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FACULTE DES SCIENCES EXACTES ET INFORMATIQUE

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N° d'ordre:

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par

Ahcene Merad

THEME

Résolution de Certaines Classes de Problèmes d'évolution avec des Conditions aux limites non locales

Devant le Jury:

Président:	N. Touafek	Prof.	Université Mohamed Seddik Benyahia de Jijel
Rapporteur:	A. Bouziani	Prof.	Université Larbi Ben M'hidi d'Oum El Bouaghi
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Examinateur:	T. Zerzaihi	Prof.	Université Mohamed Seddik Benyahia de Jijel
Examinateur:	N. Adjeroud	MCA	Université Abbes Laghrour de Khenchela

Soutenu le: 21/09/2015.

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Mr. M. Denche, Professor at the University of Constantine1.

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Contents

Introduction

Mathematical modelling and its applications to various phenomena of physics, biology and ecology often lead to problems with nonclassical boundary conditions. Boundary conditions of such a type are called nonlocal boundary conditions (boundary integral conditions). Nonlocal initial boundary-value problems are important from the point of view of their practical application to modelling and investigating of pollution processes in rivers and seas, which are caused by sewage, in various physical phenomena in the context of chemical engineering, thermoelasticity, population dynamics, heat conduction processes, plasma physics, underground-water flow, transmission theory, chemical engineering, control theory, medical science, life sciences and so forth (see [9, 52]and references therein). It is the reason why such nonlocal mixed problems gained much attention in recent years, not only in engineering but also in mathematics community. One of the first works, where nonlocal conditions were considered, is [33]. The nonlocal problem was investigated by applying the method of separation of variables and the corresponding eigenvalues and eigenfunctions were considered. First, the systematic investigation of some class of spatial nonlocal problems was carried out in [30]. Further, in the works [62, 63] resolution methods for such problems in the case of rather general elliptic equations were suggested. Note that theoretical study of nonlocal problems is connected to great difficulties. The functional analysis, the energy method and the method of singular integral equations are usually hard to apply in the investigation of problems of this type. The adopted method is based on the idea introduced in [88], and [81], presented in a form studied for the first time in [54]. This method has been used for the investigation of mixed problems related to elliptic partial differential equations [78, 79, 82], parabolic equations [11, 19, 27, 40, 41, 57, 70, 79] hyperbolic equations, [13, 15, 17, 18, 20, 22, 6, 42, 57, 65, 66, 79, 109, 111, 112], pluriparabolic equations [10, 14], plurihyperbolic equations [105], composite equations [28], mixed equations [3, 77, 12], non-classical equations[29, 8, 16, 21],operation equations [31, 32, 55, 57, 75], and transmission problems [66].

Some problems of modern physics and technology can be effectively described in terms of nonlocal (integral) problems for partial differential equations. These nonlocal conditions arise mainly when the data on the boundary cannot be measured directly.

The first paper devoted to second order partal differential equations with non local integral conditions goes back to Cannon [33]. Later, the problems with nonlocal integral conditions for parabolic equations were investigated by [75], [70], [128], [9]. Problems for elliptic equations with operator nonlocal conditions were considered by Guschin[68], Skubachevskii [115], Paneiah [110].

In recent years, much attention has been focused on the study of hyperbolic equations with purely integral conditions. Such a condition appears in cases where, for instance, direct measurement quantities are impossible and their mean values are known. Such situations take place, for example, in elastodynamics. The physical significance of integral conditions (mean, total flux, total energy, total mass, moment, ...) has served as a fundamental reason for the interest carried to this type of problems. The first investigation of hyperbolic problems goes back to Bouziani[13] in (**1996**), in which the author proved the existence, uniqueness, and continuous dependence of the solution upon the data for some hyperbolic problems with only integral boundary conditions. Later, similar problems have been studied in [11, 14, 15, 23, 104] by using the energetic method and Roth time-discretization method. We refer the reader to [6, 8, 10, 12, 13, 14, 22, 62, 104, 94, 111, 112, 113] for hyperbolic equations with Neumann and integarl conditions. For other problems with nonlocal conditions, related to other equations, we refer to[4, 8, 14, 15, 22] and references therein.

The presence of integral terms in the boundary conditions can greatly complicate the application of standard numerical techniques such as finite difference procedures, finite elements methods, spectral techniques, etc. Some other models of nonlocal boundary conditions are numerically solved by Dehghan[45, 46, 47], Bensaid et al. by the Modified Backward Euler Scheme[7], Bouziani et al. by Galerkin method[26], Rehman et al.by Method of Line (MOL)[114], Merad et al by Laplace transform technique[92, 93, 95, 96, 97, 98, 99].

So far, not much seems to have been done for obtaining an explicit solution of heat and wave equations. However, the solvability of these equations has been theoretically studied in terms of existence and uniqueness of a solution. The main tool used in this thesis is the Laplace transform and then used the numerical technique for the inverse Laplace tarnsform to obtain the numerical solution.

The Laplace transform method has been used to approximate the solution of different classes of linear partial differential equations[4, 5, 76, 89]. Suying et al.[120], established a numerical method based on the Laplace transform for solving initial problem nonlinear dynamic differential equations. The main difficulty in using the Laplace transform method consists in finding its inverse. Numerical inversion methods are then used to overcome this difficulty. There are many numerical techniques available in literature to invert Laplace transforms. In this thesis, we focus exclusively on the Stehfest inversion algorithm[117] in order to efficiently and accurately invert the Laplace tarnsform (which cannot be done analytically). We first take the Lapalce tarnsform of the equations to reduce the problem to a second order inhomogeneous ordinary differential equations with nonlocal condition. The reduced problem can be solved by the method of variation of parameter. After discretization, we use a numerical method for inverting the Laplace transform to get approximate solution.

