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The study of some multivalued differential equations and applications

Defended publicly on : In front of the jury composed of

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This thesis is dedicated

to my parents.

Contents

Notations

We list here the principal constructs that appear in the thesis.

Operations and Symbols

$:=$	Equal by definition.
$\equiv$	Identically equal.
$i.e.$	Identically equivalent.
$a.e.$	Almost every.
$s.t.$	Such that.
$resp$	Respectively.
$\langle \cdot, \cdot \rangle$	Inner product on a Hilbert space.
$\lVert \cdot \rVert$	Norm of a Hilbert space.
$\lvert \cdot \rvert$	Euclidean norm.
$\sup, \inf, \max, \min$	Supremum, Infimum, Maximum, Minimum, respectively.
$u_n \longrightarrow u$	u_n converges to u strongly.
$u_n \rightharpoonup u$	u_n converges to u weakly (in weak topology).
$u_n \xrightarrow{S} u$	$u_n \longrightarrow u$ and $u_n \in S$ for all n.
$u_n \xrightarrow{f} u$	$u_n \longrightarrow u$ and $f(u_n) \longrightarrow f(u)$ for all n.
$u.s.c$	Upper semicontinuous.
$l.s.c$	Lower semicontinuous.

Sets

$\mathbb{B}$ or $\mathbb{B}_H$	Closed unit ball of space H.
$\mathrm{co}(S)$	Convex hull of S.
$\overline{\mathrm{co}}(S)$	Closed convex hull of S.
$\mathrm{cone}(A)$	$:= \Big\{ \sum_{i=1}^{n} \lambda_i x_i : \ n \geq 1,\ x_i \in A,\ \lambda_i \geq 0 \}$.
$\mathrm{bdr}\,(S)$	Boundary of S.
$\mathrm{int}\,(S)$	Interior of S.
$\mathrm{epi}\,(f)$	Epigraph of an extended real valued function f.
$\mathrm{Dom}\,(F)$	Effectif domain of a set-valued mapping F.
$\mathrm{D}\,(f)$	Effectif domain of an extended real single-valued mapping f.
$\mathcal{R}(F)$	The range of a set-valued map F.
$\mathrm{gph}\,(F)$	Graph of a set-valued map F.
$\Gamma_0(X)$	The set of all lower semi-continuous proper convex functions on X.
$N_S^C(x)$ or $N^C(x,S)$	Clarke normal cone of S at x.
$N_S^F(x)$ or $N^F(x,S)$	Fréchet normal cone of S at x.
$N_S^P(x)$ or $N^P(x,S)$	Proximal normal cone of S at x.
$N_S^L(x)$ or $N^L(x,S)$	Mordukhovich limiting (basic) normal cone of S at x.
$\partial_P\, f(x)$	Proximal subdifferential of f at x.
$\partial_L\, f(x)$	Mordukhovich limiting (basic) subdifferential of f at x.
$\nabla\, f(x)$	Gradient vector of f at x.
$\Delta\, f(x)$	Laplacien of f.

Spaces

$\mathbb{N}$	The set of positive integers.
$\mathbb{R}$	The real line.
$\mathbb{R}_+$	The set of nonnegative numbers.
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{-\infty, +\infty\}$.
$\mathbb{R}^d$	The d-dimensional Euclidean space.
Ω	An open, bounded, connected set in $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$.
$\partial\Omega = \Gamma$	The boundary of the domain Ω.
$\bar{\Omega}$	The closure of Ω in $\mathbb{R}^d,$ i.e. $\bar{\Omega} = \Omega \cup \partial\Omega$.
$\text{mes}\,(A)$	Lebesgue measure of the measurable subset $A \subset \Gamma$.
H, V	Hilbert spaces.
$\mathcal{L}(V, X)$	The space of linear continuous operators from V to a normed space X.
$\mathcal{L}(V) \equiv \mathcal{L}(V, V)$.	
$\mathcal{LC}(V, X)$	The space of linear compact operators from V to a normed space X.
$\mathcal{LC}(V) \equiv \mathcal{LC}(V, V)$.	
J	Any interval (resp. closed set) in $\mathbb{R}$ (resp. $\mathbb{R}^2$).
$\mathcal{C}(J; H)$	The space of continuous functions defined on J with values in H.
$L^p([0, T], H)$	the spaces of measurable functions whose p-th power is integrable on J.
$L^2(\Omega)^d$	The space of mapping $: \Omega \to \mathbb{R}^d,$ with $v_i \in L^2(\Omega)^d,\ i = 1..., d$.
$W^{k,p}([0, T], H) =$	The space of mapping $v \in L^p(0, T; X)$ with $\|v^{(j)}\|_{L^p(0,T;X)} < +\infty \ \ \forall j \leq k$.
$H^k([0, T], H) \equiv W^{k,2}([0, T], H)$.	
$H_0^1([0, T], H) = W_0^{1,2}([0, T], H)$	The space of mapping $v \in H^1([0, T], H),$ with $v\|_{\partial\Omega} = 0$.

Functions and operators

$\varphi^{\circ}(x,\cdot)$	Generalized directional derivative of φ at x.
$d_S(\cdot)$ *or* $d(\cdot,S)$	Distance function.
$I_S(\cdot)$ *or* $I(\cdot,S)$	Indicator function of a set S.
$\sigma_S(\cdot)$ *or* $\sigma(\cdot,S)$	Support function of a set S.
$\mathrm{Proj}_S(\cdot)$ *or* $\mathrm{Proj}(\cdot,S)$	Metric projection onto the set S.
χ_S	Characteristic function of S.
I	the identity operator on H.
φ^*	the conjugate function.
J_λ	resolvent of the operator A.
A_λ	Yosida approximation of the operator A.
$Var(u;J)$	Variation of a function u over J.

Mapping

$\varphi : X \longrightarrow Y$	Single-valued mapping from X to Y.
$F : X \rightrightarrows Y$	Set-valued mapping from X to Y.

Introduction

Multivalued differential equations are a type of differential equation that were introduced in the 1940s to examine systems of equations with nonlinear partial drift and problems from mechanics. This theory has become increasingly significant over time and has proven successful in diverse areas, such as unilateral mechanics, mathematical economy, and non-regular electrical circuits in engineering. Multivalued differential equations are an important tool for studying variational evolutionary inequalities, especially those governed by the normal cone. It is worth noting that the sweeping process is one of the most common formulations of the evolution variational inequality problem in the existing literature. This particular process was initially presented and extensively studied by Jean Jacques Moreau in a collection of articles, notably [51, 52]. It has been demonstrated in [51] that certain processes are of great importance for mechanics, particularly in dynamics, elasto-plasticity and quasi-statics. The mathematical form of the sweeping process, as described in [51, 52], corresponds to a point that is swept by a moving closed convex set $C(t)$ in a Hilbert space H according to the following differential inclusion

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) & a.e.\ t \in [T_0, T] \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $T_0, T \in \mathbb{R}$ with $0 \leq T_0 < T$ and $N_{C(t)}(\cdot)$ denotes here the normal cone of $C(t)$ in the sense of convex analysis. The analysis of systems with external forces led to consider and analyze the following perturbed variant

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f(t, x(t)) & a.e.\ t \in [T_0, T] \\ x(T_0) = x_0 \in C(T_0), \end{cases}$$

where $f : [T_0, T] \times H \to H$ is a Carathéodory mapping, i.e., $f(t, \cdot)$ is continuous and $f(\cdot, x)$ is Bochner measurable for $[T_0, T]$ endowed with the Borel σ-field $B([T_0, T])$. By Bochner measurable mapping we mean here any limit of uniformly convergent sequence of simple mappings from $[T_0, T]$ into H with $[T_0, T]$ endowed with its Borel σ-field.

On the other hand, the degenerate sweeping process is identified by the inclusion of a linear/nonlinear operator "within" the sweeping process, i.e., differential inclusions of the form

$$\begin{cases} -\dot{u} \in N_{C(t)}(Au(t)), & \text{a.e } t \in [0, T] \\ u(0) = u_0, \; Au(0) \in C(0), \end{cases}$$

This type of differential inclusion was introduced and studied by M. Kunze and M.D.P. Monteiro Marques, specifically for the convex case where the set-valued map $C(\cdot)$ has nonempty closed and convex values. The presence of the operator A within the sweeping process makes the problem more complicated compared to the classical case where $A = Id$.

This thesis investigates two problems related to sweeping processes. The first concerns the existence of solutions for integro-differential sweeping processes where the moving sets are prox-regular. The second problem focuses on establishing the existence of solutions for degenerate sweeping processes. This work is based on [43, 44].

On the next, we provide brief review of the thesis:

Chapter 1: Mathematical background.

This chapter is devoted to elementary findings concerning several specific topics that will be usable tools in upcoming chapters. These findings include indications for convex analysis, multivalued analysis, and non-smooth analysis. We will also examine some fundamental principles related to convex sets and functions, along with an emphasis on normal cones. Following this, the chapter contains some relevant findings in functional analysis.

Chapter 2: Semi Regularization Of Prox-Regular Integro-Differential Sweeping Process

This chapter focuses on the examination of the integro-differential sweeping process, as documented in [43]. This process is defined by the following differential inclusion:

$$(P_{A,f}) : \begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + Ax(t) + \int_0^t f(t, s, x(s))ds & \text{a.e. } t \in [0, T] \\ x(0) = x_0 \in C(0). \end{cases}$$

The aim of this chapter is to show that $(P_{A,f})$ has a unique solution. We achieve this through the semi-regularization technique. Namely, we approach the differential inclusion with a penalized one, depending on a parameter whose existence is easier to establish, and then study the limit when the parameter goes to zero. More specifically, let $\lambda > 0$ and consider the following approximate sweeping process :

$$\begin{cases} -\dot{x_\lambda}(t) \in N_{C(t)}(x_\lambda(t)) + A_\lambda x_\lambda(t) + \int\limits_0^t f(t,s,x_\lambda(s))\,ds \quad \text{a.e.} \quad t \in [0,T]\,, \\ x_\lambda(0) = x_0 \in C(t). \end{cases}$$

The first important results of this chapter is Theorem 2.2 which claims the existence of solutions for $(P_{A,f})$. Furthermore, by virtue of Theorem 2.2 we obtain the uniqueness of solutions for $(P_{A,f})$ under some additional conditions. Afterward, we applied this result to obtain the existence and uniqueness result for parabolic variational inequalities.

Chapter 3: A Variant of Degenerate Sweeping Process

This chapter aims to establish the existence of solution for the following degenerate sweeping processes

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(Ax(t)) + \int\limits_0^t f(t,s,x(s))\,ds \quad \text{a.e.} \quad t \in [0,T]\,, \\ x(0) = x_0 \in C(0). \end{cases} \tag{1}$$

where $C(\cdot)$ is a set-valued map with nonempty, closed and positively α–far values, $N_{C(t)}(u(t))$ represent the Clarke normal cone to $C(t)$. $A : H \rightrightarrows H$ is set valued maximal monotone operator. Considering A a single valued linear, bounded, symmetric and β-coercive operator, $f = 0$ and $C(t)$ a convex set in the previous differential inclusion we obtain the degenerate sweeping process that was proposed by Kunze and Monteiro-Marques in [38].

To establish the existence of a solution for degenerate sweeping processes, we employ Moreau-Yosida regularization. This technique involves approximating the given differential inclusion by a penalized one, dependent on certain parameters. First, we derive uniform bounds for a sequence of approximate solutions obtained through Yosida approximation of a maximal monotone operator and by approximating the Clarke normal cone using the Clarke subdifferential of the distance function. Finally, we obtain a solution for the original problem by jointly letting the parameters tend to zero.

Chapitre 1

Mathematical background

This chapter focuses on basic results related to specific topics that will be used in subsequent chapters. We provide some reminders of theoretical backgrounds of analyses, for instance, some background on convex analysis; more precisely, we outline concepts related to convex sets and functions. This is followed by basic facts on multivalued maps, which are necessary for our study. Such as, we discuss continuity properties and provide reminders on the maximal monotone mapping. Additionally, we offer a brief introduction to differential inclusions. Afterward, we look more closely at basic notions of nonsmooth analysis, For example, we present fundamental definitions and facts related to normal cones. We also give a brief review on the properties of prox-regular sets. In the last section of this chapter, we recall some results on weak and strong convergence. In addition, we summarize without proofs some integral inequalities of Gronwall type.The proof of the results presented in this chapter can be found in standard textbooks, such as [9, 22, 62, 55, 61, 7].

1.1 Convex analysis

In this section, we present some properties of convex sets and functions as well as the subdifferential and the conjugate properties of convex functions.

Throughout this section, X denotes a real Banach space and X^* the topological dual of X.

1.1.1 Convex sets and functions

Definition 1.1. A subset $K \subset X$ is convex if it contains the line segment

$$[x, y] = \{(1-\lambda)x + \lambda y : \lambda \in [0,1]\},$$

connecting any two of its points x and y.

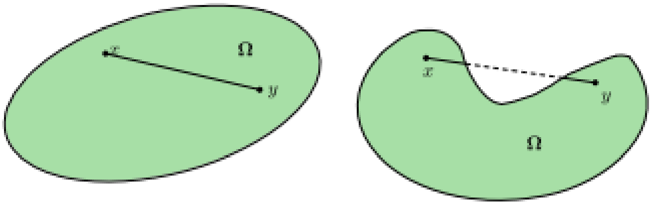

Figure 1.1: convex and non-convex sets.

The following proposition summarizes some basic properties of convex sets.

Proposition 1.2. *[9] In a subset $K \subset X$.*

1. *If K is a convex set, then $\lambda K = \{\lambda x : \ x \in K\}$ is convex, for all $\lambda \in \mathbb{R}$.*
2. *The intersection of convex sets $(K_i)_{i \in I}$ is convex.*
3. *If K_1 and K_2 are convex, then $K_1 + K_2 = \{k_1 + k_2, \ k_1 \in K_1, k_2 \in K_2\}$ the Minkowski addition of K_1 and K_2 is also convex.*
4. *If K is a convex set, then its closure $\overline{K}$ and its interior $\operatorname{int}(K)$ are convex as well.*

Definition 1.3. Consider a subset K of X. The convex hull of K is defined as the intersection of all convex subsets of X containing K. In other words, it is the smallest convex subset of X that contains K. It is described by $\operatorname{co}(K)$ and has the following characterization:

$$\operatorname{co}(K) = \left\{ \sum_{i=1}^{n} \lambda_i x_i, \ \lambda_i \geq 0, \ x_i \in K, \ \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

The closed convex hull of K is the smallest closed convex subset of X containing K. It is symbolized by $\overline{\operatorname{co}}(K)$.

Definition 1.4. A subset $K \subset X$ is known as:

1. (strongly) closed if the limit of every convergent sequence of elements of K belongs to K, in other terms
$$\{x_n\} \subset K, \ x_n \to x \ \text{ in } \ X \Longrightarrow x \in K;$$
2. weakly closed if the limit of every weakly convergent sequence of elements of K belongs to K, in other terms
$$\{x_n\} \subset K, \ x_n \rightharpoonup x \ \text{ weakly in } \ X \Longrightarrow x \in K.$$

Each weakly closed subset of X is also (strongly) closed. However, the converse is usually not true, except for convex subsets in a Banach space, as demonstrated in the following result.

Theorem 1.5 (The Mazur Theorem). *A convex subset of a Banach space X is (strongly) closed if and only if it is weakly closed.*

According to the Mazur theorem, for any sequence (x_n) converges weakly to x, we have a sequence (y_n) constructed as convex combinations of (x_n) that converges strongly to x.

Let us recall some notions concerning convex functions. For an extended real-valued function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, we denote by $\mathrm{D}(\varphi)$ the effective domain of φ, that is

$$\mathrm{D}(\varphi) := \{x \in X : \; \varphi(x) < +\infty\}.$$

The epigraph of φ is the set

$$\mathrm{epi}(\varphi) := \{(x, \lambda) \in X \times \mathbb{R} : \; \varphi(x) \leq \lambda\}.$$

The function φ is proper if its effective domain is nonempty and $\varphi(x) \neq -\infty$, for all $x \in D(\varphi)$.

Definition 1.6. Let $K \subset X$ be a convex subset. A function $\varphi : K \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be convex if

$$\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y) \;\; \forall x, y \in K, \;\; \forall \lambda \in [0, 1].$$

Next we introduce some important properties of a convex functions.

Proposition 1.7. *[22] Consider a function $\varphi : K \longrightarrow \mathbb{R} \cup \{+\infty\}$, then*

1. *φ is convex if and only if $\mathrm{epi}(\varphi)$ is a convex set in $X \times \mathbb{R}$.*
2. *If φ_1 and φ_1 are convex, then $\varphi_1 + \varphi_2$ is convex.*
3. *If φ is convex , then for every $\lambda \in \mathbb{R}$ the sublevel sets of φ defined by*

$$[\varphi(x) \leq \lambda] := \{x \in K : \varphi(x) \leq \lambda\}$$

is convex. It is essential to note that the converse is not true in general.

4. *If* $(\varphi_i)_{i\in I}$ *is a family of convex functions, then the function* φ *defined by*

$$\varphi(x) := \sup_{i\in I}\varphi_i(x)$$

is convex

Definition 1.8. A function $\varphi : X \to \mathbb{R}\cup\{+\infty\}$ is said to be upper semi-continuous (u.s.c. for short) at some point $x \in X$ if for each sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, we have

$$x_n \to x \text{ strongly in } X \Longrightarrow \limsup_{n\to+\infty}\varphi(x_n) \leq \varphi(x).$$

The function φ is upper semi-continuous if it is upper semi-continuous at every point $x \in X$.

Definition 1.9. A function $\varphi : X \to \mathbb{R}\cup\{+\infty\}$ is said to be lower semi-continuous (l.s.c.) at some point $x \in X$ if for each sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, we have

$$x_n \to x \text{ strongly in } X \Longrightarrow \liminf_{n\to+\infty}\varphi(x_n) \geq \varphi(x).$$

The function φ is lower semi-continuous if it is lower semi-continuous at every point $x \in X$.

We will now recall some fundamental facts about lower semi-continuous functions.

Proposition 1.10. *[22] Let the function* $\varphi : X \to \mathbb{R}\cup\{+\infty\}$*, then*

1. φ *is lower semi-continuous if and only if* $\text{epi}(\varphi)$ *is closed in* $X \times \mathbb{R}$*.*

2. φ *is lower semi-continuous if for any* $\lambda \in \mathbb{R}$ *the sublevel sets of* φ *defined by*

$$[\varphi \leq \lambda] := \{x \in X : \varphi(x) \leq \lambda\}$$

are closed.

3. *If* φ_1 *and* φ_2 *are lower semi-continuous functions, then* $\varphi_1 + \varphi_2$ *is lower semi-continuous as well.*

4. *If* $(\varphi_i)_{i\in I}$ *is a family of lower semi-continuous functions then the function* φ *described by*

$$\varphi(x) := \sup_{i\in I}\varphi_i(x)$$

is lower semi-continuous.

Definition 1.11. A function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be weakly lower semi-continuous at some point $x \in X$ if for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, we have

$$x_n \rightharpoonup x \text{ weakly in } X \Longrightarrow \varphi(x) \leq \liminf_{n\to+\infty} \varphi(x_n).$$

The function φ is weakly lower semi-continuous if it is weakly lower semi-continuous at every point $x \in X$.

An important property of convex lower semi-continuous functions is given by the next result.

Theorem 1.12. *Let K be a nonempty closed convex subset of X and let $\varphi : K \to \mathbb{R} \cup \{+\infty\}$ be a convex function, Then φ is lower semi-continuous if and only if it is weakly lower semi-continuous.*

Remark 1.13. The proof of Theorem 1.12 is a consequence of Mazurs theorem. It follows from this theorem that a convex continuous function $\varphi : X \to \mathbb{R}$ is weakly lower semi-continuous.

Proposition 1.14. *Let $K \subset X$ be a nonempty convex subset. Then, for $x \in X$ the distance function*

$$d_K(x) := \inf\{\|x - y\| : \, y \in K\}$$

is Lipschitz continuous with the Lipschitz constant equals 1 *and convex on X.*

Definition 1.15. Given a nonempty set $K \subset X$, we set

$$I_K(x) := \begin{cases} 0 & x \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

The function I_K is known as the indicator function of K. Notice that:

1. I_K is proper if and only if $K \neq \emptyset$.
2. I_K is convex if and only if K is a convex set.
3. I_K is lower semi-continuous if and only if K is closed.

Definition 1.16. Let $K \subset X$ be a nonempty subset. The support function $\sigma_K : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined on X as

$$\sigma_K(x^*) := \sup_{y\in K} \langle x^*, y\rangle, \quad \forall x^* \in X^*.$$

Theorem 1.17. *For the nonempty $K \subset X$ and its support function σ_K there holds*

$$u \in \overline{\text{co}} K \Longleftrightarrow \{\langle u, v\rangle \ \leq \sigma_K(v), \quad \forall v \in X\}.$$

1.1.2 Projection onto Closed Convex Sets

In the following we recall some results which concerns the characterization of the projection onto a closed convex set. We devote special attention to the case when these sets are closed convex cones.

