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Title

Log-concavity and log-convexity of sequences and sequences of polynomials

Publicly defended, on 11/05/2024, in front of the jury composed of:

Chairman	:	N. Touafek	Professor	UMSB, Jijel
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بسم الله الرحمن الرحيم

إلى عائلتي وصديقاتي الذين كانوا حاضرين بجانبي طوال هذه الرحلة.

To my family and friends who were by my side throughout this journey.

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Abstract This thesis belongs to the field of combinatorics, or to be more accurate the enumerative combinatorics and bijective combinatorics. It addresses the study of log-concavity and log-convexity of certain sequences and sequences of polynomials by a combinatorial approach in which we use the combinatorial interpretation of each one. We define first an overpartition analogues of bi^{s}nomial coefficients, and we interpret them by overpartitions and by generalized Delannoy paths. Through these interpretations we prove the property of strong log-concavity. Next, we give a q-analogue of the number of permutations with a fixed number of inversions known as "Mahonian numbers". We give the link with partitions and paths, and hence prove the log-concavity. Finally, we study the two-Catalan triangle and in particular the two-Catalan numbers. We establish the combinatorial interpretation by a subset of the set of vertically constrained Motzkin-like paths, thereafter we prove the log-convexity of two-Catalan numbers and the log-concavity of rows of two-Catalan triangle.

Key words: Bi^{s}nomial coefficients, overpartitions, over (q,t)-bi^{s}nomial coefficients, Mahonian numbers, inversions, permutations, Catalan numbers, tow-Catalan numbers, paths, tiling, log-convexity, log-concavity, q-log concavity, (q,t)-log-concavity.

ملخص

تندرج هذه الأطروحة ضمن ميدان التوافقيات، بالتحديد التوافقيات التعدادية و التوافقيات التماثلية. حيث تتناول دراسة التقعر اللوغريتمي والتحدب اللوغاريتمي لبعض المتتاليات وبعض متتاليات كثيرات الحدود وذلك باستعمال نهج توافقي نعتمد فيه على التفسير التوافقي لكل متتالية. أولا، نعرف المثيل فوق التجزئة للمعاملات بيسنوميال، ثم نقدم تفسيرا بالفوق تجزئات وبمسارات ديلانوي المعممة. ومن خلال هذان التفسيران نبرهن خاصية التقعر اللوغاريتمي.

بعد ذلك نعرف $q-$ مثيل لعدد التبديلات التي بها عدد ثابت من النازلات والمعروف بالأعداد الماهونية. ثم نقدم العلاقة بين هذا المثيل والتجزئات والمسارات، ومنه نبرهن التقعر اللوغاريتمي.

ثم أخيرا ندرس مثلث اثنان كاتالون وبالخصوص أعداد اثنان كاتالون، فنقدم تفسيرا توافقيا بمجموعة جزئية من مجموعة مسارات موتزكين لايك المقيدة عموديا، وبذلك نبرهن التحدب اللوغاريتمي لأعداد اثنان كاتلون، كما نبرهن أيضا التقعر اللوغاريتمي لأسطر مثلث اثنان كاتالون.

Contents

List of Figures

List of Tables

LIST OF CONTRIBUTIONS

1. **Y. Ghemit**, M. Ahmia, Overpartition analogues of q-bi^{s}nomial coefficients: basic properties and log-concavity, The Ramanujan Journal. **62**, 431-455 (2023). (Chapter 2).
2. **Y. Ghemit**, M. Ahmia, An Analogue of Mahonian Numbers and Log-Concavity, Annals of Combinatorics. **27**, 895-916 (2023). (Chapter 3).
3. **Y. Ghemit**, M. Ahmia, Two-Catalan numbers: combinatorial interpretation and log-convexity, Submitted (2023). (Chapter 4).

Introduction

Combinatorics is the branch of mathematics that deals with counting and arranging finite (or discrete) objects under specific rules and conditions. It is concerned with finding some identities, recurrence relations, generating function, other combinatorial interpretations and relationships with other finite objects, and often grounded in using the famous combinatorial objects and tools such as: partitions, permutations, combinations, paths, trees, $\ldots$.

Amongst the important properties that combinatorics treat, we are interested in this thesis in studying the "log-concavity" and the "log-convexity" of some particular sequences and sequences of polynomials. The study of these two properties could not be direct in many instances. Herein lies the role of bijective combinatorics, that transfers the study of log-concavity or log-convexity from the set of sequences (or sequence of polynomials) to another combinatorial structure that is in bijection with the first set, as we do in this thesis.

This thesis is structured into four chapters in the following arrangement:

In the first chapter, we present basic definitions and essential notions needed to the comprehension of this thesis. We initiate with defining some combinatorial objects: partitions, overpartitions, permutations. Next, we give definitions of some different sets of lattice paths. Then, we move on to binomial coefficients and bi^{s}nomial coefficients and their analogues, and overpartitions analogues of q-binomial coefficient. Also, we introduce the Mahonian numbers and Catalan numbers, and we finish this chapter by talking about log-concavity, log-convexity, unimodality and the directions in arithmetic triangles.

In the second chapter, we define the over q-bi^{s}nomial coefficient and (q,t)-bi^{s}nomial coefficient using overpartitions, and give some identities, recurrence relation and generating function. We also provide a lattice paths/overpartitions interpretation and by applying a bijection we get the tiling interpretation of the over (q,t)-bi^{s}nomial coefficient. Then, we study the property of log-concavity of this coefficients, basing on the combinatorial interpretation, by lattice paths on one hand, and on the other hand by partitions.

In the third chapter, we provide the q-Mahonian numbers as an analogue of the number of permutations with a fixed number of inversions known as Mahonian numbers, through counting the number of appearances of each entry of the permutation as the first element of the inversions of the permutation. We establish a link between these q-Mahonian numbers and partitions, and by means of which we provide a lattice paths and tiling interpretations. By that, we state a constructive proof of q-log-concavity of the q-Mahonian numbers.

In the fourth chapter, we present the two-Catalan triangle in which we call the coefficients of the first column of this triangle by "two-Catalan numbers". We interpret the coefficients of two-Catalan triangle by a subset of the set of vertically constrained Motzkin-like paths. Using this combinatorial interpretation, we prove the log-convexity of two-Catalan numbers. and we end by proving the log-concavity of rows of two-Catalan triangle.

CHAPTER 1

Preliminaries

We introduce in this chapter the fundamental notions and concepts of combinatorics on which the remaining chapters of the thesis are based. We start by introducing some basic combinatorial tools and objects: partitions, overpartitions and permutations. Then, we give some types of lattice paths that we need. Next, we present the binomial and bi^{s}nomial coefficients and their analogues, and the overpartition analogues of q-binomial coefficients. Moreover, we give the essential notions of Mahonian numbers and Catalan numbers. We finish this introductory chapter by presenting the main definitions of log-concavity, log-convexity and unimodality.

1.1 Combinatorial objects

1.1.1 Partitions

In 1674, Leibniz wrote a letter to Bernoulli asking about the number of ways of writing a positive integer as a sum of other positive integers. Leibniz used the term number of divulsions for the number of ways a given integer can be expressed as a sum of smaller integers, which we call now the partition of an integer into parts. About seventy years after, Euler studied the partitions in a profound way, starting by a solution he gave during a presentation he made in 1741 and published in 1751 [33], answering the question asked by Naudé in a letter he send to Euler about the number of partitions of the integer 50 into 7 distinct parts. Later, the 20th century saw an accelerated pace of research about partitions with contributions from of Rogers, Hardy, MacMahon, Ramanujan, and Rademacher [28, 3]. Since then, the partitions has been an area of research in itself, besides being a useful combinatorial object interpreted a lot of sequences, numbers and polynomials, as the q-binomial coefficient [43], more broadly the symmetric functions which are indexed as usual by partitions [48].

Definition 1.1.1 *A partition* $\eta = (\eta_1, \eta_2, \cdots, \eta_k)$ *of* n *is a non-increasing sequence of positive integers (i.e.,* $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_k$*) whose sum is* n.
- For $1 \leq i \leq k$, η_i *is called a part of* η. *Although we considered each part in a partition is positive, we sometimes allow "zero" as a part.*

Notation: *The number of parts of* η *called the* ***length of*** η *is denoted* $\ell(\eta)$*; The sum of parts of* η *called the* ***weight of*** η *is denoted* $|\eta|$.

We denote the number of partitions of n by $p(n)$. For convenience, we define $p(0) = 1$.

Example 1.1.2 *The following are the partitions of* 4*:*

$$4, 3+1, 2+2, 2+1+1, 1+1+1+1,$$

so $p(4) = 5$.

A partition into K parts with the largest part is less or equal to $N-1$ is a partition written in the following forms

$$\eta = (\eta_1, \eta_2, \ldots, \eta_K) \qquad \text{or} \qquad \eta = \eta_1\eta_2\cdots\eta_K,$$

where $N-1 \geq \eta_1 \geq \eta_2 \geq \cdots \geq \eta_K \geq 0$.

We denote the number of partitions into K parts with the largest part is less or equal to N by $p(N, K)$. And the notation $\eta \subset (N)^K$ simply means that $\eta \in p(N, K)$.

Another common expression has the same meaning as "partitions into K parts with the largest part is less or equal to N" is the following: "Partitions fitting inside an $N \times K$ rectangle". This expression cames from being the partitions can be interpreted into boxes diagram called "**The Young diagram**".

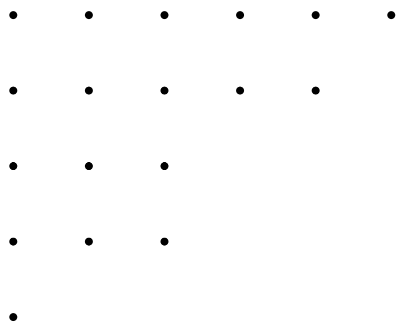


Figure 1.1: Diagram of Ferrers of $\eta = (6, 5, 3, 3, 1)$.

Ferrers diagram and Young diagram

Definition 1.1.3 *The Ferrers diagram of a partition* $\eta \in p(N, K)$ *is formally defined as the set of points* $(i_1, i_2) \in \mathbb{N}^2$ *such that* $1 \leq i_2 \leq K$ *and* $1 \leq i_1 \leq \eta_{i_2} \leq N$.

Example 1.1.4 *The Figure 1.1 shows the Ferrers diagram of the partition* $\eta = (6, 5, 3, 3, 1)$.

Definition 1.1.5 *Replacing the nodes by boxes (squares) in the Ferrers diagram, we obtain a diagram called Young diagram.*

Example 1.1.6 *The Figure 1.2 shows the Young diagram of the partition* $\eta = (5, 2, 2, 1)$.

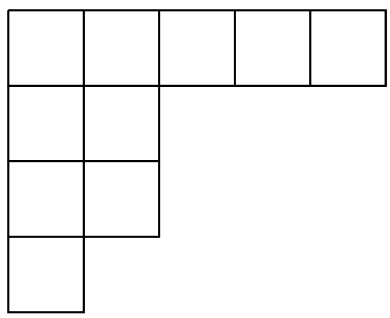
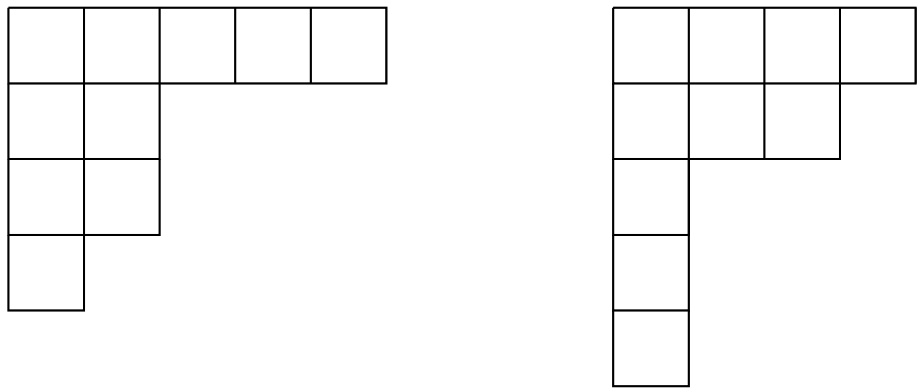


Figure 1.2: Young diagrzm of $\eta = (5, 2, 2, 1)$.

So, we can simply say that the number of boxes in the i-th row in Young diagram of a partition $\eta = (\eta_1, \eta_2, \cdots, \eta_K)$ represents the i-th part η_i, and the number of rows represents the number of parts of η.

Definition 1.1.7 *By transposing the Young diagram (resp. the Ferrers diagram) of a partition* η*, we obtain a new Young diagram (resp. Ferrers diagram) called the conjugate diagram corresponded to another partition denoted* η^c *called the conjugate partition of* η.



$\eta = (5, 2, 2, 1)$ $\eta^c = (4, 3, 1, 1, 1)$

Figure 1.3: Young diagram of $\eta = (5, 2, 2, 1)$ and its conjugate.

Let $\rho = \eta^c$ be the conjugate of the partition $\eta = (\eta_1, \eta_2, \cdots, \eta_K)$, then

$$\rho_i = \text{Card}\{j : \eta_j \geq i\}.$$

Obviously $(\eta^c)^c = \eta$.

For instance, the Figure 1.3 shows the Young diagram and its conjugate.

Let P_D be the set of partitions whose parts are all distinct, and let P_o be the set of partitions whose parts are all odd. Then

Theorem 1.1.8 *[32, **Euler**] For $n \geq 1$, the number of partitions of n into distinct parts equals to the number of partitions of n into odd parts.*

A lot of proofs of this theorem are known, the Euler's one was simple yet important, it made use of generating functions [1].

Generating functions

In order to solve the general linear recurrence problem [44], Abraham De Moivre introduced for the first time the generating functions in 1730. Since then, the theory of generating functions occupy an important place in today's mathematics by being a powerful tool of problem solving. Indeed, it makes you deal with functions instead of sequences.

Definition 1.1.9 *The generating function G of an infinite sequence $(u_n)_n$ is the formal power series in terms of a parameter z*

$$G(z) = u_0 + u_1 z + u_2 z^2 + \cdots = \sum_{n \geq 0} u_n z^n.$$

We used the word "formal" to indicate that we do not care about giving a value to z and obtaining a value for $G(z)$, not even the convergence of the series. For an example of a generating function of a famous number that is not a convergent series see [42].

Before presenting the generating functions of the number of partitions, the number of partitions into distinct parts and the number of partitions into odd parts, we give the definition of a specific type of series.

Definition 1.1.10 *The q-series or basic hypergeometric series are series involving q-shifted factorial which is the factor of the form*

$$(a)_n = (a;q)_n; = \prod_{i=1}^{n}(1 - aq^{i-1}), \quad |q| < 1,$$

with for $n = 0$

$$(a)_0 = 1.$$

And

$$(a)_\infty = (a;q)_\infty; = \prod_{i=1}^{\infty}(1 - aq^{i-1}), \quad |q| < 1.$$

The number of partitions has the generating function

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{(q)_\infty}.$$

The number of partitions into distinct parts has the generating function

$$\sum_{n=0}^{\infty} p_d(n)q^n = \prod_{k=1}^{\infty}(1+q^k).$$

The number of partitions into odd parts has the generating function

$$\sum_{n=0}^{\infty} p_o(n)q^n = \prod_{k=0}^{\infty} \frac{1}{1-q^{2k+1}}.$$

The proof of Euler's theorem (Theorem 1.1.8) is based on the fact that

$$\prod_{k=1}^{\infty}(1+q^k) = \prod_{k=1}^{\infty} \frac{1-q^{2k}}{1-q^k}. \tag{1.1.1}$$

After simplifying the right side of the equality (1.1.1) we get

$$\prod_{k=1}^{\infty}(1+q^k) = \prod_{k=0}^{\infty} \frac{1}{1-q^{2k+1}}, \tag{1.1.2}$$

which gives us that the generating function of partitions into distinct parts (the left side of (1.1.2)) is equal to that of odd parts (the right side of (1.1.2)).

1.1.2 Overpartitions

The overpartitions are a known generalization of partitions which were used under different names. Corteel and Lovejoy [27] were the first who gave the name "Overpartitions" to this generalization.

Definition 1.1.11 *[27] For a natural number n, we call an overpartition of n any non-increasing sequence of natural numbers where the final occurrence (equivalently, the first occurrence) of each number may be overlined and the sum of these numbers equals n. The number of overpartitions of an natural number n is denoted by $\overline{p}(n)$, and the set of overpartitions of non-negative integers is denoted by $\overline{\mathcal{P}}$.*

Example 1.1.12 *The 14 overpartitions of 4 are*

$$4,\overline{4},3{+}1,\overline{3}{+}1,3{+}\overline{1},\overline{3}{+}\overline{1},2{+}2,2{+}\overline{2},2{+}1{+}1,\overline{2}{+}1{+}1,2{+}1{+}\overline{1},\overline{2}{+}1{+}\overline{1},1{+}1{+}1{+}1,1{+}1{+}1{+}\overline{1}.$$

For convenience, we allow sometime (for example in Chapter 2) the 0 as a part but not as an overlined part.

Definition 1.1.13 *By seeing an overpartition as a partition in which the final occurrence of a part may be overlined, then we define the Young diagram of an overpartition to be an ordinary Young diagram in which the corners may be colored. i.e., the corner of each row is colored if and only if the part corresponded to this row is overlined.*

For instance, see Figure 1.4.

Definition 1.1.14 *By transposing the Young diagram of an overpartition η, we obtain a new Young diagram (where the corners may be overlined) called the conjugate diagram corresponded to another overpartition denoted η^c called the conjugate overpartition of η.*

The overpartition number has as generating function

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n=\prod_{n=1}^{\infty}\frac{1+q^n}{1-q^n}=\frac{(-q)_\infty}{(q)_\infty}.$$

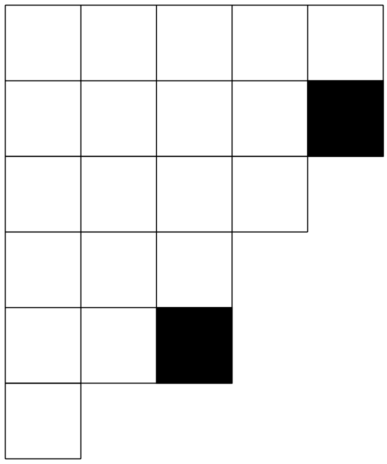


Figure 1.4: The Young diagram of $\eta = (5, \overline{5}, 4, 3, \overline{3}, 1)$.

1.1.3 Permutations

The concept of permutation has been known since ancient times, the name comes from permuting things which is rearranging them.

Let $[n]$ denote the set of natural numbers from 1 to n, i.e., the set $\{1, 2, \ldots, n\}$.

Definition 1.1.15 *A permutation π of length n is a bijection from $[n]$ to its self. The permutation is denoted by*

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$$

or for brevity it is usually denoted by $\pi = \pi_1\pi_2\cdots\pi_n$.

Definition 1.1.16 *The symmetric group denoted $\mathcal{S}_n$ is the group of all the permutations of $[n]$, with the group operation is the composition of functions.*

The cardinality of the group $\mathcal{S}_n$ is $n!$.

Example 1.1.17 *The permutations of the group S_3 are*

$$123,\ 132,\ 213,\ 231,\ 312,\ 321.$$

The group $\mathcal{S}_n$ has as identity element the permutation $\nu = \nu_1\nu_2\cdots\nu_n$ where $\nu_i = i$ for all $1 \leq i \leq n$. And for $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$, the inverse $\pi^{-1} = \alpha_1\alpha_2\cdots\alpha_n$ of the permutation π is a permutation in which for $1 \leq i \leq n$, α_i is equal to the position of the integer i in

π. For instance, the inverse of the permutation $\pi = 35214$ is $\pi^{-1} = 43152$. The inverse of a permutation is not be to confused with the backward permutation. The backward permutation of π is the permutation π' where $\pi'_i = \pi_{n+1-i}$ for $1 \leq i \leq n$.

In the following part we give the definitions of some statistics.

Definition 1.1.18 *An inversion of a permutation π is a pair (π_i, π_j) where $i < j$ and $\pi_i > \pi_j$.*

Example 1.1.19 *The 5 inversions of the permutation 25143 are: $(2,1)$, $(5,1)$, $(5,4)$, $(5,3)$ and $(4,3)$.*

The identity element ν is the only permutation with no inversion, and the permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ where $\sigma_i = n+1-i$ is the permutation of length n with the maximum number of inversions, it has exactly $\binom{n}{2}$ inversions. It is not hard to show this remark:

Remark 1.1.20 *The number of inversions of a permutation is equal to the number of inversions of its inverse.*

Now, we give another definition.

Definition 1.1.21 *[41] A backward inversion of a permutation π is a pair (π_i, π_j) where $i < j$ and $\pi_i < \pi_j$.*

A permutation of length n has $\binom{n}{2} - k$ backward inversions where k is the number of inversions in this permutation.

Definition 1.1.22 *The major index of a permutation π of length n is defined to be the sum of indices j that satisfy $\pi_j > \pi_{j+1}$.*

Example 1.1.23 *The major index of the permutation 24153 is $2+4=6$.*

Definition 1.1.24 *Let $\pi = \pi_1, \pi_2, \cdots, \pi_n \in \mathcal{S}_n$.*

- *We say that the index j where $1 \leq j \leq n-1$ is a descent of π if $\pi_j > \pi_{j+1}$.*
- *We say that the index j where $1 \leq j \leq n-1$ is an ascent of π if $\pi_j < \pi_{j+1}$.*
- *We say that the index j where $1 \leq j \leq n-1$ is an excedance of π if $\pi_j > j$.*

Example 1.1.25 *The permutation 523641 has 3 descents 1, 4 and 5, and has 2 ascents 2 and 3.*

Let $Des(\pi)$ (resp. $As(\pi)$) denote the set of all the descents of π (resp. all the ascents of π), and $des(\pi)$ (resp. $as(\pi)$) denote the cardinality of $Des(\pi)$ (resp. $As(\pi)$).

1.2 Lattice paths

A lattice path is a set of segments which are called **steps**, which they link a set of points in $\mathbb{Z} \times \mathbb{Z}$. The first point is called the initial point and the last one is called the final point. There are many types of lattice paths used in combinatorics, we are interested in this section in **North-East paths, labeled path of Gashrov, Delannoy paths, Dyck Paths, Motzkin paths** and then **Vertically constrained Motzkin-like paths**.

Definition 1.2.1 *A **North-East path** $P = (P_0, P_1, \ldots, P_{n-1})$ of length n is a set of points on a rectangular grid in $\mathbb{N} \times \mathbb{N}$, where P_0 is the initial point, and any step (P_i, P_{i+1}) is either a North step or an East step, namely the set of step vectors is $\{(1,0), (0,1)\}$.*

Let $\mathcal{P}(N, K)$ denote the set of North-East paths from $(0,0)$ to (N, K).

We denote by $a \xrightarrow{P} b$ to indicate that the path P has a as an initial vertex and b as a final vertex.

Example 1.2.2 *The path in Figure 1.5 is a path in $\mathcal{P}(3,3)$.*

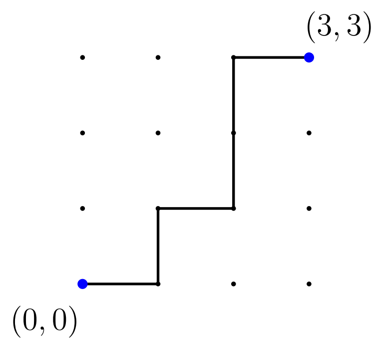


Figure 1.5: A North-East path P from $(0,0)$ to $(3,3)$.

We give now the definition of the weight associated to each North-East path.

Definition 1.2.3 *The **weight** of a North-East path P denoted w_P is the number of boxes above and to the left of P, where the boxes are the squares formed by the grid.*

Remark 1.2.4 *In fact, there is a correspondence between North-East paths and partitions, just see that each number of boxes above and to the left of the path in each row represents a part, and then, these parts give us a partition.*

For instance, the weight of the path in Figure 1.5 is 5, and this path is corresponded to the partition $\eta = (3, 2, 0)$.

