\times]0,1]$ such that $z = -\frac{1}{2}\psi'(t)$, we have*

$$\coth(t) \leq \coth(1)\,(2z+1)^{\frac{1}{p+1}}.$$

Proof.
let $z \geq 0$ and $t \in (0,1]$ such that $z = -\frac{1}{2}\psi'(t)$, then $\rho(z) = t$. Using (1.37), we have

$$\begin{aligned}2z &= -\psi'(t)\\ &= -t + a_p\frac{\coth^{p-1}(t)}{\sinh^2(t)},\\ &= -t + a_p\coth^{p-1}(t)(\coth^2(t)-1).\end{aligned}$$

Moreover, since

$$1 < \cosh(t) \leq \cosh(1),\ \ \forall t \in (0,1] \ \text{ and } \ \coth^2(t) - 1 = \frac{\coth^2(t)}{\cosh^2(t)},$$

we obtain

$$\coth^2(t) - 1 \geq \frac{1}{\cosh^2(1)} \coth^2(t), \quad \forall t \in]0,1].$$

This implies that

$$2z + 1 \geq 2z + t \geq \frac{1}{\coth^{p+1}(1)} \coth^{p+1}(t).$$

□

1.5.2 Computation of displacement step

In this section, we compute a default step size $\bar{\alpha}$ such that (x_+, y_+, s_+) defined in Algorithm 1 are strictly feasible and the proximity function (1.10) decreases sufficiently. We recall that during an inner iteration the parameter μ is fixed and from (1.5) we have

$$x_+ = \frac{x}{v}(v + \bar{\alpha} d_x), s_+ = \frac{s}{v}(v + \bar{\alpha} d_s), v_+ = \sqrt{\frac{x_+ s_+}{\mu}}.$$

The following theorem gives an upper bound for the decreasing value of the proximity gap function.

Theorem 1.5.4. *If $\bar{\alpha}$ is the default step size and $\sigma \geq 1$, then we have*

$$\Psi(v_+) - \Psi(v) \leq -\frac{\sqrt{2}\Psi(v)^{\frac{p}{2(p+1)}}}{72\,(1+8p)}. \tag{1.40}$$

Proof. From (1.37) , we have

$$\begin{aligned}
\psi''(t) = \; & 1 + a_p \left(2\frac{\coth^p(t)}{\sinh^2(t)} + (p-1)\frac{\coth^{p-2}(t)}{\sinh^4(t)} \right) \\
& = 1 + a_p \Big(2\coth^p(t)(\coth^2(t) - 1) + (p-1)\coth^{p-2}(t)(\coth^2(t) - 1)^2 \Big) \\
& = 1 + a_p \Big(2\coth^{p+2}(t) - 2\coth^p(t) + (p-1)\coth^{p+2}(t) \\
& \quad + (p-1)\coth^{p-2}(t) - 2(p-1)\coth^p(t) \Big) \\
& \leq 1 + 2a_p p \big(\coth^{p+2}(t) \big).
\end{aligned}$$

Let $t = \rho(2\sigma)$. Lemma 1.5.3 implies that

$$\begin{aligned}
\psi''(\rho(2\sigma)) &\leq 1 + 2\sinh^2(1)\coth^3(1) p (4\sigma + 1)^{\frac{p+2}{p+1}} \\
&\leq \Big(1 + 2\sinh^2(1)\coth^3(1) p \Big) (4\sigma + 1)^{\frac{p+2}{p+1}}.
\end{aligned}$$

On the other hand, from Theorem 1.2.18

$$\Psi(v_+) - \Psi(v) \leq -\frac{\sigma^2}{\psi''\left(\rho\left(2\sigma\right)\right)}.$$

It follows that

$$\begin{aligned}
\Psi(v_+) - \Psi(v) &\leq -\frac{\sigma^2}{\left(1+2\sinh^2(1)\coth^3(1)p\right)\left(4\sigma+1\right)^{\frac{p+2}{p+1}}} \\
&\leq -\frac{\sigma^2}{\left(1+2\sinh^2(1)\coth^3(1)p\right)\left(4\sigma+2\sigma\right)^{\frac{p+2}{p+1}}} \\
&\leq -\frac{\sigma^{\frac{p}{p+1}}}{36(1+8p)} \\
&\leq -\frac{\sqrt{2}\Psi(v)^{\frac{p}{2(p+1)}}}{72\left(1+8p\right)},
\end{aligned}$$

where the last inequality is obtained using Lemma 1.2.12. □

1.5.3 Iteration bound of the algorithm

We conclude this subsection by giving an upper bound for the number of inner iterations needed to return to the τ-neighbourhood, i.e. $\Psi\left(v\right) \leq \tau$ after μ-update. We set Ψ_0, as the value of $\Psi\left(v\right)$ after μ-update, whereas the subsequent values in the same outer iteration are denoted as $\Psi_k, k = 1, ..., K$, with K the total number of inner iterations in the outer iteration that can be taken.

Recall that the decrease on each inner iteration is given by (1.40), that is,

$$\Psi_{k+1} \leq \Psi_k - \frac{\sqrt{2}\Psi_k^{\frac{p}{2(p+1)}}}{72\left(1+8p\right)}.$$

Consequently, using Lemma A.0.13 for the values $t_k = \Psi_k$, $\beta = \dfrac{\sqrt{2}}{72(1+8p)}$ and $\gamma = \dfrac{p+2}{2(p+1)}$, we get the following lemma.

Lemma 1.5.5. *Let K be the total number of inner iterations in the outer iteration. Then we have*

$$K \leq \left\lceil \frac{72\sqrt{2}(1+8p)(p+1)}{(p+2)} \right\rceil \Psi_0^{\frac{p+2}{2(p+1)}},$$

where Ψ_0 is the value of $\Psi\left(v\right)$ after the $\mu-$update in outer iteration.

Using Lemma 1.5.5 and Corollary 1.3.8, we derive an upper bound for the total number of iterations.

Theorem 1.5.6. *Let Ψ_0 be the value defined in Lemma 1.2.10 and let $\tau \geq 1$. Then, the total number of iterations to obtain an approximate solution with $n\mu \leq \epsilon$ is bounded by*

$$\left[\frac{72\sqrt{2}(1+8p)(p+1)}{(p+2)}\right]\Psi_0^{\frac{p+2}{2(p+1)}}\frac{\log\frac{n}{\epsilon}}{\theta}. \tag{1.41}$$

For large-update methods with $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, the complexity of the primal-dual IPA for LO problems based on the new KF is $\mathcal{O}\left(pn^{\frac{p+2}{2(p+1)}}\log\frac{n}{\epsilon}\right)$ iterations complexity.

An interesting choice is $p = \frac{\log n}{2} - 1$, which minimizes the iteration bound. Then, it becomes $\mathcal{O}\left(\sqrt{n}\log n\log\frac{n}{\epsilon}\right)$ iterations complexity.

As for small-update methods i.e., $\tau = \mathcal{O}(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$, the substitution of these values into (1.41) does not give the best possible bound. A better bound is obtained using the following Corollary

Corollary 1.5.7. *Let θ be such that $0 < \theta < 1$. If $\Psi(v) \leq \tau$, then*

$$\Psi(v_+) \leq \frac{(\coth(1)+p)}{(1-\theta)}\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2 := \Psi_0,$$

Proof. The inequality is directly obtained using Lemma 1.2.11 and the fact that $\psi''(1) \leq 2\coth(1) + 2p$. □

Using Ψ_0 defined in the previous corollary, we get $\mathcal{O}(p^2\sqrt{n}\log\frac{n}{\epsilon})$ iterations complexity for small-update methods.

1.5.4 Numerical tests

In this section, we validate the performance of the IPA based on the KF (1.36) by providing numerical experiments. As in Sections 1.3.4 and 1.4.1, we conducted comparative numerical tests between the KFs provided in Table 1.16 on some problems from the Netlib repository alongside the seven test problems taken from Tables B.2 and B.3. Since there are parameters involved in the definition of all considered KFs, we chose two common values of these parameters: $p = 2$ and $p = 4$. This left us with 12 different KFs.

TABLE 1.16: Considered kernel functions.

Kernel function	Complexity	Ref.
$\psi_{1,p}(t) = \frac{t^2-1}{2} + \frac{t^{1-p}-1}{p-1},\ p \geq 2$	$\mathcal{O}\left(pn^{\frac{p+1}{2p}}\log\frac{n}{\epsilon}\right)$	[82]
$\psi_{2,p}(t) = \frac{t^2-1}{2} + \frac{e^{\left(\frac{1}{t^p}-1\right)}-1}{p},\ p \geq 1$	$\mathcal{O}\left(\sqrt{n}(\log n)^{\frac{p+1}{p}}n\log\frac{n}{\epsilon}\right)$	[4]
$\psi_{3,p}(t) = \frac{t^2-1}{2} + \frac{e^{p\left(\frac{1}{t}-1\right)}-1}{p},\ p \geq 1$	$\mathcal{O}\left(p^{-1}\sqrt{n}(\log n)^2\log\frac{n}{\epsilon}\right)$	[3]
$\psi_{4,p}(t) = \frac{t^2-1}{2} + \frac{4}{\pi p}\left(\tan^p\left(\frac{\pi}{2t+2}\right)-1\right), p \geq 2$	$\mathcal{O}\left(pn^{\frac{p+2}{2(p+1)}}\log\frac{n}{\epsilon}\right)$	[15]
$\psi_{5,p}(t) = \frac{t^2-1-\log t}{2} + \frac{t^{1-p}-1}{2(p-1)},\ p \geq 2$	$\mathcal{O}\left(pn^{\frac{p+1}{2p}}\log\frac{n}{\epsilon}\right)$	[14]
$\psi_{\text{new},p}(t) = \frac{t^2-1}{2} + \frac{\sinh^2(1)\coth^p(t)}{p\coth^{p-1}(1)} - \frac{\sinh^2(1)\coth(1)}{p},\ p \geq 2$	$\mathcal{O}\left(pn^{\frac{p+2}{2(p+1)}}\log\frac{n}{\epsilon}\right)$	New

Each problem was tested for multiple values of θ, $\theta \in \{0.7, 0.9, 0.95, 0.99\}$. For variable size problems EV1, EV3 and EV4, we performed Algorithm 1 with the proposed KFs for seven different sizes $n = 2m$ where $m \in \{5, 25, 50, 100, 200, 400, 1000\}$. In addition, the selected Netlib problems were tested for $\theta = 0.9$. This left us with 113 experiments for every KF. The results are summarized in tables below.

TABLE 1.17: Total number of iterations for some Netlib problems for $\theta = 0.9$.

Problem	m	n	$\psi_{1,2}$	$\psi_{1,4}$	$\psi_{2,2}$	$\psi_{2,4}$	$\psi_{3,2}$	$\psi_{3,4}$	$\psi_{4,2}$	$\psi_{4,4}$	$\psi_{5,2}$	$\psi_{5,4}$	$\psi_{\text{new},2}$	$\psi_{\text{new},4}$
afiro	27	51	115	105	101	100	117	116	114	104	108	103	109	**97**
blend	74	114	102	80	76	80	54	86	75	74	78	**52**	100	99
bore3d	233	334	**28**	**28**	**28**	**28**	**28**	**28**	**28**	**28**	**28**	**28**	**28**	**28**
degen2	444	757	**34**	**34**	36	**34**	**34**	**34**	35	**34**	**34**	**34**	**34**	**34**
degen3	1503	2604	**36**	**36**	37	**36**	**36**	**36**	**36**	**36**	**36**	**36**	**36**	**36**
nug05	210	225	**18**	–	19	**18**	**18**	**18**	**18**	19	**18**	**18**	**18**	**18**
nug06	372	486	61	58	74	56	57	60	71	**53**	**53**	56	57	62
nug07	602	931	**56**	81	64	66	68	61	73	73	69	69	63	64
nug08	912	1632	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**
qap8	912	1632	**29**	–	**29**	31	30	**29**	–	–	34	31	37	**29**
scsd1	77	760	**23**	**23**	**23**	**23**	**23**	**23**	**23**	**23**	**23**	**23**	**23**	**23**
scsd6	147	1350	**27**	**27**	**27**	**27**	**27**	**27**	**27**	**27**	**27**	**27**	**27**	**27**
scsd8	397	2750	**25**	**25**	**25**	**25**	**25**	**25**	**25**	**25**	**25**	**25**	**25**	**25**
sctap1	300	660	134	127	126	129	140	123	125	128	**121**	141	125	127
sctap2	1090	2500	76	70	75	68	71	50	84	86	**44**	54	68	83
sctap3	1480	3340	**47**	**47**	76	49	73	**47**	50	98	54	**47**	49	50

TABLE 1.18: Number of inner iterations for fixed size examples.

Example	θ	$\psi_{1,2}$	$\psi_{1,4}$	$\psi_{2,2}$	$\psi_{2,4}$	$\psi_{3,2}$	$\psi_{3,4}$	$\psi_{4,2}$	$\psi_{4,4}$	$\psi_{5,2}$	$\psi_{5,4}$	$\psi_{\text{new},2}$	$\psi_{\text{new},4}$
EF1	0.7	**17**	**17**	20	132	**17**	37	**17**	**17**	**17**	**17**	**17**	**17**
	0.9	**10**	11	28	43	13	72	**10**	12	11	11	**10**	12
	0.95	14	18	15	–	15	131	16	15	15	13	**12**	15
	0.99	30	**19**	23	–	24	31	22	22	33	21	20	24
EF2	0.7	**17**	**17**	**17**	117	**17**	23	**17**	**17**	**17**	**17**	**17**	**17**
	0.9	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	10	10	**9**	**9**
	0.95	10	**9**	**9**	–	**9**	25	**9**	**9**	10	10	**9**	**9**
	0.99	**11**	**11**	30	–	**11**	44	**11**	**11**	**11**	**11**	**11**	**11**
EF3	0.7	**18**	19	42	122	20	62	19	19	**18**	20	**18**	19
	0.9	22	**19**	31	60	21	160	21	26	28	**19**	**19**	20
	0.95	246	37	55	–	37	53	**35**	36	997	39	55	37
	0.99	81	68	**47**	–	52	106	74	54	141	63	67	58
EF4	0.7	21	23	24	131	24	96	**20**	24	21	21	**20**	24
	0.9	19	26	24	–	**16**	305	18	17	21	20	18	23
	0.95	19	23	127	–	21	50	23	21	21	22	19	**16**
	0.99	206	48	41	**36**	47	62	64	48	379	49	66	41

TABLE 1.19: Number of inner iterations for Example EV1 with different sizes $n = 2m$.

θ	m	$\psi_{1,2}$	$\psi_{1,4}$	$\psi_{2,2}$	$\psi_{2,4}$	$\psi_{3,2}$	$\psi_{3,4}$	$\psi_{4,2}$	$\psi_{4,4}$	$\psi_{5,2}$	$\psi_{5,4}$	$\psi_{\text{new},2}$	$\psi_{\text{new},4}$
$\theta = 0.7$	5	**18**	21	27	133	25	30	**18**	21	**18**	**18**	**18**	22
	25	**19**	22	28	167	26	33	**19**	22	**19**	**19**	**19**	23
	50	**20**	23	29	168	27	36	**20**	23	**20**	**20**	**20**	24
	100	**20**	23	29	168	27	36	**20**	23	**20**	**20**	**20**	24
	200	**21**	24	30	169	28	39	**21**	24	**21**	**21**	**21**	25
	400	**21**	24	30	169	28	39	**21**	24	**21**	**21**	**21**	25
	1000	**22**	25	31	170	29	42	**22**	25	**22**	**22**	**22**	26
$\theta = 0.9$	5	14	15	22	**11**	19	31	13	15	14	14	13	15
	25	15	16	23	**12**	20	32	14	16	15	15	14	16
	50	15	16	23	**12**	20	32	14	16	15	15	14	16
	100	16	17	24	**13**	21	33	15	17	16	16	15	17
	200	16	17	24	**13**	21	33	15	17	16	16	15	17
	400	16	17	24	**13**	21	33	15	17	16	16	15	17
	1000	17	18	25	**14**	22	34	16	18	18	17	16	18
$\theta = 0.95$	5	16	21	21	96	18	–	16	17	17	21	16	**14**
	25	18	22	22	104	19	–	17	18	19	22	17	**15**
	50	18	22	22	104	19	–	17	18	19	22	17	**15**
	100	18	22	22	104	19	–	17	18	19	22	17	**15**
	200	19	23	24	120	20	–	19	20	20	24	18	**17**
	400	19	23	24	120	20	–	19	20	20	24	18	**17**
	1000	19	23	24	120	20	–	19	20	20	24	18	**17**
$\theta = 0.99$	5	26	29	41	41	38	–	25	29	29	33	**22**	27
	25	26	29	41	41	38	–	25	29	29	33	**22**	27
	50	28	31	43	43	40	–	27	31	31	35	**24**	29
	100	28	31	43	43	40	–	27	31	31	35	**24**	29
	200	28	31	43	43	40	–	27	31	31	35	**24**	29
	400	28	31	43	43	40	–	27	31	31	35	**24**	29
	1000	28	31	43	43	40	–	27	31	31	35	**24**	29

TABLE 1.20: Number of inner iterations for Example EV3 with different sizes $n = 2m$.

θ	m	$\psi_{1,2}$	$\psi_{1,4}$	$\psi_{2,2}$	$\psi_{2,4}$	$\psi_{3,2}$	$\psi_{3,4}$	$\psi_{4,2}$	$\psi_{4,4}$	$\psi_{5,2}$	$\psi_{5,4}$	$\psi_{\text{new},2}$	$\psi_{\text{new},4}$
	5	**18**	**18**	50	62	**18**	36	**18**	**18**	**18**	**18**	**18**	**18**
	25	**19**	**19**	53	72	**19**	37	**19**	**19**	**19**	**19**	**19**	**19**
	50	**20**	**20**	56	79	**20**	38	**20**	**20**	**20**	**20**	**20**	**20**
$\theta = 0.7$	100	**20**	**20**	56	79	**20**	38	**20**	**20**	**20**	**20**	**20**	**20**
	200	**21**	**21**	59	83	**21**	39	**21**	**21**	**21**	**21**	**21**	**21**
	400	**21**	**21**	59	83	**21**	39	**21**	**21**	**21**	**21**	**21**	**21**
	1000	**22**	**22**	62	84	**22**	40	**22**	**22**	**22**	**22**	**22**	**22**
	5	**9**	**9**	**9**	11	**9**	**9**	**9**	**9**	10	10	**9**	**9**
	25	**10**	**10**	**10**	12	**10**	**10**	**10**	**10**	11	11	**10**	**10**
	50	**10**	**10**	**10**	12	**10**	**10**	**10**	**10**	11	11	**10**	**10**
$\theta = 0.9$	100	**11**	**11**	**11**	13	**11**	**11**	**11**	**11**	12	12	**11**	**11**
	200	**11**	**11**	**11**	13	**11**	**11**	**11**	**11**	12	12	**11**	**11**
	400	**11**	**11**	**11**	13	**11**	**11**	**11**	**11**	12	12	**11**	**11**
	1000	**12**	**12**	**12**	14	**12**	**12**	**12**	**12**	14	13	**12**	**12**
	5	10	**9**	12	23	**9**	19	10	**9**	10	10	**9**	**9**
	25	12	**11**	13	27	**11**	21	**11**	**11**	12	10	**11**	**11**
	50	12	**11**	13	27	**11**	21	**11**	**11**	12	12	**11**	**11**
$\theta = 0.95$	100	12	**11**	13	127	**11**	21	**11**	**11**	12	12	**11**	**11**
	200	13	**12**	15	38	**12**	22	**12**	**12**	13	13	**12**	**12**
	400	13	**12**	15	38	**12**	22	**12**	**12**	13	13	**12**	**12**
	1000	13	**12**	15	38	**12**	22	**12**	**12**	13	13	**12**	**12**
	5	**10**	**10**	**10**	16	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**
	25	**10**	**10**	**10**	16	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**
	50	**12**	**12**	**12**	18	**12**	**12**	**12**	**12**	13	13	**12**	**12**
$\theta = 0.99$	100	**12**	**12**	**12**	18	**12**	**12**	**12**	**12**	13	13	**12**	**12**
	200	**12**	**12**	**12**	18	**12**	**12**	**12**	**12**	13	13	**12**	**12**
	400	**12**	**12**	**12**	18	**12**	**12**	**12**	**12**	13	13	**12**	**12**
	1000	**12**	**12**	**12**	18	**12**	**12**	**12**	**12**	13	13	**12**	**12**

TABLE 1.21: Number of inner iterations for Example EV4 with different sizes $n = 2m$.

θ	m	$\psi_{1,2}$	$\psi_{1,4}$	$\psi_{2,2}$	$\psi_{2,4}$	$\psi_{3,2}$	$\psi_{3,4}$	$\psi_{4,2}$	$\psi_{4,4}$	$\psi_{5,2}$	$\psi_{5,4}$	$\psi_{\text{new},2}$	$\psi_{\text{new},4}$
$\theta = 0.7$	5	**18**	**18**	19	–	**18**	27	**18**	**18**	**18**	**18**	**18**	19
	25	**19**	**19**	20	–	**19**	28	**19**	**19**	**19**	**19**	**19**	20
	50	**20**	**20**	21	–	**20**	29	**20**	**20**	**20**	**20**	**20**	21
	100	**20**	**20**	21	–	**20**	29	**20**	**20**	**20**	**20**	**20**	21
	200	**21**	**21**	22	–	**21**	30	**21**	**21**	**21**	**21**	**21**	22
	400	**21**	**21**	22	–	**21**	30	**21**	**21**	**21**	**21**	**21**	22
	1000	**22**	**22**	23	–	**22**	31	**22**	**22**	**22**	**22**	**22**	23
$\theta = 0.9$	5	**11**	13	16	**11**	16	31	**11**	13	12	13	12	13
	25	**12**	14	17	**12**	17	32	**12**	14	13	14	13	14
	50	**12**	14	17	**12**	17	32	**12**	14	13	14	13	14
	100	**13**	15	18	**13**	18	33	**13**	15	14	15	14	15
	200	**13**	15	18	**13**	18	33	**13**	15	14	15	14	15
	400	**13**	15	18	**13**	18	33	**13**	15	14	15	14	15
	1000	**14**	16	19	**14**	19	34	**14**	16	15	16	15	16
$\theta = 0.95$	5	16	13	18	47	15	27	15	13	**12**	14	13	15
	25	17	14	19	48	16	29	16	14	**13**	15	14	16
	50	17	14	19	48	16	29	16	14	**13**	15	14	16
	100	17	14	19	48	16	29	16	14	**13**	15	14	16
	200	18	**15**	20	88	18	30	18	**15**	**15**	17	16	18
	400	18	**15**	20	88	18	30	18	**15**	**15**	17	16	18
	1000	18	**15**	20	88	18	30	18	**15**	**15**	17	16	18
$\theta = 0.99$	5	23	23	40	25	28	34	21	20	26	26	22	**18**
	25	23	23	40	25	28	34	21	20	26	26	22	**18**
	50	25	25	42	27	30	36	23	22	28	28	24	**20**
	100	25	25	42	27	30	36	23	22	28	28	24	**20**
	200	25	25	42	27	30	36	23	22	28	28	24	**20**
	400	25	25	42	27	30	36	23	22	28	28	24	**20**
	1000	25	25	42	27	30	36	23	22	28	28	24	**20**

Comments

Recall that the numerical results were obtained by performing Algorithm 1 with the KFs defined in Table 1.16 on the aforementioned test problems and some Netlib problems. For each example, we used **bold** font to highlight the best, i.e., the smallest, iteration number.

From Tables 1.17-1.21 we may conclude a few remarks.

- We encountered a problem with KFs $\psi_{1,4}$, $\psi_{2,4}$, $\psi_{3,4}$, $\psi_{4,2}$ and $\psi_{4,4}$ indicated by dash in the corresponding columns of Tables 1.17-1.19 and 1.21. Despite this, $\psi_{2,4}$ is the only one to give the smallest iteration number in EF4 and EV1 for $\theta = 0.99$ and $\theta = 0.9$ respectively.
- In all examples with variable size for $\theta = 0.7$, $\psi_{1,2}, \psi_{4,3}, \psi_{5,2}, \psi_{5,2}$ and $\psi_{\text{new},2}$ have the same smallest iterations number. Moreover, $\psi_{\text{new},2}$ has also the best iterations number in examples with fixed size for $\theta = 0.7$.
- The iteration numbers of the algorithm based on the new KF depend on the values of the parameters p. In fact, the value $p = 2$ gives better iteration numbers in general.
- $\psi_{\text{new,p}}$ outperformed the other KFs in Examples EF1, EF4 and EV1 for $\theta = 0.95$, and in Examples EV1 and EV4 for $\theta = 0.99$.
- The algorithm based on $\psi_{\text{new},4}$ achieved the best iteration numbers for almost 63% of the tested Netlib problems.
- For our KF, the obtained iteration numbers coincide with, or in worst-case, close to the best ones.

To assess the effectiveness and the superiority of our algorithm regarding the total number of iterations, we calculate, for each KF the percentage of cases where the KF has the smallest iteration number. As an illustration, we plot a histogram to compare the performance of the algorithms.

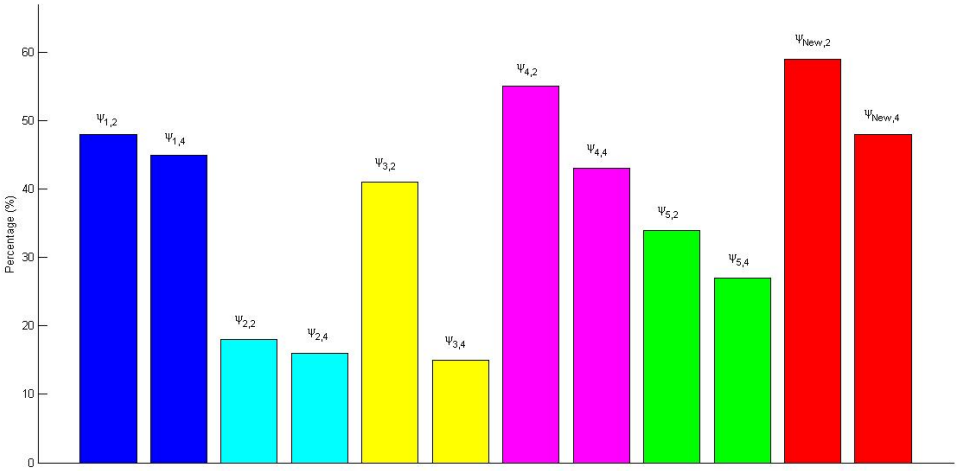

FIGURE 1.6: Performance comparison between KFs in Table 1.16.