Theorem 1.18. *Let K be a nonempty closed convex subset of a Hilbert space H. Then for any $x \in H$ there exists a unique point denoted by $\mathrm{proj}_K(x)$ of K such that*

$$d(x, K) = \|x - \mathrm{proj}_K(x)\|.$$

The point $\mathrm{proj}_K(x)$ is characterized by the following variational inequality:

$$\langle x - \mathrm{proj}_K(x), y - \mathrm{proj}_K(x)\rangle \leq 0 \ \textit{for all} \ y \in K.$$

We call $\mathrm{proj}_K(x)$ the projection of x onto K (or the nearest point of K to x).

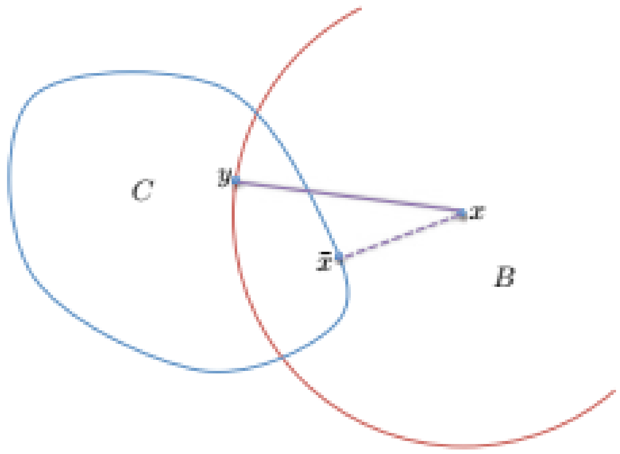

Figure 1.2: The projection of x on C is the point $\overline{x}$ of C.

Proposition 1.19. *Let K is a nonempty, closed, convex subset of X. The mapping proj_K : $x \to \mathrm{proj}_K(x)$, from X into itself, is characterized by:*

1.
$$\begin{cases} \mathrm{proj}_K(x) \in K & \textit{for all} \ x \in X, \\ \mathrm{proj}_K(x) = x & \textit{if, and only if,} \ x \in K. \end{cases}$$

2. $\mathrm{proj}_K(x)$ *is a monotone map in the sense that*

$$\langle \mathrm{proj}_K(x) - \mathrm{proj}_K(y), x - y\rangle \geq 0 \ \textit{for all} \ x, y \in X.$$

3. *The map* $\mathrm{proj}_K(x)$ *is non-expansive*

$$\|\mathrm{proj}_K(x) - \mathrm{proj}_K(y)\| \leq \|x - y\| \ \textit{for all } \ x, y \in X.$$

Definition 1.20. A subset $C \subset X$ is called a cone if and only if

$$\forall x \in C, \ \forall \lambda \geq 0 \ \text{ we have } \ \lambda x \in C.$$

1. C is a convex cone if and only if it satisfies both the properties of being a cone and a convex set.

2. The conical hull of C, denoted by $\mathrm{cone}(C)$, is the smallest convex cone that contains C.

3. The closed conical hull of C, denoted by $\overline{\mathrm{cone}}(C)$, is the smallest closed cone in X containing C.

In the field of convex analysis, the tangent cone and the normal cone are widely recognized as important types of cones.

Definition 1.21. Let C be a convex cone.

1. The dual cone C^* of C in X is defined by:

$$C^* := \{\rho \in X^* : \ \langle x, \rho \rangle \geq 0, \ \forall x \in C\}.$$

2. The polar cone C° of C in X is defined by:

$$C^\circ := \{\rho \in X : \ \langle x, \rho \rangle \leq 0, \ \forall x \in C\} = -C^*.$$

Definition 1.22. Let C be a nonempty convex subset of X and let $x \in X$, the normal cone to C at x is

$$N_C(x) = \begin{cases} \{\xi \in X^*, \langle \xi, y - x \rangle \leq 0, \ \forall y \in C\} & \text{if } x \in C; \\ \emptyset & \text{otherwise.} \end{cases}$$

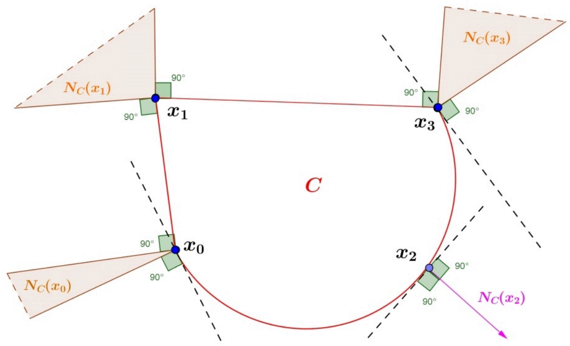

Figure 1.3: Normal cone at different points of a convex set C.

Remark 1.23. By using the concept of polarity, we can obtain the following equivalent formulation of normal cones

$$N_C(x) = \begin{cases} (C-x)^\circ = \{u \in X \ : \ \sup\langle C-x, u\rangle \leq 0\}, & \text{if } x \in C; \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma 1.24. *Let $C \subset X$ be a nonempty convex set. The following assertions hold:*

1. *$N_C(x)$ is a convex closed set containing the origin.*
2. *If $int(C) \neq \emptyset$ and for $x \in int(C)$, we have $N_C(x) = \{0\}$, which shows that the normal cone only interests with the boundary of C.*
3. *If $C, D \subset X$ are a nonempty, closed, and convex with $int(C) \cap D \neq \emptyset$, then*

$$N_{C\cap D}(x) = N_C(x) + N_D(x) \ \ \textit{for} \ \ x \in C \cap D.$$

Proposition 1.25. *Let $C \subset X$ be a nonempty convex subset. Then for each $x \in -C$ and $x, y \in X$, with $x + y \in C$, we have*

1. *$N_C(x+y) = N_{C-y}(x)$.*
2. *$-N_C(-x) = N_{-C}(x)$.*

Definition 1.26. Let $C \subset X$ be a nonempty convex subset. For each $x \in X$ the tangent cone to C at x is

$$T_C(x) = \begin{cases} \overline{\text{cone}}(C-x) = \overline{\bigcup_{\lambda\in\mathbb{R}_+} \lambda(C-x)} & \text{if } x \in C; \\ \emptyset & \text{otherwise.} \end{cases}$$

Proposition 1.27. *Let C be a nonempty convex subset of X and let $x \in X$, then*

$$T_C^\circ(x) = N_C(x) \ \ \textit{and} \ \ N_C^\circ(x) = T_C(x).$$

Proposition 1.28. *Let $S \subset X$ be a nonempty closed convex cone, and let $u, v \in X$, then*

$$S^* \in u \perp v \in S \Longleftrightarrow -u \in N_S(v).$$

1.1.3 Conjugate Convex Functions and Subdifferentials

Definition 1.29. Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semi-continuous function. Then the subdifferential of φ at $x \in \mathrm{D}(\varphi)$ is the (possibly empty) set

$$\partial\varphi(x) := \{\xi \in X : \varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle \ \text{ for every } \ y \in X\},$$

and if $x \notin \mathrm{D}(\varphi)$, the set $\partial\varphi(x) := \emptyset$. The elements of $\partial\varphi(x)$ are usually called subgradients of φ at x.

We list some elementary properties of the subdifferential of a Convex Functions.

Proposition 1.30. *Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function, then*

1. *$\partial\varphi(x)$ is a closed convex set for every $x \in X$.*
2. *Let $\alpha > 0$, then $\forall x \in \mathrm{D}(\varphi)$ we have*

$$\partial(\alpha\varphi)(x) = \alpha\partial\varphi(x).$$

3. *Let $x_0 \in \mathrm{D}(\varphi)$ and $x \in X$, then*

$$x \in \partial(\varphi)(x_0) \Longleftrightarrow (x, -1) \in N_{\mathrm{epi}(\varphi)}(x_0, \varphi(x_0)).$$

Example 1.31. for any boundary point $x \in C$,

$$\partial I_C(x) = N_C(x) = \{\xi \in X^* : \ \langle \xi, y - x \rangle \leq 0, \ \ \forall v \in C\}.$$

Recall that the set $N_C(x)$ is the normal cone of C at x. It is readily seen that

1. $\mathrm{D}(\partial I_C) = C$;
2. $\partial I_C(x) = 0$ for each $x \in \mathrm{int}(C)$.

Definition 1.32. Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The conjugate function $\varphi^* : X^* \to \mathbb{R} \cup \{+\infty\}$ associated to φ is defined by

$$\varphi^*(\xi) := \sup_{x \in X} \{\langle \xi, x \rangle - \varphi(x)\}.$$

Equivalently, we have

$$\varphi^*(\xi) := \sup_{x \in \mathrm{Dom}(\varphi)} \{\langle \xi, x \rangle - \varphi(x)\}.$$

Theorem 1.33. *Let the function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be given and $x \in X$. Then*

$$\xi \in \partial\varphi(x) \Longleftrightarrow \varphi(x) + \varphi^*(\xi) = \langle \xi, x \rangle.$$

Theorem 1.34. *If the function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous, then*

$$\xi \in \partial\varphi(x) \Longleftrightarrow x \in \partial\varphi^*(\xi).$$

Example 1.35. The conjugate function of the indicator function of C

$$I_C^*(\xi) := \sup_{x \in \mathrm{Dom}(I_C)} \{\langle \xi, x \rangle - I_C(x)\} = \sup_{x \in C} \langle \xi, x \rangle := \sigma_C(\xi),$$

is just the support function of C.

Definition 1.36. We denote by $\mathit{\Gamma}_0(X)$ the set of all functions $X \to \mathbb{R} \cup \{+\infty\}$ which are pointwise supremum of a family of functions on X of the form $x \to \langle x, x^* \rangle + \alpha$, where $x^* \in X^*$ and $\alpha \in \mathbb{R}$. Which are not the constant functions $-\infty$ and $+\infty$. Analogously, we define $\mathit{\Gamma}_0(X^*)$. Note that $\mathit{\Gamma}_0(X)$ is the set of all lower semi-continuous proper convex functions on X.

Theorem 1.37. *Let $C \in X$ be non-empty, closed and convex. Then the conjugate function of the indicator function $I_C^* \in \mathit{\Gamma}_0(X^*)$.*

Theorem 1.38. *Let $\psi \in \mathit{\Gamma}_0(X^*)$ be positively homogeneous. Then there exists a unique non-empty closed convex subset C of X such that $I_C^* = \psi$.*

1.2 Multivalued analysis

In this section, we introduce fundamental concepts and results related to set-valued maps. Subsequently, we move on to the study of maximal monotone maps. Finally, we provide a brief introduction to differential inclusions.

1.2.1 Set-valued maps

Let U and V are two linear spaces.

Definition 1.39. A multivalued map F from U to V is a map that associates with any $u \in U$ a subset $F(u) \subset V$.

1. The domain of F, denoted as $\mathrm{Dom}(F)$, is the subset of U defined by
$$\mathrm{Dom}(F) := \{u \in U \mid F(u) \neq \emptyset\}.$$
2. The graph of F is the subset of pairs (u, v) where $v \in F(u)$:
$$\mathrm{gph}(F) := \{(u, v) \in \mathrm{Dom}(F) \times V \mid v \in F(u)\}.$$
3. The range $\mathcal{R}(F)$ is, by definition, the subset
$$\mathcal{R}(F) := \bigcup_{u \in U} F(u).$$
4. The inverse of F is the multivalued map $F^{-1} : V \rightrightarrows U$ such that
$$u \in F^{-1}(v) \Longleftrightarrow v \in F(u) \Longleftrightarrow (u, v) \in \mathrm{gph}(F).$$
5. We say that a map F is proper if its domain is nonempty.

Definition 1.40. Let $F : U \rightrightarrows V$ be a set-valued map with non-empty values.

1. We say that F is upper semi-continuous (u.s.c.) at $u_0 \in U$ if for any open set M containing $F(u_0)$ there exists a neighborhood $\mathcal{V}$ of u_0 such that $F(\mathcal{V}) \subset M$. A set-valued map F is said to be upper semi-continuous if it is so at every point $u_0 \in U$.

2. A set-valued map F is called lower semi-continuous at $u_0 \in U$ if for any $v_0 \in F(u_0)$ and any neighborhood $\mathcal{V}(v_0)$ of v_0 there exists a neighborhood $\mathcal{V}(u_0)$ of u_0 such that

$$F(u) \cap \mathcal{V}(v_0) \neq \emptyset \text{ for all } u \in \mathcal{V}(u_0).$$

   A set-valued map F is said to be lower semi-continuous if it is so at every point $u \in U$.

3. A set-valued map F is said to be continuous at $u_0 \in U$ if it is both upper and lower semi-continuous at u_0. It is called continuous if it is continuous at every point $u \in U$.

4. We say that F is Lipschitzian if there exists $l \geq 0$ such that

$$F(u_1) \subset F(u_2) + l\|u_1 - u_2\|\mathbb{B}_V \text{ for all } u_1, u_2 \in U,$$

   where $\mathbb{B}_V := \{v \in V| \ \|v\| \leq 1\}$.

5. A set-valued map F is said to be locally Lipschitzian if for any $u \in U$ there exist $\epsilon > 0$ and $l > 0$ such that

$$F(u_1) \subset F(u_2) + l\|u_1 - u_2\|\mathbb{B}_V \text{ for all } u_1, u_2 \in x + \epsilon\mathbb{B}_U.$$

6. A set-valued map F has a convex image on U if $F(u)$ is a convex set for all fixed values $u \in U$.

upper semi-continuous | upper semi-continuous and convex image | upper semi-continuous, convex and closed image

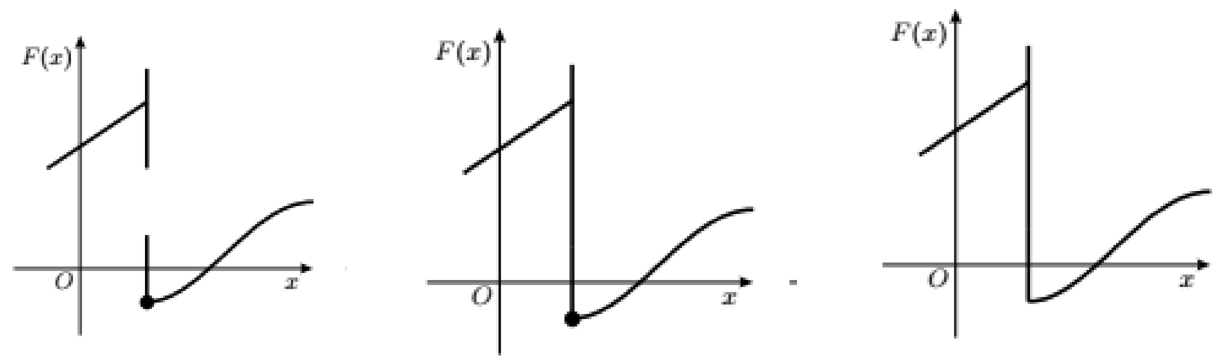

Figure 1.4: Illustration of upper semi-continuity, convexity and closedness of a set-valued function.

The Hausdorff distance (also called Pompeiu-Hausdorff distance) is a classical measure for the difference between two sets.

Definition 1.41. C_1 and C_2 are two nonempty closed subsets of U. The Hausdorff distance between C_1 and C_2 is defined as the function $d_H(\cdot,\cdot)$:

$$d_H(C_1, C_2) := \max\Big\{ \underset{v\in C_1}{d(v, C_2)}, \underset{u\in C_2}{d(u, C_1)} \Big\}.$$

Proposition 1.42. *Let C_1 and C_2 are two nonempty closed subsets of U. Then*

$$d_H(C_1, C_2) \leq \epsilon \Longleftrightarrow C_1 \subset C_2 + \epsilon\mathbb{B} \ \textit{and} \ C_2 \subset C_1 + \epsilon\mathbb{B},\ \epsilon > 0.$$

Let A be a bounded subset of H. We define the Kuratowski measure of noncompactness of A, as

$$\alpha(A) = \inf\{d > 0 : A \text{admits a finite cover by sets of diameter} \leq d\},$$

and the Hausdorff measure of non-compactness of A, as

$$\beta(A) = \inf\{r > 0 : A \text{can be covered by finitely many balls of radius} r\}.$$

The following proposition gives the main properties of the Kuratowski and Hausdorff measure of noncompactness.

Proposition 1.43. *[23] Let H be a Hilbert space and B, B_1, B_2 be bounded subsets of H. Let γ be the Kuratowski or the Hausdorff measure of non-compactness. Then,*

1. $\gamma(B) = 0$ *if and only if* $cl(B)$ *is compact.*
2. $\gamma(\lambda B) = |\lambda|\gamma(B)$ *for all* $\lambda \in \mathbb{R}$.
3. $\lambda(B_1 + B_2) \leq \lambda(B_1) + \lambda(B_2)$.
4. $B_2 \subset B_1$ *implies* $\gamma(B_2) \leq \gamma(B_1)$.
5. $\gamma(coB) = \gamma(B)$.
6. $\gamma(clB) = \gamma(B)$.
7. *If* $A : H \longrightarrow H$ *is a Lipschitz map of constant* $M \geq 0$, *then*

$$\gamma(A(B)) \leq M\gamma(B).$$

1.2.2 Maximal monotone maps

In this thesis, H always denotes a Hilbert space.

Definition 1.44. Let $A : H \rightrightarrows H$ be a set-valued map. Then A is called monotone if and only if

$$\langle u_1 - u_2, v_1 - v_2 \rangle \geq 0 \quad \forall u_1, u_2 \in \mathrm{Dom}(A),\ \forall v_i \in A(u_i),\ i = 1, 2.$$

Definition 1.45. Let $A : H \rightrightarrows H$ be monotone. Then A is maximal monotone if there exists no monotone operator $B : H \rightrightarrows H$ such that $\mathrm{gph}(B)$ properly contains $\mathrm{gph}(A)$, i.e., for every $(u, v) \in H \times H$,

$$(u, v) \in \mathrm{gph}(A) \Longleftrightarrow \forall (w, z) \in \mathrm{gph}(A) \ \ \langle u - w, v - z \rangle \geq 0.$$

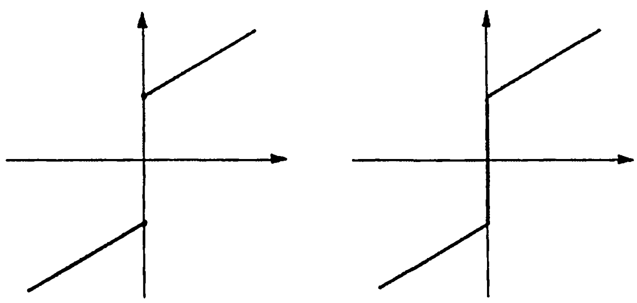

Figure 1.5: a Graph of a monotone map, b maximal monotone map.

Theorem 1.46. *(Moreau) Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then $\partial\varphi$ is a maximal monotone map.*

Proposition 1.47. *[9]*

1. *Let A be a maximal monotone operator on H. The operators A^{-1} and λA, where $\lambda > 0$, are also maximal monotone.*
2. *Let $A : H \longrightarrow H$ be monotone and continuous. Then A is maximal monotone.*
3. *Let C be a nonempty closed convex subset of H. Then Proj_C is maximal monotone.*
4. *Let C be a nonempty closed convex subset of H. Then N_C is maximal monotone.*

Proposition 1.48. *Let $A : H \rightrightarrows H$ be maximal monotone operator and let $u \in H$. Then Au is closed and convex.*

Proposition 1.49. *[9] Let $A : H \rightrightarrows H$ be maximal monotone. Then the following hold:*

(i) $\operatorname{gph}(A)$ *is sequentially closed in* $H^{\text{strong}} \times H^{\text{weak}}$*, i.e., for every sequence* $(u_n, v_n)_{n\in\mathbb{N}}$ *in* $\operatorname{gph}(A)$ *and every* $(u, v) \in H \times H$*, if* $u_n \longrightarrow u$ *and* $v_n \rightharpoonup v$*, then* $(u, v) \in \operatorname{gph}(A)$*.*

(ii) $\operatorname{gph}(A)$ *is sequentially closed in* $H^{\text{weak}} \times H^{\text{strong}}$*, i.e., for every sequence* $(u_n, v_n)_{n\in\mathbb{N}}$ *in* $\operatorname{gph}(A)$ *and every* $(u, v) \in H \times H$*, if* $u_n \rightharpoonup u$ *and* $v_n \longrightarrow v$*, then* $(u, v) \in \operatorname{gph}(A)$*.*

(iii) $\operatorname{gph}(A)$ *is closed in* $H^{\text{strong}} \times H^{\text{strong}}$*.*

(iv) A *locally bounded at every point in the interior of its domain* $\operatorname{int}(\operatorname{Dom}(A))$*.*

Now we show that a maximal monotone A can be approximated through specific single-valued Lipschitzian maps, denoted as A_λ, from H to H that is maximal monotone. These maps, known as Yosida approximations, play a crucial role.

Definition 1.50. Let A be a maximal monotone operator on H. Then, for all $\lambda > 0$, the Resolvent operator $J_\lambda : H \to H$ corresponding to $\lambda > 0$ is defined by the formula:

$$J_\lambda := (I + \lambda A)^{-1},$$

where I denotes the identity on H.

