The People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
University of Mohammed Seddik Benyahia - Jijel

Faculty of Exact Sciences and Informatics
Department of Mathematics

Thesis

Submitted by

Aya Bouabsa

In order to obtain the degree of

LMD Doctorate

Field: *Mathematics.*

Option: *Functional Analysis.*

Subject

Contribution to the study of certain evolution problems

In front of the jury composed of:

President:	*Professor*	*Nouressadat Touafek*	*University of Jijel*
Supervisor:	*Professor*	*Soumia Saïdi*	*University of Jijel*
Reviewers:	*Professor*	*Mohammed Salah Abdelouahab*	*Mila university center*
	Assoc. Prof	*Yacine Halim*	*Mila university center*
	Assoc. Prof	*Ilyas Kecis*	*University of Jijel*
	Assoc. Prof	*Sarra Maarouf*	*University of Jijel*

Academic Year **2023/2024**

Acknowledgment

I would like to express my deepest gratitude to Prof. Soumia Saïdi, my supervisor, for her unwavering support, guidance, and encouragement throughout the entire process of researching and writing this thesis. Her expertise, insightful feedback, and dedication have been invaluable.

I am also immensely thankful to the members of my thesis committee, Prof. Nouressadat Touafek, Prof. Mohammed Salah Abdelouahab, Prof. Yacine Halim, Prof. Ilyas Kecis and Prof. Sarra Maarouf for their constructive criticism, thoughtful suggestions, and valuable contributions that greatly enhanced the quality of this work.

I extend my appreciation to University of Mohammed Seddik Ben Yahia - Jijel for providing the necessary resources and facilities that facilitated the completion of this thesis.

Lastly, I am grateful to my family and friends for their patience, understanding, and continuous support during this academic endeavor. Their encouragement and belief in me have been a constant source of motivation.

Contribution to the study of certain evolution problems

Abstract In this thesis, we are interested in the well-posedness of a new class of differential inclusion driven by time-dependent subdifferential operators with integral perturbation in Hilbert space. In the first part, we handle some coupled systems by such a class and fractional differential equations. This result is obtained using Schauder's fixed point theorem. In the second part, we establish the existence and uniqueness of the solution to a system governed by a differential inclusion involving the subdifferential operator with an integral perturbation and a non-convex perturbed sweeping process. Our approach is based on a discretization method. As an application of this result, a Bolza-type problem in optimal control theory is therefore studied.

Mathematics Subject Classifications. 34A60, 26A33, 34A08, 34G25, 47H10, 49J52, 49J53, 47J35, 28B20, 45J05.

Keywords. Differential inclusion, subdifferential operator, integral perturbation, second-order, fixed point, fractional derivative, dynamical system, sweeping process, r-prox regular moving set, optimal control.

Contribution à l'étude de certains problèmes d'évolution

Résumé Dans cette thèse, nous nous intéressons à l'existence et l'unicité de la solution pour une nouvelle classe d'inclusions différentielles régies par des opérateurs sous-différentiels dépendants du temps avec une perturbation intégrale, dans un espace de Hilbert. Dans la première partie, nous traitons quelques systèmes couplés par cette classe d'inclusion differentielle et des équations différentielles fractionnaires. Ce résultat est obtenu en utilisant le théorème du point fixe de Schauder. Dans la deuxième partie, nous établissons l'existence et l'unicité de la solution pour un système gouverné par une inclusion différentielle régie par des opérateurs sous-différentiels avec une perturbation intégrale et un processus de la rafle perturbé non convexe. Notre approche est basé sur une méthode de discrétisation. Comme application de ce résultat, un problème de type Bolza en théorie du contrôle optimal à été étudié.

Mathematics Subject Classifications. 34A60, 26A33, 34A08, 34G25, 47H10, 49J52, 49J53, 47J35, 28B20, 45J05.

Mots clés. Inclusion différentielle, opérateur sous-différentiel, perturbation intégrale, second-ordre, point fixe, dérivée fractionnaire, système dynamique, processus de la rafle, r-prox regulier, contrôle optimal.

المساهمة في دراسة بعض مشاكل التطور

ملخص في هذه الاطروحة، نُثبت وجود و وحدانية الحل لصنف جديد من الاحتواءات التفاضلية بمؤثر تحت تفاضلي متعلق بالزمن و اضطراب تكاملي، في فضاء هيلبرت. في الجزء الأول، نهتم بالجمل التي تتضمن هذا الصنف من الاحتواءات التفاضلية و معادلات تفاضلية كسرية. نتبع في هذه الدراسة نظرية النقطة الثابتة لشودر. في الجزء الثاني، نثبت وجود و وحدانية الحل لجملة تتضمن احتواء تفاضلي بمؤثر تحت تفاضلي واضطراب تكاملي مع العملية الشاملة غير المحدبة. ننتهج طريقة التجزئة أو تجزئة المجال. كتطبيق للنتيجة المتحصل عليها، ندرس مشكلة من نوع بولزا في نظرية التحكم الامثل.

الكلمات المفتاحية: الاحتواء التفاضلي، المؤثر تحت التفاضلي، الاضطراب التكاملي، الدرجة الثانية، النقطة الثابتة، منتظم محلي، الجملة الديناميكية، المشتقة الكسرية، العملية الشاملة، التحكم الامثل.

Contents

1

Introduction

Differential inclusions driven by the subdifferential operators have been discussed in many works. The problem deals with the study of existence of absolutely continuous solutions for the evolution problem

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) \quad \text{a.e. } t \in I := [T_0, T], \\ x(T_0) = x_0 \in \operatorname{dom}\varphi(T_0, \cdot), \end{cases} \tag{1}$$

where $\partial\varphi(t,\cdot)$ is the subdifferential of a time-dependent proper lower semi-continuous convex function $\varphi(t,\cdot)$ of a Hilbert space H into $\mathbb{R} \cup \{+\infty\}$.

The investigation of such a class involves conditions on the Fenchel conjugate φ^* or the Yosida approximation of $\partial\varphi$ or the Moreau envelope φ_λ see e.g. [8], [22], [49], [60], [61], [77] and [78].

Among the pioneers on the study of (1) is Peralba. He has established the well-posedness of the solution for (1) (see [61]), under an assumption expressed in terms of the conjugate function $\varphi^*(t,\cdot)$ (Peralbas's assumption):

(H_2) there is $a \in W^{1,2}_{\mathbb{R}_+}(I)$ and a ρ-Lipschitz function $k : H \to \mathbb{R}_+$ verifying

$$\varphi^*(t,x) \leq \varphi^*(s,x) + k(x)|a(t) - a(s)| \text{ for all } x \in H, \text{ and } s,t \in I.$$

In particular, the sweeping process (in the convex case) is problem (1), when φ is the indicator function of a non-empty closed convex set $C(t)$, that is, $\partial\varphi(t,\cdot) = N_{C(t)}(\cdot)$.

Later, many authors have added a perturbation (single-valued or set-valued map) to the right member of the inclusion (1), see e.g. [47], [60], [66], [69], [70], [76].

In recent years, an integral perturbation has occurred in the sweeping process called, integro-differential sweeping process, which has been investigated see e.g, [16], [17], [18], [19], [40] and [52], while integro-differential inclusions for m-accretive (or maximal monotone) operators have been studied in [25], [29].

In the light of the aforementioned works, our main concern in this thesis is the class of first-order problem described by subdifferential operators with integral perturbation

$$(FOP)\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \displaystyle\int_0^t f(t,s,x(s))ds \quad \text{a.e. } t \in I := [0,1], \\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

under (H_2) above, where $f : I \times I \times H \to H$ is a suitable map.
Then, by a reduction to the appropriate first order differential inclusion and an adoption of the methods used in the study of (FOP), we handle the second-order problem

$$(SOP)\begin{cases} -\ddot{x}(t) \in \partial\varphi(t,\dot{x}(t)) + \int_0^t f(t,s,x(s),\dot{x}(s))ds \quad \text{a.e. } t \in I, \\ x(0) = x_0,\ \dot{x}(0) = v_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

So, in Section 1 and Section 2 of the third chapter, we study the well-posedness of (FOP) and (SOP) by using Schauder's fixed point theorem, this method has been also used for integro-differential with m-accretive (or maximal monotone) operator see [25], [29]. However, a discretization technique has been found in e.g. [17], [18] for integro-differential sweeping process.

Recent works on second-order problems of subdifferential type (under (H_2)) have been published in [65], [68]. See [2], [3] and [9] for some results concerning second order evolution problems with other type of subdifferentials.

Nowadays, there is an intensive activity around the significant mathematical advances in the subject of fractional differential theory and its applications, see [4], [5], [6], [12], [26], [30], [48], [62], [71], among others. Actually, coupled systems driven by evolution problems of subdifferential type and fractional differential equations have been studied in [33] and [66]. We therefore try to establish new researches in the novel setting of coupled fractional differential equations by evolution problems involving subdifferentials with integral perturbations in the last part of the chapter. The method used there is based on Schauder's fixed point approach.
The first one concerns the fractional differential inclusion with nonlocal boundary conditions

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \int_0^t f(t,s,u(s),x(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha u(t) + \lambda D^{\alpha-1}u(t) = x(t) \quad \text{a.e. } t \in I, \\ I_{0+}^\beta u(t)|_{t=0} = 0,\ u(1) = I_{0+}^\gamma u(1), \\ \ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

where $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0$, $\gamma > 0$, and $D^\alpha u$ stands for the Riemann-Liouville fractional derivative of u.
The second one deals with the fractional differential inclusion with integral boundary conditions as follows

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \int_0^t f(t,s,u(s),x(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha u(t) = x(t) \quad \text{a.e. } t \in I, \\ u(0) = 0,\ D^\alpha u(0) = b, \\ D^{\alpha-1}u(t) = \int_0^t x(s)ds + b, \\ \ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

However, related systems with maximal monotone operators (instead of subdifferentials) have been discussed in [27], [28], [33]. For fractional order boundary problems involving differential inclusions governed by m-accretive operators, we refer the reader to [25].

We are interested in the last part of the third chapter in the following m-points boundary problem, again by using schauder's fixed point theorem, we show the well-posedness of

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \int_0^t f(t,s,u(s),x(s))ds \quad \text{a.e. } t \in I, \\ \ddot{u}(t) + \gamma\dot{u}(t) = x(t) \quad \text{a.e. } t \in I, \\ u(0) = c, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \\ \ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

Sweeping process was introduced by Jean-Jacques Moreau in the seventies via a number of papers (see for instance [55], [56] and [57]), in the form

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) \quad \text{a.e. } t \in [T_0, T], \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{2}$$

where $N_{C(t)}(\cdot)$ denotes the normal cone of $C(t)$ in the sense of convex analysis. Application to elastoplasticity served as the primary source of motivation for Moreau but over time, it has been clear that the sweeping process is crucial for a wide range of applications to diverse issues in mechanics hysteresis system, traffic equilibria, social and economic models, etc.

Indeed, there is a variety of methods for determining whether solutions to (2) exist in the literature, catching-up method [54], regularization procedure [59], reduction to unconstrained differential inclusion [75]. For the first and the second techniques, the well-posedness of the solution are obtained by using Gronwall inequality, the statement

$$\frac{d}{dt}\|x(t)\|^2 = 2\langle \dot{x}(t), x(t)\rangle,$$

and the monotonocity of $N_{C(t)}(\cdot)$ (or the hypomonotonicity when the set $C(\cdot)$ is prox-regular).

In the fourth chapter, we deal with a perturbed coupled system governed by a sweeping process and a time-dependent subdifferential with integral perturbation of the form

$$(CP)\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \displaystyle\int_{T_0}^{t} f(t,s,x(s),u(s))ds \quad \text{a.e. } t \in I = [T_0,T], \\ -\dot{u}(t) \in N_{C(t)}(u(t)) + g(t,x(t),u(t)) \quad \text{a.e. } t \in I, \\ \ x(T_0) = x_0 \in \operatorname{dom}\varphi(T_0,\cdot),\ u(T_0) = u_0 \in C(T_0), \end{cases}$$

where $f : I \times I \times H \times H \to H$ and $g : I \times H \times H \to H$ are single-valued mappings.

Let us cite some examples on such a topic in the scientific literature: two sub-smooth sweeping processes have been addressed by the authors of [7], who have shown the existence of solutions for the coupled problem, while two prox-regular sweeping processes have been discussed by using an appropriate mixed catching-up technique and an application of the fixed point theorem of Schauder in [58]. In [13] a system of a closed convex sweeping process and a differential inclusion by maximal monotone operators have been discussed. Whilst, two differential inclusions by maximal monotone operators have been investigated in the recent contributions [46] and [64].

In the last part of the fourth chapter, we consider the optimization problem

$$\text{Minimize} \quad \int_0^T J_0(t,x(t),u(t),z(t),\dot{x}(t),\dot{u}(t),\dot{z}(t))dt,$$

over the set of controls $z(\cdot)$ and the corresponding solutions $(x(\cdot),u(\cdot))$ of

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \displaystyle\int_0^t f(x(s),u(s))ds \quad \text{a.e. } t \in [0,T], \\ -\dot{u}(t) \in N_{C(t)}(u(t)) + g(x(t),u(t),z(t)) \quad \text{a.e. } t \in [0,T], \\ \ z(\cdot) \in W^{1,2}_{\mathbb{R}^d}(I), \\ \ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot),\ u(0) = u_0 \in C(0), \end{cases}$$

where the map f (resp. g) depends on two (resp. three) (time)-variables, the control function $z(\cdot)$ acts in the perturbation g, and J_0 is the cost functional.

By combining the properties of the solution set of the coupled controlled system with the necessary assumptions on the cost functional J_0, we succeed to show that there are optimal solutions to the control problem of minimizing the cost functional of Bolza type.

It turns out that the class of perturbed sweeping processes including control actions and optimization (or control problems governed by maximal monotone operators) has attracted attention of many researchers, e.g. [1], [23], [24], [37], [38], [39], [41], [42], [43]. Results corresponding to minimizing a Bolza type functional have been developed by the authors in [16], for dynamical systems that are controlled by integro-differential sweeping processes.

This thesis consists of four chapters. In chapter 2, we recall some important preliminaries and auxiliary results. In chapter 3, we prove the well-posedness of (FOP), (SOP) and some fractional problems. Chapter 4, is devoted to study (CP), and an application to a optimal control theory.

Chapter 3 has been the subject of publication [14], while chapter 4 has been published in [67]. During the realization of this thesis, another manuscript has been achieved in [15].
This work was carried out within the LMPA Laboratory (Laboratory of Pure and Applied Mathematics) at the University of Jijel.

2

Preliminaries and auxiliary results

Contents

In this chapter, we give the notations, definitions and some concepts of functional analysis used throughout this thesis.

2.1 Notations, basic functions and usual spaces

In all that comes, we will adopt the following notations. Let H be a real separable Hilbert space, $\langle \cdot, \cdot \rangle$ its inner product and $\| \cdot \|$ its norm in H.
We denote by

- P_S the projection onto the non-empty subset S of H.
- $x_n \to x$ the sequence $(x_n)_{n\in\mathbb{N}}$ converges strongly to x.
- $x_n \rightharpoonup x$ the sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to x.
- $\liminf\limits_{n\to\infty} x_n$ denotes the inferior limit and of the sequence $(x_n)_{n\in\mathbb{N}}$.
- $\limsup\limits_{n\to\infty} x_n$ denotes the superior limit of the sequence $(x_n)_{n\in\mathbb{N}}$.
- $\overline{B}_H[x,r]$ the closed ball of center x and radius r of the space H.
- $\overline{B}_H$ the closed unit ball of center 0 and radius 1 of the space H.
- $\overline{co}(S)$ the closed convex hull of the set S.
- $|\cdot|$ denotes the absolute value.

Let X be a non-empty set

- $P(X)$ the power set of X.
- $\mathcal{L}(X)$ the Lebesgue σ-algebra of X.
- $\mathcal{B}(X)$ the Borel σ-algebra of X.
- $\dot{u}$ the derivative of the function u.

- $\Leftrightarrow$ the equivalence.
- $\Rightarrow$ the implication.
- $\forall$ for all.
- $\exists$ there exists ($\exists!$ there is a unique).
- a.e. almost everywhere.
- i.e. identically equivalent.
- Re real part.

Let $I := [T_0, T]$ such that $0 \leq T_0 < T < +\infty$ be an interval of $\mathbb{R}$. Let $p \in [1, +\infty[$ we denote by

- $L^p_H(I)$ the space of measurable maps $x : I \to H$ such that $\int_I \|x(t)\|^p dt < +\infty$ endowed with the usual norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$.
- $L^\infty_H(I)$ the space of measurable maps $x : I \to H$ which are essentially bounded endowed with the usual norm $\|x\|_{L^\infty_H(I)} = \inf\{c \geq 0 : \ \|x(t)\| \leq c \text{ a.e. in } I\}$.
- $W^{1,p}_H(I)$ the space of absolutely continuous in I such that $\dot{u} \in L^p_H(I)$.
- $W^{m,p}_H(I)$ the space of absolutely continuous functions in I with $D^\alpha u \in L^p_H(I)$ such that $\forall\, \alpha,\ |\alpha| \leq m$.
- $\mathcal{C}_H(I)$ the space of continuous maps $x : I \to H$ endowed with the uniform convergence norm $\|x\|_\infty = \sup\limits_{t \in I} \|x(t)\|$.

Define the distance function $d : H \times H \to \mathbb{R}_+$ by

$$d(a,b) = \|a - b\| \text{ for all } a, b \in H.$$

The distance of a to S of H is given by

$$d(a,S) = \inf_{b \in S} \|a - b\|.$$

The indicator function of a subset S of H is defined by

$$\psi_S(y) = \begin{cases} 0 & \text{if } y \in S \\ +\infty & \text{if } y \notin S. \end{cases}$$

Define the support function by

$$\delta^*(x, S) = \sup_{y \in S} \langle y, x \rangle, \qquad \forall\, x \in H.$$

Let $f : H \to \mathbb{R}$ be a function, we denote by

$$\mathrm{dom}\,(f) = \{x \in H : f(x) < +\infty\},$$

the effective domain of the function f. We say that f is proper if its effective domain is non-empty.

The Gamma function is defined by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} \exp(-t) dt \qquad \forall z \in \mathbb{C},\ Re(z) > 0.$$

2.2 Some results of functional analysis

In this section we are going to review a few basic notions in functional analysis, we define continuity, and compactness results. These results were taken from [10], [20], [44], [73] and [74].

2.2.1 Continuity of applications

Let us now give the definitions of the continuity in metric spaces. Assume that $(X, d_X), (Y, d_Y)$ are two metric spaces.

Definition 2.2.1. A function $f : X \to Y$ is continuous at a point $x_0 \in X$ if and only if

$$\forall \epsilon > 0\, \exists\, \delta > 0\, \forall x \in X,\ d_X(x_0, x) < \delta \Rightarrow d_Y(f(x_0), f(x)) < \epsilon.$$

The function f is continuous on X if it is continuous at every $x \in X$.

Proposition 2.2.1. ***(Sequentially continuous function)***

Let $f: X \to Y$ be a function, then one has

$$\Big(f \text{ continuous at a point} x_0 \in X\Big) \Leftrightarrow \Big(\forall (x_n) \subset X: \lim_{n\to\infty} x_n = x_0 \Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)\Big).$$

Definition 2.2.2. (**Lipschitz continuous function**)

A function $f: X \to Y$ is said to be Lipschitz-continuous if there exists a real constant K such that,

$$\forall x_1, x_2 \in X: \ d_Y(f(x_1), f(x_2)) \le K d_X(x_1, x_2).$$

Any such K is referred to f as a Lipschitz constant for the function f.

If $0 \le K \le 1$, then, f is called a contraction.

Definition 2.2.3. (Upper semi continuous function)

A function $f: X \to \mathbb{R}$ is called upper semi continuous at a point $x_0 \in X$ if for every real number $y > f(x_0)$ there exists a neighborhood V of x_0 such that $f(x) < y \ \forall x \in V$.

Equivalently f is upper semi continuous at x_0 if and only if

$$\limsup_{x\to x_0} f(x) \le f(x_0).$$

Definition 2.2.4. (Lower semi continuous function)

A function $f: X \to \mathbb{R}$ is called lower semi continuous at a point $x_0 \in X$ if for every real number $y < f(x_0)$ there exists a neighborhood V of x_0 such that $f(x) > y \ \forall x \in V$.

Equivalently f is lower semi continuous at x_0 if and only if

$$\liminf_{x\to x_0} f(x) \ge f(x_0).$$

Definition 2.2.5. (Absolutely continuous function)

A function $f: [c,d] \to X$ is absolutely continuous if for all $\epsilon > 0$ there is $\delta > 0$ such that for any countable collection of disjoint sub-intervals $[c_k, d_k]$ of $[c,d]$ such that $\sum_{k\in\mathbb{N}}(d_k - c_k) < \delta$, we have $\sum_{k\in\mathbb{N}} d_X(f(d_k), f(c_k)) < \epsilon$.

Theorem 2.2.1. *A function $f: [c,d] \to X$ is absolutely continuous if and only if there is an integrable function $g: [c,d] \to X$ verifying for all $s \in [c,d]$*

$$f(s) - f(c) = \int_c^s g(r) dr.$$

In this case f is derivable almost everywhere and its derivative $\dot{f} = g$ a.e.

Definition 2.2.6. (Equi-continuous family of functions)

Let $\mathcal{H}$ be a family of functions $f : X \to Y$, $\mathcal{H}$ is equi-continuous at a point $x_0 \in X$ if

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in X,\ \forall f \in \mathcal{H},\ d_X(x, x_0) < \delta \ \Rightarrow\ d_Y(f(x), f(x_0)) < \epsilon.$$

The family $\mathcal{H}$ is equi-continuous if it equi-continuous at each point of X.

2.2.2 Some results on compactness in metric spaces

Now, we turn our attention to define some properties of compactness in metric spaces.

Definition 2.2.7. A subset $A \subset X$ is called compact if every sequence in A has a subsequence converging in A. The space (X, d) is compact if X is a compact set.

Proposition 2.2.2. *Every compact set A in a metric space (X, d) is closed and bounded.*

Proposition 2.2.3. *A closed subset A of a compact set K is compact.*

Proposition 2.2.4. *If X is a finite dimensional metric space, then a subset A of X is compact if and only if it is closed and bounded.*

Proposition 2.2.5. *Let $f : A \to Y$ be continuous function. Let A be a compact set in a metric space X. Then one has, $f(A)$ is compact in Y.*

Definition 2.2.8. A relatively compact subset A of a metric space X is a subset whose closure is compact.

Definition 2.2.9. (Relatively ball compact set)

We say that a subset $A \subset X$ is relatively ball-compact if and only if its intersection with any closed ball of X is relatively compact.

Theorem 2.2.2. ***(Ascoli-Arzelà theorem)***

Let (X, d_X) be a compact metric space and let (Y, d_Y) be a complete metric space. Then, a subset $\mathcal{H}$ is relatively compact in $\mathcal{C}_Y(X)$ if the following conditions hold true

- *$\mathcal{H}$ is equi-continuous;*
- *$\forall x \in X,\ H(x) = \{f(x),\ f(\cdot) \in \mathcal{H}\}$ is relatively compact in Y.*

We shall use the following consequence of Ascoli-Arzelà theorem.

Theorem 2.2.3. *Let X be a finite dimensional Banach space. Let us consider a sequence of absolutely continuous functions $x_k : I \to X$ satisfying*

- *$\forall t \in I,\ (x_k(t))_k$ is a relatively compact subset;*
- *there exists a non-negative function $\phi(\cdot) \in L^1_{\mathbb{R}}(I)$ such that, for all $t \in I$: $\|\dot{x}_k(t)\| \leq \phi(t)$.*

Then, there exists a subsequence (denoted by $(x_k(\cdot))$) converging to an absolutely continuous function $x(\cdot) : I \to X$ in the sense that

- *$(x_k(\cdot))$ converges uniformly to $x(\cdot)$ over compact subsets of I,*
- *$(\dot{x}_k(\cdot))$ converges weakly to $\dot{x}(\cdot)$ in $L^1_X(I)$.*

Theorem 2.2.4. ***(Schauder's fixed point theorem)***
Assume K is a closed, bounded, non-empty, convex subset of a Banach space F. Let $f : K \to K$ be a map that is continuous. It follows that f has a fixed point if $f(K)$ is relatively compact.

2.2.3 Hilbert spaces

Definition 2.2.10. Let H be a vector space over the field $\mathbb{K}$. Let $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$ be a map, such that for all $x, y, z \in H$ and for all $a, b \in \mathbb{K}$, one has

- $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$;
- $\overline{\langle x, y \rangle} = \langle y, x \rangle$;

- $\langle x,x\rangle = 0 \Leftrightarrow x = 0$;
- $\langle x,x\rangle \geq 0$.

We say that $\langle\cdot,\cdot\rangle$ is the inner product on H, and the couple $(H,\langle\cdot,\cdot\rangle)$ is an inner product space.

Corollary 2.2.1. *Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space and for all $x \in H$: $\|x\| = \sqrt{\langle x,x\rangle}$. Then, $\|\cdot\|$ is a norm on H called Hilbertian norm and $(H,\|\cdot\|)$ is a normed vector space.*

Definition 2.2.11. A Hilbert space is an inner product space $(H,\langle\cdot,\cdot\rangle)$ complete with respect to its Hilbertian norm.

Remark 2.2.1. If $\mathbb{K} = \mathbb{R}$, one calls that H is a real Hilbert space.

Theorem 2.2.5. ***(Cauchy-Schwartz)***
Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space. Then for all $x,y \in H$, one has

$$|\langle x,y\rangle| \leq \|x\|\|y\|.$$

Let us give the particular case of Cauchy-Schwartz inequality in $L^2_H(I)$ and $\mathbb{R}^n$ respectively.
Let $f,g \in L^2_{\mathbb{R}}(I)$, one has

$$\int_I f(x)g(x)dx \leq (\int_I f(x)^2 dx)^{\frac{1}{2}}(\int_I g(x)^2 dx)^{\frac{1}{2}}.$$

Let $x = (x_1,x_2,\cdots,x_n), y = (y_1,y_2,\cdots,y_n) \in \mathbb{R}^n$, one has

$$|\sum_{i=1}^{n} x_i y_i| \leq \Big(\sum_{i=1}^{n} x_i^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^{n} y_i^2\Big)^{\frac{1}{2}}.$$

Lemma 2.2.1. *Let $y_j \in \mathbb{R}$, $j = 1,\cdots,m$. Then*

$$\left(\sum_{j=1}^{m} y_j\right)^2 \leq m\sum_{j=1}^{m} y_j^2.$$

Proof. Let $x_j,\, y_j \in \mathbb{R}$ $j = 1, \cdots, m$. by using the inequality above

$$\left(\sum_{j=1}^{m} x_j y_j\right)^2 \leq \left(\sum_{j=1}^{m} x_j^2\right)\left(\sum_{j=1}^{m} y_j^2\right).$$

Therefore, letting $x_j = 1$ for all $j = 1, \cdots, m$, yields the required inequality. □

Theorem 2.2.6. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Let A be a non-empty convex closed subset in H, then*

$$\forall x \in H\ \exists! y \in A:\ \|x-y\| = d(x,A) = \inf_{z\in A} \|x-z\|,$$

y is the projection of x onto A, denoted $P_A(x)$.