Supported by the performance bar graph, we can see that the parameter $p = 2$ has better percentage than $p = 4$ for all KFs. Furthermore, the algorithms based on the new KF attain the most wins among the considered algorithms. In fact, $\psi_{\text{new},2}$ has the best percentage among all KFs.

1.6 Comparative numerical tests between the new kernel functions for solving LO problems

In this section, we conduct comparative numerical experiments between all new KFs proposed in this thesis including the KFs presented in Chapters 2 and 3 (see Table 1.22). We have taken $\epsilon = 10^{-8}$, $\tau = n$, and we use several values of parameter θ namely

$$\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9, 0.99\}.$$

As in Section 1.3.4, we chose the practical step size α [54]. Our purpose is to compare the computational performance of the hyperbolic KFs provided in Table 1.22 on the eight test problems defined in Tables B.2 and B.3 with different sizes, ranging from very small to big size problems. The summary of the results is presented in tables below.

TABLE 1.22: Considered kernel functions.

Kernel function
$\psi_1(t) = \frac{t^2-1}{2} + \sinh^2(1)\left(e^{\coth(t)-\coth(1)} - 1\right)$
$\psi_{2,p}(t) = \frac{t^2-1}{2} + \frac{\sinh^2(1)}{\sinh^2(1)+p\coth^{p-1}(1)}\left(\coth^p(t) - \log t - \coth^p(1)\right),\ p = 2,3,4,\frac{\log n}{2} - 1$
$\psi_{3,p}(t) = \frac{t^2-1}{2} + \frac{\sinh^2(1)\coth^p(t)}{p\coth^{p-1}(1)} - \frac{\sinh^2(1)\coth(1)}{p},\ p = 2,3,4,\frac{\log n}{2} - 1$
$\psi_{4,p,q}(t) = \frac{t^2-1}{2} + \frac{\sinh^2(1)}{2p\coth^{p-1}(1)}\coth^p(t) - \frac{\sinh^2(1)}{2p}\coth(1) + \frac{t^{1-q}-1}{2(q-1)},\ p,q = 2,3,4,\log n$
$\psi_5(t) = \frac{t^2-1}{4} - \int_1^t \frac{\sinh(1)}{\sinh(y)}dy$

TABLE 1.23: Number of inner iterations for fixed size examples.

Ex	θ	ψ_1	$\psi_{2,2}$	$\psi_{2,3}$	$\psi_{2,4}$	$\psi_{2,\frac{\log n}{2}-1}$	$\psi_{3,2}$	$\psi_{3,3}$	$\psi_{3,4}$	$\psi_{3,\frac{\log n}{2}-1}$	$\psi_{4,2,2}$	$\psi_{4,3,2}$	$\psi_{4,4,3}$	$\psi_{4,3,3}$	$\psi_{4,4,4}$	$\psi_{4,\log n,\log n}$	ψ_5
EF 1	0.1	**188**	**188**	**188**	**188**	**188**	**188**	**188**	**188**	**188**	**188**	**188**	**188**	**188**	**188**	**188**	**188**
	0.3	**56**	**56**	**56**	**56**	**56**	**56**	**56**	**56**	**56**	**56**	**56**	**56**	**56**	**56**	**56**	**56**
	0.5	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	34
	0.7	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	33
	0.9	11	11	**10**	12	24	**10**	11	12	**10**	**10**	**10**	11	11	12	**10**	43
EF 2	0.1	**191**	**191**	**191**	**191**	**191**	**191**	**191**	**191**	**191**	**191**	**191**	**191**	**191**	**191**	**191**	**191**
	0.3	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**
	0.5	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	**29**	35
	0.7	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**	34
	0.9	**9**	**9**	**9**	**9**	11	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	33
EF 3	0.1	**192**	**192**	**192**	**192**	**192**	**192**	**192**	**192**	**192**	**192**	**192**	**192**	**192**	**192**	**192**	**192**
	0.3	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**	**57**
	0.5	**30**	**30**	**30**	**30**	**30**	**30**	**30**	32	**30**	**30**	**30**	**30**	**30**	32	**30**	36
	0.7	18	19	18	19	**17**	18	18	19	18	19	18	18	18	19	19	53
	0.9	**19**	**19**	**19**	22	33	**19**	**19**	20	32	21	**19**	**19**	**19**	26	20	53
EF 4	0.1	**196**	**196**	**196**	**196**	**196**	**196**	**196**	**196**	**196**	**196**	**196**	**196**	**196**	**196**	**196**	**196**
	0.3	**58**	**58**	**58**	**58**	**58**	**58**	**58**	**58**	**58**	**58**	**58**	**58**	**58**	**58**	**58**	**58**
	0.5	**30**	**30**	**30**	**30**	31	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	**30**	37
	0.7	24	**20**	22	24	28	21	29	24	29	**20**	21	24	21	24	**20**	46
	0.9	23	**18**	22	20	24	20	22	23	23	**18**	19	23	20	22	**18**	43

TABLE 1.24: Number of inner iterations for Example EV 1 with different sizes $n = 2m$.

θ	m	ψ_1	$\psi_{2,2}$	$\psi_{2,3}$	$\psi_{2,4}$	$\psi_{2,\frac{\log n}{2}-1}$	$\psi_{3,2}$	$\psi_{3,3}$	$\psi_{3,4}$	$\psi_{3,\frac{\log n}{2}-1}$	$\psi_{4,2,2}$	$\psi_{4,3,2}$	$\psi_{4,4,3}$	$\psi_{4,3,3}$	$\psi_{4,4,4}$	$\psi_{4,\log n,\log n}$	ψ_5
$\theta = 0.7$	5	21	**18**	21	22	**18**	**18**	21	22	21	**18**	**18**	21	21	21	**18**	37
	25	22	**19**	22	23	**19**	**19**	22	23	22	**19**	**19**	22	22	22	22	39
	50	23	**20**	23	24	**20**	**20**	23	24	23	**20**	**20**	23	23	23	24	41
	100	23	**20**	23	24	**20**	**20**	23	24	23	**20**	**20**	23	23	23	28	41
	200	24	**21**	24	25	**21**	**21**	24	25	24	**21**	**21**	24	24	24	34	43
	400	24	**21**	24	25	**21**	**21**	24	25	24	**21**	**21**	24	24	24	42	43
	1000	25	**22**	25	26	**22**	**22**	25	26	25	**22**	**22**	25	25	25	49	45
$\theta = 0.9$	5	15	14	16	16	22	**13**	15	15	14	**13**	**13**	15	**13**	15	**13**	44
	25	16	15	17	17	15	**14**	16	16	**14**	**14**	**14**	16	**14**	16	16	48
	50	16	15	17	17	16	**14**	16	16	**14**	**14**	**14**	16	**14**	16	29	48
	100	17	**15**	18	18	**15**	**15**	17	17	**15**	**15**	**15**	17	**15**	17	27	52
	200	17	**15**	18	18	16	**15**	17	17	**15**	**15**	**15**	17	**15**	17	42	52
	400	17	**15**	18	18	**15**	**15**	17	17	**15**	**15**	**15**	17	**15**	17	38	52
	1000	18	17	19	19	**16**	**16**	18	18	19	**16**	**16**	18	**16**	18	49	56
$\theta = 0.99$	5	34	24	29	30	**21**	22	29	27	**21**	23	28	34	29	29	22	47
	25	34	24	29	30	28	**22**	29	27	27	23	28	34	29	29	29	47
	50	36	27	31	32	38	**24**	31	29	27	25	30	36	31	31	25	54
	100	36	27	31	32	31	**24**	31	29	28	25	30	36	31	31	35	54
	200	36	27	31	32	27	**24**	31	29	**24**	25	30	36	31	31	35	54
	400	36	27	31	32	25	**24**	31	29	29	25	30	36	31	31	–	54
	1000	36	27	31	32	31	**24**	31	29	31	25	30	36	31	31	39	54

TABLE 1.25: Number of inner iterations for Example EV 2 with different sizes $n = 2m$.

θ	m	ψ_1	$\psi_{2,2}$	$\psi_{2,3}$	$\psi_{2,4}$	$\psi_{2,\frac{\log n}{2}-1}$	$\psi_{3,2}$	$\psi_{3,3}$	$\psi_{3,4}$	$\psi_{3,\frac{\log n}{2}-1}$	$\psi_{4,2,2}$	$\psi_{4,3,2}$	$\psi_{4,4,3}$	$\psi_{4,3,3}$	$\psi_{4,4,4}$	$\psi_{4,\log n,\log n}$	ψ_5
$\theta = 0.7$	5	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	36
	25	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	38
	50	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	20	40
	100	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	20	40
	200	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	27	42
	400	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	39	42
	1000	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	26	44
$\theta = 0.9$	5	**9**	**9**	**9**	**9**	11	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	33
	25	**10**	**10**	**10**	**10**	11	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	37
	50	**10**	**10**	**10**	**10**	11	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	37
	100	**11**	**11**	**11**	**11**	12	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	41
	200	**11**	**11**	**11**	**11**	12	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	41
	400	**11**	**11**	**11**	**11**	12	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	41
	1000	**12**	**12**	**12**	**12**	13	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	45
$\theta = 0.99$	5	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	36
	25	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	36
	50	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	44
	100	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	44
	200	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	44
	400	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	44
	1000	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	44

TABLE 1.26: Number of inner iterations for Example EV 3 with different sizes $n = 2m$.

θ	m	ψ_1	$\psi_{2,2}$	$\psi_{2,3}$	$\psi_{2,4}$	$\psi_{2,\frac{\log n}{2}-1}$	$\psi_{3,2}$	$\psi_{3,3}$	$\psi_{3,4}$	$\psi_{3,\frac{\log n}{2}-1}$	$\psi_{4,2,2}$	$\psi_{4,3,2}$	$\psi_{4,4,3}$	$\psi_{4,3,3}$	$\psi_{4,4,4}$	$\psi_{4,\log n,\log n}$	ψ_5
$\theta = 0.7$	5	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	36
	25	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	38
	50	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	20	40
	100	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	56	40
	200	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	48	42
	400	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	38	42
	1000	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	36	44
$\theta = 0.9$	5	**9**	**9**	**9**	**9**	11	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	**9**	33
	25	**10**	**10**	**10**	**10**	11	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	37
	50	**10**	**10**	**10**	**10**	11	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	37
	100	**11**	**11**	**11**	**11**	12	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	41
	200	**11**	**11**	**11**	**11**	12	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	41
	400	**11**	**11**	**11**	**11**	12	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	41
	1000	**12**	13	**12**	**12**	13	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	45
$\theta = 0.99$	5	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	36
	25	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	36
	50	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	44
	100	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	44
	200	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	44
	400	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	44
	1000	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	**12**	44

TABLE 1.27: Number of inner iterations for Example EV 4 with different sizes $n = 2m$.

θ	m	ψ_1	$\psi_{2,2}$	$\psi_{2,3}$	$\psi_{2,4}$	$\psi_{2,\frac{\log n}{2}-1}$	$\psi_{3,2}$	$\psi_{3,3}$	$\psi_{3,4}$	$\psi_{3,\frac{\log n}{2}-1}$	$\psi_{4,2,2}$	$\psi_{4,3,2}$	$\psi_{4,4,3}$	$\psi_{4,3,3}$	$\psi_{4,4,4}$	$\psi_{4,\log n,\log n}$	ψ_5
$\theta = 0.7$	5	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	**18**	36
	25	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	**19**	38
	50	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	21	40
	100	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	**20**	21	40
	200	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	28	42
	400	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	**21**	34	42
	1000	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	**22**	–	44
$\theta = 0.9$	5	12	12	12	12	12	12	12	13	**11**	**11**	12	13	12	13	12	43
	25	13	13	13	13	13	13	13	14	**12**	**12**	13	14	13	14	14	47
	50	13	13	13	13	13	13	13	14	**12**	**12**	13	14	13	14	17	47
	100	14	14	14	14	14	14	14	15	**13**	**13**	14	15	14	15	–	51
	200	14	14	14	14	14	14	14	15	14	**13**	14	15	14	15	27	51
	400	14	14	15	15	15	14	14	15	14	**13**	14	15	14	15	–	51
	1000	15	15	16	16	16	15	15	16	15	**14**	15	16	15	16	–	55
$\theta = 0.99$	5	24	21	24	22	21	22	23	**18**	22	21	26	23	26	23	22	46
	25	24	21	24	22	26	22	23	**18**	23	21	26	23	26	23	20	46
	50	26	23	26	24	27	24	25	**20**	25	23	28	25	28	25	26	54
	100	26	23	26	24	25	24	25	**20**	23	23	28	25	28	25	27	54
	200	26	23	26	24	23	24	25	**20**	24	23	28	25	28	25	32	54
	400	26	23	26	24	24	24	25	**20**	26	23	28	25	28	25	35	54
	1000	26	23	26	24	82	24	25	**20**	23	23	28	25	28	25	48	54

Comments

Recall that the numerical results were obtained by performing Algorithm 1 with the KFs defined in Table 1.22 on eight test problems. For each example, we used **bold** font to highlight the best, i.e., the smallest, iteration number.

From Tables 1.23-1.27 we may draw a few conclusions.

- The function ψ_5 never gives the smallest iteration number in examples with variable size, even for examples with fixed size it gives the smallest iteration number only for the values 0.1 and 0.3 of θ.
- For Examples EV2 and EV3 all considered KFs except $\psi_{2,\frac{\log n}{2}-1}$, $\psi_{4,\log n,\log n}$ and ψ_5, achieved the smallest iteration number for all values of θ.
- The iteration numbers of the algorithm based on the parametric KFs depend on the values of the parameters. In fact, the value $p = 2$ gives better iteration numbers for $\psi_{2,p}$ and $\psi_{3,p}$ whereas the value $p = q = 2$ outperformed the other considered values of the parameters for $\psi_{4,p,q}$.
- We can see that the algorithm based on $\psi_{4,2,2}$ attains the most wins among all hyperbolic KFs of Table 1.22, although it has worse theoretical complexity comparing with most of the considered KFs.

Chapter 2

Feasible primal-dual IPMs based on kernel functions for semidefinite optimization

In this chapter, we deal with feasible primal-dual interior-point methods (IPMs) based on kernel functions (KFs) for solving semidefinite optimization (SDO) problems. We first give a concise summary on the basic of primal-dual IPMs for SDO. Then, we present the main steps to obtain the complexity of primal-dual interior-point algorithms (IPAs) for SDO based on the class of KFs defined in Section 1.2. The approach described in Section 2.3 of this chapter is an extension from LO to SDO of the approach presented in Section 1.2. As an application, we consider the following twice parametrized hyperbolic KF

$$\psi(t) = \frac{t^2-1}{2} + \frac{a}{2p}\coth^p(t) - \frac{\sinh^2(1)}{2p}\coth(1) + \frac{t^{1-q}-1}{2(q-1)}, \ \forall t > 0,$$

where $a = \dfrac{\sinh^2(1)}{\coth^{p-1}(1)}$ and $p, q \geq 2$. This function is a combination of the prototype self-regular (SR) KF [83] and the hyperbolic KF (1.36). We prove that the complexity bounds of the new algorithm improve the results obtained by Touil and Chikouche in [98, 100] and matches the results obtained for the KF (1.36) with appropriate choices of the parameters p and q.

2.1 Introduction

SDO's prominence is attributed to its ability to offer a framework for addressing fundamental problems in diverse practical domains like economics, engineering, and operations research. Furthermore, it serves as a generalization of linear programming, providing a broader perspective on optimization problems.

Several methods were proposed to find an optimal solution for SDO problems. Primal-dual IPMs rank among the most efficient methods both theoretically and practically. IPMs were first developed by Karmarkar [53] for LO problems. After that, using the fact that LO is a special case of SDO, many primal-dual IPMs were extended to solve SDO problems, which was a significant contribution initiated by Alizadeh [1] and Nesterov and Todd [78].

The first primal-dual IPM based on the classical logarithmic barrier function was introduced by Roos et al. [92]. After that, Peng et al. presented primal-dual IPMs based on the so-called SR barrier functions, in [83] for LO and SDO. They significantly improved the theoretical complexity obtained for the classical logarithmic KF and obtained the currently best iteration bound for large-update IPMs namely, $\mathcal{O}(\sqrt{n}\log n\log\frac{n}{\epsilon})$, with n denotes the number of variables in the problem and ϵ denotes the desired accuracy. This was one of the main motivations for considering other KFs as a substitute for the classical logarithmic KF. In this context, Bai et al. [9] introduced in 2002 an IPM based on an exponential barrier term which has a finite value at the boundary of the feasible region. The growth term of this finite KF was later parametrized by El Ghami et al. [31] and the approach was extended to solve different types of optimization problems including SDO [34].

In 2004, Bai et al. [8] proposed a new class of eligible KFs which are not necessarily SR. This class includes the classical logarithmic KF and SR KFs as special cases. They presented a unified analysis of primal-dual IPMs based on eligible KFs for LO. The results obtained in [8] were successfully extended to SDO [29].

In [4, 5, 3], published in 2005 and 2007, Amini and his co-authors proposed parametrizations of the two exponential KFs proposed for the first time in [8]. They improved the iteration complexity for large-update methods from $\mathcal{O}(\sqrt{n}\log^2 n\log\frac{n}{\epsilon})$ to $\mathcal{O}(\sqrt{n}\log n\log\frac{n}{\epsilon})$. The IPMs based on these parametric exponential KFs were later extended to solve SDO [20, 85]. Upon scrutiny of the works cited above that focus on IPMs based on KFs, it can be concluded that the theoretical complexity bounds remain unaffected by the type of problem when switching from one problem type to another using, of course, the same KF.

We cannot talk about KFs without mentioning trigonometric KFs. This type of functions has been extensively explored in the literature [56, 86, 59, 43, 66, 44], starting with the work of El Ghami et al. [33] where the authors studied an IPM based on the first trigonometric KF introduced in [8]. They established that the complexity bounds for large- and small-update methods are $\mathcal{O}(n^{\frac{3}{4}}\log\frac{n}{\epsilon})$ and $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$ respectively. This function was later generalized by Bouafia et al. [15] and El Ghami el al. [87] in 2016 for LO and SDO respectively. They obtained $\mathcal{O}(\sqrt{n}\log n\log\frac{n}{\epsilon})$ iterations complexity for large-update methods which is a significant improvement from the results obtained by El Ghami et al. in [33].

Recently, Touil and Chikouche [100] introduced the first IPM based on a hyperbolic-logarithmic KF for SDO. They proved that the corresponding IPA meets $\mathcal{O}(n^{\frac{2}{3}}\log\frac{n}{\epsilon})$ iterations as the worst case complexity bound for the large-update method. In another paper [98], they presented an IPM based on a pure hyperbolic barrier term with an $\mathcal{O}(n^{\frac{3}{4}}\log\frac{n}{\epsilon})$ iteration complexity. To improve the iteration bound based on hyperbolic KFs, Guerdouh et al. in [41] proposed an IPM for solving LO problems based a generalization of the KF proposed in [98]. They showed that their algorithm enjoys the currently best-known iteration bound for both large- and small-update methods.

2.2 Preliminaries

In this section, we showcase the main steps of feasible primal-dual IPMs based on KFs for SDO. At the end, we provide a formal description of the corresponding primal-dual IPA. For more details and informations on the theory of feasible primal-dual IPMs, we refer the readers to the monograph of Peng et al. [84] and the references listed therein.

Recall that in this chapter, we are concerned about solving the SDO problem (P) which is formulated in the following standard form
Given the vector $b \in \mathbb{R}^m$ and matrices $A_i, i = 1, \ldots, m, C \in \mathbb{S}^n$, find a matrix $X \in \mathbb{S}^n$, solution of the constrained minimization problem

$$(P) \quad \min \{C \bullet X;\ A_i \bullet X = b_i,\ i = 1, \ldots, m;\ X \succeq 0\}.$$

The set of feasible solutions for (P) is defined as follows

$$\mathcal{F}_P = \{X \in \mathbb{S}^n_+ : A_i \bullet X = b_i, i = 1, \ldots, m\}.$$

The dual problem of (P) is given by

$$(D) \quad \max \left\{ b^T y;\ \sum_{i=1}^m y_i A_i + S = C;\ S \succeq 0 \right\},$$

where $y \in \mathbb{R}^m$, and the set of feasible solutions for (D) is given by

$$\mathcal{F}_D = \left\{ (y, S) \in \mathbb{R}^m \times \mathbb{S}^n_+ : \sum_{i=1}^m y_i A_i + S = C \right\}.$$

In this chapter, we assume that the matrices A_i, $i = 1, \ldots, m$ are linearly independent. Under this assumption, y is uniquely determined for a given dual feasible S. This corresponds in LO to the assumption that the constraint matrix A must have full rank. To see this, note that the linear independence of matrices $\{A_1, A_2, \ldots, A_m\}$, is equivalent to the linear independence of the vectors $\{\text{vec}(A_1), \text{vec}(A_2), \ldots, \text{vec}(A_m)\}$. Practical algorithms for ensuring full row rank of a matrix can be found in [7].

Remark 2.2.1. *If all matrices A_i, $i = 1, \ldots, m$ and C are both diagonal, SDO is reduced to LO. In this case, both X and S can be assumed to be diagonal as well, since the off-diagonal elements of X and S do not impact the objective function or constraints. In addition, any LO problem written in the standard form as presented in Section 1.1 can be formulated into a SDO problem with $C = diag(c)$, $A_i = diag(a_i), i = 1 \ldots m$, where a_i are the row vectors of A, $X = diag(x)$, and $S = diag(s)$.*

Central path

In what follows, we suppose that the problems (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists (X^0, y^0, S^0) such that

$$A_i \bullet X^0 = b_i,\ i = 1, ..., m,\ X^0 \in \mathcal{S}^n_{++},\ \sum_{i=1}^m y_i^0 A_i + S^0 = C,\ S^0 \in \mathcal{S}^n_{++}.$$

Many theoretical findings related to IPMs for LO were seamlessly extended to their SDO counterpart. For example, if the primal-dual pair (X, y, S) is feasible for both (P) and (D), then

$$b^T y \leq \text{tr}(CX),$$

serving as a direct extension of its LO counterpart. However, despite these extensions, it is acknowledged by several authors [61, 95, 104] that building a duality theory as strong as Theorem 1.2.2 in [84] for LO is unattainable for SDO. Nevertheless, a relatively weaker duality theory is established [77, 102, 95].

Theorem 2.2.2. *([84, Theorem 5.1.1]) (P) and (D) have optimal solutions $\overline{X}$ and $(\overline{y}, \overline{S})$ satisfying*

$$tr(C\overline{X}) = b^T\overline{y},$$

if both of the following conditions hold

(i) (P) is strictly feasible, that is, there exists $X \in \mathbb{S}^n_{++}$ such that

$$A_i \bullet X = tr(A_iX) = b_i,\ i = 1, \ldots, m.$$

(ii) (D) is strictly feasible, that is, there exists $(y, S) \in \mathbb{R}^m \times \mathbb{S}^n_{++}$ such that

$$\sum_{i=1}^{m} y_iA_i + S = C.$$

Under the IPC, Theorem 2.2.2 implies that (P) and (D) have optimal solutions $\overline{X}$ and $(\overline{y}, \overline{S})$ satisfying

$$\begin{aligned} C \bullet \overline{X} - b^T\overline{y} = \text{tr}(C\overline{X}) - b^T\overline{y} &= \text{tr}\left(\overline{X}(\sum_{i=1}^{m} \overline{y}_iA_i + \overline{S})\right) - b^T\overline{y} \\ &= \text{tr}(\overline{X}\,\overline{S}) + \sum_{i=1}^{m} \overline{y}_i\text{tr}(A_i\overline{X}) - b^T\overline{y} \\ &= \text{tr}(\overline{X}\,\overline{S}) + \sum_{i=1}^{m} \overline{y}_i\text{tr}(b_i) - b^T\overline{y} \\ &= \text{tr}(\overline{X}\,\overline{S}) = 0. \end{aligned}$$

Since $\overline{X}$ and $\overline{S}$ are positive semidefinite matrices, Lemma C.1.25 implies that

$$\overline{X}\,\overline{S} = 0.$$

Hence, finding an optimal solution of (P) and (D) is equivalent to solving the nonlinear system of equations

$$\begin{cases} A_i \bullet X = b_i,\ i = 1, \ldots, m, X \in \mathbb{S}^n_+, \\ \sum\limits_{i=1}^{m} y_iA_i + S = C, S \in \mathbb{S}^n_+, \\ XS = 0. \end{cases} \tag{2.1}$$

System (2.1) is called The Karuch-Kuhn-Tucker (KKT) optimality conditions. The first and second equations are called primal and dual feasibility and the last equation is called the complementarity condition for (P) and (D).

As in the LO case, the theory of IPMs for the SDO case suggests that the third equation in (2.1) has to be perturbed. Hence, replacing $XS = 0$ by the nonlinear equation $XS = \mu I$, with parameter $\mu > 0$, leads to the following system

$$\begin{cases} A_i \bullet X = b_i,\ i = 1, \ldots, m, X \in \mathbb{S}^n_{++}, \\ \sum\limits_{i=1}^{m} y_iA_i + S = C, S \in \mathbb{S}^n_{++}, \\ XS = \mu I. \end{cases} \tag{2.2}$$

The existence and uniqueness of a solution for system (2.2) are well-established in the following theorem, as documented in references [77, 64, 101].

Theorem 2.2.3. *Suppose that both (P) and (D) are strictly feasible. Then, for every positive $\mu > 0$ system (2.2) has a unique solution.*

Since the IPC holds the previous theorem implies that for each $\mu > 0$, the parameterized system (2.2) has a unique solution denoted by (X_μ, y_μ, S_μ). The solution (X_μ, y_μ, S_μ) may be interpreted as the parametric representation of a smooth curve (the central path) in terms of the parameter μ. The central path converges to the so-called analytic center of the optimal set as $\mu \to 0$. For more details about the limiting behaviour and other properties of the solution set of SDO, one can consult [13, 62, 61, 95, 104] and the references therein.