The aim of this thesis is to establish the existence, uniqueness and the continuous dependence upon the data for the solution of some linear problems. The proofs are based on an a priori estimate and the Laplace transform technique. Furthermore, we give some numerical exemples for comparaison between numerical and exact solutions. This thesis is organized as follows.

In chapter one, we give the necessary tools and some notions on the theory of the used function spaces and on the theory of Lapalce transform, as well as some important inequalities.

Chapter two is devoted to the investigation a one-dimensional parabolic problem with purely integral conditions. The existence and uniqueness of a solution are established.

In chapter three, we study a one-dimensional hyperbolic problem with purely integral conditions. We prove the existence and uniqueness of solution of the given problem.First, we establish an a priori estimate from which we deduce the uniqueness of the solution. For the solvability of the associated problem, we apply the Laplace transform technique to obtain the numerical solution using the Stehfest algorithm.

Chapter four is preserved to the study of a Telegraph equation. Thus, we establish the existence and uniqueness which are mainly based on a priori estimate and the Laplace transform technique. Afterwards, the approximate solution is obtained by the Stehfest algorithm.

In Chapter five, we prove the existence and uniqueness of solution of the pseudohyperbolic equation with nonlocal boundary conditions, the proofs besed by a priori estimate and Laplace inversion transform.

Chapter six, we study two problems. The first problem (Parabolic Integro-differential equation with purely integral conditions) is solved by the previous techniques, whereas the second problem (Hyperbolic Integro-differential equation with purely integral conditions) is treated by the same procedure.

Finally, we give a complete bibliography mainly on the treated subject and related ones.

Chapter 1

Preliminary notions

1.1 Hilbert Space

We introduce the appropriate function spaces that will be used in the remainder. Let $H = L^2(\Omega)$, where $\Omega = (0,1)$, be a Hilbert space with a norm $\|.\|_H$

Definition 1 *(i) Denote by $L^2(0,T,H)$ the set of all square measurable abstract functions $u(.,t)$ from $(0,T)$ into H equipped with the norm*

$$\|u\|_{L^2(0,T,H)} = \left(\int_0^T \|u(.,t)\|_H^2 \, dt \right)^{1/2} < \infty \tag{1.1}$$

(ii) Let $C(0,T,H)$ be the set of all continuous functions $u(.,t) : (0,T) \longrightarrow H$ with

$$\|u\|_{C(0,T,H)} = \max_{0 \le t \le T} \|u(.,t)\|_H < \infty \tag{1.2}$$

(iii) We denote by $C_0(\Omega)$ the vector space of continuous functions with compact support in Ω. Since such function are Lebesgue integrable with respect to x, we can define on $C_0(\Omega)$ the bilinear form given by

$$((u,w)) = \int_\Omega J_x^m u . J_x^m w dx, \;\; m \ge 1 \tag{1.3}$$

where

$$J_x^m u = \int_0^x \frac{(x-\zeta)^{m-1}}{(m-1)!} u(\zeta,t) \, d\zeta, \;\; \text{for } m \ge 1. \tag{1.4}$$

The bilinear form (1.3)is considered as a scalar product on $C_0(\Omega)$ is not complete.

1.2 Bouziani Space

Definition 2 *Denote by* $B_2^m(\Omega)$, $m \geq 1$ *the completion of* $C_0(\Omega)$ *for the scalar product* (1.3), *which is denoted* $(.,.)_{B_2^m(\Omega)}$, *introduced by [9]. By the norm of function* u *from* $B_2^m(\Omega)$, $m \geq 1$, *we inderstand the nonnegative number:*

$$\|u\|_{B_2^m(\Omega)} = \left(\int\limits_{\Omega} \left(J_x^m u\right)^2 dx \right)^{1/2} = \|J_x^m u\|, \ \textit{for } m \geq 1 \tag{1.5}$$

Lemma 1 *For all* $m \in \mathbb{N}^*$, *the following inequality holds:*

$$\|u\|^2_{B_2^m(\Omega)} \leq \frac{1}{2} \|u\|^2_{B_2^{m-1}(\Omega)}. \tag{1.6}$$

Proof. See [9]. ■

Corollary 2 *For all* $m \in \mathbb{N}^*$, *we have the elementary inequality*

$$\|u\|^2_{B_2^m(\Omega)} \leq \left(\frac{1}{2}\right)^m \|u\|^2_{L^2(\Omega)}. \tag{1.7}$$

Definition 3 *We denote by* $L^2(0,T;B_2^m(\Omega))$ *the space of functions which are square integrable in the Bochner sense, with the scalar product* $(u,w)_{L^2(0,T;B_2^m(\Omega))} = \int_0^T (u(.,t), w(.,t))_{B_2^m(\Omega)} dt$. *Since the space* $B_2^m(\Omega)$ *is a Hilbert space, it can be shown that* $L^2(0,T;B_2^m(\Omega))$ *is a Hilbert space as well. The set of all continuous abstract functions in* $[0,T]$ *equipped with the norm*

$$\sup_{0\leq t\leq T} \|u(.,t)\|_{B_2^m(\Omega)} \tag{1.8}$$

is denoted by $C(0,T;B_2^m(0,1))$.