Proposition 2.2.6. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Let $A \subset H$ be a closed subvector space in H and $y \in H$, then we have*

$$y = P_A(x) \Leftrightarrow \langle x-y, z\rangle = 0\ \forall z \in A.$$

2.3 Weak topology and weak star topology

We give some definitions of the weak topology and weak star topology. For more details we refer to [20] and [51].

2.3.1 Topological vector spaces

Definition 2.3.1. (Topological vector space)

Let θ be a topology defined on the vector space X such that

- $\forall\, x \in X$, $\{x\}$ is closed;
- $f : X \times X \to X$ defined by $f(x,y) = x + y$ and the function $g : \mathbb{R} \times X \to X$ defined by $g(x) = \lambda x$ are continuous with respect to $\theta \times \theta$ and $\tau_{\mathbb{R}} \times \theta$ respectively.

Then, the couple (X, θ) is called a topological vector space.

Theorem 2.3.1. *All topological vector spaces are Hausdorff.*

2.3.2 Weak topology

Let X be a set and let $(Y_i)_{i\in J}$ be a collection of topological spaces and $(f_i)_{i\in J}$ is a collection of maps such that $f_i : X \to Y_i$.
In this section, we want to define a weak topology on X that makes all the functions (f_i) $(i \in J)$ continuous.

Corollary 2.3.1. *The collection of all unions of finite intersection of sets of the form $f_i^{-1}(O_i)$ where $i \in J$ and O_i is an open set in Y_i is a topology, it is called the weak topology on X generated by $(f_i)_{i\in J}$, the functions $(f_i)_{i\in J}$ are continuous for this topology.*

Definition 2.3.2. Let $(X, \|\cdot\|)$ be a real normed vector space, we denote by $X^{'}$ the dual space of X endowed by

$$\|f\|_{X'} = \sup_{f\in\overline{B}_X} |f(x)|, \ \forall f \in X^{'}.$$

Definition 2.3.3. (The weak topology)
Let $f \in X^{'}$ and let

$$\psi_f : X \to \mathbb{R}$$
$$x \mapsto \psi_f(x) = f(x) = \langle f, x\rangle_{X',X}.$$

The weak topology that makes all ψ_f continuous is called the weak topology on X, and it is denoted by $\sigma(X, X^{'})$.

Proposition 2.3.1. *The topology $\sigma(X, X^{'})$ is Hausdorff.*

Proposition 2.3.2. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, then we have*

- *A sequence $(x_n)_{n\in\mathbb{N}}$ converges weakly to x if and only if $\forall f \in X^{'} : \lim_{n\to\infty} \langle f, x_n\rangle_{X',X} = \langle f, x\rangle_{X',X}$.*
- *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging weakly to x then $(\|x_n\|)_{n\in\mathbb{N}}$ is bounded and we have*

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

2.3.3 The weak star topology

Let $(X, \|\cdot\|)$ be a normed vector space, $X^{'}$ its dual and $X^{''}$ its bidual endowed with the norm

$$\|f\|_{X''} = \sup_{g\in \overline{B}_{X'}} |\langle f, g\rangle_{X'',X'}|.$$

In $X^{'}$ we already have two topologies the strong topology $\tau_{\|\cdot\|_{X'}}$ and the weak topology $\sigma(X^{'}, X^{''})$.

Now, we define the third topology on $X^{'}$.

Definition 2.3.4. (Weak star topology)

Let $x \in X$ and let

$$\psi_x : X^{'} \to \mathbb{R}$$
$$f \mapsto \psi_x(f) = \langle f, x\rangle_{X',X}.$$

The weak topology that makes all the function ψ_x continuous on $X^{'}$ is called the weak star topology and is denoted by $\sigma(X^{'}, X)$.

Proposition 2.3.3. *The space $(X^{'}, \sigma(X^{'}, X))$ is Hausdorff.*

Proposition 2.3.4. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $X^{'}$, then we have*

- *A sequence $(f_n)_{n\in\mathbb{N}} \subset X^{'}$ converges weakly to f if and only if*

$$\forall x \in X: \lim_{n\to\infty} \langle f_n, x\rangle_{X',X} = \langle f, x\rangle_{X',X}.$$

- *If $(f_n)_{n\in\mathbb{N}}$ converges weakly to f, then $(\|f_n\|)_{n\in\mathbb{N}}$ is bounded and we have*

$$\|f\| \leq \liminf_{n\to\infty} \|f_n\|_{X'}.$$

Proposition 2.3.5. *If X is a finite dimensional vector space then, one has*

$$\tau_{\|\cdot\|_{X'}} = \sigma(X^{'}, X^{''}) = \sigma(X^{'}, X).$$

2.4 Convex analysis

This section is devoted to the definitions and some properties of convex analysis. For more details see [10] and [34].

Definition 2.4.1. Let X be a vector space. Let $A \subset X$ be a subset of X. Then one has

$$A \text{ is convex } \Leftrightarrow \Big(\forall\, a,b \in A,\ \forall\, \lambda \in [0,1],\ \lambda a + (1-\lambda)b \in A\Big).$$

Definition 2.4.2. We define the simplex of $\mathbb{R}^n$ by

$$\Delta_n = \Big\{v = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n;\ v_i \geq 0 \text{ and } \sum_{i=1}^{n} v_i = 1\Big\}.$$

Definition 2.4.3. Let X be vector space and assume that $x_1, \cdots, x_n \in X$. An expression of the form $\sum_{i=1}^{n} v_i x_i$ is said to be a convex combination of the vectors $x_1, x_2, \cdots, x_n$ such that $v = (v_1, v_2, \cdots, v_n) \in \Delta_n$.

Definition 2.4.4. Let X be a vector space and let $A \subset X$ be a subset of X. Then, one has A is convex if and only if A is the set of its all convex combinations.

Definition 2.4.5. Let X be a vector space. Let $A \subset X$ be a subset of X. The convex hull of A which is denoted by $co(A)$, is the intersection of all convex subsets of X which contain A. In other words, $co(A)$ is the smallest convex subset of X containing A. If A is convex then $co(A) = A$.

Theorem 2.4.1. *The closed convex hull of A can be formulated as*

$$\overline{co}(A) := \{x \in X,\ \langle x, x^{'} \rangle \leq \delta^*(x^{'}, A)\ \forall\, x^{'} \in X^{'}\}.$$

Definition 2.4.6. Let X be a vector space and let $f : X \to \overline{\mathbb{R}}$. The function f is convex if and only if

$$\forall a, b \in \operatorname{dom}(f),\ \forall \lambda \in [0,1] :\ f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b).$$

Definition 2.4.7. Let A be a subset of X. Then

- ψ_A is proper if and only if $A \neq \emptyset$;

- ψ_A is convex if and only if A is convex;
- ψ_A is lower semi continuous if and only if A is closed.

2.5 Some compactness theory

In this section, we are going to expose some results which will be useful to us in our existence results, see [10], [20] and [21].

Let $(X, \|\cdot\|)$ be a Banach space.

Theorem 2.5.1. ***(Banach-Alaoglu-Bourbaki)***

The unit ball $\overline{B}_{X'}$ is weakly star compact.

Proposition 2.5.1. *Let $A \in X$ be a non-empty convex subset. Then, the subset A is weakly closed in X if and only if it is strongly closed in X.*

Theorem 2.5.2. ***(Banach-Mazur)***

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X which converges weakly to x. Then, there exists a $(y_i)_{i\in\mathbb{N}} \subset X$, such that (y_i) is a convex combination of the sequence $(x_n)_n$ and (y_i) strongly converges to x.

Proposition 2.5.2. *Let X be a separable normed vector space. Let A be a subset of $X^{'}$ which is weakly star compact. Then, $(A, \sigma_A(X^{'}, X))$ is metrizable. where $\sigma_A(X^{'}, X)$ denotes the weak star topology of $\sigma(X^{'}, X)$ on A.*

Remark 2.5.1. The space $(X^{'}, \sigma(X^{'}, X))$ is not metrizable because $X^{'}$ is not weakly star compact.

Proposition 2.5.3. ***(Reflexive spaces)***

Let X be a normed vector space. If X is reflexive, then, we identify X with $X^{''}$.

Theorem 2.5.3. *Let $(X, \|\cdot\|)$ be a Banach space. Then, X is reflexive if and only if $\overline{B}_X$ is weakly compact.*

Theorem 2.5.4. *Let X be a reflexive Banach space and let $(x_n)_n$ be a bounded sequence in X. Then there exists a subsequence $(x_{n_k})_k$ that converges in the weak topology $\sigma(X, X^{'})$.*

2.6 Some notions of measure theory

In this section, we give some results on measurability. For more details we refer to [72].

Definition 2.6.1. ($\sigma-$algebra)

Let X be an arbitrary set. A collection $\mathcal{T}$ of subsets of X is a $\sigma-$algebra on X if

- $X \in \mathcal{T}$,
- for each set B of $\mathcal{T}$, the set $X \setminus B \in \mathcal{T}$,
- $\forall\, n \in \mathbb{N}$, $B_n \in \mathcal{T}$, one has $\cup_{n\in\mathbb{N}} B_n \in \mathcal{T}$.

Then, we call the couple $(X, \mathcal{T})$ a measurable space.

Lemma 2.6.1. *Let $(\mathcal{T}_j)_{j\in J}$ be a family of $\sigma-$algebras defined on X, then $\mathcal{T} = \cap_{j\in J}\mathcal{T}_j$ is a $\sigma-$algebra on X.*

Corollary 2.6.1. *Let X be a set and let $\mathcal{A}$ be a family of subsets of X. Then there is a smallest $\sigma-$algebra on X that contains $\mathcal{A}$ called the $\sigma-$algebra generated by $\mathcal{A}$ and is denoted by $\sigma(\mathcal{A})$.*

Definition 2.6.2. (Borel $\sigma-$algebra)

The Borel $\sigma-$algebra on $\mathbb{R}^d$ is the $\sigma-$algebra on $\mathbb{R}^d$ generated by open sets of $\mathbb{R}^d$ and it is denoted by $\mathcal{B}(\mathbb{R}^d)$.

If X is a topological space, then the smallest $\sigma-$algebra on X is the Borel $\sigma-$algebra denoted $\mathcal{B}(X)$.

Definition 2.6.3. (**Measurable function**)

Let $(X,\mathcal{T}_1)$, $(Y,\mathcal{T}_2)$ be two measurable spaces. A function $f:(X,\mathcal{T}_1)\to(Y,\mathcal{T}_2)$ is said to be measurable if $\forall\, B\in\mathcal{T}_2,\ f^{-1}(B)\in\mathcal{T}_1$.

If Y is a topological space then the measurable function f is called Borel function.

Definition 2.6.4. Let $(X,\mathcal{T})$ be a measurable space. Let Y be a metric space, then an application $f:X\to Y$ is called Bochner measurable if f is a Borel function and we have $f(X)$ is separable.

Proposition 2.6.1. *All continuous functions are measurable.*

Definition 2.6.5. Let $(X,\mathcal{T})$ be a measurable space. A function $\mu:\mathcal{T}\to\overline{\mathbb{R}}_+$ such that

- $\mu(\emptyset)=0$,
- μ is $\sigma-$additive i.e. for each sequence $(A_n)_n$ of pairwise disjoint sets that belongs to $\mathcal{T}$, we have $\mu(\cup_{n\in\mathbb{N}}A_n)=\sum_{n=1}^{\infty}\mu(A_n)$.

Then the triplet $(X,\mathcal{T},\mu)$ is often called a measure space.

If X is a topological space then the measure $\mu:\mathcal{B}(X)\to\overline{\mathbb{R}}$ is called Borel measure.

Definition 2.6.6. Let $(X,\mathcal{T},\mu)$ be a measure space, a property P is said to hold almost everywhere in X if there exists a set $N\in\mathcal{T}$ with $\mu(N)=0$ and for all $x\in X\setminus N$ have the property P.

Definition 2.6.7. (Complete measure)

A measure space $(X,\mathcal{T},\mu)$ is complete if and only if $B\subset N\in\mathcal{T}$ and $\mu(N)=0\Rightarrow B\in\mathcal{T}$ (B is measurable).

Example 2.6.1. Denote by λ the Lebesgue complete measure defined by $\lambda([a,b])=|b-a|$, such that $a,b\in\mathbb{R}$, and we note by $d\lambda(t)=dt$, for $t\in X$.

Definition 2.6.8. (μ-integrable function)

A measurable function $f:(X,\mathcal{T},\mu)\to(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$ is said to be a μ-integrable function if $\int_X|f(x)|d\mu(x)<+\infty$.

Definition 2.6.9. (Bochner integrable function)

Let $(Y, \|\cdot\|_Y)$ be a Banach space and let $(X, \mathcal{T}, \mu)$ be a measure space. A Bochner measurable function $f : (X, \mathcal{T}, \mu) \to Y$ is said to be Bochner integrable if $\int_X \|f(x)\|_Y d\mu(x) < +\infty$.

Theorem 2.6.1. ***(Lebesgue Dominated convergence theorem)***

Let $(X, \mathcal{T}, \mu)$ be a measure space. Let $(Y, \|\cdot\|_Y)$ be a Banach space. Let $1 \leq p < +\infty$ and $(f_n)_n \subset L^p_Y(X)$. Assume that

- *(f_n) converges μ-almost everywhere to f in X;*
- *there exists a non-negative function $g(\cdot) \in L^p_{\mathbb{R}}(X)$ such that $\|f_n(t)\| \leq g(t)$ μ-a.e. $t \in X$.*

Then, $f(\cdot) \in L^p_Y(X)$ and $(f_n)_n$ converges to f in $L^p_Y(X)$, that is

$$\lim_{n \to \infty} \int_X \|f_n(t) - f(t)\|^p_Y d\mu(t) = 0.$$

2.7 Set valued maps

We give in this section some definitions and results concerning multi-applications and maximal monotone operators, for a detailed study we can refer the reader to [10], [11], [22] and [34].

2.7.1 Definitions and measurability of set valued maps

Definition 2.7.1. (**Set valued map**)

Let X and Y be two non-empty sets. A set valued map $F : X \rightrightarrows Y$ (or $F : X \to 2^Y$) is a map that associates with any $x \in X$ a subset $F(x) \subset Y$.

- The subset

$$D(F) = \{x \in X;\ F(x) \neq \emptyset\}$$

is called the domain of F.

- The subset

$$Im(F) = \{y \in Y;\ \exists\, x \in D(F),\ y \in F(x)\} = \cup_{x \in X} F(x)$$

is called the range of F.

- The subset

$$gph(F) = \{(x,y) \in D(F) \times Y,\ y \in F(x)\}$$

is called the graph of F.

Definition 2.7.2. Let $(X,\mathcal{T})$ be a measurable space and (Y,d) be a metric space. Let $F : X \rightrightarrows Y$ be a set valued map, we say that F is measurable if for all open set $V \in Y$, $F^{-1}(V) \in \mathcal{T}$ such that

$$F^{-1}(V) = \{x \in X;\ F(x) \cap V \neq \emptyset\}.$$

Definition 2.7.3. (**Selection**)
Let $F : X \rightrightarrows Y$ be a set valued map. A selection of F is any function $f : X \to Y$ that verifies $f(x) \in F(x)$, $\forall x \in X$.

Theorem 2.7.1. ***(Existence of measurable selections)***
Let $(X,\mathcal{T},\mu)$ be a measure space and (Y,d) be a separable complete metric space. Assume that $F : X \rightrightarrows Y$ is a closed set valued map. If F is measurable then it admits a measurable selection.
We define the set of measurable selections of F by

$$S_F = \{f : X \to Y \textit{ is measurable, } f(x) \in F(x),\ \mu\text{–a.e.}\}$$

We define the set of L^1-selections of F by

$$S_F^1 = \{f \in L_Y^1(X),\ f(x) \in F(x),\ \mu\text{–a.e.}\}$$

Proposition 2.7.1. *Let $(X,\mathcal{T},\mu)$ be a measure space and let Y be a Banach separable space. Let $\Gamma : X \rightrightarrows Y$ be a non-empty weakly compact convex multi-valued map and let $\psi : X \rightrightarrows Y$ be a non-empty closed convex set valued. If for all $z \in Y'$*

$$\delta^*(z,\psi(t)) \leq \delta^*(z,\Gamma(t))\ \ \mu - \text{a.e.}.$$

Then,

$$\psi(t) \subset \Gamma(t) \ \ \mu-\text{a.e.}.$$

2.7.2 Integrals of set valued maps

Let $(X, \|\cdot\|)$ be a normed vector space.

Definition 2.7.4. Let $F : I \subset \mathbb{R} \to X$ be a set valued map, then one has

$$\int_I F(t)dt = \left\{ \int_I f(t)dt / \ f \in S_F^1 \right\}.$$

Theorem 2.7.2. *Let $F : I \subset \mathbb{R} \to X$ be a set valued map, then one has*

- *F is integrably bounded if there exists $h \in L^1_{\mathbb{R}}(I)$ such that*

$$\forall \ f \in S_F \ \ \|f(t)\| \leq h(t) \text{ a.e. } t \in I;$$

- *F is Borel measurable if its graph is a Borel subset of $I \times X$.*

- *If F is Borel measurable and integrably bounded then $\int_I F(t)dt$ is non-empty.*

- *If F is closed and integrably bounded then $\int_I F(t)dt$ is compact.*

2.7.3 Maximal monotone operators

Definition 2.7.5. A set valued map $A : H \rightrightarrows H$ is called monotone if

$$\forall \ x_1, x_2 \in D(A), \ \forall y_i \in A(x_i), \ i = 1,2 : \ \langle y_1 - y_2, x_1 - x_2 \rangle \geq 0.$$

A monotone set valued map is maximal if there is no often monotone set valued map B whose graph contains strictly the graph of A.

Proposition 2.7.2. *Let A be a maximal monotone operator, then,*

- *Its graph is strongly weakly closed in the sense that if x_n converges to x in H and if the sequence (y_n) such that $y_n \in Ax_n$ converges weakly to y in H, then $y \in Ax$.*

2.7.4 Subdifferentials

Let X be a Banach space endowed with $\|\cdot\|$ and the dual product $\langle\cdot,\cdot\rangle$.

Definition 2.7.6. Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be a proper convex function. The subdifferential of f which is denoted by $\partial f(\cdot) : X \rightrightarrows X^{'}$ is defined for all $x \in X$ by

$$\partial f(x) = \{y \in X^{'},\ f(x) - f(z) \leq \langle y, x - z\rangle\ \forall z \in X\}.$$

- It is clear that $\partial f(\cdot)$ is a closed convex subset in $X^{'}$.
- If $f(x) = +\infty$ then $\partial f(x) = \emptyset$.

Theorem 2.7.3. *Let $f : X \to \overline{\mathbb{R}}$ be a function, we define the Fenchel conjugate function of f, $f^* : X^{'} \to \overline{\mathbb{R}}$ by*

$$f^*(y) = \sup_{x\in X}\Big(\langle y, x\rangle - f(x)\Big),\ y \in X^{'}.$$

Example 2.7.1. The Fenchel conjugate of the indicator function of S is the support function of the subset S that is

$$\psi_S^*(x) = \delta^*(x, S).$$

Theorem 2.7.4. *Let $f : X \to \overline{\mathbb{R}}$ be a proper function and let $x \in X$, then*

$$y \in \partial f(x) \Leftrightarrow f^*(y) = \langle y, x\rangle - f(x).$$

Proposition 2.7.3. *The subdifferential of a proper lower semi continuous and convex function is a maximal monotone operator.*

Definition 2.7.7. (Normal cone)

Let X be a normed vector space, and let S be a non-empty set of X, the normal cone $N_S(\cdot)$ is the subdifferential of the indicator function $\partial\psi_S(\cdot)$ and we have for all $x_0 \in X$

$$N_S(x_0) = \partial\psi_S(x_0) = \{y \in X^{'},\ \langle y, x - x_0\rangle,\ \forall\, x \in S\}.$$

If S is non-empty closed convex set then $N_S(\cdot)$ is a maximal monotone operator.

2.8 Normal cones of non-convex sets

In this part, we are going to establish some important regularity properties of prox-regular sets, see [36] and [63].

Definition 2.8.1. The proximal, the limiting and Clarke normal cones are defined by

- The proximal normal cone is defined by (see [50])

$$N_S^P(x) := \{y \in H, \langle y, z-x\rangle \leq M\|z-x\|^2 \text{ for all } z \in S \cap B_H[x,r]\}.$$

- The limiting normal cone (or Mordukhovich normal cone see [53]) is given by

$$N_S^L(x) = \{y \in H : y_n \to y \text{ weakly, } y_n \in N_S^P(x_n),\ x_n \to x \text{ strongly in } S\}.$$

- The Clarke normal cone is given by

$$N_S^C(x) = \overline{\text{co}} N_S^L(x).$$

Definition 2.8.2. (Prox regular set)

For a fixed $r > 0$, the closed set S is r-prox regular (or uniformly prox regular with constant $\frac{1}{r}$) if and only if each point x in the r-enlargement of S

$$U_r(S) = \{y \in H : d(y,S) < r\},$$

has a unique nearest point $P_S(x)$ and the mapping $P_S(\cdot)$ is continuous in $U_r(S)$.

Proposition 2.8.1. *Let S be a closed set in H. The followings are equivalent*

1) *S is r-prox-regular.*

2) *For all $x \in S$ and $z \in N_S^L(x)$, we have*

$$\langle z, y-x\rangle \leq \frac{\|z\|}{2r}\|y-x\|^2 \ \ \forall y \in S. \tag{2.8.1}$$

3) *(Hypo-monotonocity) For all $x_1, x_2 \in S, y_1 \in N_S^L(x_1), y_2 \in N_S^L(x_2)$ and $y_1, y_2 \in B_H[0,r]$, we have*

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq -\|x_1 - x_2\|^2.$$

Proposition 2.8.2. *If S is r-prox regular, then for any $x \in S$, the normal cones defined above coincide and we denote $N_S(x)$, i.e.*

$$N_S(x) = N_S^P(x) = N_S^L(x) = N_S^C(x).$$

Definition 2.8.3. (The Clarke directional derivative)

Let $f : H \to \mathbb{R}$ be Lipschitz near $x \in H$. The Clarke directional derivative of f at $x \in H$ in the direction $y \in H$ is given by (see [35])

$$f^\circ(x,y) = \limsup_{z \to x\, t \downarrow 0} \frac{f(z+ty) - f(z)}{t}.$$

Definition 2.8.4. (The Clarke subdifferential)

The Clarke subdifferential of f at $x \in H$ is given by

$$\partial^C f(x) = \{y \in H : \langle y, z\rangle \leq f^\circ(x;z)\ \forall z \in H\}.$$

Definition 2.8.5. (The proximal subdifferential)

Denote $\partial^P f(x)$ the proximal subdifferential of f at $x \in H$. The vector $z \in H$ belongs to $\partial^P f(x)$ if there exist real numbers $\alpha, \beta > 0$ where

$$f(y) - f(x) + \alpha ||y - x||^2 \geq \langle z, y - x\rangle\ \forall y \in B_H[x, \beta].$$

Proposition 2.8.3. *For all $x \in H$, one has*

$$\partial^P f(x) \subset \partial^C f(x).$$

Let us recall these useful relationships between normal cones and subdifferentials.

Proposition 2.8.4. *For all non-empty closed subset S of H and all $x \in S$, we have*

$$\partial^P d(x,S) = N_S^P(x) \cap \overline{B}_H \tag{2.8.2}$$

$$\partial^C d(x,S) \subset N_S^C(x) \cap \overline{B}_H. \tag{2.8.3}$$

If S is r-prox regular, from (2.8.2)-(2.8.3), and the equality between proximal and Clarke normal cones, it is readily seen that for any $x \in S$

$$\partial^P d(x,S) = \partial^C d(x,S).$$

2.9 Some extra results

In this part, we are going to recall and present several significant results that will be used in this thesis.

We need the following lemma (see Lemma A.5 [22]).

Lemma 2.9.1. *Let $g \in L^1_{\mathbb{R}_+}(I)$ and let $\beta \in \mathbb{R}_+$. If a continuous function $h : I \to \mathbb{R}$ satisfies*

$$\frac{1}{2}h^2(t) \leq \frac{1}{2}\beta^2 + \int_0^t g(s)h(s)ds \quad \text{for all } t \in I.$$

Thereafter, we get

$$|h(t)| \leq \beta + \int_0^t g(s)ds \quad \text{for all } t \in I.$$

We recall the following theorem taken from [32], adapted to the context of our study.

Theorem 2.9.1. *Let $f : I \times H \times H \to H$ be a measurable and integrable function. Let $u_n : I \to H$ be a sequence of measurable mappings such that $u_n(t)$ converges to $u(t)$ for all $t \in I$ and let $x_n : I \to H$ be an integrable sequence which converges weakly to x in $L^1_H(I)$.*

If $f(t,u,\cdot)$ is convex for all $(t,u) \in I \times H$, and $f(t,\cdot,\cdot)$ is lower semi-continuous for all $t \in I$. Then, one has

$$\liminf_{n\to\infty} \int_I f(t,u_n(t),x_n(t))dt \geq \int_I f(t,u(t),x(t))dt.$$

Recalling Gronwall's lemma in its discrete form.

Lemma 2.9.2. *Let $\alpha \in R_+$. Let (γ_i) and (η_i) be sequences in $\mathbb{R}_+$, such that*

$$\eta_{i+1} \leq \alpha + \sum_{k=0}^{i} \gamma_k \eta_k \quad \text{for all } i \in \mathbb{N}.$$

Then, one writes

$$\eta_{i+1} \leq \alpha \exp\left(\sum_{k=0}^{i} \gamma_k\right) \text{for all } i \in \mathbb{N}.$$

We end this section by recalling the Gronwall-like differential inequality proved in [17].