Search directions

Similar to LO case, we introduce a threshold parameter $\tau > 0$ and we fixe $\mu > 0$. Then, we define the τ-neighbourhood $\mathcal{N}(\tau, \mu)$ as follows

$$\mathcal{N}(\tau,\mu) = \{(X,y,S) \in \mathbb{S}^n_{++} \times \mathbb{R}^m \times \mathbb{S}^n_{++} : A_i \bullet X = b_i,\ 1 \le i \le m, \\ \sum_{i=1}^{m} y_i A_i + S = C, \Phi(X,S,\mu) \le \tau\},$$

with τ the radius of the neighbourhood and Φ a so-called proximity measure. Φ is used to measure the distance from the point (X, S) to (X_μ, S_μ) and will be defined later in terms of a KF.

Using Newton's approach on system (2.2), we arrive at the following system

$$\begin{cases} A_i \bullet \Delta X = 0,\ i = 1, \dots, m, \\ \sum\limits_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \\ X\Delta S + \Delta X S = \mu I - XS, \end{cases} \tag{2.3}$$

with $(\Delta X, \Delta y, \Delta S)$ is the search direction. A key requirement for IPMs for SDO is to generate symmetric ΔX and ΔS matrices. From the second equation in (2.3), ΔS is clearly symmetric. However, this is not true in general for ΔX. Various methods have been proposed to symmetrize the final equation in (2.3). In this context, we adopt the approach presented by Zhang as outlined in [107] in which he defined the linear transformation $H_P : \mathbb{R}^{n\times n} \to \mathbb{S}^n$ given by

$$H_P(M) := \frac{1}{2}\left(PMP^{-1} + (PMP^{-1})^T\right),$$

with $P \in \mathbb{R}^{n\times n}$ a non-singular matrix. P is called the scaling matrix and it determines the symmetrization strategy.

Zhang also observed that if P is invertible and M is similar to a (symmetric) positive definite matrix, then

$$H_P(M) = \mu I \Leftrightarrow M = \mu I.$$

This observation indicates that the last equation in (2.2) can be replaced by

$$H_P(XS) = \mu I.$$

Hence, we can rewrite system (2.3) as follows

$$\begin{cases} A_i \bullet \Delta X = 0,\ i = 1, \ldots, m, \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \\ H_P(X\Delta S + \Delta X S) = \mu I - H_P(XS). \end{cases}$$

The idea introduced by Zhang [107] was tested with various matrices P, resulting in different directions. See Table 2.1 for some popular choices of P.

TABLE 2.1: Choices for the scaling matrix P

Name of the direction	P	Reference
Alizadeh-Haeberly- Overton (AHO)	I	[2]
Helmberg-Kojima-Monteiro (HKM)	$X^{-\frac{1}{2}}$	[75], [64]
Helmberg-Kojima-Monteiro (HKM)	$S^{\frac{1}{2}}$	[75], [48], [64]
Nesterov-Todd (NT)	$\left(X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}}\right)^{-\frac{1}{2}}$	[96]

In this section, we consider the symmetrization scheme from which the Nesterov-Todd (NT) direction [96] is derived. The advantage of this choice is that the NT scaling technique presented in [96] transfers the primal variable X and the dual variable S into the same space: the so-called V-space. Let

$$W = X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}} = S^{-\frac{1}{2}}(S^{\frac{1}{2}}XS^{\frac{1}{2}})^{\frac{1}{2}}S^{-\frac{1}{2}},\ P_{NT} = W^{-\frac{1}{2}} \text{ and } D = P_{NT}^{-1} = W^{\frac{1}{2}}.$$

The matrix D can be used to rescale X and S to the same matrix V defined as follows

$$V = \frac{1}{\sqrt{\mu}}DSD = \frac{1}{\sqrt{\mu}}D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}}(D^{-1}XSD^{-1})^{\frac{1}{2}}. \tag{2.4}$$

In addition, the matrices D and V are both symmetric and positive definite.

Using the above notations, the NT search direction is then obtained by solving the following system

$$\begin{cases} \overline{A}_i \bullet D_X = 0,\ i = 1, \ldots, m, \\ \sum_{i=1}^{m} \Delta y_i \overline{A}_i + D_S = 0, \\ D_X + D_S = V^{-1} - V, \end{cases} \tag{2.5}$$

with

$$\overline{A_i} = \frac{1}{\sqrt{\mu}}DA_iD, i = 1, .., m, D_S = \frac{1}{\sqrt{\mu}}D\Delta SD, D_X = \frac{1}{\sqrt{\mu}}D^{-1}\Delta XD^{-1}. \tag{2.6}$$

Since the A_i, $i = 1, \ldots, m$, are linearly independent so the $\overline{A}_i$, $i = 1, \ldots, m$. Hence, system (2.5) has a unique solution $(D_X, \Delta y, D_S)$ with D_X and D_S are symmetric matrices.

Observe that

$$V^{-1} - V = -\psi_c'(V),$$

where ψ_c' denotes the matrix valued matrix function defined from $\mathbb{S}^n_{++}$ to $\mathbb{S}^n$ as follows

$$\psi_c'(V) = Q^T \text{diag}(\psi_c'(\lambda_1(V)), \psi_c'(\lambda_2(V)), ..., \psi_c'(\lambda_n(V)))Q,\ QQ^T = I,$$

with Q an orthonormal matrix that diagonalizes V. The matrix barrier function Ψ_c is defined from $\mathbb{S}^n_{++}$ to $\mathbb{R}_+$ by

$$\Psi_c(V) = \sum_{i=1}^{n} \psi_c(\lambda_i(V)). \tag{2.7}$$

The basic of kernel-based IPMs for SDO, as in LO case, is to replace ψ_c by any strictly convex function $\psi :]0, +\infty[\to [0, +\infty[$ which is minimal at $t = 1$ with $\psi(1) = 0$. The corresponding proximity function Ψ is then obtained by replacing ψ_c by ψ in (2.7). This explains the reason for calling ψ the KF of the barrier function Ψ.

Remark 2.2.4. *In this chapter, when we refer to the function ψ and its first three derivatives ψ', ψ'' and ψ''' without specifying further details, it signifies a matrix function if the argument is a matrix. However, if the argument lies in $\mathbb{R}$ it represents a univariate real valued function.*

Returning to system (2.5), it's reformulated in the following form

$$\begin{cases} \overline{A}_i \bullet D_X = 0, \ i = 1, ..., m, \\ \sum_{i=1}^{m} \Delta y_i \overline{A}_i + D_S = 0, \\ D_X + D_S = -\psi'(V). \end{cases} \tag{2.8}$$

Since ΔX and ΔS are orthogonal it is trivial to see that D_X and D_S are orthogonal, and so, $D_X \bullet D_S = 0$. Thus, we can easily verify that

$$D_X = D_S = 0_{n\times n} \Leftrightarrow \psi'(V) = 0_{n\times n} \Leftrightarrow V = I \Leftrightarrow \Psi(V) = 0 \Leftrightarrow XS = \mu I,$$

i.e., if and only if $X = X_\mu$ and $S = S_\mu$, as it should. Otherwise $\Psi(V) > 0$. Hence, if

$$(X, y, S) \neq (X_\mu, y_\mu, S_\mu),$$

then

$$(\Delta X, \Delta y, \Delta S) \neq (0_{n\times n}, 0_m, 0_{n\times n}).$$

This implies that Ψ serves as a proximity measure for closeness with respect to the $\mu-$center (X_μ, S_μ) and the inequality

$$\Psi(V) \leq \tau,$$

defines a $\tau-$neighbourhood of the $\mu-$center. Hence we can define $\Phi(X, S; \mu)$ as follows

$$\Phi(X, S; \mu) = \Psi(V).$$

We also define the norm-based proximity measure $\sigma(V)$, as follows

$$\sigma(V) = \frac{1}{2} \|D_X + D_S\| = \frac{1}{2} \|\psi'(V)\| = \frac{1}{2} \sqrt{\text{tr}\left(\psi'(V)^2\right)}. \tag{2.9}$$

By taking a step along the search direction, the new point (X_+, y_+, S_+) is then computed by

$$X_+ = X + \alpha \Delta X, \ y_+ = y + \alpha \Delta y, \ S_+ = S + \alpha \Delta S, \tag{2.10}$$

with the step size α chosen such that the strict positivity of X_+ and S_+ is guaranteed.

We end this section by providing a brief description of the algorithm corresponding to the primal-dual IPM based on KFs for SDO summarized in Algorithm 2. Starting with an interior-point (X^0, y^0, S^0) situated in a τ-neighbourhood of the given μ-center, we decrease μ to $\mu_+ := (1-\theta)\mu$ for some fixed $0 < \theta < 1$. Then, we compute the search direction $(D_X, \Delta y, D_S)$ by solving system (2.8). Using notations (2.6), we get $(\Delta X, \Delta S)$. After that, we choose an appropriate value of the step size α, which guarantees the strict positivity of the new iterate (X_+, S_+). We repeat the procedure until we find a new iterate (X_+, y_+, S_+) that again belongs to the τ-neighborhood of the current μ-center. Then, we update the parameter μ to μ_+ and we let $(X, y, S) = (X_+, y_+, S_+)$. This process is repeated until μ is small enough, more precisely until $n\mu < \epsilon$ for a certain accuracy parameter ϵ, at this stage we have found an $\epsilon-$optimal solution of (P) and (D).

Algorithm 2 : Generic Primal-Dual Feasible Interior-Point Algorithm for Semidefinite Optimization

Input

a threshold parameter $\tau \geq 1$;

an accuracy parameter $\epsilon > 0$;

a fixed barrier update parameter $\theta \in]0,1[$; (X^0, y^0, S^0) satisfy the IPC and $\mu^0 = 1$

such that $\Phi(X^0, S^0; \mu^0) = \Psi\left(V^0\right) \leq \tau$.

begin

$X := X^0; y := y^0; S := S^0; \mu := \mu^0;$

while $n\mu \geq \epsilon$

begin (outer iteration)

$\mu := (1-\theta)\mu;$

$V := \frac{1}{\sqrt{\mu}}(DXSD^{-1})^{\frac{1}{2}};$

while $\Phi(X, S; \mu) = \Psi(V) > \tau$

begin (inner iteration)

Solve system (2.8) and use (2.6) to obtain $(\Delta X, \Delta y, \Delta S)$;

Choose a suitable step size α;

$X := X + \alpha\Delta X;$

$y := y + \alpha\Delta y;$

$S := S + \alpha\Delta S;$

$V := \frac{1}{\sqrt{\mu}}(DXSD^{-1})^{\frac{1}{2}};$

end while (inner iteration)

end while (outer iteration)

end

2.3 Analysis of the interior-point algorithm based on a specific class of kernel functions

In what follows, we extend the approach presented in Section 1.2 for SDO. Let ψ be a KF that belongs to the class introduced in Section 1.2, i.e., ψ is a three times continuous differentiable KF satisfying

- **H1** For all $t \in \mathbb{R}_{++}$, $\psi'''(t) < 0$,
- **H2** For all $t > 1$, $t\psi''(t) - \psi'(t) > 0$, ($\psi$ is sqrt-convex),
- **H3** For all $t < 1$, $t\psi''(t) + \psi'(t) > 0$, ($\psi$ is e-convex),

and written in the following form

$$\psi(t) = \frac{t^2 - 1}{2} + \psi_b(t), \forall t > 0,$$

with $\psi_b'(t) < 0$ and $\psi_b''(t) \geq 0$ for all $t > 0$.

Now, we provide upper bounds for the proximity function Ψ defined in (2.7) and a lower bound for the proximity measure σ defined in (2.9) after the μ-update. These bounds will be necessary in the complexity analysis. In accordance with Algorithm 2, at the beginning of an outer iteration we have $\Psi(V) \leq \tau$ before the update of μ with the factor $(1-\theta)$. After updating μ in an outer iteration, the value of $\Psi(V)$ increases since V is divided by the factor $\sqrt{1-\theta}$. Then during the inner iteration, the value of $\Psi(V)$ decreases until it passes the threshold τ.

Theorem 2.3.1. *([29, Theorem 3.1]) Let ϱ be as defined in Definition 1.1.9. Then, for any $V \in \mathbb{S}^n_{++}$ and $\beta > 1$, we have*

$$\Psi(\beta V) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(V)}{n}\right)\right).$$

Proof. We first consider the maximization problem (MP) defined for any $z \in \mathbb{R}_+$ as follows

$$(MP)\begin{cases} \max\limits_{V} \Psi(\beta V) = \sum\limits_{i=1}^{n} \psi(\beta\lambda_i(V)), \\ \Psi(V) = \sum\limits_{i=1}^{n} \psi(\lambda_i(V)) = z. \end{cases}$$

Let the vector v be such that

$$v_i = \lambda_i(V),\ i = 1, \ldots, n.$$

Then,

$$\Psi(V) = \sum_{i=1}^{n} \psi(v_i) = \Psi(v),$$

and

$$\Psi(\beta V) = \sum_{i=1}^{n} \psi(\beta v_i) = \Psi(\beta v).$$

Therefore, applying Theorem 1.2.8, we get

$$\begin{aligned}\Psi(\beta v) &\leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right)\\ &= n\psi\left(\beta\varrho\left(\frac{\sum_{i=1}^{n}\psi(v_i)}{n}\right)\right)\\ &= n\psi\left(\beta\varrho\left(\frac{\sum_{i=1}^{n}\psi(\lambda_i(V))}{n}\right)\right)\\ &= n\psi\left(\beta\varrho\left(\frac{\Psi(V)}{n}\right)\right).\end{aligned}$$

The inequality derives since $\Psi(\beta v) = \Psi(\beta V)$. □

Remark 2.3.2. *Let $V_+ = \frac{V}{\sqrt{1-\theta}}$ and $0 \leq \theta < 1$. If we assume that $\Psi(V) \leq \tau$ just before the $\mu-$update to $(1-\theta)\mu$, we get the following upper bound*

$$\Psi(\beta V) = \Psi(V_+) \leq n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right),$$

where $\beta = \frac{1}{\sqrt{1-\theta}}$.

As a consequence, we have the following two lemmas that proofs proceed exactly as in the proofs of Lemma 1.2.10 and Lemma 1.2.11 respectively.

Lemma 2.3.3. *Let $V_+ = \frac{V}{\sqrt{1-\theta}}$ and $0 \leq \theta < 1$. If we assume that $\Psi(V) \leq \tau$ just before the $\mu-$update to $(1-\theta)\mu$, we get the following upper bound*

$$\Psi(V_+) \leq \frac{\theta n + 2\tau + 2\sqrt{2\tau n}}{2(1-\theta)} := \Psi_0.$$

Ψ_0 is an upper bound for $\Psi(V_+)$ during the process of Algorithm 2.

Lemma 2.3.4. *Let $V_+ = \frac{V}{\sqrt{1-\theta}}$ and $0 \leq \theta < 1$. If we assume that $\Psi(V) \leq \tau$ just before the $\mu-$update to $(1-\theta)\mu$, we get the following upper bound*

$$\Psi(V_+) \leq \frac{\psi''(1)}{2}\frac{\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2}{1-\theta} := \Psi_0.$$

Ψ_0 is an upper bound for $\Psi(V_+)$ during the process of Algorithm 2.

The following lemma provides a lower bound of $\sigma(V)$ in terms of the proximity function $\Psi(V)$.

Lemma 2.3.5. *Let $\sigma(V)$ be defined by (2.9). Then, for any $V \in \mathbb{S}^n_{++}$, we have*

$$\sigma(V) \geq \sqrt{\frac{\Psi(V)}{2}}.$$

Proof. Recall that the matrix barrier function $\Psi(V)$ is defined as follows

$$\Psi(V) = \sum_{i=1}^{n} \psi(\lambda_i(V)).$$

Using the first item of Proposition 1.2.2, it follows that for all $t > 0$

$$\psi(t) \leq \frac{1}{2}\psi'(t)^2.$$

Hence, using (2.9), we obtain

$$\sigma(V)^2 = \frac{1}{4}\mathrm{tr}\left(\psi'(V)^2\right) = \frac{1}{4}\sum_{i=1}^{n}\psi'(\lambda_i(V))^2 \geq \frac{1}{2}\sum_{i=1}^{n}\psi(\lambda_i(V)).$$

This completes the proof. □

Remark 2.3.6. *Throughout this chapter, we assume that $\tau \geq 1$. Using Lemma 2.3.5 and the assumption that $\Psi(V) \geq \tau$, we have*

$$\sigma(V) \geq \sqrt{\frac{1}{2}}.$$

2.3.1 Computation of the displacement step

The purpose of this subsection is to compute a default step size α such that (X_+, y_+, S_+) defined in (2.10) are strictly feasible and the proximity function decreases sufficiently.

Proposition 2.3.7. *([83, Proposition 5.2.6]) For any $X_1, X_2 \in \mathbb{S}^n_{++}$, we have*

$$\Psi\left(\left(X_1^{\frac{1}{2}}X_2X_1^{\frac{1}{2}}\right)^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\Psi(X_1) + \Psi(X_2)\right).$$

Proof. Recall that for any non-singular matrix $U \in \mathbb{R}^{n\times n}$ we have

$$\eta_i(U) = (\lambda_i(U^TU))^{\frac{1}{2}} = (\lambda_i(UU^T))^{\frac{1}{2}}, i = 1, \ldots n.$$

Hence, using the fact that the square root of a symmetric positive definite matrix is also symmetric, we get

$$\eta_i(X_1^{\frac{1}{2}}X_2^{\frac{1}{2}}) = (\lambda_i(X_1^{\frac{1}{2}}X_2X_1^{\frac{1}{2}}))^{\frac{1}{2}} = \lambda_i(X_1^{\frac{1}{2}}X_2X_1^{\frac{1}{2}})^{\frac{1}{2}}, i = 1, \ldots, n.$$

Since X_1 and X_2 are symmetric positive definite, using Lemma C.2.5 and the e-convexity of ψ (hypothesis **H3**), one gets

$$\begin{aligned}\Psi((X_1^{\frac{1}{2}}X_2X_1^{\frac{1}{2}})^{\frac{1}{2}}) &= \sum_{i=1}^{n}\psi((\lambda_i(X_1^{\frac{1}{2}}X_2X_1^{\frac{1}{2}}))^{\frac{1}{2}}) \\ &= \sum_{i=1}^{n}\psi(\eta_i(X_1^{\frac{1}{2}}X_2^{\frac{1}{2}})) \\ &\leq \sum_{i=1}^{n}\psi(\eta_i(X_1^{\frac{1}{2}})\eta_i(X_2^{\frac{1}{2}})) \\ &\leq \frac{1}{2}\sum_{i=1}^{n}\left[\psi(\eta_i^2(X_1^{\frac{1}{2}})) + \psi(\eta_i^2(X_2^{\frac{1}{2}}))\right] \\ &= \frac{1}{2}\left(\Psi(X_1) + \Psi(X_2)\right).\end{aligned}$$

□

Now, we consider the decrease in Ψ as a function of α noted f defined by

$$f(\alpha) = \Psi\left(V_+\right) - \Psi\left(V\right).$$

Using (2.4), (2.6) and (2.10), for fixed μ, we get

$$X_+ = X + \alpha\Delta X = X + \alpha\sqrt{\mu}DD_XD = \sqrt{\mu}D(V + \alpha D_X)D,$$

$$S_+ = S + \alpha\Delta S = S + \alpha\sqrt{\mu}D^{-1}D_SD^{-1} = \sqrt{\mu}D^{-1}(V + \alpha D_S)D^{-1},$$

$$V_+ = V + \alpha D_X = V + \alpha D_S,$$

and

$$V_+^2 = (V + \alpha D_X)(V + \alpha D_S),$$

with the step size α chosen such that

$$V + \alpha D_X \in \mathbb{S}_{++}^n \quad \text{and} \quad V + \alpha D_S \in \mathbb{S}_{++}^n.$$

Hence, since $V + \alpha D_X, V + \alpha D_S \in \mathbb{S}_{++}^n$, it follows that

$$V_+^2 \sim (V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}},$$

which implies that the matrix V_+ and the matrix $((V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}})^{\frac{1}{2}}$ have the same eigenvalues. Thus, we can write

$$f(\alpha) = \Psi\left(\left((V + \alpha D_X)^{\frac{1}{2}}\left(V + \alpha D_S\right)\left(V + \alpha D_X\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right) - \Psi\left(V\right).$$

Or, from Proposition 2.3.7, we get

$$\Psi\left(V_+\right) \leq \frac{1}{2}\left(\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)\right).$$

Therefore, $f(\alpha) \leq f_1(\alpha)$, where

$$f_1\left(\alpha\right) = \frac{1}{2}\left(\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)\right) - \Psi\left(V\right). \tag{2.11}$$

Differentiating the function f_1 with respect to α, we get

$$f_1'(\alpha) = \frac{1}{2}\mathrm{tr}\left(\psi'(V+\alpha D_X)D_X + \psi'(V+\alpha D_S)D_S\right),$$

and

$$f_1''(\alpha) = \frac{1}{2}\mathrm{tr}\left(\psi''(V+\alpha D_X)D_X^2 + \psi''(V+\alpha D_S)D_S^2\right). \tag{2.12}$$

Hence, using (2.9) and the last equation of (2.8) we obtain

$$f_1'(0) = -\frac{1}{2}\mathrm{tr}\left(\left(\psi'(V)\right)^2\right) = -2\sigma(V)^2.$$

In what follows, we set $\sigma(V) := \sigma$ for simplicity purposes.

Lemma 2.3.8. *([103, Lemma 5.19]) Let $f_1(\alpha)$ be as defined in (2.11). Then, we have*

$$f_1''(\alpha) \leq 2\sigma^2\psi''(\lambda_{\min}(V) - 2\alpha\sigma).$$

Proof. Recall that D_X and D_S are orthogonal, i.e., $D_X \bullet D_S = \mathrm{tr}(D_X D_S) = \mathrm{tr}(D_S D_X) = 0$ and $D_X + D_S = -\nabla\Psi(v)$. Moreover, using the definition (2.9) of σ one obtains

$$\|D_X + D_S\|^2 = \|D_X\|^2 + \|D_S\|^2 = 4\sigma^2.$$

This implies that

$$|\lambda_{\max}(D_X)| \leq 2\sigma \text{ and } |\lambda_{\max}(D_S)| \leq 2\sigma.$$

Using Lemma C.1.27 and the fact that $V, V+\alpha D_X, V+\alpha D_S \in \mathrm{S}_+^n$, one gets for all $i = 1, \ldots, n$

$$\begin{aligned}\lambda_i(V+\alpha D_X) &\geq \lambda_{\min}(V) - \alpha|\lambda_{\max}(D_X)| \geq \lambda_{\min}(V) - 2\alpha\sigma,\\ \lambda_i(V+\alpha D_S) &\geq \lambda_{\min}(V) - \alpha|\lambda_{\max}(D_S)| \geq \lambda_{\min}(V) - 2\alpha\sigma,\end{aligned}$$

which implies using the decrease of ψ'' that

$$\psi''\left(\lambda_i(V+\alpha D_X)\right) \leq \psi''\left(\lambda_{\min}(V) - 2\alpha\sigma\right),\ i = 1, \ldots, n,$$

and

$$\psi''\left(\lambda_i(V+\alpha D_S)\right) \leq \psi''\left(\lambda_{\min}(V) - 2\alpha\sigma\right)\ i = 1, \ldots, n.$$

Hence,

$$\psi''\left(V+\alpha D_X\right) \preceq \psi''\left(\lambda_{\min}(V) - 2\alpha\sigma\right) I,$$

and

$$\psi''\left(V+\alpha D_S\right) \preceq \psi''\left(\lambda_{\min}(V) - 2\alpha\sigma\right) I.$$

Using the last item of Proposition C.1.24, we obtain

$$\mathrm{tr}\Big(\psi''\left(V+\alpha D_X\right) D_X^2\Big) \leq \mathrm{tr}\Big(\psi''\left(\lambda_{\min}(V) - 2\alpha\sigma\right) D_X^2\Big),$$

and

$$\mathrm{tr}\Big(\psi''\left(V+\alpha D_S\right) D_S^2\Big) \leq \mathrm{tr}\Big(\psi''\left(\lambda_{\min}(V) - 2\alpha\sigma\right) D_S^2\Big).$$

Therefore, using (2.12) it follows that

$$\begin{aligned}
f_1''(\alpha) &= \frac{1}{2}\mathrm{tr}\Big(\psi''(V+\alpha D_X)D_X^2 + \psi''(V+\alpha D_S)D_S^2\Big)\\
&= \frac{1}{2}\left(\mathrm{tr}\left(\psi''(V+\alpha D_X)D_X^2\right) + \mathrm{tr}\left(\psi''(V+\alpha D_S)D_S^2\right)\right)\\
&\leq \frac{1}{2}\left(\mathrm{tr}\Big(\psi''\left(\lambda_{\min}(V)-2\alpha\sigma\right)D_X^2\Big) + \mathrm{tr}\Big(\psi''\left(\lambda_{\min}(V)-2\alpha\sigma\right)D_S^2\Big)\right)\\
&= \frac{1}{2}\psi''\left(\lambda_{\min}(V)-2\alpha\sigma\right)\left(\mathrm{tr}\Big(D_X^2\Big) + \mathrm{tr}\Big(D_S^2\Big)\right)\\
&= \frac{1}{2}\psi''\left(\lambda_{\min}(V)-2\alpha\sigma\right)\sum_{i=1}^{n}\Big(\lambda_i\big(D_X\big)^2 + \lambda_i\big(D_S\big)^2\Big)\\
&= 2\sigma^2\psi''\left(\lambda_{\min}(V)-2\alpha\sigma\right),
\end{aligned}$$

where the last equality is due to the fact that $\|D_X\|^2 + \|D_S\|^2 = 4\sigma^2$.

□

Putting $v_i = \lambda_i(V)$, $i = 1,\ldots,n$ and $v_1 = \lambda_{\min}(V)$, we can proceed exactly as in the LO case to obtain the following lemmas (see Lemmas 1.2.15, 1.2.16 and 1.2.17).