Corollary 3 *We deduce the continuity of the imbedding* $L^2(\Omega) \longrightarrow B_2^m(\Omega)$, *for* $m \geq 1$.

1.3 Important inequalities

Lemma 4 (**Gronwall Lemma**) *Let* $f_1(t)$, $f_2(t) \geq 0$ *be two integrable functions on* $[0,T]$, $f_2(t)$ *is nondecreasing. If*

$$f_1(\tau) \leq f_2(\tau) + c\int_0^\tau f_1(t)\,dt, \ \forall \tau \in [0,T],$$

where $c \in \mathbb{R}^+$, *then*

$$f_1(t) \leq f_2(t)\exp(ct), \ \forall t \in [0,T]. \tag{1.9}$$

Proof. The proof is the same as that of Lemma 1.3.19 in [72]. ■

Cauchy-Schwarz integral inequality:

For any u, $v \in L^2(\Omega)$, we have the following inequality:

$$\int_\Omega u(x).v(x)\,dx \leq \left(\int_\Omega u^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_\Omega v^2(x)\,dx\right)^{\frac{1}{2}}, \tag{1.10}$$

Holder's inequality:

For u, $v \in L^p(\Omega)$, we have

$$\int_\Omega u(x).v(x)\,dx \leq \left(\int_\Omega u^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_\Omega v^2(x)\,dx\right)^{\frac{1}{p}},\ p > 1. \tag{1.11}$$

This inequality is the generalization of Cauchy-Schwarz integral inequality.

Cauchy inequality with ε:

For all $\varepsilon > 0$,and for arbitrary a, b in $\mathbb{R}$ we have the inequality

$$|ab| \leq \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2. \tag{1.12}$$

Young's inequality with ε:

For all $\varepsilon > 0$,and for arbitrary a, b in $\mathbb{R}$ we have the inequality

$$|ab| \leq \frac{1}{p}|\varepsilon a|^p + \frac{p-1}{p}\left|\frac{b}{\varepsilon}\right|^{\frac{p}{p-1}} \text{ for all } p > 1, \tag{1.13}$$

which is the generalization of Cauchy inequality with ε.

1.4 Laplace Transform Method

1.4.1 Introduction

Many problems in engineering and physics can be described in terms of the evolution of solutions of linear differential equations subject to initial conditions. An important group of these problems involves constant coefficient differential equations, and equations like these can be solved very easily by using the Laplace transform.

The Laplace transform is an integral transform that changes a real variable function $f(t)$ into a function $F(s)$ of a variable s through $F(s) = \mathcal{L}(f(t)) = \int_0^\infty \exp(-st) f(t)\, dt$, where in general s is a complex variable.

The importance of the Laplace transform in the study of initial value problems for linear constant coefficient differential equations is that it replaces the operation of integrating a differential equation in $f(t)$ by much simpler algebraic operations involving $F(s)$. Unlike previous methods, where first a general solution is found, and then the constants in the complementary function are chosen to match the initial conditions, when the Laplace transform method is used the initial conditions are incorporated from the start. The task of finding the function $f(t)$ from its Laplace transform $F(s)$ is called inverting the transform, and when working with constant coefficient equations we can accomplish this by appeal to the table of Laplace transform pairs, that is, to a table listing a function $f(t)$ and its corresponding Laplace transform $F(s)$.

Laplace transform is a powerful method for solving differential equation in engineering and science. However, using the Laplace transform for solving differential equations sometimes leads to solutions in the Laplace domain that are not readily invertible to the real domain by analytical means. Numerical inversion methods are then used to convert the obtained solution from the Laplace domain into the real domain.

A very powerful technique for solving these problems is that of the Laplace transform, which literally transforms the original differential equation into an elementary algebraic expression.

Laplace transform is an efficient method for solving differential equation, partial differential equation, and integral differential equation. The main difficulty with Lapalce transform method is in inverting the Laplace domain solution into the real domain. Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; thus, a numerical inversion method (Stehfest inversion algorithm[117]).

The Laplace transformation method plays a significant role in application areas such as physics and engineering, with a growing interest in areas such as computational finance. It is a powerful tool for the solution of ordinary differential equations, as well as of partial differential equations.

1.4.2 Laplace Transform

Fundamental Ideas

Let the real function $f(t)$ be defined for $a \leq t \leq b$, and let the function $K(t,s)$ of the variables t and s be defined for $a \leq t \leq b$ and some s.When it exists, the integral $\int_a^b f(t)\ K(t,s)dt$ is a function of the single variable s. Set

$$F(s) = \int_a^b f(t)K(t,s)dt \tag{1.14}$$

The function $F(s)$ in (1.14) is called the integral transform of $f(t)$, the function $K(t,s)$ is the kernel of the transform, and s is the transform variable. The limits a and b may be finite or infinite, and when at least one limit is infinite the integral in (1.14) becomes an improper integral.

When it exists, the Laplace transform $F(s)$ of a real function $f(t)$ with domain of definition $0 \leq t < \infty$ is defined as the integral transform (1.14) with the kernel $K(t,s) = \exp(-st)$, the interval of integration $0 \leq t < \infty$, and s a complex variable such that $\operatorname{Re} s < c$ for some non negative constant c, so that

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty \exp(-st)\, f(t)\, dt, \tag{1.15}$$

Throughout the present thesis, the transform variable s will be considered to be a **positive real variable**, and c will be chosen such that the integral in (1.15) converges. However, when we consider the general problem of recovering a function $f(t)$ from its Laplace transform $F(s)$, it will be seen that s must be allowed to be a **complex variable**. The advantage of restricting s to the real variable case in this thesis is that the recovery of many useful and frequently occurring functions $f(t)$ from their Laplace transforms $F(s)$ can be accomplished in a very simple manner without the use of complex variable methods.