Lemma 2.9.3. *Let $y : I \to \mathbb{R}$ be a non-negative absolutely continuous function and let $h_1, h_2, g : I \to \mathbb{R}_+$ be non-negative integrable functions. Suppose for some $\varepsilon > 0$*

$$\dot{y}(t) \le g(t) + \varepsilon + h_1(t)y(t) + h_2(t)(y(t))^{\frac{1}{2}} \int_{T_0}^{t} (y(s))^{\frac{1}{2}} ds \text{ a.e. } t \in I.$$

Then, for all $t \in I$, we get

$$\begin{aligned}(y(t))^{\frac{1}{2}} \le{}& (y(T_0)+\varepsilon)^{\frac{1}{2}} \exp\left(\int_{T_0}^{t} (h(s)+1) ds\right) + \frac{\varepsilon^{\frac{1}{2}}}{2} \int_{T_0}^{t} \exp\left(\int_{s}^{t} (h(r)+1) dr\right) ds \\ &+ 2\left[\left(\int_{T_0}^{t} g(s) ds + \varepsilon\right)^{\frac{1}{2}} - \varepsilon^{\frac{1}{2}} \exp\left(\int_{T_0}^{t} (h(r)+1) dr\right)\right] \\ &+ 2\int_{T_0}^{t} \Big(h(s)+1\Big) \exp\left(\int_{s}^{t} (h(r)+1) dr\right) \left(\int_{T_0}^{s} g(r) dr + \varepsilon\right)^{\frac{1}{2}} ds,\end{aligned}$$

where $h(t) = \max\left(\frac{h_1(t)}{2}, \frac{h_2(t)}{2}\right)$ a.e. $t \in I$.

3

Coupled systems of subdifferential type with integral perturbation and fractional differential equations

Contents

3.1 Introduction

The focus of this chapter is the analysis of coupled systems with fractional differential equations and subdifferentials with integral perturbation. We address a new class of first-order problems formulated by

$$(FOP)\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \int_0^t f(t,s,x(s))ds \quad \text{a.e. } t \in I := [0,1], \\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

where $\partial\varphi(t,\cdot)$ stands for the subdifferential of a proper, lower semi-continuous, convex function $\varphi(t,\cdot)$ from a real separable Hilbert space H into $[0,+\infty]$, its effective domain is denoted by $\operatorname{dom}\varphi(t,\cdot)$ (for each $t \in I$). We impose an assumption that involves conjugate function of φ (see (H_2)). The map $f : I \times I \times H \to H$ is measurable, Lipschitz with respect to its third variable on bounded subsets of H, and verifying a suitable linear growth condition.

We establish the well-posedness result to (FOP) by using Schauder's fixed point theorem.

Then, we state the existence result to the second-order problem with integral perturbation

$$(SOP)\begin{cases} -\ddot{x}(t) \in \partial\varphi(t,\dot{x}(t)) + \int_0^t f(t,s,x(s),\dot{x}(s))ds \quad \text{a.e. } t \in I, \\ x(0) = x_0,\ \dot{x}(0) = v_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

by a reduction to the appropriate first-order differential inclusion and an adoption of the methods utilized in the study of (FOP).

Our goal in the chapter's last topic is to prove novel results about evolution problems involving subdifferentials with integral perturbations in the new setting of coupled fractional differential equations. The first one is concerned with nonlocal boundary conditions along with the following fractional differential inclusion

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \int_0^t f(t,s,u(s),x(s))ds \quad \text{a.e. } t \in I, \\ D^{\alpha}u(t) + \lambda D^{\alpha-1}u(t) = x(t) \quad \text{a.e. } t \in I, \\ I_{0^+}^{\beta}u(t)|_{t=0} = 0,\ u(1) = I_{0^+}^{\gamma}u(1), \\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

where $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0$, $\gamma > 0$, and $D^\alpha u$ stands for the Riemann-Liouville fractional derivative of u.

In order to apply the fixed point theorem, our method combines the existence result to (FOP) with the topological characteristics of the solution set to the fractional differential equation above (see [26]).

In the same spirit, we are able to develop further researches regarding other variants of coupled systems by relying on the previously given arguments and the structure of the solution set to some differential equations.

3.2 Main result

In this chapter, an interval of $\mathbb{R}$, $I := [0,1]$ is considered, and H is a real separable Hilbert space.

We impose these assumptions in the sequel.

Consider the map $\varphi : I \times H \to [0, +\infty]$ such that

(H_1) the function $x \mapsto \varphi(t,x)$ is convex, proper, and lower semi-continuous for each $t \in I$;

(H_2) there is $a \in W^{1,2}_{\mathbb{R}_+}(I)$ and a ρ-Lipschitz function $k : H \longrightarrow \mathbb{R}_+$, such that

$$\varphi^*(t,x) \leq \varphi^*(s,x) + k(x)|a(t) - a(s)| \text{ for every } (t,s,x) \in I \times I \times H,$$

(H_3) the set $\operatorname{dom}\varphi(t,\cdot)$ is ball-compact for all $t \in I$. This means that for any $M > 0$, the set $\{x \in \operatorname{dom}\varphi(t,\cdot) : \|x\| \leq M\}$ is compact for every $t \in I$.

(H_4) Suppose that for every $t \in I$, there is a measurable convex compact multi-valued map $X : I \rightrightarrows H$, and for every $t \in I$, $\operatorname{dom}\varphi(t,\cdot) \subset X(t) \subset M\overline{B}_H$.

Consider a map $f : I \times I \times H \to H$ such that

(i) $f(\cdot,\cdot,x)$ is measurable on $I \times I$, for every $x \in H$;

(ii) there is a function $\alpha(\cdot,\cdot) \in L^2_{\mathbb{R}_+}(I \times I)$ such that

$$\|f(t,s,x)\| \leq \alpha(t,s)(1 + \|x\|) \text{ for all } (t,s,x) \in I \times I \times H;$$

(iii) for all $\eta > 0$, there is $\beta_\eta(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that for every $(t,s) \in I \times I$, for any $x, y \in \overline{B}_H[0,\eta]$

$$||f(t,s,x) - f(t,s,y)|| \leq \beta_\eta(t)||x-y||.$$

Consider the map $f : I \times I \times H \times H \to H$ such that

(j) $f(\cdot,\cdot,x,y)$ is measurable on $I \times I$, for any $(x,y) \in H \times H$;

(jj) there is a function $\kappa(\cdot,\cdot) \in L^2_{\mathbb{R}_+}(I \times I)$ such that for any $(t,s) \in I \times I$ and for every $(x,y) \in H \times H$

$$\|f(t,s,x,y)\| \leq \kappa(t,s)(1 + \|x\| + \|y\|);$$

(jjj) for all $\eta > 0$, there is a function $\delta_\eta(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that for any $(t,s) \in I \times I$ and for all $x, y, u, v \in \overline{B}_H[0,\eta]$

$$\|f(t,s,u,x) - f(t,s,v,y)\| \leq \delta_\eta(t)(\|u-v\| + \|x-y\|).$$

The well-posedness theorem is taken from [61].

Theorem 3.2.1. *Consider the map $\varphi : I \times H \to [0,+\infty]$ that fulfills (H_1)-(H_2). Assume $x_0 \in \operatorname{dom}\varphi(0,\cdot)$. Thus, the differential inclusion*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) \quad \text{a.e. } t \in I, \\ \ \ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

admits a unique absolutely continuous solution $x(\cdot)$ on I with $x(t) \in \operatorname{dom}\varphi(t,\cdot)$ for all $t \in I$.

Let us now denote the maximal monotone operator in H associated with $\partial\varphi(t,\cdot)$, $t \in I$, by $A(t) := \partial\varphi(t,\cdot)$ such that φ fulfills (H_1) and (H_2).

Let us define the operator $\mathcal{A} : L^2_H(I) \rightrightarrows L^2_H(I)$ by

$$\mathcal{A}x = \{y \in L^2_H(I) : y(t) \in A(t)x(t) \text{ a.e.}\}.$$

Then, Theorem 3.2.1 ensures that $\mathcal{A}$ is well defined, the differential inclusion

$$-\dot{x}(t) \in A(t)x(t) = \partial\varphi(t, x(t)) \text{ a.e. } t \in I,\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot),$$

has a unique absolutely continuous solution.

According to [61], the operator $\mathcal{A}$ has the following properties.

Proposition 3.2.1. *Let $A(t) = \partial\varphi(t,\cdot)$ for every $t \in I$, where φ fulfills (H_1)-(H_2). One remarks that*

$(\mathcal{J})$ $\mathcal{A}$ is maximal monotone.

$(\mathcal{J}\mathcal{J})$ Given two sequences in $L^2_H(I)$, $(x_n)_n$ and $(y_n)_n$, such that $y_n(t) \in A(t)x_n(t)$ a.e. *$t \in I$, the sequence $(x_n)_n$ strongly converges to x, while $(y_n)_n$ converges weakly to y in $L^2_H(I)$. Then, one deduces $y(t) \in A(t)x(t)$* a.e. *$t \in I$.*

We need the following proposition [70].

Proposition 3.2.2. *Let the assumptions of Theorem* 3.2.1 *be satisfied. If $y \in L^2_H(I)$ and $x_0 \in \operatorname{dom}\varphi(0,\cdot)$, then the system*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + y(t) \quad \text{a.e. } t \in I, \\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

has a unique absolutely continuous solution $x(\cdot)$ such that its derivative $\dot{x}(\cdot)$ fulfills

$$\int_0^1 \|\dot{x}(t)\|^2 dt \leq \sigma \|y\|^2_{L^2_H(I)} + d, \tag{3.2.1}$$

where the constants d, $\sigma > 0$ are

$$d = (k^2(0) + 3(\rho+1)^2) \int_0^1 \dot{a}^2(t) dt + 2[1 + \varphi(0, x_0)], \tag{3.2.2}$$

$$\sigma = k^2(0) + 3(\rho+1)^2 + 4. \tag{3.2.3}$$

3.2.1 First-order problem of subdifferential with integral perturbation

Now, we are able to show the main result of this section concerning the well-posedness of (FOP).

Theorem 3.2.2. *Define a map $\varphi : I \times H \to [0,+\infty]$ such that (H_1)-(H_2)-(H_3) are satisfied. Assume there exists a map $f : I \times I \times H \to H$ that satisfies (i)-(ii)-(iii). Then, there is a unique absolutely continuous solution $x(\cdot)$ to (FOP) for any $x_0 \in \operatorname{dom}\varphi(0,\cdot)$. Moreover, there are constants $L, M > 0$ that depend on $\alpha(\cdot,\cdot)$, $\varphi(0,x_0)$, k, ρ, and $\dot{a}(\cdot)$ with*

$$\int_0^1 \|\dot{x}(t)\|^2 dt \leq L \text{ and } \|x(t)\| \leq M \quad \text{for every } t \in I. \tag{3.2.4}$$

Proof. **Existence.** As guaranteed by Theorem 3.2.1, let $x : I \to H$ be the unique absolutely continuous solution to the differential inclusion

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) \quad \text{a.e. } t \in I, \\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

Given the differential equation

$$\dot{z}(t) = \int_0^t \alpha(t,s)(1+z(s))ds \quad \text{a.e. } t \in I \text{ with } z(0) = \sup_{t\in I}\|x(t)\|. \tag{3.2.5}$$

Let $z : I \to \mathbb{R}_+$ be the unique absolutely continuous solution of (3.2.5). Let us define the convex $\sigma(L^2_H(I), L^2_H(I))$-compact set $\mathcal{Y}$ by

$$\mathcal{Y} := \{y \in L^2_H(I) : \|y(t)\| \leq \dot{z}(t) \text{ a.e. } t \in I\}.$$

The unique absolutely continuous solution to the differential inclusion

$$\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t,x_y(t)) + y(t) \quad \text{a.e. } t \in I, \\ x_y(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases} \tag{3.2.6}$$

guaranteed by Proposition 3.2.2 is denoted by x_y for any $y \in \mathcal{Y}$. Observe that for every $t \in I$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|x_y(t) - x(t)\|^2 &= \langle x_y(t) - x(t), \dot{x}_y(t) - \dot{x}(t)\rangle \\ &\leq \langle y(t), x(t) - x_y(t)\rangle \\ &\leq \|y(t)\|\|x_y(t) - x(t)\|, \end{aligned}$$

by monotonocity of $\partial\varphi(t,\cdot)$. The result of integration gives

$$\frac{1}{2}\|x_y(t) - x(t)\|^2 \leq \int_0^t \|y(s)\|\|x_y(s) - x(s)\|ds.$$

Applying Lemma 2.9.1 yields

$$\|x_y(t) - x(t)\| \leq \int_0^t \|y(s)\| ds.$$

The latter estimate may be simplified using (3.2.5)

$$\|x_y(t)\| \leq z(0) + \int_0^t \dot{z}(s) ds = z(t). \tag{3.2.7}$$

Assumption (ii) and (3.2.7) allow to write for each $y \in \mathcal{Y}$

$$\begin{aligned} \| \int_0^t f(t,s,x_y(s)) ds \| \leq \int_0^t \|f(t,s,x_y(s))\| ds &\leq \int_0^t \alpha(t,s)(1 + \|x_y(s)\|) ds \\ &\leq \int_0^t \alpha(t,s)(1 + z(s)) ds = \dot{z}(t). \end{aligned} \tag{3.2.8}$$

The map Ψ, for each $y \in \mathcal{Y}$, will be defined as follows:

$$\Psi(y)(t) = \int_0^t f(t,s,x_y(s)) ds \ \text{ for all } t \in I.$$

Let us use the $\sigma(L^2_H(I), L^2_H(I))$-topology to equip $\mathcal{Y}$. Remember that $\mathcal{Y}$ is convex $\sigma(L^2_H(I), L^2_H(I))$-compact. Since H is separable, $\mathcal{Y}$ is convex $\sigma(L^2_H(I), L^2_H(I))$-compact metrizable by Proposition 2.5.2. The estimate (3.2.8) proves that $\Psi(y) \in \mathcal{Y}$, then, $\Psi : \mathcal{Y} \to \mathcal{Y}$. Let's now verify that Ψ is continuous. In order to achieve this, we show that it is sequentially $\sigma(L^2_H(I), L^2_H(I))$-continuous on $\mathcal{Y}$.

Assume that a sequence $(y_n) \subset \mathcal{Y}$, $\sigma(L^2_H(I), L^2_H(I))$-converges to $y \in \mathcal{Y}$. Then, the estimate (using (3.2.1) with the same constants d and σ given by (3.2.2) and (3.2.3))

$$\sup_{n \in \mathbb{N}} \int_0^1 \|\dot{x}_{y_n}(t)\|^2 dt \leq \sigma \|\dot{z}\|^2_{L^2_{\mathbb{R}}(I)} + d = L, \tag{3.2.9}$$

is satisfied by the absolutely continuous solution x_{y_n} associated with y_n to the evolution problem

$$\begin{cases} -\dot{x}_{y_n}(t) \in \partial\varphi(t, x_{y_n}(t)) + y_n(t) & \text{a.e. } t \in I, \ y_n \in \mathcal{Y}, \\ x_{y_n}(0) = x_0 \in \operatorname{dom} \varphi(0, \cdot). \end{cases}$$

It results from the Cauchy-Schwartz inequality and the absolute continuity of x_{y_n}

$$\sup_{n \in \mathbb{N}} \|x_{y_n}(t)\| \leq \|x_0\| + L^{\frac{1}{2}} = M \quad \text{for all } \ t \in I. \tag{3.2.10}$$

Considering assumption (H_3), one may conclude that $(x_{y_n}(t))$ is relatively compact in H, for each $t \in I$, since $x_{y_n}(t) \in \operatorname{dom}\varphi(t,\cdot)$. The equi-continuity of $(x_{y_n}(\cdot))$ is evident. In fact, let $t,s \in I$, such that $|t-s| \leq \delta$, then

$$\|x_{y_n}(t) - x_{y_n}(s)\| = \|\int_s^t \dot{x}_{y_n}(z)dz\| \leq \int_s^t \|\dot{x}_{y_n}(z)\|dz \leq (L|t-s|)^{\frac{1}{2}} < \epsilon.$$

Thus for $\delta = \frac{\epsilon^2}{L}$ one has $(x_{y_n}(\cdot))$ is equi-continuous. According to Theorem 2.2.2, there exists a map $v \in \mathcal{C}_H(I)$, with $v(0) = x_0$ such that (x_{y_n}) uniformly converges in $\mathcal{C}_H(I)$ to v, up to a subsequence that we do not relabel. By combining this with (3.2.9), one can use Theorem 2.5.4 to deduce that $(\dot{x}_{y_n})$ $\sigma(L^2_H(I), L^2_H(I))$-converges to $\dot{v}$. Proposition 3.2.1 therefore yields

$$-\dot{v}(t) \in \partial\varphi(t, v(t)) + y(t) \text{ a.e. } t \in I,$$

and it follows from the uniqueness $x_y = v$. This establishes that in $\mathcal{C}_H(I)$, (x_{y_n}) uniformly converges to x_y.

Consider $g \in L^2_H(I)$. By (3.2.8), note that

$$\left|\left\langle g(t), \int_0^t f(t,s,x_{y_n}(s))ds \right\rangle\right| \leq \dot{z}(t)\|g(t)\|, \tag{3.2.11}$$

such that the map $t \mapsto \dot{z}(t)\|g(t)\| \in L^1_{\mathbb{R}}(I)$.

Making use of (ii) and (3.2.10), note that for every $n \in \mathbb{N}$

$$\|f(t,s,x_{y_n}(s))\| \leq \alpha(t,s)(1+M) \text{ for all } (t,s) \in I \times I.$$

In view of (iii) and (3.2.10), there exists $\beta_M(\cdot) \in L^2_{\mathbb{R}}(I)$ such that

$$||f(t,s,x_{y_n}(s)) - f(t,s,x_y(s))|| \leq \beta_M(t)||x_{y_n}(s) - x_y(s)|| \text{ for all } (t,s) \in I \times I.$$

Observing that $(x_{y_n}(\cdot))$ uniformly converges to $x_y(\cdot)$, Theorem 2.6.1, therefore produces

$$\begin{aligned}\|\int_0^t f(t,s,x_{y_n}(s))ds - \int_0^t f(t,s,x_y(s))ds\| &\leq \int_0^t \|f(t,s,x_{y_n}(s)) - f(t,s,x_y(s))\|ds \\ &\leq \int_0^t \beta_M(t)\|x_{y_n}(s) - x_y(s)\|ds \to 0 \text{ as } n \to \infty.\end{aligned}$$

In addition to (3.2.11), this involves

$$\lim_{n\to\infty} \int_0^1 \left\langle g(t), \int_0^t f(t,s,x_{y_n}(s))ds \right\rangle dt = \int_0^1 \left\langle g(t), \int_0^t f(t,s,x_y(s))ds \right\rangle dt,$$

adopting Theorem 2.6.1. The $\sigma(L^2_H(I), L^2_H(I))$-convergence is shown by this. Therefore, $(\Psi(y_n))$ $\sigma(L^2_H(I), L^2_H(I))$-converges to $\Psi(y)$. More precisely $\Psi : \mathcal{Y} \to \mathcal{Y}$ is continuous with regard to the $\sigma(L^2_H(I), L^2_H(I))$-topology (see Proposition 2.2.1). The fixed point theorem of Schauder (see Theorem 2.2.4) states that Ψ has a fixed point, $y = \Psi(y)$. The existence of an absolutely continuous solution to (FOP) is thus justified.

The desired estimates in (3.2.4) are obtained by passing to the limit in (3.2.9) and (3.2.10) (invoking the previous different convergences).

Uniqueness. Let (FOP) have two solutions, $x_1(\cdot)$ and $x_2(\cdot)$. The monotonicity of $\partial\varphi(t,\cdot)$ implies that

$$\frac{d}{dt}\|x_2(t) - x_1(t)\|^2 \leq \left\langle \int_0^t f(t,s,x_1(s))ds - \int_0^t f(t,s,x_2(s))ds, x_1(t) - x_2(t) \right\rangle. \tag{3.2.12}$$

As $\|x_1(t)\| \leq M$ and $\|x_2(t)\| \leq M$, in addition to (iii), there is $\beta_M(\cdot) \in L^2_{\mathbb{R}}(I)$ such that for every $t \in I$

$$||f(t,s,x_1(s)) - f(t,s,x_2(s))|| \leq \beta_M(t)||x_1(s) - x_2(s)|| \text{ for all } (t,s) \in I \times I.$$

Therefore returning to (3.2.12), it comes

$$\frac{d}{dt}\|x_2(t) - x_1(t)\|^2 \leq \|x_2(t) - x_1(t)\| \int_0^t \beta_M(t)\|x_2(s) - x_1(s)\|ds.$$

The uniqueness of the solution to (FOP) is ensured by applying Lemma 2.9.3 with $\varepsilon > 0$, which arbitrary provides $x_1 = x_2$. □

3.2.2 Second-order problem of subdifferential type with integral perturbation

One result regarding (SOP) is given in the present subsection.

Theorem 3.2.3. *Assume that the map $\varphi : I \times H \to H$ satisfies (H_1)-(H_2)-(H_4). Consider a map $f : I \times I \times H \times H \to H$ that satisfies (j)-(jj)-(jjj). Then, there exists an absolutely continuous solution for every $(x_0, v_0) \in H \times \operatorname{dom}\varphi(0,\cdot)$, $(x,v) : I \to H$*

to the coupled system

$$\begin{cases} -\dot{v}(t) \in \partial\varphi(t, v(t)) + \int_0^t f(t, s, x(s), v(s))ds \quad \text{a.e. } t \in I, \\ x(t) = x_0 + \int_0^t v(s)ds \quad \text{a.e. } t \in I, \\ v(0) = v_0 \in \operatorname{dom}\varphi(0, \cdot). \end{cases}$$

In other words, the second-order problem (SOP) *has a* $W_H^{2,2}(I)$*-solution* $x(\cdot)$*.*

Proof. Let us define f_y for every continuous map $y : I \to H$ as follows:
for each $(t, s, v) \in I \times I \times H$, $f_y(t, s, v) := f(t, s, y(s), v)$.
The measurable behavior of $f_y(\cdot, \cdot, v)$ on $I \times I$ is evident from (j). Besides, considering (jj) for all $(t, s, v) \in I \times I \times H$, one obtains

$$\|f_y(t, s, v)\| \leq \kappa(t, s)(1 + \|y(s)\| + \|v\|),$$

so that, there exists $\kappa_1(\cdot, \cdot) \in L^2_{\mathbb{R}}(I \times I)$ such that

$$\|f_y(t, s, v)\| \leq \kappa_1(t, s)(1 + \|v\|).$$

Furthermore, given (jjj) for some $\eta > 0$, there exists $\delta_\eta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that, for all $(t, s) \in I \times I$, and for all $u, v, y(s) \in \overline{B}_H[0, \eta]$,

$$\|f_y(t, s, u) - f_y(t, s, v)\| = \|f(t, s, y(s), u) - f(t, s, y(s), v)\| \leq \delta_\eta(t)\|u - v\|.$$

Thus, Theorem 3.2.2 guarantees that the evolution problem

$$(P_y) \begin{cases} -\dot{v}_y(t) \in \partial\varphi(t, v_y(t)) + \int_0^t f_y(t, s, v_y(s))ds \quad \text{a.e. } t \in I, \\ v_y(0) = v_0 \in \operatorname{dom}\varphi(0, \cdot), \end{cases}$$

has an only one, absolutely continuous solution v_y, with $\int_0^1 \|\dot{v}_y(t)\|^2 dt \leq L$ for some $L > 0$ and $\|v_y(t)\| \leq M$ for all $t \in I$, according to (H_4). The closed convex subset $\mathcal{Y}$ in the Banach space $\mathcal{C}_H(I)$ is now being considered by

$$\mathcal{Y} := \{x_g : I \to H : x_g(t) = x_0 + \int_0^t g(s)ds, \ g \in S^1_{M\overline{B}_H}\}.$$

For the closedness of $\mathcal{Y}$, we suppose that $(x_{g_n})_n$ is a sequence in $\mathcal{Y}$, which uniformly converges to some w and we show that $w \in \mathcal{Y}$.
Let $(x_{g_n}) \subset \mathcal{Y}$, one implies that $g_n \in S^1_{M\overline{B}_H}$, hence there exists $g \in S^1_{M\overline{B}_H}$ such that

$$g_n \rightharpoonup g \ \text{ in } L^1_H(I),$$

then,

$$\langle x_0 + \int_0^t g_n(s)ds, \phi\rangle \to \langle x_0 + \int_0^t g(s)ds, \phi\rangle \ \forall \phi \in H.$$

Hence $(x_{g_n}(t))_n$ converges weakly to $x_g(t)$ in H for all $t \in I$, by the uniqueness of the limit one gets $x_g = w$. The set $\mathcal{Y}$ is therefore closed in $\mathcal{C}_H(I)$.

Define the map Λ by

$$\Lambda(y)(t) = x_0 + \int_0^t v_y(s)ds, \text{ for all } t \in I \text{ and } y \in \mathcal{Y},$$

with v_y representing the unique absolutely continuous solution to (P_y). It is observed that $\Lambda(y) \in \mathcal{Y}$.

Given $v_y(t) \in \operatorname{dom}\varphi(t,\cdot)$, (H_4) implies that $v_y(t) \in X(t)$ for every $t \in I$. Keep in mind that the multi-valued map X has convex compact values and it is measurable, integrably bounded (see (H_4)). Thus, one obtains for any $y \in \mathcal{Y}$

$$\Lambda(y)(t) \in x_0 + \int_0^t X(s)ds.$$

As $s \mapsto X(s)$ is an integrably bounded multi-valued map with convex compact values, one deduces that $\Lambda(\mathcal{Y})(t)$ is relatively compact in H for each $t \in I$ since the right member of the inclusion is compact-valued by Theorem 2.7.2.

Let's prove the equi-continuity of the set $\Lambda(\mathcal{Y})$ in $\mathcal{C}_H(I)$. Let $t, s \in I$ such that $|t-s| < \delta$, let $y \in \mathcal{Y}$, then

$$\|\Lambda(y)(t) - \Lambda(y)(s)\| = \|\int_s^t v_y(z)dz\| \le M|t-s| < \epsilon,$$

so that there exists $\delta = \frac{\epsilon}{M}$ such that $\lambda(\mathcal{Y})$ is equi-continuous in $\mathcal{C}_H(I)$.

The set $\Lambda(\mathcal{Y})$ is therefore relatively compact in $\mathcal{C}_H(I)$ by applying Theorem 2.2.2.