Definition 1.51. Let A be a maximal monotone operator on H. The Yosida approximation of A corresponding to $\lambda > 0$ is defined by

$$A_\lambda := \frac{1}{\lambda}(I - J_\lambda).$$

Here, we outline the fundamental properties of the Yosida approximation for a maximal monotone operator and its resolvent map.

Proposition 1.52. *[15] Let A be a maximal monotone operator. Then for all* $\lambda > 0$

1. J_λ *is a non-expansive single-valued map from H to H, that is*

$$\|J_\lambda(u) - J_\lambda(v)\| \leq \|u - v\| \ \textit{ for all } \ u, v \in H.$$

2. *For all* $u \in H$*,* $A_\lambda(u) \in A(J_\lambda u)$*.*

3. *For all* $u \in \operatorname{Dom}(A)$*,* $J_\lambda u$ *converges to* u*.*

4. A_λ *is Lipschitz continuous, with constant* $\frac{1}{\lambda}$ *and maximal monotone.*

5. *For all* $u \in \mathrm{Dom}(A)$, $\|A_\lambda(u)\| \le \|A^\circ(u)\|$, *where* $A^\circ u = \mathrm{proj}_{A(u)}(0)$ *is the element of the closed convex set* $A(u)$ *of minimal norm, that is,*

$$\|A^\circ(u)\| := \min\Big\{\|\xi\|,\ \ \xi \in A(u)\Big\}.$$

6. *if* $u_\lambda \longrightarrow u$ *as* $\lambda \downarrow 0$ *and* $(A_\lambda u_\lambda)_\lambda$ *is bounded then* $u \in \mathrm{Dom}(A)$.

Proposition 1.53. *Let* A *be a maximal monotone operator and* $T > 0$. *Then, the extension of* A *to* $L^2([0,T];H)$ *noted by* $\mathcal{A} : L^2([0,T];H) \rightrightarrows L^2([0,T];H)$ *and defined by*

$$v(\cdot) \in \mathcal{A}u(\cdot) \Longleftrightarrow v(t) \in A(u(t)) \ a.e\ \ t \in [0,T].$$

is maximal monotone.

Lemma 1.54. *[37] Let* A *be a maximal monotone operator in* H *such that*

$$\langle Ax - Ay, x - y\rangle \ge c\|x - y\|^2$$

in $D(A) \times D(A)$ *for some* $c > 0$. *Then*

1. $\forall x \in H$ *and* $\lambda > 0$ *we have* $\lambda x + A^{-1}x = A_\lambda^{-1}x$.

2. $\forall x, y \in H$ *and* $\lambda > 0$

$$\langle A_\lambda x - A_\lambda y, x - y\rangle \ge \frac{c}{1 + \lambda c}\|x - y\|^2.$$

Lemma 1.55. *[4] Let* A *be a maximal monotone operator in* H *from Lemma 1.54 we have*

$$\langle A_\lambda x - A_\lambda y, x - y\rangle \ge \frac{c}{1 + \lambda c}\|x - y\|^2$$

in $D(A) \times D(A)$ *for some* $c > 0$, *then*

$$\langle \dot v(t), \dot x_\lambda(t)\rangle \ge \frac{c}{1 + \lambda c}\|\dot x_\lambda(t)\|^2$$

such that $v(\cdot) = A_\lambda x_\lambda(\cdot)$ *and the mapping* $x_\lambda : [0,T] \longrightarrow H$ *is absolutely continuous.*

1.2.3 Introduction to differential inclusions

Differential equations first emerged in the mid-seventeenth century when calculus was independently discovered by Newton and Leibniz, These equations frequently arise when trying to explain physical phenomena through mathematical models. Differential inclusions of the form

$$\dot{u}(t) \in F(u(t))$$

are a generalization of ordinary differential equations, where F is a set-valued map. Obviously, any process described by an ordinary differential equation

$$\dot{u} = f(u)$$

can be described by a differential inclusion with the right hand side $F(u) = \{f(u)\}$ as well. Differential inclusions are fundamental in the theory of differential equations with a discontinuous right-hand side. The study of differential inclusions is motivated by various examples, such as ordinary differential equations, control theory, evolution variational inequalities, and sweeping process. The Sweeping Process is a special type of differential inclusions, introduced and studied by J.J. Moreau [52]. To illustrate the mechanism described by this differential inclusion, consider the following example from [18], imagine a large ring enclosing a smaller ball. At time $t = 0$ the ring begins to move, the ball will either remain stationary (in case it is not hit by the ring), or be swept toward the interior of the ring. In the latter case, the velocity of the ball must point inward to the ring in order not to leave.

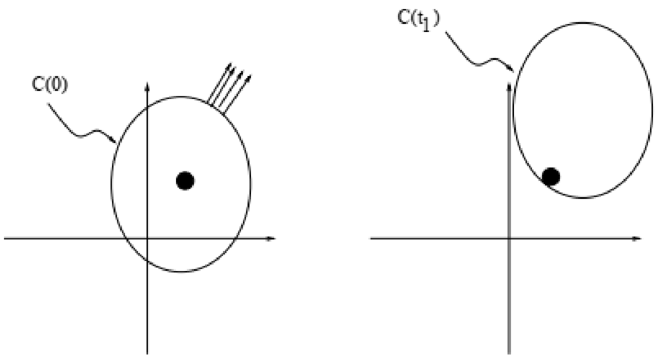

Figure 1.6: Interpretation of the sweeping process.

In more mathematical term, this becomes:

$$\begin{cases} -\dot{u}(t) \in N_{C(t)}(u(t)) & a.e.\ t \in [T_0, T] \\ u(T_0) = u_0 \in C(T_0), \end{cases} \tag{1.1}$$

where H is a Hilbert space, $C : H \rightrightarrows H$ is a set-valued mapping with nonempty closed and convex values, and $N_{C(t)}(\cdot)$ denotes here the normal cone of the convex subset $C(t)$ with initial condition $u_0 \in C(T_0)$. The differential inclusion (1.1) with the constraint $u(t) \in C(t)$ can be stated in the form of the following Variational Inequality:

$$\begin{cases} \langle -\dot{u}(t), v - u(t) \rangle \le 0, & \forall v \in C(t) \\ u(T_0) = u_0 \in C(T_0). \end{cases}$$

The differential inclusion (1.1) is also written as follows:

$$\begin{cases} -\dot{u}(t) \in \partial I_{C(t)}(u(t)) & a.e.\ t \in [T_0, T] \\ u(T_0) = u_0 \in C(T_0). \end{cases}$$

Recall that the set $\partial I_{C(t)}(u(t))$, the subdifferential of the indicator function of $C(t)$ at u.

1.3 Non-smooth analysis

In this section, We recall some basic definitions and properties of subdifferentiability for non-convex functions. Additionally, we introduce several types of normal cones, which serve as essential tools in the study of sweeping processes. Finally, we define some classes of sets that generalize the class of convex sets.

1.3.1 Subdifferential Calculus

The main goal of nonsmooth analysis is to extend differentiable tools to the nonsmooth setting; therefore, in this subsection, we will discuss various types of subdifferentials for non-convex functions and their properties.

Definition 1.56. [14] Assume that a function $\varphi : H \to \mathbb{R} \cup +\infty$ is locally Lipschitz at $\overline{u} \in H$, then

1. The Clarke subdifferential of φ at $\overline{u}$ is defined by

$$\partial^C \varphi(\overline{u}) := \{v \in H \ : \ \langle v, h \rangle \le \varphi^\circ(\overline{u}, h), \quad \text{for all } h \in H\},$$

where

$$\varphi^\circ(\overline{u},h) := \limsup_{(t,u)\to(0^+,\overline{u})} t^{-1}[\varphi(u+th)-\varphi(u)]$$

is the generalized directional derivative of the function φ at $\overline{u}$ in the direction $h \in H$.

2. The Fréchet subdifferential of φ at $\overline{u}$ is denoted $\partial^F\varphi(\overline{u})$ and defined by

$$\partial^F\varphi(\overline{u}) := \{\xi \in H,\ \forall\epsilon > 0,\ \exists\delta > 0 :\ \langle \xi, u-\overline{u}\rangle \leq \varphi(u)-\varphi(\overline{u})+\epsilon\|u-\overline{u}\| \ \text{ for all, } u \in \overline{u}+\delta\mathbb{B}\}.$$

3. The limiting subdifferential (also called basic subdifferential or Mordukhovich subdifferential) of φ at $\overline{u}$ is denoted $\partial_L\varphi(\overline{u})$ and defined by

$$\partial^L\varphi(\overline{u}) := \{v \in H\ :\ \exists\overline{u}_n \to \overline{u}, \exists\zeta_n \to^{w*} \zeta \text{ with } \zeta_n \in \partial^F\varphi(\overline{u}_n)\}.$$

4. The proximal subdifferential of φ at $\overline{u}$ is denoted by $\partial^P f(\overline{u})$ and defined as

$$\partial^P\varphi(\overline{u}) := \{\xi \in H,\ \exists\sigma,\delta > 0 :\ \langle \xi, u-\overline{u}\rangle \leq \varphi(u)-\varphi(\overline{u})+\sigma\|u-\overline{u}\|^2 \ \text{ for all, } u \in \overline{u}+\delta\mathbb{B}\}.$$

Proposition 1.57. *Let $\varphi : H \to \mathbb{R}$ be a locally Lipschitz function. Then $\partial^C\varphi : H \rightrightarrows H$ is upper semicontinuous from H into H^{weak}.*

Proposition 1.58. *We always has the following inclusions:*

$$\partial^P\varphi(\overline{u}) \subset \partial^F\varphi(\overline{u}) \subset \partial^L\varphi(\overline{u}) \subset \partial^C\varphi(\overline{u}).$$

Remark 1.59. For any convex continuous function φ one has:

$$\partial^P\varphi(\overline{u}) = \partial^F\varphi(\overline{u}) = \partial^L\varphi(\overline{u}) = \partial^C\varphi(\overline{u}) = \partial\varphi(\overline{u}).$$

1.3.2 Normal Cones

In this subsection, we will discuss the definition and properties of several concepts of normal cones to non-convex sets.

Definition 1.60 (The Clarke tangent cone)**.** Let C be a nonempty closed subset of H and $u \in C$. The Clarke tangent cone of a subset C at some point u is defined by

$$T^C(C;u) = \{v \in H : \forall t_n \downarrow 0, \forall u_n \to u, \text{ with } u_n \in C,\ \ \exists v_n \to v \text{ s.t. } u_n + t_n v_n \in C\ \ \forall n \in \mathbb{N}\}.$$

Definition 1.61 (The Clarke normal cone)**.** we define $N^C(C;u)$, the Clarke normal cone to C at u, as follows:

$$N^C(C;u) := \{\zeta \in H : \ \langle \zeta, v\rangle \leq 0,\ \forall v \in T^C(C;u)\}.$$

Definition 1.62 (The Proximal Normal Cone)**.** Let C be a nonempty closed subset of H and $u \in C$. We define the proximal normal cone to C at u as

$$N_C^P(u) = \{v \in H : \ \exists r > 0 \text{ such that } u \in \operatorname{Proj}_S(u + rv)\}. \tag{1.2}$$

When $u \notin C$, the proximal normal cone $N^P(C;u)$ is undefined.

Remark 1.63*.* When u belongs to C and is such that $u \notin \operatorname{Proj}_C(v)$, for all $v \notin C$ i.e., there is no point v outside of C such that $u \in \operatorname{Proj}_C(v)$ (which is the case when $u \in \operatorname{int}(C)$) we set $N^P(C;u) = \{0\}$.

Proposition 1.64. *The proximal normal cone is analytically characterized by the following:*

$$\begin{aligned} \xi \in N^P(C;u) &\Longleftrightarrow \Big\{\exists \delta, \sigma > 0;\ \langle \xi, v-u\rangle \leq \sigma\|v-u\|^2, \quad \textit{for all}\, v \in (u+\delta\mathbb{B}) \cap S\Big\} \\ &\Longleftrightarrow \Big\{\exists \sigma = \sigma(\xi,u) > 0;\ \langle \xi, v-u\rangle \leq \sigma\|v-u\|^2, \quad \textit{for all}\, v \in C\Big\}. \end{aligned}$$

Definition 1.65 (The Fréchet Normal Cone)**.** [14] Let C be a nonempty closed subset of H and $u \in C$.The Fréchet Normal Cone of a subset C at some point C is defined by

$$N^F(C;u) := \Big\{\forall \epsilon > 0,\ \exists \sigma > 0;\ \langle \xi, v-u\rangle \leq \epsilon\|v-u\|^2 \text{ for all } v \in (u+\sigma\mathbb{B}) \cap C\Big\}.$$

with $N^F(C;u) := \emptyset$ whenever $u \notin C$.

The Fréchet normal cone and the proximal normal cone suffer from instability, i.e., the Fréchet normal cone (the same with the proximal normal cone) may vary widely as its point base varies. This instability is a problem in applications of Nonsmooth Analysis, as it requires exclusion. The limiting normal cone, or Mordukhovich normal cone, is defined to address this issue.

Definition 1.66 (The Limiting Normal Cone or Mordukhovich Normal Cone)**.** [14] Let C be a nonempty closed subset of H and $u \in C$. We define the limiting normal cone by

$$\begin{aligned} N^L(C;u) &:= \Big\{\xi \in H,\ \exists \xi_n \rightharpoonup \xi \text{ weakly and } \ \xi_n \in N^P(C;u_n);\ u_n \to u \text{ in } C\Big\} \\ &:= \Big\{\xi \in H,\ \exists \xi_n \rightharpoonup \xi \text{ weakly and } \ \xi_n \in N^F(C;u_n);\ u_n \to u \text{ in } C\Big\}. \end{aligned}$$

We set $N^L(u;C) := \emptyset$ if $u \notin C$.

Proposition 1.67. *The following inclusions always hold true*

$$N^P(u;C) \subset N^F(u;C) \subset N^L(u;C) \subset N^C(u;C).$$

By convention, we set $N^C(u;C) = N^P(u;C) = N^F(u;C) = N^L(u;C) = \{0\}$ *if* $u \in \mathrm{int}(C)$.

One interesting relationship between the normal cone concept and the subdifferential concept of the distance function is as follows:

Proposition 1.68. *Let C be a nonempty closed subset of H and $u \in C$ Then*

$$\partial_P d_C(u) = N_C^P(u) \cap \mathbb{B}_H \quad \textit{and} \quad N_C^P(u) = \mathbb{R}_+ \partial_P d_C(u), \textit{ for all } u \in C. \tag{1.3}$$

1.3.3 Some classes of sets

In this section, we recall some classes of sets that generalize the class of convex sets.

Positively α–far sets

The concept of positively α–far sets was introduced in [25] and subsequently extensively studied in [31].

Definition 1.69. Let $\alpha \in]0,1]$ and $\rho \in]0,+\infty]$. Let C be a nonempty closed subset of H. We say that the Clarke subdifferential of the distance function $d(\cdot,C)$ ensures that the origin remains at least α–far from the open ρ-tube around, defined as C,

$$U_\rho(C) := \{x \in H : 0 < d(x;C) < \rho\},$$

if the following inequality holds

$$0 < \alpha < \inf_{x \in U_\rho(C)} d(0, \partial d(\cdot, C)(x)). \tag{1.4}$$

Prox-regular sets

The proximal normal cone is the right concept to use for defining the prox-regularity of a set C by requiring in (1.2) that the constant r be uniform for all the unit proximal normal vectors of r. Sets satisfying this property are known as (uniformly) prox-regular sets.

Definition 1.70. Let $r \in]0, +\infty]$. A nonempty closed set C is said to be r-prox-regular (or uniformly prox-regular with constant r) if each point u in the open r-enlargement of C

$$\mathcal{U}_r(C) := \{u \in H \mid d_C(u) < r\}$$

has a unique nearest point $\text{proj}_C(u)$ and the mapping $\text{proj}_C(\cdot)$ is continuous over $\mathcal{U}_r(C)$. It is evident that the r-prox-regularity of C with $r = +\infty$ corresponds to its convexity. This class of sets was first established by Federer [24] in the finite-dimensional framework under the name "positively reached sets".

The next theorem provides some useful properties of prox-regular sets. For further properties of prox-regular sets, refer to [21, 58].

Theorem 1.71. *Let $C \subset H$ be a nonempty closed subset and let $r \in]0; +\infty]$. The subsequent properties are equivalent:*

1. *The set C is r-prox-regular.*

2. *For all $u \in C$ and $\xi \in N^P(C; u)$, we have*

$$\langle \xi, v - u \rangle \leq \frac{\|\xi\|}{2r} \|v - u\|^2 \ \ \forall v \in C.$$

3. *For all $u_1, u_2 \in C$, for all $\vartheta_1 \in N^P(C; u_1) \cap \mathbb{B}$, and for all $\vartheta_2 \in N^P(C; u_2) \cap \mathbb{B}$, we have*

$$\langle \vartheta_1 - \vartheta_2, u_1 - u_2 \rangle \geq -\frac{1}{r} \|u_1 - u_2\|^2.$$

Remark 1.72. The property (3) of the last theorem means that the set-valued map $N^P(C; \cdot) \cap \mathbb{B}$ is hypomonotone.

We use the notation $\text{proj}_C(u)$ instead of $\text{Proj}_C(u)$ when this set has a unique point.

Proposition 1.73. *Let the subset $C \subset H$ be a nonempty and closed, and let $r \in]0, \infty]$. If C is uniformly r-prox-regular, then the following assertions are hold:*

1. *For all $u \in H$ with $d_C(u) < r$, $\text{proj}_C(u)$ exists.*

2. *For all $u \in C$, one has $N^C(u; C) = N^P(u; C) = N^F(u; C) = N^L(u; C)$.*

3. *The Clarke and the Proximal subdifferentials of d_C coincide for all points $u \in H$ with $d_C(u) < r$.*

1.4 Some useful results of functional analysis

In this section, we examine fundamental concepts and theorems relevant to our work. These results cover the strong and weak convergence theorem and specific integral inequalities of Gronwall type.

1.4.1 strong and weak convergence theorems

In this subsection, we present some basic properties of weak and strong convergence in a Banach space X and its topological dual X^*.

Proposition 1.74. *Let $\{x_n\}$ be a sequence in X. Then*

1. $x_n \rightharpoonup x \Longleftrightarrow \langle \vartheta, x_n \rangle \to \langle \vartheta, x \rangle, \ \ \forall \vartheta \in X^*$.
2. *If $x_n \to x$ then $x_n \rightharpoonup x$.*
3. *If $x_n \rightharpoonup x$, then (x_n) is bounded and*

$$\|x\| \leq \liminf_{n \to +\infty} \|x_n\|.$$

4. *If $x_n \rightharpoonup x$ in X and $\vartheta_n \to \vartheta$ in X^*, then $\langle \vartheta_n, x_n \rangle \to \langle \vartheta, x \rangle$.*

Theorem 1.75 (Lebesgue dominated convergence theorem)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1(\Omega)$ assumed to satisfy the following two properties.*

1. *Almost everywhere in Ω, f_n converge to f.*
2. *There exists a function $g \in L^1(\Omega)$ such that for all $n \in \mathbb{N}$,*

$$\|f_n(x)\| \leq g(x).$$

Then $f \in L^1(\Omega)$ and

$$\|f_n(x) - f(x)\| \longrightarrow 0.$$

Definition 1.76. A function $f : [a; b] \to H$ is said to be absolutely continuous if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $]a_n; b_n[$ are pairwise disjoint subintervals of $[a; b]$

$$\sum_{n\geq 0}(b_n - a_n) < \delta \Rightarrow \sum_{n\geq 0} \|f(a_n) - f(b_n)\| < \varepsilon.$$

Furthermore, the function $f : [a, b] \longrightarrow H$ is absolutely continuous if and only if

$$f(b) - f(a) = \int_a^b f'(s)ds.$$

If f is Lipschitz-continuous, then f is obviously absolutely continuous.

Definition 1.77. Let a function $f : [T_0, T] \longrightarrow H$, a subinterval $J \subset [T_0, T]$, we define the variation of f on I by the following expression

$$var(f, J) := \sup \left\{ \sum_{i=1}^{n} ||f(t_i) - f(t_{i-1})||,\ n \in \mathbb{N},\ t_i \in J,\ t_0 < t_1 < ... < t_n \right\}.$$

We state that a function f has bounded variation on the interval $[T_0, T]$ if there exists a constant M such that the variation of f over any subinterval J satisfies $var(f, J) < M$.