Definition 1.2.5 *[35] A **labeled path of Gashrov** is a North-East lattice path with $n-1$ steps, k of them are vertical, such that a horizontal step $i-1$ units above the initial point of this path is labeled with an integer between 1 and i, and a vertical step $i-1$ units to the right of the initial point of this path is labeled with an integer between 1 and i.*

We denote such paths by $\mathcal{P}_G(n, k)$.

Example 1.2.6 *The path in Figure 1.6 is a path in $\mathcal{P}_G(4, 1)$.*

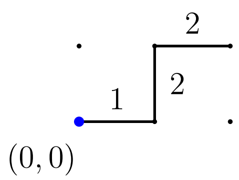


Figure 1.6: A labeled path of Gasharov.

Definition 1.2.7 *[9] A **Delannoy path** denoted P is a path on a rectangular grid starts from $(0, 0)$ and ends on (m, n), and any step (P_i, P_{i+1}) is either an East step, or a North step, or a North-East step, namely the set of step vectors is $\{(1, 0), (0, 1), (1, 1)\}$. $\mathcal{D}(m, n)$ is the set of all Delannoy paths from $(0, 0)$ to (m, n).*

Example 1.2.8 *The path of Figure 1.7 is a path in $\mathcal{D}(3, 3)$.*

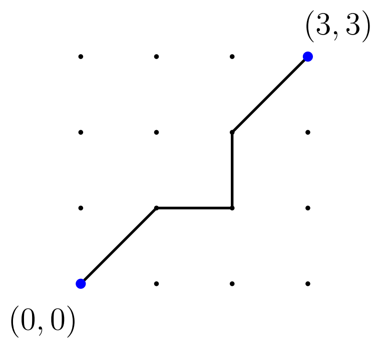


Figure 1.7: A path P in $\mathcal{D}(3, 3)$.

Dousse and Kim [30] defined the weight of a Delannoy path as follows.

Definition 1.2.9 *The **weight** of a Delannoy path P denoted $wt(P)$ is defined to be the sum of the weights of its steps, where the weight of any of its step p_k starts from (i_1, i_2) is*

$$wt(p_k) = \begin{cases} 0, & \text{if it goes to } (i_1+1, i_2), \\ i_1, & \text{if it goes to } (i_1, i_2+1), \\ i_1+1, & \text{if it goes to } (i_1+1, i_2+1). \end{cases}$$

The number of North-East steps in P is denoted by the same authors by $d(P)$.

Definition 1.2.10 *A **Dyck path** P of length n is a path from $(0,0)$ to $(2n,0)$ the upper right quarter-plane (Q) using only North-East step and South-East step, namely the set of step vectors is $\{(1,1),(1,-1)\}$.*

Example 1.2.11 *The path in Figure 1.8 is a Dyck path of length 3.*

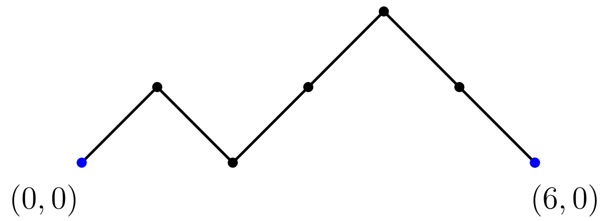


Figure 1.8: A Dyck path P of length 3.

Definition 1.2.12 *A **Motzkin path** P of length n is a path from $(0,0)$ to $(n,0)$ on the upper right quarter-plane (Q) using only East step, North-East step and South-East step, namely the set of step vectors is $\mathcal{M} = \{(1,0),(1,1),(1,-1)\}$.*

The path of Figure 1.9 is a Motzkin path of length 7.

Irvine et al. [40] added the vertical steps North and South (that is $\{(0,1),(0,-1)\}$) to the set of steps of Motzkin paths, with the constraint :"no consecutive vertical steps are allowed".

Definition 1.2.13 *A **vertically constrained lattice paths** P is a path on the discrete Cartesian plane $\mathbb{Z} \times \mathbb{Z}$, using East step, North-East step, South-East step, North step and South step, namely the set of step vectors is $\mathcal{A} = \{(1,0),(1,1),(1,-1),(0,1),(0,-1)\}$, with no consecutive vertical steps are allowed.*

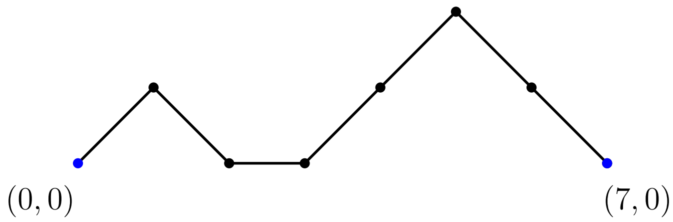


Figure 1.9: A Motzkin path of length 7.

Then, the authors distinguished the following four classes of vertically constrained lattice paths:

- $\mathbf{A^H}$ the set of partially directed, vertically constrained lattice paths using step vectors in $\mathcal{A}$ in the half-plane. Let $A^H(n_1, n_2)$ denote the number of paths of type $\mathbf{A^H}$ from $(0, 0)$ to (n_1, n_2).

- $\mathbf{A_R^H}$ the set of partially directed, vertically constrained lattice paths using step vectors in $\mathcal{A}$ in the half-plane, in which the leading step is restricted to $\mathcal{M}$. Let $A_R^H(n_1, n_2)$ denote the number of paths of type $\mathbf{A_R^H}$ from $(0, 0)$ to (n_1, n_2).

- $\mathbf{A^Q}$ the set of partially directed, vertically constrained lattice paths using step vectors in $\mathcal{A}$ restricted to the quarter-plane. Let $A^Q(n_1, n_2)$ denote the number of paths of type $\mathbf{A^Q}$ from $(0, 0)$ to (n_1, n_2).

- $\mathbf{A_R^Q}$ the set of partially directed, vertically constrained lattice paths using step vectors in $\mathcal{A}$ restricted to the quarter-plane, in which the leading step is restricted to $\mathcal{M}$. Let $A_R^Q(n_1, n_2)$ denote the number of paths of type $\mathbf{A_R^Q}$ from $(0, 0)$ to (n_1, n_2).

Among the four previous sets of vertically constrained lattice paths, we are interested in this thesis (exactly in chapter four) in the last one $\mathbf{A_R^Q}$. Figure 1.10 shows for instance a path in $\mathbf{A_R^Q}$.

1.3 Binomial and bi^{s}nomial coefficients

Pascal triangle is one of the famous triangle in arithmetic and combinatorics, but actually, Pascal was not the first who studied this triangle, the triangle was already studied in China, India and by AL-Karaji and Al-Khayyam, this is why sometimes Pascal triangle is called

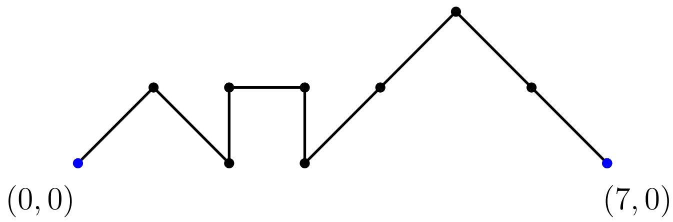


Figure 1.10: A path in $A_R^Q(7,0)$.

Al-Karaji triangle. The coefficients of Pascal triangle are called **Binomial coefficients**, and they are defined as follows.

1.3.1 Binomial coefficient

Let N and K be two integers. The binomial coefficient is defined for $N \geq K \geq 0$ as

$$\binom{N}{K} = \frac{N!}{K!(N-K)!},$$

with the convention $\binom{N}{K} = 0$ for either $K < 0$ or $K > N$.

The binomial coefficient satisfies the following properties:

- **The symmetry relation**
$$\binom{N}{K} = \binom{N}{N-K}.$$
- **The recurrence relation**
$$\binom{N}{K} = \binom{N-1}{K} + \binom{N-1}{K-1}.$$
- **Vandermonde's convolution formula**
$$\binom{N_1+N_2}{K} = \sum_{j=0}^{K} \binom{N_1}{j}\binom{N_2}{K-j}.$$

The binomial coefficients have as generating function

$$\sum_{K\geq 0} \binom{N}{K} x^K = (1+x)^N.$$

These coefficients build us the Pascal triangle. The first values of Pascal triangle are given in Table 1.1.

N/K	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Table 1.1: Pascal Triangle.

Combinatorial interpretation: The binomial coefficient has a various combinatorial interpretations, the most famous one is that $\binom{N}{K}$ counts the number of distinct ways to choose K elements from N elements. And by Remark 1.2.4, the binomial coefficient counts the number of North-East paths from $(0,0)$ to $(N-K, K)$.

1.3.2 Bi^{s}nomial coefficient

The bi^{s}nomial coefficients are obtained by the multinomial expansion:

$$(1 + x + \cdots + x^s)^N = \sum_{K \geq 0} \binom{N}{K}_s x^K, \tag{1.3.1}$$

where $\binom{N}{K}_1 = \binom{N}{K}$ is the binomial coefficient, and $\binom{N}{K}_s = 0$ for $K > sN$ or $K < 0$. For more about these coefficients see Andrews and Baxter [4], Belbachir et al. [13] and Belbachir and Igueroufa [15].

The bi^{s}nomial coefficients satisfy the following properties:

- Expression in terms of binomial coefficients,

$$\binom{N}{K}_s = \sum_{j_1+j_2+\cdots+j_s=K} \binom{N}{j_1}_s \binom{j_1}{j_2}_s \cdots \binom{j_{s-1}}{j_s}_s.$$

- Symmetry relation

$$\binom{N}{K}_s = \binom{N}{sN-K}_s. \tag{1.3.2}$$

n/k	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1	1										
2	1	2	3	2	1								
3	1	3	6	7	6	3	1						
4	1	4	10	16	19	16	10	4	1				
5	1	5	15	30	45	51	45	30	15	5	1		
6	1	6	21	50	90	126	141	126	90	50	21	6	1

Table 1.2: Triangle of trinomial coefficients $s = 2$.

- The two recurrence relations
  - The longitudinal recurrence relation
$$\binom{N}{K}_s = \sum_{j=0}^{s} \binom{N-1}{K-j}_s. \tag{1.3.3}$$
  - The diagonal recurrence relation
$$\binom{N}{K}_s = \sum_{j=0}^{s} \binom{N}{j} \binom{N}{K-j}_{s-1}.$$

The values of the bi^{s}nomial coefficients build us the s-Pascal triangle as shown in Table 1.2. We find the first values of the 2-Pascal triangle in OEIS [56] as A027907.

Combinatorial interpretations:

-Bondarenko [22] gave a combinatorial interpretation of $\binom{N}{K}_s$ as the number of ways to distribute K balls among N boxes where each box contains at most s balls, for $0 \leq K \leq sN$;

- Bazeniar et al. [10] showed that bi^{s}nomial coefficient $\binom{N}{K}_s$ counts the number of North-East paths from $(0, 0)$ to $(K, N-1)$ taking at most s consecutive steps in the East direction. We denote by $\mathcal{P}^{(s)}(N-1, K)$ the set of such paths.

1.4 q-Analogue of binomial and bi^{s}nomial coefficients

The q-analogue of a mathematical expression is a generalization of this expression contains a new parameter $q \in \mathbb{R}$ so that we can obtain the original expression by taking the limit $q \to 1$.

The first q-analogues were the basic hypergeometric series. For more information see Exton [34].

Noting that

$$\lim_{q\to 1}\frac{1-q^n}{1-q}=n,$$

the q-analogue of an integer n was defined to be

$$[n]_q=1+q+\cdots+q^n=\frac{1-q^n}{1-q}=n.$$

And it satisfies

$$[n]_q=[k]_q+q^k[n-k]_q=[n-k]_q+q^{n-k}[k]_q.$$

Using the definition of the q-analogue of an integer, we define the q-analogue of the factorial known as q-factorial, by

$$[n]_q!=[1]_q[2]_q\cdots[n]_q.$$

From Definition 1.1.10, we can write the q-factorial of n as follows,

$$[n]_q!=\frac{(q)_n}{(1-q)^n}.$$

1.4.1 q-Binomial coefficient

Definition 1.4.1 *The q-binomial coefficient or the Gaussian polynomial is defined as*

$$\begin{bmatrix}N+K\\K\end{bmatrix}_q=\frac{[N+K]_q!}{[N]_q![K]_q!}=\frac{(q)_{N+K}}{(q)_N(q)_K}.$$

In other words, we can rewrite the definition of q-binomial coefficient is as follows

Definition 1.4.2 *The q-binomial coefficient is defined also by*

$$\begin{bmatrix}N\\K\end{bmatrix}_q=\prod_{i=1}^{K}\frac{1-q^{N-i+1}}{1-q^i},\quad 1\leq K\leq N,$$

with $\left[\begin{smallmatrix}N\\0\end{smallmatrix}\right]_q=1$ and $\left[\begin{smallmatrix}0\\K\end{smallmatrix}\right]_q=\delta_{K,0}$, where $\delta_{K,0}$ is the Kronecker delta.

The q-binomial coefficient satisfies the following properties:

- **The symmetry relation**

$$\begin{bmatrix}N+K\\K\end{bmatrix}_q=\begin{bmatrix}N+K\\N\end{bmatrix}_q.$$

- **The recurrence relations**

$$\begin{bmatrix} N+K \\ K \end{bmatrix}_q = \begin{bmatrix} N+K-1 \\ K \end{bmatrix}_q + q^N \begin{bmatrix} N+K-1 \\ K-1 \end{bmatrix}_q,$$
$$\begin{bmatrix} N+K \\ K \end{bmatrix}_q = \begin{bmatrix} N+K-1 \\ K-1 \end{bmatrix}_q + q^K \begin{bmatrix} N+K-1 \\ K \end{bmatrix}_q.$$

The q-binomial coefficient can be written in terms of the two symmetric functions "the elementary and the complete symmetric functions".

Definition 1.4.3 *The k-th elementary symmetric function is the sum of all the products of k distinct variables x_i where $i \in \{1, 2, \cdots, n\}$, that is*

$$e_k(n) := e_k(x_1, x_2, \cdots, x_n) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

with $e_0(n) = 1$, and $e_k(n) = 0$ if $n < k$ or $k < 0$.

For instance,

$$e_3(4) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4.$$

Definition 1.4.4 *The k-th complete symmetric function is the sum of all the products of n variables x_i where $i \in \{1, 2, \cdots, n\}$, that is*

$$h_k(n) := h_k(x_1, x_2, \cdots, x_n) = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

with $h_0 = 1$.

For instance,

$$h_2(3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3.$$

From the previous two definitions, one can easily obtain the following recurrence relations.

Proposition 1.4.5 *For a nonnegative integer k and a positive integer n,*

- $e_k(n) = e_k(n-1) + x_n e_{k-1}(n-1)$,
- $h_k(n) = h_k(n-1) + x_n h_{k-1}(n)$.

From the Proposition above one can easily conclude.

Corollary 1.4.6 *For two nonnegative integers K and N,*

- $\binom{N}{K} = e_K(\underbrace{1, 1, \cdots, 1}_{N\ times})$,
- $\begin{bmatrix} N \\ K \end{bmatrix}_q = q^{-\binom{K}{2}} e_K(1, q, \cdots, q^{N-1})$,
- $\begin{bmatrix} N+K \\ K \end{bmatrix}_q = h_K(1, q, \cdots, q^N)$.

Hence, we have the following corollary.

Corollary 1.4.7 *For two nonnegative integers K and N,*

- $\begin{bmatrix} N \\ K \end{bmatrix}_q$ *is the generating function for the number of partitions whose Ferrers diagram is in a $K \times (N-K)$ rectangle (Knuth [43]). That is*

$$\begin{bmatrix} N \\ K \end{bmatrix}_q = \sum_{\eta \subset (N-K)^K} q^{|\eta|}, \tag{1.4.1}$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_K)$ with $N - K \geq \eta_1 \geq \eta_2 \geq \cdots \geq \eta_K \geq 0$, $|\eta| = \sum \eta_i$.

- $\begin{bmatrix} N \\ K \end{bmatrix}_q$ *is the generating function for the number of partitions on K distinct parts where the largest part is less or equal to $N-1$. That is*

$$\begin{bmatrix} N \\ K \end{bmatrix}_q = q^{-K(K-1)/2} \sum_{\eta \subset (N-1)^K} q^{|\eta|}, \tag{1.4.2}$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_K)$ with $N - 1 \geq \eta_1 \geq \eta_2 \geq \cdots \geq \eta_K \geq 0$, $|\eta| = \sum \eta_i$.

The q-binomial coefficient can also be interpreted by lattice paths instead of partitions as follows.

Corollary 1.4.8 *For two nonnegative integers K and N,*

- $\begin{bmatrix} N \\ K \end{bmatrix}_q$ *is the generating function for the number of North-East paths from $(0,0)$ to $(N-K, K)$. That is*

$$\begin{bmatrix} N \\ K \end{bmatrix}_q = \sum_{P \in \mathcal{P}(N-K,K)} q^{w_P}. \tag{1.4.3}$$

- $\begin{bmatrix} N \\ K \end{bmatrix}_q$ *is the generating function for the number of North-East paths from $(0,0)$ to $(N-1, K)$ with no consecutive North steps are allowed. That is*

$$\begin{bmatrix} N \\ K \end{bmatrix}_q = q^{-K(K-1)/2} \sum_{P \in \mathcal{P}^{(1)}(N-1,K)} q^{w_P}. \tag{1.4.4}$$

N/K	0	1	2	3	4
0	1				
1	1	1	1		
2	1	$1+q$	$1+q+q^2$	$q+q^2$	q^2
3	1	$1+q+q^2$	$1+q+2q^2+q^3+q^4$	$\cdots$	$\cdots$
4	1	$1+q+q^2+q^3$	$1+q+2q^2+2q^3+2q^4+\cdots$	$\cdots$	

Table 1.3: Table values of q-trinomial coefficients.

1.4.2 q-Bi^{s}nomial coefficient

The q-bi^{s}nomial coefficients are an analogue of bi^{s}nomial coefficients. As reference, see [4, 10, 12].

Definition 1.4.9 *The q-bi^{s}nomial coefficient [12] is defined as follows:*

$$\begin{bmatrix} N \\ K \end{bmatrix}_q^{(s)} = \sum_{j_1+\cdots+j_s=K} \begin{bmatrix} N \\ j_1 \end{bmatrix}_q^{(1)} \cdots \begin{bmatrix} N \\ j_s \end{bmatrix}_q^{(1)} (-1)^K \left(e^{i\frac{2\pi}{s+1}}\right)^{-\sum_{r=1}^{s} r j_r},$$

where $\begin{bmatrix} N \\ K \end{bmatrix}_q^{(1)} = q^{\binom{K}{2}} \begin{bmatrix} N \\ K \end{bmatrix}_q$.

And we have these two relations

$$\begin{bmatrix} N \\ K \end{bmatrix}_q^{(s)} = \sum_{i=0}^{s} q^{(N-1)i} \begin{bmatrix} N-1 \\ K-i \end{bmatrix}_q^{(s)}, \tag{1.4.5}$$

$$\begin{bmatrix} N \\ K \end{bmatrix}_q^{(s)} = \sum_{i=0}^{s} q^{K-i} \begin{bmatrix} N-1 \\ K-i \end{bmatrix}_q^{(s)}. \tag{1.4.6}$$

That leads us to build according to the value of s, the triangle of q-bi^{s}nomial coefficients. For instance, Table 1.3 shows the triangle of q-bi^{2}nomial (or q-trinomial) coefficients.

The q-bi^{s}nomial coefficients have as generating function:

$$\sum_{K\geq 0} \begin{bmatrix} N \\ K \end{bmatrix}_q^{(s)} x^K = \prod_{i=0}^{N-1} \left(1+q^i x+\cdots+q^{is}x^s\right) = \prod_{i=0}^{N-1} \left(\frac{1-q^{i(s+1)}x^{s+1}}{1-q^i x}\right), \tag{1.4.7}$$

Bazeniar *et al.* [10] proved that the q-bi^{s}nomial coefficients have an interesting interpretation using partitions, basing on the generalized elementary symmetric function that they defined as follows.

Definition 1.4.10 *For a positive integer $s \geq 1$, the generalized elementary symmetric function is defined by*

$$E_k^{(s)}(n) = E_k^{(s)}(x_1, x_2, \cdots, x_n) = \sum_{\substack{i_1+i_2+\cdots i_n=k \\ 0\leq i_1,i_2,\cdots,i_n\leq s}} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

with $E_0^{(s)}(n) = 1$, $E_k^{(s)}(n) = 0$ if $sn < k$ or $k < 0$.

For instance,

$$E_2^{(2)}(3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3.$$

The same previous authors [10] proved that the generalized elementary symmetric function satisfies the following recurrence.

Proposition 1.4.11 *For a positive integer s,*

$$E_k^{(s)}(n) = \sum_{j=0}^{s} x_n^j E_{k-j}^{(s)}(n-1).$$

Then, the authors [10] concluded the following result, and then deduced the relationship between q-bi^{s}nomial coefficients, partitions and lattice paths.

Corollary 1.4.12 *[10]*

- $E_k^{(s)}(1, 1, \cdots, 1) = \binom{n}{k}_s$,
- $E_k^{(s)}(1, q, \cdots, q^{n-1}) = \begin{bmatrix} n \\ k \end{bmatrix}_q^{(s)}$.

Proposition 1.4.13 *[10] For a positive integer s,*

$$\begin{bmatrix} N \\ K \end{bmatrix}_q^{(s)} = \sum_{\eta\subset(N-1)^K} q^{|\eta|}, \tag{1.4.8}$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_K)$ with $N-1 \geq \eta_1 \geq \eta_2 \geq \cdots \geq \eta_K \geq 0$, $|\eta| = \sum \eta_i$ and $\eta_i \neq \eta_{i+s}$ for all $i \geq 1$.

Corollary 1.4.14 *[10] For a positive integer s,*

The q-bi^{s}nomial coefficient $\begin{bmatrix} N \\ K \end{bmatrix}_q^{(s)}$ is the generating function for the number of North-East paths from $(0,0)$ to $(N-1, K)$ taking at most s successive steps in the North direction. That is

$$\begin{bmatrix} N \\ K \end{bmatrix}_q^{(s)} = \sum_{P\in\mathcal{P}^{(s)}(N-1,K)} q^{w_P}. \tag{1.4.9}$$

1.5 Overpartitions analogues of q-binomial coefficients

1.5.1 Over q-binomial coefficient

Dousse and Kim [30] defined the over q-binomial coefficient $\overline{\begin{bmatrix} M+N \\ M \end{bmatrix}}_q$ as the generating function for the number of overpartitions such that the number of parts $\leq M$ (they don't consider the "0" as a part) and each part $\leq N$, then they give the following expression for over q-binomial coefficients

Theorem 1.5.1 *Let M and N be two positive integers, then*

$$\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_q = \sum_{i=0}^{minN,M} q^{\frac{i(i+1)}{2}} \frac{(q)_{N+M-i}}{(q)_i (q)_{N-i} (q)_{M-i}}.$$

The over q-binomial coefficient satisfies the following properties,

- **The symmetry relation**

$$\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_q = \overline{\begin{bmatrix} N+M \\ N \end{bmatrix}}_q,$$

- **The two recurrence relations**

$$\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_q = \overline{\begin{bmatrix} N+M-1 \\ M-1 \end{bmatrix}}_q + q^M \overline{\begin{bmatrix} N+M-1 \\ M \end{bmatrix}}_q + q^M \overline{\begin{bmatrix} N+M-2 \\ M-1 \end{bmatrix}}_q$$

and

$$\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_q = \overline{\begin{bmatrix} N+M-1 \\ M \end{bmatrix}}_q + q^N \overline{\begin{bmatrix} N+M-1 \\ M-1 \end{bmatrix}}_q + q^N \overline{\begin{bmatrix} N+M-2 \\ M-1 \end{bmatrix}}_q.$$

The over q-binomial coefficients satisfy the following partition interpretation.

Proposition 1.5.2 *Let N and M be two positive integers, then*

$$\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_q = \sum_{i\geq 0} \overline{p}(N,M,i) q^i,$$

where $\overline{p}(N,M,i)$ counts the number of overpartitions of i, such that the number of parts $\leq M$ and each part $\leq N$.