Verifying the continuity of $\Lambda : \mathcal{Y} \to \mathcal{Y}$ is the last requirement.

Assume that the sequence $(y_n)_n \subset \mathcal{Y}$ converges uniformly to y in $\mathcal{Y}$. Then, for every n, the absolutely continuous solution to

$$\begin{cases} -\dot{v}_{y_n}(t) \in \partial\varphi(t, v_{y_n}(t)) + \int_0^t f(t,s,y_n(s),v_{y_n}(s))ds & \text{a.e. } t \in I, \\ v_{y_n}(0) = v_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

v_{y_n} associated with y_n fulfills $\int_0^1 \|\dot{v}_{y_n}(t)\|^2 dt \le L$ and $\|v_{y_n}(t)\| \le M$, $t \in I$.

Considering assumption (H_4), one can infer that $(v_{y_n}(t))$ is relatively compact in H, for each $t \in I$, since $v_{y_n}(t) \in \operatorname{dom}\varphi(t,\cdot)$. The equi-continuity of $(v_{y_n}(\cdot))$ is evident. Indeed, let $t,s \in I$ such that $|t-s| < \delta$

$$\|v_{y_n}(t) - v_{y_n}(s)\| = \|\int_s^t \dot{v}_{y_n}(z)dz\| \leq (L|t-s|)^{\frac{1}{2}} < \epsilon,$$

so that, there exists $\delta = \frac{\epsilon^2}{L}$ such that $(v_{y_n}(\cdot))$ is equi-continuous. According Theorem 2.2.2, there exists a map $v \in \mathcal{C}_H(I)$, such that (v_{y_n}) uniformly converges to v in $\mathcal{C}_H(I)$, with $v(0) = v_0$, up to a subsequence that we do not relabel, since $\sup_n \int_0^1 \|\dot{v}_{y_n}(t)\|^2 dt \leq L$, one can derive the conclusion that $(\dot{v}_{y_n})$ $\sigma(L^2_H(I), L^2_H(I))$-converges to $\dot{v}$ by applying Theorem 2.5.4.
According to (jj), observe that

$$\|f(t,s,y_n(s),v_{y_n}(s))\| \leq \kappa(t,s)(1+\|x_0\|+2M) \text{ for all } (t,s) \in I \times I.$$

Remark that, for any $g \in L^2_H(I)$

$$\left|\left\langle g(t), \int_0^t f(t,s,y_n(s),v_{y_n}(s))ds \right\rangle\right| \leq (1+\|x_0\|+2M)\|g(t)\| \int_0^t \kappa(t,s)ds, \quad (3.2.13)$$

where the map $t \mapsto (1+\|x_0\|+2M)\|g(t)\| \int_0^t \kappa(t,s)ds$ is integrable.
Set that $M_1 = M + \|x_0\|$. The map, $\delta_{M_1}(\cdot) \in L^2_{\mathbb{R}}(I)$ exists in view of (jjj), so that

$$||f(t,s,y_n(s),v_{y_n}(s)) - f(t,s,y(s),v(s))|| \leq \delta_{M_1}(t)(||y_n(s)-y(s)|| + ||v_{y_n}(s)-v(s)||)$$

for all $(t,s) \in I \times I$. Then, Theorem 2.6.1 entails

$$\begin{aligned}
&\|\int_0^t f(t,s,y_n(s),v_{y_n}(s))ds - \int_0^t f(t,s,y(s),v(s))ds\| \\
&\leq \int_0^t \|f(t,s,y_n(s),v_{y_n}(s)) - f(t,s,y(s),v(s))\|ds \\
&\leq \int_0^t \delta_{M_1}(t)(||y_n(s)-y(s)|| + ||v_{y_n}(s)-v(s)||)ds \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Taking into account (3.2.13), one easily gets

$$\lim_{n\to\infty} \int_0^1 \left\langle g(t), \int_0^t f(t,s,y_n(s),v_{y_n}(s))ds \right\rangle dt = \int_0^1 \left\langle g(t), \int_0^t f(t,s,y(s),v(s))ds \right\rangle dt,$$

applying Theorem 2.6.1. The $\sigma(L^2_H(I), L^2_H(I))$-convergence is therefore justified.
As a result, Proposition 3.2.1 ensures

$$-\dot{v}(t) \in \partial\varphi(t,v(t)) + \int_0^t f(t,s,y(s),v(s))ds \quad \text{a.e. } t \in I,$$

and it yields $v_y = v$ via uniqueness. This establishes that in $\mathcal{C}_H(I)$, (v_{y_n}) uniformly converges to v_y. Returning to the map Λ, we have for each $t \in I$

$$\Lambda(y_n)(t) - \Lambda(y)(t) = \int_0^t v_{y_n}(s)ds - \int_0^t v_y(s)ds.$$

Since $\|v_{y_n}(\cdot) - v_y(\cdot)\|_\infty \to 0$, and $\|v_{y_n}(\cdot) - v_y(\cdot)\|_\infty \leq 2M$, one obtains

$$\sup_{t\in I} \|\Lambda(y_n)(t) - \Lambda(y)(t)\| \leq \|v_{y_n}(\cdot) - v_y(\cdot)\|_\infty \to 0 \text{ as } n \to \infty.$$

Therefore, $\Lambda : \mathcal{Y} \to \mathcal{Y}$ is continuous (see Proposition 2.2.1). The fixed point theorem of Schauder (see Theorem 2.2.4) affirms that the map Λ admits a fixed point, $y = \Lambda(y)$, with

$$\begin{cases} -\dot{v}_y(t) \in \partial\varphi(t, v_y(t)) + \int_0^t f(t,s,y(s),v_y(s))ds \quad \text{a.e. } t \in I, \\ y(t) = \Lambda(y)(t) = x_0 + \int_0^t v_y(s)ds, \quad t \in I, \\ v_y(0) = v_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

As a result, $W_H^{2,2}(I)$-solution to (SOP) is justified. □

3.3 Some coupled problems with fractional derivatives

3.3.1 Riemann-Liouville fractional derivative coupled with subdifferentials

Here are some important definitions and properties from [48] and [71].

Definition 3.3.1. Consider $f : I \to H$. The fractional Bochner integral of order $\alpha > 0$ is

$$I^\alpha_{a^+} f(t) := \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau, \ \tau > a.$$

The following lemma comes from [62].

Lemma 3.3.1. *Let $f \in L^1_H(I)$. One has*

if $0 < \alpha < 1$, then $I^\alpha f$ exists a.e. on I and one has $I^\alpha f \in L^1_H(I)$.

If $\alpha \geq 1$, then $I^\alpha f \in \mathcal{C}_H(I)$.

Definition 3.3.2. (Riemann-Liouville fractional derivative) Let $f \in L^1_H(I)$, for an order $\alpha > 0$, the Riemann-Liouville fractional derivative is

$$D^\alpha f(t) := D^\alpha_{0^+} f(t) = \frac{d^n}{dt^n} I^{n-\alpha}_{0^+} f(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(s)ds, \quad \text{where } n = [\alpha]+1.$$

The set $W^{\alpha,1}_H(I)$ is given by

$$W^{\alpha,1}_H(I) := \{u \in \mathcal{C}_H(I) : \; D^{\alpha-1}u \in \mathcal{C}_H(I), \; D^\alpha u \in L^1_H(I)\}.$$

3.3.1.1 Coupled systems with nonlocal boundary conditions

The Green function and its properties are discussed in [26],

Lemma 3.3.2. *Assume that* $\gamma > 0$, $\beta \in [0, 2-\alpha]$, $\alpha \in]1, 2]$, *and* $\lambda \geq 0$. *Assume that the function* $G_1 : I \times I \to \mathbb{R}$ *is defined by*

$$G_1(t,s) = \phi(s) I^{\alpha-1}_{0^+}(\exp(-\lambda t)) + \begin{cases} \exp(\lambda s) I^{\alpha-1}_{s^+}(\exp(-\lambda t)), & 0 \leq s \leq t \leq 1, \\ 0 & 0 \leq t \leq s \leq 1, \end{cases}$$

where

$$\phi(s) = \frac{\exp(\lambda s)}{\mu_0} \Big[\Big(I^{\alpha-1+\gamma}_{s^+}(\exp(-\lambda t)) \Big)(1) - \Big(I^{\alpha-1}_{s^+}(\exp(-\lambda t)) \Big)(1) \Big]$$

with

$$\mu_0 = \Big(I^{\alpha-1}_{0^+}(\exp(-\lambda t)) \Big)(1) - \Big(I^{\alpha-1+\gamma}_{0^+}(\exp(-\lambda t)) \Big)(1).$$

Then,

(A) *the estimate is fulfilled*

$$|G_1(t,s)| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{1+\Gamma(\gamma+1)}{|\mu_0| \Gamma(\alpha)\Gamma(\gamma+1)} + 1 \right) = M_{G_1}.$$

(B) *If* $u \in W^{\alpha,1}_H(I)$ *verifies*

$$\begin{cases} D^\alpha u(t) + \lambda D^{\alpha-1} u(t) = f(t) \quad \text{a.e. } t \in I, \; f \in L^1_H(I), \\ I^\beta_{0^+} u(t)|_{t=0} = 0, \; u(1) = I^\gamma_{0^+} u(1), \end{cases}$$

then, one has

$$u(t) = \int_0^1 G_1(t,s) f(s) ds \;\; \text{for all } t \in I.$$

(C) Let $f \in L^1_H(I)$ and let $u_f : I \to H$ be the function given by

$$u_f(t) := \int_0^1 G_1(t,s)f(s)ds \quad t \in I.$$

Thus, one has

$$I^\beta_{0+}u_f(t)|_{t=0} = 0, \text{ and } u_f(1) = (I^\gamma_{0+}u_f)(1).$$

Furthermore, $u_f \in W^{\alpha,1}_H(I)$ and

$$(D^\alpha u_f)(t) + \lambda(D^{\alpha-1}u_f)(t) = f(t) \quad \text{for all } t \in I.$$

According to [26], the solution set is described as follows.

Theorem 3.3.1. *Given a measurable multi-valued map $X : I \rightrightarrows H$, with convex compact values, let $X(t) \subset \nu\overline{B}_H$ for all $t \in I$ and $\nu > 0$. Consequently, the set of $W^{\alpha,1}_H(I)$-solutions for*

$$\begin{cases} D^\alpha u(t) + \lambda D^{\alpha-1}u(t) = f(t),\ f \in S^1_X, & \text{a.e. } t \in I, \\ I^\beta_{0+}u(t)|_{t=0} = 0,\ u(1) = I^\gamma_{0+}u(1), \end{cases}$$

is an equi-continuous, convex, compact subset of $\mathcal{C}_H(I)$. Furthermore,

$$\{u_f : I \to H : u_f(t) = \int_0^1 G_1(t,s)f(s)ds,\ f \in S^1_X, t \in I\},$$

characterizes the solution set.

It is worth to point out a crucial remark from [26].

Remark 3.3.1. If the multi-valued map $X : I \rightrightarrows H$ is measurable, convex, weakly compact and bounded, then the set

$$\{u_f : I \to H : u_f(t) = \int_0^1 G_1(t,s)f(s)ds,\ f \in S^1_X, t \in I\},$$

is equi-continuous, convex, weakly compact subset in $\mathcal{C}_H(I)$.

We are now prepared to establish a novel result regarding a coupled system that has nonlocal boundary conditions.

Theorem 3.3.2. *Consider a map* $\varphi : I \times H \to [0, +\infty]$ *verifying* (H_1)-(H_2)-(H_4). *Assume that there exists a map* $f : I \times I \times H \times H \to H$ *that satisfies* (j)-(jj)-(jjj). *Then, there are an absolutely continuous map* $x : I \to H$ *and a* $W_H^{\alpha,1}(I)$ *map* $u : I \to H$ *such that*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + \int_0^t f(t, s, u(s), x(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha u(t) + \lambda D^{\alpha-1}u(t) = x(t), \;\; t \in I, \\ I_{0+}^\beta u(t)|_{t=0} = 0, \; u(1) = I_{0+}^\gamma u(1) \\ \;\; x(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot). \end{cases}$$

Proof. The set $\mathcal{Y}$ defined as follows

$$\mathcal{Y} := \{u_f : I \to H : u_f(t) = \int_0^1 G_1(t,s)f(s)ds, \; f \in S^1_{M\overline{B}_H}, \; t \in I\}$$

is closed, convex, bounded and equi-continuous in $\mathcal{C}_H(I)$, by Remark 3.3.1.
Let us define the map f_y for every $y \in \mathcal{Y}$ and for each $(t, s, v) \in I \times I \times H$ by $f_y(t, s, v) := f(t, s, y(s), v)$, which satisfies (i)-(ii)-(iii). Thus, according to Theorem 3.2.2

$$(P_y) \begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x_y(t)) + \int_0^t f_y(t, s, x_y(s))ds \quad \text{a.e. } t \in I, \\ \;\; x_y(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot), \end{cases}$$

has a unique absolutely continuous solution x_y with $\int_0^1 \|\dot{x}_y(t)\|^2 dt \le L$, for $L > 0$. Since $x_y(t) \in \operatorname{dom}\varphi(t, \cdot)$, one gets by assumption (H_4) that $\|x_y(t)\| \le M$ for all $t \in I$. The map Λ on $\mathcal{Y}$ is therefore defined by

$$\Lambda(y)(t) = \int_0^1 G_1(t,s)x_y(s)ds, \;\; t \in I,$$

where x_y represents the unique absolutely continuous solution to (P_y) for each $y \in \mathcal{Y}$. It is then observed that $\Lambda(y) \in \mathcal{Y}$. Given $x_y(t) \in \operatorname{dom}\varphi(t, \cdot)$ for all $t \in I$, (H_4) implies that $x_y(t) \in X(t)$, where $X(t)$ is convex compact. Thus, one obtains for each $y \in \mathcal{Y}$

$$\Lambda(y) \in \mathcal{Z} := \{u_f : I \to H : u_f(t) = \int_0^1 G_1(t,s)f(s)ds, \; f \in S^1_X, t \in I\},$$

so that, by Theorem 3.3.1, $\mathcal{Z}$ is convex compact in $\mathcal{C}_H(I)$, with $\Lambda(\mathcal{Y}) \subset \mathcal{Z} \subset \mathcal{Y}$ (see (H_4)). This proves the relative compactness of $\Lambda(\mathcal{Y})$. Proving the continuity of Λ

on $\mathcal{Y}$ is sufficient.

In order to prove that the sequence of solutions x_{y_n} associated to y_n of

$$\begin{cases} -\dot{x}_{y_n}(t) \in \partial\varphi(t, x_{y_n}(t)) + \int_0^t f_{y_n}(t,s,x_{y_n}(s))ds \quad \text{a.e. } t \in I, \\ x_{y_n}(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

uniformly converges to x_y solution to (P_y), let $(y_n) \subset \mathcal{Y}$ be a sequence that uniformly converges to $y \in \mathcal{Y}$.

By considering assumption (H_4), it can be inferred that, for any $t \in I$, $(x_{y_n}(t))$ is relatively compact in H. The equi-continuity of $(x_{y_n}(\cdot))$ is evident. According to Theorem 2.2.2, there exists a map $v \in \mathcal{C}_H(I)$, such that (x_{y_n}) uniformly converges in $\mathcal{C}_H(I)$ to v with $v(0) = x_0$, up to a subsequence that we do not relabel. Furthermore, one may deduce from Theorem 2.5.4 that $(\dot{x}_{y_n})$ $\sigma(L^2_H(I), L^2_H(I))$-converges to $\dot{v}$ since $\sup_n \int_0^1 \|\dot{x}_{y_n}(t)\|^2 dt \leq L$.

Observing from (jj) and Lemma 3.3.2 (A),

$$\|f(t,s,y_n(s),x_{y_n}(s))\| \leq \kappa(t,s)(1 + MM_{G_1} + M) \text{ for all } (t,s) \in I \times I.$$

Remark that

$$\left| \left\langle g(t), \int_0^t f(t,s,y_n(s),x_{y_n}(s))ds \right\rangle \right| \leq (1 + MM_{G_1} + M)\|g(t)\| \int_0^t \kappa(t,s)ds, \quad (3.3.1)$$

for any $g \in L^2_H(I)$, where the map $t \mapsto (1 + MM_{G_1} + M)\|g(t)\| \int_0^t \kappa(t,s)ds$ is integrable.

Set $M_1 = \max(M, MM_{G_1})$, then, from (jjj), there exists $\delta_{M_1}(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that

$$||f(t,s,y_n(s),x_{y_n}(s)) - f(t,s,y(s),v(s))|| \leq \delta_{M_1}(t)(||y_n(s) - y(s)|| + ||x_{y_n}(s) - v(s)||),$$

for all $(t,s) \in I \times I$. Because $(x_{y_n}(\cdot))$ (resp. (y_n)) uniformly converges to $v(\cdot)$ (resp. y), Theorem 2.6.1, therefore provides

$$\begin{aligned} &\| \int_0^t f(t,s,y_n(s),x_{y_n}(s))ds - \int_0^t f(t,s,y(s),v(s))ds \| \\ &\leq \int_0^t \|f(t,s,y_n(s),x_{y_n}(s)) - f(t,s,y(s),v(s))\| ds \\ &\leq \int_0^t \delta_{M_1}(t)(||y_n(s) - y(s)|| + ||x_{y_n}(s) - v(s)||)ds \to 0 \text{ as } n \to \infty. \end{aligned}$$

Combining this with (3.3.1), yields

$$\lim_{n\to\infty}\int_0^1 \left\langle g(t), \int_0^t f(t,s,y_n(s),x_{y_n}(s))ds \right\rangle dt = \int_0^1 \left\langle g(t), \int_0^t f(t,s,y(s),v(s))ds \right\rangle dt,$$

applying Theorem 2.6.1. The $\sigma(L^2_H(I), L^2_H(I))$-convergence is therefore justified. As a result, Proposition 3.2.1 gives

$$-\dot{v}(t) \in \partial\varphi(t,v(t)) + \int_0^t f(t,s,y(s),v(s))ds \quad \text{a.e. } t \in I,$$

and by uniqueness, it results $x_y = v$.

Returning to the Λ map, for any $t \in I$, one has

$$\begin{aligned}\|\Lambda(y_n)(t) - \Lambda(y)(t)\| &= \| \int_0^1 G_1(t,s)x_{y_n}(s)ds - \int_0^1 G_1(t,s)x_y(s)ds\| \\ &\le M_{G_1}\int_0^1 \|x_{y_n}(s) - x_y(s)\|ds.\end{aligned}$$

It follows that since $\|x_{y_n}(\cdot) - x_y(\cdot)\|_\infty \to 0$ and (x_{y_n}) is uniformly bounded

$$\sup_{t\in I}\|\Lambda(y_n)(t) - \Lambda(y)(t)\| \le M_{G_1}\|x_{y_n}(\cdot) - x_y(\cdot)\|_\infty \to 0 \text{ as } n \to \infty.$$

Therefore, according to Proposition 2.2.1, $\Lambda : \mathcal{Y} \to \mathcal{Y}$ is continuous. The fixed point theorem of Schauder (see Theorem 2.2.4) gives that the map Λ has a fixed point, $y = \Lambda(y)$ with

$$y(t) = \Lambda(y)(t) = \int_0^1 G_1(t,s)x_y(s)ds, \ \ t \in I,$$

such that x_y solution to (P_y).

As a consequence, a map $y \in W^{\alpha,1}_H(I)$ and an absolutely continuous map $x_y \in \mathcal{C}_H(I)$ exist for which

$$\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t,x_y(t)) + \int_0^t f(t,s,y(s),x_y(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha y(t) + \lambda D^{\alpha-1}y(t) = x_y(t) \quad \text{a.e. } t \in I, \\ I^\beta_{0^+}y(t)|_{t=0} = 0, \ y(1) = I^\gamma_{0^+}y(1), \\ \ \ x_y(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

□

3.3.1.2 Coupled systems with integral boundary conditions

Here are some important results from [30].

Lemma 3.3.3. *Assume that $f \in L^1_H(I)$, $b \in H$, and $\alpha \in]1,2]$. Then, the map $u_f : I \to H$ which is defined by*

$$u_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},\ t \in I$$

is the problem's unique $W^{\alpha,1}_H(I)$-solution

$$\begin{cases} D^\alpha u(t) = f(t) \quad \text{a.e. } t \in I, \\ u(0) = 0,\ D^\alpha u(0) = b \\ D^{\alpha-1} u(t) = \int_0^t f(s)ds + b. \end{cases}$$

Lemma 3.3.4. *Suppose $b \in H$. Let $X : I \rightrightarrows H$ be a multi-valued map with convex compact values that is measurable and integrably bounded. Then, the fractional differential inclusion's $W^{\alpha,1}_H(I)$-solution set of*

$$\begin{cases} D^\alpha u(t) \in X(t) \quad \text{a.e. } t \in I, \\ u(0) = 0,\ D^\alpha u(0) = b, \end{cases}$$

is bounded convex equi-continuous and compact in $\mathcal{C}_H(I)$. In addition, the $W^{\alpha,1}_H(I)$-solution set is characterized by

$$\{u_f : I \to H,\ u_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},\ f \in S^1_X,\ t \in I\}.$$

Remark 3.3.2. If the multi-valued map $X : I \rightrightarrows H$ is measurable convex, weakly compact and bounded, then the set

$$\{u_f : I \to H,\ u_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},\ f \in S^1_X,\ t \in I\},$$

is equi-continuous convex weakly compact subset in $\mathcal{C}_H(I)$.

We may now establish a new theorem for a coupled system that has integral boundary conditions.

Theorem 3.3.3. *Suppose $b \in H$. Assume that the map $\varphi : I \times H \to [0,+\infty]$ satisfies (H_1)-(H_2)-(H_4). Consider a map $f : I \times I \times H \times H \to H$ that satisfies (j)-(jj)-(jjj). Then, there is an absolutely continuous map $x : I \to H$ and a $W_H^{\alpha,1}(I)$ map $u : I \to H$ such that*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \int_0^t f(t,s,u(s),x(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha u(t) = x(t) \quad \text{a.e. } t \in I, \\ u(0) = 0,\ D^\alpha u(0) = b, \\ D^{\alpha-1}u(t) = \int_0^t x(s)ds + b, \\ \ \ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

Proof. The set $\mathcal{Y}$ defined as follows

$$\mathcal{Y} := \{u_f : I \to H,\ u_f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},\ f \in S^1_{M\overline{B}_H},\ t \in I\},$$

is closed convex bounded and equi-continuous in $\mathcal{C}_H(I)$, according to Remark 3.3.2. Let us define the map f_y for every $y \in \mathcal{Y}$ and for each $(t,s,v) \in I \times I \times H$ by $f_y(t,s,v) := f(t,s,y(s),v)$, which satisfies (i)-(ii)-(iii). As a result, duo to Theorem 3.2.2, the evolution problem

$$(P_y)\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t,x_y(t)) + \int_0^t f_y(t,s,x_y(s))ds \quad \text{a.e. } t \in I, \\ \ \ x_y(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

has a unique absolutely continuous solution x_y, with $\int_0^1 \|\dot{x}_y(t)\|^2 dt \le L$ for $L > 0$. Since $x_y(t) \in \operatorname{dom}\varphi(t,\cdot)$, from (H_4), one has for every $t \in I$ $\|x_y(t)\| \le M$ for all $t \in I$. The map Λ on $\mathcal{Y}$ is therefore defined by

$$\Lambda(y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} x_y(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},$$

noting that $\Lambda(y) \in \mathcal{Y}$ for all $t \in I$. Given $x_y(t) \in \operatorname{dom}\varphi(t,\cdot)$, (H_4) implies that $x_y(t) \in X(t)$, where $X(t)$ is convex compact. Thus, one obtains for any $y \in \mathcal{Y}$

$$\Lambda(y) \in \mathcal{Z} := \{u_f : I \to H : u_f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},\ f \in S^1_X, t \in I\},$$

such that, according to Lemma 3.3.4, $\mathcal{Z}$ is convex compact in $\mathcal{C}_H(I)$, with $\Lambda(\mathcal{Y}) \subset \mathcal{Z} \subset \mathcal{Y}$ (see (H_4)). This establishes the relative compactness of $\Lambda(\mathcal{Y})$ in $\mathcal{C}_H(I)$.

Proving the continuity of Λ on $\mathcal{Y}$ is sufficient.

Let $(y_n) \subset \mathcal{Y}$ be a sequence that converges uniformly to $y \in \mathcal{Y}$ and demonstrate that the sequence of x_{y_n} solutions associated to y_n for

$$\begin{cases} -\dot{x}_{y_n}(t) \in \partial\varphi(t, x_{y_n}(t)) + \int_0^t f_{y_n}(t,s,x_{y_n}(s))ds \quad \text{a.e. } t \in I, \\ x_{y_n}(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

converges uniformly to the x_y solution to (P_y) by using the same reasoning as in the proof of Theorem 3.3.2.

The continuity of $\Lambda : \mathcal{Y} \to \mathcal{Y}$ can be easily inferred from this. It may be shown that the map Λ has a fixed point, $y = \Lambda(y)$, by using Schauder's fixed point theorem (see Theorem 2.2.4) with

$$y(t) = \Lambda(y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x_y(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},$$

with x_y solution to (P_y).

As a consequence, a map $y \in W_H^{\alpha,1}(I)$ and an absolutely continuous map $x_y \in \mathcal{C}_H(I)$ exist, for which

$$\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x(t)) + \displaystyle\int_0^t f(t,s,y(s),x_y(s))ds \quad \text{a.e. } t \in I, \\ D^\alpha y(t) = x_y(t) \quad \text{a.e. } t \in I, \\ y(0) = 0,\ D^\alpha y(0) = b, \\ D^{\alpha-1} y(t) = \int_0^t x_y(s)ds + b, \\ x_y(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

□

3.3.2 Second-order differential equation coupled with sub-differentials

The Green function and its properties are introduced in [31].