Corollary 1.78. *Let f be an absolutely continuous function on $[T_0, T]$. Then f is of bounded variation on $[T_0, T]$.*

Theorem 1.79. *[45] Consider a sequence u_n of functions from the interval $I = [0, T]$ to a Hilbert space H. Assume that u_n is uniformly bounded in norm and in variation, i.e., that there exist $L, M > 0$ such that:*

$$\|u_n(t)\| \leq L \quad (t \in I, n \in \mathbb{N}),$$

$$var(u_n, I) \leq M \quad (n \in \mathbb{N}).$$

Then, there exists a subsequence (u_{n_k}) of (u_n) which converges weakly to some function $u : I \longrightarrow H$ with $Var(u, I) \leq M$

$$u_{n_k}(t) \rightharpoonup u(t) \quad (t \in I, k \in \mathbb{N}).$$

Lemma 1.80. *[34] Let $(x_n(t))_n \in \mathbb{N}$ be a sequence of absolutely continuous functions from $[T_0, T]$ into H with $x_n(T_0) = x_0^n$. Assume that for all $n \in \mathbb{N}$*

$$\|\dot{x}_n(t)\| \leq \phi(t) \ \ \textit{for all} \ \ t \in [T_0, T]$$

where $\phi \in L^1([T_0, T]$ and that $x_0^n \to x_0$ as $n \to +\infty$. Then, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ and an absolutely continuous function x such that

1. *$x_{n_k}(t) \rightharpoonup x(t)$ in H as $k \to +\infty$ for all $t \in [T_0, T]$.*

2. $x_{n_k}(t) \rightharpoonup x(t)$ *in* $L^1([T_0, T])$ *as* $k \to +\infty$.

3. $\dot{x}_{n_k}(t) \rightharpoonup \dot{x}(t)$ *in* $L^1([T_0, T])$ *as* $k \to +\infty$.

4. $\|\dot{x}(t)\| \leq \phi(t)$ *for all* $t \in [T_0, T]$.

Proposition 1.81. *[7] Let us consider a sequence of absolutely continuous function* x_K *from an interval* I *of* $\mathbb{R}$ *to a Banach space* X *satisfying*

1. $\forall t \in I$, $\{x_k(t)\}_k$ *is a relatively compact subset of* X,

2. *there exists a positive function* $c \in L_1(I)$ *such that, for almost all* $t \in I$ $\|\dot{x}_k(t)\| \leq c(t)$.

Then there exist a subsequence (again denoted by x_K*) converging to an absolutely continuous function* x *from* I *to* X *in the sense that*

1. x_K *converges uniformly to* x *over compact subsets of* I.

2. $\dot{x}_k$ *converges weakly to* $\dot{x}$ *in* $L_1(I, X)$.

Lemma 1.82. *Let* $u : [T_0, T] \longrightarrow H$ *be an absolutely continuous function. Then*

1. $\frac{1}{2}\left(\frac{d}{dt}||u(t)||^2\right) = \langle \dot{u}(t), u(t)\rangle$.

2. $\int_{T_0}^{T} \langle \dot{u}(t), u(t)\rangle = \frac{1}{2}||u(T)||^2 - \frac{1}{2}||u(T_0)||^2$.

1.4.2 Integral inequalities of Gronwall type

Theorem 1.83. *[6] Let* ψ, φ, ϕ *be continuous and non-negative functions in* $[a, b]$, *and let the function* ϕ *also be non-decreasing in* $[a, b]$. *Then the inequality*

$$\varphi(t) \leq \phi(t) + \int_a^t \psi(s)\varphi(s)ds, \qquad t \in [a, b]$$

implies that

$$\varphi(t) \leq \phi(t) \exp\Big[\int_a^t \psi(s)ds\Big], \qquad t \in [a, b].$$

Lemma 1.84. *[7] Let* α, β *two positive numbers and* $\varphi : [0; T] \longrightarrow \mathbb{R}$ *be an absolutely continuous function. Assume that*

$$\dot{\varphi}(t) + \beta\varphi(t) \leq \alpha \ \ \textit{for a.e} \quad t \in [T_0; T]$$

Then, for all $t \in [T_0; T]$

$$\varphi(t) \leq \varphi(T_0)(-\beta(t-T_0)) + \frac{\alpha}{\beta}(1-\exp(-\beta(t-T_0))).$$

Lemma 1.85 (Gronwall-like differential inequality)**.** *[13] Let* $\rho : [T_0, T] \longrightarrow \mathbb{R}$ *be an absolutely continuous non-negative function and let* $K_1, K_2, \varepsilon : [T_0, T] \longrightarrow \mathbb{R}_+$ *be non-negative Lebesgue integrable functions that satisfies for some* $\epsilon > 0$

$$\dot{\rho}(t) \leq \varepsilon(t) + \epsilon + K_1(t)\rho(t) + K_2(t)\sqrt{\rho(t)} \int_{T_0}^{t} \sqrt{\rho(s)}\, ds, \quad \text{a.e. } t \in [T_0, T]. \tag{1.5}$$

Then for all $t \in [T_0, T]$*, one has*

$$\begin{aligned} \sqrt{\rho(t)} \leq{}& \sqrt{\rho(T_0)+\epsilon}\, \exp\left(\int_{T_0}^{t} (K(s)+1)\, ds\right) + \frac{\sqrt{\epsilon}}{2} \int_{T_0}^{t} \exp\left(\int_{s}^{t} (K(\tau)+1)\, d\tau\right) ds \\ &+ 2\left(\sqrt{\int_{T_0}^{t} \varepsilon(s)\, ds + \epsilon} - \sqrt{\epsilon}\, \exp\left(\int_{T_0}^{t} (K(\tau)+1)\, d\tau\right)\right) \\ &+ 2\int_{T_0}^{t} (K(s)+1) \exp\left(\int_{s}^{t} (K(\tau)+1)\, d\tau\right) \sqrt{\int_{T_0}^{s} \varepsilon(\tau)\, d\tau + \epsilon}\; ds, \end{aligned}$$

where $K(t) := \max\left\{\frac{K_1(t)}{2}, \frac{K_2(t)}{2}\right\}$ *for* $t \in [T_0, T]$*.*

Chapter Summary

This chapter is devoted to the preliminary results of some specific topics that we need in the subsequent chapters. Thus, we make some theoretical reminders of analysis, such as, for example, some points on convex analysis, particularly, we present some properties of convex sets and functions as well as conjugate and subdifferential properties of convex functions. Subsequently, we provide some tools which concern the characterization of the projection onto a closed convex set. This is followed by some reminders on set-valued mappings that are necessary for the study of differential inclusions. Next, we introduce some concepts of various semi-continuities. We recall some results on maximal monotone operators. Finally, an introduction to differential inclusions is presented. In the next section, we look closely at some basic notions of nonsmooth analysis, for example, we present some definitions and properties of normal cones. Subsequently, we give a short review on the properties of some class of sets. In the final section of this chapter, we present some results covering the strong and weak convergence theorem and specific integral inequalities of Gronwall type.

Chapitre 2

Semi Regularization Of Prox-Regular Integro-Differential Sweeping Process

Let H be a Hilbert space, T be a non-negative real number. In this chapter, which is based on [43]. we study the existence and uniqueness of solutions of a perturbed differential inclusion governed by a non-convex sweeping process by using a semi-regularization technique. In this technique we approach the integro-differential sweeping process of Volterra type associated with maximal monotone operators of following form,

$$(P_{A,f}) : \begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + Ax(t) + \int\limits_0^t f(t,s,x(s))ds, \qquad \text{a.e. } t \in [0,T] \\ x(0) = x_0 \in C(0), \end{cases} \tag{2.1}$$

by a penalized one, depending on a parameter. This makes it easier to prove the existence of a solution. Subsequently, as the solution is established, the parameter can be taken to zero to obtain the desired result.

This Chapter is organized as follows. In Section 1 we recall the main assumptions that will be used throughout the chapter. In Section 2, we present our main existence and uniqueness result. In Section 3, we provide an example in parabolic-variational inequalities with Volterra-type operators.

2.1 Technical assumptions

In this section, we gather the hypotheses used along the chapter to enhance readability.

($\mathcal{H}_1$) The set-valued mapping $C : [0,T] \rightrightarrows H$ has non-empty, closed and r-prox-regular values for some constant $r \in]0,+\infty]$, and there exists an absolutely continuous function $v(\cdot) : [0,T] \to \mathbb{R}$ such that

$$C(t) \subset C(s) + |v(t) - v(s)|\mathbb{B}, \quad \forall t,s \in [0,T].$$

($\mathcal{H}_2$) The set-valued mapping $A : H \rightrightarrows H$ is a maximal monotone operator.

($\mathcal{H}_3$) $f : Q_\Delta \times H \longrightarrow H$ is a measurable mapping such that, there is a non-negative function $\beta(\cdot,\cdot) \in L^1(Q_\Delta, \mathbb{R}_+)$ satisfies:

$$\|f(t,s,x)\| \leq \beta(t,s)(1+\|x\|), \text{ for all } (t,s) \in Q_\Delta \text{ and for each } x \in \mathcal{R}(C(t)),$$

where

$$Q_\Delta := \{(t,s) \in [0,T] \times [0,T] : s \leq t\}.$$

($\mathcal{H}_4$) For any real $\eta > 0$, there exists a non-negative function $L^\eta(\cdot) \in L^1([0,T], \mathbb{R}_+)$ such that

$$\|f(t,s,x) - f(t,s,y)\| \leq L^\eta(t)\|x-y\|,$$

for all $(t,s) \in Q_\Delta$ and for each $(x,y) \in B[0,\eta] \times B[0,\eta]$.

2.2 Existence and uniqueness results

We now proceed to prove the main results about the existence and uniqueness of the solution. But first we need the following auxiliary theorem, which is proved in [13].

Theorem 2.1. *Assume, in addition to* ($\mathcal{H}_1$),($\mathcal{H}_2$), ($\mathcal{H}_3$) *and* ($\mathcal{H}_4$) *that:*

1. $g : [0,T] \times H \longrightarrow H$ *is a measurable mapping such that, there exists a non-negative function* $\beta_1(\cdot) \in L^1([0,T], \mathbb{R})$ *satisfies:*

$$\|g(t,x)\| \leq \beta_1(t)(1+\|x\|), \text{ for all } t \in [0,T] \text{ and for any } x \in \mathcal{R}(C(t)).$$

2. *For each real* $\eta > 0$, *there is a non-negative function* $L_1^\eta(\cdot) \in L^1([0,T],\mathbb{R})$ *such that*

$$\|g(t,x) - g(t,y)\| \leq L_1^\eta(t)\|x-y\|,$$

for any $t \in [0,T]$ *and for any* $(x,y) \in B[0,\eta] \times B[0,\eta]$.

Then for any $x_0 \in C(0)$ *there exists a unique absolutely continuous solution* $x(\cdot)$ *for the following differential inclusion*

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + g(t,x(t)) + \int\limits_0^t f(t,s,x(s))ds, & a.e.\ t \in [0,T], \\ x(0) = x_0 \in C(0). \end{cases}$$

Moreover, for almost every $t \in [0,T]$, *one has*

$$\left\|\dot{x}(t) + g(t,x(t)) + \int\limits_0^t f(t,s,x(s))ds\right\| \leq \left\|g(t,x(t)) + \int\limits_0^t f(t,s,x(s))ds\right\| + |\dot{v}(t)|.$$

We now present the primary outcome of this chapter, which establishes the existence solution of $(P_{A,f})$.

Theorem 2.2. *Let the assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ *and* $(\mathcal{H}_4)$ *hold. Under the following additional conditions:*

1. $\mathcal{R}(C(t)) \subset \mathrm{Dom}(A)$,

2. *there exists a non-negative functions* $\alpha(\cdot), \delta(\cdot) \in L^2([0,T],\mathbb{R}_+)$ *with* $\alpha(\cdot)$ *is a continuous function, such that*

$$\|A^\circ x\| \leq \alpha(t)\|x\| + \delta(t), \ \textit{for a.e. } t \in [0,T] \ \textit{ and } x \in C(t),$$

3. *there exists* $\gamma_1(\cdot), \gamma_2(\cdot) \in L^1([0,T],\mathbb{R}_+)$ *such that*

$$\beta(t,s) \leq \gamma_1(t).\gamma_2(s) \ \textit{ for a.e. } t,s \in [0,T],$$

for each $x_0 \in C(0)$, *the differential inclusion* (2.1) *admits, at least, an absolutely continuous solution* $x(\cdot) : [0,T] \to H$.

Proof. The proof of the existence of the solution is divided into several steps.

Step 1. A family of approximate solutions

Fix any $\lambda > 0$ and consider the following approximate problem

$$\begin{cases} -\dot{x}_\lambda(t) \in N_{C(t)}(x_\lambda(t)) + A_\lambda x_\lambda(t) + \displaystyle\int_0^t f(t,s,x_\lambda(s))\,ds, & \text{a.e. } t \in [0,T], \\ x_\lambda(0) = x_0 \in C(0). \end{cases} \tag{2.2}$$

From Theorem 2.1, it results that for any $\lambda > 0$, the integro-differential sweeping process (2.2) has a unique absolutely continuous solution $x_\lambda(\cdot)$ on $[0,T]$. Furthermore, for almost every $t \in [0,T]$

$$\|\dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds\| \leq \|A_\lambda x_\lambda(t)\| + \int_0^t \|f(t,s,x_\lambda(s))\|\,ds + |\dot{v}(t)|. \tag{2.3}$$

Step 2. An upper bound of the norm of the approximate solutions $x_\lambda(\cdot)$

From the assumption (2), one has

$$\begin{aligned} \|A_\lambda(x_\lambda(t))\| &\leq \|A^\circ(x_\lambda(t))\| \\ &\leq \alpha(t)\|x_\lambda(t)\| + \delta(t). \end{aligned} \tag{2.4}$$

In addition, from the assumptions $(\mathcal{H}_3)$ and (3) we have

$$\begin{aligned} \int_0^t \|f(t,s,x_\lambda(s))\|\,ds &\leq \int_0^t \beta(t,s)(1 + \|x_\lambda(s)\|)\,ds \\ &\leq \int_0^t \beta(t,s)\,ds + \int_0^t \beta(t,s)\|x_\lambda(s)\|\,ds \\ &\leq \int_0^t \beta(t,s)\,ds + \gamma_1(t)\int_0^t \gamma_2(s)\|x_\lambda(s)\|\,ds, \quad \text{a.e. } t \in [0,T]. \end{aligned} \tag{2.5}$$

Therefore, from (2.3) and by utilize (2.4) and (2.5)

$$\begin{aligned}
\|\dot{x}_\lambda(t)\| &= \|\dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds - A_\lambda x_\lambda(t) - \int_0^t f(t,s,x_\lambda(s))\,ds\| \\
&\le \|\dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds|| + ||A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds\| \\
&\le 2||A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds\| + |\dot{v}(t)| \\
&\le 2\|A_\lambda x_\lambda(t)\| + 2\int_0^t \|f(t,s,x_\lambda(s))\|\,ds + |\dot{v}(t)| \\
&\le 2\alpha(t)\|x_\lambda(t)\| + 2\gamma_1(t)\int_0^t \gamma_2(s)\|x_\lambda(s)\|\,ds + 2\delta(t) + 2\int_0^t \beta(t,s)\,ds + |\dot{v}(t)|.
\end{aligned}$$

On the other hand, we note that

$$\begin{aligned}
\|x_\lambda(t)\| &= \|x_0 + \int_0^t \dot{x}_\lambda(t)\,ds\| \\
&\le \|x_0\| + \int_0^t \|\dot{x}_\lambda(t)\|\,ds \\
&\le \|x_0\| + \int_0^t \Big(2\alpha(\tau)\|x_\lambda(\tau)\| + 2\,\gamma_1(\tau)\int_0^\tau \gamma_2(s)\|x_\lambda(s)\|\,ds + 2\,\delta(\tau) + 2\int_0^\tau \beta(\tau,s)\,ds + |\dot{v}(\tau)|\Big)\,d\tau \\
&= \|x_0\| + \mathit{\Gamma}(t) + 2\int_0^t \alpha(\tau)\|x_\lambda(\tau)\|d\tau + 2\int_0^t \gamma_1(\tau)\Big(\int_0^\tau \gamma_2(s)\|x_\lambda(s)\|\,ds\Big)d\tau,
\end{aligned}$$

where

$$\mathit{\Gamma}(t) := 2\int_0^t \delta(\tau)d\tau + 2\int_0^t\int_0^\tau \beta(\tau,s)\,ds d\tau + \int_0^t |\dot{v}(\tau)|d\tau,$$

as a result of the fact

$$\int_0^t \gamma_1(\tau)\Big(\int_0^\tau \gamma_2(s)\|x_\lambda(s)\|\,ds\Big)d\tau \le \|\gamma_1\|_{L^1([0,T],\mathbb{R}_+)}\int_0^t \gamma_2(s)\|x_\lambda(s)\|\,ds,$$

we get

$$\|x_\lambda(t)\| \leq \|x_0\| + \Gamma(t) + \int_0^t \omega(\tau)\|x_\lambda(\tau)\|\, d\tau,$$

where

$$\omega(t) := 2\alpha(t) + 2\|\gamma_1\|_{L^1([0,T],\mathbb{R}_+)}\gamma_2(t).$$

Therefore, by applying Gronwall's inequality as mentioned in Theorem 1.83, we can conclude that

$$\|x_\lambda(t)\| \leq (\|x_0\| + \Gamma(t)) \exp\left(\int_0^t \omega(\tau)d\tau\right) \leq (\|x_0\| + \Gamma(t)) \exp\left(\int_0^T \omega(\tau)d\tau\right) =: M,$$

which signifies the boundedness of $x_\lambda(\cdot)$ independently of λ on $[0,T]$.

Step 3. The convergence of the sequence $(x_\lambda)_\lambda$

It suffices to show that $(x_\lambda)_\lambda$ is a Cauchy sequence in the Banach space $(C([0,T],H), \|\cdot\|_\infty)$, in another words

$$\lim_{\lambda,\mu\to\infty} \|x_\lambda(\cdot) - x_\mu(\cdot)\|_\infty = 0,$$

in a manner that

$$\lim_{\lambda,\mu\to\infty} \|x_\lambda(\cdot) - x_\mu(\cdot)\|_\infty = \sup_{t\in[0,T]} \|x_\lambda(t) - x_\mu(t)\|.$$

Let us establish

$$\xi(t) := M\alpha(t) + \delta(t) + (M+1)\int_0^t \beta(t,s)ds,$$

moreover $\xi \in L^2([0,T],\mathbb{R}_+)$, and for almost every $t \in [0,T]$ we obtain

$$\|A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\, ds\| \leq \xi(t),$$

and

$$\|\dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\, ds\| \leq \xi(t) + |\dot{v}(t)|. \tag{2.6}$$

Indeed,

$$
\begin{aligned}
\|A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds\| &\le \|Ax_\lambda(t)\| + \int_0^t \|f(t,s,x_\lambda(s))\|\,ds \\
&\le \|A^\circ x_\lambda(t)\| + \int_0^t (1+\|x_\lambda(s)\|)\beta(t,s)\,ds \\
&\le \alpha(t)\|x_\lambda(s)\| + \delta(t) + \int_0^t (1+\|x_\lambda(s)\|)\beta(t,s)\,ds \\
&\le M\alpha(t) + (1+M)\int_0^t \beta(t,s)\,ds + \delta(t) \\
&=: \xi(t).
\end{aligned}
$$

As a consequence

$$
\begin{aligned}
\|\dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds\| &\le \|A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds\| + |\dot{v}(t)| \\
&\le \xi(t) + |\dot{v}(t)|.
\end{aligned}
$$

which implies that

$$
-\frac{1}{\xi(t)+|\dot{v}(t)|}\Big(\dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds\Big) \in N_{C(t)}(x_\lambda(t)) \cap \mathbb{B}. \tag{2.7}
$$

Let now $\lambda, \mu > 0$. Since the sets $C(t)$ are r-prox-regular hence, by using the hypomonotonicity property given in (3) of Theorem 1.71 and the inclusion (2.7), one has for almost all $t \in [0,T]$

$$
\begin{aligned}
\Big\langle -\frac{1}{\xi(t)+|\dot{v}(t)|}\Big(\dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds - \dot{x}_\mu(t) - A_\mu x_\mu(t) - \int_0^t f(t,s,x_\mu(s))\,ds\Big) \\
, x_\lambda(t) - x_\mu(t)\Big\rangle \ge -\frac{1}{r}\|x_\lambda(t) - x_\mu(t)\|^2,
\end{aligned}
$$

then

$$\Big\langle \dot{x}_\lambda(t) + A_\lambda x_\lambda(t) + \int_0^t f(t,s,x_\lambda(s))\,ds - \dot{x}_\mu(t) - A_\mu x_\mu(t) - \int_0^t f(t,s,x_\mu(s))\,ds, x_\lambda(t) - x_\mu(t) \Big\rangle$$
$$\leq \frac{\xi(t) + |\dot{v}(t)|}{r} \|x_\lambda(t) - x_\mu(t)\|^2. \tag{2.8}$$

It is obvious that $x_\lambda(t) = J_\lambda(x_\lambda(t)) + \lambda A_\lambda(x_\lambda(t))$. Additionally, the inclusion $A_\lambda x_\lambda(t) \in A(J_\lambda x_\lambda(t))$ remains valid by virtue of Proposition 1.52. Notably, the operator A is monotone. Consequently,

$$\langle\, A_\lambda x_\lambda(t) - A_\mu x_\mu(t), J_\lambda x_\lambda(t) - J_\mu x_\mu(t) \,\rangle \geq 0.$$