Lemma 2.3.9. *If the step size α satisfies the inequality*

$$\psi'\left(\lambda_{\min}\left(V\right)\right) - \psi'\left(\lambda_{\min}\left(V\right) - 2\alpha\sigma\right) \leq 2\sigma, \tag{2.13}$$

then

$$f_1'(\alpha) \leq 0.$$

Lemma 2.3.10. *Let ρ be the function defined in Definition 1.1.9. Then the largest possible value of the step size α^* satisfying (2.13) is given by*

$$\alpha^* = \frac{\rho\left(\sigma\right) - \rho\left(2\sigma\right)}{2\sigma}.$$

Lemma 2.3.11. *Let ρ be the function defined in Definition 1.1.9 and α^* be as defined in Lemma 2.3.10. Then, we have*

$$\alpha^* \geq \frac{1}{\psi''\left(\rho\left(2\sigma\right)\right)}.$$

Theorem 2.3.12. *Let us set $\bar{\alpha} = \dfrac{1}{\psi''\left(\rho\left(2\sigma\right)\right)}$, as the default step size. Then*

$$f(\bar{\alpha}) \leq -\sigma^2\bar{\alpha} = -\frac{\sigma^2}{\psi''\left(\rho\left(2\sigma\right)\right)}.$$

Proof. As in the LO case, we define the univariate function g as follow

$$g(\alpha) = -2\alpha\sigma^2 + \alpha\sigma\psi'(\lambda_{\min}(V)) + \frac{1}{2}\Big(\psi\left(\lambda_{\min}(V)-2\alpha\sigma\right) - \psi(\lambda_{\min}(V))\Big),\ \forall\alpha\in[0,\alpha^*].$$

Differentiating g twice, we get

$$\begin{aligned}
g'(\alpha) &= -2\sigma^2 - \sigma\Big(\psi'(\lambda_{\min}(V)-2\alpha\sigma) - \psi'(\lambda_{\min}(V))\Big)\\
g''(\alpha) &= 2\sigma^2\psi''(\lambda_{\min}(V)-2\alpha\sigma).
\end{aligned}$$

Obviously,

$$g(0) = f_1(0) = 0,\ g'(0) = f_1'(0) = -2\sigma^2 < 0.$$

On the other hand, from Lemma 2.3.8 we have

$$f_1''(\alpha) \leq g''(\alpha).$$

Therefore, $f_1'(\alpha) \leq g'(\alpha)$ and $f_1(\alpha) \leq g(\alpha)$. Taking $\alpha \in [0, \alpha^*]$, we obtain

$$g'(\alpha) = -2\sigma^2 - \sigma\Big(\psi'(\lambda_{\min}(V) - 2\alpha\sigma) - \psi'(\lambda_{\min}(V))\Big) \leq 0.$$

Due to the increase of g'' in α, using Lemma A.0.11 we arrive at

$$f(\alpha) \leq f_1(\alpha) \leq g(\alpha) \leq \frac{1}{2}\alpha g'(0) = -\alpha\sigma^2.$$

□

2.4 A primal-dual IPM for SDO based on a bi-parametrized kernel function

In this section, we provides a new primal-dual IPA for solving SDO problems based on a new KF. The latter combines the popular SR KF and the hyperbolic KF (1.36). The complexity analysis shows that by choosing special values of the parameters, the so far best-known iteration bounds for large- and small-update methods are derived. Moreover, preliminary numerical experiments showcased the practical efficiency of the algorithm based on the new KF comparing with some other existing KFs. The results presented in this section are parts of a submitted paper [40].

2.4.1 The new kernel function and its properties

In 2001, Peng et al. [83] proposed an IPM based on a SR function defined for all $t > 0$ as follows

$$\psi_{\text{SR}}(t) = \frac{t^2-1}{2} + \frac{t^{1-q}-1}{q-1},\ q \geq 2.$$

In 2016, Bouafia et al. [15] introduced an IPM based on a KF with a parameterized trigonometric barrier term defined as follows

$$\psi_{\text{Trigo}}(t) = \frac{t^2-1}{2} + \frac{4}{\pi p}\left(\tan^p\left(\frac{\pi}{2t+2}\right) - 1\right), p \geq 2.$$

Later on, Bouafia and Adnane [17] combined the SR function ψ_{SR} with the parametric trigonometric KF ψ_{Trigo} to propose the following twice parametric KF

$$\psi_{\text{Trigo/SR}}(t) = t^2 + \frac{t^{1-q}}{q-1} - \frac{q}{q-1} + \frac{4}{\pi p}\left(\tan^p\left(\frac{\pi}{2t+2}\right) - 1\right), p, q \geq 2.$$

After that, Guerdouh et al. [41] proposed a primal-dual IPM based on the following parametric hyperbolic KF

$$\psi_{\text{Hyper}}(t) = \frac{t^2-1}{2} + \frac{\sinh^2(1)\coth^p(t)}{p\coth^{p-1}(1)} - \frac{\sinh^2(1)\coth(1)}{p},\ \ p \geq 2.$$

Motivated by the last two works, we introduce a new efficient twice parameterized KF

$$\psi_{\text{Hyper/SR}} = \frac{t^2-1}{2} + \frac{a}{2p}\coth^p(t) - \frac{\sinh^2(1)}{2p}\coth(1) + \frac{t^{1-q}-1}{2(q-1)},\ \forall t>0, \quad (2.14)$$

where $a = \dfrac{\sinh^2(1)}{\coth^{p-1}(1)}$ and $p,q \geq 2$. To simplify the notations, we denote $\psi_{\text{Trigo/SR}}$ by ψ. For convenience of reference, we give its first three derivatives with respect to t as follows:

$$\psi'(t) = t - a\frac{\coth^{p-1}(t)}{2\sinh^2(t)} - \frac{1}{2t^q}, \quad (2.15)$$

$$\psi''(t) = 1 + \frac{a}{2}\left(2\frac{\coth^p(t)}{\sinh^2(t)} + (p-1)\frac{\coth^{p-2}(t)}{\sinh^4(t)}\right) + \frac{q}{2t^{q+1}} > 1, \quad (2.16)$$

and

$$\psi'''(t) = -\frac{a}{2}\left(4\frac{\coth^{p+1}(t)}{\sinh^2(t)} + (6p-4)\frac{\coth^{p-1}(t)}{\sinh^4(t)} + (p-1)(p-2)\frac{\coth^{p-3}(t)}{\sinh^6(t)}\right) - \frac{q(q+1)}{2t^{q+2}} < 0.$$

We can easily verify that $\psi'(1) = \psi(1) = 0$ and $\lim\limits_{t\to 0^+}\psi(t) = \lim\limits_{t\to+\infty}\psi(t) = +\infty$. This implies that ψ is a KF verifying **H1**. Furthermore, we can write

$$\psi(t) = \frac{t^2-1}{2} + \psi_b(t),$$

with

$$\psi_b(t) = \frac{a}{2p}\coth^p(t) - \frac{\sinh^2(1)}{2p}\coth(1) + \frac{t^{1-q}-1}{2(q-1)}.$$

From (2.15) and (2.16), it follows that

$$\psi_b'(t) < 0 \text{ and } \psi_b''(t) > 0, \forall t > 0.$$

The following lemma provided some key properties of ψ including the e-convexity. In addition, it also implies that ψ defined in (2.14) belongs to the class of KFs defined in Sections 1.2 and 2.3.

Lemma 2.4.1. *Let ψ be the function defined in (2.14). Then, we have*

(i) $t\psi''(t) - \psi'(t) > 0,\ \forall t > 0.$

(ii) $t\psi''(t) + \psi'(t) > 0,\ \forall t > 0.$

Proof. From (2.15) and (2.16), it follows that for all $t > 0$

$$t\psi''(t) - \psi'(t) = \frac{a}{2}\left((p-1)t\frac{\coth^{p-2}(t)}{\sinh^4(t)} + \frac{\coth^{p-1}(t)}{\sinh^2(t)}(2t\coth(t)+1)\right) + \frac{q+1}{2t^q},$$

and

$$t\psi''(t)+\psi'(t)=2t+\frac{a}{2}\left((p-1)t\frac{\coth^{p-2}(t)}{\sinh^4(t)}+\frac{\coth^{p-1}(t)}{\sinh^2(t)}(2t\coth(t)-1)\right)+\frac{q-1}{2t^q}.$$

$t\psi''(t)-\psi'(t)$ is obviously strictly positive for all $t>0$. Using (1.29) of Lemma 1.3.1, we obtain the second item of the lemma. □

Let ϱ and ρ be as defined in Definition 1.1.9. Then we have the following lemma.

Lemma 2.4.2. *For all $(z,t)\in[0,+\infty[\times]0,1]$ such that $z=-\frac{1}{2}\psi'(t)$, one has*

(i) $\coth(t)\leq\coth(1)\,(4z+1)^{\frac{1}{p+1}}$.

(ii) $\frac{1}{t}\leq(4z+2)^{\frac{1}{q}}$.

Proof. Let $(z,t)\in[0,+\infty[\times]0,1]$ such that $z=-\frac{1}{2}\psi'(t)$. Then, (2.15) implies that

$$\begin{aligned}2z&=-\psi'(t)\\&=-t+a\frac{\coth^{p-1}(t)}{2\sinh^2(t)}+\frac{1}{2t^q}\\&=-t+a\coth^{p-1}(t)\frac{\coth^2(t)-1}{2}+\frac{1}{2t^q}\\&=-t+a\coth^{p-1}(t)\frac{\coth^2(t)}{2\cosh^2(t)}+\frac{1}{2t^q},\end{aligned}\tag{2.17}$$

where the last equality is due to the fact that $\coth^2(t)-1=\dfrac{\coth^2(t)}{\cosh^2(t)}$. Since the function $t\mapsto\cosh(t)$ is a monotonically increasing and $\dfrac{1}{t^q}\geq 1$ for all $0<t\leq 1$, we obtain

$$\frac{\coth^{p+1}(t)}{2\coth^{p+1}(1)}=\frac{a\coth^{p+1}(t)}{2\cosh^2(1)}\leq\frac{a\coth^{p+1}(t)}{2\cosh^2(t)}=2z+t-\frac{1}{2t^q}\leq\left(2z+\frac{1}{2}\right).$$

Therefore,

$$\coth^{p+1}(t)\leq\coth^{p+1}(1)\,(4z+1),$$

which proves the first item. For the second item, from (2.17), we have

$$0\leq\frac{a\coth^{p+1}(t)}{2\cosh^2(1)}=2z+t-\frac{1}{2t^q},$$

which implies that

$$\frac{1}{t}\leq(4z+2)^{\frac{1}{q}}.$$

This completes the proof. □

2.4.2 Analysis of the algorithm

Now, we would like to have a default step size $\bar{\alpha}$ such that (X_+, y_+, S_+) defined in Algorithm 2 are strictly feasible and the proximity function (2.7) decreases sufficiently. We recall that during an inner iteration the parameter μ is fixed. Using (2.4), (2.6) and (2.10), for fixed μ, we get

$$X_+ = \sqrt{\mu}D(V + \bar{\alpha}D_X)D, S_+ = \sqrt{\mu}D^{-1}(V + \bar{\alpha}D_S)D^{-1}, V_+ = V + \bar{\alpha}D_X = V + \bar{\alpha}D_S.$$

We then present an upper bound for the decreasing value of the proximity in the inner iteration in the following theorem

Theorem 2.4.3. *If $\bar{\alpha}$ is the default step size and $\sigma \geq 1$, then we have*

$$f(\bar{\alpha}) \leq -\frac{\Psi(V)^{\frac{pq-p-2}{2(p+1)q}}}{100\sqrt{2}\left(1 + (\sinh^2(1)\coth^3(1))p + \frac{q}{2}\right)}.$$

Proof. From (2.15) , we have

$$\begin{aligned}
\psi''(t) =& 1 + \frac{a}{2}\left(2\frac{\coth^p(t)}{\sinh^2(t)} + (p-1)\frac{\coth^{p-2}(t)}{\sinh^4(t)}\right) + \frac{q}{2t^{q+1}},\\
=& 1 + \frac{a}{2}\left(2\coth^p(t)\left(\coth^2(t) - 1\right) + (p-1)\coth^{p-2}(t)\left(\coth^2(t) - 1\right)^2\right) + \frac{q}{2t^{q+1}}\\
=& 1 + \frac{a}{2}\left(2\coth^{p+2}(t) - 2\coth^p(t) + (p-1)\coth^{p+2}(t)\right.\\
&\left. + (p-1)\coth^{p-2}(t) - 2(p-1)\coth^p(t)\right) + \frac{q}{2t^{q+1}}\\
\leq& 1 + ap\coth^{p+2}(t) + \frac{q}{2t^{q+1}}.
\end{aligned}$$

Let $t = \rho(2\sigma)$. Lemma 2.4.2 implies that

$$\begin{aligned}
\psi''(\rho(2\sigma)) &\leq 1 + (\sinh^2(1)\coth^3(1))p(4\sigma+1)^{\frac{p+2}{p+1}} + \frac{q}{2}(4\sigma+2)^{\frac{q+1}{q}}\\
&\leq \left(1 + (\sinh^2(1)\coth^3(1))p + \frac{q}{2}\right)(8\sigma+2)^{\frac{(p+2)(q+1)}{(p+1)q}}\\
&\leq \left(1 + (\sinh^2(1)\coth^3(1))p + \frac{q}{2}\right)(8\sigma+2\sigma)^{\frac{(p+2)(q+1)}{(p+1)q}}.
\end{aligned}$$

The last inequality is due to Remark 2.3.6. Hence, using Theorem 2.3.12 and Lemma 2.3.5, we get

$$\begin{aligned}
f(\bar{\alpha}) &\leq -\frac{\sigma^2}{\psi''(\rho(2\sigma))}\\
&\leq -\frac{\sigma^2}{\left(1 + (\sinh^2(1)\coth^3(1))p + \frac{q}{2}\right)(10\sigma)^{\frac{(p+2)(q+1)}{(p+1)q}}}\\
&\leq -\frac{\sigma^{\frac{pq-p-2}{(p+1)q}}}{100\left(1 + (\sinh^2(1)\coth^3(1))p + \frac{q}{2}\right)}\\
&\leq -\frac{\Psi(V)^{\frac{pq-p-2}{2(p+1)q}}}{100\sqrt{2}\left(1 + (\sinh^2(1)\coth^3(1))p + \frac{q}{2}\right)},
\end{aligned}$$

which completes the proof. □

Iteration complexity

Now, we compute how many inner iterations are required to return to the situation where $\Psi(V) \leq \tau$ after μ-update. Let us define the value of $\Psi(V)$ after μ-update as Ψ_0, and the subsequent values in the same outer iteration as $\Psi_i, i = 1, ..., K$, where K stands for the total number of inner iterations in the outer iteration. The decrease on each inner iteration is given by

$$\Psi_{i+1} \leq \Psi_i - \frac{1}{100\sqrt{2}\left(1 + (\sinh^2(1)\coth^3(1))p + \frac{q}{2}\right)} \Psi_i^{\frac{pq-p-2}{2(p+1)q}}.$$

We arrive at the final result of this section which summarizes the complexity bounds for large and small-update methods.

Theorem 2.4.4. *Let Ψ_0 be the value defined in Lemma 2.3.3 and let $\tau \geq 1$. Then, the total number of iterations to obtain an approximate solution with $n\mu \leq \epsilon$ is bounded by*

$$\left(\frac{100\sqrt{2}\left(2 + \sinh^2(1)\coth^3(1)(p+1) + q\right)(p+1)q}{(p+2)(q+1)}\right)\Psi_0^{\frac{(p+2)(q+1)}{2(p+1)q}} \frac{\log\frac{n}{\epsilon}}{\theta}.$$

Proof. Using Lemma A.0.13 we get

$$K \leq \left(\frac{100\sqrt{2}\left(2 + \sinh^2(1)\coth^3(1)(p+1) + q\right)(p+1)q}{(p+2)(q+1)}\right)\Psi_0^{\frac{(p+2)(q+1)}{2(p+1)q}},$$

with K the number of inner iterations in an outer iteration. The result then follows from Corollary 1.3.8. □

For large-update method with $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, the complexity of the primal-dual IPA for SDO problems based on the new KF is
$\mathcal{O}\left((p+q)n^{\frac{(p+2)(q+1)}{2(p+1)q}} \log\frac{n}{\epsilon}\right)$ iterations complexity.
Taking $p = q = \log n$, the iteration bound becomes $\mathcal{O}\left(\sqrt{n}\log n \log\frac{n}{\epsilon}\right)$ iterations complexity.

For small-update method i.e., $\tau = \mathcal{O}(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$, we get
$\mathcal{O}((p+q)\sqrt{n}\log\frac{n}{\epsilon})$ iterations complexity for small-update methods.

2.4.3 Numerical tests

In this section, we consolidate our theoretical analysis by performing some preliminary numerical experiments. We have taken $\epsilon = 10^{-8}$, $\tau = \sqrt{n}$, and $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

TABLE 2.2: Considered kernel functions.

Kernel function	Ref.
$\psi_c(t) = \frac{t^2-1}{2} - \log t$	[92]
$\psi_{1,p}(t) = t - 1 + \frac{t^{1-p}-1}{p-1}$, $p = 2,3,4$	[103]
$\psi_2(t) = \frac{t^2-1}{2} - \log t + \lambda \tan^2(\pi \frac{1-t}{4t+2})$, $\lambda = \frac{8}{25}\pi$	[19]
$\psi_3(t) = t^2 - 2t + \frac{1}{\sin(\frac{\pi t}{t+1})}$	[59]
$\psi_{4,p}(t) = \frac{t^2-1-\log t}{2} + \frac{e^{\frac{1}{t^p}-1}-1}{2p}$, $p = 3,4$	[24]
$\psi_5(t) = \frac{1+2\coth(1)}{2\sinh^2(1)}(t^2-1) + \coth^2(t) - \coth^2(1) - \log t$	[100]
$\psi_6(t) = \frac{t^2-1}{2\sinh(1)^2} + \coth(t) - \coth(1)$	[98]
$\psi_{7,p}(t) = \frac{t^2-1}{2} + \frac{(e-1)^{p+1}}{pe(e^t-1)^p} - \frac{e-1}{e}$, $p = 2,3,4$	[65]
$\psi_8(t) = \frac{t^2-1}{2} + \sinh^2(1)\left(e^{\coth(t)-\coth(1)} - 1\right)$	[39]
$\psi_{\text{Trigo},p}(t) = \frac{t^2-1}{2} + \frac{4}{\pi p}\left(\tan^p\left(\frac{\pi}{2t+2}\right) - 1\right)$, $p = 2,3,4$	[15]
$\psi_{\text{Trigo/SR},p,q}(t) = t^2 + \frac{t^{1-q}}{q-1} - \frac{q}{q-1} + \frac{4}{\pi p}\left(\tan^p\left(\frac{\pi}{2t+2}\right) - 1\right)$, $p,q = 2,3,4$	[17]
$\psi_{\text{SR},p}(t) = \frac{t^2-1}{2} + \frac{t^{1-p}-1}{p-1}$, $p = 2$.	[83]
$\psi_{\text{Hyper},p}(t) = \frac{t^2-1}{2} + \frac{\sinh^2(1)}{p}\left(\frac{\coth^p(t)}{\coth^{p-1}(1)} - \coth(1)\right)$, $p = 2,3,4$	[41]
$\psi_{\text{Hyper/SR},p,q}(t) = \frac{t^2-1}{2} + \frac{a}{2p}\coth^p(t) - \frac{\sinh^2(1)}{2p}\coth(1) + \frac{t^{1-q}-1}{2(q-1)}$, $p,q = 2,3,4$	New

We conducted comparative numerical tests between the algorithms based on the KFs provided in Table 2.2 on the test problem SDP1 taken from Table B.4. The summary of results is given in the following table.

TABLE 2.3: Number of inner iterations for $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$.

KF	$\theta = 0.1$				$\theta = 0.3$				$\theta = 0.5$				$\theta = 0.7$				$\theta = 0.9$			
m	5	10	15	20	5	10	15	20	5	10	15	20	5	10	15	20	5	10	15	20
ψ_c	192	202	**202**	212	58	**61**	61	63	**30**	32	32	32	**17**	**18**	**18**	**18**	**15**	17	17	17
$\psi_{1,2}$	202	213	242	298	60	59	63	96	31	**31**	33	33	18	**18**	31	31	16	**16**	**16**	18
$\psi_{1,3}$	203	272	291	334	60	87	147	172	31	**31**	**31**	34	18	**18**	32	50	16	**16**	**16**	18
$\psi_{1,4}$	203	350	398	435	60	152	244	234	32	35	33	33	18	35	123	103	16	**16**	**16**	**16**
ψ_2	626	686	690	670	530	552	553	556	33	65	67	67	504	510	512	512	15	558	558	558
ψ_3	193	205	204	214	**57**	**61**	61	**61**	**30**	32	**31**	**31**	18	**18**	**18**	**18**	**15**	17	17	17
$\psi_{4,3}$	198	211	210	222	59	63	63	86	31	32	32	32	18	19	19	26	**15**	**16**	**16**	**16**
$\psi_{4,4}$	200	213	212	212	59	63	63	63	31	33	33	33	18	19	19	19	16	**16**	**16**	**16**
ψ_5	699	677	678	688	547	557	555	557	50	52	529	530	502	503	505	510	37	558	558	585
ψ_6	193	205	204	**206**	**57**	**61**	63	63	**30**	32	32	32	**17**	**18**	**18**	**18**	**15**	17	17	17
$\psi_{7,2}$	192	689	683	702	**57**	539	546	547	**30**	34	34	34	**17**	–	–	–	**15**	17	–	–
$\psi_{7,3}$	664	681	655	679	520	538	535	544	34	49	49	49	–	–	–	–	**15**	–	–	–
$\psi_{7,4}$	667	664	695	698	546	548	554	556	44	56	56	56	–	–	–	–	–	–	–	–
ψ_8	192	664	697	702	**57**	527	540	546	**30**	34	34	34	**17**	504	506	512	**15**	17	17	35
$\psi_{\text{Trigo},2}$	192	204	679	654	**57**	**61**	150	537	**30**	**31**	35	35	**17**	**18**	76	50	**15**	17	17	17
$\psi_{\text{Trigo},3}$	192	648	699	663	**57**	529	537	546	**30**	34	34	34	**17**	54	71	514	**15**	17	17	35
$\psi_{\text{Trigo},4}$	597	655	679	689	**57**	530	546	545	**30**	37	37	37	**17**	502	504	501	**15**	17	60	–
$\psi_{\text{Trigo/SR},2,2}$	664	688	686	706	544	547	556	559	34	526	528	523	502	–	590	515	–	–	–	–
$\psi_{\text{Trigo/SR},3,2}$	669	687	697	699	543	546	561	560	34	527	529	527	509	508	533	601	–	561	–	–
$\psi_{\text{Trigo/SR},4,3}$	676	685	684	706	548	555	551	565	37	526	527	528	510	508	588	–	502	–	–	–
$\psi_{\text{SR},2}$	193	204	203	213	**57**	**61**	**60**	64	**30**	32	**31**	33	**17**	**18**	**18**	**18**	**15**	**16**	**16**	**16**
$\psi_{\text{Hyper/SR},2,2}$	**191**	202	207	209	**57**	**61**	63	208	**30**	**31**	**31**	**31**	**17**	**18**	**18**	**18**	**15**	**16**	**16**	**16**
$\psi_{\text{Hyper/SR},3,2}$	192	202	207	209	**57**	70	61	419	**30**	**31**	**31**	**31**	**17**	**18**	**18**	**18**	**15**	**16**	**16**	**16**
$\psi_{\text{Hyper/SR},4,3}$	**191**	**201**	205	210	**57**	66	414	125	**30**	**31**	**31**	**31**	**17**	**18**	**18**	**18**	**15**	**16**	**16**	**16**
$\psi_{\text{Hyper},2}$	192	675	657	649	**57**	589	533	548	**30**	35	37	37	**17**	505	511	507	**15**	17	17	17
$\psi_{\text{Hyper},3}$	192	682	679	674	**57**	535	529	547	**30**	34	34	34	**17**	73	109	109	**15**	17	35	503
$\psi_{\text{Hyper},4}$	647	658	669	688	511	537	552	550	**30**	34	34	36	**17**	503	503	503	**15**	35	40	–

Comments

Recall that the numerical results were obtained by performing Algorithm 2 with the KFs defined in Table 2.2. We used **bold** font to highlight the best, i.e., the smallest, iteration number. The dash (–) in Table 2.3 indicates that the algorithm requires a huge number of iterations to obtain the optimal solution. By comparing the results in Table 2.3, we can easily see that the algorithm based on our KF attains the most wins, on average, among all considered algorithms. In particular, $\psi_{\text{Hyper/SR}}$ far outperformed ψ_{Hyper}. In fact, in all experiments $\psi_{\text{Hyper/SR}}$ doesn't exceed 209 iterations except in two cases, while ψ_{Hyper} takes more than 500 iterations 28 times. This confirms that adding the SR barrier term to the hyperbolic barrier one has affected positively the number of iterations of the algorithm without reducing the performance of the SR KF. But this is not always satisfied, as we can see for the trigonometric barrier KF (compare the number of iterations of ψ_{Trigo} and $\psi_{\text{Trigo/SR}}$ in Table 2.3) .

2.5 Comparative numerical tests between the new kernel functions for solving SDO problems

In this section, we conduct comparative numerical experiments between the new KFs provided in Table 1.22 for solving SDO problems. We have taken $\epsilon = 10^{-8}$, $\tau = \sqrt{n}$, and $\theta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. As in Section 2.4.3, we chose the practical step size α [54]. Our purpose is to compare the computational performance of the new hyperbolic KFs provided in Table 1.22 on the two test problems defined in Table B.4 with different sizes $m \in \{5, 10, 15, 20\}$. The summary of results is given in the following tables.

TABLE 2.4: Number of inner iterations for Example SDP1.