Lemma 3.3.5. *Assume* $0 < \eta_1 < \eta_2 < \cdots < \eta_m - 2 < 1$, $\gamma > 0$. *Let* $\alpha_i \in \mathbb{R}$ $(i = 1, \cdots, m-2)$ *and* $m > 3$ *be an integer number fulfill the condition*

$$\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma\eta_i) \neq 0.$$

Let $G_2 : I \times I \to \mathbb{R}$ be the function

$$G_2(t,s) = \begin{cases} \frac{1}{\gamma}(1-\exp(-\gamma(t-s))) + \frac{A}{\gamma}(1-\exp(-\gamma t))\psi(s) & 0 \le s \le t \le 1 \\ \frac{A}{\gamma}(1-\exp(-\gamma t))\psi(s) & t < s \le 1, \end{cases}$$

such that

$$\psi(s) = \begin{cases} 1-\exp(-\gamma(1-s)) - \sum_{i=1}^{m-2}\alpha_i(1-\exp(-\gamma(\eta_i - s))), & 0 \le s < \eta_1 \\ 1-\exp(-\gamma(1-s)) - \sum_{i=2}^{m-2}\alpha_i(1-\exp(-\gamma(\eta_i - s))), & \eta_1 \le s \le \eta_2 \\ \ldots\ldots. & \\ 1-\exp(-\gamma(1-s)), & \eta_{m-2} \le s \le 1, \end{cases}$$

and

$$A = \left(\sum_{i=1}^{m-2}\alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2}\alpha_i \exp(-\gamma\eta_i)\right)^{-1}.$$

Then the estimate is fulfilled.

(a) For all $(t,s) \in I \times I$,

$$|G_2(t,s)| \le M_{G_2}$$

with

$$M_{G_2} := \max\{\gamma^{-1}, 1\}\left[1 + |A|\Big(1 + \sum_{i=1}^{m-2}|\alpha_i|\Big)\right].$$

(b) If $u \in W_H^{2,1}(I)$ such that $u(0) = c$ and $u(1) = \sum_{i=1}^{m-2}\alpha_i u(\eta_i)$, then

$$u(t) = e_c(t) + \int_0^1 G_2(t,s)(\ddot{u}(s) + \gamma\dot{u}(s))ds, \quad \forall t \in I,$$

where

$$e_c(t) = c + A(1 - \sum_{i=1}^{m-2}\alpha_i)(1-\exp(-\gamma t))c.$$

(c) Let $u_f : I \to H$ be the function defined by

$$u_f(t) = e_c(t) + \int_0^1 G_2(t,s)f(s)ds \quad \forall t \in I$$

with $f \in L_H^1(I)$. Then, one has

$$u_f(0) = c \quad u_f(1) = \sum_{i=1}^{m-2}\alpha_i u_f(\eta_i).$$

(d) The function u_f is derivable on I, $\dot{u}_f$ is scalarly derivable, and its weak derivative $\ddot{u}_f$ satisfies

$$\ddot{u}_f(t) + \gamma \dot{u}_f(t) = f(t) \quad \text{a.e. } t \in I,$$

if $f \in L^1_H(I)$.

Proposition 3.3.1. *Consider $f \in L^1_H(I)$. Then, there is only one solution to the m-points boundary problem*

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) = f(t) \quad \text{a.e. } t \in I, \\ u(0) = c, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases}$$

$u_f \in W^{2,1}_H(I)$, such that

$$u_f(t) = e_c(t) + \int_0^1 G_2(t,s) f(s) ds, \ t \in I.$$

Proposition 3.3.2. *Let $X : I \rightrightarrows H$ be a multi-valued map that is measurable, integrably bounded with convex and compact values. Consequently, the set of $W^{2,1}_H(I)$-solutions for*

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) = f(t) \quad \text{a.e. } t \in I, \ f \in S^1_X, \\ u(0) = c, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases}$$

is equi-continuous, compact, convex, and bounded in $\mathcal{C}_H(I)$. Furthermore this set is characterized by

$$\{u_f : I \to H, \ u_f(t) = e_c(t) + \int_0^1 G_2(t,s) f(s) ds, \ t \in I, \ f \in S^1_X\}.$$

We may now establish a new theorem for a coupled system with m-points boundary conditions.

Theorem 3.3.4. *Given a map $\varphi : I \times H \to [0, +\infty]$, satisfying (H_1)-(H_2)-(H_4). Assume that there exists a map $f : I \times I \times H \times H \to H$ satisfying (j)-(jj)-(jjj). Then, an absolutely continuous map $x : I \to H$ and a $W^{2,1}_H(I)$ map $u : I \to H$ fulfill*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + \int_0^t f(t,s,u(s),x(s)) ds \quad \text{a.e. } t \in I, \\ \ddot{u}(t) + \gamma \dot{u}(t) = x(t) \quad \text{a.e. } t \in I, \\ u(0) = c, \ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \\ x(0) = x_0 \in \operatorname{dom} \varphi(0, \cdot). \end{cases}$$

Proof. The set $\mathcal{Y}$ defined as follows

$$\mathcal{Y} := \{u_f : I \to H,\ u_f(t) = e_c(t) + \int_0^1 G_2(t,s)f(s)ds,\ t \in I,\ f \in S_X^1\},$$

where X is the multi-valued map with convex compact values that is measurable and integrably bounded, as given in (H_4), is convex, compact, bounded, and equicontinuous in $\mathcal{C}_H(I)$ according to Proposition 3.3.2.
The map Λ on $\mathcal{Y}$ is therefore defined by

$$\Lambda(y)(t) = e_c(t) + \int_0^1 G_2(t,s)x_y(s)ds$$

where x_y denotes the unique absolutely continuous solution to

$$(P_y)\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x_y(t)) + \int_0^t f(t,s,y(s),x_y(s))ds \quad \text{a.e. } t \in I, \\ x_y(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

By arguing the same reasoning as in Theorem 3.3.2, it can be proved the continuity of $\Lambda : \mathcal{Y} \to \mathcal{Y}$. The fixed point theorem of Schauder (see Theorem 2.2.4) tells that the map Λ admits a fixed point, $y = \Lambda(y)$, via application with

$$y(t) = \Lambda(y)(t) = e_c(t) + \int_0^1 G_2(t,s)x_y(s)ds,$$

with x_y solution to (P_y). As a consequence, there exist an absolutely continuous map $x_y : I \to H$ and a $W_H^{2,1}(I)$ map $y : I \to H$ such that

$$\begin{cases} -\dot{x}_y(t) \in \partial\varphi(t, x_y(t)) + \int_0^t f(t,s,y(s),x_y(s))ds \quad \text{a.e. } t \in I, \\ \ddot{y}(t) + \gamma\dot{y}(t) = x_y(t) \quad \text{a.e. } t \in I, \\ y(0) = c,\ y(1) = \sum_{i=1}^{m-2} \alpha_i y(\eta_i), \\ x_y(0) \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

□

4

A coupled problem described by time-dependent subdifferential operator and non-convex perturbed sweeping process

Contents

4.1 Introduction

The goal of this chapter is to examine the existence and uniqueness of a solution for a coupled problem that is described by a non-convex perturbed sweeping process and a time-dependent subdifferential operator.

Let $I := [T_0, T]$ be an interval of $\mathbb{R}$. The normal cone to a non-empty closed subset $C(t)$ of H that is r-prox-regular is denoted by $N_{C(t)}$. Let $\partial\varphi(t,\cdot)$ represent the subdifferential of an extended-real-valued proper, lower semi-continuous, convex function $\varphi(t,\cdot)$. Then, the suggested system class is expressed by

$$(CP)\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \displaystyle\int_{T_0}^{t} f(t,s,x(s),u(s))ds \quad \text{a.e. } t \in I, \\ -\dot{u}(t) \in N_{C(t)}(u(t)) + g(t,x(t),u(t)) \quad \text{a.e. } t \in I, \\ \quad (x(T_0),u(T_0)) = (x_0,u_0) \in \operatorname{dom}\varphi(T_0,\cdot) \times C(T_0), \end{cases}$$

under suitable conditions on the single valued mappings $f : I \times I \times H \times H \to H$ and $g : I \times H \times H \to H$. Our method is based on discretizing the interval I, which yields an algorithm. This algorithm consists of two sequences of solutions, each one is a solution to a differential inclusion containing a perturbation that varies on time. Constructing a sequence (x_n,u_n) of functions that are absolutely continuous is the fundamental idea. These sequences are proved to converge to a couple of absolutely continuous functions (x,u) by using an argument based on Cauchy's criterion. This is then proved to be the solution of the original system, namely (CP). As well, the hypomonotonicity of the normal cone, the monotonicity of the subdifferential, and the Lipschitz behavior of the single-valued mappings f and g all confirm the uniqueness.

After that, we focus on the concept of optimal solution, we are concerned with the following controlled problem

$$\text{Minimize} \quad \int_0^T J_0(t,x(t),u(t),z(t),\dot{x}(t),\dot{u}(t),\dot{z}(t))dt,$$

over the set of controls $z(\cdot)$ and the corresponding solutions $(x(\cdot),u(\cdot))$ of (CP), where J_0 is the cost functional, the control function $z(\cdot)$ appears in the perturbation g, and the map f (resp. g) depends on two (resp. three) (time)-variables.

4.2 Auxiliary results

We are going to recall several significant results related to the existence and uniqueness theory of this chapter.

Regarding an evolution problem with a single-valued perturbation that depends only on time, we need the following result [70].

Proposition 4.2.1. *Given a map* $\varphi : I \times H \to [0,+\infty]$ *satisfying* (H_1)*-*(H_2)*. Given* $h \in L^2_H(I)$ *and* $x_0 \in \operatorname{dom}\varphi(T_0,\cdot)$*, then there is a unique absolutely continuous solution* $x(\cdot)$

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + h(t) & \text{a.e. } t \in I \\ x(T_0) = x_0 \in \operatorname{dom}\varphi(T_0,\cdot), \end{cases}$$

that fulfills

$$\int_{T_0}^{T} ||\dot{x}(t)||^2 dt \leq \sigma \int_{T_0}^{T} ||h(t)||^2 dt + d_*, \tag{4.2.1}$$

where $d_*,\sigma \in \mathbb{R}_+$ *are defined by*

$$\begin{aligned} d_* &= [k^2(0)+3(\rho+1)^2]||\dot{a}||^2_{L^2_{\mathbb{R}}(I)} + 2[T - T_0 + \varphi(T_0,x_0) - \varphi(T,x(T))], \\ \sigma &= k^2(0)+3(\rho+1)^2+4. \end{aligned}$$

We recall finally an important proposition concerning the perturbed sweeping process (see [45]).

Proposition 4.2.2. *Consider a multi-valued map* $C : I \rightrightarrows H$*, such that* $C(t)$ *is a non-empty closed subset of* H *that is* r*-prox regular for every* $t \in I$*, an absolutely continuous function* $v : I \to \mathbb{R}$ *exists, such that for all* $x \in H$ *and* $t,s \in I$*,*

$$|d(x,C(t)) - d(x,C(s))| \leq |v(t) - v(s)|.$$

Given $h \in L^1_H(I)$*, then there is a unique absolutely continuous solution* $x(\cdot)$ *to*

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + h(t) & \text{a.e. } t \in I, \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

that fulfills

$$||\dot{x}(t) + h(t)|| \leq ||h(t)|| + |\dot{v}(t)| \text{ a.e. } t \in I. \tag{4.2.2}$$

4.3 Main result

In the reminder, we will impose the following assumptions:

Given the multi-valued $C : I \rightrightarrows H$ such that

(H_3) $C(\cdot)$ is a non-empty closed subset of H which is r-prox regular;

(H_4) there exists $v \in W^{1,2}_{\mathbb{R}}(I)$ such that for any $x \in H$, and $t, s \in I$

$$|d(x, C(t)) - d(x, C(s))| \leq |v(t) - v(s)|.$$

Let $f : I \times I \times H \times H \to H$ be a map such that

(j) the map $f(\cdot, \cdot, x, u)$ is measurable on $I \times I$ for each $(x, u) \in H \times H$;

(jj) there exists a function $\alpha(\cdot, \cdot) \in L^2_{\mathbb{R}_+}(I \times I)$ such that for all $(t, s) \in I \times I$ and for all $x, u \in H$, one has

$$||f(t, s, x, u)|| \leq \alpha(t, s)(1 + ||x|| + ||u||);$$

(jjj) the map $f(t, s, \cdot, \cdot)$ is continuous on $H \times H$, and for all $\eta > 0$, there exists a function $\beta_\eta(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that for all $(t, s) \in I \times I$ and for any $x, y, u, v \in \overline{B}_H[0, \eta]$

$$||f(t, s, x, u) - f(t, s, y, v)|| \leq \beta_\eta(t)(||x - y|| + ||u - v||).$$

Let $g : I \times H \times H \to H$ be a map such that

(i) the map $g(\cdot, x, u)$ is Lebesgue measurable on I, for each $(x, u) \in H \times H$;

(ii) there exists a function $\gamma(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that for all $t \in I$ and for all $x, u \in H$, one has

$$||g(t, x, u)|| \leq \gamma(t)(1 + ||x|| + ||u||);$$

(iii) the map $g(t, \cdot, \cdot)$ is continuous on $H \times H$, and for every $\eta > 0$, there exists a function $\zeta_\eta(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that for all $t \in I$ and for any $x, y, u, v \in \overline{B}_H[0, \eta]$

$$||g(t, x, u) - g(t, y, v)|| \leq \zeta_\eta(t)(||x - y|| + ||u - v||).$$

It is worth to emphasize that in our proof, we follow some ideas and arguments developed in [17], [45], [69]. However, many computations have to be checked carefully because the corresponding algorithm is difficult.

Theorem 4.3.1. *Consider a map $\varphi : I \times H \to [0,+\infty]$ that satisfies (H_1)-(H_2). Given a multi-valued map $C : I \rightrightarrows H$, let (H_3)-(H_4) hold true. Assume there exists a map $f : I \times I \times H \times H \to H$ that satisfies (j)-(jj)-(jjj). Let $g : I \times H \times H \to H$ be a map fulfilling (i)-(ii)-(iii). Then, there exists a unique absolutely continuous solution $(x,u) : I \to H \times H$ to (CP), for any $(x_0,u_0) \in \operatorname{dom}\varphi(T_0,\cdot) \times C(T_0)$. Also, one has the following estimations*

$$\begin{aligned}\int_{T_0}^{T} ||\dot{x}(t)||^2 dt &\leq d + 2\sigma(T-T_0)\int_{T_0}^{T}\int_{T_0}^{t} \alpha^2(t,s)(1+||x(s)||+||u(s)||)^2 ds dt,\\ &\leq d + 2\sigma(T-T_0)\int_{T_0}^{T}\int_{T_0}^{t} \alpha^2(t,s)(1+K+\xi)^2 ds dt, \qquad (4.3.1)\end{aligned}$$

$$\|\dot{u}(t)\| \leq 2\|g(t,x(t),u(t))\| + |\dot{v}(t)| \leq 2\gamma(t)(1+K+\xi) + |\dot{v}(t)|, \qquad (4.3.2)$$

with

$$\begin{aligned} d &= (k^2(0)+3(\rho+1)^2)\int_{T_0}^{T} \dot{a}^2(t)dt + 2[T-T_0+\varphi(T_0,x_0)]\\ \sigma &= k^2(0)+3(\rho+1)^2+4,\end{aligned}$$

K, $\xi > 0$ that depend on I, x_0, u_0, ρ, $\varphi(T_0,x_0)$, $k(\cdot)$, $\dot{v}(\cdot)$, $\dot{a}(\cdot)$, $\gamma(\cdot)$, $\alpha(\cdot,\cdot)$.

Proof. **Part 1: Existence of the solution.**

Step 1. Construction of the sequence of couples $(x_n(\cdot),u_n(\cdot))$.

Let $n \in \mathbb{N}^*$ and define a subdivision of $I := [T_0,T]$ by

$$t_i^n = T_0 + i\frac{T-T_0}{n} \ (0 \leq i \leq n).$$

Set $(x_0^n,u_0^n) = (x_0,u_0)$, and let the dynamical system be

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \phi^{n,0}(t) \quad \text{a.e. } t \in [t_0^n,t_1^n],\\ -\dot{u}(t) \in N_{C(t)}(u(t)) + g(t,x_0^n,u_0^n) \quad \text{a.e. } t \in [t_0^n,t_1^n],\\ (x(t_0^n),u(t_0^n)) = (x_0^n,u_0^n) \in \operatorname{dom}\varphi(t_0^n,\cdot)\times C(T_0),\end{cases}$$

where for all $t \in [t_0^n,t_1^n]$ the function $\phi^{n,0}$ is given by

$$\phi^{n,0}(t) = \int_{T_0}^{t} f(t,s,x_0^n,u_0^n)ds.$$

Notice that $g(\cdot,x_0^n,u_0^n) \in L_H^1([t_0^n,t_1^n])$ is evident from assumption (ii). In addition, we also have $\phi^{n,0}(\cdot) \in L_H^2([t_0^n,t_1^n])$. Actually, for every $t \in [t_0^n,t_1^n]$, one obtains from

(jj)

$$\begin{aligned}\int_{T_0}^{t_1^n}||\phi^{n,0}(t)||^2dt &= \int_{T_0}^{t_1^n}||\int_{T_0}^{t} f(t,s,x_0^n,u_0^n)ds||^2dt\\ &\le \int_{T_0}^{t_1^n}\left(\int_{T_0}^{t}||f(t,s,x_0^n,u_0^n)||ds\right)^2dt\\ &\le (1+||x_0^n||+||u_0^n||)^2\int_{T_0}^{t_1^n}\left(\int_{T_0}^{t}\alpha(t,s)ds\right)^2dt.\end{aligned}$$

Cauchy-Schwartz inequality is used to obtain

$$\begin{aligned}\int_{T_0}^{t_1^n}||\phi^{n,0}(t)||^2dt &\le 2(1+2||x_0^n||^2+2||u_0^n||^2)\int_{T_0}^{t_1^n}\left((t-T_0)\int_{T_0}^{t}\alpha^2(t,s)ds\right)dt\\ &\le 2(T-T_0)(1+2||x_0^n||^2+2||u_0^n||^2)\int_{T_0}^{t_1^n}\int_{T_0}^{t_1^n}\alpha^2(t,s)dsdt<\infty,\end{aligned}$$

due to $\alpha(\cdot,\cdot)\in L^2_{\mathbb{R}}(I\times I)$.

Propositions 4.2.1-4.2.2 show that there is only one absolutely continuous solution $(x^{n,0}(\cdot),u^{n,0}(\cdot)):[t_0^n,t_1^n]\to H\times H$ to our system. Remark that $(x^{n,0}(t),u^{n,0}(t))\in \operatorname{dom}\varphi(t,\cdot)\times C(t)$ for all $t\in[t_0^n,t_1^n]$. Putting $(x_1^n,u_1^n)=(x^{n,0}(t_1^n),u^{n,0}(t_1^n))$. In view of (4.2.1)-(4.2.2), consequently, it entails that

$$\begin{aligned}\int_{t_0^n}^{t_1^n}||\dot{x}^{n,0}(t)||^2dt &\le \sigma\int_{t_0^n}^{t_1^n}||\phi^{n,0}(t)||^2dt+d_0^n,\\ ||\dot{u}^{n,0}(t)+g(t,x_0^n,u_0^n)|| &\le ||g(t,x_0^n,u_0^n)||+|\dot{v}(t)| \text{ a.e. } t\in[t_0^n,t_1^n],\end{aligned}$$

where

$$\begin{aligned}\sigma &= k^2(0)+3(\rho+1)^2+4,\\ d_0^n &= [k^2(0)+3(\rho+1)^2]\int_{t_0^n}^{t_1^n}\dot{a}^2(t)dt+2[(t_1^n-t_0^n)+\varphi(t_0^n,x_0^n)-\varphi(t_1^n,x_1^n)].\end{aligned}$$

Let the dynamical system be

$$\begin{cases}-\dot{x}(t)\in\partial\varphi(t,x(t))+\phi^{n,1}(t) \quad \text{a.e. } t\in[t_1^n,t_2^n],\\ -\dot{u}(t)\in N_{C(t)}u(t))+g(t,x_1^n,u_1^n) \quad \text{a.e. } t\in[t_1^n,t_2^n],\\ (x(t_1^n),u(t_1^n))=(x_1^n,u_1^n)\in\operatorname{dom}\varphi(t_1^n,\cdot)\times C(t_1^n),\end{cases}$$

where for any $t\in[t_1^n,t_2^n]$ the function $\phi^{n,1}$ is given by

$$\phi^{n,1}(t)=\int_{T_0}^{t_1^n}f(t,s,x_0^n,u_0^n)ds+\int_{t_1^n}^{t}f(t,s,x_1^n,u_1^n)ds.$$

Assumption (ii) makes evident that $g(\cdot, x_1^n, u_1^n) \in L_H^1([t_1^n, t_2^n])$. Meanwhile, we have $\phi^{n,1}(\cdot) \in L_H^2([t_1^n, t_2^n])$. Actually, for every $t \in [t_1^n, t_2^n]$, one obtains via (jj)

$$\begin{aligned}
&\int_{t_1^n}^{t_2^n} ||\phi^{n,1}(t)||^2 dt \\
&= \int_{t_1^n}^{t_2^n} || \int_{T_0}^{t_1^n} f(t,s,x_0^n,u_0^n)ds + \int_{t_1^n}^{t} f(t,s,x_1^n,u_1^n)ds||^2 dt \\
&\leq 2\int_{t_1^n}^{t_2^n} || \int_{T_0}^{t_1^n} f(t,s,x_0^n,u_0^n)ds||^2 dt + 2\int_{t_1^n}^{t_2^n} || \int_{t_1^n}^{t} f(t,s,x_1^n,u_1^n)ds||^2 dt \\
&\leq 2\int_{t_1^n}^{t_2^n} \left(\int_{T_0}^{t_1^n} ||f(t,s,x_0^n,u_0^n)||ds \right)^2 dt + 2\int_{t_1^n}^{t_2^n} \left(\int_{t_1^n}^{t} ||f(t,s,x_1^n,u_1^n)||ds \right)^2 dt \\
&\leq 2(1+||x_0^n||+||u_0^n||)^2 \int_{t_1^n}^{t_2^n} \left(\int_{T_0}^{t_1^n} \alpha(t,s)ds \right)^2 dt \\
&\quad + 2(1+||x_1^n||+||u_1^n||)^2 \int_{t_1^n}^{t_2^n} \left(\int_{t_1^n}^{t} \alpha(t,s)ds \right)^2 dt \\
&\leq 4\sum_{j=0}^{1}(1+2||x_j^n||^2+2||u_j^n||^2) \int_{t_1^n}^{t_2^n} \left(\int_{t_j^n}^{t_{j+1}^n} \alpha(t,s)ds \right)^2 dt.
\end{aligned}$$

Observing that $(t_{j+1}^n - t_j^n) \leq T - T_0$ and applying Cauchy-Schwartz inequality, one derives

$$\begin{aligned}
&\int_{t_1^n}^{t_2^n} ||\phi^{n,1}(t)||^2 dt \\
&\leq 4\sum_{j=0}^{1}(1+2||x_j^n||^2+2||u_j^n||^2) \int_{t_1^n}^{t_2^n} (t_{j+1}^n - t_j^n) \left(\int_{t_j^n}^{t_{j+1}^n} \alpha^2(t,s)ds \right) dt \\
&\leq 4(T-T_0)\sum_{j=0}^{1}(1+2||x_j^n||^2+2||u_j^n||^2) \int_{t_1^n}^{t_2^n} \int_{t_j^n}^{t_{j+1}^n} \alpha^2(t,s)dsdt < \infty,
\end{aligned}$$

because $\alpha(\cdot,\cdot) \in L_{\mathbb{R}}^2(I \times I)$. There is only one absolutely continuous solution in the view of Propositions 4.2.1-4.2.2, $(x^{n,1}(\cdot), u^{n,1}(\cdot)) : [t_1^n, t_2^n] \to H \times H$ to our system with $(x^{n,1}(t_1^n), u^{n,1}(t_1^n)) = (x^{n,0}(t_1^n), u^{n,0}(t_1^n))$. Notice that $(x^{n,1}(t), u^{n,1}(t)) \in \operatorname{dom}\varphi(t,\cdot) \times C(t)$ for all $t \in [t_1^n, t_2^n]$. Put $(x_2^n, u_2^n) = (x^{n,1}(t_2^n), u^{n,1}(t_2^n))$. By conducting (4.2.1)-(4.2.2), it remains certain that

$$\int_{t_1^n}^{t_2^n} ||\dot{x}^{n,1}(t)||^2 dt \leq \sigma \int_{t_1^n}^{t_2^n} ||\phi^{n,1}(t)||^2 dt + d_1^n,$$

$$||\dot{u}^{n,1}(t) + g(t,x_1^n,u_1^n)|| \leq ||g(t,x_1^n,u_1^n)|| + |\dot{v}(t)| \text{ a.e. } t \in [t_1^n, t_2^n],$$

where

$$d_1^n = [k^2(0) + 3(\rho+1)^2] \int_{t_1^n}^{t_2^n} \dot{a}^2(t)dt + 2[(t_2^n - t_1^n) + \varphi(t_1^n, x_1^n) - \varphi(t_2^n, x_2^n)].$$

In a similar way, set $(x_i^n, u_i^n) = (x^{n,i-1}(t_i^n), u^{n,i-1}(t_i^n))$, for each $i \in \{2, \cdots, n-1\}$, and take the dynamical system into consideration

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + \phi^{n,i}(t) \quad \text{a.e. } t \in [t_i^n, t_{i+1}^n], \\ -\dot{u}(t) \in N_{C(t)} u(t)) + g(t, x_i^n, u_i^n) \quad \text{a.e. } t \in [t_i^n, t_{i+1}^n], \\ (x(t_i^n), u(t_i^n)) = (x_i^n, u_i^n) \in \operatorname{dom}\varphi(t_i^n, \cdot) \times C(t_i^n), \end{cases}$$

where for any $t \in [t_i^n, t_{i+1}^n]$ the function $\phi^{n,i}$ is defined by

$$\phi^{n,i}(t) = \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(t, s, x_j^n, u_j^n) ds + \int_{t_i^n}^{t} f(t, s, x_i^n, u_i^n) ds.$$

It is evident by assumption (ii) that $g(\cdot, x_i^n, u_i^n) \in L_H^1([t_i^n, t_{i+1}^n])$. Moreover, one has $\phi^{n,i}(\cdot) \in L_H^2([t_i^n, t_{i+1}^n])$. Indeed, one obtains for any $t \in [t_i^n, t_{i+1}^n]$

$$\begin{aligned} ||\phi^{n,i}(t)|| &= ||\sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(t, s, x_j^n, u_j^n) ds + \int_{t_i^n}^{t} f(t, s, x_i^n, u_i^n) ds|| \\ &\leq \sum_{j=0}^{i-1} ||\int_{t_j^n}^{t_{j+1}^n} f(t, s, x_j^n, u_j^n) ds|| + ||\int_{t_i^n}^{t} f(t, s, x_i^n, u_i^n) ds|| \\ &\leq \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} ||f(t, s, x_j^n, u_j^n)|| ds + \int_{t_i^n}^{t_{i+1}^n} ||f(t, s, x_i^n, u_i^n)|| ds \\ &\leq \sum_{j=0}^{i} \int_{t_j^n}^{t_{j+1}^n} ||f(t, s, x_j^n, u_j^n)|| ds. \end{aligned} \tag{4.3.3}$$