The over q-binomial coefficients have also the following asymptotic behaviour:

$$\lim_{N\to\infty} \overline{\begin{bmatrix} N \\ i \end{bmatrix}}_q = \frac{(-q)_i}{(q)_i},$$

when N goes to infinity, the restriction on the number of parts disappears.

1.5.2 Over (q,t)-binomial coefficient

Dousse and Kim in [31] added one more parameter counting the number of overlined part, by defining the over (q,t)-binomial coefficient $\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_{q,t}$ as follows.

Definition 1.5.3 *For two positive integers M and N,*

$$\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_{q,t} = \sum_{k,i\geq 0} \overline{p}(N,M,k,i)t^k q^i,$$

where $\overline{p}(N,M,k,i)$ counts the number of overpartitions of i, with k overlined parts such that the number of parts $\leq M$ and each part $\leq N$.

By setting $t=0$, i.e., no part is overlined, then we obtain the q-binomial, and by setting $t=1$, we obtain the over q-binomial coefficients.

The over (q,t)-binomial coefficient satisfies the following properties,

- **The symmetry relation**

$$\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_{q,t} = \overline{\begin{bmatrix} N+M \\ N \end{bmatrix}}_{q,t}.$$

- **The two recurrence relations**

$$\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_{q,t} = \overline{\begin{bmatrix} N+M-1 \\ M-1 \end{bmatrix}}_{q,t} + q^M \overline{\begin{bmatrix} N+M-1 \\ M \end{bmatrix}}_{q,t} + tq^M \overline{\begin{bmatrix} N+M-2 \\ M-1 \end{bmatrix}}_{q,t},$$

and

$$\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_{q,t} = \overline{\begin{bmatrix} N+M-1 \\ M \end{bmatrix}}_{q,t} + q^N \overline{\begin{bmatrix} N+M-1 \\ M-1 \end{bmatrix}}_{q,t} + tq^N \overline{\begin{bmatrix} N+M-2 \\ M-1 \end{bmatrix}}_{q,t}.$$

Combinatorial interpretation: Dousse and Kim [30] proved that the over (q,t)-binomial coefficients $\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_{q,t}$ are generating functions for Delannoy paths from $(0,0)$ to (N,M) as in the following proposition.

Proposition 1.5.4 *[30]*

$$\overline{\begin{bmatrix} N+M \\ M \end{bmatrix}}_{q,t} = \sum_{P\in\mathcal{D}(N,M)} t^{d(P)} q^{wP)}.$$

where $d(P)$ and $wt(P)$ are defined in Definition 1.2.9.

The limiting behavior of over (q,t)-binomial coefficients is as follows,

$$\lim_{N\to\infty} \overline{\begin{bmatrix} N \\ i \end{bmatrix}}_{q,t} = \frac{(-tq)_i}{(q)_i},$$

when N goes to infinity, the restriction on the size of the largest part disappears.

1.6 Mahonian numbers and Catalan numbers

1.6.1 Mahonian numbers

Let $m \geq 1$ and $h \geq 0$ be two integers. Let $S_{inv}(m,h)$ be the set of all permutations of length m where each permutation has exactly h inversions. We denote by $i_m(h)$ the cardinality of $S_{inv}(m,h)$, that is $i_m(h) = |S_{inv}(m,h)|$.

This number $i_m(h)$ was introduced for the first time by Rodrigues [52]. After, MacMahon [49] showed that this number $i_m(h)$, which now carries his name "Mahonian number", is equal to the number of permutations of length m with major index equals h.

The generating function of $i_m(h)$ is as follows [50].

Theorem 1.6.1 *For positive integer m,*

$$\sum_{h=0}^{\binom{m}{2}} i_m(h) z^h = (1+z)(1+z+z^2)\cdots(1+z+\cdots+z^{m-1}). \tag{1.6.1}$$

For more details, we refer the reader to [19].

These Mahonian numbers are also studied by Ahmia and Belbachir [6] under the name of "**Rising binomial coefficients**", they gave the following combinatorial interpretation.

Combinatorial interpretation: $i_m(h)$ counts the number of different ways of distributing "h" balls over "$m-1$" boxes where the i^{th} box can receive up to "i" balls.

That leads us to easily prove the following relation.

Theorem 1.6.2 *[41] For $0 \leq h \leq \binom{m}{2}$, $i_m(h)$ satisfies the symmetry relation:*

$$i_m(h) = i_m\left(\binom{m}{2} - h\right) \tag{1.6.2}$$

and the longitudinal recurrence:

$$i_m(h) = \sum_{k=0}^{m-1} i_{m-1}(h-k). \tag{1.6.3}$$

The first values of the Mahonian triangle can be found in OEIS [56] as $A008302$.

As an illustration, Table 1.4 introduces the "**Mahonian triangle**".

m/ h	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1															
2	1	1														
3	1	2	2	1												
4	1	3	5	6	5	3	1									
5	1	4	9	15	20	23	20	15	9	4	1					
6	1	5	14	29	49	71	90	101	101	90	71	49	29	14	5	1

Table 1.4: The Mahonian triangle.

Remark 1.6.3 *The Mahonian numbers can be seen also as the coefficients of q-factorial polynomials as follows*

$$[m]_q! = \sum_{h=0}^{\binom{m}{2}} i_m(h)q^h. \tag{1.6.4}$$

1.6.2 Catalan numbers

The Catalan numbers are considered as one of the well-known sequences in combinatorics, this sequence has been the subject of many studies from the ancient time, it take its name from the mathematician Eugène Charles Catalan.

Definition 1.6.4 *The Catalan numbers are a sequence of natural integers which has the explicit formula*

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

For more details on these numbers, we refer the reader to [45].

The first values of Catalan numbers given by A000108 in OEIS [56] are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670,$$
$$129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650,$$
$$1289904147324, 4861946401452, 18367353072152, 69533550916004, 263747951750360,$$
$$1002242216651368, 3814986502092304, \ldots$$

The Catalan number satisfies the recursion

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

And they have as generating function

$$C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Combinatorial interpretations: The Catalan numbers have a various combinatorial interpretations [26, 29, 53], we mention from them:

- The number of triangulations of $(n+2)$-gone.
- The binary bracketing of $n+1$ terms.
- The Catalan number C_n counts the number of rooted plane trees with n edge (for more details see [29]).
- The Catalan number C_n counts the number of Dyck Paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ and never falling below the x-axis, or similarly, the number of lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ and $(1,0)$, never rising above the line $y=x$.

1.7 Log-concavity, log-convexity and unimodality

The concept of log-concavity, log-convexity and unimodality of sequences occupies a notable place in the field of combinatorics, algebra and geometry. Many methods arise under this, to study those three properties, for instance see [21, 47, 57, 60].

Definition 1.7.1 *We say that a positive sequence* $(u_i)_i$ *is log-concave if*

$$u_{i-1}u_{i+1} \leq u_i^2,$$

for all $i \geq 1$.

And we have the following equivalent definition.

Definition 1.7.2 *We say that a positive sequence $(u_i)_i$ is log-concave if*

$$u_{j-1}u_{i+1} \leq u_j u_i. \tag{1.7.1}$$

for $1 \leq j \leq i$.

Example 1.7.3
- *For n fixed, the binomial coefficients $\left\{\binom{n}{k}\right\}_k$ form a log-concave sequence, because*

$$\binom{n}{k-1}\binom{n}{k+1} = \binom{n}{k}^2 \frac{k(n-k)}{k+1(n-k+1)} < \binom{n}{k}^2;$$

- *The bi^{s}nomial coefficients for k and n: $\left\{\binom{n}{k}_s\right\}_k$ and $\left\{\binom{n}{k}_s\right\}_n$ are log-concave [11].*

Definition 1.7.4 *We say that a positive sequence $(u_k)_k$ is log-convex if for $i \geq 1$:*

$$u_{i-1}u_{i+1} \geq u_i^2.$$

And we have the following equivalent definition.

Definition 1.7.5 *We say that a positive sequence $(u_i)_i$ is log-convex if*

$$u_{j-1}u_{i+1} \geq u_j u_i. \tag{1.7.2}$$

for $1 \leq j \leq i$.

Example 1.7.6
- *The Catalan numbers $(C_n)_n$ form a log-convex sequence, because*

$$C_{n-1}C_{n+1} = \frac{(2n+2)(2n+1)}{2(2n-1)(n+2)}C_n^2 > C_n^2;$$

- *The Motzkin numbers $M_n = \sum_{k\geq 0}\binom{n}{2k}C_k$ are log-convex [47];*
- *The central binomial coefficients $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$ are log-convex, because*

$$\binom{2n-2}{n-1}\binom{2n+2}{n+1} = \binom{2n}{n}^2 \frac{(2n+2)(2n+1)}{2(2n-1)(n+1)} > \binom{2n}{n}^2.$$

Definition 1.7.7 *We say that a sequence of real numbers $(u_k)_{k=0}^n$ is unimodal if there exists two integers k_1, k_2 $(k_1 \leq k_2)$ such that a_k is increasing for $0 \leq k \leq k_1$, constant for $k_1 \leq k \leq k_2$, decreasing for $k_2 \leq k \leq n$.*
The integers l, $k_1 \leq l \leq k_2$ called the modes of the sequence (u_k).
If $k_1 = k_2$ we say that (u_k) has a pick, else we say that it has a plateau with $(k_2 - k_1 + 1)$ elements.

Theorem 1.7.8 *[20] A log-concave sequence is unimodal.*

Definition 1.7.9 *We say that a polynomial $p(x) = u_0 + u_1x + u_2x^2 + \cdots + u_nx^n$ is unimodal if the sequence $(u_0, \ldots, u_n)$ is unimodal.*

Definition 1.7.10 *We say that a polynomial*

$$Q(x, y) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} u_{i,j}x^iy^j$$

is doubly unimodal if

- *for all $0 \leq i \leq n_1$, the polynomial coefficient of x^i in $Q(x, y)$ is unimodal in y,*
- *for all $0 \leq j \leq n_2$, the polynomial coefficient of y^j in $Q(x, y)$ is unimodal in x, i.e., the sequence $(u_{i,j})_i$ is unimodal.*

Definition 1.7.11 *We say that a sequence of polynomials $(Q_i(t))_i$ is t-log-concave (resp. strongly t-log-concave) if*
$Q_i(t)^2 - Q_{i-1}(t)Q_{i+1}(t) \geq_t 0$, for all $i \geq 1$, i.e., $Q_i(t)^2 - Q_{i-1}(t)Q_{i+1}(t)$ has nonnegative coefficients as a polynomial in t, for all $i \geq 1$ (resp. $Q_j(t)Q_i(t) - Q_{j-1}(t)Q_{i+1}(t) \geq_t 0$, for all $i \geq j \geq 1$).

For more details, see [57].

Example 1.7.12

- *The q-binomial $\begin{bmatrix} N \\ M \end{bmatrix}_q$ is strongly q-log-concave [23];*
- *$\left(\begin{bmatrix} N \\ M \end{bmatrix}_q^{(s)}\right)_M$ and $\left(\begin{bmatrix} N \\ M \end{bmatrix}_q^{(s)}\right)_N$ are strongly q-log-concave.*

Remark 1.7.13

- *Every strongly t-log-concave sequence is a strongly log-concave. But the converse is not true in general (see, Sagan [55]).*
- *Every t-log-concave sequence is a log-concave sequence for each fixed positive number t.*

Definition 1.7.14 *We say that a sequence of polynomials $(P_i(q, t))_i$ is (q, t)-log-concave (resp. strongly (q, t)-log-concave) if*
$P_i(q, t)^2 - P_{i-1}(q, t)P_{i+1}(q, t) \geq_{q,t} 0$, for all $i \geq 1$, i.e., $P_i(q, t)^2 - P_{i-1}(t)P_{i+1}(q, t)$ has nonnegative coefficients as a polynomial in q and t, for all $i \geq 1$ (resp. $P_j(q, t)P_i(q, t) - P_{j-1}(t)P_{i+1}(q, t) \geq_{q,t} 0$, for all $i \geq j \geq 1$).

1.8 Directions in arithmetic triangles

In this section, we define the notion of directions in arithmetic triangles.

Definition 1.8.1 *Let $\{a(n,k)\}_{0\leq k\leq n}$ be a triangle of positive integers. Then the sequence $\{a(n-\alpha k, u+\beta k)\}_k$ is the sequence of integers crossing for each $n\in\mathbb{N}$, the transversal of direction (β,α) initialized at position u.*

Taking the example of Pascal triangle, the sequence $\{\binom{n-\alpha k}{u+\beta k}\}_k$ is the sequence of integers crossing for each $n\in\mathbb{N}$, the transversal of direction (β,α) initialized at position u. If $u=0$ we are in the principal diagonal. When $u\geq 1$ we speak about the intermediate rays of order u ($u=0,1,\ldots,\beta-1$).

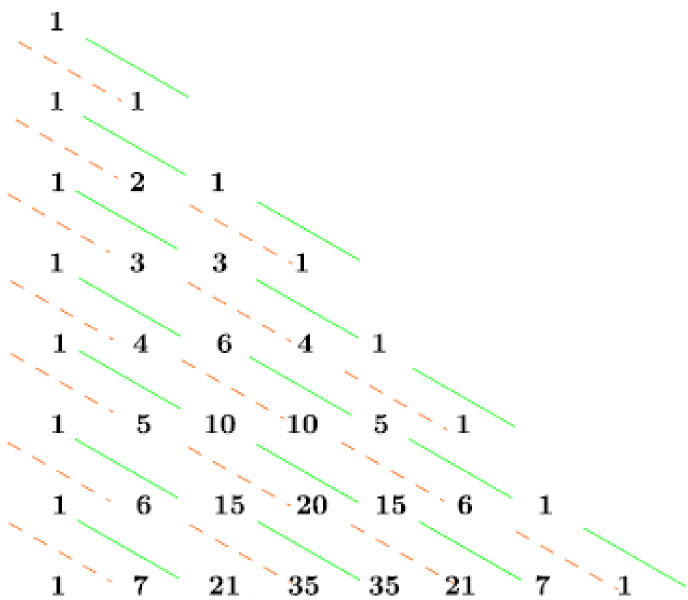


Figure 1.11: Illustration of the direction $(2,-1)$ in the Pascal triangle.

CHAPTER 2

Overpartition analogues of q-bi^{s}nomial coefficients and log-concavity

In this chapter, we study an overpartition analogues of q-bi^{s}nomial coefficients by defining first these analogues, which we call over q-bi^{s}nomial coefficients and over (q,t)-bi^{s}nomial coefficients, and giving then the corresponding interpretation by lattice paths. Next, we study some basic properties. Finally we prove the (q,t)-log-concavity (resp. the q-log-concavity) of over (q,t)-bi^{s}nomial (resp. over q-bi^{s}nomial) coefficients, and extend the Dousse and Kim's results on the (q,t)-log-concavity of over (q,t)-binomial coefficients.

We consider in this chapter the "0" as a part but not an overlined part.

2.1 Over q-bi^{s}nomial coefficients

2.1.1 q-Bi^{s}nomial coefficients and partitions

Recall the result of Bazeniar *et al.* [10] given in Proposition 1.4.13

$$\begin{bmatrix} M \\ K \end{bmatrix}_q^{(s)} = \sum_{\eta \subset (M-1)^K} q^{|\eta|}, \tag{2.1.1}$$

such that $\eta = (\eta_1, \eta_2, \dots, \eta_K)$ where $M-1 \geq \eta_1 \geq \eta_2 \geq \cdots \geq \eta_K \geq 0$, $|\eta| = \sum \eta_i$ and $\eta_i \neq \eta_{i+s}$ for all $i \geq 1$.

In the following, the number of partitions of n fitting inside $(M-1) \times K$ rectangle where the number of successive parts is at most s, that number is denoted $p_s(M-1, K, n)$. Then, Andrews [2] showed that this number verifies

$$\sum_{K\geq 0}\sum_{n\geq 0} p_s(M-1,K,n)x^K q^n = \prod_{n=0}^{M-1}\left(\frac{1-q^{n(s+1)}x^{s+1}}{1-q^n x}\right). \qquad (2.1.2)$$

Hence, we get the following result according to (1.4.7) and (2.1.2).

Corollary 2.1.1 *The q-bi^{s}nomial coefficient is the generating function of the number $p_s(M-1,K,n)$. That is*

$$\begin{bmatrix} M \\ K \end{bmatrix}_q^{(s)} := \sum_{n\geq 0} p_s(M-1,K,n)q^n. \qquad (2.1.3)$$

2.1.2 Over q-bi^{s}nomial coefficients

Motivated by (2.1.3), in the following we introduce the definition of our overpartition analogue of $\begin{bmatrix} M \\ K \end{bmatrix}_q^{(s)}$.

Definition 2.1.2 *We define the over q-bi^{s}nomial coefficients $\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(s)}$ as follows:*

$$\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(s)} := \sum_{n\geq 0} \overline{p}(M-1,K,n)q^n, \qquad (2.1.4)$$

where $\overline{p}_s(M-1,K,n)$ denotes the number of overpartitions of n into K parts in which the largest part is $\leq M-1$ and each part can be repeated at most s times, fitting inside an $(M-1)\times K$ rectangle.

These over q-bi^{s}nomial coefficients satisfy the subsequent recursion.

Theorem 2.1.3 *For $M \geq 1$, $0 \leq K \leq sM$:*

$$\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(s)} = \overline{\begin{bmatrix} M-1 \\ K \end{bmatrix}}_q^{(s)} + 2\sum_{j=1}^{s} q^{(M-1)j}\overline{\begin{bmatrix} M-1 \\ K-j \end{bmatrix}}_q^{(s)}. \qquad (2.1.5)$$

Proof For any overpartition into K parts with the largest part is less or equal to $M-1$, in which each part can be repeated at most s times.

So, we have the subsequent cases: either the largest part $< M-1$ which corresponds to (2.1.4) to $\overline{\begin{bmatrix} M-1 \\ K \end{bmatrix}}_q^{(s)}$, or the j largest parts, for $1 \leq j \leq s$, are equal to $M-1$ or the $j-1$ largest parts are equal to $M-1$ after the part $\overline{M-1}$, then we get for both $q^{(M-1)j}\overline{\begin{bmatrix} M-1 \\ K-j \end{bmatrix}}_q^{(s)}$, which gives us $2\sum_{j=1}^{s} q^{(M-1)j}\overline{\begin{bmatrix} M-1 \\ K-j \end{bmatrix}}_q^{(s)}$ and hence the equality (2.1.5). ∎

M/K	0	1	2	3
0	1			
1	1	1	1	
2	1	$1+2q$	$1+2q+2q^2$	$2q+2q^2$
3	1	$1+2q+2q^2$	$1+2q+4q^2+4q^3+2q^4$	$2q+4q^2+4q^3$ $+6q^4+4q^5$
4	1	$1+2q+2q^2+2q^3$	$1+2q+4q^2+6q^3+6q^4$ $+4q^5+2q^6$	$\dots$

Table 2.1: Triangle of $\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(2)}$.

Due to (2.1.5), we build Table 2.1 of the first values of over q-trinomial coefficients $\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(2)}$.

According to the previous recurrence relation (2.1.5), we have the subsequent generating function of $\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(s)}$.

Theorem 2.1.4 *Let $\overline{g}_{M,s}(x;q)$ be the generating function of $\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(s)}$, so*

$$\overline{g}_{M,s}(x;q) := \sum_{K\geq 0} \overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(s)} x^K =$$
$$\begin{cases} 1, & \textit{if} \quad M=0, \\ 1+x+\cdots+x^s, & \textit{if} \quad M=1, \\ (1+x+\cdots+x^s)\prod_{j=1}^{M-1}(1+2(q^jx)+\cdots+2(q^jx)^s) & \textit{else.} \end{cases}$$

Proof We start the proof for $M \geq 2$ (because it is obvious for $M=0$ and $M=1$). By (2.1.5) we obtain

$$\begin{aligned} \overline{g}_{M,s}(x;q) = \sum_{K\geq 0} \overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(s)} x^K &= \sum_{K\geq 0} x^K \left(\overline{\begin{bmatrix} M-1 \\ K \end{bmatrix}}_q^{(s)} + 2\sum_{j=1}^{s} q^{(M-1)j} \overline{\begin{bmatrix} M-1 \\ K-j \end{bmatrix}}_q^{(s)} \right) \\ &= \sum_{K\geq 0} \overline{\begin{bmatrix} M-1 \\ K \end{bmatrix}}_q^{(s)} x^K + 2\sum_{j=1}^{s} q^{(M-1)j} \sum_{K\geq 0} \overline{\begin{bmatrix} M-1 \\ K-j \end{bmatrix}}_q^{(s)} x^K \\ &= \overline{g}_{M-1,s}(x;q) + 2\sum_{j=1}^{s} q^{(M-1)j} x^j \overline{g}_{M-1,s}(x;q) \\ &= \overline{g}_{M-1,s}(x;q) \left[1 + 2\sum_{j=1}^{s} \left(xq^{M-1} \right)^j \right]. \end{aligned}$$

Iterating, we get

$$\overline{g}_{M,s}(x;q) = (1 + x + \cdots + x^s) \prod_{j=1}^{M-1} (1 + 2(q^j x) + \cdots + 2(q^j x)^s).$$

∎

Using the theorem above, we show that $\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(s)}$ verifies the subsequent identity.

Theorem 2.1.5 *For $M + N > 1$, we have*

$$\sum_{i=0}^{s} \overline{\begin{bmatrix} M+N \\ K-i \end{bmatrix}}_q^{(s)} q^{(M-1)i} = \sum_{j=0}^{K} \overline{\begin{bmatrix} M \\ K-j \end{bmatrix}}_q^{(s)} \overline{\begin{bmatrix} N+1 \\ j \end{bmatrix}}_q^{(s)} q^{(M-1)j}.$$

Proof

$$\begin{aligned}
\sum_{K\geq 0} \overline{\begin{bmatrix} M+N \\ K \end{bmatrix}}_q^{(s)} x^K &= (1 + x + \cdots + x^s) \prod_{j=1}^{M+N-1} \left(1 + 2(q^j x) + \cdots + 2(q^j x)^s\right) \\
&= (1 + x + \cdots + x^s) \prod_{j=1}^{M-1} \left(1 + 2(q^j x) + \cdots + 2(q^j x)^s\right) \\
&\times \prod_{j=M}^{M+N-1} \left(1 + 2(q^j x) + \cdots + 2(q^j x)^s\right) \\
&= \sum_{A\geq 0} \overline{\begin{bmatrix} M \\ A \end{bmatrix}}_q^{(s)} x^A \prod_{j=1}^{N} \left(1 + 2(q^j(q^{M-1}x)) + \cdots + 2(q^j(q^{M-1}x))^s\right).
\end{aligned}$$

Multiplying both sides by $(1 + q^{M-1}x + \cdots + (q^{M-1}x)^s)$, we find

$$(1 + q^{M-1}x + \cdots + (q^{M-1}x)^s) \sum_{K\geq 0} \overline{\begin{bmatrix} M+N \\ K \end{bmatrix}}_q^{(s)} x^K = \sum_{A\geq 0} \overline{\begin{bmatrix} M \\ A \end{bmatrix}}_q^{(s)} x^A \sum_{B\geq 0} \overline{\begin{bmatrix} N+1 \\ B \end{bmatrix}}_q^{(s)} q^{(M-1)H} x^B.$$

Hence, we conclude that

$$\sum_{K\geq 0} x^K \sum_{i=0}^{s} \overline{\begin{bmatrix} M+N \\ K-i \end{bmatrix}}_q^{(s)} q^{(M-1)i} = \sum_{K\geq 0} x^K \sum_{j=0}^{K} \overline{\begin{bmatrix} M \\ K-j \end{bmatrix}}_q^{(s)} \overline{\begin{bmatrix} N+1 \\ j \end{bmatrix}}_q^{(s)} q^{(M-1)j}.$$

The proof follows by comparing the coefficients of x^K on both sides of this equation. ∎

Now, what if we take the limit as $s \to \infty$, here it would be no constraint about the equal parts, and therefore we obtain.