The reason for interest in integral transforms in general, and the Laplace transform in particular, will become clear when the solution of initial value problems for differential equations is considered. It will then be seen that the Laplace transform replaces integrations with respect to t by simple algebraic operations involving $F(s)$. So, provided $f(t)$ can be recovered from $F(s)$ in a simple manner, the solution of an initial value problem can be found by means of straightforward algebraic operations.

Clearly, the kernel $\exp(-st)$ will only decrease as t increases if $s > 0$, and the Laplace transform of $f(t)$ will only be defined for functions $f(t)$ that decrease sufficiently rapidly as $t \to \infty$ for the integral in (1.14) to exist. In general, if the function to be transformed is denoted by a lower case letter such as f, then its Laplace transform will be denoted by the corresponding uppercase letter F, as in (1.14). It is convenient to denote the Laplace transform operation by the symbol $\mathcal{L}$, so that symbolically: $F(s) = \mathcal{L}\{f(t)\}$.

1.4.3 Formal definition of the Laplace transform

Let $f(t)$ be defined for $0 \leq t < \infty$. Then, when the improper integral exists, the Laplace transform $F(s)$ of $f(t)$,written symbolically $F(s) = \mathcal{L}\{f(t)\}$, is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty \exp(-st)\, f(t)\, dt,$$

1.4.4 Laplace transform pair and inverse transform

The two functions $f(t)$ and $F(s)$ are called a Laplace transform pair, and for all ordinary functions, given $F(s)$ the corresponding function $f(t)$ is determined uniquely, just as $f(t)$ determines $F(s)$ uniquely. This relationship is expressed symbolically by using the symbol $\mathcal{L}^{-1}$ to denote the operation of finding a function $f(t)$ with a given Laplace transform $F(s)$.This process is called finding the inverse Laplace transform of $F(s)$. In terms of the foregoing example, we have $\mathcal{L}\{\exp(at)\} = 1/(s-a)$ and $\mathcal{L}^{-1}\{1/(s-a)\} = \exp(at)$. This is a particular case of the general result that, by definition, the inverse Laplace transform acting on the Laplace transform of the function returns the original function, so we can write

$$\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t). \tag{1.16}$$

1.4.5 Numerical inversion of Laplace transform

Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; therefore a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by H. Hassanzadeh et al. [69]. In this work we use the

Stehfest's algorithm [117] that is easy to implement. This numerical technique was first introduced by Graver [61] and its algorithm then offered by [117].Stehfest's algorithm approximates the time domain solution as

$$u(x,t) \approx \frac{\ln 2}{t} \sum_{n=1}^{2m} \beta_n U\left(x; \frac{n \ln 2}{t}\right), \tag{1.17}$$

where m is a positive integer,

$$\beta_n = (-1)^{n+m} \sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)} \frac{k^m (2k)!}{(m-k)!k!(k-1)!(n-k)!(2k-n)!}, \tag{1.18}$$

and $[q]$ denotes the integer part of the real number q.

1.5 Existence of Laplace transform Method

A sufficient condition for the existence of the Laplace transform of a function $f(t)$ is that the absolute value of $f(t)$ can be bounded for all $t \geq 0$ by

$$|f(t)| \leq M \exp(\alpha t), \tag{1.19}$$

for some constants M and α. This means that if numbers M and α can be found such that

$$|\exp(-st) f(t)| \leq M \exp((\alpha - s)t),$$

then

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^\infty \exp(-st) f(t)\, dt \\ &\leq M \int_0^\infty \exp((\alpha - s)t)\, dt \\ &= \frac{M}{s-\alpha} \end{aligned}$$

The integral on the right will be convergent provided that $s > \alpha > 0$, so when this is true the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ will exist. It should be clearly understood that (1.19) is only a sufficient condition for the existence of a Laplace transform, and not a necessary one, because Laplace transforms can be found for functions that do not satisfy condition (1.19). For example, the function $f(t) = t^{-1/4}$ does not satisfy condition (1.19).

The preceding inequality implies that when $\mathcal{L}\{f(t)\}$ exists, $F(s)$ must be such that $\lim_{s\to\infty} F(s) = 0$.

In addition, the condition $\mathcal{L}\{f(t)\} \leq M/(s-\alpha)$ implies that $F(s)$ cannot be the Laplace transform of on ordinary function $f(t)$ unless $F(s) \to 0$ as $s \to \infty$.

The Laplace transform is a linear operation, and the consequence of this important and useful property is expressed in the following theorem.

1.5.1 Fundamental linearity property

Theorem 5 ***(Linearity of the Laplace transform)*** *Let the functions $f_1(t)$, $f_2(t)$, ..., $f_n(t)$ have Laplace transforms, and let c_1, c_2, ..., c_n be any set of arbitrary constants. Then*

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t) + \quad + c_n f_n(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} + \quad c_n \mathcal{L}\{f_n(t)\} \tag{1.20}$$

This theorem has many applications and its use is essential when working with the Laplace transform. The process of finding an inverse Laplace transform involves reversing the foregoing argument and seeking a function $f(t)$ that has the required Laplace transform $F(s)$.Where possible, this is accomplished by simplifying the algebraic structure of $F(s)$ to the point at which it can be recognized as the sum of the Laplace transforms of known functions of t.