Then, (jj) allow to deduce that

$$\begin{aligned} ||\phi^{n,i}(t)||^2 &\leq \left(\sum_{j=0}^{i} \int_{t_j^n}^{t_{j+1}^n} ||f(t, s, x_j^n, u_j^n)|| ds \right)^2 \\ &\leq \left(\sum_{j=0}^{i} (1 + ||x_j^n|| + ||u_j^n||) \int_{t_j^n}^{t_{j+1}^n} \alpha(t, s) ds \right)^2. \end{aligned}$$

Now, put $y_j = (1 + ||x_j^n|| + ||u_j^n||) \int_{t_j^n}^{t_{j+1}^n} \alpha(t, s) ds$ in Lemma 2.2.1, that leads to

$$||\phi^{n,i}(t)||^2 \leq (i+1) \sum_{j=0}^{i} (1 + ||x_j^n|| + ||u_j^n||)^2 \left(\int_{t_j^n}^{t_{j+1}^n} \alpha(t, s) ds \right)^2.$$

Cauchy-Schwartz inequality is used to obtain

$$||\phi^{n,i}(t)||^2 \leq (i+1) \sum_{j=0}^{i} (1 + ||x_j^n|| + ||u_j^n||)^2 (t_{j+1}^n - t_j^n) \left(\int_{t_j^n}^{t_{j+1}^n} \alpha^2(t, s) ds \right).$$

As $(i+1)(t_{j+1}^n - t_j^n) \leq T - T_0$, one gets

$$\begin{aligned} ||\phi^{n,i}(t)||^2 &\leq (T-T_0)\sum_{j=0}^{i}(1+||x_j^n||+||u_j^n||)^2\left(\int_{t_j^n}^{t_{j+1}^n}\alpha^2(t,s)ds\right) \\ &\leq 2(T-T_0)\sum_{j=0}^{i}(1+2||x_j^n||^2+2||u_j^n||^2)\left(\int_{t_j^n}^{t_{j+1}^n}\alpha^2(t,s)ds\right), \end{aligned} \tag{4.3.4}$$

as a result

$$\begin{aligned} &\int_{t_i^n}^{t_{i+1}^n}||\phi^{n,i}(t)||^2dt \\ &\leq 2(T-T_0)\int_{t_i^n}^{t_{i+1}^n}\sum_{j=0}^{i}(1+2||x_j^n||^2+2||u_j^n||^2)\left(\int_{t_j^n}^{t_{j+1}^n}\alpha^2(t,s)ds\right)dt, \\ &\leq 2(T-T_0)\sum_{j=0}^{i}(1+2||x_j^n||^2+2||u_j^n||^2)\left(\int_{t_i^n}^{t_{i+1}^n}\int_{t_j^n}^{t_{j+1}^n}\alpha^2(t,s)dsdt\right) < \infty, \end{aligned}$$

since $\int\sum = \sum\int$ and that $\alpha(\cdot,\cdot) \in L^2_{\mathbb{R}}(I\times I)$. Given Propositions 4.2.1-4.2.2, there is a unique absolutely continuous solution for each $i \in \{2,\cdots,n-1\}$, $(x^{n,i}(\cdot),u^{n,i}(\cdot)) : [t_i^n,t_{i+1}^n] \to H\times H$ to our system with $(x^{n,i}(t_i^n),u^{n,i}(t_i^n)) = (x^{n,i-1}(t_i^n),u^{n,i-1}(t_i^n))$, and $(x^{n,i}(t),u^{n,i}(t)) \in \operatorname{dom}\varphi(t,\cdot)\times C(t)$ for all $t \in [t_i^n,t_{i+1}^n]$. Putting $(x_{i+1}^n,u_{i+1}^n) = (x^{n,i}(t_{i+1}^n),u^{n,i}(t_{i+1}^n))$, from (4.2.1)-(4.2.2), one gets

$$\int_{t_i^n}^{t_{i+1}^n}||\dot{x}^{n,i}(t)||^2dt \leq \sigma\int_{t_i^n}^{t_{i+1}^n}||\phi^{n,i}(t)||^2dt + d_i^n, \tag{4.3.5}$$

$$||\dot{u}^{n,i}(t)+g(t,x_i^n,u_i^n)|| \leq ||g(t,x_i^n,u_i^n)||+|\dot{v}(t)| \text{ a.e. } t\in[t_i^n,t_{i+1}^n], \tag{4.3.6}$$

where

$$d_i^n = [k^2(0)+3(\rho+1)^2]\int_{t_i^n}^{t_{i+1}^n}\dot{a}^2(t)dt + 2[(t_{i+1}^n-t_i^n)+\varphi(t_i^n,x_i^n)-\varphi(t_{i+1}^n,x_{i+1}^n)].$$

Define the maps $x_n,u_n,\phi_n : I\to H$, $\theta_n : I\to I$, for all n by

$$\phi_n(t) = \phi^{n,i}(t),\ x_n(t) = x^{n,i}(t),\ u_n(t) = u^{n,i}(t)\ \ \forall t\in[t_i^n,t_{i+1}^n]\,,\ i\in\{0,\cdots,n-1\}.$$

$$\begin{cases} \theta_n(T_0) = T_0 \\ \theta_n(t) = t_i^n \qquad \text{if } t \in]t_i^n,t_{i+1}^n],\ i\in\{0,\cdots,n-1\}. \end{cases}$$

It is evident that the couple $(x_n(\cdot),u_n(\cdot))$ is absolutely continuous on I for every n, and

$$\begin{cases} -\dot{x}_n(t) \in \partial\varphi(t,x_n(t)) + \int_{T_0}^{t} f(t,s,x_n(\theta_n(s)),u_n(\theta_n(s)))ds \text{ a.e. } t\in I, \\ -\dot{u}_n(t) \in N_{C(t)}(u_n(t)) + g(t,x_n(\theta_n(t)),u_n(\theta_n(t))) \text{ a.e. } t\in I, \\ (x_n(T_0),u_n(T_0)) = (x_0,u_0). \end{cases} \tag{4.3.7}$$

Moreover, from (4.3.6), we have for almost every $t \in I$

$$||\dot{u}_n(t) + g(t, x_n(\theta_n(t)), u_n(\theta_n(t)))|| \leq ||g(t, x_n(\theta_n(t)), u_n(\theta_n(t)))|| + |\dot{v}(t)|. \quad (4.3.8)$$

Observe that the map ϕ_n can be written as follows

$$\phi_n(t) = \int_{T_0}^{t} f(t, s, x_n(\theta_n(s)), u_n(\theta_n(s)))ds, \forall t \in I. \quad (4.3.9)$$

Note that

$$\phi^{n,0}(t) = \int_{T_0}^{t} f(t, s, x_0^n, u_0^n)ds,\ t \in [t_0^n, t_1^n]$$
$$\phi^{n,1}(t) = \int_{T_0}^{t_1^n} f(t, s, x_0^n, u_0^n)ds + \int_{t_1^n}^{t} f(t, s, x_1^n, u_1^n)ds,\ t \in [t_1^n, t_2^n]$$
$$\cdots$$
$$\phi^{n,i}(t) = \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f(t, s, x_j^n, u_j^n)ds + \int_{t_i^n}^{t} f(t, s, x_i^n, u_i^n)ds,\ t \in [t_i^n, t_{i+1}^n].$$

Furthermore, it is evident for $k = 0, \cdots, i$ $(i = 0, \cdots, n-1)$ from (4.3.4) (see also (4.3.3)) that

$$||\phi^{n,k}(t)||^2 \leq 2(T - T_0) \sum_{j=0}^{i} (1 + 2||x_j^n||^2 + 2||u_j^n||^2) \left(\int_{t_j^n}^{t_{j+1}^n} \alpha^2(t, s)ds \right),\ t \in [t_k^n, t_{k+1}^n]. \quad (4.3.10)$$

Thus,

$$\begin{aligned}
&\int_{T_0}^{t_{i+1}^n} ||\phi_n(t)||^2 dt \\
&= \sum_{j=0}^{i} \int_{t_j^n}^{t_{j+1}^n} ||\phi^{n,j}(t)||^2 dt \\
&= \int_{T_0}^{t_1^n} ||\phi^{n,0}(t)||^2 dt + \int_{t_1^n}^{t_2^n} ||\phi^{n,1}(t)||^2 dt + \cdots + \int_{t_i^n}^{t_{i+1}^n} ||\phi^{n,i}(t)||^2 dt \\
&\leq 2(T - T_0) \int_{T_0}^{t_{i+1}^n} \sum_{j=0}^{i} (1 + 2||x_j^n||^2 + 2||u_j^n||^2) \left(\int_{t_j^n}^{t_{j+1}^n} \alpha^2(t, s)ds \right) dt, \\
&\leq 2(T - T_0) \sum_{j=0}^{i} (1 + 2||x_j^n||^2 + 2||u_j^n||^2) \left(\int_{T_0}^{t_{i+1}^n} \int_{t_j^n}^{t_{j+1}^n} \alpha^2(t, s)dsdt \right), \quad (4.3.11)
\end{aligned}$$

making use (4.3.10) and realizing that $\int \sum = \sum \int$.

We return to (4.3.5) with the help of (4.3.11)

$$\begin{aligned}
&\int_{T_0}^{t_{i+1}^n} ||\dot{x}_n(t)||^2 dt \\
&\leq 2\sigma(T-T_0)\sum_{j=0}^{i}(1+2||x_j^n||^2+2||u_j^n||^2)\left(\int_{T_0}^{t_{i+1}^n}\int_{t_j^n}^{t_{j+1}^n}\alpha^2(t,s)dsdt\right)+\sum_{j=0}^{i} d_j^n, \\
&\leq 2\sigma(T-T_0)\sum_{j=0}^{i}(1+2||x_j^n||^2+2||u_j^n||^2)\left(\int_{T_0}^{t_{i+1}^n}\int_{t_j^n}^{t_{j+1}^n}\alpha^2(t,s)dsdt\right)+d, \qquad (4.3.12)
\end{aligned}$$

where

$$d=[k^2(0)+3(\rho+1)^2]\int_{T_0}^{T}\dot{a}^2(t)dt+2[(T-T_0)+\varphi(T_0,x_0)], \qquad (4.3.13)$$

since $-\varphi(t_{i+1}^n,x_{i+1}^n)\leq 0$. It follows from the Cauchy-Schwartz inequality and the absolute continuity of (x_n) that

$$\begin{aligned}
||x_n(t_{i+1}^n)-x_n(T_0)||^2 &= ||\int_{T_0}^{t_{i+1}^n}\dot{x}_n(t)dt||^2 \\
&\leq (t_{i+1}^n-T_0)\int_{T_0}^{t_{i+1}^n}||\dot{x}_n(t)||^2 dt \\
&\leq (T-T_0)\int_{T_0}^{t_{i+1}^n}||\dot{x}_n(t)||^2 dt,
\end{aligned}$$

for $i=0,\cdots,n-1$. Along with (4.3.12),

$$\begin{aligned}
&||x_n(t_{i+1}^n)-x_n(T_0)||^2 \\
&\leq 2\sigma(T-T_0)^2\sum_{j=0}^{i}(1+2||x_j^n||^2+2||u_j^n||^2)\left(\int_{T_0}^{t_{i+1}^n}\int_{t_j^n}^{t_{j+1}^n}\alpha^2(t,s)dsdt\right)+d(T-T_0),
\end{aligned}$$

that gives

$$\begin{aligned}
||x_{i+1}^n||^2 &\leq 2||x_{i+1}^n-x_0^n||^2+2||x_0||^2 \\
&\leq 4\sigma(T-T_0)^2\sum_{j=0}^{i}(1+2||x_j^n||^2+2||u_j^n||^2)\left(\int_{T_0}^{t_{i+1}^n}\int_{t_j^n}^{t_{j+1}^n}\alpha^2(t,s)dsdt\right) \\
&\quad +2||x_0||^2+2d(T-T_0) \\
&\leq \sigma_1\sum_{j=0}^{i}(||x_j^n||^2+||u_j^n||^2)\int_{T_0}^{t_{i+1}^n}\int_{t_j^n}^{t_{j+1}^n}\alpha^2(t,s)dsdt+\sigma_2, \qquad (4.3.14)
\end{aligned}$$

such that

$$\begin{aligned}
\sigma_1 &= 8\sigma(T-T_0)^2 \\
\sigma_2 &= 2||x_0||^2+4\sigma(T-T_0)^2\int_{T_0}^{T}\int_{T_0}^{T}\alpha^2(t,s)dsdt+2d(T-T_0).
\end{aligned}$$

In view of (4.3.8) once more, we can obtain for each $i \in \{0,\cdots,n\}$

$$||\dot{u}_n(t)|| \leq 2||g(t,x_n(t_i^n),u_n(t_i^n))|| + |\dot{v}(t)|, \text{ a.e. } t \in [t_i^n, t_{i+1}^n],$$

which implies

$$||u_n(t_{i+1}^n)|| \leq ||u_n(t_i^n)|| + 2\int_{t_i^n}^{t_{i+1}^n} ||g(t,x_n(t_i^n),u_n(t_i^n))||dt + \int_{t_i^n}^{t_{i+1}^n} |\dot{v}(t)|dt.$$

Iterating

$$||u_{i+1}^n|| \leq ||u_0|| + 2\sum_{j=0}^{i}\int_{t_j^n}^{t_{j+1}^n} ||g(t,x_j^n,u_j^n)||dt + \int_{T_0}^{t_{i+1}^n} |\dot{v}(t)|dt.$$

Consequently

$$\begin{aligned}
&||u_{i+1}^n||^2 \\
&\leq 2||u_0||^2 + 16\Big(\sum_{j=0}^{i}\int_{t_j^n}^{t_{j+1}^n} ||g(t,x_j^n,u_j^n)||dt\Big)^2 + 4\Big(\int_{T_0}^{t_{i+1}^n} |\dot{v}(t)|dt\Big)^2 \\
&\leq 2||u_0||^2 + 16\Big(\sum_{j=0}^{i}(1+||x_j^n||+||u_j^n||)\int_{t_j^n}^{t_{j+1}^n} \gamma(t)dt\Big)^2 + 4\Big(\int_{T_0}^{t_{i+1}^n} |\dot{v}(t)|dt\Big)^2,
\end{aligned}$$

with the help of (ii). Now, take $y_j = (1+||x_j^n||+||u_j^n||)\int_{t_j^n}^{t_{j+1}^n}\gamma(t)dt$ in Lemma 2.2.1

$$\begin{aligned}
&||u_{i+1}^n||^2 \\
&\leq 2||u_0||^2 + 16(i+1)\sum_{j=0}^{i}(1+||x_j^n||+||u_j^n||)^2\Big(\int_{t_j^n}^{t_{j+1}^n} \gamma(t)dt\Big)^2 \\
&\quad + 4\Big(\int_{T_0}^{t_{i+1}^n} |\dot{v}(t)|dt\Big)^2.
\end{aligned}$$

By applying Cauchy-Schwartz inequality, it follows

$$\begin{aligned}
||u_{i+1}^n||^2 &\leq 2||u_0||^2 + 16(i+1)\sum_{j=0}^{i}(1+||x_j^n||+||u_j^n||)^2(t_{j+1}^n - t_j^n)\Big(\int_{t_j^n}^{t_{j+1}^n} \gamma^2(t)dt\Big) \\
&\quad + 4(t_{i+1}^n - T_0)\Big(\int_{T_0}^{t_{i+1}^n} |\dot{v}(t)|^2dt\Big).
\end{aligned}$$

Since $(i+1)(t_{j+1}^n - t_j^n) \leq (T-T_0)$, it is shown that

$$\begin{aligned}
||u_{i+1}^n||^2 &\leq 2||u_0||^2 + 32(T-T_0)\sum_{j=0}^{i}(1+2||x_j^n||^2+2||u_j^n||^2)\Big(\int_{t_j^n}^{t_{j+1}^n} \gamma^2(t)dt\Big) \\
&\quad + 4(T-T_0)\Big(\int_{T_0}^{t_{i+1}^n} |\dot{v}(t)|^2dt\Big).
\end{aligned}$$

Thus

$$||u_{i+1}^n||^2 \leq \sigma_3 \sum_{j=0}^{i} (||x_j^n||^2 + ||u_j^n||^2) \int_{t_j^n}^{t_{j+1}^n} \gamma^2(t)dt + \sigma_4, \tag{4.3.15}$$

such that

$$\begin{aligned}\sigma_3 =& 64(T-T_0)\\ \sigma_4 =& 2||u_0||^2 + 32(T-T_0)\int_{T_0}^{T} \gamma^2(t)dt + 4(T-T_0)\int_{T_0}^{T} |\dot{v}(t)|^2 dt.\end{aligned}$$

Summing (4.3.14)-(4.3.15) member to member, one obtains

$$\begin{aligned}&||x_{i+1}^n||^2 + ||u_{i+1}^n||^2\\ &\leq \sum_{j=0}^{i} (||x_j^n||^2 + ||u_j^n||^2)\left(\sigma_1 \int_{T_0}^{t_{i+1}^n}\int_{t_j^n}^{t_{j+1}^n} \alpha^2(t,s)dsdt + \sigma_3 \int_{t_j^n}^{t_{j+1}^n} \gamma^2(t)dt\right) + \sigma_5,\\ &\leq \sum_{j=0}^{i} (||x_j^n||^2 + ||u_j^n||^2)\left(\sigma_1 \int_{T_0}^{T}\int_{t_j^n}^{t_{j+1}^n} \alpha^2(t,s)dsdt + \sigma_3 \int_{t_j^n}^{t_{j+1}^n} \gamma^2(t)dt\right) + \sigma_5,\end{aligned}$$

such that $\sigma_5 = \sigma_2 + \sigma_4$. Lemma 2.9.2, clearly yields

$$||x_{i+1}^n||^2 + ||u_{i+1}^n||^2 \leq M, \tag{4.3.16}$$

for the constant

$$M := \sigma_5 \exp\left(\sigma_1 \int_{T_0}^{T}\int_{T_0}^{T} \alpha^2(t,s)dsdt + \sigma_3 \int_{T_0}^{T} \gamma^2(t)dt\right),$$

observing that

$$\begin{aligned}\sum_{j=0}^{i} \int_{T_0}^{T}\int_{t_j^n}^{t_{j+1}^n} \alpha^2(t,s)dsdt &= \int_{T_0}^{T}\left(\sum_{j=0}^{i}\int_{t_j^n}^{t_{j+1}^n} \alpha^2(t,s)ds\right)dt\\ &= \int_{T_0}^{T}\int_{T_0}^{t_{i+1}^n} \alpha^2(t,s)dsdt\\ &\leq \int_{T_0}^{T}\int_{T_0}^{T} \alpha^2(t,s)dsdt.\end{aligned}$$

We return to (4.3.11) and (4.3.16), for $i = n-1$ which can be written

$$\begin{aligned}&\int_{T_0}^{T} ||\phi_n(t)||^2 dt\\ &\leq 2(T-T_0)\sum_{j=0}^{n-1}(1 + 2||x_j^n||^2 + 2||u_j^n||^2)\left(\int_{T_0}^{T}\int_{t_j^n}^{t_{j+1}^n} \alpha^2(t,s)dsdt\right),\\ &\leq 2(T-T_0)(1+2M)\left(\int_{T_0}^{T}\int_{T_0}^{T} \alpha^2(t,s)dsdt\right) < \infty,\end{aligned} \tag{4.3.17}$$

summing (4.3.5), for $i=0$ to $i=n-1$, it results

$$\int_{T_0}^{T} ||\dot{x}_n(t)||^2 dt \leq \sigma \int_{T_0}^{T} ||\phi_n(t)||^2 dt + d_n^n, \tag{4.3.18}$$

where

$$d_n^n = [k^2(0)+3(\rho+1)^2]\int_{T_0}^{T} \dot{a}^2(t)dt + 2[(T-T_0)+\varphi(T_0,x_0)-\varphi(T,x_n(T))].$$

Observe that $-\varphi(T,x_n(T)) \leq 0$, it follows from (4.3.17)-(4.3.18),

$$\int_{T_0}^{T} ||\dot{x}_n(t)||^2 dt \leq \sigma \int_{T_0}^{T} ||\phi_n(t)||^2 dt + d < \infty, \tag{4.3.19}$$

hence

$$\sup_{n\in\mathbb{N}} ||\dot{x}_n(\cdot)||_{L^2_H(I)} < \xi_1 < +\infty. \tag{4.3.20}$$

We combine (ii), (4.3.8) and (4.3.16) to obtain for almost all t and for any n

$$||g(t,x_n(\theta_n(t)),u_n(\theta_n(t)))|| \leq (1+2M^{\frac{1}{2}})\gamma(t), \tag{4.3.21}$$

and

$$||\dot{u}_n(t)+g(t,x_n(\theta_n(t)),u_n(\theta_n(t)))|| \leq \kappa(t), \tag{4.3.22}$$

thus

$$||\dot{u}_n(t)|| \leq \psi(t), \tag{4.3.23}$$

noting that the maps defined by κ and ψ are

$$\kappa(t) = (1+2M^{\frac{1}{2}})\gamma(t)+|\dot{v}(t)|,\ \psi(t) = 2(1+2M^{\frac{1}{2}})\gamma(t)+|\dot{v}(t)|,\ t\in I.$$

Step 2. Let us prove that $(u_n(\cdot))$ and $(x_n(\cdot))$ converge.

We will show that the uniform Cauchy's criterion on I is satisfied for the sequences $(u_n(\cdot))_n$ and $(x_n(\cdot))_n$. Let p and q be arbitrary integers, and that for a.e. $t\in I$

$$-\dot{u}_p(t)-g(t,x_p(\theta_p(t)),u_p(\theta_p(t))) \in N_{C(t)}(u_p(t)),$$
$$-\dot{u}_q(t)-g(t,x_q(\theta_q(t)),u_q(\theta_q(t))) \in N_{C(t)}(u_q(t)).$$

Using (2.8.1) and (4.3.22) in combination with these last inclusions, one has

$$\langle \dot{u}_p(t)+g(t,x_p(\theta_p(t)),u_p(\theta_p(t)))-\dot{u}_q(t)-g(t,x_q(\theta_q(t)),u_q(\theta_q(t))),u_p(t)-u_q(t)\rangle$$
$$\leq \frac{\kappa(t)}{r}||u_p(t)-u_q(t)||^2.$$

Simplifying,

$$\begin{aligned}
&\langle \dot{u}_p(t)-\dot{u}_q(t), u_p(t)-u_q(t)\rangle \\
&\leq \frac{\kappa(t)}{r}||u_p(t)-u_q(t)||^2 \\
&\quad + \langle g(t,x_p(\theta_p(t)),u_p(\theta_p(t))) - g(t,x_q(\theta_q(t)),u_q(\theta_q(t))), u_q(t)-u_p(t)\rangle. \qquad (4.3.24)
\end{aligned}$$

The absolute continuity of $u_n(\cdot)$ and (4.3.23), ensure that, there is a $K>0$ such that for any n and for all $t\in I$

$$||u_n(t)|| \leq K. \qquad (4.3.25)$$

The absolute continuity of $x_n(\cdot)$ and (4.3.20), guarantee that there exists $\xi>0$ such that for any n and for all $t\in I$,

$$||x_n(t)|| \leq \xi. \qquad (4.3.26)$$

Set $K_1=\max(K,\xi)$. According to (iii), a non-negative function $\zeta_{K_1}(\cdot)\in L^2_{\mathbb{R}}(I)$ then exists such that for a.e. $t\in I$,

$$\begin{aligned}
&||g(t,x_p(\theta_p(t)),u_p(\theta_p(t))) - g(t,x_q(\theta_q(t)),u_q(\theta_q(t)))|| \\
&\leq \zeta_{K_1}(t)\Big(||x_p(\theta_p(t))-x_q(\theta_q(t))|| + ||u_p(\theta_p(t))-u_q(\theta_q(t))||\Big). \qquad (4.3.27)
\end{aligned}$$

Remark that for any $t\in I$

$$||x_p(\theta_p(t))-x_q(\theta_q(t))|| \leq ||x_q(\theta_q(t))-x_q(t)|| + ||x_q(t)-x_p(t)|| + ||x_p(t)-x_p(\theta_p(t))||.$$

The absolute continuity of x_p for each p, the construction of θ_p that is, for any $t\in I$ and any p, $0\leq t-\theta_p(t)\leq (T-T_0)/p$, along with (4.3.20) yield for any $t\in I$

$$\begin{aligned}
||x_p(t)-x_p(\theta_p(t))|| &\leq \int_{\theta_p(t)}^t ||\dot{x}_p(s)||ds \\
&\leq (t-\theta_p(t))^{\frac{1}{2}}\left(\int_{T_0}^T ||\dot{x}_p(s)||^2 ds\right)^{\frac{1}{2}} \leq \left(\frac{T-T_0}{p}\right)^{\frac{1}{2}}\xi_1.
\end{aligned}$$

For any $t\in I$, we conclude

$$||x_p(\theta_p(t))-x_q(\theta_q(t))|| \leq ||x_q(t)-x_p(t)|| + \xi_1\left[\left(\frac{T-T_0}{p}\right)^{\frac{1}{2}} + \left(\frac{T-T_0}{q}\right)^{\frac{1}{2}}\right]. \qquad (4.3.28)$$

Once more by the absolute continuity of u_p, one can now obtain for any p and any $t \in I$

$$||u_p(t) - u_p(\theta_p(t))|| \leq \int_{\theta_p(t)}^{t} ||\dot{u}_p(s)||ds \leq \int_{\theta_p(t)}^{t} \psi(s)ds,$$

with the help of (4.3.23).