Corollary 2.1.6 *For $M > 0$,*

$$\lim_{s\longrightarrow\infty} \overline{\begin{bmatrix} M+1 \\ K \end{bmatrix}}_q^{(s)} = \overline{\begin{bmatrix} M+K \\ K \end{bmatrix}}_q.$$

Due to Theorem 2.1.4 and relation (2.1.4), we have for each positive integer M

$$\sum_{n\geq 0}\sum_{K\geq 0} \overline{p}_s(M,K,n)x^K q^n = (1+x+\cdots+x^s)\prod_{j=1}^{M}(1+2(q^jx)+\cdots+2(q^jx)^s). \tag{2.1.6}$$

Adding and subtracting the number 1 to and from $(1+2(q^jx)+\cdots+2(q^jx)^s)$ in the right-hand side of (2.1.6), we obtain

$$\sum_{n\geq 0}\sum_{K\geq 0} \overline{p}_s(M,K,n)x^K q^n = \left(\frac{1-x^{s+1}}{1-x}\right)\prod_{n=1}^{M}\left(\frac{1-2x^{s+1}q^{n(s+1)}+xq^n}{1-xq^n}\right). \tag{2.1.7}$$

Now, if M geos to ∞, we obtain.

Corollary 2.1.7 *Let $\overline{p}_s(K,n)$ be the number of overpartitions of n into K parts such that the number of repeated parts is at most s. Then,*

$$\sum_{n\geq 0}\sum_{K\geq 0} \overline{p}_s(K,n)x^K q^n = \left(\frac{1-x^{s+1}}{1-x}\right)\prod_{n=1}^{+\infty}\left(\frac{1-2x^{s+1}q^{n(s+1)}+xq^n}{1-xq^n}\right).$$

Moreover, by taking the limit as $s\to\infty$ in (2.1.7), and by relation (2.1.4) and Corollary 2.1.6, we get the generating function of over q-binomial coefficients.

Proposition 2.1.8 *For a positive integer M,*

$$\sum_{K\geq 0} \overline{\begin{bmatrix} M+K \\ K \end{bmatrix}}_q x^K = \frac{(-xq;q)_M}{(1-x)(xq;q)_M}. \tag{2.1.8}$$

Furthermore, using Proposition 2.1.8 and [30, Theorem 1.2], it is not difficult to get the subsequent identity due to Dousse and Kim [30, Proposition 3.1] for over q-binomial coefficient.

$$\frac{(-xq;q)_M}{(xq;q)_M} = 1+\sum_{K\geq 1} x^K q^K\left(\overline{\begin{bmatrix} M+K-1 \\ K \end{bmatrix}}_q + \overline{\begin{bmatrix} M+K-2 \\ K-1 \end{bmatrix}}_q\right). \tag{2.1.9}$$

2.1.3 Interpretation by lattice paths

The over (q,t)-binomial coefficients $\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t}$ are interpreted by Dousse and Kim [30] by Delannoy paths (see Definition 1.2.7) as detailed in the first chapter. In this subsection we prove in similar manner that the over q-bi^{s}nomial coefficients are generating functions for a specific type of paths we call "*generelized delannoy paths*" defined as follows.

Definition 2.1.9 *We define the generalized Delannoy paths to be the paths start form $(0,0)$ to $(M-1,K)$ on a rectangular grid, in which the allowed steps are East, North and North-East step, where the number of consecutive North steps is not exceeding s and not exceeding $s-1$ if there is a North-East step just before. Let $\mathcal{D}^{(s)}_{M,K}$ denote the set of such paths.*
For a path $\mathcal{D} \in \mathcal{D}^{(s)}_{M,K}$, the weight of each of steps p_l of $\mathcal{D}$, where this step starts form (i_1,i_2) is defined as

$$w_{p_l} = \begin{cases} 0, & \text{if it goes to } (i_1+1, i_2), \\ i_1, & \text{if it goes to } (i_1, i_2+1), \\ i_1+1, & \text{if it goes to } (i_1+1, i_2+1). \end{cases}$$

And we define the weight of $\mathcal{D}$ as

$$w_{\mathcal{D}} = \sum_l w_{p_l}. \tag{2.1.10}$$

It should be noted that by considering the half box above the North-East step as a whole box, the weight $w_{\mathcal{D}}$ of $\mathcal{D}$ actually counts the number of the boxes above and to the left of $\mathcal{D}$.

Now, we can see the over q-bi^{s}nomial coefficients as generating functions of *generalized Delannoy paths* by mapping North-East steps to overlined parts.

Proposition 2.1.10 *For nonnegative integers K, M and s,*

$$\overline{\begin{bmatrix} M \\ K \end{bmatrix}}^{(s)}_q = \sum_{\mathcal{D} \in \mathcal{D}^{(s)}_{M,K}} q^{w_{\mathcal{D}}}. \tag{2.1.11}$$

2.2 Over (q,t)-bi^{s}nomial coefficients

Here, we take into consideration the number of overlined parts in $\overline{\begin{bmatrix} M \\ K \end{bmatrix}}^{(s)}_q$, for that, we define one more parameter to count it.

Definition 2.2.1 *The over (q,t)-bi^{s}nomial coefficient is defined as*

$$\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t}^{(s)} := \sum_{k,n\geq 0} \overline{p}_s(M-1,K,k,n)t^k q^n,$$

where $\overline{p}_s(M-1,K,k,n)$ counts the number of overpartitions of n with k overlined parts, in which the number of repeated parts is at most s, fitting inside an $(M-1)\times K$ rectangle.

Remark 2.2.2 *We can observe that by setting $t=0$, that is no part is overlined, we get the q-bi^{s}nomial coefficients, and by setting $t=1$ we get the over q-bi^{s}nomial coefficients defined in the previous section.*

2.2.1 Interpretation by lattice paths

In fact, the interpretation of $\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t}^{(s)}$ does not differ from that of over q-bi^{s}nomial coefficients by *generalixed Delannoy paths*, we add just for $\mathcal{D}\in\mathcal{D}_{M,K}^{(s)}$, the weight $O(\mathcal{D})$ counts the number of North-East steps in $\mathcal{D}$.

Hence, a generalization of Proposition 2.1.10 is as follows

Proposition 2.2.3 *For non-negative integers K, M and s,*

$$\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t}^{(s)} = \sum_{\mathcal{D}\in\mathcal{D}_{M,K}^{(s)}} t^{O(\mathcal{D})} q^{w_{\mathcal{D}}}. \tag{2.2.1}$$

Furthermore, the generalization of recurrence relation of Theorem 2.1.3 is as follows.

Theorem 2.2.4 *For $M\geq 2$, $0\leq K\leq sM$,*

$$\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t}^{(s)} = \overline{\begin{bmatrix} M-1 \\ K \end{bmatrix}}_{q,t}^{(s)} + (1+t)\sum_{j=1}^{s} q^{(M-1)j} \overline{\begin{bmatrix} M-1 \\ K-j \end{bmatrix}}_{q,t}^{(s)}. \tag{2.2.2}$$

Proof We take here the variable t calculating the number of overlined parts into account, so let $\mathcal{D}\in\mathcal{D}_{M,K}^{(s)}$, we have either the last step of $\mathcal{D}$ is an East step, hence, it is generated by $\overline{\begin{bmatrix} M-1 \\ K \end{bmatrix}}_{q,t}^{(s)}$, or $\mathcal{D}$ has j successive North steps at the end, $1\leq j\leq s$, then it is generated by $q^{(M-1)j}\overline{\begin{bmatrix} M-1 \\ K-j \end{bmatrix}}_{q,t}^{(s)}$ or $\mathcal{D}$ has $j-1$ successive North steps at the end after an North-East step, and in this case $\mathcal{D}$ is generated by $tq^{(M-1)j}\overline{\begin{bmatrix} M-1 \\ K-j \end{bmatrix}}_{q,t}^{(s)}$, and since j can take any value between 1 and s, we get (2.2.2). (See Figure 2.1). ∎

Remark 2.2.5 *It should be noted that: by taking $t=0$ in (2.2.2) we get the recursion (1.4.5) of q-bi^{s}nomial coefficients, and by taking $t=1$, we get the recursion (2.1.5) of over q-bi^{s}nomial coefficients.*

The recursion (2.2.2) leads us to build the triangle of over (q,t)-bi^{s}nomial coefficients, see Table 2.2.

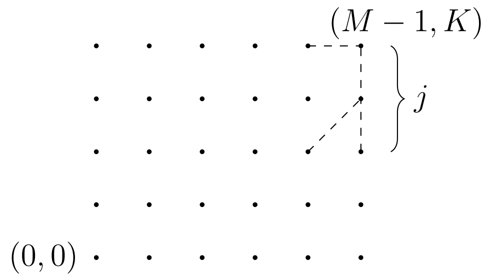


Figure 2.1: The decomposition of the path.

M/K	0	1	2	$\cdots$
0	1			
1	1	1	1	
2	1	$1+(1+t)q$	$1+(1+t)q+(1+t)q^2$	
3	1	$1+(1+t)q+(1+t)q^2$	$1+(1+t)q+2(1+t)q^2$	

Table 2.2: Triangle of over (q,t)-trinomial coefficients.

2.2.2 Interpretation by tiling

Here, we give the combinatorial interpretation of over (q,t)-bi^{s}nomial coefficient by a specific type of tiling.

Definition 2.2.6 *Let $\mathcal{L}^{(s)}_{M,K}$ be the set of weighted tilings of an $(M+K-1)\times 1$-board in which we use only: at most K blue squares and red regtangles with an area of two squares (square pair), and at most $M-1$ yellow squares and red rectangles, where the number of all squares and rectangles is exactly $M+K-1$, and the number of successive blue squares is at most s, and it is at most $s-1$ after a red rectangle.*

We give now the definition of the weight of a tiling, basing on the calculation the indices of blue squares and red rectangles as follows:

Definition 2.2.7 *Let T be a tiling in $\mathcal{L}^{(s)}_{M,K}$, the weight $w(L)$ of L is defined as*

$$w_L = t^{n_r} q^n,$$

where n_r is the number of all red rectangles in L, and n is the sum of indices of all blue squares and red rectangles in L where the calculation of the indices is as follows:

We denote each blue square by b_i, where i equals to the number of yellow squares and red rectangles to the left of that blue square in L, and we denote each red rectangle by r_j, where j is equal to the number of yellow squares and red rectangles to the left of that red rectangle plus itself.

Calculating the weight w_L will be more clear by looking at the following illustrative example of a weighted tiling L in $L_{6,4}^{(2)}$ as shown in Figure 2.2, in which we calculate the weight as follows $w_L = t^2 q^{2+2+4+5} = t^2 q^{13}$.

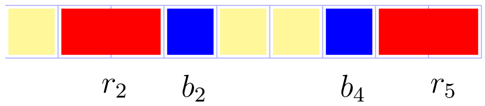


Figure 2.2: The tiling L of weight t^2q^{13}.

Finally, we have

Proposition 2.2.8 *For non-negative integers K, M and s,*

$$\overline{\begin{bmatrix} M \\ K \end{bmatrix}}^{(s)}_{q,t} = \sum_{L \in \mathcal{L}^{(s)}_{M,K}} w_L. \tag{2.2.3}$$

Proof This tiling interpretation is in bijection with paths interpretation of over (q,t)-bi^{s}nomial coefficients. Each yellow square is corresponded to an East step, and each blue square to a North step and the red rectangle to a North-East step, and vice versa. Moreover, this bijection is weight-preserving. Indeed, by looking to the definition of the weight of a step of a path, where this step starts form (i_1, i_2) is defined as

$$w_{p_k} = \begin{cases} 0, & \text{if it goes to } (i_1+1, i_2), \\ i_1, & \text{if it goes to } (i_1, i_2+1), \\ i_1+1, & \text{if it goes to } (i_1+1, i_2+1). \end{cases}$$

In other words, if the step is an East step that its weight is equal to 0, if it is a North step then its weight is equal to i_1 which represents the number of East steps before (or to the left of) this North step, and if it is a North-East step then its weight is equal to i_1+1, which represents the number of East steps before (or to the left of) this North step $+1$. Then the wight of the path is the sum of the weights of its steps, which is exactly equal to n of Definition 2.2.7. ∎

2.3 Log-concavity property

The study of the log-concavity in many instances is based on the use of the combinatorial interpretation. For instance: Butler [23] proved the strong q-log-concavity of q-binomial coefficients, based on their interpretation by partitions. In other hand, Sagan [54] showed the same property using the lattice path's approach. While Gasharov [35] proved the log-concavity of the Eulerian polynomial using a different type of paths that are labeled in each step called "labeled paths of Gasharov" (defined in Definition 1.2.5).

Recently, Dousse and Kim [31] generalized the method of Butler to prove the log-concavity of over (q,t)-binomial coefficients, and Baseniar *et al.* [11] generalized that of Sagan for q-bi^{s}nomial coefficients. In this section, motivated by these works, we use both previous methods in proving the log-concavity for our over analogues.

For a pair of paths $(\mathcal{D}_1, \mathcal{D}_2) \in \mathcal{D}^{(s)}_{M,K} \times \mathcal{D}^{(s)}_{M,K}$, let $w_{\mathcal{D}_1\mathcal{D}_2}$ the weight of $(\mathcal{D}_1, \mathcal{D}_2)$ be the sum of weight of $\mathcal{D}_1$ and the weight of $\mathcal{D}_2$, and $O(\mathcal{D}_1, \mathcal{D}_2)$ be the sum of $O(\mathcal{D}_1)$ and $O(\mathcal{D}_2)$, that is

$$w_{\mathcal{D}_1\mathcal{D}_2} = w_{\mathcal{D}_1} + w_{\mathcal{D}_2}, \qquad O(\mathcal{D}_1\mathcal{D}_2) = O(\mathcal{D}_1) + O(\mathcal{D}_2),$$

and so

$$q^{w_{\mathcal{D}_1\mathcal{D}_2}} = q^{w_{\mathcal{D}_1}} q^{w_{\mathcal{D}_2}}, \qquad t^{O(\mathcal{D}_1\mathcal{D}_2)} = t^{O(\mathcal{D}_1)} t^{O(\mathcal{D}_2)}.$$

Before introducing our results, we give the definition of the involution that the proof is based on.

Definition 2.3.1 *Let a_1, a_2, b_1 and b_2 be four points of the lattice. For two paths $\mathcal{D}_1$ from a_1 to b_1 and $\mathcal{D}_2$ from a_2 to b_2, the involution Γ_s is defined as*

$$\Gamma_s(\mathcal{D}_1, \mathcal{D}_2) = (Q_1, Q_2)$$

such that

- *if $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$ then $Q_1 = \mathcal{D}_1$ and $Q_2 = \mathcal{D}_2$,*
- *if $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$ then, Γ_s switches the portions of $\mathcal{D}_1$ and $\mathcal{D}_2$ after b_0 as in Figure 2.3, that is*

$$Q_1 = a_1 \xrightarrow{\mathcal{D}_1} b_0 \xrightarrow{\mathcal{D}_2} b_2 \text{ and } Q_2 = a_2 \xrightarrow{\mathcal{D}_2} b_0 \xrightarrow{\mathcal{D}_1} b_1,$$

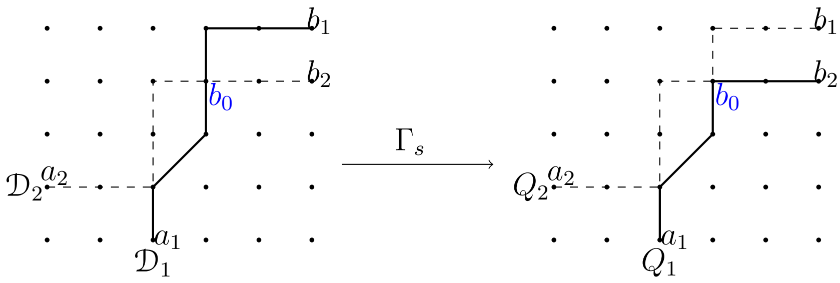


Figure 2.3: The involution Γ_s.

where b_0 is the last intersection vertex of $\mathcal{D}_1$ and $\mathcal{D}_2$ such that the number of vertical steps of $\mathcal{D}_1$ and $\mathcal{D}_2$ all together, in the vertical level of b_0, does not exceed s, and does not exceed $s-1$ if there is a North-East step of $\mathcal{D}_1$ or $\mathcal{D}_2$ just before b_0.

And we have the two following maps.

Definition 2.3.2 *Let $\mathcal{T}_d$ be the following translation*

$$\mathcal{T}_d(\mathcal{D}_1, \mathcal{D}_2) = (Q_1, Q_2), \tag{2.3.1}$$

where Q_1 is the path $\mathcal{D}_1$ after a translation in the direction d by one unit, where $d = W$ (West direction) or $d = S$ (South direction), and $Q_2 = \mathcal{D}_2$.

Definition 2.3.3 *For two paths $\mathcal{D}_1$ and $\mathcal{D}_2$ let θ be the following involution*

$$\theta(\mathcal{D}_1, \mathcal{D}_2) = (Q_1, Q_2),$$

where Q_1 is the path $\mathcal{D}_2$ translated to start at the initial point of $\mathcal{D}_1$, and Q_2 is the path $\mathcal{D}_1$ translated to start at the initial point of $\mathcal{D}_2$. For an illustrative example see Figure 2.4.

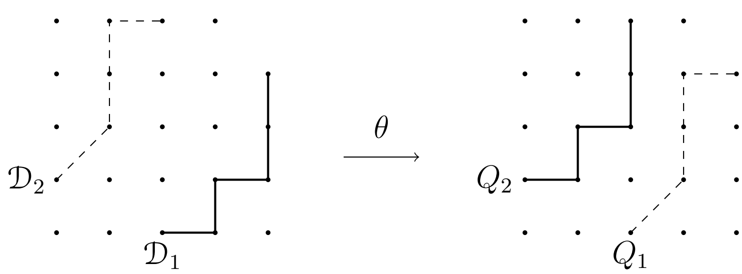


Figure 2.4: The involution θ.

Now, in the next theorem, we start establishing our first result:

Theorem 2.3.4 *For $K \geq H$,*

$$\overline{\begin{bmatrix} M \\ H \end{bmatrix}_{q,t}}^{(s)} \overline{\begin{bmatrix} M \\ K \end{bmatrix}_{q,t}}^{(s)} - \overline{\begin{bmatrix} M \\ H-1 \end{bmatrix}_{q,t}}^{(s)} \overline{\begin{bmatrix} M \\ K+1 \end{bmatrix}_{q,t}}^{(s)} \geq_{q,t} 0,$$

i.e., $\left(\overline{\begin{bmatrix} M \\ K \end{bmatrix}_{q,t}}^{(s)} \right)_K$ *satisfies the strong (q,t)-log-concavity.*

Proof in path's approach Let us first prove that

$$\left(\overline{\begin{bmatrix} M \\ K \end{bmatrix}_{q,t}}^{(s)} \right)^2 - \overline{\begin{bmatrix} M \\ K-1 \end{bmatrix}_{q,t}}^{(s)} \overline{\begin{bmatrix} M \\ K+1 \end{bmatrix}_{q,t}}^{(s)} \tag{2.3.2}$$

has nonnegative coefficients as a polynomial in t and q. Let us fix a pair of initial vertices $a_1 = (0,0)$ and $a_2 = (0,1)$, and a pair of final ones b_1 and b_2. And let us assume that $\mathcal{D}_1$ starts from a_1 and $\mathcal{D}_2$ from a_2. Let us define the sign $(-1)^{\mathcal{D}_1\mathcal{D}_2}$ of a pair of paths $(\mathcal{D}_1, \mathcal{D}_2)$ as $(-1)^{\mathcal{D}_1\mathcal{D}_2} = +1$ if $a_1 \xrightarrow{\mathcal{D}_1} b_1$ and $a_2 \xrightarrow{\mathcal{D}_2} b_2$, and $(-1)^{\mathcal{D}_1\mathcal{D}_2} = -1$ if $a_1 \xrightarrow{\mathcal{D}_1} b_2$ and $a_2 \xrightarrow{\mathcal{D}_2} b_1$. We have

$$\left(\overline{\begin{bmatrix} M \\ K \end{bmatrix}_{q,t}}^{(s)} \right)^2 - \overline{\begin{bmatrix} M \\ K-1 \end{bmatrix}_{q,t}}^{(s)} \overline{\begin{bmatrix} M \\ K+1 \end{bmatrix}_{q,t}}^{(s)} = \sum_{\mathcal{D}_1,\mathcal{D}_2} (-1)^{\mathcal{D}_1\mathcal{D}_2} t^{O(\mathcal{D}_1\mathcal{D}_2)} q^{w_{\mathcal{D}_1\mathcal{D}_2}}, \tag{2.3.3}$$

where the sum is over all pairs $(\mathcal{D}_1, \mathcal{D}_2)$.

Hence, for each pair of paths $(\mathcal{D}_1, \mathcal{D}_2)$ such that $(-1)^{\mathcal{D}_1\mathcal{D}_2} = -1$, $\Gamma_s(\mathcal{D}_1, \mathcal{D}_2) = (Q_1, Q_2)$, then $(-1)^{Q_1Q_2}$ is necessarily equal to $+1$. Furthermore, the sum of the number of boxes above and to the left of $\mathcal{D}_1$ and the number of boxes above and to the left of $\mathcal{D}_2$ after the application of the involution Γ_s is the same as before the application of Γ_s, because $\mathcal{D}_1$ and $\mathcal{D}_2$ departs from vertices that are in the same vertical line, and the boxes lost by one of the two paths are gained by the other. Thereby

$$w_{\mathcal{D}_1\mathcal{D}_2 = }w_{Q_1Q_2}, \qquad O(\mathcal{D}_1\mathcal{D}_2) = O(Q_1Q_2).$$

Consequently we obtain that the relation (2.3.3) has nonnegative coefficients as a polynomial in q and t.

To show that

$$\overline{\begin{bmatrix} M \\ H \end{bmatrix}_{q,t}}^{(s)} \overline{\begin{bmatrix} M \\ K \end{bmatrix}_{q,t}}^{(s)} - \overline{\begin{bmatrix} M \\ H-1 \end{bmatrix}_{q,t}}^{(s)} \overline{\begin{bmatrix} M \\ K+1 \end{bmatrix}_{q,t}}^{(s)}$$

has nonnegative coefficients as a polynomial in q and t where $0 < K \leq H$, we only need to take the initial vertex $a_2 = (0, K - H + 1)$. ∎

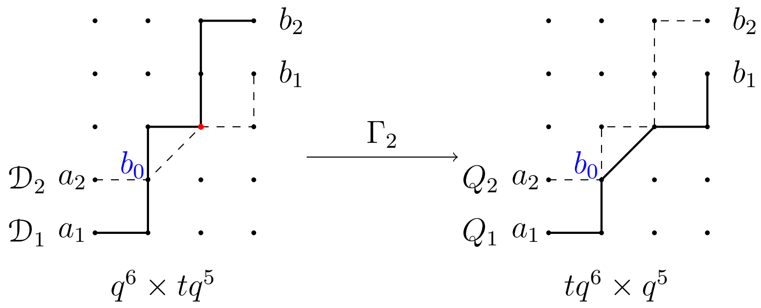


Figure 2.5: Two paths before and after applying Γ_2

As an illustration of the proof, let us take $s = 2$, $M = 4$ and $K = 3$, and take the pair of paths $(\mathcal{D}_1, \mathcal{D}_2)$ corresponded to $q^6 \times tq^5$ of $\overline{\begin{bmatrix}4\\4\end{bmatrix}_q}^{(2)} \overline{\begin{bmatrix}4\\2\end{bmatrix}_q}^{(2)}$ as in Figure 2.5. Indeed, $\mathcal{D}_1$ corresponds to the partition $(2, 2, 1, 1)$ of 6 and $\mathcal{D}_2$ corresponds to the partition $(3, \overline{2})$ of 5.

We can easily observe that the red intersection vertex can not be the vertex of switching because we have $s = 2$, so the vertex of switching is b_0, and the two paths we get after applying Γ_2 are Q_1 which is associated to tq^6 (corresponds to the partition $(3, \overline{2}, 1)$) of $\overline{\begin{bmatrix}4\\3\end{bmatrix}_q}^{(2)}$, and Q_2 which is associated to q^5 (corresponds to the partition $(2, 2, 1)$) of $\overline{\begin{bmatrix}4\\3\end{bmatrix}_q}^{(2)}$.