KF	$\theta = 0.1$				$\theta = 0.3$				$\theta = 0.5$				$\theta = 0.7$				$\theta = 0.9$			
m	5	10	15	20	5	10	15	20	5	10	15	20	5	10	15	20	5	10	15	20
ψ_1	192	664	697	702	**57**	527	540	546	**30**	34	34	34	**17**	504	506	512	**15**	17	17	35
$\psi_{2,2}$	198	211	211	211	59	63	63	63	**30**	32	32	32	18	19	19	19	16	**16**	**16**	**16**
$\psi_{2,3}$	197	210	210	210	59	63	63	63	**30**	32	32	32	18	19	19	19	16	**16**	**16**	**16**
$\psi_{2,4}$	197	210	210	210	59	63	63	63	**30**	32	32	32	18	19	19	19	16	**16**	**16**	**16**
$\psi_{2,\frac{\log n}{2}-1}$	199	212	212	211	59	63	63	63	31	33	33	33	18	19	19	19	16	**16**	**16**	**16**
$\psi_{3,2}$	192	675	657	649	**57**	589	533	548	**30**	35	37	37	**17**	505	511	507	**15**	17	17	17
$\psi_{3,3}$	192	682	679	674	**57**	535	529	547	**30**	34	34	34	**17**	73	109	109	**15**	17	35	503
$\psi_{3,4}$	647	658	669	688	511	537	552	550	**30**	34	34	36	**17**	503	503	503	**15**	35	40	–
$\psi_{3,\frac{\log n}{2}-1}$	197	209	209	209	58	62	62	**62**	**30**	32	32	32	18	19	19	19	16	**16**	**16**	**16**
$\psi_{4,2,2}$	**191**	202	207	**209**	**57**	**61**	63	208	**30**	**31**	**31**	**31**	**17**	**18**	**18**	**18**	**15**	**16**	**16**	**16**
$\psi_{4,3,2}$	192	202	207	**209**	**57**	70	**61**	419	**30**	**31**	**31**	**31**	**17**	**18**	**18**	**18**	**15**	**16**	**16**	**16**
$\psi_{4,4,3}$	**191**	**201**	**205**	210	**57**	66	414	125	**30**	**31**	**31**	**31**	**17**	**18**	**18**	**18**	**15**	**16**	**16**	**16**
$\psi_{4,\log n,\log n}$	197	209	209	**209**	58	62	62	**62**	**30**	32	32	32	18	19	19	19	**15**	**16**	**16**	**16**
ψ_5	202	216	216	216	60	64	64	64	31	33	33	33	18	19	19	19	16	**16**	**16**	**16**

TABLE 2.5: Number of inner iterations for Example SDP2.

KF	$\theta = 0.1$				$\theta = 0.3$				$\theta = 0.5$				$\theta = 0.7$				$\theta = 0.9$			
m	5	10	15	20	5	10	15	20	5	10	15	20	5	10	15	20	5	10	15	20
ψ_1	197	210	210	210	**58**	**62**	**62**	**62**	**30**	32	**32**	**32**	30	30	30	33	19	19	19	22
$\psi_{2,2}$	197	210	210	210	**58**	63	63	63	**30**	32	**32**	**32**	19	**19**	21	21	18	18	18	20
$\psi_{2,3}$	197	210	210	210	59	**62**	**62**	**62**	**30**	32	**32**	**32**	27	29	30	30	19	19	19	22
$\psi_{2,4}$	197	210	210	210	59	**62**	**62**	**62**	**30**	32	**32**	**32**	30	30	32	32	22	22	22	22
$\psi_{2,\frac{\log n}{2}-1}$	198	211	211	211	59	63	63	63	**30**	33	**32**	**32**	**18**	**19**	**19**	**19**	**16**	18	18	**18**
$\psi_{3,2}$	197	210	210	210	59	**62**	**62**	**62**	**30**	32	**32**	**32**	24	24	26	26	18	18	18	20
$\psi_{3,3}$	**196**	210	210	210	59	**62**	**62**	**62**	**30**	32	**32**	**32**	30	30	32	32	18	18	18	18
$\psi_{3,4}$	**196**	**209**	**209**	**209**	65	65	65	65	**30**	32	**32**	**32**	33	33	33	36	22	22	22	22
$\psi_{3,\frac{\log n}{2}-1}$	197	209	209	209	**58**	**62**	**62**	**62**	**30**	32	**32**	**32**	20	20	21	21	17	**16**	**16**	**18**
$\psi_{4,2,2}$	**196**	**209**	211	211	**58**	**62**	**62**	64	**30**	32	33	33	**18**	–	149	–	–	68	69	–
$\psi_{4,3,2}$	**196**	**209**	211	211	**58**	**62**	**62**	64	**30**	**31**	33	33	19	–	85	84	–	–	–	–
$\psi_{4,4,3}$	**196**	**209**	211	211	**58**	**62**	**62**	64	**30**	**31**	**32**	34	**18**	**19**	–	109	–	–	–	209
$\psi_{4,\log n,\log n}$	**196**	**209**	211	211	**58**	**62**	**62**	64	**30**	**31**	**32**	34	**18**	506	–	108	–	74	–	–
ψ_5	202	215	215	215	60	64	64	64	31	33	33	33	20	20	20	21	18	18	18	**18**

Comments

Recall that the numerical results were obtained by performing Algorithm 2 with the KFs defined in Table 1.22. We used **bold** font to highlight the best, i.e., the smallest, iteration number. By comparing the results in Tables 2.4 and 2.5, we can easily see that the algorithm based on $\psi_{4,p,q}$ attains the most wins, on average, among all considered hyperbolic KFs. In particular, $\psi_{4,4,3}$ achieved the smallest iteration number in 26 out of 40 cases. On the other hand, ψ_5 exhibits poorer performance when compared to the other functions under consideration.

Chapter 3

Infeasible primal-dual interior-point algorithm for linear optimization based on a kernel function

In this chapter, we analyze a full-Newton step infeasible interior-point algorithm (IIPA) for solving linear optimization (LO) problems based on the following hyperbolic kernel function (KF)

$$\psi(t) = \frac{t^2-1}{4} - \int_1^t \frac{\sinh(1)}{2\sinh(y)}dy, \ \forall t > 0. \tag{3.1}$$

This is the first KF whose barrier term is the integral of a hyperbolic function. See [18, 37] for other KFs with similar forms.

Unlike feasible IPAs, this algorithm doesn't need a feasible starting point. In addition, the algorithm avoids a big-M or a two-phase approach. Each main iteration of this algorithm involves a sequence of actions, including a feasibility step, and a series of centrality steps.

The feasibility search directions are computed using the hyperbolic KF (3.1); however, the centring search directions are obtained using the classical KF. Using some mild properties, the complexity analysis for the primal-dual infeasible interior-point method (IIPM) based on the proximity function induced by ψ indicates that the iteration bound of the algorithm matches the currently best iteration bound for IIPMs. We consolidate these theoretical results by performing some numerical experiments. The results of this chapter were published in [38].

3.1 Introduction

The LO problem has gained growing attention in academic literature both from the theoretical and computational points of view. This model has found extensive use in modelling diverse problems across fields like economics, engineering, and operations research.

The first modelisation of an economic problem in the form of a LO problem was made by the Russian mathematician L.V. Kantorovich (1939, [52]) and the general

formulation was given later by the American mathematician G.B. Dantzig in his work [22].

Various methods were proposed to solve LO problems. IPMs, that were first developed by Karmarkar [53], are among the most popular methods. They are based on using Newton's method in a careful and controlled manner.

Initially, IPAs that necessitated a feasible initial-point were investigated [63, 75]. However, it is important to note that a feasible initial point is not generally available. Lustig [70] offered a solution to this issue when he introduced the first infeasible-start algorithm. His method was further refined in the predictor-corrector algorithm of Mehrotra [74]. After that, Roos [90] presented a new IIPA, which uses only full-Newton steps. Some variants on Roos's approach were conducted by Liu and Sun [67], Liu et al. [68] and Mansouri and Roos [71].

Salahi et al. [94] introduced a new IIPM for LO based on a specific self-regular KF. Recently, Kheirfam and Haghighi [57, 58] and Moslemi and Kheirfam [76] studied the complexity of trigonometric proximity based IIPMs for LO and SDO problems.

An alternative approach to determine new search directions was proposed by Rigó and Darvay [23]. They presented a novel method that relies on an algebraic transformation of the centering equation of the system, which defines the central path.

3.2 Preliminaries

In this section, we present a brief description of the basics of IIPMs using KFs for LO. We introduce crucial ideas and necessary tools, such as initial point, perturbed problem, central path, and proximity measure. Let's start with recalling the standard LO problem

$$(P)\begin{cases}\min c^T x\\ Ax = b,\\ x \geq 0,\end{cases}$$

and its dual problem

$$(D)\begin{cases}\max b^T y\\ A^T y + s = c,\\ s \geq 0,\end{cases}$$

where $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = m < n$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are given.

As discussed in Section 1.1, in feasible IPMs, a triple (x, y, s) is called an $\epsilon-$ solution of (P) and (D) if

$$x^T s = 0.$$

On the other hand, in IIPMs, a triple (x, y, s) is called an $\epsilon-$ solution of (P) and (D) if the norms of the residual vectors $r_b = b - Ax$ and $r_c = c - A^T y - s$ do not exceed ϵ, with $x^T s \leq \epsilon$. In other words, a feasible triple (x, y, s) is called an ϵ-solution of (P) and (D) if

$$\max\{\|r_b\|, \|r_c\|, x^T s\} \leq \epsilon.$$

Following the usual approach of IIPMs, we choose $x^0 > 0$ and $s^0 > 0$ such that $x^0 s^0 = \mu^0 e$ for some positive number μ^0, while ${r_b}^0$ and ${r_c}^0$ denote the initial residual

vectors. In this chapter, we select the initial iterates as follows:

$$x^0 = s^0 = \xi e, \;\; y^0 = 0, \;\; \mu^0 = \xi^2, \tag{3.2}$$

with μ^0 the initial duality gap and ξ satisfies the following inequality

$$\|x^* + s^*\|_\infty \leq \xi, \text{ for some optimal solution } (x^*, y^*, s^*) \text{ of } (P) \text{ and } (D).$$

3.2.1 The perturbed problems

Following [90], for any ν with $0 < \nu \leq 1$, we consider the pair of perturbed problems (P_ν) and (D_ν)

$$(P_\nu) \begin{cases} \min(c - \nu {r_c}^0)^T x \\ Ax = b - \nu {r_b}^0, \\ x \geq 0, \end{cases}$$

$$(D_\nu) \begin{cases} \max(b - \nu {r_b}^0)^T y \\ A^T y + s = c - \nu {r_c}^0, \\ s \geq 0. \end{cases}$$

Note that if $\nu = 1$ then (P_ν) and (D_ν) satisfy the interior-point condition (IPC), i.e., (P_ν) has a feasible solution $x > 0$ and (D_ν) has a solution (y, s) with $s > 0$. This implies that the triple $(x, y, s) = (x^0, y^0, s^0)$ yields a strictly feasible solution of the pair of problems (P_ν) and (D_ν).

Theorem 3.2.1. *([106, Theorem 5.13]) The original problems (P) and (D) are feasible if and only if for each ν satisfying $0 < \nu \leq 1$ the perturbed problems (P_ν) and (D_ν) satisfy the IPC.*

Proof. We start with the first implication. Let's suppose that (P) and (D) are both feasible. Let $\bar{x}$ be a feasible solution of (P) and $(\bar{y}, \bar{s})$ a feasible solution of (D). This implies that $\bar{x} \geq 0, \bar{s} \geq 0$ and

$$A\bar{x} = b, A^T\bar{y} + \bar{s} = c.$$

Now, let's define the triplet (x, y, s) as follows

$$x = (1-\nu)\bar{x} + \nu x^0, y = (1-\nu)\bar{y} + \nu y^0, s = (1-\nu)\bar{s} + \nu s^0,$$

with $0 < \nu \leq 1$. x and s are obviously strictly positive since $\bar{x}, \bar{s}$ are positive and x^0, s^0 are strictly positive. In addition,

$$Ax = (1-\nu)A\bar{x} + \nu Ax^0 = (1-\nu)b + \nu Ax^0 = b - \nu(b - Ax^0),$$

which means that x is feasible for (P_ν). In a similar way we have

$$\begin{aligned} A^T y + s &= (1-\nu)(A^T\bar{y} + \bar{s}) + \nu(A^T y^0 + s^0) \\ &= (1-\nu)c + \nu(A^T y^0 + s^0) \\ &= c - \nu(c - A^T y^0 - s^0), \end{aligned}$$

indicating that (y, s) is feasible for (D_ν). Hence, (P_ν) and (D_ν) verify the IPC.
For the inverse implication, we suppose that for each $0 < \nu \leq 1$, (P_ν) and (D_ν) verify

the IPC. This implies that for each $0 < \nu \leq 1$, (P_ν) and (D_ν) are feasible. Letting ν go to zero, it can be deduced that both (P) and (D) are feasible. □

3.2.2 The central path of the perturbed problems

In what follows, we assume that (P) and (D) are feasible. Then, Theorem 3.2.1 implies that for every $0 < \nu \leq 1$, the perturbed problems (P_ν) and (D_ν) satisfy the IPC. Therefore, using Theorem 1.1.1, it follows that there exists an optimal solution for the pair of problems (P_ν) and (D_ν). In addition, finding an optimal solution of (P_ν) and (D_ν) is equivalent to solving the non-linear system of equations

$$\begin{cases} b - Ax = \nu r_b{}^0, x \geq 0, \\ c - A^T y - s = \nu r_c{}^0, s \geq 0, \\ xs = \mu e, \quad \mu > 0. \end{cases} \tag{3.3}$$

The set of unique solutions

$$\{(x(\mu,\nu), y(\mu,\nu), s(\mu,\nu)) : \mu > 0,\ 0 < \nu \leq 1\},$$

defines the central path with $(x(\mu,\nu), y(\mu,\nu), s(\mu,\nu))$ denote the μ-centers of (P_ν) and (D_ν). In the sequel, the parameters μ and ν always satisfy the relation $\mu = \nu\mu^0$ and we denote $(x(\mu,\nu), y(\mu,\nu), s(\mu,\nu)) = (x(\nu), y(\nu), s(\nu))$ for simplicity purposes. It can be noticed from system (3.3) that the parameters ν and μ control the feasibility and the optimality, respectively. In addition, taking $\nu = 0$ and thus $\mu = 0$, system (3.3) coincides with the KKT system (1.1) in Section 1.1.

3.2.3 An iteration of the algorithm

Before we define the search directions, we showcase the outline of one iteration of the IIPA. Starting by the initialization defined in (3.2), each main iteration consists of a so-called feasibility step, a μ-update and a few centrality steps. The feasibility step provides iterates (x^f, y^f, s^f) that are strictly feasible for (P_{ν^+}) and (D_{ν^+}), with $\nu^+ = (1-\theta)\nu$ and $\theta \in]0,1[$. These iterates belong to the quadratic convergence region with respect to the μ^+-center of (P_{ν^+}) and (D_{ν^+}), with $\mu^+ = \xi^2\nu^+$. After that, we apply a limited number of centering steps (at most 5 centering steps). These centering steps produces iterates (x^+, y^+, s^+) that are strictly feasible for (P_{ν^+}) and (D_{ν^+}), and such that $\delta(x^+, s^+; \mu^+) \leq \tau$. This process is repeated until the duality gap $x^T s$ and the norms of the residual vectors r_b and r_c are less than some prescribed accuracy parameter ϵ.

A graphical illustration of an iteration of the algorithm is given by Figure 3.1. The straight lines depict the central paths of the pairs (P_ν) and (D_ν) and (P_{ν^+}) and (D_{ν^+}). The dark gray circles represents the $\tau-$neighbourhoods of the μ and μ_+-centers. The proximity measure $\delta(x,s;\mu)$, denoted by $\Phi(x,s;\mu)$ in the graphical illustration, will be defined later in the centrality step.

The region in light gray shows the quadratically convergent region of the μ^+-center of (P_{ν^+}) and (D_{ν^+}). The quadratically convergent region of the μ^+-center is the set of primal-dual (feasible) pairs (x^f, s^f) satisfying

$$\delta(x^f, s^f; \mu^+) \leq \tau^f.$$

The Newton steps are illustrated by the arrows and the iterates by the circles. Each iteration starts at a point inside the $\tau-$neighbourhood of the μ-centers of (P_ν) and (D_ν). After a feasibility step, one gets iterates inside the light gray region. Using a few centering steps, one obtains iterates in the dark gray neighbourhood of the μ_+-center of (P_{ν^+}) and (D_{ν^+}).

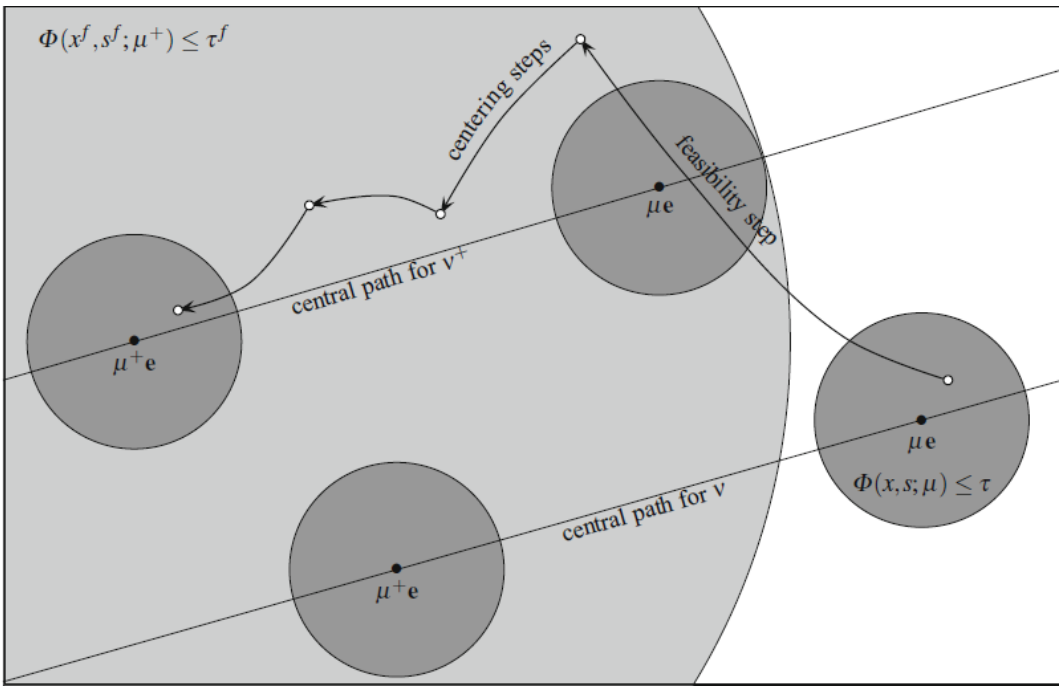

FIGURE 3.1: An illustration of an iteration of the algorithm

3.2.4 Feasibility step

To generate iterates (x^f, y^f, s^f) that are strictly feasible for (P_{ν^+}) and (D_{ν^+}), we have to solve the following system of equations

$$\begin{cases} A\Delta^f x = \theta\nu r_b^0, \\ A^T\Delta^f y + \Delta^f s = \theta\nu r_c^0, \\ s\Delta^f x + x\Delta^f s = \mu e - xs. \end{cases} \tag{3.4}$$

This system has a unique solution $(\Delta^f x, \Delta^f y, \Delta^f s)$. The feasible new iterates are then defined by

$$x^f = x + \Delta^f x, \quad y^f = y + \Delta^f y, \quad s^f = s + \Delta^f s. \tag{3.5}$$

Let the scaled search directions d_x^f and d_s^f be defined as follows:

$$v = \sqrt{\frac{xs}{\mu}}, \; d_x^f = \frac{v\Delta^f x}{x}, \; d_s^f = \frac{v\Delta^f s}{s}. \tag{3.6}$$

Using these definitions, system (3.4) is then rewritten in the following form

$$\begin{cases} \overline{A}d_x^f = \theta v r_b^0, \\ \overline{A}^T \frac{\Delta^f y}{\mu} + d_s^f = \theta v v s^{-1} r_c^0, \\ d_x^f + d_s^f = v^{-1} - v, \end{cases} \tag{3.7}$$

where $\overline{A} = AV^{-1}X$, $V = \text{diag}(v)$ and $X = \text{diag}(x)$.
Observe that the right-hand side in the last equation of (3.7) is equal to minus gradient of the classical logarithmic scaled barrier (proximity) function

$$\Psi(v) = \sum_{i=1}^{n} \psi_c(v_i), \tag{3.8}$$

where

$$\psi_c(t) = \frac{t^2 - 1}{2} - \log t,$$

is the so-called KF of the barrier function $\Psi(v)$.
Coming back to system (3.7), we can convert it to

$$\begin{cases} \overline{A}d_x^f = \theta v r_b^0, \\ \overline{A}^T \frac{\Delta^f y}{\mu} + d_s^f = \theta v v s^{-1} r_c^0, \\ d_x^f + d_s^f = -\nabla\Psi(v). \end{cases} \tag{3.9}$$

Similarly to Section 1.1, we replace ψ_c by the KF ψ defined in (3.1). The corresponding proximity function Ψ is then obtained by replacing ψ_c by ψ in (3.8). We also define a new proximity mesure based on this KF, as follows:

$$\sigma(x, s; \mu) = \sigma(v) = \left\| d_x^f + d_s^f \right\| = \|\nabla\Psi(v)\| = \left\| \frac{\sinh(e)}{2\sinh(v)} - \frac{v}{2} \right\|. \tag{3.10}$$

We can easily verify that

$$\sigma(v) = 0 \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow v = e.$$

This implies that σ can be indeed considered as a proximity measure.

3.2.5 Centrality step

After the feasibility step, we perform some centring steps to get new iterates in the desired τ- neighbourhood of the μ^+-center. Starting with $(x, y, s) = (x^f, y^f, s^f)$, the centring step is obtained by taking full steps with the search direction $(\Delta x, \Delta y, \Delta s)$ solution of the following Newton system

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ s\Delta x + x\Delta s = \mu e - xs. \end{cases} \tag{3.11}$$

Following this centrality step, the new point (x^+, y^+, s^+) is then computed according to

$$x^+ = x + \Delta x,\ y^+ = y + \Delta y,\ s^+ = s + \Delta s. \tag{3.12}$$

For convenience, we introduce the following scaled vector v and scaled search directions d_x and d_s

$$v = \sqrt{\frac{xs}{\mu}},\ \ d_x = \frac{v\Delta x}{x},\ \ d_s = \frac{v\Delta s}{s}. \tag{3.13}$$

Using (3.13), system (3.11) can be transformed into the following form

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T \frac{\Delta y}{\mu} + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{3.14}$$

where $\overline{A} = AV^{-1}X,\ \ V = \text{diag}(v)$ and $X = \text{diag}(x)$. The third equation in (3.14) is called the scaled centering equation.

Defining the proximity mesure δ as follows:

$$\delta(x, s; \mu) := \delta(v) := \frac{1}{2}\|v^{-1} - v\|. \tag{3.15}$$

This proximity mesure was first used by Jansen et al. [51] and later, with some minor modifications, by Roos et al. [92].

Note that

$$v = e \Leftrightarrow xs = \mu e.$$

This means that the variance vector v is the all-one vector if and only if the iterates x and (y, s) are the μ-centers. In addition,

$$\delta(v) = 0 \Leftrightarrow v = e,$$

which implies that δ vanishes only at the μ-center. Alternatively, it can be expressed as

$$\delta(v) = 0 \Leftrightarrow v = e \Leftrightarrow xs = \mu e.$$

As a consequence, we can use δ to measure the distance between an iterate (x, y, s) and the μ-center of the perturbed problem pair (P_ν) and (D_ν).

We end this section by providing a brief description of the algorithm corresponding to the primal-dual IIPM based on KFs summarized in Algorithm 3. Starting with (x^0, y^0, s^0) defined in (3.2), we solve system (3.9) and use (3.6) and (3.5) to obtain (x^f, y^f, s^f) that are strictly feasible for (P_{ν^+}) and (D_{ν^+}), with $\nu^+ = (1-\theta)\nu$ and $\theta \in]0, 1[$. The iterate (x^f, y^f, s^f) verifies $\delta(x^f, s^f, \mu^+) \leq \frac{1}{\sqrt[4]{2}}$. Then, we reduce μ to $\mu_+ := (1-\theta)\mu$ for some fixed $0 < \theta < 1$. After that, we perform some centring steps. In each centering step, we solve the system (3.14) to obtain the search direction $(d_x, \Delta y, d_s)$. Then, we use notations (3.13) to obtain $(\Delta x, \Delta s)$. The new iterate (x_+, y_+, s_+) is computed using (3.12). The centring step is repeated until $\delta(v_+) \leq \tau$. Then, we let $(x, y, s) = (x_+, y_+, s_+)$. This procedure is repeated until the duality gap $x^T s$ and the norms of the residual vectors r_b and r_c are less than some prescribed accuracy parameter ϵ. In this case, an ϵ-approximate optimal solution of problems (P) and (D) is found.

Algorithm 3 : Primal-Dual Infeasible Interior-Point Algorithm for Linear Optimization

Input
a threshold parameter $\tau > 0$;
an accuracy parameter $\epsilon > 0$;
a fixed barrier update parameter $\theta \in]0,1[$;
an initialization parameter $\xi > 0$.

begin
$x := \xi e,\ y := 0,\ s := \xi e,\ \nu := 1;$

while $\max(x^T s, \|b - Ax\|, \|c - A^T y - s\|) \geq \epsilon$

begin
feasibility step:

Solve system (3.7) to get $(d_x^f, \Delta^f y, d_s^f)$;

Use (3.6) to obtain $(\Delta^f x, \Delta^f s)$;

$(x, y, s) := (x, y, s) + (\Delta^f x, \Delta^f y, \Delta^f s)$;

$\mu-$update:

$\mu := (1 - \theta)\mu$;

centering steps:
while $\delta(x, s; \mu) > \tau$

Solve system (3.14) to get $(d_x, \Delta y, d_s)$;

Use (3.13) to obtain $(\Delta x, \Delta s)$;

$(x, y, s) := (x, y, s) + (\Delta x, \Delta y, \Delta s)$;

end while (inner iteration)

end while (outer iteration)

end

3.3 Analysis of the algorithm

In this section, we prove that the IIPM based on the new KF (3.1) is well defined. Initially, we present some technical lemmas that we need in the complexity analysis of the algorithm. Subsequently, we establish the strict feasibility of the iterates derived from the feasibility step. Following this, we establish an upper limit on the number of iterations necessary for the algorithm to achieve an optimal solution.

3.3.1 Technical lemmas

The algorithm's analysis relies heavily on the proximity measure δ defined in (3.15). In what follows, we denote $\delta(x, s; \mu)$ simply by δ. Then, we recall the following two pivotal lemmas. The first lemma provides a simple bound on the coordinates of the vector v defined in (3.13) in terms of the proximity measure δ. Whereas the second lemma describes the improvement in the proximity after a feasible Newton step.