Therefore, it is evident that

$$||u_p(\theta_p(t)) - u_q(\theta_q(t))|| \leq ||u_q(t) - u_p(t)|| + \left(\int_{\theta_q(t)}^{t} \psi(s)ds + \int_{\theta_p(t)}^{t} \psi(s)ds\right). \quad (4.3.29)$$

A combination of (4.3.25), (4.3.28)-(4.3.29), and (4.3.27), lead to

$$\begin{aligned}
&\langle g(t, x_p(\theta_p(t)), u_p(\theta_p(t))) - g(t, x_q(\theta_q(t)), u_q(\theta_q(t))), u_q(t) - u_p(t)\rangle \\
&\leq \zeta_{K_1}(t)||u_q(t) - u_p(t)||^2 + \zeta_{K_1}(t)||u_q(t) - u_p(t)||||x_q(t) - x_p(t)|| \\
&\quad + 2K\zeta_{K_1}(t)\left(\xi_1\left[\left(\frac{T-T_0}{p}\right)^{\frac{1}{2}} + \left(\frac{T-T_0}{q}\right)^{\frac{1}{2}}\right] + \int_{\theta_q(t)}^{t} \psi(s)ds + \int_{\theta_p(t)}^{t} \psi(s)ds\right).
\end{aligned}$$

We then go back to (4.3.24) to deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||u_p(t) - u_q(t)||^2 \leq \left(\zeta_{K_1}(t) + \frac{\kappa(t)}{r}\right)||u_p(t) - u_q(t)||^2 + L_{p,q}(t) \\
&+ \zeta_{K_1}(t)||u_q(t) - u_p(t)||||x_q(t) - x_p(t)|| \\
&\leq \left(\frac{3}{2}\zeta_{K_1}(t) + \frac{\kappa(t)}{r}\right)\left(||u_p(t) - u_q(t)||^2 + ||x_q(t) - x_p(t)||^2\right) + L_{p,q}(t), \qquad (4.3.30)
\end{aligned}$$

seeing that, for each $a, b \in \mathbb{R}$, $ab \leq \frac{1}{2}(a^2 + b^2)$, where the map $L_{p,q} : I \to \mathbb{R}_+$ is given by

$$L_{p,q}(t) = 2K\zeta_{K_1}(t)\left(\xi_1\left[\left(\frac{T-T_0}{p}\right)^{\frac{1}{2}} + \left(\frac{T-T_0}{q}\right)^{\frac{1}{2}}\right] + \int_{\theta_q(t)}^{t} \psi(s)ds + \int_{\theta_p(t)}^{t} \psi(s)ds\right),$$

for all $t \in I$. Since, by assumption, $\zeta_{K_1}(\cdot), \psi(\cdot) \in L^2_{\mathbb{R}}(I)$, and $\theta_p(t), \theta_q(t) \to t$ when $p, q \to +\infty$, one derives

$$\lim_{p,q\to\infty} L_{p,q}(t) = 0 \text{ a.e. } t \in I.$$

Also, $|L_{p,q}(t)| \leq 4K\zeta_{K_1}(t)\left(\xi_1(T-T_0)^{\frac{1}{2}} + \int_{T_0}^{T} \psi(s)ds\right)$ for every $t \in I$, according to Theorem 2.6.1, it comes

$$\lim_{p,q\to\infty} \int_{T_0}^{T} L_{p,q}(t)dt = 0. \quad (4.3.31)$$

Let p and q denote two arbitrary integers. Remember that for a.e. $t \in I$

$$-\dot{x}_p(t) - \int_{T_0}^t f(t,s,x_p(\theta_p(s)),u_p(\theta_p(s)))ds \in \partial\varphi(t,x_p(t)),$$
$$-\dot{x}_q(t) - \int_{T_0}^t f(t,s,x_q(\theta_q(s)),u_q(\theta_q(s)))ds \in \partial\varphi(t,x_q(t)).$$

The monotony of $\partial\varphi(t,\cdot)$ produces

$$\begin{aligned}&\langle \dot{x}_p(t) - \dot{x}_q(t), x_p(t) - x_q(t)\rangle \\ &\le \Big\langle \int_{T_0}^t f(t,s,x_q(\theta_q(s)),u_q(\theta_q(s)))ds \\ &\quad - \int_{T_0}^t f(t,s,x_p(\theta_p(s)),u_p(\theta_p(s)))ds, x_p(t) - x_q(t)\Big\rangle.\end{aligned} \tag{4.3.32}$$

Taking into consideration (4.3.25)-(4.3.26), choose $K_1 = \max(K,\xi)$, and by (jjj), there is a function $\beta_{K_1}(\cdot) \in L^2_{\mathbb{R}_+}(I)$ such that for a.e. $t \in I$,

$$\begin{aligned}&||f(t,s,x_p(\theta_p(s)),u_p(\theta_p(s))) - f(t,s,x_q(\theta_q(s)),u_q(\theta_q(s)))|| \\ &\le \beta_{K_1}(t)\Big(||x_p(\theta_p(s)) - x_q(\theta_q(s))|| + ||u_p(\theta_p(s)) - u_q(\theta_q(s))||\Big).\end{aligned}$$

Making use of (4.3.28)-(4.3.29) and (4.3.26), Thus, it comes

$$\begin{aligned}&\Big\langle \int_{T_0}^t f(t,s,x_q(\theta_q(s)),u_q(\theta_q(s)))ds - \int_{T_0}^t f(t,s,x_p(\theta_p(s)),u_p(\theta_p(s)))ds, x_p(t) - x_q(t)\Big\rangle \\ &\le \Big(\int_{T_0}^t ||f(t,s,x_q(\theta_q(s)),u_q(\theta_q(s))) - f(t,s,x_p(\theta_p(s)),u_p(\theta_p(s)))||ds\Big)||x_p(t) - x_q(t)|| \\ &\le \beta_{K_1}(t)||x_q(t) - x_p(t)||\int_{T_0}^t \Big(||u_q(s) - u_p(s)|| + ||x_q(s) - x_p(s)||\Big)ds \\ &\quad + 2\xi\beta_{K_1}(t)\int_{T_0}^t\Big(\xi_1\Big[\Big(\frac{T-T_0}{p}\Big)^{\frac{1}{2}} + \Big(\frac{T-T_0}{q}\Big)^{\frac{1}{2}}\Big] + \int_{\theta_q(s)}^s \psi(r)dr + \int_{\theta_p(s)}^s \psi(r)dr\Big)ds.\end{aligned}$$

We return to (4.3.32), it results

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||x_p(t) - x_q(t)||^2 = \langle \dot{x}_p(t) - \dot{x}_q(t), x_p(t) - x_q(t)\rangle \\ &\le \beta_{K_1}(t)||x_q(t) - x_p(t)||\int_{T_0}^t\Big(||u_q(s) - u_p(s)|| + ||x_q(s) - x_p(s)||\Big)ds + F_{p,q}(t) \\ &\le \sqrt{2}\beta_{K_1}(t)||x_q(t) - x_p(t)||\int_{T_0}^t\Big(||u_q(s) - u_p(s)||^2 + ||x_q(s) - x_p(s)||^2\Big)^{\frac{1}{2}}ds \\ &\quad + F_{p,q}(t),\end{aligned}$$

observing that $a+b \leq \sqrt{2}(a^2+b^2)^{\frac{1}{2}}$ with $a,\, b \geq 0$, and $F_{p,q} : I \to \mathbb{R}$ is given by

$$F_{p,q}(t) \\ = 2\xi\beta_{K_1}(t)\int_{T_0}^{t}\left(\xi_1\left[\left(\frac{T-T_0}{p}\right)^{\frac{1}{2}}+\left(\frac{T-T_0}{q}\right)^{\frac{1}{2}}\right]+\int_{\theta_q(s)}^{s}\psi(r)dr+\int_{\theta_p(s)}^{s}\psi(r)dr\right)ds,$$

for every $t \in I$.

Further, note that

$$||x_q(t)-x_p(t)|| \leq (||u_q(t)-u_p(t)||^2+||x_q(t)-x_p(t)||^2\Big)^{\frac{1}{2}},$$

due to $a \leq (a^2+b^2)^{\frac{1}{2}}$ for $a,\, b \geq 0$. Setting $y(t) = ||u_q(t)-u_p(t)||^2+||x_q(t)-x_p(t)||^2$, $t \in I$, one gets

$$\frac{1}{2}\frac{d}{dt}||x_p(t)-x_q(t)||^2 \leq \sqrt{2}\beta_{K_1}(t)(y(t))^{\frac{1}{2}}\int_{T_0}^{t}(y(s))^{\frac{1}{2}}ds+F_{p,q}(t). \tag{4.3.33}$$

Since, by assumption $\beta_{K_1}(\cdot), \psi(\cdot) \in L^2_{\mathbb{R}}(I)$, and $\theta_p(t), \theta_q(t) \to t$ when $p,q \to +\infty$, then (see (4.3.31)), one deduces

$$\lim_{p,q\to\infty} F_{p,q}(t) = 0 \text{ a.e. } t \in I.$$

Also, $|F_{p,q}(t)| \leq 4\xi\beta_{K_1}(t)\left(\xi_1(T-T_0)^{\frac{3}{2}}+\int_{T_0}^{T}\int_{T_0}^{T}\psi(r)drds\right)$ for all $t \in I$.
Theorem 2.6.1 leads to the conclusion that

$$\lim_{p,q\to\infty}\int_{T_0}^{T}F_{p,q}(t)dt = 0. \tag{4.3.34}$$

Summing (4.3.33) and (4.3.30), member to member gives us

$$\dot{y}(t) \leq (3\zeta_{K_1}(t)+\frac{2\kappa(t)}{r}\Big)y(t)+2\sqrt{2}\beta_{K_1}(t)(y(t))^{\frac{1}{2}}\int_{T_0}^{t}(y(s))^{\frac{1}{2}}ds+g(t).$$

For $\varepsilon > 0$, set

$$y(t) = ||u_p(t)-u_q(t)||^2+||x_p(t)-x_q(t)||^2,$$

$$h_1(t) = 3\zeta_{K_1}(t)+\frac{2\kappa(t)}{r},\; h_2(t) = 2\sqrt{2}\beta_{K_1}(t),$$

$$h(t) = \max\left(\frac{h_1(t)}{2},\frac{h_2(t)}{2}\right),\; g(t) = 2(L_{p,q}(t)+F_{p,q}(t)),\text{ for almost all } t \in I.$$

Therefore, Lemma 2.9.3 allows to conclude

$$(y(t))^{\frac{1}{2}} \leq (y(T_0)+\varepsilon)^{\frac{1}{2}} \exp\left(\int_{T_0}^{t}(h(s)+1)ds\right) + \frac{\varepsilon^{\frac{1}{2}}}{2}\int_{T_0}^{t}\exp\left(\int_{s}^{t}(h(r)+1)dr\right)ds$$

$$+2\left[\left(\int_{T_0}^{t} g(s)ds+\varepsilon\right)^{\frac{1}{2}} - \varepsilon^{\frac{1}{2}}\exp\left(\int_{T_0}^{t}(h(r)+1)dr\right)\right]$$

$$+2\int_{T_0}^{t}\left(h(s)+1\right)\exp\left(\int_{s}^{t}(h(r)+1)dr\right)\left(\int_{T_0}^{s} g(r)dr+\varepsilon\right)^{\frac{1}{2}} ds,$$

for every $t \in I$.

According to (4.3.31) and (4.3.34), $||u_p(T_0)-u_q(T_0)||=0$, $||x_p(T_0)-x_q(T_0)||=0$, with $\varepsilon \to 0$, it results

$$\lim_{p,q\to\infty} ||u_p(\cdot)-u_q(\cdot)||_\infty = 0 \text{ and } \lim_{p,q\to\infty} ||x_p(\cdot)-x_q(\cdot)||_\infty = 0.$$

In other words, $(u_n(\cdot))$ uniformly converges on I to some map $u(\cdot)\in \mathcal{C}_H(I)$, as guaranteed by the uniform Cauchy's criterion. Taking (4.3.23) into consideration, one can infer that $(\dot{u}_n(\cdot))$ converges weakly in $L^1_H(I)$ to a map $h \in L^1_H(I)$, and for any $t \in I$, one has

$$\int_{T_0}^{t} \dot{u}_n(s)ds \rightharpoonup \int_{T_0}^{t} h(s)ds \text{ in } H.$$

As $(u_n(t))$ strongly converges to $u(t)$ in H, it follows that $u(t) = u_0 + \int_{T_0}^{t} h(s)ds$. This proves that for all $t\in I$, $u(\cdot)$ is absolutely continuous with $\dot{u} = h$ a.e., and

$$\dot{u}_n(\cdot) \rightharpoonup \dot{u}(\cdot) \text{ in } L^1_H(I). \tag{4.3.35}$$

Furthermore, one infers from (4.3.25)

$$||u(t)|| \leq K,\ t\in I. \tag{4.3.36}$$

Moreover, (4.3.23) and the absolute continuity of $(u_n(\cdot))$ produce for $T_0 \leq \tau \leq t \leq T$,

$$||u_n(t)-u_n(\tau)|| = ||\int_{\tau}^{t}\dot{u}_n(s)ds|| \leq \int_{\tau}^{t}\psi(s)ds.$$

As

$$||u_n(\theta_n(t))-u(t)|| \leq ||u_n(\theta_n(t))-u_n(t)|| + ||u_n(t)-u(t)||$$
$$\leq \int_{\theta_n(t)}^{t}\psi(s)ds + ||u_n(t)-u(t)||,$$

and $\theta_n(t) \to t$ by construction, it follows that

$$||u_n(\theta_n(t)) - u(t)|| \longrightarrow 0, \text{ as } n \to \infty, \text{ for any } \ t \in I. \tag{4.3.37}$$

Now, the aforementioned uniform Cauchy's criterion ensures that

$$(x_n(\cdot)) \text{ uniformly converges on } I \text{ to some map } x(\cdot) \in \mathcal{C}_H(I), \tag{4.3.38}$$

furthermore, one infers from (4.3.26)

$$||x(t)|| \leq \xi, \ t \in I. \tag{4.3.39}$$

Meanwhile, the absolute continuity of $(x_n(\cdot))$ and (4.3.20) produce for $T_0 \leq r \leq t \leq T$,

$$||x_n(t) - x_n(r)|| = || \int_r^t \dot{x}_n(\tau) d\tau || \leq (t-r)^{\frac{1}{2}} (\int_{T_0}^T ||\dot{x}_n(\tau)||^2 d\tau)^{\frac{1}{2}} \leq (t-r)^{\frac{1}{2}} \xi_1.$$

As

$$\begin{aligned} ||x_n(\theta_n(t)) - x(t)|| &\leq ||x_n(\theta_n(t)) - x_n(t)|| + ||x_n(t) - x(t)|| \\ &\leq (t - \theta_n(t))^{\frac{1}{2}} \xi_1 + ||x_n(t) - x(t)||, \end{aligned}$$

and $\theta_n(t) \to t$ by construction, then

$$||x_n(\theta_n(t)) - x(t)|| \longrightarrow 0, \text{ as } n \to \infty, \text{ for any } \ t \in I. \tag{4.3.40}$$

We note that for each n, one has by (jj), (4.3.25), and (4.3.26)

$$||f(t,s,x_n(\theta_n(s)), u_n(\theta_n(s)))|| \leq (1 + K + \xi)\alpha(t,s) \text{ a.e. } t \in I, \tag{4.3.41}$$

where, for any $t \in I$, the map given by $s \mapsto (1+K+\xi)\alpha(t,s)$ is integrable. Along with (4.3.37) and (4.3.40), $f(t,s,\cdot,\cdot)$ is considered to be continuous, therefore for almost every $t \in I$

$$||f(t,s,x_n(\theta_n(s)), u_n(\theta_n(s))) - f(t,s,x(s),u(s))|| \to 0 \text{ as } n \to \infty.$$

For every $t \in I$, we remark that

$$\begin{aligned} &|| \int_{T_0}^t f(t,s,x_n(\theta_n(s)), u_n(\theta_n(s))) ds - \int_{T_0}^t f(t,s,x(s),u(s)) ds || \\ &= || \int_{T_0}^t \Big(f(t,s,x_n(\theta_n(s)), u_n(\theta_n(s))) - f(t,s,x(s),u(s)) \Big) ds || \\ &\leq \int_{T_0}^t ||f(t,s,x_n(\theta_n(s)), u_n(\theta_n(s))) - f(t,s,x(s),u(s))|| ds. \end{aligned}$$

In addition to (4.3.41), Theorem 2.6.1 implies that

$$\lim_{n\to\infty} || \int_{T_0}^{t} f(t,s,x_n(\theta_n(s)),u_n(\theta_n(s)))ds - \int_{T_0}^{t} f(t,s,x(s),u(s))ds|| = 0. \qquad (4.3.42)$$

Observe that the mapping ϕ_n determined in (4.3.9) fulfills

$$||\phi_n(t)|| \leq (1+K+\xi)\int_{T_0}^{t} \alpha(t,s)ds \text{ a.e. } t \in I,$$

employing (4.3.41). Since $\alpha \in L^2_{\mathbb{R}_+}(I\times I)$, the function $\Delta : I \to \mathbb{R}$ such that $\Delta(t) = (1+K+\xi)\int_{T_0}^{T} \alpha(t,s)ds$ is square-integrable. By using this in conjunction with (4.3.42), Theorem 2.6.1 yields

$$\lim_{n\to\infty} \int_{T_0}^{T} || \int_{T_0}^{t} f(t,s,x_n(\theta_n(s)),u_n(\theta_n(s)))ds - \int_{T_0}^{t} f(t,s,x(s),u(s))ds||^2 dt = 0,$$

that is,

$$\phi_n(\cdot) \longrightarrow \phi(\cdot) \text{ in } L^2_H(I), \qquad (4.3.43)$$

where the function $\phi : I \to H$ is $\phi(t) = \displaystyle\int_{T_0}^{t} f(t,s,x(s),u(s))ds$, $t \in I$. Now, note that $(\dot{x}_n(\cdot))$ is bounded in $L^2_H(I)$, as indicated in (4.3.20), so that up to a subsequence that we do not relabel, we can assume that $(\dot{x}_n(\cdot))_n$ converges weakly in $L^2_H(I)$ to some map $w(\cdot) \in L^2_H(I)$. For any integer n, $(x_n(\cdot))_n$ is absolutely continuous. Therefore, for each $y \in H$ and for $T_0 \leq r \leq t \leq T$, one has

$$\int_{T_0}^{T} \langle y\mathbf{1}_{[r,t]}(\tau), \dot{x}_n(\tau)\rangle d\tau = \langle y, x_n(t) - x_n(r)\rangle.$$

Passing to the limit of equality provides (using (4.3.38))

$$\langle y, \int_{r}^{t} w(\tau)d\tau\rangle = \langle y, x(t) - x(r)\rangle.$$

Thus, for each $r,t \in I$: $r \leq t$, we obtain $\int_r^t w(\tau)d\tau = x(t) - x(r)$. Therefore $x(\cdot)$ is absolutely continuous and $w(\cdot) = \dot{x}(\cdot)$ a.e.. As a result, $\dot{x} \in L^2_H(I)$ and

$$\dot{x}_n(\cdot) \rightharpoonup \dot{x}(\cdot) \quad \text{in } L^2_H(I). \qquad (4.3.44)$$

Step 3. First, let us prove the differential inclusion:

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + \int_{T_0}^{t} f(t,s,x(s),u(s))ds \quad \text{a.e. } t \in I. \qquad (4.3.45)$$

Remember that for any n, (4.3.7) holds true. Proposition 3.2.1 gives that $\mathcal{A}$ is a maximal monotone operator, consequently, the differential inclusion (4.3.45) holds true in conjunction with the previous modes of convergence (4.3.38), (4.3.43), and (4.3.44). Now let's confirm the velocity estimates. By considering (4.3.43) and (4.3.44), one can pass to the inferior limit on n in (4.3.19) and obtain

$$\int_{T_0}^{T} ||\dot{x}(t)||^2 dt \leq d + \sigma \int_{T_0}^{T} || \int_{T_0}^{t} f(t,s,x(s),u(s))ds||^2 dt.$$

By (jj) and the Cauchy-Schwartz inequality, we write

$$\int_{T_0}^{T} ||\dot{x}(t)||^2 dt \leq d + 2\sigma(T-T_0) \int_{T_0}^{T} \int_{T_0}^{t} \alpha^2(t,s)(1+||x(s)||+||u(s)||)^2 ds dt.$$

Using (4.3.36) in conjunction with (4.3.39), the latter inequality is then combined to get (4.3.1).

Let's now prove the system's second differential inclusion of (CP)

$$\dot{u}(t) + g(t,x(t),u(t)) \in -N_{C(t)}(u(t)) \text{ a.e. } t \in I. \tag{4.3.46}$$

From (iii), (4.3.37), and (4.3.40), it can be inferred that

$$g(t,x_n(\theta_n(t)),u_n(\theta_n(t))) \to g(t,x(t),u(t)),\ t \in I.$$

Theorem 2.6.1 gives

$$g(\cdot,x_n(\theta_n(\cdot)),u_n(\theta_n(\cdot))) \to g(\cdot,x(\cdot),u(\cdot)) \text{ in } L^1_H(I), \tag{4.3.47}$$

with the help of (4.3.21). As well, we obtain by (ii), (4.3.36), (4.3.39)

$$||g(t,x(t),u(t))|| \leq \gamma(t)(1+K+\xi) \text{ a.e. } t \in I.$$

Mazur's lemma allow to deduce that, given (4.3.35), and (4.3.47), there is a sequence $(y_n(\cdot))$ that strongly converges in $L^1_H(I)$ to $\dot{u}(\cdot)+g(\cdot,x(\cdot),u(\cdot))$ that is, for every $t \in I$ and every $n \in \mathbb{N}$

$$y_n(t) \in \text{co}\{\dot{u}_p(t) + g(t,x_p(\theta_p(t)),u_p(\theta_p(t))) :\ p \geq n\}.$$

Thus, we could extract a subsequence so that

$$y_n(t) \to \dot{u}(t) + g(t,x(t),u(t)) \text{ a.e. } t \in I,$$

then for almost every $t \in I$

$$\dot{u}(t)+g(t,x(t),u(t)) \in \bigcap_n \overline{\text{co}}\{\dot{u}_p(t)+g(t,x_p(\theta_p(t)),u_p(\theta_p(t))) : \, p \geq n\}.$$

Hence, for all $t \in I$, for almost $z \in H$,

$$\langle z, \dot{u}(t)+g(t,x(t),u(t))\rangle \leq \inf_n \sup_{p\geq n} \langle z, \dot{u}_p(t)+g(t,x_p(\theta_p(t)),u_p(\theta_p(t)))\rangle.$$

Given (2.8.2), (4.3.7), and (4.3.22), it can be inferred that

$$\begin{aligned}\langle z, \dot{u}(t)+g(t,x(t),u(t))\rangle &\leq \kappa(t) \limsup_n \delta^*(-\partial^P d(u_n(t),C(t)),z)\\ &\leq \kappa(t) \limsup_n \delta^*(-\partial^C d(u_n(t),C(t)),z).\end{aligned}$$

However, for all $t \in I$, $\delta^*(-\partial^C d(\cdot, C(t)),z)$ is upper semi-continuous on H, for almost all $t \in I$, for all $z \in H$, it follows that

$$\langle z, \dot{u}(t)+g(t,x(t),u(t))\rangle \leq \kappa(t)\delta^*(-\partial^C d(u(t),C(t)),z).$$

For all $t \in I$, the Clarke subdifferential $\partial^C d(u(t),C(t))$ is closed convex, by using Proposition 2.7.1, one obtains

$$\dot{u}(t)+g(t,x(t),u(t)) \in -\kappa(t)\partial^C d(u(t),C(t)) \text{ a.e. } t \in I,$$

in conjunction with the inclusion (2.8.3) leads (4.3.46).

Proposition 4.2.2 entails for a.e. $t \in I$

$$||\dot{u}(t)+g(t,x(t),u(t))|| \leq ||g(t,x(t),u(t))|| + |\dot{v}(t)|.$$

In view of (ii) produces

$$||\dot{u}(t)+g(t,x(t),u(t))|| \leq \gamma(t)(1+||x(t)||+||u(t)||)+|\dot{v}(t)|. \tag{4.3.48}$$

Then, it comes

$$||\dot{u}(t)|| \leq 2\gamma(t)(1+||x(t)||+||u(t)||)+|\dot{v}(t)|.$$

The latter inequality, (4.3.36) and (4.3.39) ensure the validity of the estimate (4.3.2). By making use of (4.3.45)-(4.3.46) in conjunction with $(x(T_0),u(T_0)) = (x_0,u_0)$, the existence of a $(x,u) : I \to H \times H$ solution to (CP) is guaranteed.

Part 2: Uniqueness of the solution. Let us consider two solutions to (CP) (x_1,u_1), (x_2,u_2). In particular, since the absolutely continuous mappings x_i and u_i

are bounded on I, we can select a real constant $S > 0$, such that for every $i = 1,2$, $||x_i(t)|| \leq S$ and $||u_i(t)|| \leq S$ for all $t \in I$. Estimate (4.3.48) states that one has for $i = 1,2$, for almost all $t \in I$

$$||\dot{u}_i(t) + g(t, x_i(t), u_i(t))|| \leq (1+2S)\gamma(t) + |\dot{v}(t)|.$$

Given the hypomonotony of the normal cone and the Lipschitz property of g in (iii), we write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u_2(t)-u_1(t)||^2 \leq & \Big(\zeta_S(t) + \frac{\kappa_1(t)}{r}\Big)||u_2(t)-u_1(t)||^2 \\ & + \zeta_S(t)||u_2(t)-u_1(t)||||x_2(t)-x_1(t)|| \\ \leq & \Big(\frac{3}{2}\zeta_S(t) + \frac{\kappa_1(t)}{r}\Big)\Big(||u_2(t)-u_1(t)||^2 + ||x_2(t)-x_1(t)||^2\Big),\end{aligned} \tag{4.3.49}$$

as in (4.3.30), where $\kappa_1(t) = (1+2S)\gamma(t) + |\dot{v}(t)|$ for each $t \in I$.
Additionally, because $\partial\varphi(t,\cdot)$ is monotone and f has the Lipschitz property (jjj), it leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|x_2(t)-x_1(t)\|^2 \\ &\leq \sqrt{2}\beta_S(t)\|x_2(t)-x_1(t)\| \int_{T_0}^{t} \Big(\|u_2(s)-u_1(s)\|^2 + \|x_2(s)-x_1(s)\|^2\Big)^{\frac{1}{2}} ds \\ &\leq \sqrt{2}\beta_S(t)\Big(y(t)\Big)^{\frac{1}{2}} \int_{T_0}^{t} \Big(y(s)\Big)^{\frac{1}{2}} ds,\end{aligned} \tag{4.3.50}$$

as in (4.3.33) with $y(t) = \|u_2(t)-u_1(t)\|^2 + \|x_2(t)-x_1(t)\|^2$ for any $t \in I$. Summing (4.3.49)-(4.3.50) member to member yields

$$\dot{y}(t) \leq \Big(3\zeta_S(t) + \frac{2\kappa_1(t)}{r}\Big)y(t) + 2\sqrt{2}\beta_S(t)\Big(y(t)\Big)^{\frac{1}{2}} \int_{T_0}^{t} \Big(y(s)\Big)^{\frac{1}{2}} ds,$$

for any $t \in I$. Using Lemma 2.9.3, one obtains $(x_1,u_1) = (x_2,u_2)$ for any $\varepsilon > 0$. The solution is therefore unique. Thus, the theorem's proof is complete. □

4.4 Application to optimal control theory

We provide an application of optimal control theory in this section. Our primary goal is to minimize the following Bolza type functional

$$\min L_0[x,u,z], \tag{4.4.1}$$

subject to the set of controls z and the appropriate solutions (x,u) to the dynamical system

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \int_0^t f(x(s),u(s))ds \quad \text{a.e. } t \in [0,T], \\ -\dot{u}(t) \in N_{C(t)}(u(t)) + g(x(t),u(t),z(t)) \quad \text{a.e. } t \in [0,T], \\ z(\cdot) \in W^{1,2}_{\mathbb{R}^d}(I), \\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot),\ u(0) = u_0 \in C(0), \end{cases} \tag{4.4.2}$$

whereas $L_0[x,u,z] = \int_0^T J_0(t,x(t),u(t),z(t),\dot{x}(t),\dot{u}(t),\dot{z}(t))dt$. The cost functional $J_0 : [0,T]\times\mathbb{R}^{4n+2d} \to [0,+\infty[$ and the mappings $f:\mathbb{R}^{2n}\to\mathbb{R}^n$, $g:\mathbb{R}^{2n+d}\to\mathbb{R}^n$ fulfill adequate assumptions.