Furthermore,

$$\begin{aligned}
\langle\, A_\lambda x_\lambda(t) - A_\mu x_\mu(t), x_\lambda(t) - x_\mu(t) \,\rangle &= \langle\, A_\lambda x_\lambda(t) - A_\mu x_\mu(t), J_\lambda x_\lambda(t) + \lambda A_\lambda x_\lambda(t) - J_\mu x_\mu(t) - \mu A_\mu x_\mu(t) \,\rangle \\
&\geq \langle\, A_\lambda x_\lambda(t) - A_\mu x_\mu(t), \lambda A x_\lambda(t) - \mu A_\mu x_\mu(t) \,\rangle \\
&+ \langle\, A_\lambda x_\lambda(t) - A_\mu x_\mu(t), J_\lambda x_\lambda(t) - J_\mu x_\mu(t) \,\rangle \\
&\geq \langle\, A_\lambda x_\lambda(t) - A_\mu x_\mu(t), \lambda A x_\lambda(t) - \mu A_\mu x_\mu(t) \,\rangle \\
&\geq \lambda \|\, A_\lambda x_\lambda(t)\|^2 + \mu \|A_\mu x_\mu(t)\|^2 - \lambda \|A_\lambda x_\lambda(t)\| \|A_\mu x_\mu(t)\| \\
&- \mu \|A_\mu x_\mu(t)\| \|A_\lambda x_\lambda(t)\|\,,
\end{aligned}$$

however,

$$\begin{aligned}
0 &\leq \left(\sqrt{\lambda} \|A_\lambda x_\lambda(t)\| - \frac{\sqrt{\lambda}}{2} \|A_\mu x_\mu(t)\| \right)^2 \\
&= \lambda \|A_\lambda x_\lambda(t)\|^2 + \frac{\lambda}{4} \|A_\mu x_\mu(t)\|^2 - \lambda \|A_\lambda x_\lambda(t)\| \|A_\mu x_\mu(t)\|
\end{aligned}$$

and

$$\begin{aligned}
0 &\leq \left(\sqrt{\mu} \|A_\mu x_\mu(t)\| - \frac{\sqrt{\mu}}{2} \|A_\lambda x_\lambda(t)\| \right)^2 \\
&= \mu \|A_\mu x_\mu(t)\|^2 + \frac{\mu}{4} \|A_\mu x_\mu(t)\|^2 - \mu \|A_\mu x_\mu(t)\| \|A_\lambda x_\lambda(t)\|,
\end{aligned}$$

for this reason

$$-\lambda \|A_\lambda x_\lambda(t)\| \|A_\mu x_\mu(t)\| \geq -\lambda \|A_\lambda x_\lambda(t)\|^2 - \frac{\lambda}{4} \|A_\mu x_\mu(t)\|^2, \tag{2.9}$$

and

$$-\mu\|A_\mu x_\mu(t)\|\|A_\lambda x_\lambda(t)\| \geq -\mu\|A_\mu x_\mu(t)\|^2 - \frac{\mu}{4}\|A_\lambda x_\lambda(t)\|^2. \tag{2.10}$$

By virtue of (2.9), (2.10) and the assumption (2), we obtain

$$\begin{aligned}\langle A_\lambda x_\lambda(t) - A_\mu x_\mu(t), x_\lambda(t) - x_\mu(t)\rangle &\geq -\frac{1}{4}\big(\lambda\|A_\mu x_\mu(t)\|^2 + \mu\|A_\lambda x_\lambda(t)\|^2\big)\\ &\geq -\frac{1}{4}(\lambda+\mu)\Big(M\alpha(t)+\delta(t)\Big)^2\end{aligned}$$

this implies

$$\langle A_\lambda x_\lambda(t) - A_\mu x_\mu(t), x_\lambda(t) - x_\mu(t)\rangle \geq -\frac{1}{4}(\lambda+\mu)\xi^2(t). \tag{2.11}$$

On the other hand, as mentioned by the assumption $(\mathcal{H}_4)$ one has

$$\begin{aligned}&\Big\langle \int_0^t f(t,s,x_\lambda(s))ds - \int_0^t f(t,s,x_\mu(s))ds, x_\lambda(t) - x_\mu(t)\Big\rangle\\ &= -\Big\langle \int_0^t f(t,s,x_\mu(s)) - f(t,s,x_\lambda(s))ds, x_\lambda(t) - x_\mu(t)\Big\rangle\\ &\geq -\Big\|\int_0^t f(t,s,x_\mu(s)) - f(t,s,x_\lambda(s))ds\Big\|\,\Big\|x_\lambda(t) - x_\mu(t)\Big\|\\ &\geq -L^\eta(t)\|x_\lambda(t) - x_\mu(t)\|\int_0^t \|x_\lambda(s) - x_\mu(s)\|\,ds. \end{aligned}\tag{2.12}$$

From (2.8), we have

$$\begin{aligned}\langle \dot{x}_\lambda(t) - \dot{x}_\mu(t), x_\lambda(t) - x_\mu(t)\rangle \leq{}& \frac{\xi(t)+|\dot{v}(t)|}{r}\|x_\lambda(t) - x_\mu(t)\|^2 - \langle A_\lambda x_\lambda(t) - A_\mu x_\mu(t), x_\lambda(t) - x_\mu(t)\rangle\\ &-\Big\langle \int_0^t f(t,s,x_\lambda(s))ds - \int_0^t f(t,s,x_\mu(s))ds, x_\lambda(t) - x_\mu(t)\Big\rangle\end{aligned}$$

true to Lemma 1.82, we get

$$\begin{aligned}\frac{d}{dt}\|x_\lambda(t) - x_\mu(t)\|^2 \leq{}& 2\frac{\xi(t)+|\dot{v}(t)|}{r}\|x_\lambda(t) - x_\mu(t)\|^2 - 2\langle A_\lambda x_\lambda(t) - A_\mu x_\mu(t), x_\lambda(t) - x_\mu(t)\rangle\\ &-2\Big\langle \int_0^t f(t,s,x_\lambda(s))ds - \int_0^t f(t,s,x_\mu(s))ds, x_\lambda(t) - x_\mu(t)\Big\rangle.\end{aligned}$$

Associating this last inequality with (2.11) and (2.12), appear to us

$$\frac{d}{dt}\|x_\lambda(t)-x_\mu(t)\|^2 \leq \frac{1}{2}(\lambda+\mu)\xi^2(t)+2\frac{\xi(t)+|\dot{v}(t)|}{r}\|x_\lambda(t)-x_\mu(t)\|^2 + 2L^\eta(t)\|x_\lambda(t)-x_\mu(t)\|\int_0^t \|x_\lambda(s)-x_\mu(s)\|\,ds\,,$$

by applying Lemma 1.85 with

$$\rho(t)=\|x_\lambda(t)-x_\mu(t)\|^2,\ K_1(t)=2\frac{\xi(t)+\dot{v}(t)}{r},\ K_2(t)=2L^\eta(t),$$

$$\varepsilon(t)=\frac{1}{2}(\lambda+\mu)\xi^2(t),\ \epsilon>0$$

and considering the equality $x_\lambda(0)=x_\mu(0)=x_0$, we can conclude that, for each $t\in[0,T]$

$$\begin{aligned}\|x_\lambda(t)-x_\mu(t)\| \leq & \sqrt{\epsilon}\exp\left(\int_0^t (K(s)+1)\,ds\right)+\frac{\sqrt{\epsilon}}{2}\int_0^t \exp\left(\int_s^t (K(\tau)+1)\,d\tau\right)ds\\ &+2\left(\sqrt{\int_0^t \varepsilon(s)\,ds+\epsilon}-\sqrt{\epsilon}\exp\left(\int_0^t (K(\tau)+1)\,d\tau\right)\right)\\ &+2\int_0^t (K(s)+1)\exp\left(\int_s^t (K(\tau)+1)\,d\tau\right)\sqrt{\int_0^s \varepsilon(\tau)\,d\tau+\epsilon}\,ds,\end{aligned}$$

where $K(t):=\max\left\{\frac{K_1(t)}{2},\frac{K_2(t)}{2}\right\}$, almost every $t\in[0,T]$.
By taking $\epsilon\longrightarrow 0$, we conclude that

$$\lim_{\lambda,\mu\to\infty}\|x_\lambda(\cdot)-x_\mu(\cdot)\|_\infty=0.$$

As a result, the sequence $(x_\lambda(\cdot))_{\lambda>0}$ is a Cauchy sequence in $C([0,T];H)$, and therefore, it converges uniformly to a sequence $x(\cdot)\in C([0,T];H)$ as $\lambda\downarrow 0$. Furthermore, $x(t)\in C(t)$ due to $x_\lambda(t)\in C(t)$ for every $\lambda>0$, and $C(t)$ is closed subset. Additionally, $x(t)\in \mathrm{Dom}(A)$ based on the property (6) of Proposition 1.52, and the inequality $\|A^0x\|\leq\alpha(t)M+\delta(t)$.

Step 4. x(·) is a solution of $(P_{A,f})$

For the reason that

$$\|\dot{x}_\lambda(t)\| \leq 2\xi(t) + |\dot{v}(t)| \quad \text{for almost every} \quad t \in [0,T],$$

there exists a subsequence $(\dot{x}_{\lambda_n})_n$ converges weakly to some $g(\cdot) \in L^1([0,T];H)$, in other words

$$\int_0^T \langle \dot{x}_n(s), h(s)\rangle\, ds \longrightarrow \int_0^T \langle g(s), h(s)\rangle\, ds, \quad \forall\, h \in L^\infty([0,T];H).$$

Now fix any $t \in [0,T]$, for all $z \in H$ we find

$$\int_0^T \big\langle \dot{x}_n(s), z\chi_{[0,t]}(s)\big\rangle\, ds = \int_0^t \langle \dot{x}_n(s), z\rangle\, ds = \Big\langle \int_0^t \dot{x}_n(s)ds, z\Big\rangle.$$

Furthermore

$$\int_0^T \big\langle g(s), z\chi_{[0,t]}(s)\big\rangle\, ds = \int_0^t \langle g(s), z\rangle\, ds = \langle \int_0^t g(s)ds, z\rangle,$$

it follows from this that

$$\int_0^t \dot{x}_n(s)\, ds \text{ converges weakly to } \int_0^t g(s)\, ds \;\text{ in }\; H.$$

Consequently

$$x_n(t) = x_n(0) + \int_0^t \dot{x}_n(s)\, ds \;\text{ converges weakly to } x(0) + \int_0^t g(s)\, ds \;\text{ in }\; H.$$

Account of the fact that

$$x_n(\cdot) \;\text{ converges uniformly to }\; x(\cdot),$$

we establish that

$$x(t) = x_0 + \int_0^t g(s)ds,$$

which expresses the absolute continuous property of $x(\cdot)$, furthermore $\dot{x}(\cdot) = g(\cdot)$ a.e. $t \in [0,T]$, and in related manner

$$\|x(t)\| \leq M_1 := \|x_0\| + \int_0^T g(s)ds. \tag{2.13}$$

Across the continuity property of $x \mapsto f(t,s,x)$ and the uniform convergence of $x_{\lambda_n}(\cdot)$ to $x(\cdot)$ we obtain

$$\lim_{n\to+\infty} f(t,s,x_{\lambda_n}(s)) \longrightarrow f(t,s,x(s)). \tag{2.14}$$

We establish for each $t \in [0,T]$,

$$\phi_n(t) := \int_0^t f(t,s,x_{\lambda_n}(s))\,ds, \text{ and } \phi(t) := \int_0^t f(t,s,x(s))\,ds.$$

Additionally, let us set $\eta_0 := \max\{M, M_1\}$. Therefore,

$$(x(t), x_\lambda(t)) \in B[0,\eta_0] \times B[0,\eta_0], \text{ for all } t \in [0,T].$$

Thus, through assumption $(\mathcal{H}_4)$ there exists $L^{\eta_0}(\cdot) \in L^1([0,T];\mathbb{R}_+)$ such that

$$\int_0^T \|\phi_n(t) - \phi(t)\|dt \leq \int_0^T L^{\eta_0}(t) \int_0^t \|x_{\lambda_n}(s) - x(s)\|ds\,dt. \tag{2.15}$$

Recognize that, for each $(t,s) \in Q_\Delta$

$$L^{\eta_0}(t) \int_0^t \|x_{\lambda_n}(s) - x(s)\|ds \leq 2\eta_0 T L^{\eta_0}(t)\,dt. \tag{2.16}$$

Furthermore, by combining (2.14), (2.15) and (2.16) and applying the Lebesgue dominated convergence theorem, we establish

$$\phi_n(\cdot) \text{ converges strongly to } \phi(\cdot) \text{ in } L^1([0,T];H).$$

On the other hand, one has

$$\|A_{\lambda_n} x_{\lambda_n}(t)\| \leq M\alpha(t) + \delta(t) \leq \xi(t) \quad \text{for a.e. } t \in [0,T],$$

for this reason, there exist a subsequence, still symbolized by $(A_{\lambda_n} x_{\lambda_n}(\cdot))_n$ and $\vartheta \in L^1([0,T];H)$ such that

$$A_{\lambda_n} x_{\lambda_n} \text{ converges weakly to } \vartheta \text{ in } L^1([0,T];H).$$

Therefore we obtain

$$\Psi_n(\cdot) := \dot{x}_{\lambda_n} + A_{\lambda_n} x_{\lambda_n} + \phi_{\lambda_n} \text{ converges weakly to } \Psi(\cdot) := \dot{x} + \vartheta + \phi \text{ in } L^1([0,T];H).$$

By Mazur's Theorem 1.5, for any $n \in \mathbb{N}$, there exists a sequence of convex combinations in the form of

$$\left(\sum_{k=n}^{T(n)} S_{k,n}\Psi_k\right)_n \quad \text{with} \quad S_{k,n} \geq 0 \;\text{ and }\; \sum_{k=n}^{T(n)} S_{k,n} = 1,$$

such that

$$\left(\sum_{k=n}^{T(n)} S_{k,n}\Psi_k\right)_n \quad \text{converges strongly to} \quad \Psi \;\text{ in }\; L^1([0,T];H).$$

By extracting a subsequence, we can assume the existence of a negligible set $\mathcal{N} \subset [0,T]$ such that

$$\left(\sum_{k=n}^{T(n)} S_{k,n}\Psi_k(t)\right)_n \quad \text{converges strongly to} \quad \Psi \;\text{ in }\; H, \qquad \forall t \in [0,T]\backslash\mathcal{N}.$$

and such that for all $n \in \mathbb{N}$

$$-\Psi_n(t) \in N_{C(t)}(x_{\lambda_n}(t)) \qquad \text{a.e.} \quad t \in [0,T]\,.$$

Fix any $t \in [0,T] \setminus \mathcal{N}$, through the prox-regularity of $C(t)$ and by applying Theorem 1.71, one has

$$\langle \Psi_k(t), y - x_{\lambda_k}(t)\rangle \geq -\frac{\xi(t) + |\dot{v}(t)|}{2r}\|y - x_{\lambda_k}(t)\|^2, \quad \forall y \in C(t). \tag{2.17}$$

Therefore, for all $y \in C(t)$

$$\sum_{k=n}^{T(n)} S_{k,n}\,\langle \Psi_k(t), y - x_{\lambda_k}(t)\rangle \geq -\frac{\xi(t) + |\dot{v}(t)|}{2r}\sum_{k=n}^{T(n)} S_{k,n}\|y - x_{\lambda_k}(t)\|^2. \tag{2.18}$$

Notice that

$$\begin{aligned}\left|\sum_{k=n}^{T(n)} S_{k,n}\,\langle \Psi_k(t), x(t) - x_{\lambda_k}(t)\rangle\right| &\leq \sum_{k=n}^{T(n)} S_{k,n}\|\Psi_k(t)\|\|x(t) - x_{\lambda_k}(t)\| \\ &\leq (\xi(t) + |\dot{v}(t)|)\sum_{k=n}^{T(n)} S_{k,n}\|x(t) - x_{\lambda_k}(t)\|,\end{aligned}$$

as a result

$$\sum_{k=n}^{T(n)} S_{k,n}\,\langle \Psi_k(t), x(t) - x_{\lambda_k}(t)\rangle \underset{n\to\infty}{\longrightarrow} 0, \tag{2.19}$$

because it is easily discovered that

$$\sum_{k=n}^{T(n)} S_{k,n}\|x(t) - x_{\lambda_k}(t)\| \underset{n\to\infty}{\longrightarrow} 0$$

due to

$$\|x(t) - x_{\lambda_n}(t)\| \underset{n\to\infty}{\longrightarrow} 0$$

and

$$\sum_{k=n}^{T(n)} S_{k,n} = 1.$$

As a consequence of the convergence (2.19) and the equality

$$\begin{aligned}\sum_{k=n}^{T(n)} S_{k,n} \left\langle \Psi_k(t), y - x_{\lambda_k}(t) \right\rangle &= \sum_{k=n}^{T(n)} S_{k,n} \left\langle \Psi_k(t), y - x(t) + x(t) - x_{\lambda_k}(t) \right\rangle \\ &= \left\langle \sum_{k=n}^{T(n)} S_{k,n} \Psi_k(t), y - x(t) \right\rangle \\ &+ \sum_{k=n}^{T(n)} S_{k,n} \left\langle \Psi_k(t), x(t) - x_{\lambda_k}(t) \right\rangle, \end{aligned} \tag{2.20}$$

we achieve

$$\sum_{k=n}^{T(n)} S_{k,n} \left\langle \Psi_k(t), y - x_{\lambda_k}(t) \right\rangle \underset{n\to\infty}{\longrightarrow} \left\langle \Psi(t), y - x(t) \right\rangle. \tag{2.21}$$

On the other hand, since $x_{\lambda_n}(t) \underset{n\to\infty}{\longrightarrow} x(t)$ we have

$$\sum_{k=n}^{T(n)} S_{k,n} \|y - x_{\lambda_k}(t)\|^2 \longrightarrow \|y - x(t)\|^2, \tag{2.22}$$

Passing to the limit on n in the inequality (2.18), we obtain by virtue of (2.21) and (2.22)

$$\langle \Psi(t), y - x(t) \rangle \geq -\frac{\xi(t) + |\dot{v}(t)|}{2r} \|y - x(t)\|^2, \ \forall y \in C(t),$$

leading to

$$\Psi(t) = \dot{x}(t) + \vartheta(t) + \int_0^t f(t, s, x(s))ds \in N_{C(t)}(x(t)). \tag{2.23}$$

Immediately, for the purpose of complete the proof of the Theorem, let us show that

$$\vartheta(t) \in Ax(t) \quad \text{for a.e. } t \in [0, T].$$

In light of this, let us remember that

$$A_{\lambda_n} x_{\lambda_n}(t) \in A(J_{\lambda_n} x_{\lambda_n}(t)) \quad \text{for a.e. } t \in [0,T],$$

as well as,

$$A_{\lambda_n} x_{\lambda_n} \quad \text{converges weakly to} \quad \vartheta \ \text{ in } L^2([0,T];H).$$

Furthermore,

$$(J_{\lambda_n} x_{\lambda_n})_n \quad \text{converges weakly to} \quad x \ \text{ in } L^2([0,T];H).$$

Indeed,

$$\begin{aligned}
\|J_{\lambda_n} x_{\lambda_n}(t) - x(t)\| &= \|J_{\lambda_n} x_{\lambda_n}(t) - x_{\lambda_n}(t) + x_{\lambda_n}(t) - x(t)\| \\
&\leq \|J_{\lambda_n} x_{\lambda_n}(t) - x_{\lambda_n}(t)\| + \|x_{\lambda_n}(t) - x(t)\| \\
&\leq \lambda_n \| A_\lambda x_\lambda(t)\| + \|x_{\lambda_n}(t) - x(t)\| \\
&\leq \lambda_n \xi(t) + \|x_{\lambda_n}(t) - x(t)\| \underset{n\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

since $\lambda_n \longrightarrow 0$ and $x_{\lambda_n} \longrightarrow x$ as $n \to +\infty$. Consequently, we have

$$\begin{cases}
A_{\lambda_n} x_{\lambda_n}(\cdot) \in \mathcal{A}(J_{\lambda_n} x_{\lambda_n}(\cdot)) \text{ in } L^2([0,T];H) \\
A_{\lambda_n} x_{\lambda_n}(\cdot) \xrightarrow{w} \vartheta(\cdot) \text{ in } L^2([0,T];H) \\
J_{\lambda_n} x_{\lambda_n} \xrightarrow{\|\cdot\|} x(\cdot) \text{ in } L^2([0,T];H),
\end{cases}$$

where $\mathcal{A}$ represents the extension of A presented in Proposition 1.53. linking this last three properties with the strong-weak closeness of $\mathcal{A}$ in $L^2([0,T];H)$ (see Proposition 1.49) we conclude that

$$\vartheta(\cdot) \in \mathcal{A}(x(\cdot)) \text{ in } L^2([0,T];H) \Leftrightarrow \vartheta(t) \in Ax(t) \quad \text{for a.e. } t \in [0,T]. \tag{2.24}$$

By combining (2.23) and (3.10), we conclude that

$$\dot{x}(t) \in -N_{C(t)}(x(t)) - Ax(t) - \int_0^t f(t,s,x(s))\,ds \quad \text{for a.e. } t \in [0,T]\,,$$

which completes the proof of Theorem. □

By imposing additional conditions, we can establish a uniqueness result. The following theorem is formulated within this framework.