Lemma 3.3.1. *([92, Lemma II.60]) Let $\rho(\delta) := \delta + \sqrt{1+\delta^2}$. Then*

$$\frac{1}{\rho(\delta)} \leq v_i \leq \rho(\delta),\ i = 1, \ldots, n.$$

Proof. From the definition of v, we can conclude that

$$v_i \geq 0,\ i = 1, \ldots, n.$$

In addition

$$\delta(v) = \frac{1}{2}\left\|v^{-1} - v\right\| = \frac{1}{2}\left\|\frac{e - v^2}{v}\right\|.$$

Using some simple calculations as well as the Hölder inequality, we get

$$-2\delta v_i \leq 1 - v_i^2 \leq 2\delta v_i,$$

which implies that

$$(v_i - \delta)^2 - 1 - \delta^2 \leq 0 \leq (v_i + \delta)^2 - 1 - \delta^2,$$

and

$$(v_i - \delta)^2 \leq 1 + \delta^2 \leq (v_i + \delta)^2.$$

Thus,

$$v_i - \delta \leq |v_i - \delta| \leq \sqrt{1+\delta^2} \leq v_i + \delta.$$

Therefore, it follows that

$$v_i \leq \sqrt{1+\delta^2} + \delta = \rho(\delta),$$

and

$$v_i \geq \sqrt{1+\delta^2} - \delta = \frac{1}{\sqrt{1+\delta^2}+\delta} = \frac{1}{\rho(\delta)}.$$

□

Lemma 3.3.2. *([92, Lemma II.51]) If $\delta \leq 1$, then the primal-dual Newton step is feasible, i.e. x^+ and s^+ are nonnegative and $(x^+)^T s^+ = n\mu$. Moreover, if $\delta < 1$, then*

$$\delta(x^+, s^+; \mu) \leq \frac{\delta^2}{\sqrt{2(1-\delta^4)}}.$$

Remark 3.3.3. *The previous lemma presents a sharper version of Lemma II.49 in [92]. The latter was used in the complexity analysis of the IIPA proposed in [90].*

A direct consequence of this lemma is the following corollary.

Corollary 3.3.4 ([58]). *If* $\delta \leq \frac{1}{\sqrt[4]{2}}$ *then* $\delta(x^+, s^+; \mu) \leq \delta^2$ *and*

$$\delta(x^+, s^+; \mu^+) \leq \left(\frac{1}{\sqrt[4]{2}}\right)^{2^k}.$$

Proof. The first part of the corollary is a direct consequence of Lemma 3.3.2. For the second part, recall that after the feasibility step we perform centering steps in order to get iterates (x^+, y^+, s^+) that satisfy ${x^+}^T s^+ = n\mu^+$ and

$$\delta(x^+, s^+; \mu^+) \leq \tau.$$

Assuming that $\delta = \delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt[4]{2}}$, after k centering steps we get iterates (x^+, y^+, s^+) that are still feasible for (P_{ν^+}) and (D_{ν^+}) and that satisfy

$$\delta(x^+, s^+; \mu^+) \leq \left(\frac{1}{\sqrt[4]{2}}\right)^{2^k}.$$

□

Remark 3.3.5. *From Corollary 3.3.4, it follows that* $\delta(x^+, s^+; \mu^+) \leq \tau$ *will holds after*

$$2 + [\log_2(\log_2 \frac{1}{\tau})]$$

centering steps.

Recall that ψ is defined for all $t > 0$ as follows

$$\psi(t) = \frac{t^2 - 1}{4} - \int_1^t \frac{\sinh(1)}{2\sinh(y)} dy.$$

For all $t > 0$, differentiating ψ two times we get

$$\psi'(t) = \frac{t}{2} - \frac{\sinh(1)}{2\sinh(t)},$$

$$\psi''(t) = \frac{1}{2} + \frac{\sinh(1)\cosh(t)}{2\sinh^2(t)} \geq 1.$$

Clearly, $\psi'(1) = \psi(1) = 0$, $\lim_{t \to 0^+} \psi(t) = \lim_{t \to +\infty} \psi(t) = +\infty$ and $\psi''(t) > 0, \forall t > 0$. Thus, according to Definition 1.1.5, ψ is indeed a KF.

The complexity analysis of our algorithm relies heavily on the following lemma.

Lemma 3.3.6. *One has*

$$\left|\frac{t}{2} - \frac{\sinh(1)}{2\sinh(t)}\right| \leq \left|t - \frac{1}{t}\right|, \ \forall t > 0. \tag{3.16}$$

Proof. It's clear that (3.16) is satisfied for $t = 1$. Let g be the function defined for $t > 0$ as follows:

$$\begin{aligned} g(t) &= \left|\frac{t}{2} - \frac{\sinh(1)}{2\sinh(t)}\right| - \left|t - \frac{1}{t}\right| \\ &= \begin{cases} \frac{1}{t} - \frac{t}{2} - \frac{\sinh(1)}{2\sinh(t)}, & \forall t \geq 1, \\ \frac{t}{2} + \frac{\sinh(1)}{2\sinh(t)} - \frac{1}{t}, & \forall t \leq 1. \end{cases} \end{aligned}$$

For this function, we have

$$g'(t) = \begin{cases} \frac{\sinh(1)\coth(t)}{2\sinh(t)} - \left(\frac{1}{t^2} + \frac{1}{2}\right), & \forall t > 1, \\ \left(\frac{1}{t^2} + \frac{1}{2}\right) - \frac{\sinh(1)\coth(t)}{2\sinh(t)}, & \forall t < 1. \end{cases}$$

An important observation is that $\lim_{t\to 0^+} g(t) = \lim_{t\to+\infty} g(t) = -\infty$ and $g(1) = 0$. So, to prove the inequality (3.16), it suffices to verify that

$$g'(t) > 0, \ \forall 0 < t < 1, \ \text{ and } \ g'(t) < 0, \ \forall t > 1.$$

We start by the case $t > 1$. Since the function $t \mapsto \sinh(t)$ is monotonically increasing, this implies that

$$g'(t) < \frac{t^2(\coth(t) - 1) - 2}{2t^2} =: \frac{h(t)}{2t^2}.$$

By deriving h, and using inequality (1.28) from Lemma 1.3.1, we can easily prove that $h(t) < 0, \ \forall t > 1$.
Now, we move to the second case, i.e., $t < 1$. We rewrite g' as follows:

$$\begin{aligned} g'(t) =& \frac{(t^2+2)\sinh^2(t) - \sinh(1)t^2\cosh(t)}{2t^2\sinh^2(t)} \\ =& \frac{t^2\sinh^2(t) + \left(2\sinh^2(t) - \sinh(1)t^2\cosh(t)\right)}{2t^2\sinh^2(t)} \\ >& \frac{t^2\sinh^2(t) + \left(2\sinh^2(t) - \sinh(1)\cosh(1)t^2\right)}{2t^2\sinh^2(t)} \\ >& \frac{t^2\sinh^2(t) + 2\left(\sinh^2(t) - t^2\right)}{2t^2\sinh^2(t)} > 0, \end{aligned}$$

where the inequalities are obtained using the increase of the hyperbolic cosine function on $[0, +\infty[$, the fact that $\sinh(1)\cosh(1) < 2$, and that $\sinh(t) \geq t$, for all $t > 0$. This completes the proof. □

Remember that, in this chapter, we retain the logarithmic barrier function and its proximity measure δ, despite introducing a new KF to define the feasibility step. The previous lemma provides an important feature of ψ which enables us to provide an upper bound for the proximity measure σ in terms of δ in the following result

Lemma 3.3.7. *One has*

$$\sigma(v) \leq 2\delta(v).$$

Proof. The inequality is directly obtained using Lemma 3.3.6 and definitions (3.10) and (3.15). □

3.3.2 Analysis of the feasibility step

In this subsection, we demonstrate that following the feasibility step, the new iterates (x^f, y^f, s^f) are strictly positive and fall within the quadratically convergent region of the μ^+-center of (P_{v^+}) and (D_{v^+}), i.e.

$$\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt[4]{2}}.$$

Recall that from (3.9)

$$d_x^f + d_s^f = -\nabla\Psi(v) = \left(\frac{\sinh(e)}{2\sinh(v)} - \frac{v}{2}\right).$$

Hence, using (3.5) and (3.6) we may write

$$\begin{aligned} x^f s^f &= \frac{xs}{v^2}(v + d_x^f)(v + d_s^f) \\ &= \mu(v^2 + v(d_x^f + d_s^f) + d_x^f d_s^f) \\ &= \mu\left(v^2 + v\left(\frac{\sinh(e)}{2\sinh(v)} - \frac{v}{2}\right) + d_x^f d_s^f\right) \\ &= \mu\left(\frac{v^2}{2} + \frac{v\sinh(e)}{2\sinh(v)} + d_x^f d_s^f\right). \end{aligned} \tag{3.17}$$

In the next lemma, we provide a sufficient condition to guarantee the strict feasibility of the feasibility step.

Lemma 3.3.8. *The new iterates (x^f, y^f, s^f) are strictly feasible if*

$$\frac{v^2}{2} + \frac{v\sinh(e)}{2\sinh(v)} + d_x^f d_s^f > 0. \tag{3.18}$$

Proof. Let us define

$$x^\alpha = x + \alpha\Delta^f x, \quad y^\alpha = y + \alpha\Delta^f y, \quad s^\alpha = s + \alpha\Delta^f s,$$

with $0 \leq \alpha \leq 1$ a step length. Taking $\alpha = 0$ ($\alpha = 1$), we get $x^0 = x$ ($x^1 = x^f$) respectively and $x^0 s^0 > 0$. Hence, using notations (3.6), (3.5) and inequality (3.18),

we have

$$\begin{aligned}
x^\alpha s^\alpha &= \mu(v+\alpha d_x^f)(v+\alpha d_s^f)\\
&= \mu\left(v^2+\alpha v(d_x^f+d_s^f)+\alpha^2 d_x^f d_s^f\right)\\
&= \mu\left(v^2+\alpha\left(\frac{v\sinh(e)}{2\sinh(v)}-\frac{v^2}{2}\right)+\alpha^2 d_x^f d_s^f\right)\\
&> \mu\left(v^2+\alpha\left(\frac{v\sinh(e)}{2\sinh(v)}-\frac{v^2}{2}\right)+\alpha^2\left(-\frac{v^2}{2}-\frac{v\sinh(e)}{2\sinh(v)}\right)\right)\\
&> \mu\left(v^2+\alpha\frac{v\sinh(e)}{2\sinh(v)}-\alpha\frac{v^2}{2}-\alpha^2\frac{v^2}{2}-\alpha^2\frac{v\sinh(e)}{2\sinh(v)}\right)\\
&> \mu\left(v^2+\alpha\frac{v\sinh(e)}{2\sinh(v)}-\alpha\frac{v^2}{2}-\alpha\frac{v^2}{2}-\alpha^2\frac{v\sinh(e)}{2\sinh(v)}\right)\\
&= \mu\left((1-\alpha)v^2+\alpha(1-\alpha)\frac{v\sinh(e)}{2\sinh(v)}\right).
\end{aligned}$$

Since $\dfrac{v\sinh(e)}{2\sinh(v)} \geq 0,\ \forall v>0$, it follows that

$$\left((1-\alpha)v^2+\alpha(1-\alpha)\frac{v\sinh(e)}{2\sinh(v)}\right) \geq 0.$$

Hence, we may conclude that for every $\alpha \in [0,1]$, $x^\alpha s^\alpha > 0$ which means that none of the components of x^α and s^α vanishes. Putting $\alpha = 1$, we obtain $x^f > 0$ and $s^f > 0$. □

In the rest of the chapter, we denote

$$\omega := \|(\omega_1, ..., \omega_n)\|,$$

and

$$\delta(x^f, s^f; \mu^+) := \delta(v^f) = \frac{1}{2}\|(v^f)^{-1} - v^f\|,$$

where

$$\omega_i = \frac{1}{2}\sqrt{|d_{x_i}^f|^2 + |d_{s_i}^f|^2} \text{ and } v^f = \sqrt{\frac{x^f s^f}{\mu^+}}.$$

Remark 3.3.9. *One has*

$$|d_{x_i}^f d_{s_i}^f| \leq \frac{1}{2}\left(|d_{x_i}^f|^2 + |d_{s_i}^f|^2\right) = 2{\omega_i}^2 \leq 2\omega^2,\ i = 1, \ldots, n.$$

In the subsequent lemma, we present an upper bound for δ after the feasibility step.

Lemma 3.3.10. *If* $\dfrac{v^2}{2} + \dfrac{v\sinh(e)}{2\sinh(v)} + d_x^f d_s^f > 0$, *then*

$$\delta(v^f) \leq \frac{\theta\sqrt{n} + 4\delta\rho(\delta) + 2\omega^2}{2\sqrt{(1-\theta)\left(\frac{1}{\rho(\delta)^2} - 2\delta\rho(\delta) - 2\omega^2\right)}}.$$

Proof. From (3.15) we have

$$\delta(v^f) = \frac{1}{2}\|(v^f)^{-1} - v^f\| \leq \frac{1}{2v^f_{\min}}\|e - (v^f)^2\|. \tag{3.19}$$

Using (3.17) and Remark 3.3.9, it follows that

$$\begin{aligned}
\|e - (v^f)^2\| \leq & \frac{1}{1-\theta}\left\|(1-\theta)e - \frac{v^2}{2} - \frac{v\sinh(e)}{2\sinh(v)} - d_x^f d_s^f\right\| \\
= & \frac{1}{1-\theta}\left\|(e - v^2) - \theta e + v\left(\frac{v}{2} - \frac{\sinh(e)}{2\sinh(v)}\right) - d_x^f d_s^f\right\| \\
= & \frac{1}{1-\theta}\left\|v(v^{-1} - v) - \theta e + v\left(\frac{v}{2} - \frac{\sinh(e)}{2\sinh(v)}\right) - d_x^f d_s^f\right\| \\
\leq & \frac{1}{1-\theta}\left(\|v(v^{-1} - v)\| + \theta\sqrt{n} + \left\|v\left(\frac{v}{2} - \frac{\sinh(e)}{2\sinh(v)}\right)\right\| + 2\omega^2\right).
\end{aligned} \tag{3.20}$$

Furthermore, using (3.15), Lemma 3.3.1 and Lemma 3.3.6 we get the following inequalities

$$\begin{aligned}
\left\|v\left(\frac{1}{v} - v\right)\right\|^2 &= \sum_{i=1}^{n}\left|v_i\left(\frac{1}{v_i} - v_i\right)\right|^2 \\
&\leq \rho(\delta)^2\sum_{i=1}^{n}\left|\frac{1}{v_i} - v_i\right|^2 \\
&= 4\rho(\delta)^2\delta^2,
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\left\|v\left(\frac{v}{2} - \frac{\sinh(e)}{2\sinh(v)}\right)\right\|^2 &= \sum_{i=1}^{n}\left|v_i\left(\frac{v_i}{2} - \frac{\sinh(1)}{2\sinh(v_i)}\right)\right|^2 \\
&\leq \rho(\delta)^2\sum_{i=1}^{n}\left|\frac{1}{v_i} - v_i\right|^2 \\
&= 4\rho(\delta)^2\delta^2.
\end{aligned} \tag{3.22}$$

Substituting inequalities (3.21) and (3.22) into (3.20) produces the following inequality

$$\left\|e - (v^f)^2\right\| \leq \frac{1}{1-\theta}\left(4\rho(\delta)\delta + \theta\sqrt{n} + 2\omega^2\right). \tag{3.23}$$

In addition, using the definition of v^f and (3.17), we have

$$\begin{aligned}
(v^f_{\min})^2 &= \min_i \frac{1}{1-\theta}\left(\frac{{v_i}^2}{2} + \frac{v_i \sinh(1)}{2\sinh(v_i)} + d^f_{x_i} d^f_{s_i}\right) \\
&= \min_i \frac{1}{1-\theta}\left(\frac{{v_i}^2}{2} + \frac{{v_i}^2}{2} - \frac{{v_i}^2}{2} + \frac{v_i \sinh(1)}{2\sinh(v_i)} + d^f_{x_i} d^f_{s_i}\right) \\
&= \min_i \frac{1}{1-\theta}\left({v_i}^2 - \left(\frac{{v_i}^2}{2} - \frac{v_i \sinh(1)}{2\sinh(v_i)}\right) + d^f_{x_i} d^f_{s_i}\right) \\
&\geq \frac{1}{1-\theta}\left(v^2_{\min} - \left\|\frac{v^2}{2} - \frac{v\sinh(e)}{2\sinh(v)}\right\| - 2\omega^2\right) \\
&\geq \frac{1}{1-\theta}\left(\frac{1}{\rho(\delta)^2} - 2\rho(\delta)\delta - 2\omega^2\right), \qquad (3.24)
\end{aligned}$$

where the last inequality results from Lemma 3.3.1 with (3.22). Hence, by substituting (3.23) and (3.24) into (3.19), we arrive at the desired inequality. □

Corollary 3.3.11. *Let $n \geq 2$, $\delta \leq \tau$. Choosing $\tau = \frac{1}{16}$, $\omega \leq \frac{1}{2\sqrt{2}}$ and $\theta = \frac{\alpha}{4\sqrt{n}}$ with $\alpha \leq 1$, we have*

1. *the iterates (x^f, y^f, s^f) obtained after feasibility step are strictly feasible, i.e., $x^f s^f > 0$.*
2. *after the feasibility step, the new iterates (x^f, y^f, s^f) are within the region where the Newton process targeting at the μ^+-centers of (P_{v^+}) and (D_{v^+}), is quadratically convergent, i.e., $\delta(v^f) \leq \frac{1}{\sqrt[4]{2}}$.*

Proof. Recall from Lemma 3.3.8 that the new iterates (x^f, y^f, s^f) are strictly feasible if

$$\frac{v^2}{2} + \frac{v\sinh(e)}{2\sinh(v)} + d^f_x d^f_s > 0. \qquad (3.25)$$

Thus, we need to verify that the inequality (3.25) is satisfied. Using (3.24), we get

$$\begin{aligned}
\min_i \left(\frac{{v_i}^2}{2} + \frac{v_i \sinh(1)}{2\sinh(v_i)} + d^f_{x_i} d^f_{s_i}\right) &\geq \left(\frac{1}{\rho(\delta)^2} - 2\rho(\delta)\delta - 2\omega^2\right) \\
&\geq \left(\frac{1}{\rho(\tau)^2} - 2\rho(\tau)\tau - 2\omega^2\right) \\
&\simeq 0.4995 > 0,
\end{aligned}$$

where the second inequality is obtained using the fact that $\rho(\delta)$ is monotonically increasing with respect to δ. Therefore, for all $i = 1, \ldots, n$

$$0 < \min_i \left(\frac{{v_i}^2}{2} + \frac{v_i \sinh(1)}{2\sinh(v_i)} + d^f_{x_i} d^f_{s_i}\right) \leq \left(\frac{{v_i}^2}{2} + \frac{v_i \sinh(1)}{2\sinh(v_i)} + d^f_{x_i} d^f_{s_i}\right),$$

which implies that inequality (3.25) is satisfied. Thus, the first item is proved. Concerning the second item, using (3.25) and Lemma 3.3.10 we have

$$\delta(v^f) \leq \frac{\frac{1}{4}+\frac{1}{4}\left(\frac{1}{16}+\sqrt{1+\frac{1}{16^2}}\right)+\frac{1}{4}}{2\sqrt{\left(1-\frac{1}{4\sqrt{2}}\right)\left(\frac{1}{\left(\frac{1}{16}+\sqrt{1+\frac{1}{16^2}}\right)^2}-\frac{\left(\frac{1}{16}+\sqrt{1+\frac{1}{16^2}}\right)}{8}-\frac{1}{4}\right)}}$$
$$\leq 0.5974 \leq \frac{1}{\sqrt[4]{2}}.$$

□

In the following part of this chapter, let's assume that

$$\tau = \frac{1}{16},\ \omega \leq \frac{1}{2\sqrt{2}} \text{ and } \theta = \frac{\alpha}{4\sqrt{n}} \text{ with } \alpha \leq 1.$$

3.3.3 Upper bounds for ω and $\|q\|$

In this subsection, we will adopt the same approach as outlined in Section 4.3 of [90]. Let $\mathcal{L}$ be the null space of the matrix $\bar{A}$. It should be noted that the affine space

$$\{\xi \in \mathbb{R}^n : \bar{A}\xi = \theta v r_b{}^0\},$$

is equal to $d_x^f + \mathcal{L}$. Moreover, the row space of $\bar{A}$ is equal to the orthogonal complement $\mathcal{L}^\perp$ of $\mathcal{L}$ defined as follows:

$$\mathcal{L}^\perp = \left\{\bar{A}^T z : z \in \mathbb{R}^m\right\}.$$

A clear observation is that $d_s^f \in \theta v v s^{-1} r_c{}^0 + \mathcal{L}^\perp$. Additionally, $\mathcal{L}^\perp \cap \mathcal{L} = \{0\}$. As a result, the affine spaces $d_x^f + \mathcal{L}$ and $d_s^f + \mathcal{L}^\perp$ have a unique common point q, i.e., q is the solution of the system

$$\begin{cases} \overline{A}q = \theta v r_b{}^0, \\ \overline{A}^T z + q = \theta v v s^{-1} r_c{}^0. \end{cases} \tag{3.26}$$

Lemma 3.3.12. *Let q be the unique solution of (3.26). Then,*

$$\omega \leq \frac{1}{2}\sqrt{\|q\|^2 + (\|q\| + \sigma(v))^2}.$$

Proof. The lemma is obtained using the same arguments as in Lemma 4.6 in [90] with $r = -\nabla\Psi(v)$. For $q = 0$, the inequality is satisfied with equality.

Assuming that $q \neq 0$, let us denote

$$r = d_x^f + d_s^f = v^{-1} - v = -\nabla\Psi(v).$$

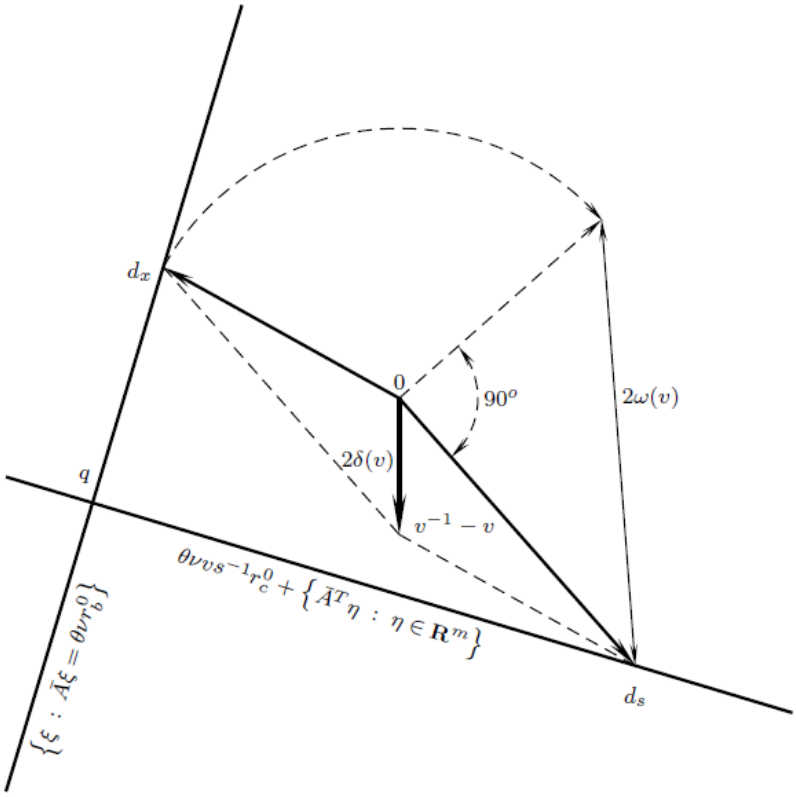

FIGURE 3.2: Geometric interpretation of $\omega := \omega(v)$

Considering that

$$\mathcal{L} \oplus \mathcal{L}^{\perp} = \mathbb{R}^n,$$

and that $r, q \in \mathbb{R}^n$, we can write

$$r = r_1 + r_2,\ q = q_1 + q_2,$$

with $q_1, r_1 \in \mathcal{L}$ and $q_2, r_2 \in \mathcal{L}^{\perp}$.
However, since $d_x^f - q \in \mathcal{L}$ and $d_s^f - q \in \mathcal{L}^{\perp}$, it follows that there exists $l_1 \in \mathcal{L}$ and $l_2 \in \mathcal{L}^{\perp}$ such that

$$d_x^f = q + l_1,\ d_s^f = q + l_2.$$

This implies that

$$r = 2q + l_1 + l_2 = 2q_1 + 2q_2 + l_1 + l_2 = r_1 + r_2.$$

As the decomposition $\mathcal{L} + \mathcal{L}^{\perp} = \mathbb{R}^n$ is unique, we can deduce that

$$l_1 = r_1 - 2q_1,\ l_2 = r_2 - 2q_2.$$

Therefore, we get

$$d_x^f = q + r_1 - 2q_1 = (r_1 - q_1) + q_2,$$

and

$$d_s^f = q + r_2 - 2q_2 = (r_2 - q_2) + q_1.$$

Using the fact that $\mathcal{L}$ and $\mathcal{L}^{\perp}$ are orthogonal, we obtain

$$4\omega(v)^2 = \|d_x^f\|^2 + \|d_s^f\|^2 = \|r_1 - q_1\|^2 + \|q_2\|^2 + \|q_1\|^2 + \|r_2 - q_2\|^2 = \|q - r\|^2 + \|q\|^2.$$

Since $\|r\| = \sigma(v)$, we deduce that the maximum value of the right-hand side $\|q - r\|^2 + \|q\|^2$ is attained when

$$r = \frac{-\sigma(v)q}{\|q\|}.$$

Consequently, we derive that

$$4\omega(v)^2 \leq \left\| \left(1 + \frac{\sigma(v)}{\|q\|}\right) q \right\|^2 + \|q\|^2 = \|q\|^2 + (\|q\| + \sigma(v))^2.$$

□

Lemma 3.3.13. *([90, Lemma 4.7]) One has*

$$\sqrt{\mu}\|q\| \leq \theta v \xi \sqrt{e^T \left(\frac{x}{s} + \frac{s}{x}\right)}.$$

Proof. Recall that $\overline{A} = AV^{-1}X$ with $V = \text{diag}(v)$ and $X = \text{diag}(x)$. We can express $\overline{A}$ as follows

$$\overline{A} = \sqrt{\mu} AD,$$

with

$$D = \text{diag}\left(\frac{xv^{-1}}{\sqrt{\mu}}\right) = \text{diag}\left(\sqrt{\frac{x}{s}}\right) = \text{diag}\left(\sqrt{\mu} v s^{-1}\right).$$

For now, let us put

$$r_b = \theta v r_b^0, \; r_c = \theta v r_c^0.$$

Using these notations, system (3.26) is rewritten in the following form

$$\begin{cases} \sqrt{\mu} ADq = r_b, \\ \sqrt{\mu} DA^T z + q = \frac{1}{\mu} D r_c. \end{cases}$$