The dynamical system (4.4.2) can be represented in the following way

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + y(t) \quad \text{a.e. } t \in [0,T], \\ \dot{y}(t) = f(x(t),u(t)) \quad \text{a.e. } t \in [0,T], \\ -\dot{u}(t) \in N_{C(t)}(u(t)) + g(x(t),u(t),z(t)) \quad \text{a.e. } t \in [0,T], \\ z(\cdot) \in W^{1,2}_{\mathbb{R}^d}(I), \\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot),\ u(0) = u_0 \in C(0),\ y(0) = 0. \end{cases} \tag{4.4.3}$$

The appropriate running cost $J : [0,T]\times\mathbb{R}^{6n+2d} \to [0,+\infty[$

$$J(t,x(t),y(t),u(t),z(t),\dot{x}(t),\dot{y}(t),\dot{u}(t),\dot{z}(t)) := J_0(t,x(t),u(t),z(t),\dot{x}(t),\dot{u}(t),\dot{z}(t)), \tag{4.4.4}$$

is associated to (4.4.3), in order to reformulate the optimal control problem as follows

$$\min L[x,y,u,z],$$

subject to the set of solutions (x,y,u,z) to the dynamical system (4.4.3), as well as

$$L[x,y,u,z] = \int_0^T J(t,x(t),y(t),u(t),z(t),\dot{x}(t),\dot{y}(t),\dot{u}(t),\dot{z}(t))dt.$$

We may now establish the main theorem of this section.

Theorem 4.4.1. *Suppose that* $\varphi : I \times \mathbb{R}^n \to [0, +\infty]$ *is a map which fulfills* (H_1)-(H_2). *Given a multi-valued map* $C : I \rightrightarrows \mathbb{R}^n$, *let* (H_3)-(H_4) *hold true. Given two continuous mappings,* $f : \mathbb{R}^{2n} \to \mathbb{R}^n$ *and* $g : \mathbb{R}^{2n+d} \to \mathbb{R}^n$ *such that*

$(j)'$ *a constant* $\alpha > 0$ *exists such that, for any* $x, u \in \mathbb{R}^n$, *one has*

$$||f(x,u)|| \leq \alpha(1 + ||x|| + ||u||);$$

$(jj)'$ *a constant* $\beta > 0$ *exists for each* $\eta > 0$ *such that for each* $x, y, u, v \in \overline{B}_{\mathbb{R}^n}[0, \eta]$, *one has*

$$||f(x,u) - f(y,v)|| \leq \beta(||x - y|| + ||u - v||);$$

$(i)'$ *a constant* $\gamma > 0$ *exists such that, for every pair of values* $x, u \in \mathbb{R}^n$, $w \in \mathbb{R}^d$, *one has*

$$||g(x,u,w)|| \leq \gamma(1 + ||x|| + ||u||);$$

$(ii)'$ *for each* $\eta > 0$, *there is a constant* $\zeta > 0$, *for every* $w \in \mathbb{R}^d$ *and* $x, y, u, v \in \overline{B}_{\mathbb{R}^n}[0, \eta]$

$$||g(x,u,w) - g(y,v,w)|| \leq \zeta(||x - y|| + ||u - v||).$$

Assume that the cost functional $J_0 : I \times \mathbb{R}^{4n+2d} \to [0, +\infty[$ *from* (4.4.4) *is a measurable map and, for every* $t \in I$, $J_0(t, \cdot)$ *is lower semi-continuous on* $\mathbb{R}^{4n+2d}$. *Let us further assume that the cost functional* J_0 *is convex with respect to the velocity variables* $\dot{x}$, $\dot{u}$, *and* $\dot{z}$. *Furthermore, we assume that* $(z_n(\cdot))$ *is bounded in* $W^{1,2}_{\mathbb{R}^d}(I)$ *along a minimizing sequence* $(x_n(\cdot), u_n(\cdot), z_n(\cdot))$ *of* (4.4.1). *Hence, there is an optimal solution for problem* (4.4.1).

Proof. First, note that Theorem 4.3.1 guarantees the existence, uniqueness of the solution $(x(\cdot), u(\cdot))$, and suitable estimates see (4.3.1), (4.3.2) for a fixed $z(\cdot) \in W^{1,2}_{\mathbb{R}^d}(I)$. Take the minimizing sequence $(x_n(\cdot), u_n(\cdot), z_n(\cdot))$ of the problem (4.4.1), that is,

$$\lim_{n\to\infty}\int_0^T J_0(t,x_n(t),u_n(t),z_n(t),\dot{x}_n(t),\dot{u}_n(t),\dot{z}_n(t))dt$$
$$=\inf_{(v,w,h)}\int_0^T J_0(t,v(t),w(t),h(t),\dot{v}(t),\dot{w}(t),\dot{h}(t))dt, \tag{4.4.5}$$

where the set of controls $h(\cdot)$ and the corresponding solutions $(v(\cdot),w(\cdot))$ to the dynamical system (4.4.2) are taken over by the inf. The sequence $(\dot{z}_n(\cdot))$ is weakly compact in $L^2_{\mathbb{R}^d}(I)$ since the sequence $(z_n(\cdot))$ is bounded in $W^{1,2}_{\mathbb{R}^d}(I)$ by assumption. Therefore, by applying Theorem 2.2.3, we can show that there exists a map $z(\cdot)\in W^{1,2}_{\mathbb{R}^d}(I)$ such that $(z_n(\cdot))$ uniformly converges to $z(\cdot)$ and $(\dot{z}_n(\cdot))$ converges weakly to $\dot{z}(\cdot)$ in $L^2_{\mathbb{R}^d}(I)$.

One has $\sup\limits_{n\in\mathbb{N}}\|\dot{u}_n(\cdot)\|_{L^2_{\mathbb{R}^n}(I)}<+\infty$ (see (4.3.2)) according to Theorem 4.3.1, and the sequence $(u_n(\cdot))$ is equi-continuous and uniformly bounded on I. Once more, by Theorem 2.2.3, there is a map $u(\cdot)\in W^{1,2}_{\mathbb{R}^n}(I)$ such that $(u_n(\cdot))$ uniformly converges to $u(\cdot)$, and $\dot{u}(\cdot)$ converges weakly to $\dot{u}(\cdot)$ in $L^2_{\mathbb{R}^n}(I)$.

In addition, a map $x(\cdot)\in W^{1,2}_{\mathbb{R}^n}(I)$ exists in the spirit of (4.3.1), such that $(x_n(\cdot))$ converges uniformly to $x(\cdot)$ and $(\dot{x}_n(\cdot))$ converges weakly to $\dot{x}(\cdot)$ in $L^2_{\mathbb{R}^n}(I)$.

Let $y_n(t)=\int_0^t f(x_n(s),u_n(s))ds$, for all $t\in I$, and for each n. Since the function f is continuous, the above convergence modes leads to

$$f(x_n(s),u_n(s))\to f(x(s),u(s)) \text{ a.e. } s\in I.$$

Taking $(j)'$ into consideration, along with the uniform boundedness of $(u_n(\cdot))$ and $(x_n(\cdot))$ on I, Theorem 2.6.1 gives

$$y_n(t)\to y(t), \text{ as } n\to\infty \text{ a.e. } t\in I, \tag{4.4.6}$$

with $y:I\to\mathbb{R}^n$ is given by $y(t)=\int_0^t f(x(s),u(s))ds$, $t\in I$.

Now, from $(j)'$ and the uniform boundedness of $(u_n(\cdot))$ and $(x_n(\cdot))$ on I, there is $S>0$ such that, for any $t\in I$, $||y_n(t)||\le S$. In conjunction with (4.4.6), thus, Theorem 2.6.1 leads

$$y_n(\cdot)\to y(\cdot) \text{ in } L^2_{\mathbb{R}^n}(I) \text{ as } n\to\infty.$$

An application of Theorem 2.9.1, one can infer

$$L[x,y,u,z] = L_0[x,u,z] \leq \liminf_{n\to\infty} L_0[x_n,u_n,z_n] = \liminf_{n\to\infty} L[x_n,y_n,u_n,z_n],$$

thus,

$$\begin{aligned}&\int_0^T J_0(t,x(t),u(t),z(t),\dot{x}(t),\dot{u}(t),\dot{z}(t))dt\\ &\leq \liminf_{n\to\infty}\int_0^T J_0(t,x_n(t),u_n(t),z_n(t),\dot{x}_n(t),\dot{u}_n(t),\dot{z}_n(t))dt,\end{aligned}$$

where J_0 is defined in (4.4.4). According to (4.4.5), it can be concluded that

$$\begin{aligned}&\inf_{(v,w,h)}\int_0^T J_0(t,v(t),w(t),h(t),\dot{v}(t),\dot{w}(t),\dot{h}(t))dt\\ &=\int_0^T J_0(t,x(t),u(t),z(t),\dot{x}(t),\dot{u}(t),\dot{z}(t))dt.\end{aligned}$$

Note that for all $n\in\mathbb{N}$, $(x_n(\cdot),u_n(\cdot))$ is the unique solution related to the control map $z_n(\cdot)$ to the system

$$\begin{cases} -\dot{x}_n(t)\in\partial\varphi(t,x_n(t))+\int_0^t f(x_n(s),u_n(s))ds \quad \text{a.e. } t\in I,\\ -\dot{u}_n(t)\in N_{C(t)}(u_n(t))+g(x_n(t),u_n(t),z_n(t)) \quad \text{a.e. } t\in I,\\ \quad z_n(\cdot)\in W^{1,2}_{\mathbb{R}^d}(I),\\ \quad x_n(0)=x_0\in\operatorname{dom}\varphi(0,\cdot),\ u_n(0)=u_0\in C(0).\end{cases}$$

In conjunction with the modes of convergence mentioned above, thus, Proposition 3.2.1 produces

$$-\dot{x}(t)\in\partial\varphi(t,x(t))+\int_0^t f(x(s),u(s))ds \text{ a.e. } t\in I.$$

Observe that g is continuous. According to the previously mentioned modes of convergence, it can be deduced that for a.e., $t\in I$

$$\lim_{n\to\infty}\|g(x_n(t),u_n(t),z_n(t))-g(x(t),u(t),z(t))\|=0.$$

In order to prove the second differential inclusion in (4.4.2), we then argue as in **Step 3** of Theorem 4.3.1 proof. Recall that for every n, there is $\eta(\cdot)\in L^2_{\mathbb{R}}(I)$ such that

$$||\dot{u}_n(t)+g(x_n(t),u_n(t),z_n(t))||\leq\eta(t) \text{ for any } t\in I.$$

For all $t \in I$, the Clarke subdifferential $\partial^C d(u(t), C(t))$ is closed convex. This leads to

$$\dot{u}(t) + g(x(t), u(t), z(t)) \in -\eta(t)\partial^C d(u(t), C(t)) \subset -N_{C(t)}(u(t)) \text{ a.e. } t \in I,$$

with the help of the inclusion (2.8.3). As a result, for the dynamical system (4.4.2), the unique solution associated with the control map $z(\cdot)$ is $(x(\cdot), u(\cdot))$. That is completes the theorem's proof. □

Conclusion and future researches

In this thesis, we have been interested in some evolutionary problems for a particular class of differential inclusions in Hilbert spaces. Using Schauder's fixed point theorem, we have proved in the third chapter the well-posedness of a new differential inclusion of subdifferential type with integral perturbation. Then, we have successfully proved theorems regarding coupled systems with fractional differential equations. The ideas used there can be extended to the case of other fractional systems and could provide some light on the analysis of fractional order systems in optimal control theory.

A novel dynamical system coupled by two first-order differential inclusions has been handled in the fourth chapter. The first differential inclusion in the system, is a non-convex perturbed sweeping process, while the second one is driven by time-dependent subdifferential operators with integral perturbation. We follow a discretization approach in our development. The corresponding well-posedness result has been then used in optimal control theory. Among open problems are: the dynamical system of first-order mixed partially bounded variation sweeping process involving differential inclusion of subdifferential type with new applications and the study of numerical solution. Also, we aim in our future contributions to generalize the established results to a more general setting: in Banach spaces for example.

Bibliography

[1] L. Adam, J. Outrata, On optimal control of a sweeping process coupled with an ordinary differential equation, Discrete Contin. Dyn. Syst. Ser. B, 19 (2014), 2709-2738.

[2] S. Adly, H. Attouch, Finite convergence of proximal-gradient inertial algorithms combining dry friction with Hessian-driven damping, SIAM J. Optim., 30 (3) (2020), 2134-2162.

[3] S. Adly, H. Attouch, A. Cabot, Finite time stabililization of nonlinear oscillators subject to dry friction, Nonsmooth mechanics and analysis, Adv. Mech. Math, 12 (2006), 289-304.

[4] R.P. Agarwal, B. Ahmad, A. Alsaedi, N. Shahzad, Dimension of the solution set for fractional differential inclusion, J. Nonlinear Convex Anal. 14 (2) (2013), 314-323.

[5] R.P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu, Fuzzy fractional integral equations under compactness type condition, Fract. Calc. Appl. Anal. 15 (4) (2012), 572-590.

[6] B. Ahmad, J. Nieto, Riemann-Liouville fractional differential equations with fractional boundary conditions, Fixed Point Theory 13 (2) (2012), 329-336.

[7] M. Aissous, F. Nacry, V. A. T. Nguyen, First and second order state-dependent bounded subsmooth sweeping processes, Linear Nonlinear Anal., 6 (2020), 447-472.

[8] H. Attouch, D. Damlamian, Problèmes d'évolution dans les Hilberts et applications, J. Math. Pures Appl. 54 (1975), 53-74.

[9] H. Attouch, P.E. Maingé, P. Redont, A second-order differential system with hessian driven damping; application to non-elastic shock laws, Differ. Equ. Appl., 4 (1) (2012), 27-65.

[10] J. P. Aubin, A. Cellina, Differential inclusions set-valued maps and Viability theory. Springer-Verlag, Berlin, (1984).

[11] R. J. Aumann, Integrals of Set-valued functions, J. Math. Anal. Appl., 12 (1) (1965), 1-12.

[12] M. Benchohra, J. Graef, F.Z. Mostefai, Weak solutions for boundary-value problems with nonlinear fractional differential inclusions, Nonlinear Dyn. Syst. Theory 11 (3) (2011), 227-237.

[13] M. Benguessoum, D. Azzam-Laouir, C. Castaing, On a time and state dependent maximal monotone operator coupled with a sweeping process with perturbations, Set-Valued Var. Anal., 29 (2021), 191-219.

[14] A. Bouabsa, S. Saïdi, Coupled systems of subdifferential type with integral perturbation and fractional differential equations. Adv. Theory Nonlinear Anal. Appl., 7 (1) (2023), 253–271.

[15] A. Bouabsa, S. Saïdi, On a system involving an integro-differential inclusion with subdifferential and caputo fractional derivative, Matematiche (Catania), 78 (2) (2023), 289–315.

[16] A. Bouach, T. Haddad, B.S. Mordukhovich, Optimal control of nonconvex integro-differential sweeping processes, J. Differential Equations, 329 (2022), 255-317.

[17] A. Bouach, T. Haddad, L. Thibault, Nonconvex integro-differential sweeping process with applications, SIAM J. Control Optim., 60 (2022), 2971-2995.

[18] A. Bouach, T. Haddad, L. Thibault, On the discretization of truncated integro-differential sweeping process and optimal control, J. Optim. Theory Appl., 193 (1-3) (2022), 785-830.

[19] Y. Brenier, W. Gangbo, G. Savare, M. Westdickenberg, Sticky particle dynamics with interactions, J. Math. Pures Appl. 99 (2013), 577-617.

[20] H. Brézis, Analyse fonctionnelle, théorie et applications, Masson, Paris, (1983).

[21] H. Brézis, Functional analysis, Sobolev spaces and partial differential equations., New York: Springer, (2011).

[22] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, Lecture Notes in Math., North-Holland, (1973).

[23] L. M. Briceño-Arias, N. D. Hoang, J. Peypouquet, Existence, stability and optimality for optimal control problems governed by maximal monotone operators, J. Differential Equations, 260 (2016), 733-757.

[24] T. H. Cao, B. S. Mordukhovich, Optimal control of a nonconvex perturbed sweeping process, J. Differential Equations, 266 (2019), 1003-1050.

[25] C. Castaing, C. Godet-Thobie, M.D.P. Monteiro Marques, A. Salvadori, Evolution problems with m-accretive operators and perturbations, Mathematics 10 (3) (2022), 317.

[26] C. Castaing, C. Godet-Thobie, P.D. Phung, L.T. Truong, On fractional differential inclusions with nonlocal boundary conditions, Fract. Calc. Appl. Anal. 22 (2019), 444-478.

[27] C. Castaing, C. Godet-Thobie, P.D. Phung, L.T. Truong, Fractional order of evolution inclusion coupled with a time and state dependent maximal monotone operator, Mathematics 8 (9) (2020), 1395.

[28] C. Castaing, C. Godet-Thobie, S. Saïdi, On fractional evolution inclusion coupled with a time and state dependent maximal monotone operator, Set-Valued Var. Anal., 30 (2) (2022), 621-656.

[29] C. Castaing, C. Godet-Thobie, S. Saïdi, M.D.P. Monteiro Marques, Various perturbations of time dependent maximal monotone/accretive operators in evolution inclusions with applications, Appl. Math. Optim., 87 (24) (2023).

[30] C. Castaing, C. Godet-Thobie, L.T. Truong, F.Z. Mostefai, On a fractional differential inclusion in Banach space under weak compactness condition, Adv. Math. Econ. 20 (2016), 23-75.

[31] C. Castaing, C. Godet-Thobie, L.T. Truong, B. Satco, Optimal control problems governed by a second order ordinary differential equation with m-point boundary condition, Adv. Math. Econ. 18 (2014), 1-59.

[32] C. Castaing, P. Raynaud de Fitte, M. Valadier, Young measures on topological spaces with applications in control theory and probability theory, Kluwer Academic Publishers, Dordrecht, (2004).

[33] C. Castaing, S. Saïdi, Lipschitz perturbation to evolution inclusion driven by time-dependent maximal monotone operators, Topol. Methods Nonlinear Anal., 58 (2) (2021), 677-712.

[34] C. Castaing, M. Valadier, Convex analysis and measurable multifunctions, Lecture Notes in Math, 580, Springer-Verlag Berlin Heidelberg, (1977).

[35] F. H. Clarke, Optimization and nonsmooth analysis, Wiley Inter science, New York, (1983).

[36] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, P. R. Wolenski, Nonsmooth analysis and control theory, Springer-Verlag, New York, Inc., (1998).

[37] G. Colombo, R. Henrion, N. D. Hoang, B. S. Mordukhovich, Optimal control of the sweeping process, Dyn. Contin. Discrete Impuls. Syst. Ser. B, 19 (2012), 117-159.

[38] G. Colombo, R. Henrion, N. D. Hoang, B. S. Mordukhovich, Discrete approximations of a controlled sweeping process, Set-Valued Var. Anal., 23 (2015), 69-86.

[39] G. Colombo, R. Henrion, N. D. Hoang, B. S. Mordukhovich, Optimal control of the sweeping process over polyhedral controlled sets, J. Differential Equations, 260 (2016), 3397-3447.

[40] G. Colombo, C. Kozaily, Existence and uniqueness of solutions for an integral perturbation of Moreau‘s sweeping process, J. Convex Anal. 27 (2020), 227-236.

[41] G. Colombo, M. Palladino, The minimum time function for the controlled Moreau's sweeping process, SIAM J. Control Optim., 54 (2016), 2036-2062.

[42] M. d. R. de Pinho, M. M. A. Ferreira, G. V. Smirnov, Optimal control involving sweeping processes, Set-Valued Var. Anal., 27 (2019), 523-548.

[43] M. d. R. de Pinho, M. M. A. Ferreira, G. V. Smirnov, Optimal control with sweeping processes: Numerical method, J. Optim. Theory Appl., 185 (2020), 845-858.

[44] K. Deimling, Non linear functional analysis, Springer, Berlin, (1985).

[45] J. F. Edmond, L. Thibault, Relaxation of an optimal control problem involving a perturbed sweeping process, Math. Program., Ser. B, 104 (2005), 347-373.

[46] F. Fennour, S. Saïdi, A minimization problem subject to a coupled system by maximal monotone operators, Bol. Soc. Mat. Mex. 29 (3) (2023), 78.

[47] S. Guillaume, A. Syam, On a time-dependent subdifferential evolution inclusion with a nonconvex upper-semicontinuous perturbation, Electron. J. Qual. Theory Differ. Equ. 11 (2005), 1-22.

[48] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, Math. Studies 204, North Holland: Amsterdam, The Netherlands, 2006.

[49] M. Kubo, Characterisation of a class of evolution operators generated by time dependent subdifferential, Funkcial. Ekvac., 32 (1989), 301-321.

[50] B. K. Le, On properties of differential inclusions with prox-regular sets, Pac. J. Optim. 13 (2017), 17-27.

[51] D. Lecomte, weak topologies, (2006).

[52] M. Mansour, I. Kecis, Nonconvex, T. Haddad, Integro-differential sweeping processes involving maximal monotone operators, Hacet. J. Math. Stat. 52 (6) (2023), 1677-1960.

[53] B. S. Mordukhovich, Y. Shao, Nonsmooth sequential analysis in Asplund spaces, Trans. Amer. Math. Soc., 348 (1996), 1235-1280.

[54] J. J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space, J. Differential Equations 26 (1977), 347-374.

[55] J. J. Moreau, Rafle par un convexe variable, I, Sém. Anal. Convexe, Montpellier, Vol. 1, 1971, Exposé No. 15.

[56] J. J. Moreau, Rafle par un convexe variable, II, Sém. Anal. Convexe, Montpellier, Vol. 2, 1972, Exposé No. 3.

[57] J. J. Moreau, Rétraction d'une multiapplication, Sém. Anal. convexe, Montpellier, 1972, Exposé 13.

[58] F. Nacry, J. Noel, L. Thibault, On first and second order state-dependent sweeping processes, Pure Appl. Funct. Anal., 6 (2021), 1453-1493.

[59] F. Nacry, L. Thibault, Regularization of sweeping process, old and new, Pure Appl. Funct. Anal., 4 (2019), 59-117.

[60] M. Otani, Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, Cauchy problems, J. Differential Equations, 46 (1982), 268-299.

[61] J. C. Peralba, Équations d'évolution dans un espace de Hilbert, associées à des opérateurs sous-Différentiels, Ph. D thesis, University of Montpellier II in Montpellier, 1973.

[62] P.H. Phung, L.X. Truong, On a fractional differential inclusion with integral boundary conditions in Banach space, Fract. Calc. Appl. Anal. 16 (3) (2013), 538-558.

[63] R. A. Poliquin, R. T. Rockafellar, L. Thibault, Local differentiability of distance functions, Trans. Amer. Math. Soc., 352 (2000), 5231-5249.

[64] S. Saïdi, Coupled problems driven by time and state-dependent maximal monotone operators, Numer. Algebra Control Optim. 10.3934/naco.2023006 (to apear)

[65] S. Saïdi, Set-valued perturbation to second-order evolution problems with time-dependent subdifferential operators, Asian-Eur. J. Math., 15 (7) (2022), 1-20.

[66] S. Saïdi, Some results associated to first-order set-valued evolution problems with subdifferentials, J. Nonlinear Var. Anal. 5 (2) (2021), 227-250.

[67] S. Saïdi, A. Bouabsa, A coupled problem described by time-dependent subdifferential operator and non-convex perturbed sweeping process. Evol. Equ. Control Theory, 12 (4) (2023), 1145–1173.

[68] S. Saïdi, F. Fennour, Second-order problems involving time-dependent subdifferential operators and application to control, Math. Control Relat. Fields., 13 (3) (2023), 873-894.

[69] S. Saïdi, L. Thibault, M. Yarou, Relaxation of optimal control problems involving time dependent subdifferential operators, Numer. Funct. Anal. Optim., 34 (2013), 1156-1186.

[70] S. Saïdi, M. F. Yarou, Set-valued perturbation for time dependent subdifferential operator, Topol. Methods Nonlinear Anal., 46 (2015), 447-470.

[71] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives: Theory and applications, Gordon and Breach, New York, (1993).

[72] L. Schwartz, Analyse III. Hermann, Paris, (1998).

[73] W. Sutherland, Introduction to metric and topological spaces, Oxford University Press, (2009).

[74] Y. Sonntag, Topologie et analyse fonctionnelle, ellipses, édition marketing S.A, (1998).

[75] L. Thibault, Sweeping process with regular and nonregular sets, J. Differential Equations 193 (2003), 1-26.

[76] A.A. Tolstonogov, Properties of attainable sets of evolution inclusions and control systems of subdifferential type, Sib. Math. J. 45(4) 2004 763-784.

[77] J. Watanabe, On certain nonlinear evolution equations, J. Math. Soc. Japan 25 (1973), 446-463.

[78] S. Yotsutani, Evolution equations associated with the subdifferentials, J. Math. Soc. Japan 31 (1978), 623-646.