Theorem 2.3. *Let the assumptions* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ *and* $(\mathcal{H}_4)$ *hold and assume that*

$$\mathcal{R}(C) \subset \operatorname{int}(\operatorname{Dom}(A)). \tag{2.25}$$

Then, for given initial condition $x_0 \in C(0)$, *the problem* (2.1) *has only one absolutely continuous solution.*

Proof. Let $x_1(\cdot)$, $x_2(\cdot)$ be two solutions of (2.1) satisfying

$$x_1(0) = x_2(0) = x_0 \in C(0) \subset \mathcal{R}(C).$$

Since A is maximal monotone, it follows that A is locally bounded in $\operatorname{int}(\operatorname{Dom}(A))$, as stated in Proposition 1.49. In other words, there exists $\rho > 0$ and $R > 0$ such that $B(x_0, \rho) \subset \operatorname{int}(\operatorname{Dom}(A))$,

$$\|\omega\| \leq R, \quad \text{for all} \quad \omega \in A(y), \quad \text{and all} \quad y \in B(x_0, \rho). \tag{2.26}$$

The continuity of $x_i(\cdot)$, $i = 1, 2$ on $[0, T]$ implies that

$$\forall \epsilon > 0, \ \exists\, 0 < T' < T \ \text{ such that,} \ \ \|x_i(t) - x_i(0)\| < \epsilon, \quad \text{for all} \ \ t \in [0, T'],$$

more particularly, for $\epsilon = \rho$, one obtains

$$x_i([0, T']) \subset B(x_0, \rho), \quad i = 1, 2.$$

It follows from (2.26) that

$$\|\omega\| \leq R, \quad \text{for all} \quad \omega \in A(x_i(t)), \quad \text{and all} \quad t \in [0, T'], \quad i = 1, 2. \tag{2.27}$$

Let $g_i(\cdot) \in -Ax_i(\cdot)$, $i = 1, 2$ such that, for almost every $t \in [0, T']$,

$$-\dot{x}_i(t) \in N_{C(t)}(x_i(t)) + g_i(t) + \int_0^t f(t, s, x_i(s))ds, \quad i = 1, 2.$$

Therefore, as claimed by Theorem 2.1, one has

$$\begin{aligned} \|\dot{x}_i(t) + g_i(t) + \int_0^t f(t, s, x_i(s))\, ds\| &\leq |\dot{v}(t)| + \|g_i(t)\| + \int_0^t \|f(t, s, x_i(s))\|\, ds \\ &\leq |\dot{v}(t)| + R + \mathcal{K}(t), \end{aligned}$$

where $\mathcal{K}(t) := (1+\rho+\|x_0\|)\int\limits_0^t \beta(t,s)ds.$ and

$$\|g_i(t) + \int\limits_0^t f(t,s,x_i(s))\,ds\| \leq R + \mathcal{K}(t).$$

Therefore,

$$-\frac{1}{R+\mathcal{K}(t)+|\dot{v}(t)|}\left(\dot{x}_i(t) + g_i(t) + \int\limits_0^t f(t,s,x_i(s))\,ds\right) \in N_{C(t)}(x_i(t)) \cap \mathbb{B}, \quad i = 1,2. \tag{2.28}$$

As long as $C(t)$ is r-prox-regularity, and by utilizing the hypomonotonicity property stated in (3) of Theorem 1.71, along with the inclusion (2.28), we deduce that for almost every $t \in [0,T']$

$$\Big\langle -\frac{1}{R+\mathcal{K}(t)+|\dot{v}(t)|}\Big(\dot{x}_1(t) + g_1(t) + \int\limits_0^t f(t,s,x_1(s))ds - \dot{x}_2(t) - g_2(t) - \int\limits_0^t f(t,s,x_2(s))ds\Big)$$
$$, x_1(t) - x_2(t)\Big\rangle \geq -\frac{1}{r}\Big\|x_1(t) - x_2(t)\Big\|^2.$$

Then,

$$\Big\langle \dot{x}_1(t) + g_1(t) + \int\limits_0^t f(t,s,x_1(s))ds - \dot{x}_2(t) - g_2(t) - \int\limits_0^t f(t,s,x_2(s)), x_1(t) - x_2(t)\Big\rangle$$
$$\leq \frac{1}{r}\Big(R + \mathcal{K}(t) + |\dot{v}(t)|\Big)\Big\|x_1(t) - x_2(t)\Big\|^2. \tag{2.29}$$

which implies that

$$\begin{aligned}\Big\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\Big\rangle \leq &-\Big\langle g_1(t) - g_2(t), x_1(t) - x_2(t)\Big\rangle \\ &+ \Big\langle \int\limits_0^t (f(t,s,x_2(s)) - f(t,s,x_1(s)))ds, x_1(t) - x_2(t)\Big\rangle \\ &+ \frac{1}{r}\Big(R + \mathcal{K}(t) + |\dot{v}(t)|\Big)\Big\|x_1(t) - x_2(t)\Big\|^2.\end{aligned} \tag{2.30}$$

Recalling that the operator A is monotone, then

$$\langle g_1(t) - g_2(t), x_1(t) - x_2(t)\rangle \geq 0. \tag{2.31}$$

Through the assumption $(\mathcal{H}_4)$, we have

$$\langle \int_0^t (f(t,s,x_2(s)) - f(t,s,x_1(s)))ds, x_1(t)-x_2(t)\rangle \leq L^{\eta_1}(t)\|x_1(t)-x_2(t)\| \int_0^t \|x_1(t)-x_2(t)\|$$

where $\eta_1 = \rho + \|x_0\|$.
Combining this last inequality with (2.30) and (2.31), we get

$$\begin{aligned}\langle\, \dot{x_1}(t) - \dot{x_2}(t), x_1(t)-x_2(t)\,\rangle \leq & L^{\eta}(t)\|x_1(t)-x_2(t)\| \int_0^t \|x_1(s)-x_2(s)\|\,ds \\ & + \frac{1}{r}\big(R+\mathcal{K}(t)+|\dot{v}(t)|\big)\|x_1(t)-x_2(t)\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dt}\|x_1(t)-x_2(t)\|^2 \leq & 2L^{\eta_1}(t)\|x_1(t)-x_2(t)\| \int_0^t \|x_1(s)-x_2(s)\|\,ds \\ & + \frac{2}{r}\big(R+\mathcal{K}(t)+|\dot{v}(t)|\big)\|x_1(t)-x_2(t)\|^2.\end{aligned}$$

Applying Lemma 1.85 such that

$$\epsilon > 0,\ \varepsilon(t) = 0,\ p(t) = \|x_1(t)-x_2(t)\|^2,$$

$$K_2(t) = 2L^{\eta_1}(t) \text{ and } K_1 = \frac{2}{r}\Big(R+\mathcal{K}(t)+|\dot{v}(t)|\Big),$$

we deduce that

$$x_1(\cdot) = x_2(\cdot) \text{ on } [0,T']. \tag{2.32}$$

Let us set

$$E_\tau := \{t \in [0,\tau]; x_1(t) \neq x_2(t)\}$$

where $\tau \in [0,T]$ is such that $x_1(\tau) \neq x_2(\tau)$. It is obvious that

$$E_\tau \subset]0,\tau].$$

Furthermore, through (2.32) and $T' > 0$ one has

$$\varrho := \inf E_\tau \in]0,\tau].$$

Therefore

$$x_1(t) = x_2(t) \quad \text{for all } \ t \in [0, \varrho[.$$

Letting t tending to ϱ we obtain

$$x_1(\varrho) = x_2(\varrho)$$

as a consequence of the continuity of $x_i(\cdot)$, $i = 1, 2$. For this reason

$$0 < \varrho < \tau$$

due to $x_1(\tau) \neq x_2(\tau)$. With the similar argument as discussed above, there exists some $T' > 0$ such that

$$x_1(\cdot) = x_2(\cdot) \ \text{ on } \ [0, \varrho + T'].$$

This entails a contradiction with the definition of $\varrho := \inf E_\tau$. Consequently

$$x_1(\cdot) = x_2(\cdot) \ \text{ on } \ [0, T].$$

□

Remark 2.4. In Theorem 2.3, we can also weaken the condition $\mathcal{R}(C) \subset \text{int}(\text{Dom}(A))$ by assuming that A is locally bounded on $\mathcal{R}(C)$, in other words, for all $x \in \mathcal{R}(C)$, there exists $K > 0$, $\rho > 0$ such that A is bounded by K in $B(x_0, \rho) \cap \mathcal{R}(C)$.

2.3 Parabolic variational inequalities with Volterra type operators

In this section, we present the connection between integro-differential sweeping process and the parabolic variational inequalities with Volterra type operators. Our example completes that in [2].

Consider a bounded subset Ω of $\mathbb{R}^n$. We define spaces as follows:

1. Hilbert space H is given by $H = L^2(\Omega)$.

2. $U = H^2(\Omega) \cap H_0^1(\Omega) = \{u \in H_0^1(\Omega) \mid \Delta u \in L^2(\Omega)\}$.

Now, let us introduce the functions:

1. $\phi \in L^2(0, T; U)$.

2. $M \in L^2(0, T; \mathbb{R})$.

3. $B : [0, T] \to L^\infty(\Omega)$.

Additionally, we suppose that $\phi(\cdot)$ is k-Lipschitz continuous with respect to the supremum norm.
For almost every $t \in [0, T]$, we have the following

1. $C_1(t) := \{v \in U : v \geq \phi(t) \text{ a.e. on } \Omega \text{ and } \|\Delta v\| \leq M(t)\}$ is closed and convex set.

2. $C_2(t) := \{v \in U : v \leq \phi(t) - 1 \text{ a.e. on } \Omega \text{ and } \|\Delta v\| \leq M(t)\}$ is closed and convex set.

3. $C(t) := C_1(t) \cup C_2(t)$.

We consider the following parabolic variational inequalities with a moving obstacle : find a function $x(t) \in C(t)$ such that there exists a positive constant $\delta_t > 0$ satisfying

$$\int_\Omega \dot{x}(t)(v(t) - x(t))dy + \int_\Omega \nabla x(t)(\nabla v(t) - \nabla x(t))dy$$
$$+ \int_\Omega \Big(\int_0^t B(t-s)x(s)ds\Big)(v(t) - x(t))dy \geq -\delta_t \|v(t) - x(t)\|^2, \quad \forall v(t) \in C(t), \quad (2.33)$$

where the initial value is prescribed as

$$x(0) = x_0 \in C(0).$$

Our goal is to demonstrate the existence and uniqueness of solutions for parabolic variational inequalities involving Volterra-type operators. To this end, we will establish the equivalence between a parabolic variational inequalities and an integro-differential sweeping process of Volterra type. Namely, We consider the following differential inclusion:

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + Ax(t) + \int\limits_0^t f(t, s, x(s))ds \qquad \text{a.e. } t \in [0, T] \\ x(0) = x_0 \in C(0), \end{cases} \quad (2.34)$$

Suppose that assumptions of Theorem 1.79 and Theorem 2.3 are satisfied.

Before going on , it is crucial to first introduce the subsequent theorem, known as the Green's identity

Theorem 2.5. *Let $\Omega \in \mathbb{R}^n$, for any $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega \nabla u \nabla v + \int_\Omega v \Delta u = \int_{\partial\Omega} v(\nabla u \cdot \overrightarrow{n})d\sigma,$$

where $\overrightarrow{n}$ is the outward pointing unit normal on the surface element of $d\sigma$.
Note that when $v \in H_0^1(\Omega)$, Green's identity reads

$$\int_\Omega \nabla u \nabla v + \int_\Omega v \Delta u = 0 \tag{2.35}$$

due to $v \in H_0^1(\Omega) \Rightarrow v|_{\partial\Omega} = 0$.

It is evident that for each $t \in [0, T]$, the set $C(t)$ is closed and prox-regular (but non-convex). This conclusion follows from the fact that $C_1(t)$ and $C_2(t)$ are two disjoint closed convex sets. Moreover, we have for all $v_1 \in C_1(t)$, $v_2 \in C_2(t)$

$$\|v_1 - v_2\|_H = \left(\int_\Omega \|v_1 - v_2\|^2 \right)^{\frac{1}{2}} \geq \sqrt{\mathrm{m}(\Omega)},$$

where $\mathrm{m}(\Omega)$ represents the volume of Ω. Additionally, $C(\cdot)$ is k-Lipschitz continuous in view of the fact that $\phi(\cdot)$ is k-Lipschitz continuous. Then $C(\cdot)$ is absolutely continuous.

Consider the linear unbounded operator $A : \mathrm{Dom}(A) \subset H \to H$ defined by

$$\begin{cases} \mathrm{Dom}(A) = H^2(\Omega) \cap H_0^1(\Omega) := U, \\ \overline{\mathrm{Dom}(A)} = H, \\ Ax(t) = -\Delta x(t), \end{cases}$$

where Δ is the Laplace operator. Then A is a self-adjoint maximal monotone operator. Indeed,

1. A is monotone. For every $x \in \mathrm{Dom}(A)$ we have

$$\langle Ax, x \rangle_{L^2(\Omega)} = \int_\Omega (-\Delta x)x = \int_\Omega |\nabla x|^2 \geq 0$$

2. A is maximal monotone. We have that $\mathcal{R}(I + A) = \mathcal{H}$ (see [17]).

3. A is self-adjoint. Taking into account that if A is a maximal monotone symmetric operator, then A is self-adjoint. It is enough to verify that A is symmetric. For every $x, y \in \mathrm{Dom}(A)$ we have

$$\langle Ax, y \rangle_{L^2(\Omega)} = \int_\Omega (-\Delta x)y = \int_\Omega \nabla x \cdot \nabla y$$

and

$$\langle x, Ay \rangle_{L^2(\Omega)} = \int_\Omega x(-\Delta y) = \int_\Omega \nabla x \cdot \nabla y,$$

thus, $\langle Ax, y\rangle_{L^2(\Omega)} = \langle x, Ay\rangle_{L^2(\Omega)}$.

Therefore, through (2.35) we obtain for all $v, x \in U$

$$\begin{aligned}\int_\Omega \nabla x(t)(\nabla v(t) - \nabla x(t))dy &= \int_\Omega \nabla x(t)\nabla(v(t) - x(t))dy \\ &= \int_\Omega -\Delta x(t)(v(t) - x(t))dy \\ &= \int_\Omega Ax(t)(v(t) - x(t))dy.\end{aligned}$$

Hence, the inequality (2.33) can be expressed as

$$\int_\Omega \Big(\dot{x}(t) + Ax(t) + \int_0^t B(t-s)x(s)ds\Big)(v(t) - x(t))dy \geq -\delta_t\|v(t) - x(t)\|^2. \tag{2.36}$$

We remark that the variational inequality (2.33) can be written as a subdifferential inclusion. For this purpose we use the indicator function $I_{C(t)}$. By the Clarke subdifferential of $I_{C(t)}$ at $x(t)$ it follows that the problem of the variational inequality (2.33) with initial condition $x(0) = x_0$ is equivalent to

$$\dot{x}(t) + Ax(t) + \int_0^t B(t-s)x(s)ds \in -\partial I_{C(t)}(x(t)).$$

Then the problem (2.33) can be rewritten as follows

$$\dot{x}(t) + Ax(t) + \int_0^t B(t-s)x(s)ds \in -N_{C(t)}(x(t)). \tag{2.37}$$

Assume that the operator B satisfies the following condition $B \in C(0,T; L^\infty(\Omega))$, then the function

$$f(t,s,v) := B(t-s)v \text{ for all } (t,s) \in Q_\Delta \text{ and } v \in \mathcal{H},$$

satisfies the assumptions $(\mathcal{H}_3)$-$(\mathcal{H}_4)$ with

$$\beta(t,s) = \|B(t-s)\|_{L^\infty(\Omega)} \text{ and } L(t) = \sup_{t\in[0,T]} \|B(t)\|_{L^\infty(\Omega)} \text{ for all } (t,s) \in Q_\Delta.$$

Furthermore, all the assumptions of Theorem 2.3 are satisfied. Consequently, for an initial condition $x_0 \in C(0)$, there exists an absolutely continuous solution $x(\cdot)$ to (2.1), and subsequently to (2.33).
In addition, if $M(\cdot)$ is a constant function, then A is locally bounded by M. through Remark 2.4, one can deduce the uniqueness of solutions.

Chapter Summary

In this chapter, we demonstrated the well-posedness of a perturbed differential inclusion, governed by a nonconvex sweeping process in a Hilbert space. This sweeping process is perturbed by a sum of an integral forcing term, which depends on two specific time variables, and a maximal monotone operator. We used a regularization technique along with a Gronwall-like inequality for this purpose. Afterward, we applied this result to obtain the existence and uniqueness result for parabolic variational inequalities.

Chapitre 3

Moreau-Yosida Regularization of Degenerate Intgro-Differential Sweeping Process

In this chapter, our primary focus is on establishing the existence of solutions for integro-differential degenerate sweeping processes described by

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(Ax(t)) + \int\limits_0^t f(t,s,x(s))\,ds \quad \text{a.e.} \quad t \in [0,T]\,, \\ x(0) = x_0 \in D(A). \end{cases} \tag{3.1}$$

Here, $C(t) : [0,T] \rightrightarrows H$ represents a set-valued mapping with nonempty closed and positively α–far values in the separable Hilbert space H, the set $N_{C(t)}(u(t))$ corresponds to the Clarke normal cone to $C(t)$. The set-valued mapping $A : H \rightrightarrows H$ is maximal monotone operator. Additionally, the moving set $t \longmapsto C(t)$ varies in a Lipschitz continuous way with respect to the Hausdorff distance.

We observe that this problem was proposed by Kunze and Monteiro Marques in [38, 37]. Specifically, they considered the case where A is a set-valued maximal monotone, strongly monotone operator, $f = 0$, and the convex set-valued map $C(t) : [0,T] \rightrightarrows H$ varies Lipschitz continuously with respect to the Hausdorff distance. In their work, they employed a discretization technique based on the surjectivity of the sum of two maximal monotone operators, one of which is the normal cone.

This chapter is structured as follows. Our main result, presented in Theorem 3.10, establishes the existence of solutions for the degenerate sweeping process using the Moreau-Yosida regularization technique. Specifically, we consider the case where the moving sets are positively α–far and vary in a Lipschitz continuous way with respect to the Hausdorff distance.

3.1 Assumptions on data

In this section, we will gather the main assumptions that will be used throughout the chapter.
Hypotheses on the set-valued map $C : [0,T] \rightrightarrows H$: is a set-valued map with nonempty and closed values.

($\mathcal{H}_1^C$) The sets $C(t)$ move in Lipschitz way, that is, there exist a constant K such that for all $t, s \in [0,T]$,
$$d_H(C(t), C(s)) \leq K|t-s|.$$

($\mathcal{H}_2^C$) There exist two constants $\alpha \in]0,1]$ and $\rho \in]0,+\infty]$ such that
$$0 < \alpha \leq \inf_{x \in U_\rho(C(t))} d(0, \partial d(x, C(t)) \text{ a.e. } t \in [0,T],$$
where $U_\rho(C(t)) = \{x \in H, 0 < d(x, C(t) < \rho\}$ is the ρ-tube around $C(t)$.

($\mathcal{H}_3^C$) For a.e. $t \in [0,T]$ the set $C(t)$ is ball compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in H.

($\mathcal{H}_4^C$) For a.e. $t \in [0,T]$ the set $C(t)$ is r-uniformly prox-regular for some $r > 0$.

Hypotheses on the set–valued map $A : H \rightrightarrows H$**:** is a maximal monotone operator in H, such that

($\mathcal{H}_1^A$) for some $c > 0$.
$$\langle Ax - Ay, x - y \rangle \geq c\|x - y\|^2$$
in $D(A) \times D(A)$.

($\mathcal{H}_2^A$) If $\mu_n \longrightarrow 0^+$, $u_n \longrightarrow u$ in $C([0,T], H)$ and $v_n = A_{\mu_n} u_n \rightharpoonup v$ in $L^2([0,T], H)$, then even $v_n \longrightarrow v$ in $L^2([0,T], H)$.

Hypotheses on $f : Q_\Delta \times H \longrightarrow H$**:** is a function satisfying:

($\mathcal{H}_1^f$) for every $x \in H$, the map $(t,s) \mapsto f(t,s,x)$ is Bochner measurable.