On the other hand, we can also prove the Theorem 2.3.4 by using the overpartition's approach.

Before starting the proof, let define first some sets that we need in the comprehension of the proof:

$$\overline{\mathcal{P}} := \{\text{the set of overpartitions}\},$$

$$\overline{\mathcal{P}^{(s)}} := \{\eta \in \overline{\mathcal{P}} : \text{ the number of successive equal parts is at most } s\},$$

$$\overline{\mathcal{P}^{(s)}}(M, K) := \{\eta \in \overline{\mathcal{P}^{(s)}} : \eta \text{ fits inside } (M-1) \times K\text{regtangle}\},$$

$$\overline{\mathcal{P}_{diff_s}} := \{\eta \in \overline{\mathcal{P}} : \eta_j - \eta_{j+1} \leq s, \quad \text{for all } j \in \{1, \cdots, K\}\},$$

$$\overline{\mathcal{P}_{diff_s}}(M, K) := \{\eta \in \overline{\mathcal{P}_{diff_s}} : \eta \text{ fits inside } (M-1) \times K\text{regtangle}\}.$$

And for a partition η, we denote the number of overlined parts in η by $O(\eta)$.

Proof in overpartition's approach The objective of the proof is to establish a map that must be injective as follows

$$\Gamma_s : \overline{\mathcal{P}^{(s)}}(M, K+1) \times \overline{\mathcal{P}^{(s)}}(M, H-1) \to \overline{\mathcal{P}^{(s)}}(M, K) \times \overline{\mathcal{P}^{(s)}}(M, H)$$
$$(\eta, \rho) \mapsto (\mu, \lambda),$$

where $|\eta| + |\rho| = |\mu| + |\lambda|$ and $O(\eta) + O(\rho) = O(\eta) + O(\eta)$.

Let us define these two maps:

The involution $\mathcal{F}$

$$\mathcal{F} : \overline{\mathcal{P}} \times \overline{\mathcal{P}} \to \overline{\mathcal{P}} \times \overline{\mathcal{P}}$$
$$(\eta, \rho) \mapsto (\eta^c, \rho^c),$$

and the map $\mathcal{B}^{(s)}_{K,H}$

$$\mathcal{B}^{(s)}_{K,H} : \overline{\mathcal{P}_{diff_s}} \times \overline{\mathcal{P}_{diff_s}} \to \overline{\mathcal{P}_{diff_s}} \times \overline{\mathcal{P}_{diff_s}}$$
$$(\eta, \rho) \mapsto (\gamma, \tau),$$

such that

$$\gamma = (\rho_1 + (K-H+1), \ldots, \rho_J + (K-H+1), \eta_{J+1}, \eta_{J+2}, \ldots),$$
$$\tau = (\eta_1 - (K-H+1), \ldots, \eta_J - (K-H+1), \rho_{J+1}, \rho_{J+2}, \ldots),$$

where J is the largest integer which verifies

$$\eta_J - \rho_{J+1} \geq \begin{cases} K-H+1, & \text{if } \eta_J \text{ is not overlined,} \\ K-H+2, & \text{if } \eta_J \text{ is overlined,} \end{cases} \tag{2.3.4}$$

and

$$\eta_{J+1} \geq \rho_J + (K-H+1) - s, \text{if } \eta_J - \rho_J < K-H+1, \tag{2.3.5}$$

such that we take $\rho_{j+1} = 0$ if $\eta_j > 0$ but the number of parts in ρ is less then $j+1$. If no such J exists, Let $J = 0$.

With the consideration that $\eta_i - (K-H+1)$ (resp. $\rho_i + (K-H+1)$) in τ (resp. γ) follows η_i (resp. ρ_i) in η (resp. ρ) in being or not being overlined.

Now, we define Γ_s to be the restriction of $\mathcal{F} \circ \mathcal{B}^{(s)}_{K,H} \circ \mathcal{F}$ in $\overline{\mathcal{P}^{(s)}}(M, K+1) \times \overline{\mathcal{P}^{(s)}}(M, H-1)$. To show the injectivity of Γ_s, we prove that

(1) $\mathcal{B}^{(s)}_{K,H}$ is an involution on $\overline{\mathcal{P}_{diff_s}} \times \overline{\mathcal{P}_{diff_s}}$,

(2) $\mathcal{B}^{(s)}_{K,H}\left(\overline{\mathcal{P}_{diff_s}}(K+1,M)\times\overline{\mathcal{P}_{diff_s}}(H-1,M)\right)\subset\overline{\mathcal{P}_{diff_s}}(K,M)\times\overline{\mathcal{P}_{diff_s}}(H,M)$,

(3) $\Gamma_s\left(\overline{\mathcal{P}^{(s)}}(M,K+1)\times\overline{\mathcal{P}^{(s)}}(M,H-1)\right)\subset\overline{\mathcal{P}^{(s)}}(M,K)\times\overline{\mathcal{P}^{(s)}}(M,H)$.

Before starting checking the three points above, we verify that $\mathcal{B}^{(s)}_{K,H}$ is well defined, that is for $(\eta,\rho)\in\overline{\mathcal{P}_{diff_s}}\times\overline{\mathcal{P}_{diff_s}}$ such that $\mathcal{B}^{(s)}_{K,H}(\eta,\rho)=(\gamma,\tau)$ hence $\gamma,\tau\in\overline{\mathcal{P}_{diff_s}}\times\overline{\mathcal{P}_{diff_s}}$.
We have $(\rho_1+(K-H+1),\ldots,\rho_J+(K-H+1))$, $(\eta_{J+1},\eta_{J+2},\ldots)$, $(\eta_1-(K-H+1),\ldots,\eta_J-(K-H+1))$ and $(\rho_{J+1},\rho_{J+2},\ldots)$ when they are seen separately, are in $\overline{\mathcal{P}_{diff_s}}$, it remains to check that γ and τ are overpartitions and $\gamma_i-\gamma_{i+1}\le s$ and $\mathcal{T}_j-\tau_{j+1}\le s\ \forall j\in\{1,\cdots,K\}$, i.e., checking that:

$$\rho_J+(K-H+1)\ge\begin{cases}\eta_{J+1}+1, & \text{if } \rho_J+(K-H+1) \text{ is overlined},\\ \eta_{J+1}, & \text{if } \rho_J+(K-H+1) \text{ is not overlined},\end{cases}\tag{2.3.6}$$

$$\eta_J-(K-H+1)\ge\begin{cases}\rho_{J+1}+1, & \text{if } \eta_J-(K-H+1) \text{ is overlined},\\ \rho_{J+1}, & \text{if } \eta_J-(K-H+1) \text{ is not overlined},\end{cases}\tag{2.3.7}$$

and

$$\rho_J+(K-H+1)-\eta_{J+1}\le s,\tag{2.3.8}$$

$$\eta_J-(K-H+1)-\rho_{J+1}\le s.\tag{2.3.9}$$

By (2.3.4), inequality (2.3.7) is obvious.

And concerning (2.3.6), distinguish case **(a)** and case **(b)**:

(a)

If

$$\rho_{J+2}+(K-H+1)\ge\begin{cases}\eta_{J+1}, & \text{if } \eta_{J+1} \text{ is overlined},\\ \eta_{J+1}+1, & \text{if } \eta_{J+1} \text{ is not overlined}.\end{cases}$$

Here, we know that if ρ_J is overlined then $\rho_J\ge\rho_{J+2}+1$, and if ρ_J is not overlined then $\rho_J\ge\rho_{J+2}$.

Hence, the verification of (2.3.6) is done.

(b) If

$$\eta_{J+1}\ge\begin{cases}\rho_{J+2}+(K-H+2), & \text{if } \eta_{J+1} \text{ is overlined},\\ \rho_{J+2}+(K-H+1), & \text{if } \eta_{J+1} \text{ is not overlined},\end{cases}$$

with $\eta_{J+1} - \rho_{J+1} < K - H + 1$ and $\eta_{J+2} < \rho_{J+1} + (K - H + 1) - s$.
And since $\eta \in \overline{\mathcal{P}_{diff_s}}$, so

$$\eta_{J+1} \leq \eta_{J+2} + s < \rho_{J+1} + (K - H + 1).$$

Hence

$$\eta_{J+1} + 1 \leq \rho_J + (K - H + 1).$$

Which gives us (2.3.6).

For checking the inequality (2.3.8), we have either

- $\eta_J - \rho_J < K - H + 1$, and here

$$\rho_J + (K - H + 1) - \eta_{J+1} \leq s.$$

- or $\eta_J - \rho_J \geq K - H + 1$, so

$$\rho_J + (K - H + 1) - \eta_{J+1} \leq \eta_J - \eta_{J+1} \leq s.$$

And then (2.3.8) holds.

Concerning the inequality (2.3.9), we have also either:

- $\eta_J - \rho_J < K - H + 1$, hence

$$\begin{aligned} \eta_J - (\rho_{J+1} + (K - H + 1)) &\leq \eta_J - \rho_J - (K - H + 1) + s \\ &\leq s. \end{aligned}$$

- or $\eta_J - \rho_J \geq K - H + 1$, we distinguish again the two previous cases **(a)** and **(b)**:

(a)

If

$$\rho_{J+2} + (K - H + 1) \geq \begin{cases} \eta_{J+1}, & \text{if } \eta_{J+1} \text{ is overlined,} \\ \eta_{J+1} + 1, & \text{if } \eta_{J+1} \text{ is not overlined.} \end{cases}$$

Hence

$$\begin{aligned} \eta_J - (\rho_{J+1} + (K - H + 1)) &\leq \eta_J - (\rho_{J+2} + (K - H + 1)) \\ &< \eta_J - \eta_{J+1} \\ &\leq s. \end{aligned}$$

(b)

If

$$\eta_{J+1} \geq \begin{cases} \rho_{J+2} + (K - H + 2), & \text{if } \eta_{J+1} \text{ is overlined,} \\ \rho_{J+2} + (K - H + 1), & \text{if } \eta_{J+1} \text{ is not overlined,} \end{cases}$$

with $\eta_{J+1} - \rho_{J+1} < K - H + 1$ and $\eta_{J+2} < \rho_{J+1} + (K - H + 1) - s$.

Hence

$$\eta_J - (\rho_{J+1} + (K - H + 1)) \leq \eta_J - \eta_{J+1} \leq s.$$

Let us back to check quickly the three points **(1)**, **(2)** and **(3)**.

For the point **(1)** we have that $\mathcal{B}^{(s)}_{K,H}$ is an involution on $\overline{\mathcal{P}_{diff_s}} \times \overline{\mathcal{P}_{diff_s}}$ following the method Dousse and Kim [31, Theorem 1.5] use in showing their map $\mathcal{A}$ is an involution.

For the point **(2)**, simply apply the definition of our map $\mathcal{B}^{(s)}_{K,H}$.

It remains to check the point **(3)**, we have the following

$$\begin{aligned} \mathcal{F}\left(\overline{\mathcal{P}^{(s)}}(M, K+1) \times \overline{\mathcal{P}^{(s)}}(M, H-1)\right) &= \overline{\mathcal{P}_{diff_s}}(K+1, M) \times \overline{\mathcal{P}_{diff_s}}(H-1, M), \\ \mathcal{B}^{(s)}_{K,H}\left(\overline{\mathcal{P}_{diff_s}}(K+1, M) \times \overline{\mathcal{P}_{diff_s}}(H-1, M)\right) &\subset \overline{\mathcal{P}_{diff_s}}(K, M) \times \overline{\mathcal{P}_{diff_s}}(H, M), \\ \mathcal{F}\left(\overline{\mathcal{P}_{diff_s}}(K, M) \times \overline{\mathcal{P}_{diff_s}}(H, M)\right) &= \overline{\mathcal{P}^{(s)}}(M, K) \times \overline{\mathcal{P}^{(s)}}(M, H). \end{aligned}$$

And this completes our proof. ∎

To see the application of the injection Γ_s, let us give the following illustrative example. Let $s = 2$, $M = 4$, and $K = H = 3$. Let us take $(\eta, \rho) \in \overline{\mathcal{P}^{(s)}}(3, 4) \times \overline{\mathcal{P}^{(s)}}(3, 2)$ which corresponds to (P_1, P_2) of the illustrative example of the first proof, i.e. $\eta = (2, 2, 1, 1)$ and $\rho = (3, \overline{2})$. We apply $\Gamma_2 = \mathcal{F} \circ \mathcal{B}^{(2)}_{3,3} \circ \mathcal{F}$ on (η, ρ) as follows

$$\mathcal{F}(\eta, \rho) = \left((4, 2, 0), (2, \overline{2}, 1)\right),$$

then, $J = 1$ because it satisfies (2.3.4) and (2.3.5) but $j = 2$ and $j = 3$ do not. Consequently

$$\mathcal{B}^{(2)}_{3,3}\left((4, 2, 0), (2, \overline{2}, 1)\right) = \left((3, 2, 0), (3, \overline{2}, 1)\right),$$

hence,

$$\mathcal{F}\left((3, 2, 0), (3, \overline{2}, 1)\right) = \left((2, 2, 1), (3, \overline{2}, 1)\right).$$

Thereby,

$$\Gamma_2(\eta, \rho) = \left((2, 2, 1), (3, \overline{2}, 1)\right) \in \overline{\mathcal{P}^{(2)}}(3, 3) \times \overline{\mathcal{P}^{(2)}}(3, 3).$$

The resulting pair corresponds to (Q_2, Q_1) in the illustrative example of the first proof.

For $s \to \infty$ the inequality (2.3.5) is always satisfied, then we can say that we recover the map of Dousse-Kim.

Furthermore, we can see that:

If you look closely at both proofs, you will be able to notice that the two previous approaches of the two proofs are basically bijective combinatorially. This appears clearly in the illustrative examples of the two proofs, the switching is the same in the two. It differs only in that the first is in path's approach and the second in overpartition's approach.

On the other hand, following the steps of path's approach proof, Theorem 2.3.5 of Dousse and Kim about the strong (q,t)-log-concavity of over (q,t)-binomial coefficients can also be proved in this way.

Theorem 2.3.5 *[31, Theorem 1.5] For all* $0 < H \leq K < M$,

$$\overline{\begin{bmatrix} M \\ H \end{bmatrix}}_{q,t} \overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t} - \overline{\begin{bmatrix} M \\ H-1 \end{bmatrix}}_{q,t} \overline{\begin{bmatrix} M \\ K+1 \end{bmatrix}}_{q,t} \geq_{q,t} 0,$$

i.e., $\left(\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t}\right)_K$ *satisfies the strong* (q,t)*-log-concavity.*

In the following, we see that the over q-bi^{s}nomial coefficients remain strongly (q,t)-log-concave for M.

Theorem 2.3.6 *For* $M \geq N$,

$$\overline{\begin{bmatrix} M \\ K \end{bmatrix}}^{(s)}_{q,t} \overline{\begin{bmatrix} N \\ K \end{bmatrix}}^{(s)}_{q,t} - \overline{\begin{bmatrix} N-1 \\ K \end{bmatrix}}^{(s)}_{q,t} \overline{\begin{bmatrix} M+1 \\ K \end{bmatrix}}^{(s)}_{q,t} \geq_{q,t} 0, \tag{2.3.10}$$

i.e., $\left(\overline{\begin{bmatrix} M \\ K \end{bmatrix}}^{(s)}_{q,t}\right)_M$ *satisfies the strong* (q,t)*-log-concavity.*

Proof We have

$$\overline{\begin{bmatrix} M \\ K \end{bmatrix}}^{(s)}_{q,t} \overline{\begin{bmatrix} N \\ K \end{bmatrix}}^{(s)}_{q,t} - \overline{\begin{bmatrix} N-1 \\ K \end{bmatrix}}^{(s)}_{q,t} \overline{\begin{bmatrix} M+1 \\ K \end{bmatrix}}^{(s)}_{q,t} = \sum_{P_1,P_2} (-1)^{P_1P_2} t^{O(P_1P_2)} q^{w_{P_1P_2}}. \tag{2.3.11}$$

Let $a_1 = (0,0)$, $a_2 = (M-N+1, 0)$, $b_1 = (M-1, K)$ and $b_2 = (M, K)$.

Hence, for each pair of path (P_1, P_2) such that $(-1)^{P_1P_2} = -1$, $\Gamma_s(P_1, P_2) = (Q_1, Q_2)$, then $(-1)^{Q_1Q_2}$ is necessarily equal to $+1$. And we have $O(P_1P_2) = O(Q_1Q_2)$. Furthermore, $w_{P_1P_2} = w_{Q_1Q_2}$, because the sum of the number of boxes above P_1 and the number of boxes above P_2 after the application of the involution Γ_s is the same as before the application of Γ_s because P_1 and P_2 departs from vertices that are in the same horizontal line.

As an illustration of the proof, let us take $s = 2$, $M = 3$ and $K = 3$, as in Figure 2.6, and take the pair of paths (P_1, P_2) correspond to $t^2q^6 \times q^2$ of $\overline{\begin{bmatrix}4\\3\end{bmatrix}}_q^{(2)} \overline{\begin{bmatrix}2\\3\end{bmatrix}}_q^{(2)}$.

The vertex of switching is b_0, and the two Paths we get after applying Γ_2 are Q_1 which is associated to tq^5 of $\overline{\begin{bmatrix}3\\3\end{bmatrix}}_q^{(2)}$, and Q_2 which is associated to tq^3 of $\overline{\begin{bmatrix}3\\3\end{bmatrix}}_q^{(2)}$.

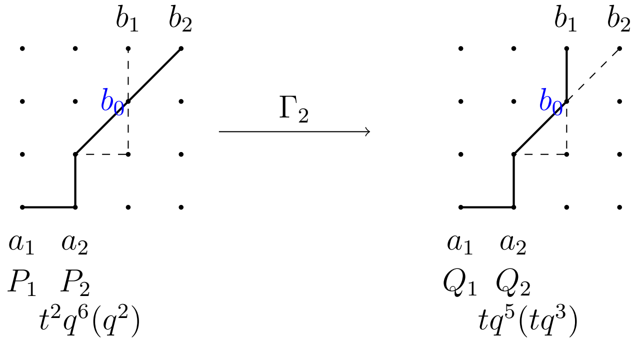


Figure 2.6: Two paths before and after applying Γ_2.

Next, we have the following result about the strong (q, t)-log-concavity for both M and K are fixed.

Theorem 2.3.7 *For $sN \geq M \geq N$ and $H \geq K$, and $M + K = N + H$*

$$\overline{\begin{bmatrix}M\\K\end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix}N\\H\end{bmatrix}}_{q,t}^{(s)} - \overline{\begin{bmatrix}M+1\\K-1\end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix}N-1\\H+1\end{bmatrix}}_{q,t}^{(s)} \geq_{q,t} 0,$$

so, $\left(\overline{\begin{bmatrix}M+j\\K-j\end{bmatrix}}_{q,t}^{(s)}\right)_{j \leq K}$ satisfies the strong (q, t)-log-concavity.

Proof For $sN \geq M \geq N$ and $H \geq K$, and $M + K = N + H$:

$$\overline{\begin{bmatrix}M\\K\end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix}N\\H\end{bmatrix}}_{q,t}^{(s)} - \overline{\begin{bmatrix}M+1\\K-1\end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix}N-1\\H+1\end{bmatrix}}_{q,t}^{(s)} = \sum_{P_1,P_2} (-1)^{P_1P_2} t^{O(P_1P_2)} q^{w_{P_1P_2}} \tag{2.3.12}$$

Let $a_1 = (0, 0)$, $a_2 = (M - N + 1, K - H - 1)$, $b_1 = (M - 1, K)$, and $b_2 = (M, K - 1)$, and in order to preserve the wight before and after the changing, we recall the maps in 2.3.1, 2.3.2 and 2.3.3, and we have the subsequent compositions:

$$(P_1, P_2) \xrightarrow{\mathfrak{T}_S\, \Gamma_s\, \mathfrak{T}_S^{-1}} (Q_1, Q_2) \xrightarrow{\theta\, \mathfrak{T}_W^{M-N+1} \mathfrak{T}_S^{K-H}\, \Gamma_s\, \mathfrak{T}_W^{N-M-1} \mathfrak{T}_S^{H-K}} (Q'_1, Q'_2)\,. \tag{2.3.13}$$

Here, for each pair of path (P_1, P_2) such that $(-1)^{P_1P_2} = -1$, we get (Q'_1, Q'_2), then $(-1)^{Q_1Q_2}$ is necessarily equal to $+1$. Consequently, the expression in (2.3.12) has nonnegative coefficients as a polynomial in q and t. ∎

As an illustration of the proof, let us take $s = 2$, $M = 4$, $N = 3$, $H = 3$ and $K = 2$. Figure 2.7 shows the result pair of paths after we apply the composition above on the pair of paths associated to the term $q^2 \times tq^2$.

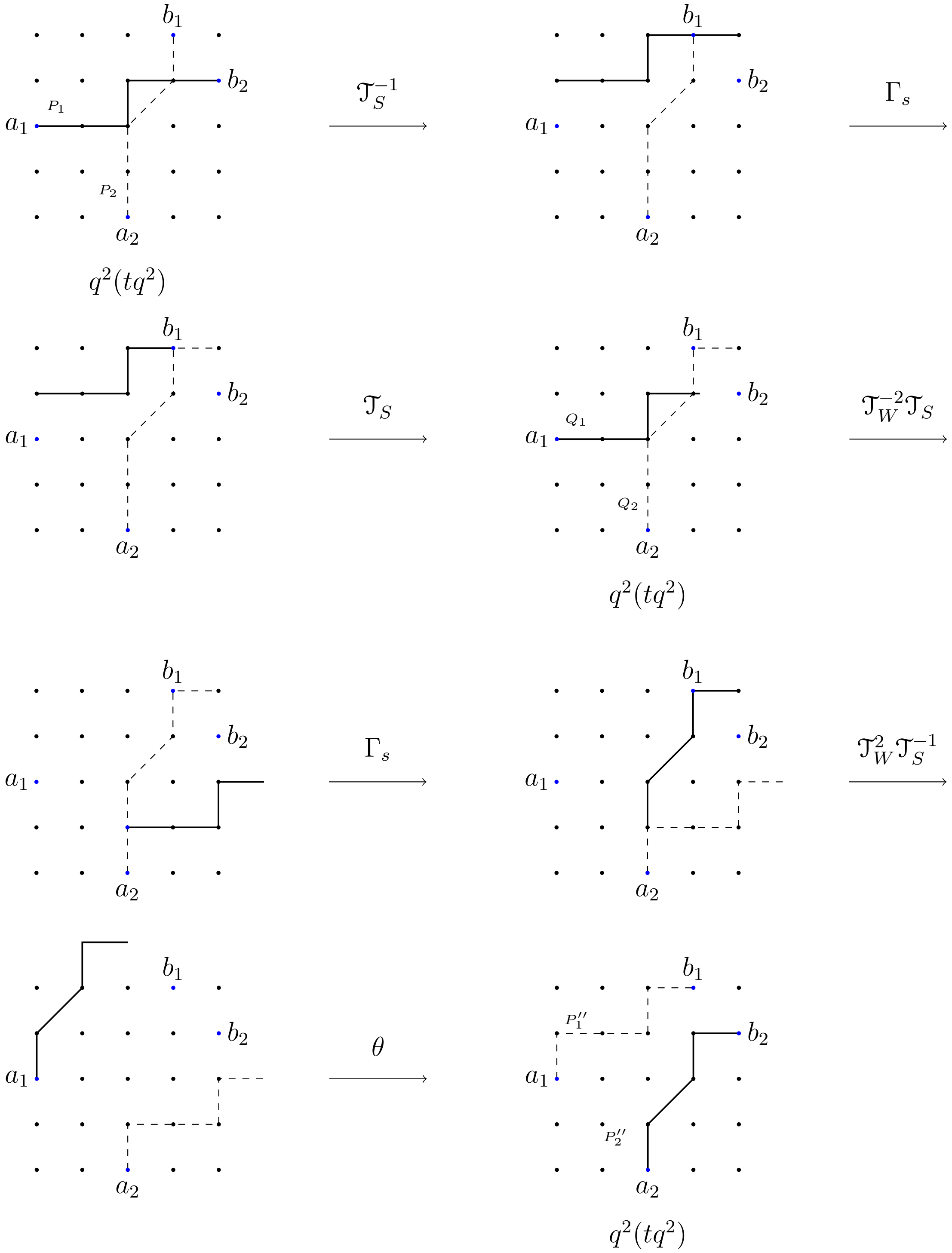


Figure 2.7: The composition (2.3.13).