1.5.2 Infinite Series

For a general infinite series, $\sum_{n=0}^{\infty} a_n t^n$, it is not possible to obtain the Laplace transform of the series by taking the transform term by term.

Theorem 6 *If*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n$$

converges for $t \geq 0$, with

$$|a_n| \leq \frac{K\alpha^n}{n!},$$

for all n sufficiently large and $\alpha > 0$, $K > 0$, then

$$\mathcal{L}(f(t)) = \sum_{n=0}^{\infty} a_n \mathcal{L}(t^n) = \sum_{n=0}^{\infty} \frac{a_n n!}{s^{n+1}}, \ (\mathrm{Re}(s) > \alpha) \tag{1.21}$$

1.5.3 Gamma function

This provides a way of representing the factorial $n!$ in terms of an integral, and it is our first encounter with a special case of the Gamma function that will be required later. The gamma function, denoted by $\Gamma(x)$ for $x > 0$, is defined by

$$\Gamma(x) = \int_0^\infty \exp(-t)\, t^{x-1} dt$$

In terms of the earlier notation, when the restriction that n is an integer is removed, and n is replaced by a positive real variable x, we have

$$\Gamma(x+1) = \int_0^\infty \exp(-t)\, t^{x} dt = I(x,1),$$

but

$$I(x,1) = xI(x-1,1) = x\Gamma(x) \quad \text{for } x > 0,$$

So, combining results shows that the gamma function satisfies the fundamental relation

$$\Gamma(x+1) = x\Gamma(x) \quad \text{for } x > 0.$$

It is easily seen from this that

$$\Gamma(n+1) = n! \quad \text{for } n = 0, 1, 2, \ldots,$$

so as $\Gamma(x)$ is defined for all positive x the gamma function provides a generalization of the factorial function $n!$ for positive non-integer values of n.It will be seen later that the gamma function, which belongs to the general class of functions called higher transcendental functions, occursf requently throughout mathematics.

1.5.4 Continuity Requirements

Discontinuous Functions

Because the Laplace transform is defined in terms of an integral, it is possible to find Laplace transforms of discontinuous functions. Suppose, for example, that a function $g(t)$ is discontinuous at $t = a$. Then, provided it converges, the integral defining the Laplace transform of $g(t)$ is given by

$$\mathcal{L}\{g(t)\} = \lim_{\varepsilon\to 0} \int_0^{a-\varepsilon} \exp(-t)\, g(t)\, dt + \lim_{\delta\to 0} \int_{a-\delta}^{\infty} \exp(-t)\, g(t)\, dt$$

where ε and δ are both positive. For simplicity, the upper limit in the first integral is usually denoted by a_- and the lower limit in the second integral by a_+. These are, respectively, the limits of integration to the left and to the right of $t = a$.

The Laplace transform of a function $f(t)$ has been defined. A condition has been given that ensures the existence of the transform, and the concept of a Laplace transform pair has been introduced. The transform has been shown to have the fundamental property of linearity, and some simple transform pairs have been found directly from the definition.

The Heaviside unit step function $H(t-a)$, which jumps from zero for $0 \le t < a$ to unity for $t > a$, has been introduced and used.

Definition 4 *A function f has a **jump discontinuity** at a point t_0 if both limits*

$$\lim_{t \to t_0^-} f(t) = f\left(t_0^-\right) \text{ and } \lim_{t \to t_0^+} f(t) = f\left(t_0^+\right)$$

exist (as finite numbers).

The class of functions for which we consider the Laplace transform defined will have the following property.

Definition 5 *A function f is **piecewise continuous** on the interval $[0, \infty)$ if*

1) $\lim_{t \to 0^+} f(t) = f(0^+)$ exists,

2) f is continuous on every finite interval $(0, b)$ except possibly at a finite number of points $\tau_1,\ \tau_2,\ ...,\ \tau_n$ in $(0, b)$ at which f has a jump discontinuity.

An important consequence of piecewise continuity is that on each subinterval the function f is also bounded. That is to say,

$$|f(t)| \le M_i,\ \tau_i < t < \tau_{i+1},\ i = 1, 2,\ n-1,$$

for finite constants M_i.

In order to integrate piecewise continuous functions from 0 to b, one simply integrates f over each of the subintervals and takes the sum of these integrals, that is,

$$\int_0^b f(t)\,dt = \int_0^{\tau_1} f(t)\,dt + \int_{\tau_1}^{\tau_2} f(t)\,dt + \ ...\ + \int_{\tau_n}^{b} f(t)\,dt$$

This can be done since the function f is both continuous and bounded on each subinterval, and thus on each subinterval it has a well defined Riemann integral.

1.5.5 Operational Properties of the Laplace Transform

To use the Laplace transform in order to solve initial-value problems for linear differential equations and systems, it is necessary to establish a number of fundamental properties of the transform known as its operational properties. This name is given to properties of the transform itself that relate to the way it operates on any function $f(t)$ that is transformed, rather than to the effect these properties of the transform have on specific functions $f(t)$.

This means that operational properties are general properties of the Laplace transform that are not specific to any particular function $f(t)$ or to its transform $F(s)$. An important example of an operational property has already been encountered in Theorem 7.1, where the linearity property of the transform was established.