($\mathcal{H}_2^f$) There exists a non-negative function $\beta(\cdot,\cdot) \in L^1(Q_\Delta, \mathbb{R}_+)$ such that
$$\|f(t,s,x)\| \leq \beta(t,s)(1 + \|x\|) \text{ for all } (t,s) \in Q_\Delta \text{ and for any } x \in \mathcal{R}(C),$$
where
$$Q_\Delta := \{(t,s) \in [0,T] \times [0,T] : s \leq t\}.$$

($\mathcal{H}_3^f$) for each real $\eta > 0$ there exists a non-negative function $L^\eta(\cdot) \in L^1([0,T], \mathbb{R}_+^*)$ such that for all $(t,s) \in Q_\Delta$ and for any $(x,y) \in B[0,\eta] \times B[0,\eta]$,

$$\|f(t,s,x) - f(t,s,y)\| \leq L^\eta(t)\|x-y\|,$$

Hypotheses on $F : [0,T] \times H \rightrightarrows H$: F is a set-valued map with nonempty closed and convex values satisfying:

($\mathcal{H}_1^F$) For every $x \in H$, $F(\cdot, x)$ is measurable.

($\mathcal{H}_2^F$) For every $t \in [0,T]$, $F(\cdot, x)$ is upper semi continuous from H into H^{weak}.

($\mathcal{H}_3^F$) There exist $c, d \in L^1(0,T)$ such that

$$d(0, F(t,v)) := \inf\{\|w\| : w \in F(t,v)\} \leq c(t)\|v\| + d(t),$$

for all $v \in H$ and a.e $t \in [0,T]$.

($\mathcal{H}_4^F$) For a.e. $t \in [0,T]$ and $A \subset H$ bounded,

$$\gamma(F(t,A) \leq K(t)\gamma(A),$$

for some $K \in L^1(0,T)$ with $k(t) < +\infty$ for all $t \in [0,T]$ where $\gamma = \alpha$ or $\gamma = \beta$ is either the Kuratowski or the Hausdorff measure of non–compactness.

3.2 Preliminary tools

In this section, we provide preliminary results that will be utilized in the subsequent section. Since $-d(\cdot, C)$ has a directional derivative that coincides with the Clarke directional derivative of $-d(\cdot, C)$ whenever $x \notin C$ we obtain the following lemma.

Lemma 3.1. *Let $C \subset H$ be a closed set, $x \notin C$ and $v \in H$. Then*

$$\lim_{h \downarrow 0} = \frac{d(x+hv, C) - d(x,C)}{h} = \min_{z \in \partial d(x,C)} \langle z, v \rangle.$$

Lemma 3.2. *Assume that ($\mathcal{H}_3^C$) holds, let $t \in [0,T]$ and $x \notin C(t)$. Then,*

$$\partial d(x, C(t)) = \frac{x - \text{cl co Proj}_{C(t)}(x)}{d(x, C(t))}.$$

Lemma 3.3. *[34] Let $A : H \rightrightarrows H$ be a map satisfying $(\mathcal{H}_1^A)$, and let $x, y : [0,T] \to H$ be two absolutely continuous functions. Additionally, consider $C : [0,T] \rightrightarrows H$, a set-valued map with nonempty closed values satisfying $(\mathcal{H}_1^C)$. Define $z_\lambda(t) := A_\lambda(x(t))$ for all $t \in [0,T]$. Then,*

1. *The function $t \to d_{C(t)}(z_\lambda(t))$ is absolutely continuous over $[0,T]$.*

2. *For all $t \in]0,T[$,*

$$\limsup_{s\downarrow 0} \frac{d_{C(t+s)}(z_\lambda(t+s)) - d_{C(t)}(z_\lambda(t))}{s} \leq K + \limsup_{s\downarrow 0} \frac{d_{C(t)}(z_\lambda(t+s)) - d_{C(t)}(z_\lambda(t))}{s}$$

   where K is the constant given by $(\mathcal{H}_1^C)$.

3. *For all $t \in]0,T[$, where $\dot{z}_\lambda(t)$ exists,*

$$\limsup_{s\downarrow 0} \frac{d_{C(t)}(z_\lambda(t+s)) - d_{C(t)}(z_\lambda(t))}{s} \leq \max_{y \in \partial d(z_\lambda(t), C(t))} \langle y, \dot{z}_\lambda(t) \rangle$$

4. *for all $\{t \in]0,T[: z_\lambda(t) \notin C(t)\}$, where $\dot{z}_\lambda(t)$ exists,*

$$\lim_{s\downarrow 0} \frac{d_{C(t)}(z_\lambda(t+s)) - d_{C(t)}(z_\lambda(t))}{s} = \min_{y \in \partial d(z_\lambda(t), C(t))} \langle y, \dot{z}_\lambda(t) \rangle$$

5. *For every $x \in H$ the set-valued map $t \rightrightarrows \partial d(\cdot, C(t))(z_\lambda(t))$ is measurable.*

Proposition 3.4. *[34] Assume that $(\mathcal{H}_1^A)$, $(\mathcal{H}_1^C)$ and $(\mathcal{H}_3^C)$ hold. Then, the set-valued map $G_\lambda : [0,T] \times H \rightrightarrows H$ defined by $G_\lambda(t,x) := \frac{1}{2}\partial d(A_\lambda(x), C(t))$ satisfies:*

1. *for all $x \in H$ and all $t \in [0,T]$, $G_\lambda(t,x) = A_\lambda(x) - cl\ co\ Proj_{C(t)}(A_\lambda(x))$.*

2. *For every $x \in H$, the set-valued map $G_\lambda(\cdot, x)$ is measurable.*

3. *For every $t \in [0,T]$, $G_\lambda(\cdot, x)$ is upper semi-continuous from H into H^{weak}.*

4. *For every $t \in [0,T]$, and $B \subset H$ bounded, $\gamma(G_\lambda(t,B)) \leq M\gamma(B)$, where $\gamma = \alpha$ or $\gamma = \beta$ is the Kuratowski or the Hausdorff measure of non-compactness of B.*

5. *Let $A_\lambda(x_0) \in C(0)$, then for all $t \in [0,T]$ and $x \in H$*

$$\|G_\lambda(t,x)\| := \sup\{\|w\| : w \in G(t,x)\} \leq \frac{1}{\lambda}\|x - x_0\| + Kt,$$

   where K is the constant given by $(\mathcal{H}_1^C)$.

Proposition 3.5. *[7] Let $F : [0,T] \times H \longmapsto H$ be a set–valued map satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$. Let (u_n), (f_n) be measurable functions such that*

1. *(u_n) converges almost everywhere on $[0,T]$ to a function $u : [0,T] \longrightarrow H$.*
2. *(f_n) converges weakly in $L_1([0,T], H)$ to $f : [0,T] \longrightarrow H$.*
3. *For all n, $f_n(t) \in F(t, u_n(t))$ a.e. $t \in [0,T]$.*

Then $f(t) \in F(t, u(t))$ a.e. on $[0,T]$.

The following existence result for integro-differential inclusions was established by P Pérez–Aros, M Torres–Valdebenito, E Vilches in [57].

Theorem 3.6. *Let H be a separable Hilbert space and $I = [0,T]$ for some $T > 0$. Assume that $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$, $(\mathcal{H}_3^F)$, $(\mathcal{H}_4^F)$ and $(\mathcal{H}_1^f)$, $(\mathcal{H}_2^f)$, $(\mathcal{H}_3^f)$ holds, Then, the differential inclusion*

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) + \int\limits_0^t f(t, s, x(s))\, ds \quad \text{a.e.} \quad t \in [0,T], \\ x(0) = x_0 \in H \end{cases} \tag{3.2}$$

has at least one absolutely continuous solution x.

3.3 Main results

In this section, we establish the existence of Lipschitz continuous solutions to the degenerate sweeping process (3.1) using a Moreau-Yosida regularization approach.

Let $\mu > 0$ and consider the following degenerate sweeping process

$$\begin{cases} \dot{x}_\mu(t) \in -N_{C(t)}(A_\mu x_\mu(t)) + \int\limits_0^t f(t, s, x_\mu(s))\, ds \quad \text{a.e.} \quad t \in [0,T], \\ A_\mu(x_0) \in C(0). \end{cases} \tag{3.3}$$

Let $\lambda > 0$, $\mu > 0$ and consider the following differential inclusion

$$\begin{cases} \dot{x}_{\lambda,\mu}(t) \in -\frac{1}{2\lambda} \partial d^2_{C(t)}(A_\mu x_{\lambda,\mu}(t)) + \int\limits_0^t f(t, s, x_{\lambda,\mu}(s))\, ds \quad \text{a.e.} \quad t \in [0,T], \\ A_\mu(x_0) \in C(0). \end{cases} \tag{3.4}$$

where $A_\mu(x_0) \in C(0)$.

The following proposition is a direct consequence of Theorem 3.6 and Proposition 3.4.

Proposition 3.7. *fix an arbitrary $\mu > 0$. Assume that $(\mathcal{H}_1^C)$ and $(\mathcal{H}_3^C)$, $(\mathcal{H}_2^F)$, $(\mathcal{H}_3^F)$ hold. Then, for every $\lambda > 0$ there exists at least one absolutely continuous solution $x_{\lambda,\mu}$ of (3.4).*

assumption 1

There exists $M \geq 0$ such that, for all $t \in [0, T]$

$$\sup_{y \in \Omega(t)} \|f(t, s, y)\| \leq M,$$

where $\Omega_\mu(t) = \{y \in H; \quad A_\mu y \in \text{Proj}_{C(t)}(A_\mu x)\}$.

In all what follows $\varphi_{\lambda,\mu}(t) = d_{C(t)}(A_\mu(x_{\lambda,\mu}(t))$ for any $t \in [0, T]$.

Proposition 3.8. *Assume, in addition to the hypotheses of Proposition 3.7, that $(\mathcal{H}_1^A)$, and Assumption 1 holds, then if $\mu \in \left]0, \frac{1}{c}\right]$ and $\lambda < \lambda^*$ for a.e $t \in [0, T]$, we have the following estimate:*

$$\dot{\varphi}_{\lambda,\mu}(t) \leq K - \frac{\alpha^2 c}{(1 + \mu c)\lambda} \varphi_{\lambda,\mu}(t) + \frac{(1 + \mu c)L^n(t)}{\mu c} \int_0^t \varphi_{\lambda,\mu}(s)\, ds + \frac{MT}{\mu}, \tag{3.5}$$

where λ^* is given by:

$$\lambda^* = \frac{1}{2} \min\left\{ \frac{\alpha^2 c^2 \mu}{(1 + \mu c)^2 T \max\limits_{t \in [0,T]} L_n(t)}, \frac{\rho \alpha^2 c^2 \mu}{(\mu K + MT)(1 + \mu c)c + \rho(1 + \mu c)^2 T \max\limits_{t \in [0,T]} L_n(t)} \right\}.$$

Consequently,

$$\begin{aligned} \varphi_{\lambda,\mu}(t) &\leq \frac{(\mu K + MT)(1 + \mu c)c\lambda}{\alpha^2 c^2 \mu - (1 + \mu c)^2 T L_n(t)\lambda} \text{ for all } t \in [0, T] \\ &\leq \frac{2(K + MTc)\lambda}{\alpha^2 c - 4T L_n(t)\lambda}. \end{aligned} \tag{3.6}$$

Proof. According to Proposition 3.7, the function $x_{\lambda,\mu}$ is absolutely continuous. Moreover, due to $(\mathcal{H}_1^C)$, for every $t, s \in [0, T]$,

$$|\varphi_{\lambda,\mu}(t) - \varphi_{\lambda,\mu}(s)| \leq K|t - s| + \frac{1}{\mu}\|x_{\lambda,\mu}(t) - x_{\lambda,\mu}(s)\|.$$

Hence, $\varphi_{\lambda,\mu}$ is absolute continuous.

Let firstly take $t \in [0,T]$ where $\varphi_{\lambda,\mu}(t) \in]0,\rho[$ and $\dot{x}_\lambda(t)$ exists. Then, we suppose that $A_\mu y_{\lambda,\mu}(t) \in proj_{C(t)}(A_\mu x_{\lambda,\mu}(t))$, and by using Lemma 3.3, we get

$$\varphi_{\lambda,\mu}(t+h) - \varphi_{\lambda,\mu}(t) \leq |d_{C(t+h)}(z_\mu(t+h)) - d_{C(t)}(z_\mu(t+h))| + |d_{C(t)}(z_\mu(t+h)) - d_{C(t)}(z_\mu(t))|$$
$$\frac{\varphi_{\lambda,\mu}(t+h) - \varphi_{\lambda,\mu}(t)}{h} \leq K + \frac{|d_{C(t)}(z_\mu(t+h)) - d_{C(t)}(z_\mu(t))|}{h}.$$

By taking the limit, we obtain the following result:

$$\begin{aligned}
\dot{\varphi}_{\lambda,\mu}(t) &\leq K + \lim_{h\to 0}\frac{d_{C(t)}(z_\mu(t+h)) - d_{C(t)}(z_\mu(t))}{h} \\
&\leq K + \min_{w\in\partial d_{C(t)}(z_\mu(t))}\langle w, \dot{z}_\mu(t)\rangle \\
&\leq K - \frac{\alpha^2 c}{(1+\mu c)\lambda}\varphi_{\lambda,\mu}(t) + \frac{1}{\mu}\int_0^t \|f(t,s,x_{\lambda,\mu}(s)) - f(t,s,y_{\lambda,\mu}(s))\|\,ds + \frac{1}{\mu}\int_0^t \|f(t,s,y_{\lambda,\mu}(s))\|\,ds \\
&\leq K - \frac{\alpha^2 c}{(1+\mu c)\lambda}\varphi_{\lambda,\mu}(t) + \frac{L^n(t)}{\mu}\int_0^t \|x_{\lambda,\mu}(s) - y_{\lambda,\mu}(s)\|\,ds + \frac{MT}{\mu} \\
&\leq K - \frac{\alpha^2 c}{(1+\mu c)\lambda}\varphi_{\lambda,\mu}(t) + \frac{(1+\mu c)L^n(t)}{\mu c}\int_0^t \|A_{\lambda,\mu}(x_{\lambda,\mu}(s)) - A_{\lambda,\mu}(y_{\lambda,\mu}(s))\|\,ds + \frac{MT}{\mu} \\
&\leq K - \frac{\alpha^2 c}{(1+\mu c)\lambda}\varphi_{\lambda,\mu}(t) + \frac{(1+\mu c)L^n(t)}{\mu c}\int_0^t \varphi_{\lambda,\mu}(s)\,ds + \frac{MT}{\mu}.
\end{aligned}$$

Moreover, let $t \in \varphi_{\lambda,\mu}^{-1}(\{0\})$ where $\dot{x}_{\lambda,\mu}(t)$ exists, then according to $(\mathcal{H}_1^C)$, and the identity $\partial d_S^2(x) = 2d_S(x)\partial d_S(x)$, we obtain the following:

$$\begin{aligned}
\dot{\varphi}_{\lambda,\mu}(t) &= \lim_{h\downarrow 0}\frac{1}{h}\Big(d_{C(t+h)}(z_\mu(t+h)) - d_{C(t)}(z_\mu(t+h)) + d_{C(t)}(z_\mu(t+h)) - d_{C(t)}(z_\mu(t))\Big)\\
&\le k + \lim_{h\downarrow 0}\frac{1}{h}\Big(d_{C(t)}(z_\mu(t+h)) - d_{C(t)}(z_\mu(t))\Big)\\
&\le K + \frac{1}{\mu}\|\dot{x}_{\lambda,\mu}(t)\|\\
&\le K + \frac{1}{\mu\lambda}\varphi_{\lambda,\mu}(t) + \frac{1}{\mu}\int_0^t \|f(t,s,x_{\lambda,\mu}(s))\|\,ds\\
&\le K + \frac{1}{\mu\lambda}\varphi_{\lambda,\mu}(t) + \frac{MT}{\mu}\\
&= K - \frac{\alpha^2 c}{(1+\mu c)\lambda}\varphi_{\lambda,\mu}(t) + \frac{(1+\mu c)L^n(t)}{\mu c}\int_0^t \varphi_{\lambda,\mu}(s)\,ds + \frac{MT}{\mu}.
\end{aligned}$$

Claim 1: $\varphi^{-1}_{\lambda,\mu}(]-\infty,\rho[) = [0,T]$.

Proof of claim 1. we have that $\varphi_{\lambda,\mu}(t) < \rho$ for all $t \in [0,T]$. Otherwise, since $\varphi^{-1}_{\lambda,\mu}(]-\infty,\rho[)$ is open relative to $[0,T]$ and $0 \in \varphi^{-1}_{\lambda,\mu}(]-\infty,\rho[)$, there would exist $t^* \in]0,T]$ such that

$$[0,t^*[\subseteq \varphi^{-1}_{\lambda,\mu}(]-\infty,\rho[) \text{ and } \varphi_{\lambda,\mu}(t^*) = \rho.$$

Hence,

$$\dot{\varphi}_{\lambda,\mu}(t) \le K - \frac{\alpha^2 c}{(1+\mu c)\lambda}\varphi_{\lambda,\mu}(t) + \frac{(1+\mu c)L^n(t)}{\mu c}\int_0^t \varphi_{\lambda,\mu}(s)\,ds + \frac{MT}{\mu} \quad \text{a.e } t \in [0,t^*[.$$

Let us define $\phi_{\lambda,\mu}(t) = \max_{t\in[0,T]}\varphi_{\lambda,\mu}(t)$. Then, we have the following estimates

$$\dot{\phi}_{\lambda,\mu}(t) \le K + \left(\frac{(1+\mu c)TL_n(t)}{\mu c} - \frac{\alpha^2 c}{(1+\mu c)\lambda}\right)\phi_{\lambda,\mu}(t) + \frac{MT}{\mu} \quad \text{a.e } t \in [0,t^*[,$$

$$\dot{\phi}_{\lambda,\mu}(t) \le K + \frac{MT}{\mu} - \left(\frac{\alpha^2 c}{(1+\mu c)\lambda} - \frac{(1+\mu c)TL_n(t)}{\mu c}\right)\phi_{\lambda,\mu}(t).$$

Suppose that

$$\frac{\alpha^2 c}{(1+\mu c)\lambda} - \frac{(1+\mu c)TL_n(t)}{\mu c} > 0,$$

which implies that

$$\lambda < \frac{\alpha^2 c^2 \mu}{(1+\mu c)^2 T L_n(t)}.$$

Additionally, suppose that

$$\frac{(\mu K + MT)(1+\mu c)c\lambda}{\alpha^2 c^2 \mu - (1+\mu c)^2 T L_n(t)\lambda} < \rho.$$

Then, by Gronwall's inequality, for every $t \in [0, t^*[$, we have

$$\begin{aligned}
\phi_{\lambda,\mu}(t) &\le \frac{(\mu K + MT)(1+\mu c)c\lambda}{\alpha^2 c^2 \mu - (1+\mu c)^2 T L_n(t)\lambda}\left(1 - \exp\left(-\frac{\alpha^2 c^2 \mu - (1+\mu c)^2 T L_n(t)\lambda}{(1+\mu c)\lambda \mu c} t\right)\right) \\
&\le \frac{(\mu K + MT)(1+\mu c)c\lambda}{\alpha^2 c^2 \mu - (1+\mu c)^2 T L_n(t)\lambda} \\
&\le \frac{(\mu K + MT)(1+\mu c)c}{\alpha^2 c^2 \mu} \times \frac{\lambda}{\left(1 - \frac{(1+\mu c)^2 T L_n(t)}{\alpha^2 c^2 \mu}\lambda\right)} \\
&\le \frac{(\mu K + MT)(1+\mu c)}{\alpha^2 c \mu} \times \frac{\lambda}{\left(1 - \frac{\lambda}{2\lambda^*}\right)} \\
&\le 2\frac{(\mu K + MT)(1+\mu c)}{\alpha^2 c \mu}\lambda^* \\
&< \rho
\end{aligned}$$

This implies that $\varphi_{\lambda,\mu}(t^*) < \rho$, which is impossible. □

Thus, we have proved that $\varphi_{\lambda,\mu}$ satisfies (3.5) and (3.6). This completes the proof of the proposition. □

As a corollary of the previous proposition, we obtain that x_λ is uniformly Lipschitz continuous.