Since we can obtain the over q-bi^{s}nomial coefficient and the q-bi^{s}nomial coefficient by taking $t = 1$ and $t = 0$ respectively, the three previous results remain valid for the over q-bi^{s}nomial coefficient and the q-bi^{s}nomial coefficient, noting that those of q-bi^{s}nomial coefficient are due to Bazeniar *et al.* [10, Theorem 1].

Corollary 2.3.8 *The sequences of polynomials* $\left(\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(s)}\right)_K$, $\left(\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_q^{(s)}\right)_M$, $\left(\overline{\begin{bmatrix} M+j \\ K-j \end{bmatrix}}_q^{(s)}\right)_{j\leq K}$, $\left(\begin{bmatrix} M \\ K \end{bmatrix}_q^{(s)}\right)_K$, $\left(\begin{bmatrix} M \\ K \end{bmatrix}_q^{(s)}\right)_M$ *and* $\left(\begin{bmatrix} M+j \\ K-j \end{bmatrix}_q^{(s)}\right)_{j\leq K}$ *satisfy the strong q-log-concavity.*

Furthermore, we have two more generalized results of the (q, t)-log-concavity of over (q, t)-bi^{s}nomial coefficients.

Theorem 2.3.9 *For $H \leq K$, $N \leq M$ and $s \geq 1$:*

(i) $\overline{\begin{bmatrix} M \\ H \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N \\ K \end{bmatrix}}_{q,t}^{(s)} \geq_{q,t} \overline{\begin{bmatrix} M \\ H-1 \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N \\ K+1 \end{bmatrix}}_{q,t}^{(s)}$;

(ii) $\overline{\begin{bmatrix} M \\ H \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N \\ K \end{bmatrix}}_{q,t}^{(s)} \geq_{q,t} \overline{\begin{bmatrix} M+1 \\ H \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N-1 \\ K \end{bmatrix}}_{q,t}^{(s)}$.

Proof **(i)** Following the method of the proof of Theorem 2.3.4, we simply choose $a_2 = (N - M, K - H + 1)$, $b_1 = (N - 1, K)$ and $b_2 = (N - 1, K + 1)$.

(ii) Following the method of the proof of Theorem Theorem 2.3.6, we simply choose $a_2 = (M - N + 1, H - K)$, $b_1 = (M - 1, H)$ and $b_2 = (M, H)$.

By Su and Wang's approach [58, Theorem 4] and Theorem 2.3.9, we establish the strong (q, t)-log-concavity of rays crossing the triangle of over (q, t)-bi^{s}nomial coefficients for some directions, as follows.

Theorem 2.3.10 *The following sequence of polynomials*

$$\left(\overline{\begin{bmatrix} M_0 + ja \\ K_0 - jb \end{bmatrix}}_{q,t}^{(s)}\right)_{j\geq 0}$$

satisfies the strong (q, t)-log-concavity, where $K_0 \leq sM_0$ and $0 < ab$.

Proof Since $0 < ab$ (i.e., they have the same sign), then it suffices to take a and b both positive. We set $K = K_0 - bj$, $H = K_0 - bi$, $M = M_0 + aj$, $N = M_0 + ai$ for $j \geq i$, with $K \leq H$, $N \leq M$.

By repeating using (i) and (ii) of Theorem 2.3.9:

$$\overline{\begin{bmatrix} M+a \\ K-b \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N-a \\ H+b \end{bmatrix}}_{q,t}^{(s)} \leq_{q,t} \overline{\begin{bmatrix} M+a-1 \\ K-b \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N-a+1 \\ H+b \end{bmatrix}}_{q,t}^{(s)} \leq_{q,t} \cdots$$
$$\cdots \leq_{q,t} \overline{\begin{bmatrix} M \\ K-b \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N \\ H+b \end{bmatrix}}_{q,t}^{(s)},$$

and

$$\overline{\begin{bmatrix} M \\ K-b \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N \\ H+b \end{bmatrix}}_{q,t}^{(s)} \leq_{q,t} \overline{\begin{bmatrix} M \\ K-b+1 \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N \\ H+b-1 \end{bmatrix}}_{q,t}^{(s)} \leq_{q,t} \cdots$$
$$\cdots \leq_{q,t} \overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N \\ H \end{bmatrix}}_{q,t}^{(s)}.$$

Hence,

$$\overline{\begin{bmatrix} M+a \\ K-b \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} N-a \\ H+b \end{bmatrix}}_{q,t}^{(s)} \leq_{q,t} \overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t}^{(s)} \overline{\begin{bmatrix} M \\ H \end{bmatrix}}_{q,t}^{(s)}.$$

∎

We end this chapter by two results.

We take $s \to \infty$ in Theorem 2.3.6, Theorem 2.3.7 and Theorem 2.3.10, we get:

Theorem 2.3.11 *The sequences of polynomials* $\left(\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t}\right)_M$, $\left(\overline{\begin{bmatrix} M+j \\ K-j \end{bmatrix}}_{q,t}\right)_{j\leq K}$ *and* $\left(\overline{\begin{bmatrix} M_0+ja \\ K_0-jb \end{bmatrix}}_{q,t}\right)_{j\geq 0}$ *where* $ab>0$*, satisfy the strong* (q,t)*-log-concavity.*

We take $t=1$ in the previous theorem above, we get:

Corollary 2.3.12 *The sequences of polynomials* $\left(\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q}\right)_M$, $\left(\overline{\begin{bmatrix} M+j \\ K-j \end{bmatrix}}_{q}\right)_{j\leq K}$ *and* $\left(\overline{\begin{bmatrix} M_0+ja \\ K_0-jb \end{bmatrix}}_{q}\right)_{j\geq 0}$ *where* $ab>0$*, . satisfy the strong* q*-log-concavity.*

CHAPTER 3

An analogue of Mahonian numbers and log-concavity

In this chapter, we define a q-analogue of the number of permutations $i_m(h)$ of length m having h inversions known as Mahonian numbers. We investigate useful properties and some combinatorial interpretations by lattice paths/partitions and tilings. Furthermore, we give a constructive proof of the strong q-log-concavity of the q-Mahonian numbers. In particular for $q = 1$, we obtain a constructive proof of the log-concavity of the Mahonian numbers.

3.1 Definition and combinatorial interpretation

3.1.1 Definition and some identities

Recall $S_{inv}(m, h)$ is the set of permutations of length m with h inversions.

Let π be a permutation of length m. We denote by a_j^π the number of appearances of the entry $j + 1$ as first element of the inversions of π, for $0 \leq j \leq m - 1$. Thus, the number of inversions of π is the sum of a_j^π where $0 \leq j \leq m - 1$, that is $\sum_{j=0}^{m-1} a_j^\pi$.

Now, we define a new weight w_π (the weight of π) as follows

$$w_\pi := \sum_{j=0}^{m-1} j \times a_j^\pi. \tag{3.1.1}$$

Here, we define our a q-analogue of the Mahonian numbers $i_m(h)$ basing on the definition of the weight of a permutation which was just mentioned.

Definition 3.1.1 *For $m \geq 1$ and $0 \leq h \leq \binom{m}{2}$, our q-Mahonian number denoted $p^q_{inv}(m,h)$ or $i_q(n,k)$ is defined as follows*

$$p^q_{inv}(m,h) = \sum_{\pi \in S_{inv}(m,h)} q^{w_\pi}, \tag{3.1.2}$$

Now, let $P_{inv}(m,h)$ denote the following set:

$$P_{inv}(m,h) = \{\pi \in S_{inv}(m,h) : w_\pi = N\}. \tag{3.1.3}$$

We denote by $p_{inv}(m,h;N)$ be the cardinality of $P_{inv}(m,h)$, that is

$$p_{inv}(m,h;N) = |P_{inv}(m,h)|.$$

So, due to this last notation we can rewrite an equivalent expression to that of Definition 3.1.1.

Proposition 3.1.2 *For $m \geq 1$ and $0 \leq h \leq \binom{m}{2}$,*

$$p^q_{inv}(m,h) = \sum_{N \geq 0} p_{inv}(m,h;N) q^N.$$

And we have the subsequent recursion.

Theorem 3.1.3 *Let m be a positive integer such that $m > 1$, then*

$$p^q_{inv}(m,h) = \sum_{i=0}^{min(m-1,h)} q^{i(m-1)} p^q_{inv}(m-1, h-i), \tag{3.1.4}$$

where $p^q_{inv}(m,0) = 1$.

Proof For a permutation $\pi = \pi_1\pi_2\cdots\pi_m$ in $S_{inv}(m,h)$. So, there are m possibilities for the positions of m in π (from π_1 to π_m). If m is in the position $m-i$, i.e. $\pi_{m-i} = m$ for $0 \leq i \leq m-1$, then m is the first element of i inversions and not a second element of any inversion, thus $a^\pi_{m-1} = i$. This count corresponds to the inclusion of the term $q^{i(m-1)} p^q_{inv}(m-1, h-i)$ in the formula. ∎

Due to the recurrence relation (3.1.4), we can establish Table 3.1 of the triangle of $p^q_{inv}(m,h)$.

By Theorem 3.1.3, we can easily get the subsequent corollary.

m/h	0	1	2	3	4
1	1				
2	1	q			
3	1	$q+q^2$	q^3+q^4	q^5	
4	1	$q+q^2+q^3$	$q^3+2q^4+q^5+q^6$	$\cdots$	$\cdots$
5	1	$q+q^2+q^3+q^4$	$q^3+2q^4+2q^5+2q^6+q^7+q^8$	$\cdots$	

Table 3.1: Table of first values of q-Mahonian numbers.

Corollary 3.1.4 *Let $m > 1$, then*

$$p^q_{inv}(m,h) = p^q_{inv}(m-1,h) + q^{m-1}p^q_{inv}(m,h-1) - q^{m(m-1)}p^q_{inv}(m-1,h-m).$$

We can obtain the row generating function of $p^q_{inv}(m,h)$ using the recursion (3.1.4).

Theorem 3.1.5 *The q-Mahonian numbers have as generating function:*

$$\sum_{h=0}^{\binom{m}{2}} p^q_{inv}(m,h)x^h = \prod_{j=0}^{m-1}\left(1+q^jx+\cdots+(q^jx)^j\right). \tag{3.1.5}$$

Proof Let

$$g_m(x;q) = \sum_{h=0}^{\binom{m}{2}} p^q_{inv}(m,h)x^h$$

Hence,

$$\begin{aligned} g_m(x;q) := \sum_{h=0}^{\binom{m}{2}} p^q_{inv}(m,h)x^h &= \sum_{h=0}^{\binom{m}{2}}\sum_{j=0}^{m-1} q^{(m-1)j}p^q_{inv}(m-1,h-j)x^h \\ &= \sum_{j=0}^{m-1}(q^{(m-1)}x)^j \sum_{h=0}^{\binom{m}{2}} p^q_{inv}(m-1,h-j)x^{h-j}. \end{aligned}$$

And since $p^q_{inv}(m-1,h-j) = 0$ if $h-j<0$ or $h-j>\binom{m-1}{2}$, then

$$\begin{aligned} g_m(x;q) &= \sum_{j=0}^{m-1}(q^{(m-1)}x)^j \sum_{h=j}^{\binom{m-1}{2}+j} p_{inv}^q(m-1,h-j)x^{h-j} \\ &= \sum_{j=0}^{m-1}(q^{(m-1)}x)^j \sum_{h=0}^{\binom{m-1}{2}} p_{inv}^q(m-1,h)x^h \\ &= g_{m-1}(x;q)\sum_{j=0}^{m-1}(q^{(m-1)}x)^j. \end{aligned}$$

So, we get a relation of recurrence that write $g_m(x;q)$ in terms of $g_{m-1}(x;q)$. Thus, using the induction hypothesis we get the equality (3.1.5) ∎

Now, we can say that the relation in following theorem is considered as a q-analogue of the relation (1.6.2).

Theorem 3.1.6 *For $m > 1$,*

$$p_{inv}^q(m,h) = q^{\frac{m(m-1)(2n-1)}{6}} p_{inv}^{\frac{1}{q}}\left(m, \binom{m}{2} - h\right).$$

Proof We consider the following bijection

$$\mathcal{B} : \mathcal{S}_n \to \mathcal{S}_n$$
$$\pi \mapsto \pi',$$

where π' is the backward permutation of π, i.e., $\pi'_j = \pi_{m+1-j}$ for all $1 \leq j \leq m$.

Hence, if (π_j, π_l) is a backward inversion of π, so (π_l, π_j) is an inversion of π', and vice versa.

Thereby, the number of inversions of π' is $\binom{m}{2} - h$.

Moreover, $a_l^{\pi'} = l - a_l^{\pi}$, for $0 \leq l \leq m-1$.

Then, if the weight of π equals M, so by (3.1.1)

$$w_{\pi'} = \sum_{l=0}^{m-1} l^2 - M = \frac{m(m-1)(2n-1)}{6} - M.$$

Hence, by (3.1.2) we get the identity. ∎

In the following, we want to write our analogue in term of Gaussian polynomial. First let give some identities.

Theorem 3.1.7 *We have*

$$\prod_{i=1}^{\infty}\left(1-q^{i^2}x^i\right)=1+\sum_{h\geq 1}x^h\left[\sum_{\eta\in\mathcal{P}_D(h)}(-1)^{\ell(\eta)}q^{\sum_{j=1}^{\ell(\eta)}\eta_j^2}\right],$$

where $\mathcal{P}_D(h)$ is the set of partitions into distinct parts of the integer h, and $\ell(\eta)$ is the length of $\eta=(\eta_1,\eta_2,\ldots)$.

Proof

$$\begin{aligned}\prod_{j=1}^{\infty}\left(1-q^{j^2}x^j\right)&=\left(1-q^{1^2}x^1\right)\left(1-q^{2^2}x^2\right)\left(1-q^{3^2}x^3\right)\cdots\\&=1+(-1)^1q^{1^2}x^1+(-1)^1q^{2^2}x^2+\left((-1)^1q^{3^2}+(-1)^2q^{1^2+2^2}\right)x^3+\cdots\\&=1+(-1)^{\ell(1)}q^{1^2}x^1+(-1)^{\ell(2)}q^{2^2}x^2+\left((-1)^{\ell(3)}q^{3^2}+(-1)^{\ell(2,1)}q^{1^2+2^2}\right)x^3+\cdots\end{aligned}$$

We notice that, for each x^h, the exponents of q are the sum of squares of the distinct parts of h. Hence

$$\prod_{i=1}^{\infty}\left(1-q^{i^2}x^i\right)=1+\sum_{h\geq 1}x^h\left[\sum_{\eta\in\mathcal{P}_D(h)}(-1)^{\ell(\eta)}q^{\sum_{j=1}^{\ell(\eta)}\eta_j^2}\right].$$

■

Here, we get Euler's pentagonal number theorem merely take $q=1$ in the previous theorem

Theorem 3.1.8 (Euler) *[39, 41]*

$$\prod_{i=1}^{\infty}\left(1-x^i\right)=1+\sum_{h\geq 1}(-1)^h\left(x^{h(3k+1)/2}+x^{h(3k-1)/2}\right).$$

Following the same steps of the proof of Theorem 3.1.7, we get the subsequent product.

Corollary 3.1.9

$$\prod_{i=1}^{m}\left(1-q^{i^2}x^i\right)=1+\sum_{h\geq 1}x^h\left[\sum_{\eta\in\mathcal{P}_D(h,m)}(-1)^{\ell(\eta)}q^{\sum_{j=1}^{\ell(\eta)}\eta_j^2}\right],$$

where $\mathcal{P}_D(h,m)$ is the set of partitions of h into distinct parts where the largest part is at most m.

Moreover

Corollary 3.1.10

$$\prod_{i=1}^{\infty}\left(1-q^{i^2}\right)=1+\sum_{h\geq 1}\left[\sum_{\eta\in\mathcal{P}_D(h)}(-1)^{\ell(\eta)}q^{\sum_{j=1}^{\ell(\eta)}\eta_j^2}\right].$$

Here, we achieve our objective of these last lines, which is finding the link between our analogue and Gaussian polynomial.

Theorem 3.1.11 *For* $1\leq h\leq\binom{m}{2}$,

$$p_{inv}^{q}(m,h)=\begin{bmatrix}m+h-1\\ h\end{bmatrix}_q+\sum_{i=1}^{h}\begin{bmatrix}m+h-i-1\\ h-i\end{bmatrix}_q\sum_{\eta\in\mathcal{P}_D(j,m)}(-1)^{\ell(\eta)}q^{\sum_{j=1}^{\ell(\eta)}\eta_j^2-i}.$$

Proof We can write the product in (3.1.5) as

$$\begin{aligned}\sum_{h=0}^{\binom{m}{2}}p_{inv}^{q}(m,h)x^h&=\prod_{i=1}^{m}\left(\frac{1-(q^{i-1}x)^i}{1-q^{i-1}x}\right)\\&=\prod_{i=1}^{m}\left(1-(q^{i-1}x)^i\right)\prod_{i=1}^{m}\left(1-q^{i-1}x\right)^{-1}\\&=\prod_{i=1}^{m}\left(1-(q^{i-1}x)^i\right)\sum_{h\geq 0}\begin{bmatrix}m+h-1\\ h\end{bmatrix}_q x^h.\end{aligned}$$

And letting $x\mapsto xq^{-1}$ in Corollary 3.1.9, gives us

$$\prod_{i=1}^{m}\left(1-q^{i(i-1)}x^i\right)=1+\sum_{h\geq 1}x^h\left[\sum_{\eta\in\mathcal{P}_D(h,m)}(-1)^{\ell(\eta)}q^{\sum_{j=1}^{\ell(\eta)}\eta_j^2-h}\right].$$

Hence,

$$\begin{aligned}\sum_{h=0}^{\binom{m}{2}}p_{inv}^{q}(m,h)x^h&=\left(1+\sum_{h\geq 1}x^h\left[\sum_{\eta\in\mathcal{P}_D(h,m)}(-1)^{\ell(\eta)}q^{\sum_{j=1}^{\ell(\eta)}\eta_j^2-h}\right]\right)\sum_{h\geq 0}\begin{bmatrix}m+h-1\\ h\end{bmatrix}_q x^h\\&=\sum_{h\geq 0}\begin{bmatrix}m+h-1\\ h\end{bmatrix}_q x^h+\sum_{h\geq 1}x^h\left(\sum_{i=1}^{h}\begin{bmatrix}m+h-i-1\\ h-i\end{bmatrix}_q\sum_{\eta\in\mathcal{P}_D(i,m)}(-1)^{\ell(\eta)}q^{\sum_{j=1}^{\ell(\eta)}\eta_j^2-i}\right)\\&=1+\sum_{h\geq 1}x^h\left(\begin{bmatrix}m+h-1\\ h\end{bmatrix}_q+\sum_{i=1}^{h}\begin{bmatrix}m+h-i-1\\ h-i\end{bmatrix}_q\sum_{\eta\in\mathcal{P}_D(i,m)}(-1)^{\ell(\eta)}q^{\sum_{j=1}^{\ell(\eta)}\eta_j^2-i}\right)\end{aligned}$$

∎

If we take $q=1$ in the theorem above, we get the Knuth's explicit formula [44] for the hth term of the Mahonian numbers when $h\leq m$.

Corollary 3.1.12 *[44, Knuth] or [51, Netto] Let $h \leq m$, then*

$$i_m(h) = \binom{m+h-1}{h} + \sum_{i=1}^{\infty}(-1)^i\binom{m+h-u_i-i-1}{h-u_i-i} + \sum_{i=1}^{\infty}(-1)^i\binom{m+h-u_i-1}{h-u_i},$$

where $u_i = \frac{i(3i-1)}{2}$ is the ith pentagonal number.

Since the largest value of h in Theorem 3.1.11 is $\binom{m}{2}$, we can conclude.

Corollary 3.1.13 *Let $h > \binom{m}{2}$, then*

$$\begin{bmatrix} m+h-1 \\ h \end{bmatrix}_q = \sum_{i=1}^{h} \begin{bmatrix} m+h-i-1 \\ h-i \end{bmatrix}_q \sum_{\eta \in \mathcal{P}_D(i,m)} (-1)^{\ell(\eta)+1} q^{\sum_{j=1}^{\ell(\eta)} \eta_j^2 - i}.$$

3.1.2 Interpretations by paths and partitions

In this subsection, we give a combinatorial interpretation for our q-analogue of Mahonian numbers by a specific type of lattice paths and then by partitions.

Definition 3.1.14 *We denote by $\mathcal{P}^I_{m,h}$ the subset of $\mathcal{P}(m,h)$ (the set of North-East paths starting from $(0,0)$ to $(m-1,h)$ defined in Definition 1.2.1) such that for each $0 \leq i \leq m-1$, it is at most i North steps in the vertical level i. As an illustration, see Figure 3.1.*

Now, we define the weight of a path in $\mathcal{P}^I_{m,h}$.

Definition 3.1.15 *Let $Q \in \mathcal{P}^I_{m,h}$, we define the weight of Q denoted $wt(Q)$ as*

$$wt(Q) = \sum_{i=0}^{m-1} i \times n_N(i),$$

where $n_N(i)$ denotes the number of North steps in the vertical level i. In other words, the weight of Q is merely the number of the boxes above and to the left of the path Q.

For instance, Figure 3.1 shows the weight of a path Q where $wt(Q) = 0\times0+1\times1+1\times2+2\times3 = 9$, which is exactly the number of boxes above this path.

Now, let us interpret the q-analogue of the number of permutations with h inversions, by such type of lattice paths.

Theorem 3.1.16 *Let $m \geq 1$, $0 \leq h \leq \binom{m}{2}$, then*

$$p^q_{inv}(m,h) = \sum_{Q \in \mathcal{P}^I_{m,h}} q^{wt(Q)}.$$

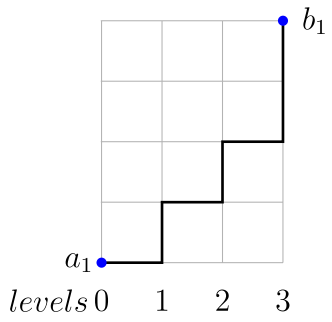


Figure 3.1: An example of a path in $\mathcal{Q}^I_{m,h}$.

Proof From a path $Q \in \mathcal{P}^I_{m,h}$, we obtain the permutation corresponded to Q following this procedure:

The initial vertex $a_1 = (0,0)$ corresponds to the entry 1 of the permutation, then we follow the rule: each East step is equivalent to adding the following entry, and number of North steps in each vertical level i is equivalent to that the entry i is moved $n_N(i)$ positions to the left, in the subsequent manner:

First, we said that the initial vertex $a_1 = (0,0)$ corresponds to the entry 1, then Q has obligatory an East step (because in the vertical level 0 there are 0 North steps allowed), so we add the entry 2 to the right of 1 at the vertex $(1,0)$. Here, we have two cases: either the next step is an East step, and then we add at the vertex $(2,0)$ the entry 3 to the right of 12 we get 123, or the next step is a North step and here the entry 2 is moved one position to the left (one position because we have one North step) we get 21 at the vertex $(1,1)$ and then obligatory the next step is an East step in which the entry 3 is added. We repeat that in each vertex, so at a certain vertex after an East step we have $j_1 j_2 \cdots j_{i-1} i$, then, if the next step is an East step we only add $i+1$ to the right of $j_1 j_2 \cdots j_{i-1} i$, we obtain $j_1 j_2 \cdots j_i i(i+1)$, and if the next steps are l successive North steps before an East step, where $1 \leq l \leq i-1$, here i is moved one position in each North step to the left, that is, i would be moved l positions to the left, then we have the East step, and so on. For instance, see Figure 3.2.