It follows that

$$\mu AD^2 A^T z = AD^2 r_c - r_b,$$

which implies that

$$z = \frac{1}{\mu}(AD^2A^T)^{-1}(AD^2 r_c - r_b).$$

Hence,

$$q = \frac{1}{\mu}\left(Dr_c - DA^T(AD^2A^T)^{-1}(AD^2 r_c - r_b)\right).$$

We can easily verify that

$$q_1 = \left(Dr_c - DA^T(AD^2A^T)^{-1}AD^2 r_c\right) = \left(I - DA^T(AD^2A^T)^{-1}AD\right) Dr_c,$$

is the orthogonal projection of Dr_c onto the null space of AD.
Let $(\bar{x}, \bar{y}, \bar{s})$ be such that $A\bar{x} = b$ and $A^T\bar{y} + \bar{s} = c$. Then we may rewrite r_b and r_c as

follows

$$r_b = \theta\nu r_b^0 = \theta\nu(b - Ax^0) = \theta\nu A(\bar{x} - x^0),$$
$$r_c = \theta\nu r_c^0 = \theta\nu(c - A^T y^0 - s^0) = \theta\nu(A^T(\bar{y} - y^0) + \bar{s} - s^0).$$

$DA^T(\bar{y} - y^0)$ belongs to the row space of AD, which is orthogonal to the null space of AD, hence

$$\|q_1\| \leq \theta\nu\|D(\bar{s} - s^0)\|.$$

Moreover, the vector

$$q_2 = DA^T(AD^2A^T)^{-1}r_b = \theta\nu DA^T(AD^2A^T)^{-1}AD(D^{-1}(\bar{x} - x^0)),$$

is the orthogonal projection of $\theta\nu D^{-1}(\bar{x} - x^0)$ onto the row space of AD. Therefore,

$$\|q_2\| \leq \theta\nu\|D^{-1}(\bar{x} - x^0)\|.$$

Hence, using the fact that $\sqrt{\mu}q = q_1 + q_2$ q_1 and q_2 are orthogonal, we get

$$\sqrt{\mu}\|q\| = \sqrt{\|q_1\|^2 + \|q_2\|^2} \leq \theta\nu\sqrt{\|D^{-1}(\bar{x} - x^0)\|^2 + \|D(\bar{s} - s^0)\|^2}.$$

Taking $\bar{x} = x^*$ and $\bar{s} = s^*$, and using the fact that

$$\|x^* + s^*\|_\infty \leq \xi,$$

it follows that the entries of the vectors $x^0 - \bar{x}$ and $s^0 - \bar{s}$ satisfy

$$0 \leq x^0 - \bar{x} \leq \xi e,\ 0 \leq s^0 - \bar{s} \leq \xi e.$$

As a consequence, we obtain

$$\sqrt{\|D^{-1}(\bar{x} - x^0)\|^2 + \|D(\bar{s} - s^0)\|^2} \leq \xi\sqrt{\|D^{-1}e\|^2 + \|De\|^2} = \xi\sqrt{e^T\left(\frac{x}{s} + \frac{s}{x}\right)},$$

which implies that

$$\sqrt{\mu}\|q\| \leq \theta\nu\sqrt{\|D^{-1}(\bar{x} - x^0)\|^2 + \|D(\bar{s} - s^0)\|^2} \leq \theta\nu\xi\sqrt{e^T\left(\frac{x}{s} + \frac{s}{x}\right)}.$$

□

Corollary 3.3.14. *([58, Corollary 3.10]) Let $\tau = \frac{1}{16}$ and $\delta(v) \leq \tau$. Then*

$$\sqrt{\frac{x}{s}} \leq \sqrt{2}\frac{x(\mu,\nu)}{\sqrt{\mu}},\ \sqrt{\frac{s}{x}} \leq \sqrt{2}\frac{s(\mu,\nu)}{\sqrt{\mu}}.$$

Recall that $\delta(v^f) \leq \frac{1}{\sqrt[4]{2}}$ holds if $\omega \leq \frac{1}{2\sqrt{2}}$. Using Lemma 3.3.12 and Lemma 3.3.7, this will certainly holds if

$$\|q\|^2 + (\|q\| + 2\delta)^2 \leq \frac{1}{2}.$$

Or from Lemma 3.3.13 and Corollary 3.3.14, it follows that

$$\mu\|q\| \leq \theta\nu\xi\sqrt{2}\sqrt{\|x(\mu,\nu)\|^2 + \|s(\mu,\nu)\|^2}.$$

As in [90], using $\mu = \mu^0\nu = \nu\xi^2$ and $\theta = \frac{\alpha}{4\sqrt{n}}$, we obtain the following upper bound for the norm q

$$\begin{aligned}\|q\| &\leq \frac{\sqrt{2}\alpha}{4\xi\sqrt{n}}\sqrt{\|x(\mu,\nu)\|^2 + \|s(\mu,\nu)\|^2} \\ &= \frac{\alpha}{2\xi\sqrt{2n}}\sqrt{\|x(\mu,\nu)\|^2 + \|s(\mu,\nu)\|^2}.\end{aligned}$$

Let us denote

$$\kappa(\xi,\nu) = \frac{\sqrt{\|x(\mu,\nu)\|^2 + \|s(\mu,\nu)\|^2}}{\xi\sqrt{2n}};\ 0 < \nu \leq 1,\ \mu = \mu^0\nu,$$

and

$$\bar{\kappa}(\xi) = \max_{0<\nu\leq 1} \kappa(\xi,\nu).$$

We rewrite the upper bound for the norm q as follows

$$\|q\| \leq \frac{1}{2}\alpha\bar{\kappa}(\xi).$$

After some calculations, we conclude that

$$\delta(v^f) \leq \frac{1}{\sqrt[4]{2}} \text{ if } \|q\| \leq 0.4336.$$

However since $\|q\| \leq \frac{1}{2}\alpha\bar{\kappa}(\xi)$, the latter inequality will be satisfied if

$$\alpha := \frac{0.8672}{\bar{\kappa}(\xi)}. \tag{3.27}$$

The following result gives a range for the parameter $\bar{\kappa}(\xi)$. We follow the technics from Section 4.6 of [90].

Lemma 3.3.15. *One has*

$$1 \leq \bar{\kappa}(\xi) \leq \sqrt{2n}.$$

Proof. From initialization (3.2), we get $\kappa(\xi) = 1$. This yields the left-hand side of the inequality. For the other side, let x^* be an optimal solution of (P) and (y^*, s^*) an optimal solution of (D). For simplicity, we denote $x = x(\mu,\nu)$, $y = y(\mu,\nu)$ and $s = s(\mu,\nu)$. Hence, the triple (x, y, s) is the unique solution of the following system

$$\begin{cases} A(x^* - x - \nu x^* + \nu\xi e) = 0, x \geq 0, \\ A^T(y^* - y - \nu y^*) = s - s^* + \nu s^* - \nu\xi e, s \geq 0, \\ xs = \nu\xi^2 e, \quad \mu > 0. \end{cases}$$

Using the fact that the row space of a matrix and its null space are orthogonal, we get

$$((1-\nu)x^* - x + \nu\xi e)^T(s - (1-\nu)s^* - \nu\xi e) = 0. \tag{3.28}$$

Let us define

$$\begin{aligned} a_1 &= (1-\nu)x^* + \nu\xi e \geq \nu\xi e, \\ a_2 &= (1-\nu)s^* + \nu\xi e \geq \nu\xi e. \end{aligned}$$

From (3.28), we get

$$a_1^T a_2 = a_1^T s + a_2^T x. \tag{3.29}$$

Therefore, since ${x^*}^T s^* = 0$ and $x^* + s^* \leq \xi e$ we can obtain

$$\begin{aligned} a_1^T a_2 + x^T s &= ((1-\nu)x^* + \nu\xi e)^T((1-\nu)s^* + \nu\xi e) + \nu\xi^2 n \\ &\leq 2\nu\xi^2 n. \end{aligned} \tag{3.30}$$

In addition, we can easily verify that

$$\begin{aligned} a_1^T s + a_2^T x &= ((1-\nu)x^* + \nu\xi e)^T s + ((1-\nu)s^* + \nu\xi e)^T x \\ &\geq \nu\xi e^T(x+s) = \nu\xi(\|x\|_1 + \|s\|_1). \end{aligned} \tag{3.31}$$

Using (3.29), (3.30) and (3.31) it follows that

$$(\|x\|_1 + \|s\|_1) \leq 2\xi n.$$

Using the equivalence between the Euclidean norm and the 1-norm, we get the final inequality. □

3.3.4 Iteration bound

We arrive at the final result of this section which summarizes the complexity bound. As we found in the previous sections, starting from an iterate (x, y, s) satisfying $\delta(x, s; \mu) \leq \tau$ with τ and θ defined previously, the new iterate (x^f, y^f, s^f) is strictly feasible and $\delta(x^f, s^f; \mu^+) \leq \frac{1}{\sqrt[4]{2}}$. Moreover, according to Remark 3.3.5, the number of centrality steps needed to obtain iterates (x^+, y^+, s^+) satisfying $\delta(x^+, s^+; \mu^+) \leq \tau$ is at most 5. Therefore, the total number of main iterations is bounded by

$$\frac{1}{\theta} \log \frac{\max\{(x^0)^T s^0, \|r_b^0\|, \|r_c^0\|\}}{\xi}.$$

Let us recall that $\theta = \dfrac{\alpha}{4\sqrt{n}}$. Thus, using (3.27) and the fact that $(x^0)^T s^0 = n\xi^2$, we obtain the following upper bound for the total number of iterations

$$25\sqrt{n}\bar{\kappa}(\xi) \log \frac{\max\{n\xi^2, \|r_b^0\|, \|r_c^0\|\}}{\xi}.$$

From Lemma 3.3.15, we can state the final result of this section which summarizes the complexity bound.

Theorem 3.3.16. *Let (P) and (D) be feasible and $\xi > 0$ such that*

$$\|x^* + s^*\|_\infty \leq \xi$$

for some optimal solutions x^* of (P) and (y^*, s^*) of (D). Then, the algorithm finds an $\epsilon-$solution of (P) and (D) after at most

$$25\sqrt{2}n \log \frac{\max\{n\xi^2, \|{r_b}^0\|, \|{r_c}^0\|\}}{\epsilon}$$

iterations.

3.4 Numerical tests

In this section, we showcase the performance of the proposed algorithm outlined in Algorithm 3 by performing some preliminary numerical tests.

3.4.1 Comparison with other algorithms

First, we compare the practical performance of our algorithm with six other IIPAs. For each algorithm, we choose the suitable theoretical value of the parameter θ that guarantees its convergence. The algorithms with their theoretical values of θ are summarized in Table 3.1.

TABLE 3.1: Considered algorithms

Algorithms	Alg. 1	Alg. 2	Alg. 3	Alg. 4	Alg. 5	Alg. 6	Alg. 7
Ref.	[67]	[68]	[58]	[76]	[91]	[57]	New
θ	$\frac{1}{8\sqrt{2}n}$	$\frac{5}{36n}$	$\frac{1}{7\sqrt{2}n}$	$\frac{1}{16n}$	$\frac{1}{8n}$	$\frac{1}{22n}$	$\frac{1}{5\sqrt{2}n}$

Since there is a parameter p involved in the definition of Algorithm 3, we choose $p = \frac{1}{2}$. All the considered algorithms were implemented in MATLAB and were tested on 10 problems from the Netlib repository. We set $\epsilon = 10^{-4}$, $\tau = \frac{1}{16}$, and $(x^0, y^0, s^0) = (\xi e, 0, \xi e)$, with ξ chosen such that $\|x^* + s^*\|_\infty \leq \xi$. The results are summarized in the table below. For each example, we used **bold** font to highlight the best, i.e., the smallest, iteration number.

TABLE 3.2: Total number of iterations for some Netlib problems

Problem	ξ	Alg. 1	Alg. 2	Alg. 3	Alg. 4	Alg. 5	Alg. 6	Alg. 7
afiro	500	11199	9378	12898	20854	10421	28679	**9209**
adlittle	3500	36121	30249	41596	67238	33612	92458	**29707**
blend	89	22456	18805	25860	41802	20896	57482	**18468**
sc50a	350	16942	14187	19510	31541	15764	43373	**13932**
sc50b	325	16842	14104	19396	31356	15672	43119	**13851**
sc105	720	38471	32217	44301	71609	35798	98568	**31640**
sc205	200	76642	64185	88255	142650	71318	196149	**63036**
scagr7	500	49743	41657	57281	92589	46287	127316	**40911**
share2b	400	31736	26577	36546	59074	29531	81230	**26101**
stocfor1	6300	45112	37778	51948	83971	41978	115464	**37102**

Based on the results presented in Table 3.2, we can clearly see that our algorithm outperformed all the other algorithms with a difference in iteration numbers that can amount up to 79614. In fact, our algorithm achieved the best iteration number in all realized experiments.

3.4.2 Comparison with SeDuMi solver

In this subsection, we compare our algorithm (Alg. 7) with the SeDuMi solver using the average CPU time. The latter is the time needed to obtain an optimal solution. The comparison was done on a set of seven Netlib problems. While implementing our algorithm, we maintain the same values of the parameters ϵ and τ and the same starting point used in the previous subsection. As for θ, we used fixed values $\theta \in \{0.4, 0.5, 0.6, 0.7\}$ because they perform better than the theoretical value $\theta = \dfrac{1}{5\sqrt{2n}}$. The results are summarized in Table 3.3. For each example, we used **bold** font to highlight the best, i.e., the shortest CPU time.

TABLE 3.3: Average CPU time measured in seconds for seven Netlib problems

Problem	afiro	adlittle	blend	sc50a	sc50b	sc105	share2b
Alg. 7	**0.0253**	0.16	**0.0846**	0.0878	**0.0585**	0.1957	0.1747
SeDuMi	0.0302	**0.1585**	0.0869	**0.0619**	0.1252	**0.1750**	**0.1411**

From Table 3.3, it becomes clear that both our algorithm and SeDuMi solver take similar time to obtain an optimal solution.

Conclusions

Conclusions and remarks

In this thesis, our focus lied in the analysis of complexity and numerical implementation of IPMs for LO and SDO problems. Specifically, we explored the concept of both feasible and infeasible IPMs using KFs to define search directions.

In Chapter 1, we investigated feasible primal-dual IPMs based on KFs for LO. The chapter begun with a summary of primal-dual IPMs. Section 2 outlined the steps to determine the complexity of primal-dual IPAs based on a specific class of KFs, drawing inspiration from Bai et al.'s work [8]. We then applied this methodology to three new hyperbolic KFs: the exponential hyperbolic KF [39], the parameterized hyperbolic logarithmic KF [42] which can be considered a generalization of the KF proposed in [100] and the pure hyperbolic KF [41] which is a generalization, up to a multiplicative constant, of the KF introduced in [98]. For each function, we studied the primal-dual interior-point algorithm and derived complexity bounds for both large- and small-update methods. Additionally, we presented practical performance comparisons with existing interior-point algorithms based on KFs.

Chapter 2 extended the exploration of primal-dual feasible IPMs based on KFs, as discussed in Chapter 1, to SDO. The chapter provided a concise summary of primal-dual IPMs for SDO and outlined the steps to determine complexity based on the same class of KFs studied in Chapter 1. The exposed procedure was applied to a twice-parametrized KF, a combination of the prototype self-regular KF introduced in [83] and the hyperbolic KF introduced in [41]. The chapter concluded with numerical experiments showcasing the practical performance of the IPA based on the new parametrized KF in solving SDO problems.

Chapter 3 shifted focus to a full-Newton step IIPA for solving LO problems based on a new hyperbolic KF. In contrast to the feasible IPAs in Chapter 1, this algorithm does not require a feasible starting point and avoids a big-M or a two-phase approach. Each main iteration involved a feasibility step and some centrality steps. Feasibility search directions used the hyperbolic KF, while centring search directions used the classical KF. Under general conditions, we guaranteed the convergence to an optimal solution. The complexity analysis for the primal-dual IIPM based on the corresponding proximity function, under mild properties, indicated that the iteration bound matched the best iteration bound for IIPMs. We supported these theoretical results with numerical experiments, comparing the IIPA based on the considered KF with established IIPAs and evaluating against the SeDuMi solver.

Directions for further research

Future research might focus on the following questions.

- Is there a KF that exhibits equivalent complexity for both large-update and small-update methods?

- Can dual (or primal) IPMs for LO be developed using the new class of KFs? If so, how can these results be extended to SDO problems?
- Can the KFs used in feasible IPMs be used to define IIPMs?

Chapter A

Convex analysis

Definition A.0.1. *A function $\gamma : C \mapsto \mathbb{R}$, with C a convex subset of $\mathbb{R}^n$, is called a convex if it satisfies the inequality*

$$\gamma(\theta\xi_1 + (1-\theta)\xi_2) \leq \theta\gamma(\xi_1) + (1-\theta)\gamma(\xi_2),$$

for any $\xi_1, \xi_2 \in C$ and $\theta \in [0,1]$.

Definition A.0.2. *A function $\gamma : C \mapsto \mathbb{R}$, with C a convex subset of $\mathbb{R}^n$, is called a concave function if it satisfies the inequality*

$$\gamma(\theta\xi_1 + (1-\theta)\xi_2) \geq \theta\gamma(\xi_1) + (1-\theta)\gamma(\xi_2),$$

for any $\xi_1, \xi_2 \in C$ and $\theta \in [0,1]$.

The following proposition presents some characterisations of convex and concave functions defined on an interval of $\mathbb{R}$.

Proposition A.0.3. *Let I be an interval of $\mathbb{R}$ and $\gamma : I \mapsto \mathbb{R}$.*

- *If γ is continuous on I, γ is convex if and only if for all $\xi_1, \xi_2 \in I$*

$$\gamma\left(\frac{\xi_1 + \xi_2}{2}\right) \leq \frac{1}{2}\left(\gamma(\xi_1) + \gamma(\xi_2)\right). \tag{A.1}$$

- *If γ is continuous on I, γ is concave if and only if for all $\xi_1, \xi_2 \in I$*

$$\gamma\left(\frac{\xi_1 + \xi_2}{2}\right) \geq \frac{1}{2}\left(\gamma(\xi_1) + \gamma(\xi_2)\right).$$

- *If γ is differentiable on I, γ is convex if and only if γ' is increasing on I.*
- *If γ is differentiable on I, γ is concave if and only if γ' is decreasing on I.*
- *If γ is twice differentiable on I, γ is convex if and only if $\gamma''(\xi) \geq 0$ for all $\xi \in I$.*
- *If γ is twice differentiable on I, γ is concave if and only if $\gamma''(\xi) \leq 0$ for all $\xi \in I$.*

Remark A.0.4. *We can see from the first item of the previous proposition that property (A.1), called midpoint convexity, means convexity in the context of continuity. For a detailed proof, see [80, Theorem 1.1.4] or visit* https://wikipedie.ovh/articles/f/o/

n/Fonction_convexe.html for another proof based on the density of dyadic rationals in the real line.

Theorem A.0.5. *A local minimizer of a convex function is also a global minimizer.*

Theorem A.0.6. *A local maximizer of a concave function is also a global maximizer.*

Theorem A.0.7. *Let γ be a twice differentiable function on an interval I of $\mathbb{R}$ containing ξ_0.*

- *If $\gamma'(\xi_0) = 0$ and $\gamma''(\xi) \leq 0,\ \forall \xi \in I$, then, γ admits a global maximum on ξ_0.*
- *If $\gamma'(\xi_0) = 0$ and $\gamma''(\xi) \geq 0\ \forall \xi \in I$, then, γ admits a global minimum on ξ_0.*

Corollary A.0.8. *Let γ be a twice differentiable function on I, an interval of $\mathbb{R}$ containing ξ_0.*

- *If $\gamma'(\xi_0) = 0$ and γ is convex, then, γ admits a global minimum on ξ_0.*
- *If $\gamma'(\xi_0) = 0$ and γ is concave then, γ admits a global maximum on ξ_0.*

Lemma A.0.9. *([84, Lemma 2.1.2]) Let ψ be a twice differentiable function on $\mathbb{R}_{++}$. Then, the following properties are equivalent*

(i) *the function $\xi \to \psi(e^{\xi})$ is convex, i.e., ψ is exponentially convex or shortly e-convex.*

(ii) $\psi^{'}(t) + t\psi^{''}(t) \geq 0,\ \forall t \in \mathbb{R}_{++}$.

(iii) $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}\left(\psi(t_1) + \psi(t_2)\right),\ \forall t_1, t_2 \in \mathbb{R}_{++}$.

Proof. We divide the proof into two parts: we first prove that $(i) \Leftrightarrow (ii)$, then we prove that $(i) \Leftrightarrow (iii)$. The third equivalence is immediate.
For the first equivalence, let's define the function $\gamma_1 : \mathbb{R} \mapsto \mathbb{R}_+$ as follows

$$\gamma_1(\xi) = \psi(e^{\xi}).$$

Using the next-to-last item of Proposition A.0.3, we have

$$(i) \Leftrightarrow \gamma_1^{''}(\xi) = e^{\xi}\psi^{'}(e^{\xi}) + e^{2\xi}\psi^{''}(e^{\xi}) \geq 0,\ \forall \xi > 0.$$

Setting $t = e^{\xi}$, we get

$$\begin{aligned}(i) &\Leftrightarrow t\psi^{'}(t) + t^2\psi^{''}(t) \geq 0,\ \forall t > 0,\\ &\Leftrightarrow \psi^{'}(t) + t\psi^{''}(t) \geq 0,\ \forall t > 0.\end{aligned}$$

For the second equivalence, using the first item of Proposition A.0.3 on γ_1, we obtain

$$(i) \Leftrightarrow \psi(e^{\frac{\xi_1+\xi_2}{2}}) \leq \frac{1}{2}\left(\psi(e_1^{\xi}) + \psi(e_2^{\xi})\right),\ \forall \xi_1\xi_2 \in \mathbb{R}.$$

Setting $t_1 = e_1^{\xi}$ and $t_2 = e_2^{\xi}$ we arrive at the second equivalence, which completes the proof. □

Lemma A.0.10. *([8, Lemma 2.2.3]) Let ψ be a twice differentiable function on $\mathbb{R}_{++}$. Then, the following three properties are equivalent*

(i) the function $\xi \to \psi(\sqrt{\xi})$ is convex, i.e., ψ is square root convex or shortly sqrt-convex.

(ii) $t\psi^{''}(t) - \psi^{'}(t) \geq 0,\ \forall t \in \mathbb{R}_{++}$.

(iii) $\psi(\sqrt{\frac{t_1^2+t_2^2}{2}}) \leq \frac{1}{2}\left(\psi(t_1) + \psi(t_2)\right),\ \forall t_1, t_2 \in \mathbb{R}_{++}$.