Corollary 3.9. *Under the assumption of Proposition 3.8, for every $\lambda > 0$, the function x_λ is Lipschitz continuous with the following Lipschitz constant*

$$\frac{(\mu K + MT)(1+\mu c)c\lambda}{\alpha^2 c^2 \mu - (1+\mu c)^2 T \max_{t\in[0,T]} L_n(t)\lambda}\left(1 + \frac{(1+\mu c)T \max_{t\in[0,T]} L_n(t)}{c}\lambda\right) + TM.$$

Indeed

$$\begin{aligned}\|\dot{x}_{\lambda,\mu}\| &\leq \frac{1}{\lambda}\varphi_{\lambda,\mu}(t) + \int_0^t \|f(t,s,x_{\lambda,\mu}(s))\|\,ds \\ &\leq \frac{1}{\lambda}\varphi_{\lambda,\mu}(t) + \frac{L_n(t)(1+\mu c)}{c}\int_0^t \varphi_{\lambda,\mu}(s)\,ds + TM \\ &\leq \frac{(\mu K + MT)(1+\mu c)c\lambda}{\alpha^2 c^2 \mu - (1+\mu c)^2 T L_n(t)\lambda}\left(1 + \frac{(1+\mu c)TL_n(t)}{c}\lambda\right) + TM.\end{aligned}$$

On the other hand, we note that

$$\|x_{\lambda,\mu}(t) - x_0\| \leq \int_0^t \|\dot{x}_{\lambda,\mu}\|\,ds.$$

Then,

$$\begin{aligned}\|x_{\lambda,\mu}(t) - x_0\| &\leq \int_0^t \left(\frac{(\mu K + MT)(1+\mu c)c\lambda}{\alpha^2 c^2 \mu - (1+\mu c)^2 T L_n(t)\lambda}\left(1 + \frac{(1+\mu c)TL_n(t)}{c}\lambda\right) + TM\right) ds \\ &\leq \left(\frac{(\mu K + MT)(1+\mu c)c\lambda}{\alpha^2 c^2 \mu - (1+\mu c)^2 T \max L_n(t)\lambda}\left(1 + \frac{(1+\mu c)T\max L_n(t)}{c}\lambda\right) + TM\right)(t-0).\end{aligned}$$

Let $(\lambda_n)_n$ be a sequence converging to 0^+ with $\lambda_n < \lambda^*$ for all $n \in \mathbb{N}$. Considering both Proposition 3.7 and Lemma 1.80 , the following result establishes the existence of a subsequence $(\lambda_{n_k})_k$ of $(\lambda_n)_n$ such that $(x_{\lambda_{n_k},\mu})_k$ converges (in the sense of Lemma 1.80) to a solution of (3.3) over the interval $[0,T]$.

Theorem 3.10. *Assume that* $(\mathcal{H}_1^C)$, $(\mathcal{H}_2^C)$, $(\mathcal{H}_3^C)$, $(\mathcal{H}_1^f)$, $(\mathcal{H}_2^f)$, $(\mathcal{H}_3^f)$, $(\mathcal{H}_1^A)$, *and assumption 1 holds. Then, there exists at least a Lipschitz continuous solution* x *of* (3.3).

Proof. As a consequence of Proposition 3.8, we obtain

$$\begin{aligned}
\|\dot{x}_{\lambda,\mu}(t)\| &\leq \frac{1}{\lambda}\varphi_{\lambda,\mu}(t) + \int_0^t \|f(t,s,x_{\lambda,\mu}(s))\|\,ds \\
&\leq \frac{1}{\lambda}\varphi_{\lambda,\mu}(t) + \frac{L_n(t)(1+\mu c)}{c}\int_0^t \varphi_{\lambda,\mu}(s)\,ds + TM \\
&\leq \frac{(\mu K + MT)(1+\mu c)c\lambda}{\alpha^2 c^2\mu - (1+\mu c)^2 TL_n(t)\lambda}\left(1 + \frac{(1+\mu c)TL_n(t)}{c}\lambda\right) + TM \\
&\leq 2\frac{(\mu K + MT)(1+\mu c)}{\alpha^2 c\mu}\left(1 + \frac{(1+\mu c)TL_n(t)}{c}\lambda^*\right) + TM \\
&\leq 4\frac{(K + McT)}{\alpha^2 c}\left(1 + \frac{2TL_n(t)}{c}\lambda^*\right) + TM.
\end{aligned}$$

Hence, the sequences $(x_{\lambda_n,\mu})_n$, and $(A_\mu x_{\lambda_n,\mu})_n$ satisfy the hypotheses of Lemma 1.80 over the interval $[0,T]$, with

$$\psi(t) = 4\frac{(K + McT)}{\alpha^2 c}\left(1 + \frac{2TL_n(t)}{c}\lambda^*\right) + TM,$$

and

$$\phi(t) = \frac{1}{\mu}\psi(t).$$

respectively. Consequently, there exist subsequences $(x_{\lambda_{n_k},\mu})_k$ and $(A_\mu x_{\lambda_{n_k},\mu})_k$ of $(x_{\lambda_n,\mu})_n$ and$(A_\mu x_{\lambda_n,\mu})_n$, respectively and functions $x_\mu : [0,T] \to H$ and $A_\mu x_\mu : [0,T] \to H$ satisfying the hypotheses of Lemma 1.80. For simplicity, we write x_k and $A_\mu x_k$ instead of $x_{\lambda_{n_k},\mu}$ and $A_\mu x_{\lambda_{n_k},\mu}$ respectively, for all $k \in \mathbb{N}$.

Claim 1. $(A_\mu(x_k(t)))_k$ and $(x_k(t))_k$ are relatively compact in H for all $t \in [0,T]$.

Proof of claim 1. For all $t \in [0;T]$, let us consider $y_k(t) \in P_{C(t)}(A_\mu x_k(t))$ (the projection exists due to the ball compactness of C), then we have

$$\|A_\mu x_k(t) - y_k(t)\| = d_{C(t)}(A_\mu x_k(t)).$$

Thus

$$\begin{aligned}\|y_k(t)\| &\le d_{C(t)}(A_\mu x_k(t)) + \|A_\mu x_k(t)\| \\ &\le 4\frac{(K+McT)}{\alpha^2 c}\lambda^*_{n_k} + \frac{1}{\mu}\|x_k(t) - x_0\| + \|A_\mu x_0\| \\ &\le r := 4\frac{(K+McT)}{\alpha^2 c}\lambda^*_{n_k}\Big(1 + c + 2TL_n(t)\lambda^*\Big)(t-0) + TMc(t-0) + \|A_\mu x_0\|.\end{aligned}$$

Furthermore, since $(A_\mu x_k(t) - y_k(t))$ converges to 0, we have

$$\gamma(\{x_k(t), k \in \mathbb{N}\}) = \gamma(\{y_k(t), k \in \mathbb{N}\}). \tag{3.7}$$

To establish this equality, consider the following:
On one hand,

$$\begin{aligned}\gamma(\{A_\mu x_k(t), k \in \mathbb{N}\}) &\le \gamma(\{A_\mu x_k(t) - y_k(t), k \in \mathbb{N}\}) + \gamma(\{y_k(t), k \in \mathbb{N}\}) \\ &= \gamma(\{y_k(t), k \in \mathbb{N}\})\end{aligned}$$

On the other hand,

$$\begin{aligned}\gamma(\{y_k(t), k \in \mathbb{N}\}) &\le \gamma(\{y_k(t) - A_\mu x_k(t), k \in \mathbb{N}\}) + \gamma(\{A_\mu x_k(t), k \in \mathbb{N}\}) \\ &= \gamma(\{A_\mu x_k(t), k \in \mathbb{N}\}),\end{aligned}$$

which shows (3.7). Therefore, we conclude

$$\gamma(\{A_\mu x_k(t), k \in \mathbb{N}\}) = \gamma(\{y_k(t), k \in \mathbb{N}\}) \le \gamma(\{C(t) \cap r(t)\mathbb{B}\}) \le K(t)\gamma(\{y\}) = 0.$$

This completes the result. □

Claim 2: $A_\mu x(t) \in C(t)$ for all $t \in [0;T]$.

Proof of claim 2. As a result of the weak convergence $x_k(t) \rightharpoonup x(t)$ for all $t \in [0;T]$ due to (1) of Lemma 1.80 and Claim 1, we obtain that

$$x_k(t) \longrightarrow x(t) \;\text{ for all }\; t \in [0;T].$$

Therefore, due to $(\mathcal{H}_1^C)$ and Proposition 3.8, we have

$$\begin{aligned} d_{C(t)}(A_\mu x(t)) =& d_{C(t)}(A_\mu x(t)) + d_{C(t)}(A_\mu x_k(t)) - d_{C(t)}(A_\mu x_k(t)) \\ &\leq \liminf_{k \longrightarrow +\infty} d_{C(t)}(A_\mu x_k(t)) + \frac{1}{\mu}\|x_k(t) - x(t)\| \\ &\leq \liminf_{k \longrightarrow +\infty} \Big(4\frac{K + MTc}{\alpha^2} + c\|x_k(t) - x(t)\|\Big) \\ &= 0, \end{aligned}$$

as claimed. □

Now we prove that x is a solution of (3.3). Define

$$\tilde{F}(t,x) = -\bar{\text{co}}(\omega \partial d_{C(t)}(A_\mu x(t)) \cup \{0\}) + \int_0^t f(t,s,x(s))\,ds \text{ for } (t,x) \in [0,T] \times H,$$

where

$$\omega := 4\frac{(K + McT)}{\alpha^2 c}.$$

Therefore, for a.e. $t \in [0,T]$

$$\begin{aligned} \dot{x}_k(t) &\in -\frac{1}{2\lambda_{n_k}} \partial d^2_{C(t)}(A_\mu x_k(t)) + \int_0^t f(t,s,x_k(s))\,ds \\ &= -\frac{d_{C(t)}(A_\mu x_k(t))}{\lambda_{n_k}} \partial d_{C(t)}(A_\mu x_k(t)) + \int_0^t f(t,s,x_k(s))\,ds \\ &\subseteq \tilde{F}(t, x_k(t)), \end{aligned}$$

where we have used Proposition 3.8.

Claim 3: $\tilde{F}$ has closed convex values and satisfies:

1. For each $x \in H$, $\tilde{F}(\cdot, x)$ is measurable.
2. For all $t \in [0,T]$, $\tilde{F}(t,\cdot)$ is upper semicontinuous from H into H^{weak}.
3. If $x \in C(t)$ then $\tilde{F}(t,x) = -\omega \partial d_{C(t)}(A_\mu x(t)) + \int_0^t f(t,s,x(s))\,ds$.

Proof of claim 3. Let us define

$$G(t,x) := -\omega \partial d_{C(t)}(A_\mu x) \cup \{0\} \text{ for any } t \in [0,T] \text{ and } x \in H.$$

We note that $G(\cdot, x)$ is measurable as the union of two measurable set valued maps. Let us define

$$\Gamma(t) := \tilde{F}(t, x).$$

Then, Γ takes weakly compact convex values.
Fix any $d \in H$, by virtue of [[30], Proposition 2.2.39], it is enough to verify that the support function $t \longmapsto \sigma(d, \Gamma(t)) := \sup_{\nu \in \Gamma(t)} \{\langle \nu, d \rangle : \quad \nu \in \Gamma(t)\}$ is measurable. Therefore,

$$\sigma(d, \Gamma(t)) := \sup\{\langle \nu, d \rangle, \nu \in \Gamma(t)\} = \sup\{\langle \nu, d \rangle, \quad \nu \in G(t, x) + \int_0^t f(t, s, x(s))\, ds\}$$

is measurable because $G(\cdot, x)$ and $f(\cdot, \cdot, x)$ are measurable. Hence (1) holds. Assertion (2) follows directly from [[5], Theorem 17.27 and 17.3]. Finally, if $A_\mu(x) \in C(t)$ then $0 \in \partial d_{C(t)}(A_\mu x)$. Hence, using the fact that the subdifferential of a locally Lipschitz function is closed and convex,

$$\begin{aligned}
\tilde{F}(t, x) &= -\bar{\text{co}}(\omega \partial d_{C(t)}(A_\mu x(t)) \cup \{0\}) + \int_0^t f(t, s, x(s))\, ds \\
&= -\bar{\text{co}}(\omega \partial d_{C(t)}(A_\mu x(t))) + \int_0^t f(t, s, x(s))\, ds \\
&= \omega \partial d_{C(t)}(A_\mu x(t)) + \int_0^t f(t, s, x(s))\, ds,
\end{aligned}$$

which shows (3). □

Summarizing, we have

1. For each $x \in H, \tilde{F}(\cdot; x)$ is measurable.
2. For all $t \in [0; T]; \tilde{F}(t;)$ is upper semicontinuous from H into H^{weak}.
3. $\dot{x}_k(t) \rightharpoonup \dot{x}(t)$ as $k \longrightarrow +\infty$ for all $t \in [0; T]$.
4. $x_k(t) \longrightarrow x(t)$ as $k \longrightarrow +\infty$ for all $t \in [0; T]$.
5. $\dot{x}_k(t) \in \tilde{F}(t, x_k(t))$.

These conditions and the Convergence Theorem (see [7] for more details) implies that

$$\begin{cases} \dot{x}(t) \in \tilde{F}(t, x(t)) \quad \text{a.e.} \quad t \in [0, T], \\ x(0) = x_0 \in C(0). \end{cases} \tag{3.8}$$

which, according to Claim 3, implies that x is a solution of

$$\begin{cases} \dot{x}_{\lambda,\mu}(t) \in \omega \partial d_{C(t)}(A_\mu x_\mu(t)) + \int\limits_0^t f(t,s,x_\mu(s))\, ds \quad \text{a.e.} \quad t \in [0,T]\,, \\ x_{\lambda,\mu}(0) = x_0 \in C(0). \end{cases} \tag{3.9}$$

Therefore, by virtue of Proposition 1.68 and Claim 2, x is a solution of (3.3).

To find a solution of (3.1) in the limit, we fix sequence $\mu_n \in]0; \frac{1}{c}]$ with $\mu_n \longrightarrow 0^+$, and denote by x_n and z_n the functions x_{μ_n} and $A_{\mu_n} x_{\mu_n}$ respectively.

To solve (3.1), we need to demonstrate that the solution of (3.9) also satisfies (3.1) for all $v(t) \in A(x(t)) \cap C(t)$ and $x(t) \in D(A)$.

Let $x(\cdot)$ be any solution of the unconstrained differential inclusion (3.9). We know that $\partial d_{C(t)}(v(t)) \subset N_{C(t)}(v(t))$ for all $v(t) \in C(t)$, Since $x(t) \in D(A)$ a.e. in $[0;T]$ ([7], Proposition 1.2) and $v(t) \in C(t)$ (according to Claim 2), it is enough to show that $v(t) \in A(x)$ for all $t \in [0,T]$.

Let us recall that

$$A_{\mu_n} x_{\mu_n}(t) \in A(J_{\mu_n} x_{\mu_n}(t)), \text{ for a.e. } t \in [0,T],$$

and

$$v_n := A_{\mu_n} x_{\mu_n} \text{ converges weakly to } v \text{ in } L^2([0,T],H).$$

In addition,

$$J_{\mu_n} x_{\mu_n} \text{ converges strongly to } x \text{ in } L^2([0,T],H).$$

Indeed,

$$\begin{aligned} \|J_{\mu_n} x_{\mu_n}(t) - x(t)\| &\leq \|J_{\mu_n} x_{\mu_n}(t) - x_{\mu_n}(t)\| + \|x_{\mu_n}(t) - x(t)\| \\ &\leq \mu_n \| A_\mu x_\mu(t)\| + \|x_{\mu_n}(t) - x(t)\| \\ &\leq \mu_n \phi(t) + \|x_{\mu_n}(t) - x(t)\| \longrightarrow 0 \text{ as } n \longrightarrow +\infty. \end{aligned}$$

Consequently, by using the fact that the operator A is sequentially strongly-weakly closed, we can conclude that

$$v(t) \in Ax(t) \text{ for a.e. } t \in [0,T]. \tag{3.10}$$

Hence, we deduce that

$$\dot{x}(t) \in -N_{C(t)}(Ax(t)) + \int_0^t f(t,s,x(s))\, ds, \quad \text{for a.e. } t \in [0,T]\,.$$

Which complete the proof of the theorem.

□

Chapter Summary

In this chapter, based on the Moreau-Yosida regularization technique, we proved a theorem concerning the existence of a solution under the Lipschitz continuity of positively α–far sets for a new variant of the degenerate sweeping process.

Conclusion

In this thesis, we employ tools from non-smooth, multivalued, and variational analysis to explore differential variational inequalities, with a specific focus on sweeping processes. These processes are a type of differential inclusion involving normal cones.

In chapter 1, we introduce essential definitions and technical results that are relevant throughout our work. These include discussions on convexity, multivalued mappings, and non-smooth analysis.

In chapter 2, by using the semi-regularization method we have investigate the existence and uniqueness of solution for an integro-differential sweeping process with uniformly prox-regular sets in a Hilbert space that vary in an absolutely continuous way with respect to the Hausdorff distance.

In chapter 3, we will focus on a new variant of degenerate sweeping process with positively α–far sets in a separable Hilbert spaces. We have established the existence of this form through the utilization of the Moreau-Yosida regularization.

It is worth noting that this work also raises unresolved questions in the theory of sweeping processes, which could serve as valuable topics for future research in this field.

Future Directions

In the future, we plan to continue our research on the following issues. We will demonstrate the existence of solutions to two variants of the Volterra-type integro-differential sweeping processes under specific conditions.

Integro-Differential Sweeping Process with Bounded Variation Moving Sets:

We will establish the existence of solutions for the Volterra-type integro-differential sweeping process under the absolute continuity in time t of the closed sets C(t) and under their (uniform) prox-regularity. An existence result where the prox-regular sets C(t) have bounded variation (BV) would be the subject of future work.

Optimal Control of the integro-differential Sweeping Process and Applications:

Another unexplored research topic is the optimal control of integro-differential sweeping processes, which is of particular interest to practitioners. We hope to address these challenges in the future.

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Abstract

In this thesis, our objective is to study the well-posedness (in the sens of the existence and uniqueness of solution) for two variant of the well known moreau's sweeping process. Those problems can be formulated as a constrained differential inclusions involving the normal cone.

The thesis is organized into two main parts. First, we use a semi-regularization method in conjunction with a Gronwall-like inequality to establish results concerning the existence and uniqueness of solutions for Volterra-integro-differential sweeping processes associated with uniformly prox-regular sets. This processes is characterized by

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + Ax(t) + \int\limits_0^t f(t,s,x(s))ds \qquad \text{a.e. } t \in [0,T] \\ x(0) = x_0 \in C(0). \end{cases}$$

Secondly, we prove the existence of solutions for degenerate sweeping process associated with positively α–far sets and described by

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(Ax(t)) + \int\limits_0^t f(t,s,x(s))\, ds \quad \text{a.e.} \quad t \in [0,T], \\ x(0) = x_0 \in C(0). \end{cases}$$

To demonstrate the existence of solutions, we utilize the Moreau-Yosida regularization.

Keywords: Differential inclusions, sweeping process, normal cone, Moreau-Yosida regularization, Volterra integro-differential equation, maximal monotone operator, degenerate sweeping processes.

Résumé

Dans cette thèse, notre objectif est d'étudier le caractère bien posé (au sens de l'existence et de l'unicité de la solution) pour deux variantes du processus bien connu de Moreau. Ces problèmes peuvent être formulés comme des inclusions différentielles avec contraintes impliquant le cône normal.

La thèse est divisée en deux parties principales.. Premièrement, nous utilisons une méthode de semi-régularisation en conjonction avec une inégalité de Gronwall-like pour établir des résultats concernant l'existence et l'unicité des solutions pour Volterra-integro-différentiel processus de rafle associés à des ensembles uniformément prox-réguliers. Ce processus est caractérisé par

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + Ax(t) + \int\limits_0^t f(t,s,x(s))ds \quad \text{a.e. } t \in [0,T] \\ x(0) = x_0 \in C(0). \end{cases} \tag{3.11}$$

Deuxièmement, nous prouvons lexistence de solutions pour processus de rafle dégénéré associés à des ensembles $\alpha-$ positivement far et décrit par

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(Ax(t)) + \int\limits_0^t f(t,s,x(s))\,ds \quad \text{a.e.} \quad t \in [0,T]\,, \\ x(0) = x_0 \in C(0). \end{cases}$$

Pour démontrer l'existence de solutions, nous utilisons une régularisation de Moreau-Yosida.

Mots-clés : Inclusions différentielles , processus de rafle, cône normal, régularisation de Moreau-Yosida, équation intégro-différentielle de Volterra, opérateur maximal monotone , processus de rafle dégénéré.

ملخص

نهدف في هذه الأطروحة إلى دراسة جودة الحل (بمعنى وجود الحل وتفرده) لنوعين مختلفين من عملية مورو الشاملة. يمكننا صياغة هاتين المسألتين على شكل احتوائيات تفاضلية مقيدة تتضمن المخروط العادي.

تنقسم الأطروحة إلى جزأين رئيسيين. أولاً، نستخدم طريقة شبه تنظيمية بالاقتران مع متباينة شبيهة بمتباينة غرونوال لإثبات النتائج المتعلقة بوجود وتفرد حلول عملية مورو الشاملة التكاملية-التفاضلية من نوع فولتيرا المرتبطة بمجموعات الأقرب انتظاما. تعرف هذه العمليات ب

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}\big(x(t)\big) + Ax(t) + \int_0^t f\big(t,s,x(s)\big)\,ds \quad \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in C(t). \end{cases}$$

ثانيًا، نبرهن وجود حلول عملية مورو الشاملة المتدهورة ذات المجموعات الموجبة بتقارب الفا، والتي توصف ب

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}\big(Ax(t)\big) + \int_0^t f\big(t,s,x(s)\big)\,ds \quad \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in C(t). \end{cases}$$

لإثبات وجود الحلول، نستخدم تعديل مورو-يوسيدا.

الكلمات المفتاحية: عملية مورو الشاملة، مخروط عادي ، معادلة فولتيرا التكاملية-التفاضلية، احتوائيات تفاضلية، عملية مورو الشاملة المتدهورة، عامل الرتابة الأعلى، تعديل مورو-يوسيدا.