Inversely, for a permutation $\pi \in S_{inv}(m,h)$, it should be noted that for $0 \leq i \leq m-1$, $0 \leq a^\pi_i \leq i$, because a^π_i counts the number of appearances of the entry i as a first element of the inversions of π. As an example, the entry 4 is either not a first element of any inversion, or it is the first element of an inversion with 3, 2 or 1, or with two of them, or with the three together, so $a^\pi_3 \in \{0,1,2,3\}$.

Hence, any permutation $\pi \in S_{inv}(m,h)$ corresponds to a path $Q \in \mathcal{P}^I_{m,h}$, such that for $0 \le i \le m-1$, each vertical level i corresponds to the entry $i+1$ of the permutation π, and $n_N(i)$ is equal to a^π_i, which means knowing the number of North steps in each level i, so the path will be automatically constructed. Furthermore

$$w_\pi = \sum_{i=1}^{m} i \times a_i^\pi = \sum_{i=0}^{m-1} i \times n_N(i) = wt(Q).$$

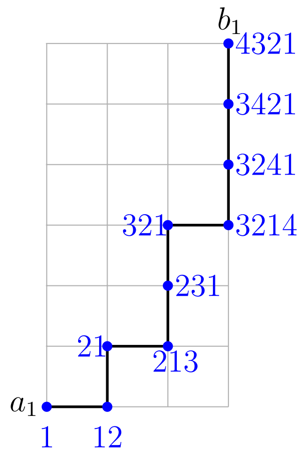


Figure 3.2: The path Q corresponded to $\pi = 4321$.

Here, we could interpret the number of permutations of length m with h inversions by lattice paths by letting $q \to 1$ in the theorem above.

Corollary 3.1.17 *The number of permutations of length m with h inversions (Mahonian number) $i_m(h)$ counts the number of paths from $a_1 = (0,0)$ to $b_1 = (m-1,h)$ where the number of North steps is at most i in each vertical level i.*

Since there is a correspondence between such type of paths (which are a subset of North-East paths introduced in the first chapter) and partitions (see Remark 1.2.4), the following proposition shows how we can interpret the q-Mahonian numbers by partitions.

Proposition 3.1.18 *For a fixed m and h, the q-Mahonian number is a polynomial in q. This polynomial is the generating function of the number of partitions into h parts in which each part i can be used at most i times and the largest part $\le m-1$, that is*

$$p^q_{inv}(m,h) = \sum_{\eta \subset (m-1)^h} q^{|\eta|}, \tag{3.1.6}$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_h)$ with $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_h \leq m-1$, $|\ \eta\ |= \sum \eta_i$ and the number i can be repeated at most i times.

Proof Since the set $Q \in \mathcal{P}^I_{m,h}$ is a subset of the set of North-East lattice paths and by Remark 1.2.4: each number of boxes above and to the left of a path in $\mathcal{P}^I_{m,h}$ in each row represents a part, and then the weight of the obtained partition, which is the sum of the parts, is the total number of boxes above and to the left of this path Q, i.e., the the weight of Q. ∎

Example 3.1.19 *Figure 3.3, represents a path and the corresponded partition associated to q^5. Indeed, the partition corresponded to the path is $(2,2,1)$, and $|(2,2,1)| = 5$.*

Consequently,

Corollary 3.1.20 *For $m \geq 1$ and $0 \leq h \leq \binom{m}{2}$,*

$$p^q_{inv}(m,h) = \sum_{N \geq 1} p^I(m,h,N) q^N,$$

where $p^I(m,h,N)$ counts the number of partitions of N into h parts, and the largest part is less or equal to $m-1$, such that each part i can be repeated at most i times.

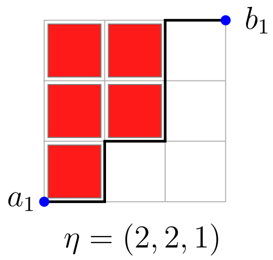


Figure 3.3: The partition corresponds to a path of weight equals 5.

Since we have such type of North-East paths, we can also obtain:

3.1.3 Interpretation by tiling

Definition 3.1.21 *Let $L^I_{m,h}$ be the set of weighted tilings of an $(m+h-1) \times 1$-board in which we use only $m-1$ green squares and h orange squares where the number of successive orange squares is at most i if there are i green squares before.*

We give now the definition of the weight of a tiling L.

Definition 3.1.22 *Let L be a tiling in $L^I_{m,h}$, We define the weight w_L of L by:*

$$w_L = \sum_i w_{o_i},$$

where o_i denotes the the weight of the ith orange square counts the number of green squares to the left of that orange square.

Furthermore, we have:

Proposition 3.1.23 *For $m \geq 1$ and $0 \leq h \leq \binom{m}{2}$,*

$$p^q_{inv}(m,h) = \sum_{L \in L^I_{m,h}} q^{w_L}.$$

Proof This tiling interpretation is in bijection with paths interpretation. Each green square is corresponded to an East step, and each orange square to a North step, and vice versa. Moreover, this bijection is weight-preserving. Indeed, the weight of a path is equal to the total number of boxes above and to the left of this path, or in other words, the sum of the number of boxes in each row. And since each row is associated to a North step. So, in each row, by calculating the number of East steps before the North step associated to this row, we obtain the number of boxes in this row. ∎

As an example, in Figure 3.4 we have a tiling T of q^5 and its corresponded path. The weight here is calculated as $w_{o_1} + w_{o_2} + w_{o_3} = 1 + 2 + 2 = 5$.

Accordingly, if we set $q = 1$ above, we get:

Corollary 3.1.24 *The number of permutations of length m having h inversions $i_m(h)$ is counting the number of tilings of size $(m + h - 1) \times 1$-board taking only $m - 1$ green squares and h orange squares where the number of successive orange squares is at most i if there are i green squares before.*

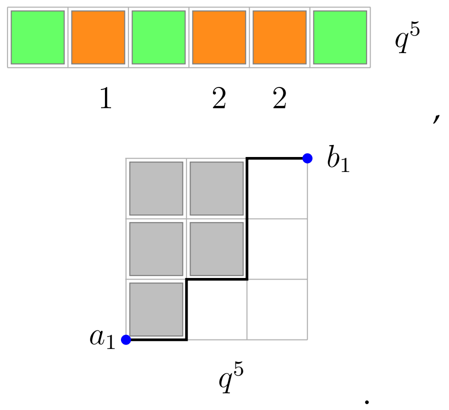


Figure 3.4: A tiling of weight 5 and its corresponded path.

3.2 Log-concavity property

Sagan in [54], Bazeniar et al. in [11] and Gasharov in [35] studied the log-concavity using paths as already mentioned in Section 2.3. Bóna [18] showed that the number $i_m(h)$ satisfies the log-concavity in h, by using an induction hypothesis about injection over m, but this injection is non-constructive. In this section.

Motivated by all these works we construct an injection to prove the strong q-log-concavity of our analogue of $i_m(h)$, and then by taking $q = 1$ we obtain a constructive injection that gives us the log-concavity of $i_m(h)$ instead of the non-constructive one of Bóna.

Theorem 3.2.1 *For* $\binom{m}{2} > l \geq h > 0$,

$$p^q_{inv}(m,h)p^q_{inv}(m,l) - p^q_{inv}(m,h-1)p^q_{inv}(m,l+1) \geq_q 0,$$

i.e., the sequence of polynomials $(p^q_{inv}(m,h))_h$ *satisfies the strong q-log-concavity.*

Proof What we will do in this proof is to construct an injective map

$$\mathcal{G}_{m,h,l} : S_{inv}(m,l+1) \times S_{inv}(m,h-1) \to S_{inv}(m,h) \times S_{inv}(m,l)$$
$$(\pi,\tau) \mapsto (\beta,\alpha),$$

such that $w_\pi + w_\tau = w_\beta + w_\alpha$.

For $(\pi,\tau) \in S_{inv}(m,l+1) \times S_{inv}(m,h-1)$, let $\mathcal{G}_{m,h,l}$ be defined as

$$\mathcal{G}_{m,h,l}(\pi,\tau) = (\beta,\alpha),$$

such that:

Let J be the largest integer verifying:

$$\sum_{j\geq J} a_j^{\pi} \geq \sum_{j\geq J+1} a^{\tau} j + 1 \tag{3.2.1}$$

and

$$\sum_{j\geq J} a_j^{\tau} + 1 - \sum_{j\geq J+1} a_j^{\pi} \leq J, \quad \text{if} \sum_{j\geq J} a_j^{\pi} < \sum_{j\geq J} a_j^{\tau} + 1. \tag{3.2.2}$$

Then, (β, α) is obtained by some modifications on τ and π as follows:

- β is obtained by the subsequent changes on τ: for $j+1$ iterating from m to $J+2$ in decreasing order, the entry $j+1$ is shifted a_j^{τ} positions to the right. The entry $j+1 = J+1$, $j+1$ is shifted $\left(\sum_{i=1}^{J} a_i^{\pi} - \sum_{i=1}^{J} a_i^{\tau} - (l-h+1)\right)$ positions to the left. After, for $j+1$ starting from $J+2$ to m, the entry $j+1$ is shifted m_j^{π} positions to the left.

- α is obtained by the subsequent changes on π: for $j+1$ iterating from m to $J+2$ in decreasing order, the entry $j+1$ is shifted a_j^{π} positions to the right. The entry $j+1 = J+1$, $j+1$ is shifted $\left(\sum_{i=1}^{J} a_i^{\pi} - \sum_{i=1}^{J} a_i^{\tau} - (l-h+1)\right)$ positions to the right. For $j+1$ starting from $J+2$ to m, the entry $j+1$ is shifted m_j^{τ} positions to the left.

Let check first that the map $\mathcal{G}_{m,h,l}$ is well defined, it suffices to verify:

$$0 \leq \sum_{j=1}^{J} a_j^{\pi} - \sum_{j=1}^{J} m_j^{\tau} - (l-h+1) \leq a_J^{\tau} \tag{3.2.3}$$

and

$$m_J^{\tau} + \sum_{j=1}^{J} a_j^{\pi} - \sum_{j=1}^{J} m_j^{\tau} - (l-h+1) \leq J. \tag{3.2.4}$$

The amount of the left side of (3.2.3) can be written as

$$\begin{aligned}\sum_{j=1}^{J} a_j^{\pi} - \sum_{j=1}^{J} m_j^{\tau} - (l-h+1) &= \sum_{j=1}^{m-1} a_j^{\pi} - \sum_{j=J+1}^{m-1} a_j^{\pi} - \sum_{j=1}^{m-1} m_j^{\tau} + \sum_{j=J+1}^{m-1} m_j^{\tau} - l + h - 1 \\ &= 1 - \sum_{j=J+1}^{m-1} a^{\pi} + \sum_{j=J+1}^{m-1} m_j^{\tau}. \end{aligned} \tag{3.2.5}$$

Hence, (3.2.4) will be written as

$$m_J^\tau + \sum_{j=1}^{J} a_j^\pi - \sum_{j=1}^{J} m_j^\tau - (l-h+1) = m_j^\tau + 1 - \sum_{j=J+1}^{m-1} a^\pi + \sum_{j=J+1}^{m-1} m_j^\tau$$
$$= 1 - \sum_{j=J+1}^{m-1} a^\pi + \sum_{j=J}^{m-1} m_j^\tau. \quad (3.2.6)$$

By (3.2.5) and (3.2.1), we get the right inequality of (3.2.3),

$$1 - \sum_{j=J+1}^{m-1} a^\pi + \sum_{j=J+1}^{m-1} m_j^\tau \le a_J^\pi.$$

We turn now to the left inequality of relation (3.2.3), we have here the case **(a)** and the case **(b)**:

(a): If $\sum_{j=J+1}^{m-1} a_j^\pi < \sum_{j=J+2}^{m-1} a_j^\tau + 1$:

$$1 - \sum_{j=J+1}^{m-1} a^\pi + \sum_{j=J+1}^{m-1} m_j^\tau > 1 - \sum_{j=J+2}^{m-1} a_j^\tau - 1 + \sum_{j=J+1}^{m-1} m_j^\tau$$
$$= m_{J+1}^\tau \ge 0.$$

(b): If $\sum_{j=J+1}^{m-1} a_j^\pi \ge \sum_{j=J+2}^{m-1} a_j^\tau + 1$ and $\sum_{j=J+1}^{m-1} a_j^\pi < \sum_{j=J+1}^{m-1} a_j^\tau + 1$ but $\sum_{j=J+1}^{m-1} a_j^\tau + 1 - \sum_{j=J+2}^{m-1} a_j^\pi > J+1$:

$$1 - \sum_{j=J+1}^{m-1} a_j^\pi + \sum_{j=J+1}^{m-1} a_j^\tau > J + 1 - m_{J+1}^\pi \ge 0.$$

Let move on now to the inequality (3.2.4), to prove it we distinguish two cases:

- If $\sum_{j=J}^{m-1} a_j^\pi < \sum_{j=J}^{m-1} a_j^\tau + 1$, we use the simplified amount in (3.2.6), then the inequality (3.2.4) is already hold by the second condition (3.2.2).
- If $\sum_{j=J}^{m-1} a_j^\pi \ge \sum_{j=J}^{m-1} a_j^\tau + 1$, hence

$$-\sum_{j=J+1}^{m-1} a_j^\pi + \sum_{j=J+1}^{m-1} a_j^\tau + 1 \le m_J^\pi \le J.$$

Verifying now that $(\beta, \tau) \in S_{inv}(m,h) \times S_{inv}(m,l)$.

The number of inversions in β is:

$$\sum_{j=1}^{J} a_j^\tau + \left(\sum_{j=0}^{J} a_j^\pi - \sum_{j=0}^{J} a_j^\tau - (l-h-1) \right) + \sum_{J+1}^{m-1} a_j^\tau = -l + h - 1 + \sum_{j=1}^{m-1} a_j^\tau$$
$$= -l + h - 1 + l + 1$$
$$= h.$$

And the number of inversions of α is:

$$\sum_{j=1}^{J} a_j^{\pi} - \left(\sum_{j=0}^{J} a_j^{\pi} - \sum_{j=0}^{J} a_j^{\tau} - (l-h-1)\right) + \sum_{J+1}^{m-1} a_j^{\tau} = l-h+1+\sum_{j=1}^{m-1} a_j^{\tau}$$
$$= l-h+1+h-1$$
$$= l.$$

Hence, ($\beta \in S_{inv}(m,h)$ and $\tau \in S_{inv}(m,l)$.

Also, we have

$$\begin{aligned} w_\beta + w_\alpha &= \sum_{j=1}^{J} a_j^{\tau}\cdot j + \left(\sum_{j=0}^{J} a_j^{\pi} - \sum_{j=0}^{J} a_j^{\tau} - (l-h-1)\right)\cdot J + \sum_{j=J+1}^{m-1} a_j^{\pi}\cdot j \\ &+ \sum_{j=1}^{J} a_j^{\pi}\cdot j - \left(\sum_{j=0}^{J} a_j^{\pi} - \sum_{j=0}^{J} a_j^{\tau} - (l-h-1)\right)\cdot J + \sum_{j=J+1}^{m-1} a_j^{\tau}\cdot j \\ &= \sum_{j=1}^{m-1} a_j^{\pi}\cdot j + \sum_{j=1}^{m-1} a_j^{\tau}\cdot j \\ &= w_\pi + w_\tau. \end{aligned}$$

Regarding the injectivity, we have $\mathcal{G}_{m,h,l}^{-1} = \mathcal{G}_{m,l+1,h-1}$, and thus our map $\mathcal{G}_{m,h,l}$ is injective. ∎

To see the application of our map $\mathcal{G}_{m,h,l}$ more clearly, here is the following illustrative example. For $m=4$, $l=5$ and $h=4$.

Let $\pi = 4321 \in S_{inv}(4,6)$, and $\tau = 4123 \in S_{inv}(4,3)$. Indeed π has as inversions: $(4,3)$, $(4,2)$, $(4,1)$, $(3,2)$, $(3,1)$ and $(2,1)$. And τ has as inversions: $(4,1)$, $(4,2)$ and $(4,3)$.

Hence,

$a_1^{\pi} = 1$, $a_2^{\pi} = 2$ and $a_3^{\pi} = 3$,

and

$a_1^{\tau} = 0$, $a_2^{\tau} = 0$ and $a_3^{\tau} = 3$.

We start by finding J:

Let us start by the index 3:

- $a_3^{\pi} \geq a_4^{\tau} + 1$ such that $a_4^{\tau} = 0$, hence (3.2.1) is verified.
- $a_3^{\pi} < a_3^{\tau} + 1$ but $a_3^{\tau} + 1 - a_4^{\pi} = 4 > 3$, hence (3.2.2) is not verified.

Thus, we move on to the index 2:

- $a_2^{\pi} + a_3^{\pi} \geq a_3^{\tau} + 1$, hence (3.2.1) is verified.
- $a_2^{\pi} + a_3^{\pi} < a_2^{\tau} + a_3^{\pi} + 1$ and $a_2^{\tau} + a_3^{\tau} + 1 - a_3^{\pi} = 1 \leq 3$, hence (3.2.2) is verified.

Accordingly, $J = 2$. Now, we obtain $(\beta, \alpha) = \mathcal{G}_{4,4,5}(\pi, \tau)$ as follows

$$\pi = 4123 \to 1234 \to 1324 \to 4132 = \beta,$$

$$\tau = 4321 \to 3214 \to 2314 \to 4231 = \alpha.$$

Thus,

$$\mathcal{G}_{4,4,5}(4321, 4123) = (4132, 4231),$$

such that $\beta \in S_{inv}(4,4)$ and $\alpha \in S_{inv}(4,5)$ has 5 inversions. Indeed, the inversions of β are : $(4,1)$, $(4,3)$, $(4,2)$ and $(3,2)$.
And the inversions of α are: $(4,2)$, $(4,3)$, $(4,1)$, $(2,1)$ and $(3,1)$.

If we take $q \to 1$, the proof above becomes the combinatorial proof of the log-concavity of $i_m(h)$, and furthermore with a constructed injection.

Corollary 3.2.2 *For $\binom{m}{2} > l \geq h > 0$,*

$$i_m(h)i_m(l) - i_m(h-1)i_m(l+1) \geq 0.$$

i.e., the sequence $(i_m(h))_h$ satisfies the log-concavity.

We have seen in Subsection 3.1.2, that any permutation in $S_{inv}(m,h)$ can be interpreted by a path in $\mathcal{P}^I_{m,h}$, thereby we can prove Theorem 3.2.1 using lattice paths, by a similar way to that of Sagan's proof [54] by choosing an appropriate involution as we did in Chapter 2.

Definition 3.2.3 *Let a_1, a_2, b_1 and b_2 four point of the lattice. For two paths Q_1 from a_1 to b_1 and Q_2 from a_2 to b_2, the involution G^I is defined as*

$$G^I(Q_1, Q_2) = (Q'_1, Q'_2)$$

such that

- *if $Q_1 \cap Q_2 = \emptyset$ then $Q'_1 = Q_1$ and $Q'_2 = Q_2$,*
- *if $Q_1 \cap Q_2 \neq \emptyset$ then, G^I_s switches the portions of Q_1 and Q_2 after b_0 as in Figure 3.5, that is*

$$Q'_1 = a_1 \xrightarrow{Q_1} b_0 \xrightarrow{Q_2} b_2 \text{ and } Q'_2 = a_2 \xrightarrow{Q_2} b_0 \xrightarrow{Q_1} b_1,$$

where b_0 is the last intersection vertex of Q_1 and Q_2 such that in the vertical level i of b_0 the number of vertical steps of Q_1 and Q_2 all together does not exceed i.

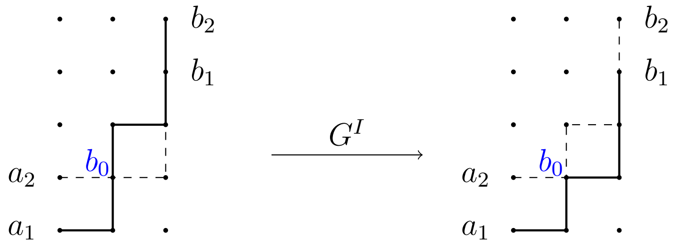


Figure 3.5: The application of G^I.

CHAPTER 4

Two-Catalan numbers and log-convexity

We study in this section the two-Catalan triangle and in particular the two-Catalan numbers, with a recurrence relation and a combinatorial interpretation that leads us to prove the log-convexity of those numbers. Furthermore we demonstrate that the rows of two-Catalan triangle form a log-concave sequence.

4.1 Definition and combinatorial interpretation

For an odd integer s, Belbachir and Igueroufa [14] introduced the s-Catalan number as

$$C_n^{(s)} = \binom{2n}{sn}_s - \binom{2n}{sn+1}_s, \qquad (4.1.1)$$

where $\binom{2n}{sn}_s$ is the central bi^{s}nomial coefficients. Linz [46] generalized the definition of s-Catalan numbers for all positive integers s, and he gave a combinatorial description for these numbers in terms of Littlewood-Richardson coefficients. In this chapter we consider the case of $s = 2$ of the s-Catalan number defined by the relation (4.1.1).

4.1.1 Definition and recurrence relation

Definition 4.1.1 *Let n and k be two positive integers. We define the coefficients of two-Catalan triangle as follows*

$$C_{n,k}^{(2)} := \binom{2n}{2n+k}_2 - \binom{2n}{2n+k+1}_2,$$

for $0 \leq k \leq 2n$.

In particular, we call the coefficients of two-Catalan triangle for $k = 0$ *by "the two-Catalan numbers" denoted* $C_n^{(2)}$,

$$C_n^{(2)} := C_{n,0}^{(2)} = \binom{2n}{2n}_2 - \binom{2n}{2n+1}_2,$$

where $\binom{2n}{2n}_2$ *is the central trinomial coefficient.*

This definition leads us to the following proposition.

Proposition 4.1.2 *The coefficients of the two-Catalan triangle satisfy*

$$\begin{cases} C_{n+1,0}^{(2)} = C_{n,0}^{(2)} + C_{n,1}^{(2)} + C_{n,2}^{(2)}, & (4.1.2) \\ C_{n+1,1}^{(2)} = C_{n,0}^{(2)} + 3C_{n,1}^{(2)} + 2C_{n,2}^{(2)} + C_{n,3}^{(2)}, & (4.1.3) \\ C_{n+1,k}^{(2)} = C_{n,k-2}^{(2)} + 2C_{n,k-1}^{(2)} + 3C_{n,k}^{(2)} + 2C_{n,k+1}^{(2)} + C_{n,k+2}^{(2)}, \text{ for } k \geq 2, & (4.1.4) \end{cases}$$

where $C_{0,0}^{(2)} = 1$ *and* $C_{n,k}^{(2)} = 0$ *unless* $2n \geq k \geq 0$.

Proof From Definition 4.1.1, and by applying the recurrence relation (1.3.3) twice in succession on left side of (4.1.4) we obtain, for $k \geq 2$,

$$\begin{aligned} C_{n+1,k}^{(2)} &= \binom{2n+2}{2n+k+2}_2 - \binom{2n}{2n+2+k+3}_2 = \binom{2n+1}{2n+k}_2 - \binom{2n+1}{2n+k+3}_2 \\ &= \binom{2n}{2n+k-2}_2 + \binom{2n}{2n+k-1}_2 + \binom{2n}{2n+k}_2 - \binom{2n}{2n+k+1}_2 - \binom{2n}{2n+k+2}_2 \\ &\quad - \binom{2n}{2n+k+3}_2 \qquad (4.1.5) \\ &= C_{n,k-2}^{(2)} + 2C_{n,k-1}^{(2)} + 3C_{n,k}^{(2)} + 2C_{n,k+1}^{(2)} + C_{n,k+2}^{(2)}. \end{aligned}$$

To prove (4.1.2) and (4.1.3), it suffices to use the symmetry property (1.3.2) on the right side of (4.1.5) on the term $\binom{2n}{2n+k-2}_2$ for $k = 1$, and on the terms $\binom{2n}{2n+k-2}_2$ and $\binom{2n}{2n+k-1}_2$ for $k = 0$, we obtain respectively:

$$\binom{2n}{2n}_2 + 2\binom{2n}{2n+1}_2 - \binom{2n}{2n+2}_2 - \binom{2n}{2n+3}_2 - \binom{2n}{2n+4}_2$$

and

$$\binom{2n}{2n}_2 - \binom{2n}{2n+3}_2,$$

which are the right side of $C_{n+1,1}^{(2)}$ and $C_{n+1,0}^{(2)}$ respectively after simplifying. ∎

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	**1**										
1	**1**	1	1								
2	**3**	6	6	3	1						
3	**15**	36	40	29	15	5	1				
4	**91**	232	280	238	154	76	28	7	1		
5	**603**	1585	2025	1890	1398	837	405	155	45	9	1

Table 4.1: Two-Catalan Triangle.