Some operational properties, such as the scaling and shift theorems that will be proved later, save effort when finding the Laplace transform of a function or inverting a transform, whereas others such as the transform of a derivative are essential when applying the Laplace transform to solve initial-value problems for differential equations.

The way derivatives transform is used to find how the homogeneous part of a linear differential equation is transformed, and we will see later that it also shows how the initial conditions for the differential equation enter into the transformed equation. Laplace transform pairs are needed when transforming the nonhomogeneous term in the differential equation.

1.5.6 Transforming Derivatives

Theorem 7 ***(Transform of a derivative)*** *Let $f(t)$ be continuous on $0 \leq t < \infty$, and let $f'(t)$ be piecewise continuous on every finite interval contained in $[0, \infty)$. Then*

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0). \tag{1.22}$$

where $\mathcal{L}\{f(t)\} = F(s)$.

Theorem 8 ***(Transform of a higher derivative)*** *Let $f(t)$ be continuous on $0 \leq t < \infty$, and let $f'(t)$, $f''(t)$, $,f^{(n-1)}(t)$ be piecewise continuous on every finite interval contained in $[0, \infty)$. Then*

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \quad - s f^{(n-2)}(0) - f^{(n-1)}(0)\,. \tag{1.23}$$

where $\mathcal{L}\{f(t)\} = F(s)$.

1.5.7 Transform of f' when is discontinuous at $t = a$

Theorem 9 *Let $f(t)$ be continuous on $0 \leq t < a$ and on $a \leq t < \infty$, and let it have a simple jump discontinuity at $t = a$ with the value $f_-(a)$ to the immediate left of a at $t = a_-$ and the value $f_+(a)$ to the immediate right of a at $t = a_+$. Then, if $\mathcal{L}\{f(t)\} = F(s)$,*

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) + [f_-(a) - f_+(a)] \exp(-as). \tag{1.24}$$

1.5.8 The $s-$Shift Theorem

Theorem 10 *(**The first shift theorem or the $s-$shift theorem**) Let $\mathcal{L}\{f(t)\} = F(s)$ for $s > \gamma$. Then, the Laplace tarnsform of $\exp(at) f(t)$ is obtained from $F(s)$ by replacing s by $s - a$, where $s - a > \gamma$. Thus*

$$\mathcal{L}\{\exp(at) f(t)\} = F(s-a), \text{ for } s - a > \gamma.$$

Conversely, we have the inverse transform

$$\mathcal{L}^{-1}\{F(s-a)\} = \exp(at) f(t).$$

1.5.9 The $t-$Shift Theorem

Theorem 11 *(**The second shift theorem or the $t-$shift theorem**) Let $\mathcal{L}\{f(t)\} = F(s)$. Then*

$$\mathcal{L}\{H(t-a) f(t-a)\} = \exp(-as) F(s)$$

and, conversely,

$$\mathcal{L}^{-1}\{\exp(-as) F(s)\} = H(t-a) f(t-a)$$

1.5.10 Differentiation and Integration of The Laplace Transform

Theorem 12 *Let f be piecewise continuous on $[0, \infty)$ of exponential order α and $\mathcal{L}(f(t)) = F(s)$. Then*

$$\frac{d^n}{ds^n} F(s) = \mathcal{L}\left((-1)^n t^n f(t)\right), \quad n = 1, 2, 3, \quad (s > \alpha). \tag{1.25}$$

Theorem 13 *If f is piecewise continuous on $[0,\infty)$ and exponential order α, with $F(s) = \mathcal{L}(f(t))$ and such that $\lim\limits_{t\to 0^+} \frac{f(t)}{t}$ exists, then*

$$\int_s^\infty F(s)\,dx = \mathcal{L}\left(\frac{f(t)}{t}\right) \qquad (s > \alpha) \tag{1.26}$$

1.5.11 Scaling a Transform

Theorem 14 *Let $\mathcal{L}\{f(t)\} = F(s)$. Then, if $k > 0$,*

$$\mathcal{L}\{f(kt)\} = \frac{1}{k}F(\frac{s}{k}).$$

1.5.12 Transforming a Periodic Function

Theorem 15 *Let $f(t)$ be a periodic function with period T such that $\int_0^T \exp(-st)\,f(t)\,dt$ is finite. Then*

$$\mathcal{L}\{f(t)\} = \frac{1}{1-\exp(-Ts)}\int_0^T \exp(-st)\,f(t)\,dt, \text{ for } s > 0.$$

1.5.13 The Convolution Operation

Let the functions $f(t)$ and $g(t)$ be defined for $t \geq 0$.Then, the convolution of the functions f and g denoted by $(f*g)(t)$, and in abbreviated form by $(f*g)$, is defined as the integral

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau. \tag{1.27}$$

1.5.14 The Convolution Theorem

Theorem 16 *Let $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Then*

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)*g(t)\}\,F(s)G(s), \tag{1.28}$$

or, equivalently,

$$\mathcal{L}\left\{\int_0^t f(\tau)\,g(t-\tau)\,d\tau\right\} = F(s)G(s).$$

Conversely,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

Note: A more rigorous proof of the convolution theorem can be found in any standard treatise (see Doetsch, 1950) on Laplace transforms. The convolution operation has the following properties:

1) **Commutative**

$$f(t) * g(t) = g(t) * f(t) \tag{1.29}$$

2) **Associative**

$$f(t) * \{g(t) * h(t)\} = \{f(t) * g(t)\} * h(t) \tag{1.30}$$

3) **Distributive**

$$\begin{aligned} f(t) * \{ag(t) * bh(t)\} &= af(t) * g(t) + bf(t) * h(t) \\ f(t) * \{ag(t)\} &= \{af(t)\} * g(t) = a\{f(t) * g(t)\} \\ \mathcal{L}\{f_1(t) * f_2(t) * f_3(t) ... * f_n(t)\} &= F_1(s)F_2(s)...F_n(s) \\ \mathcal{L}\{f^{*n}\} &= \{F(s)\}^n \end{aligned}$$

where a and b are constants. $f^{*n} = f * f * ... * f$ is sometimes called the nth convolution.