Proof. In a similar way to the proof of the previous lemma, we start by proving that $(i) \Leftrightarrow (ii)$, then we prove that $(i) \Leftrightarrow (iii)$.
For the first equivalence, we define the function $\gamma_2 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as follows

$$\gamma_2(\xi) = \psi(\sqrt{\xi}).$$

Using the next-to-last item of Proposition A.0.3, we have

$$(i) \Leftrightarrow \gamma_2^{''}(\xi) = \frac{1}{4\xi^{\frac{3}{2}}}\left(\sqrt{\xi}\psi^{''}(\sqrt{\xi}) - \psi^{'}(\sqrt{\xi}\right) \geq 0,\ \forall \xi > 0.$$

Setting $t = \sqrt{\xi}$, we get

$$\begin{aligned}(i) &\Leftrightarrow \frac{1}{4t^3}(t\psi^{''}(t) - \psi^{'}(t)) \geq 0,\ \forall t > 0,\\ &\Leftrightarrow \psi^{'}(t) - t\psi^{''}(t) \geq 0,\ \forall t > 0.\end{aligned}$$

For the other equivalence, from the first item of Proposition A.0.3, γ_2 is convex if and only if for all $\xi_1, \xi_2 \in \mathbb{R}_+$

$$\psi\Big(\sqrt{\frac{\xi_1 + \xi_2}{2}}\Big) \leq \frac{1}{2}\left(\psi(\sqrt{\xi_1}) + \psi(\sqrt{\xi_2})\right).$$

By putting $t_1 = \sqrt{\xi_1}$ and $t_2 = \sqrt{\xi_2}$, we arrive at

$$\psi(\sqrt{\frac{t_1^2 + t_2^2}{2}}) \leq \frac{1}{2}\left(\psi(t_1) + \psi(t_2)\right).$$

This completes the proof. □

Lemma A.0.11. *([83, Lemma 12]) Let g be a twice continuous differentiable convex function on $\mathbb{R}_+$ with $g(0) = 0$, and let g attain its (global) minimum at $t^* > 0$. If g'' is increasing on $[0, t^*]$, then*

$$g\left(t\right) \leq \frac{g^{'}\left(0\right)}{2}t,\ \ 0 \leq t \leq t^*.$$

Proof. Since g is twice differentiable on $\mathbb{R}_+$, we can rewrite for all $t > 0$

$$\begin{aligned}g(t) &= \int_0^t g'(\xi)d\xi\\ g(t) &= \int_0^t \left(g'(\xi) - g'(0) + g'(0)\right)d\xi\\ &= g'(0)t + \int_0^t \int_0^{\xi} g''(\zeta)d\zeta d\xi\\ &\leq g'(0)t + \int_0^t \xi g''(\xi)d\xi,\end{aligned}$$

where the last inequality is obtained using the increasing of g''. Integrating by parts, we get

$$\begin{aligned} g(t) &\leq g'(0)t + (\xi g'(\xi))|_0^t - \int_0^t g'(\xi)d\xi \\ &\leq g'(0)t - \int_0^t dg'(\xi) \\ &= g'(0)t - g(t), \end{aligned}$$

the last inequality holds because

$$g'(t) \leq g'(t^*) = 0, \ \forall 0 \leq t \leq t^*.$$

This inequality is due to the convexity of g and the fact that t^* is a critical point. □

Lemma A.0.12. *([83, Lemma 2.1]) If $\alpha \in [0,1]$, then*

$$(1+t)^\alpha \leq 1 + \alpha t, \ \forall t \geq -1.$$

Proof. The desired inequality is trivial for $\alpha = 0$. For $\alpha \in]0,1]$, let us define the function l as follows

$$l(t) = (1+t)^\alpha - 1 - \alpha t, \ \forall t \geq -1.$$

Deriving l twice, we obtain

$$l'(t) = \alpha(1+t)^{\alpha-1} - \alpha,$$

$$l''(t) = \alpha(\alpha-1)(1+t)^{\alpha-2}.$$

Clearly l'' is negative on $[-1, +\infty[$ since $\alpha \leq 1$. Hence, using Proposition A.0.3 l is concave. In addition, $l'(0) = 0$. Therefore using Corollary A.0.8, it follows that the function l reaches its maximal value at the point $t = 0$. Finally, since $l(0) = 0$, the desired inequality is obtained. □

Lemma A.0.13. *([83, Lemma 14]) Let $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \ k = 0, 1, \ldots, K-1, \tag{A.2}$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then,

$$K \leq \frac{t_0^\gamma}{\beta\gamma}.$$

Proof. Since the sequence $t_0, t_1, \ldots, t_K$ is positive, (A.2) implies that for all $k = 0, 1, \ldots, K-1$

$$0 < t_{k+1}^\gamma \leq (t_k - \beta t_k^{1-\gamma})^\gamma = t_k^\gamma (1 - \beta t_k^{-\gamma})^\gamma. \tag{A.3}$$

Using Lemma A.0.12 with $\alpha = \gamma$ and $t = -\beta t_k^{-\gamma}$, we obtain

$$t_k^\gamma (1 - \beta t_k^{-\gamma})^\gamma \leq t_k^\gamma (1 - \gamma\beta t_k^{-\gamma}) = t_k^\gamma - \beta\gamma, \ k = 0, 1, \ldots, K-1.$$

Thus, using this inequality in (A.3) we get

$$0 < t_{k+1}^\gamma \leq t_k^\gamma - \beta\gamma, \ k = 0, 1, \ldots, K-1.$$

Consequently,

$$t_1^\gamma \leq t_0^\gamma - \beta\gamma$$
$$t_2^\gamma \leq t_1^\gamma - \beta\gamma \leq t_0^\gamma - 2\beta\gamma.$$

This implies that, for all $k = 1, \ldots, K$

$$0 < t_k^\gamma \leq t_0^\gamma - k\beta\gamma.$$

In particular when $k = K$, we get

$$t_0^\gamma - K\beta\gamma > 0,$$

and thus,

$$K \leq \frac{t_0^\gamma}{\beta\gamma}.$$

□

Chapter B

Examples for numerical tests

TABLE B.1: Some Netlib-standard problems

Name	Rows	Columns	Nonzeros	Optimal value
adlittle	57	97	465	2.2549496316E + 05
afiro	28	32	88	4.6475314286E + 02
blend	75	83	521	3.0812149846E + 01
bore3d	234	315	1525	1.3730803942E + 03
degen2	445	534	4449	1.4351780000E + 03
degen3	1504	1818	26230	9.8729400000E + 02
sc50a	51	48	131	6.4575077059E + 01
sc50b	51	48	119	7.00000000000E + 01
sc105	106	103	281	5.2202061212E + 01
sc205	206	203	552	5.2202061212E + 01
scagr7	130	140	553	2.3313892548E + 08
scsd1	78	760	3148	8.6666666743E + 00
scsd6	148	1350	5666	5.0500000078E + 01
scsd8	398	2750	11334	9.0499999993E + 02
sctap1	301	480	2052	1.4122500000E + 03
sctap2	1091	1880	8124	1.7248071429E + 03
sctap3	118	225	1182	1.4240000000E + 03
share2b	97	79	730	4.1573224074E + 02
stocfor1	118	111	474	4.1131976219E + 04

TABLE B.2: List of linear programming examples with fixed size

Name	Input data A, b, c	Initialization	Ref.
EF1	$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$	$x^0 = \begin{pmatrix} 0.5 \\ 0.27 \\ 0.14 \\ 0.09 \end{pmatrix}$, $y^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $s^0 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$	[6]
EF2	$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 8 \\ 7 \\ 3 \end{pmatrix}$, $c = \begin{pmatrix} -4 \\ -5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$x^0 = \begin{pmatrix} 2.85 \\ 1.9 \\ 0.4 \\ 0.35 \\ 1.1 \end{pmatrix}$, $y^0 = \begin{pmatrix} -1.2 \\ -1.8 \\ -0.5 \end{pmatrix}$, $s^0 = \begin{pmatrix} 0.2 \\ 0.3 \\ 1.2 \\ 1.8 \\ 0.5 \end{pmatrix}$	[6]
EF3	$A = \begin{pmatrix} 2 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $c = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$x^0 = \begin{pmatrix} 0.06757 \\ 0.13258 \\ 0.13302 \\ 0.26774 \\ 0.13302 \\ 0.2664 \end{pmatrix}$, $y^0 = \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix}$, $s^0 = \begin{pmatrix} 10 \\ 4 \\ 6 \\ 1 \\ 5 \\ 1 \end{pmatrix}$	[6]
EF4	$A = \begin{pmatrix} 0 & 1 & 2 & -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & -2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \\ 1 & 3 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$x^0 = \begin{pmatrix} 0.1819 \\ 0.0699 \\ 0.063 \\ 0.1105 \\ 0.2012 \\ 0.6732 \\ 1.1885 \\ 2.835 \\ 2.1912 \end{pmatrix}$, $y^0 = \begin{pmatrix} -1.3843 \\ -0.8751 \\ -0.4241 \\ -0.4463 \\ -3.0424 \end{pmatrix}$, $s^0 = \begin{pmatrix} 4.9398 \\ 13.1544 \\ 14.7156 \\ 9.1788 \\ 4.5072 \\ 1.3843 \\ 0.8751 \\ 0.4241 \\ 0.4463 \end{pmatrix}$	[6]

TABLE B.3: List of linear programming examples with variable size $n = 2m$

Name	Input data A, b, c	Initialization	Ref.
EV1	For $i, j = 1, ..., m$. $A(i,j) = \begin{cases} 1 & \text{if } i = j \text{ or } j = i+m, \\ 0 & \text{otherwise,} \end{cases}$ $c(i) = -1$, $c(i+m) = 0$ and $b(i) = 2$	For $i = 1, ..., m$. $x^0 = e$, $y^0(i) = -2$ $s^0(i) = 1$, $s^0(i+m) = 2$	[15]
EV2	For $i, j = 1, ..., m$. $A(i,j) = \begin{cases} 1 & \text{if } i = j \text{ or } j = i+m, \\ 0 & \text{otherwise,} \end{cases}$ $c = -e$ and $b(i) = 2$	For $i = 1, ..., m$ $x^0 = s^0 = e$ and $y^0(i) = -2$	[6]
EV3	For $i, j = 1, ..., m$. $A(i,j) = \begin{cases} 1 & \text{if } i = j \text{ or } j = i+m, \\ 0 & \text{otherwise,} \end{cases}$ $c = -e$ and $b(i) = 2$	For $i = 1, ..., m$, $x^0(i) = 1.5$, $x^0(i+m) = 0.5$, $y^0(i) = -2$, and $s^0 = e$.	[97]
EV4	For $i, j = 1, ..., m$. $A(i,j) = \begin{cases} 1 & \text{if } i = j \text{ or } j = i+m, \\ 0 & \text{otherwise,} \end{cases}$ for $i = 1, ..., m$. $c(i) = \frac{1}{4}$, $c(i+m) = \frac{3}{4}$ and $b(i) = 2$	For $i = 1, ..., m$. $x^0(i) = \frac{5}{4}$, $x^0(i+m) = \frac{3}{4}$, $y^0(i) = \frac{-7}{12}$, and $s^0(i) = \frac{5}{6}$, $s^0(i+m) = \frac{4}{3}$,	New

TABLE B.4: List of semidefinite programming examples with variable size $n = 2m$

Name	Input data	Initialization	Ref.
SDP1	$A_k(i,j) = \begin{cases} 1 & \text{if } i = j = k, \\ 1 & \text{if } i = j = m + k, \\ 0 & \text{otherwise}, \end{cases}$ $C = -I, n = 2m, b(i) = 2, i = 1, ..., m.$	$S^0 = X^0 = I,$ $y^0(i) = -2, i = 1, \ldots, m.$	[65]
SDP2	$A_k(i,j) = \begin{cases} 1 & \text{if } i = j = k, \\ 1 & \text{if } i = j = m + k, \\ 0 & \text{otherwise}, \end{cases}$ $C = -I, n = 2m, b(i) = 2, i = 1, ..., m.$	$X^0(i,j) = \begin{cases} 1.5 & \text{if } i \leq j, \\ 0.5 & \text{if } i > j, \end{cases}$ $y^0(i) = -2,\ i = 1, ..., m,$ and $S^0 = I.$	[97]

Chapter C

Matrix analysis

For more details on matrix analysis, we refer the readers to [49].

C.1 Matrices

Definition C.1.1 (Rank). *Let $A \in \mathbb{R}^{n\times m}$. Its rank, denoted* $\operatorname{rank}(A)$*, is a nonnegative integer defined as the largest number of columns of A that constitutes a linearly independent set.*

Proposition C.1.2. $\operatorname{rank}(A^T) = \operatorname{rank}(A)$.

Definition C.1.3 (Matrix multiplication). *Let $A \in \mathbb{R}^{k\times m}$ and $B \in \mathbb{R}^{m\times n}$. Then $AB \in \mathbb{R}^{k\times n}$ is defined as*

$$(AB)_{i,j} = \sum_{l=1}^{m} A_{i,l}B_{l,j}.$$

Note that this product is not commutative, even if $k = n$. However it has some other good properties.

Proposition C.1.4. *Let $A, B, C \in \mathbb{R}^{m\times m}$. The matrix multiplication is:*

(i) Associative: $(AB)C = A(BC)$.

(ii) Distributive over matrix addition: $A(B + C) = AB + AC$.

(iii) Scalar multiplication: $\beta(AB) = (\beta A)B = A(\beta B)$, $\forall \beta \in \mathbb{R}$.

(iv) Commutative by transpose: $(AB)^T = B^T A^T$.

Proposition C.1.5. *Let $A, B \in \mathbb{R}^{m\times m}$, then*

$$\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B)),$$

and

$$\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B).$$

Definition C.1.6 (Eigenvalues and eigenvectors). *Let $A \in \mathbb{R}^{n\times n}$. If $x \in \mathbb{R}^n - \{0\}$ and $\lambda \in \mathbb{R}$ satisfy $Ax = \lambda x$, then λ is called eigenvalues of A and x is called an eigenvector of A associated with λ.*

Remark C.1.7. *A is nonsingular if and only if it does not admit 0 as eigenvalue.*

Definition C.1.8 (Singular value)**.** *A singular value of a real matrix A is the positive square root of an eigenvalue of the symmetric matrix AA^T or A^TA.*

Definition C.1.9. *Let $A \in \mathbb{R}^{n\times n}$. The trace of A denoted $\mathrm{tr}(A)$ is defined as follows*

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{i,i},$$

with $a_{i,i}$ denotes the entry on the ith row and ith column of A.

Proposition C.1.10. *For all $A, B, C \in \mathbb{R}^{n\times n}$ and $\alpha, \beta \in \mathbb{R}$, one has*

- $\mathrm{tr}(\alpha A + \beta B) = \alpha\mathrm{tr}(A) + \beta\mathrm{tr}(B)$.
- $\mathrm{tr}(A^T) = \mathrm{tr}(A)$.
- $\mathrm{tr}(A^2) \leq \mathrm{tr}(A^TA)$.
- $\mathrm{tr}(A^TB) = \mathrm{tr}(AB^T) = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} A_{i,j}B_{i,j}$.
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
- *If A and B are symmetric then,* $\mathrm{tr}(AB) \leq \frac{1}{2}\mathrm{tr}(A^2 + B^2)$.
- $\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA) \neq \mathrm{tr}(ACB)$.
- *If B is nonsingular* $\mathrm{tr}(B^{-1}AB) = \mathrm{tr}(A)$.

Proposition C.1.11. *Let $A \in \mathbb{S}^n$. Then*

$$\sum_{i=1}^{n} \lambda_i(A) = \mathrm{tr}(A) \text{ and } \prod_{i=1}^{n} \lambda_i(A) = \det(A).$$

Definition C.1.12 (Hadamard product)**.** *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$. Then the Hadamard product of A and B, denoted $A \circ B \in \mathbb{R}^{m\times n}$ is defined as*

$$(A \circ B)_{i,j} = A_{i,j}B_{i,j}.$$

From the above definition, we can see that the Hadamard product of two matrices of the same dimension is the component wise product.

Definition C.1.13 (Frobenius inner product and Frobenius norm)**.** *For any $A, B \in \mathbb{R}^{n\times m}$, the Frobenius inner product is defined as follows*

$$A \bullet B = \mathrm{tr}(A^TB) = \mathrm{tr}(B^TA) = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{i,j}B_{i,j}.$$

Its associated Frobenius norm denoted $\|\cdot\|_F$ is defined as follows

$$\|A\|_F = \sqrt{A \bullet A}.$$

Proposition C.1.14. *(i) For $A \in \mathbb{R}^{n\times m}$, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$, we have*

$$u^TAv = A \bullet uv^T.$$

(ii) For $A \in \mathbb{S}^n$,

$$\|A\|_F = \sqrt{\text{tr}(A^2)} = \sum_{i=1}^{n} \lambda_i^2(A),$$

with $\lambda_i, i = 1, \ldots, n$, *the eigenvalues of* A.

Theorem C.1.15 (Singular value decomposition)**.** *Let us consider a matrix* $A \in \mathbb{R}^{n\times m}$. *There exists a factorization of the form*

$$A = U\Sigma V^T,$$

where U *and* V *are* $n \times n$ *and* $m \times m$ *unitary matrix and* Σ *is an* $n \times m$ *diagonal matrix with nonnegative real numbers on the diagonal. Such a factorization is called the singular value decomposition of* A.

Proposition C.1.16. *The rank of a matrix* $A \in \mathbb{R}^{n\times m}$ *equals the number of its non-zero singular values.*

Definition C.1.17 (Inverse and singular matrix)**.** *The inverse of a square matrix* A, *denoted by* A^{-1} *is a matrix* X *such that* $AX = XA = I$. *If* A^{-1} *exists,* A *is nonsingular, otherwise it is singular.*

Corollary C.1.18.

- *The rank of a matrix* $A \in \mathbb{R}^{n\times m}$ *satisfies* $\text{rank}(A) \leq \min\{n, m\}$
- *We say that a matrix* A *has full rank if*

$$\text{rank}(A) = \min\{n, m\}.$$

With this notion, we can alternatively define a square matrix as nonsingular if it has full rank.

The following theorem establishes a connection between the rank of the matrix system and the number of solutions of the system.

Theorem C.1.19 (Rouché–Kronecker–Capelli)**.** *Let* $Ax = b$ *be a system of linear equations with* $A \in \mathbb{R}^{m\times n}$, $b \in R^m$ *and* $A^* = [A|b]$ *be its augmented matrix. The system has solutions, i.e., is consistent if and only if*

$$\text{rank}(A) = \text{rank}(A^*).$$

Moreover if $\text{rank}(A) = n$, *the solution is unique. Otherwise, there are infinitely many solutions.*

Corollary C.1.20. *The maximum number of linearly independent solutions of the consistent linear system* $Ax = b$ *is* $n - \text{rank}(A)$.

Positive semidefinite and Positive definite symmetric matrices and their properties

Theorem C.1.21. *([49]) A symmetric matrix* $A \in \mathbb{S}^n$, *is positive semidefinite denoted* $A \succeq 0$ *if* A *satisfies any one of the following equivalent conditions:*

- $x^T Ax \geq 0$ *for all* $x \in \mathbb{R}^n$;
- *All its eigenvalues are nonnegative;*
- *All the principal minors of* A *are nonnnegative;*

- *There exists a matrix* $B \in \mathbb{R}^{m\times n}$ *such that* $A = BB^T$.

Theorem C.1.22. *([49]) A symmetric matrix* $A \in \mathbb{S}^n$*, is positive definite denoted* $A \succ 0$ *if* A *satisfies any one of the following equivalent conditions:*

- $x^T Ax > 0$ *for all* $x \in \mathbb{R}^n - \{0\}$;
- *All its eigenvalues are positive;*
- *All the principal minors of* A *are positive;*
- *There exists* $B \in \mathbb{R}^{n\times n}$ *with* B *has full column rank such that* $A = BB^T$.

Definition C.1.23. *A matrix* $A \in \mathbb{S}^n$*, is negative semidefinite (resp. definite), which is denoted by* $A \preceq 0$ *(resp.* $A \prec 0$*) if* $-A$ *is positive semidefinite (resp. definite).*

Proposition C.1.24. *1. Let* $A, B \in \mathbb{S}^n_+$*, then*

- $A \succeq B \Leftrightarrow A - B \succeq 0$.
- $A + B \succeq B$.
- $A^{\frac{1}{2}} B A^{\frac{1}{2}} \succeq 0$.
- $\mathrm{tr}(AB) \leq \mathrm{tr}(A)\mathrm{tr}(B)$.
- $\mathrm{tr}(AB) \geq 0$.

2. *Let* $A, B \in \mathbb{S}^n$*. We have the following equivalences*

$$A \succeq 0 \Leftrightarrow \mathrm{tr}(A^T B) \geq 0, \forall B \succeq 0,$$

$$A \succ 0 \Leftrightarrow \mathrm{tr}(A^T B) > 0, \forall B \succeq 0.$$

3. *Let* $B \in \mathbb{R}^{n\times n}$ *be a non singular matrix, then*

$$A \in \mathbb{S}^n_+ \text{ (resp. } \mathbb{S}^n_{++} \text{)} \Leftrightarrow B^T AB \in \mathbb{S}^n_+ \text{ (resp. } \mathbb{S}^n_{++} \text{).}$$

4. *Let* $A, B \in \mathbb{S}^n$.

- *If* $A \succeq 0$*, then* $\|A\|_F \leq \mathrm{tr}(A)$ *and* $n(\det(A))^{\frac{1}{n}} \leq \mathrm{tr}(A)$.
- $A \succeq B \Leftrightarrow C^T AC \succeq C^T BC$, $\forall C \in \mathbb{R}^{n\times n}$.
- $A \succeq I$ *then* A *is non singular and* $I \succeq A^{-1}$.
- $B \succeq A \succ 0$ *then* B *is non singular* $(B \succ 0)$ *and* $A^{-1} \succeq B^{-1}$.
- *If* $C, D \in \mathbb{S}^n$ *such that* $C - A \succeq 0$ *and* $D - B \succeq 0$ *then*

$$\mathrm{tr}(AB) \leq \mathrm{tr}(CD).$$

In particular, if $C = A$ *then* $\mathrm{tr}(AB) \leq \mathrm{tr}(AD)$.

Lemma C.1.25. *([47, Lemma 1.2.3]) Let* $A, B \in \mathbb{S}^n_+$*. Then*

$$\mathrm{tr}(AB) = 0 \text{ if and only if } AB = 0.$$

Lemma C.1.26. *Let* $A, B \in \mathbb{S}^n_{++}$*. Then*

$$AB \sim A^{\frac{1}{2}} B A^{\frac{1}{2}}.$$

Lemma C.1.27. *([103, Lemma 5.18]) Let* $A, A + B \in \mathbb{S}^n_+$*. Then one has*

$$\lambda_i(A + B) \geq \lambda_{\min}(A) - |\lambda_{\max}(B)|, i = 1, \ldots, n.$$

C.2 Matrix Functions

Definition C.2.1 (Matrix function). *Let $V \in \mathbb{S}^n_{++}$, with*

$$V = Q^T \text{diag}\,(\lambda_1(V), \ldots, \lambda_n(V))Q,$$

where Q is an orthonormal matrix ($Q^T = Q^{-1}$) that diagonalizes V. Then, for given real function ψ defined for all $t > 0$, the matrix function $\psi(V) : \mathbb{S}^n_{++} \to \mathbb{S}^n$ is defined as follows

$$\psi(V) = Q^T \text{diag}(\psi(\lambda_1(V)), \ldots, \psi(\lambda_n(V)))Q. \tag{C.1}$$

Furthermore, the proximity function (measure) for SDO $\Psi : \mathbb{S}^n_{++} \to \mathbb{R}_+$ is defined as follows

$$\Psi(V) := \text{tr}(\psi(V)) = \sum_{i=1}^{n} \psi(\lambda_i(V)). \tag{C.2}$$

Remark C.2.2. *If the function ψ is differentiable on the interval $]0, +\infty[$ such that*

$$\psi'(t) > 0, \forall t > 0,$$

we can obtain the definition of the matrix function ψ' by replacing $\psi(\lambda_i(V))$ in (C.1) by $\psi'(\lambda_i(V))$ for each $i = 1, \ldots, n$, i.e.,

$$\psi'(V) = Q^t \text{diag}(\psi'(\lambda_1(V)), \psi'(\lambda_2(V)), ..., \psi'(\lambda_n(V)))Q, .$$

Definition C.2.3. *A matrix $M(t)$ is said to be a matrix of functions if each entry of $M(t)$ is a function of t, i.e.,*

$$M(t) = [M_{ij}(t)].$$

The standard ideas of continuity, differentiability, and integrability can be seamlessly expanded to matrices of functions by considering each entry individually. This approach involves interpreting the matrix operations entry-wise.

Proposition C.2.4. *Suppose that the matrix-valued functions $M(t), N(t)$ are differentiable with respect to t. Then, one has*

- $\frac{\mathrm{d}}{\mathrm{d}t}(M(t)) := \frac{\mathrm{d}}{\mathrm{d}t}\left[M_{ij}(t)\right] = M'(t).$
- $\frac{\mathrm{d}}{\mathrm{d}t}(\text{tr}(M(t))) := \text{tr}(\frac{\mathrm{d}}{\mathrm{d}t}M(t)) = \text{tr}(M'(t)).$
- $\frac{\mathrm{d}}{\mathrm{d}t}(\text{tr}(\psi(M(t)))) := \text{tr}\left[\psi'(M(t))M'(t)\right].$
- $\frac{\mathrm{d}}{\mathrm{d}t}((M(t)N(t))) := M'(t)N(t) + N'(t)M(t).$

Lemma C.2.5. *([49, Lemma 3.3.14 (c)]) Let $A, B \in \mathbb{S}^n$ be two nonsingular matrices and f a real-valued function such that $t \mapsto f(e^t)$ is a convex function. One has*

$$\sum_{i=1}^{n} f(\eta_i(AB)) \leq \sum_{i=1}^{n} f(\eta_i(A)\eta_i(B)),$$

where $\eta_i(A)$ and $\eta_i(B)$ denote the singular values of A and B respectively.

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ملخص:

موضوع هذه الأطروحة يقع ضمن مجال البرمجة الخطية و البرمجة الخطيةٌ نصف المعرفة. الهدف هو دراسة طرق النقاط الداخلية الأولية –الثنوية لحل مسائل البرمجة الخطية. تعتمد هذه الطرق على استعمال دوال نواة زائدية جديدة لتحديد فئة جديدة من اتجاهات الانتقال. سيتم حساب تكلفة للخوارزمية، وسيتم تمديد هذه الطرق لحل مسائل البرمجة الخطية نصف المعرفة. على وجه الخصوص، نحقق في مفهوم طرق النقاط الداخلية القابلة وغير القابلة للتنفيذ التي تعتمد على الدالة النواة لتحديد اتجاهات الانتقال. أولاً، نتعامل مع طرق النقاط الداخلية القابلة للتنفيذ لحل مسائل البرمجة الخطية. تعتمد هذه الطرق على عدة دوال نواة زائدية جديدة. ثم نمدد طرق النقاط الداخلية القابلة للتنفيذ لحل مسائل البرمجة الخطية نصف المعرفة. بعد ذلك، نقدم طريقة النقاط الداخلية الغير قابلة للتنفيذ بخطوة كاملة لحل مسائل البرمجة الخطية بناءً على دالة نواة زائدية Newton

الكلمات المفتاحية : البرمجة الخطية، البرمجة الخطيةٌ نصف المعرفة، طرق النقاط الداخلية قابلة للتنفيذ، طرق النقاط الداخلية غير قابلة للتنفيذ، خطوة كاملة Newton، تحليل تكلفة الخوارزمية.

Résumé :

Le sujet de cette thèse relève du domaine de l'optimisation linéaire et de l'optimisation semi-définie. L'objectif est d'étudier les méthodes de points-intérieurs primale-duale pour résoudre des problèmes d'optimisation linéaire. Ces méthodes se fondent sur l'introduction de nouvelles fonctions noyaux hyperboliques pour déterminer une nouvelle classe de directions de déplacement. L'analyse de la complexité sera établie, et une extension au cas de l'optimisation semi-définie sera abordée. En particulier, nous examinons le concept de méthodes de points intérieurs réalisables et non réalisables qui reposent sur des fonctions noyaux pour définir les directions de déplacement. Nous commençons par traiter des méthodes de points intérieurs primale-duale réalisables pour résoudre des problèmes d'optimisation linéaire. Ces méthodes sont basées sur de nouvelles fonctions noyaux hyperboliques. Ensuite, nous étendons les méthodes de points intérieurs primale-duale réalisables pour résoudre des problèmes d'optimisation semi-définie. Enfin, nous présentons une méthode de points-intérieurs non réalisable basée sur une fonction noyau hyperbolique pour résoudre des problèmes d'optimisation linéaire en utilisant une itération complète de Newton.

Mots clés : Programmation linéaire, Programmation semi-définie linéaire, Méthode de points-intérieurs réalisable, Méthode de points-intérieurs non réalisable, Fonction noyau, Méthode complète de Newton, Analyse de la complexité

Abstract :

This thesis topic falls within the realm of linear optimization and semidefinite optimization. The objective is to study primal-dual interior-point methods for solving linear optimization problems. These methods are based on introducing new hyperbolic kernel functions to determine new class of search directions. The analysis of the complexity will be established, and an extension to the semidefinite optimization case will be addressed. In particular, we investigate the concept of feasible and infeasible interior-point methods that rely on kernel functions to define the search directions. We first deal with feasible primal-dual interior-point methods for solving linear optimization problems. These methods are based on new hyperbolic kernel functions. Then, we extend primal-dual feasible interior-point methods to solve semidefinite optimization problems. After that, we present a full-Newton step infeasible interior-point method for solving linear optimization problems based on a hyperbolic kernel function.

Keywords : Linear programming, Semidefinite programming, Feasible interior-point methods, Infeasible interior-point methods, Kernel function, full-Newton step, Complexity analysis.