Table 4.1 gives us the first values of the two-Catalan triangle.

The two-catalan numbers equal also the Riordan numbers of even indices [17], which are given by the formula

$$r_n = \frac{1}{n+1}\sum_{k=1}^{n-1}\binom{n+1}{k}\binom{n-k-1}{k-1}.$$

They are related to the Catalan numbers by the relation [17]

$$r_n = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}C_j.$$

The Riordan numbers have many combinatorial interpretations, see [17]. And they have as generating function

$$R(x) = \sum_{n\geq 0} r_n x^n = \frac{1+x-\sqrt{1-2x-3x^2}}{2x(1+x)}.$$

The first few Riordan numbers r_n are $\mathbf{1}, 0, \mathbf{1}, 1, \mathbf{3}, 6, \mathbf{15}, 36, \mathbf{91}, 232, \mathbf{603}$. See OEIS [56, A005043].

Then the two-catalan numbers satisfy the following identity:

$$C_n^{(2)} = \sum_{j=0}^{2n}(-1)^j\binom{2n}{j}C_j. \tag{4.1.6}$$

4.1.2 Combinatorial interpretation

According to [17], we have the following combinatorial interpretation for the two-Catalan numbers.

Corollary 4.1.3 *The two-Catalan number $C_n^{(2)}$ counts the number of short bushes with $2n$ edges in which no vertex has outdegree one (i.e., each internal node has at least two edges).*

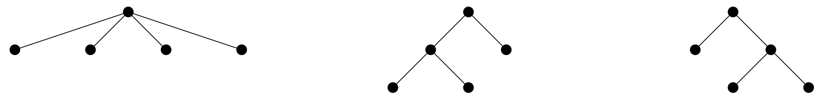


Figure 4.1: The short bushes for $n = 2$.

As an example, the Figure 4.1 shows the three short bushes for $n = 2$.

Irvine et al. [40] introduced the vertically constrained Motzkin-like paths as it is explained in Section 1.2 of Chapter 1. Inspired by these paths, we define a subset of vertically constrained Motzkin-like paths, and in the same way we prove in Theorem 4.1.5 that our paths satisfy the recurrence relation of the coefficients of two-Catalan triangle.

Definition 4.1.4 *Let $\mathcal{A}^Q_{R,2}$ be the set of vertically constrained Motzkin-like paths from $(0,0)$ in the upper right quarter-plane (Q) in which the leading step is not a vertical step, satisfying the condition that in each point (i,j), if $j = 1$ no horizontal step is allowed on the horizontal level $y = 0$ (the x-axis) on the level just before this point, and if $j = 0$ no horizontal step is allowed on the horizontal levels $y = 0$ and $y = 1$ on the level just before this point. We denote by $\mathcal{A}^Q_{R,2}(n,k)$ the set of paths of type $\mathcal{A}^Q_{R,2}$ from $(0,0)$ to (n,k), and by $a^Q_{R,2}(n,k)$ the cardinality of $\mathcal{A}^Q_{R,2}(n,k)$, i.e., $a^Q_{R,2}(n,k)$ counts the number of paths in $\mathcal{A}^Q_{R,2}(n,k)$.*

For instance, the paths of $\mathcal{A}^Q_{R,2}(2,0)$ are shown in Figure 4.2.

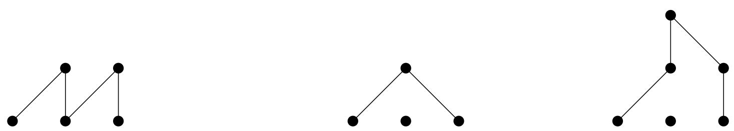


Figure 4.2: The paths of $\mathcal{A}^Q_{R,2}(2,0)$.

Theorem 4.1.5 *Let n and k be two positive integers, then*

$$C^{(2)}_{n,k} = a^Q_{R,2}(n,k).$$

Proof What we are going to do is to prove that $a^Q_{R,2}(n,k)$ satisfies the same recurrence relation of Proposition 4.1.2, and then conclude the equality $a^Q_{R,2}(n,k) = C^{(2)}_{n,k}$. For $k \geq 2$, there are five possible cases as it is illustrated in Figure 4.3. For $k = 1$ (resp. for $k = 0$) and as no horizontal step is allowed on the x-axis (resp. on the horizontal levels $y = 0$ and $y = 1$), we delete any horizontal step there, and of course any step extends below the x-axis because we should not forget that our paths are restricted to the upper right quarter-plane (Q) as it is illustrated in Figure 4.4 (resp. in Figure 4.5). $\blacksquare$

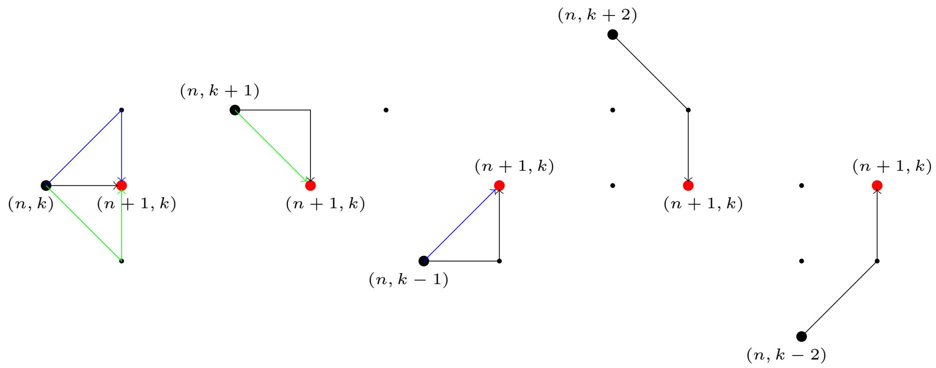


Figure 4.3: The five possible cases for $k \geq 2$.

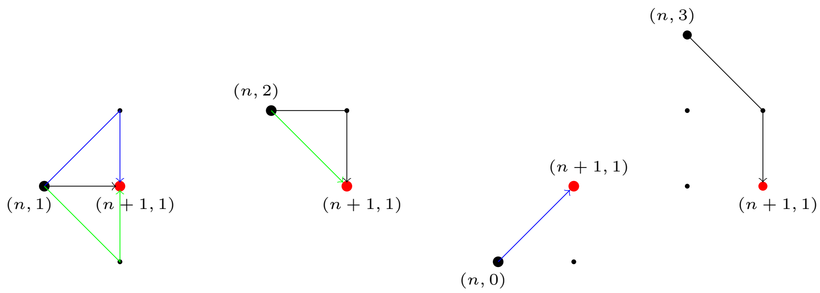


Figure 4.4: The four possible cases for $k = 1$.

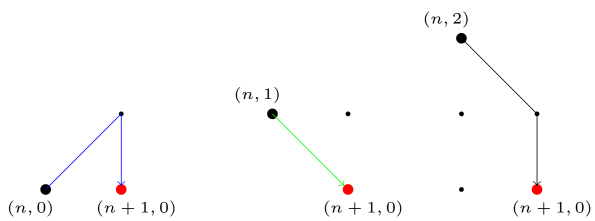


Figure 4.5: The three possible cases for $k = 0$.

From the previous theorem and Definition 4.1.4, we immediately obtain the following combinatorial interpretation.

Corollary 4.1.6 *The two-Catalan numbers $C_n^{(2)}$ counts the number of paths in $\mathcal{A}_{R,2}^{Q}(n, 0)$.*

4.2 Log-convexity of two-Catalan numbers

The first result dealing with unimodality of bi^{s}nomial coefficients is due to Belbachir and Szalay [16] who proved that any ray intersecting Pascal's triangle provides a unimodal sequence. Then, Ahmia and Belbachir in [7, 8, 5] established respectively the strong log-convexity, the unimodality and log-concavity properties for the bi^{s}nomial coefficients.

By constructing injective using combinatorial interpretations by paths, Callan [24] proved the log-convexity of the Motzkin numbers, Liu and Wang [47] proved the same property for the Catalan numbers and Sun and Wang [59] also did the same for the Catalan-like numbers. Then, in a similar way Chen et al. [25] gave a combinatorial proof of the log-convexity of sequences in Riordan arrays. Motivated by these works, we give in this section an injective proof for the log-convexity of the two-Catalan numbers.

Theorem 4.2.1 *The sequence of two-Catalan numbers $C_n^{(2)}$ is log-convex.*

Proof We will construct an injection Γ from $\mathcal{A}^Q_{R,2}(n,0)\times\mathcal{A}^Q_{R,2}(n,0)$ to $\mathcal{A}^Q_{R,2}(n+1,0)\times\mathcal{A}^Q_{R,2}(n-1,0)$. For two paths $(P_1,P_2)\in\mathcal{A}^Q_{R,2}(n,0)\times\mathcal{A}^Q_{R,2}(n,0)$ such that P_1 starts at $(0,0)$ and P_2 starts at $(1,0)$ and in a way inspired by Callan's method, we define our "encounter" in two cases

- Not between $y=0$ and $y=1$, the encounter is:
  - either a lattice point that is common between P_1 and P_2 such there is at most one vertical step linked to this point. For instance, in Figure 4.6a the encounter is an intersection at a lattice point, and in Figure 4.6b the encounter is not the red point because there are two vertical steps linked two this red point but is the green point,
  - or the intersection of two diagonal steps as shown in Figure 4.6c,
  - or a pair of flatsteps forming the top and bottom of a unit square as in Figure 4.6d.
- Between $y=0$ and $y=1$: the encounter is either as in the situation of Figure 4.6a or Figure 4.6b or Figure 4.6c, but the situation of the Figure 4.6d can not be existed, this case is replaced by: the encounter is the shape of "a flatstep of the first path with a North-East step followed by a South step" as shown in Figure 4.8d′.

Obviously, at least one such encounter exists. Now we consider the first encounter in the two previous cases and we define the application Γ in each case as follows

- The first encounter is not between $y = 0$ and $y = 1$:
  - In the situation of Figure 4.6a and Figure 4.6b, switch the paths to the right of the common lattice point as shown in Figure 4.7a and Figure 4.7b respectively.
  - In the situation of Figure 4.6c, swing the two diagonal steps so that they become horizontal steps, and then the paths to the right will be switched as shown in Figure 4.7c.
  - In the situation of Figure 4.6d, change the lower horizontal step to a North-East step and the upper one to a South-East step and then the paths to the right will be switched as shown in Figure 4.7d.
- The first encounter is between $y = 0$ and $y = 1$:
  - If the first encounter is lattice point that is common between P_1 and P_2 such there is at most one vertical step linked to this point. This situation is the same as that of Figure 4.6a and Figure 4.6b, (switch the paths to the right of the common lattice point as shown in Figure 4.7a and Figure 4.7b respectively).
  - In the situation of Figure 4.8c', swing the diagonal step of the first path so that it becomes an horizontal step, and complete the diagonal step of the second path by a South step, and then the paths to the right will be switched as shown in Figure 4.9c'.
  - In the situation of Figure 4.8d', change the flatstep of the first path to a South-East step and then, remove the South step of the second path, then, the paths to the right will be switched as shown in Figure 4.9d'.

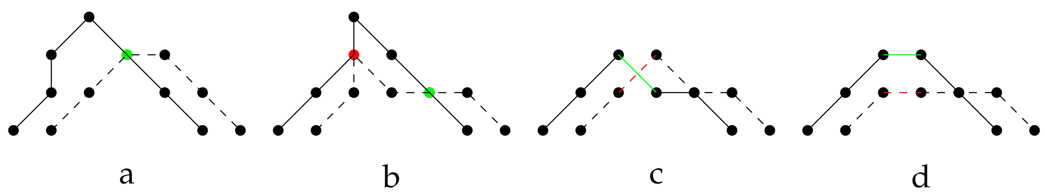


Figure 4.6: The first encounter is not between $y = 0$ and $y = 1$.

In all cases, the resulting pair of paths are in $\mathcal{A}^Q_{R,2}(n+1,0) \times \mathcal{A}^Q_{R,2}(n-1,0)$. Furthermore, the location of the first encounter will remain invariant after applying Γ, thus the mapping is reversible and then Γ is an injection. ∎

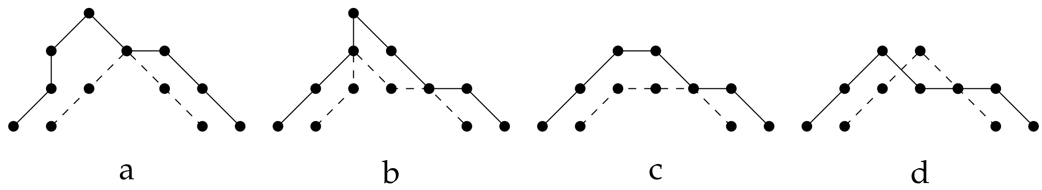


Figure 4.7: The application of Γ_2 on the paths in Figure 4.6.

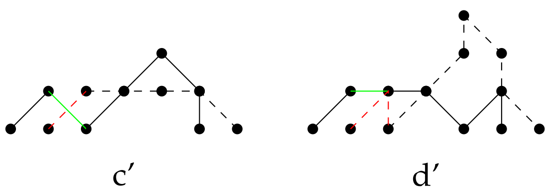


Figure 4.8: The two special cases when the first encounter is between $y = 0$ and $y = 1$.

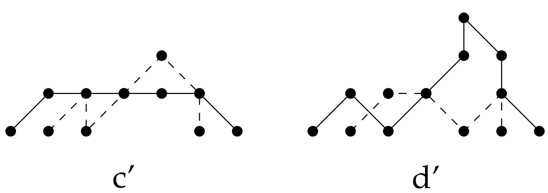


Figure 4.9: The application of Γ on the two above special cases given in Figure 4.8.

We know that the Riordan numbers of even indices equal to the two-catalan numbers. So from the previous theorem, we obtain the following result.

Corollary 4.2.2 *The sequence of Riordan numbers of even indices $\{r_{2n}\}_{n\geq 0}$ is log-convex.*

4.3 Log-concavity of rows of two-Catalan triangle

In this section, we prove the log-concavity of the rows of two-Catalan triangle.

Theorem 4.3.1 *The row sequence of two-Catalan triangle $\left(C_{n,k}^{(2)}\right)_{0\leq k\leq 2n}$ is log-concave.*

Proof To show that $\left(C_{n,k}^{(2)}\right)_{0\leq k\leq 2n}$ is log-concave in k, it suffices to prove that

$$\left(C_{n,k}^{(2)}\right)^2 - C_{n,k-1}^{(2)}C_{n,k+1}^{(2)} \geq 0$$

for any $k \geq 0$, which will be done by induction on n. It is clear for $n = 0$. Thus, we suppose that it follows for $1 \leq n \leq m$. Then for $n = m+1$ and $0 \leq k \leq 2(m+1)$ we have that

$$\begin{aligned}
&\left(C_{m+1,k}^{(2)}\right)^2 - C_{m+1,k-1}^{(2)}C_{m+1,k+1}^{(2)} = \Bigg[\left(C_{m,k-2}^{(2)}\right)^2 + 4\left(C_{m,k-1}^{(2)}\right)^2 + 9\left(C_{m,k}^{(2)}\right)^2 + 4\left(C_{m,k+1}^{(2)}\right)^2 \\
&+ \left(C_{m,k+2}^{(2)}\right)^2 + 4C_{m,k-2}^{(2)}C_{m,k-1}^{(2)} + 6C_{m,k-2}^{(2)}C_{m,k}^{(2)} + 4C_{m,k-2}^{(2)}C_{m,k+1}^{(2)} + 2C_{m,k-2}^{(2)}C_{m,k+2}^{(2)} \\
&+12C_{m,k-1}^{(2)}C_{m,k}^{(2)} + 8C_{m,k-1}^{(2)}C_{m,k+1}^{(2)} + 4C_{m,k-1}^{(2)}C_{m,k+2}^{(2)} + 12C_{m,k}^{(2)}C_{m,k+1}^{(2)} \\
&+6C_{m,k}^{(2)}C_{m,k+2}^{(2)} + 4C_{m,k+1}^{(2)}C_{m,k+2}^{(2)}\Bigg] - \Bigg[C_{m,k-3}^{(2)}C_{m,k-1}^{(2)} + 4C_{m,k-2}^{(2)}C_{m,k}^{(2)} + 9C_{m,k-1}^{(2)}C_{m,k+1}^{(2)} \\
&+4C_{m,k}^{(2)}C_{m,k+2}^{(2)} + C_{m,k+1}^{(2)}C_{m,k+3}^{(2)} + 2C_{m,k-3}^{(2)}C_{m,k}^{(2)} + 2C_{m,k-2}^{(2)}C_{m,k-1}^{(2)} + 3C_{m,k-3}^{(2)}C_{m,k+1}^{(2)} \\
&+ 3\left(C_{m,k-1}^{(2)}\right)^2 + 2C_{m,k-3}^{(2)}C_{m,k+2}^{(2)} + 2C_{m,k-1}^{(2)}C_{m,k}^{(2)} + C_{m,k-3}^{(2)}C_{m,k+3}^{(2)} + C_{m,k-1}^{(2)}C_{m,k+1}^{(2)} \\
&+ 6C_{m,k-2}^{(2)}C_{m,k+1}^{(2)} + 6C_{m,k-1}^{(2)}C_{m,k}^{(2)} + 4C_{m,k-2}^{(2)}C_{m,k+2}^{(2)} + 4\left(C_{m,k}^{(2)}\right)^2 + 2C_{m,k-2}^{(2)}C_{m,k+3}^{(2)} \\
&+ 2C_{m,k}^{(2)}C_{m,k+1}^{(2)} + 6C_{m,k-1}^{(2)}C_{m,k+2}^{(2)} + 6C_{m,k}^{(2)}C_{m,k+1}^{(2)} + 3C_{m,k-1}^{(2)}C_{m,k+3}^{(2)} + 3\left(C_{m,k+1}^{(2)}\right)^2 \\
&+2C_{m,k}^{(2)}C_{m,k+3}^{(2)} + 2C_{m,k+1}^{(2)}C_{m,k+2}^{(2)}\Bigg].
\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}
&\left(C_{m+1,k}^{(2)}\right)^2 - C_{m+1,k-1}^{(2)}C_{m+1,k+1}^{(2)} = \left(\left(C_{m,k-2}^{(2)}\right)^2 - C_{m,k-3}^{(2)}C_{m,k-1}^{(2)}\right) \\
&+ \left(\left(C_{m,k-1}^{(2)}\right)^2 - C_{m,k-2}^{(2)}C_{m,k}^{(2)}\right) + \left(\left(C_{m,k+1}^{(2)}\right)^2 - C_{m,k}^{(2)}C_{m,k+2}^{(2)}\right) \\
&+ 5\left(\left(C_{m,k}^{(2)}\right)^2 - C_{m,k-1}^{(2)}C_{m,k+1}^{(2)}\right) + \left(\left(C_{m,k+2}^{(2)}\right)^2 - C_{m,k+1}^{(2)}C_{m,k+3}^{(2)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ 2\left(C^{(2)}_{m,k-2}C^{(2)}_{m,k-1} - C^{(2)}_{m,k-3}C^{(2)}_{m,k}\right) + 3\left(C^{(2)}_{m,k-2}C^{(2)}_{m,k} - C^{(2)}_{m,k-3}C^{(2)}_{m,k+3}\right)\\
&+ 2\left(C^{(2)}_{m,k-2}C^{(2)}_{m,k+1} - C^{(2)}_{m,k-3}C^{(2)}_{m,k+2}\right) + 4\left(C^{(2)}_{m,k-1}C^{(2)}_{m,k} - C^{(2)}_{m,k-2}C^{(2)}_{m,k+1}\right)\\
&+ \left(C^{(2)}_{m,k-2}C^{(2)}_{m,k+2} - C^{(2)}_{m,k-3}C^{(2)}_{m,k+3}\right) + 3\left(C^{(2)}_{m,k-1}C^{(2)}_{m,k+1} - C^{(2)}_{m,k-2}C^{(2)}_{m,k+2}\right)\\
&+ 2\left(C^{(2)}_{m,k-1}C^{(2)}_{m,k+2} - C^{(2)}_{m,k-2}C^{(2)}_{m,k+3}\right) + 4\left(C^{(2)}_{m,k}C^{(2)}_{m,k+1} - C^{(2)}_{m,k-1}C^{(2)}_{m,k+2}\right)\\
&+ 3\left(C^{(2)}_{m,k}C^{(2)}_{m,k+2} - C^{(2)}_{m,k-1}C^{(2)}_{m,k+3}\right) + 2\left(C^{(2)}_{m,k+1}C^{(2)}_{m,k+2} - C^{(2)}_{m,k}C^{(2)}_{m,k+3}\right) \geq 0
\end{aligned}
$$

since $\left(C^{(2)}_{m,k}\right)_{0\leq k\leq 2m}$ is log-concave. This completes the proof. ∎

Conclusion

Throughout this thesis, we investigated some combinatorial sequences focusing on studying the log-concavity or the log-convexity property, using the bijective combinatorics, and that by finding the appropriate combinatorial interpretation of each sequence.

Firstly, we introduce the over (q,t)-bi^{s}nomial coefficients $\left(\overline{\begin{bmatrix} M \\ K \end{bmatrix}}_{q,t}^{(s)}\right)_N$ as a generating function for the number of overpartitions in which no part appears more than s times, fitting inside the $(M-1)\times K$ rectangle. We study some basic properties including a combinatorial interpretation of these coefficients by three bijective approaches: by overpartitions, by generalized Delannoy paths and by tilling. Basing on the two first approachs we proved the log-concavity of these the sequence of polynomials for M fixed, K fixed and for M and K fixed.

Secondly, we gave a q-analogue of Mahonian numbers that we called "q-Mahonian numbers" noted $p^q_{inv}(m,h)$, and we provided some identities. We established combinatorial interpretations by partitions and paths and consequently by tiling. We have thereby been able to prove combinatorially that the q-analogue of Mahonian numbers is a strong q-log-concave sequence of polynomials by constructing an injection.

Finally, we presented the two-Catalan triangle and we focused on the coefficients of the first column which we called "two-Catalan numbers". We gave a recurrence relation by which we could interpret the two-Catalan numbers by a set of vertically constrained Motzkin-like paths in the upper right quarter-plane (Q) in which the leading step is not a vertical step, satisfying the condition that in each point (i,j), no horizontal step is allowed on the horizontal levels from $y=0$ to $y=1-j$. From that, we proved the log-convexity of these numbers by constructing an injection. Furthermore we proved the log-concavity of rows of two-Catalan triangle.

As research perspectives, we plan to study the following points:

- Log-concavity and unimodality of q-bi^{s}nomial as polynomial in q;

- Log-convexity of central Mahonian numbers $i_n(n-1)$ for $n \geq 1$;

- Log-convexity of s-Catalan numbers for $s \geq 3$.

At the end, it would be very appealing to investigate the following conjectures:

Conjecture 1 *For every positive integers K, n and s, the over (q,t)-bi^{s}nomial coefficient $\overline{\begin{bmatrix} N \\ K \end{bmatrix}}^{(s)}_{q,t}$ is doubly unimodal as a polynomial in q and t.*

Conjecture 2 *For every positive integers K, n and s, the over q-bi^{s}nomial coefficient $\overline{\begin{bmatrix} N \\ K \end{bmatrix}}^{(s)}_{q}$ is unimodal as a polynomial in q.*

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