Remark 1 *By virtue of* (1.30)*and* (1.29)*, it is clear that the set of all Laplace transformable functions forms a commutative semigroup with respect to the operation* $*$*. The set of all Laplace transformable functions does not form a group because* $f * g^{-1}$ *does not, in general, have a Laplace transform.*

1.5.15 Transforming an Integral

Theorem 17 ***(The transform of an integral)*** *Let* f *be piecewise continuous on* $[0, \infty)$ *of exponential order* α *and* $\mathcal{L}(f(t)) = F(s)$*. Then*

$$\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = \frac{F(s)}{s}, \quad (s > \alpha).$$

and, conversely,

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\, d\tau$$

1.5.16 Relating Initial Values and the Transform

Theorem 18 ***(The initial value theorem)*** *Let $\mathcal{L}\{f(t)\} = F(s)$ be the Laplace transform of an $n-$ time differentiable function $f(t)$. Then*

$$f^{(r)}(t) = \lim_{s\to\infty} \left\{ s^{r+1}F(s) - s^r f(0) - s^{r-1}f'(0) - \quad - sf^{(n-1)}(0) \right\}, \ r = \overline{0, n}$$

In particular,

$$\begin{aligned} f(0) &= \lim_{s\to\infty} \{sF(s)\}, \ f'(0) = \lim_{s\to\infty} \left\{ s^2F(s) - sf(0) \right\} \\ f''(t) &= \lim_{s\to\infty} \left\{ s^3F(s) - s^2f(0) - sf'(0) \right\} \end{aligned}$$

Chapter 2

Solvability of parabolic problem with purely integral conditions

2.1 Position of the Problem

In the recent years, heat equation with purely integral conditions takes an important wide surface in the research in many branches of physics problems.

Let be the rectangular domain $Q = \Omega \times I = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$, we consider parabolic equation of determining a function $v = v(x,t)$ satisfying

$$\frac{\partial v}{\partial t} - \alpha \frac{\partial^2 v}{\partial x^2} = g(x,t)\,,\ 0 < x < 1,\ 0 < t \leq T, \tag{2.1}$$

subject to the initial conditions

$$v(x,0) = \Phi(x)\,,\ 0 < x < 1, \tag{2.2}$$

and the nonlocal boundary conditions

$$\begin{aligned} \int_0^1 v(x,t)\,dx &= E(t),\ 0 < t \leq T, \\ \int_0^1 xv(x,t)\,dx &= M(t),\ 0 < t \leq T. \end{aligned} \tag{2.3}$$

where g, Φ, E, and M are known functions, α and T are known positive constants.

2.2 Reformulation of the problem

Since integral boundary conditions are inhomogeneous, it is convenient to convert problem $(2.1)-(2.3)$ to an equivalent problem with homogeneous integral conditions. For this, we introducing a new unknown function

$$u\left(x,t\right)=v(x,t)-w(x,t), \tag{2.4}$$

where

$$w(x,t)=E(t)+6\left(3x^{2}-2x\right).\left(2M\left(t\right)-E(t)\right). \tag{2.5}$$

Problems $(2.1)-(2.3)$ with inhomogeneous integral conditions (2.3) can be equivalently reduced to the problem of finding a function u satisfying

$$\frac{\partial u}{\partial t}-\alpha\frac{\partial^{2}u}{\partial x^{2}}=f\left(x,t\right),\ 0\leq x\leq1,\ 0<t\leq T, \tag{2.6}$$

initial conditions

$$u(x,0)=\varphi\left(x\right),\ 0\leq x\leq1, \tag{2.7}$$

and purely nonlocal conditions

$$\begin{aligned}\int_{0}^{1}u\left(x,t\right)dx &= 0,\ 0<t\leq T,\\ \int_{0}^{1}xu\left(x,t\right)dx &= 0,\ 0<t\leq T,\end{aligned} \tag{2.8}$$

where

$$f\left(x,t\right)=g(x,t)-\left(\frac{\partial w}{\partial t}-\alpha\frac{\partial^{2}w}{\partial x^{2}}\right), \tag{2.9}$$

and

$$\varphi\left(x\right)=\Phi\left(x\right)-w\left(x,0\right). \tag{2.10}$$

Hance, instead of looking for v , we simply look for u. The solution of problem$(2.1)-(2.3)$will be obtained by the relations(2.4) and (2.5).

2.3 Uniqueness and continuous dependence of the solution

We first establish an a priori estimates. Then, the uniqueness and continuous dependence of the solution with respect to the data are immediate consequences.

Theorem 19 *If $u(x,t)$ is a solution of problem $(2.6)-(2.8)$ and $f \in C\left(\overline{Q}\right)$, then we have the a priori estimates*

$$\begin{aligned} \|u(.,\tau)\|^2_{L^2(0,1)} &\leq c_1 \left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \|\varphi\|^2_{L^2(0,1)} \right), \\ \left\| \frac{\partial u(.,\tau)}{\partial t} \right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} &\leq c_2 \left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \|\varphi\|^2_{L^2(0,1)} \right) \end{aligned} \tag{2.11}$$

where $c_1 = \frac{\max(1,\alpha)}{\alpha}$, $c_2 = \max(1,\alpha)$ and $0 \leq \tau \leq T$.

Proof. Taking the scalar product in $B_2^1(0,1)$ of equation (2.6) and $\frac{\partial u}{\partial t}$, and integrating over $(0,\tau)$, we have

$$\begin{aligned} & \int_0^\tau \left(\frac{\partial u(.,t)}{\partial t}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt - \alpha \int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt \\ = & \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt. \end{aligned} \tag{2.12}$$

The integration by parts on the left-hand sid of (2.12) we obtain

$$\int_0^\tau \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{B_2^1(0,1)} dt + \frac{\alpha}{2} \|u(.,\tau)\|^2_{L^2(0,1)} - \frac{\alpha}{2} \|\varphi\|^2_{L^2(0,1)} = \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt. \tag{2.13}$$

by the **Cauchy inequality**, the right-hand side of (2.13) is bounded by

$$\frac{1}{2} \int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \frac{1}{2} \int_0^\tau \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{B_2^1(0,1)} dt. \tag{2.14}$$

Substitution of (2.14) into (2.13) yields

$$\int_0^\tau \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{B_2^2(0,1)} dt + \alpha \|u(.,\tau)\|^2_{L^2(0,1)} \leq \int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \alpha \|\varphi\|^2_{L^2(0,1)}. \tag{2.15}$$

From (2.15), we obtain estimates (2.11). ■

Corollary 20 *If problem $(2.6)-(2.8)$ has a solution, then this solution is unique and depends continuously on (f,φ).*

2.4 Existence of solution

Laplace transform is an efficient method for solving many differential equations and partial differential equations,The main difficulty with Laplace transform method is in inverting the Laplace domain solution into the real domain. In this section we shall apply the Laplace transform technique to find solutions of partial differential equations.

Suppose that $v(x,t)$ is defined and is of exponential order for $t \geq 0$ i.e. there exists A, $\gamma > 0$ and $t_0 > 0$ such that $|f(t)| \leq A\exp(\gamma t)$ for $t \geq t_0$. Than the Laplace transform $V(x,s)$, exists and is given by

$$V(x,s) = \mathcal{L}\{v(x,t); t \longrightarrow s\} = \int_0^{\infty} v(x,t)\exp(-st)\,dt, \tag{2.16}$$

where s is **positive réel parameter**. Taking the Laplace transforms on both sides of (2.1), we have

$$sV(x,s) - \alpha\frac{d^2}{dx^2}[V(x,s)] = G(x,s) + \Phi(x), \tag{2.17}$$

where $G(x,s) = \mathcal{L}\{g(x,t); t \longrightarrow s\}$. Similarly,we have

$$\begin{aligned}\int_0^1 V(x,s)\,dx &= A(s),\\ \int_0^1 xV(x,s)\,dx &= B(s).\end{aligned} \tag{2.18}$$

where $A(s) = \mathcal{L}\{E(t); t \longrightarrow s\}$ and $B(s) = \mathcal{L}\{M(t); t \longrightarrow s\}$.Thus, the considered equation is reduced to the boundary-value problem governed by second order inhomogeneous ordinary differential equation. We obtain a general solution of (2.17) as

$$\begin{aligned}V(x,s) &= -\sqrt{\frac{\alpha}{s}}\int_0^x [G(\tau,s)+\Phi(\tau)]\sinh\left(\sqrt{\frac{s}{\alpha}}[x-\tau]\right)d\tau + \\ &\quad C_1(s)\exp\left(-\sqrt{\frac{s}{\alpha}}x\right) + C_2(s)\exp\left(\sqrt{\frac{s}{\alpha}}x\right)\end{aligned} \tag{2.19}$$

where $C_1$and C_2 are arbitrary functions of s. Substituting (2.19) into (2.18), we have

$$\begin{aligned}&C_1(s)\int_0^1 \exp\left(-\sqrt{\frac{s}{\alpha}}x\right)dx + C_2(s)\int_0^1 \exp\left(\sqrt{\frac{s}{\alpha}}x\right)dx\\ =\ &\sqrt{\frac{\alpha}{s}}\int_0^1\left[[F(\tau,s)+\varphi(\tau)]\int_\tau^1 \sinh\left(\sqrt{\frac{s}{\alpha}}[x-\tau]\right)dx\right]d\tau + A(s),\\ &C_1(s)\int_0^1 x\exp\left(-\sqrt{\frac{s}{\alpha}}x\right)dx + C_2(s)\int_0^1 x\exp\left(\sqrt{\frac{s}{\alpha}}x\right)dx\\ =\ &\sqrt{\frac{\alpha}{s}}\int_0^1\left[[G(\tau,s)+\Phi(\tau)]\int_\tau^1 x\sinh\left(\sqrt{\frac{s}{\alpha}}[x-\tau]\right)dx\right]d\tau + B(s).\end{aligned} \tag{2.20}$$

where

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{2.21}$$

and

$$\begin{aligned}
a_{11}(s) &= \int_0^1 \exp\left(-\sqrt{\frac{s}{\alpha}}x\right)dx,\ a_{12}(s) = \int_0^1 \exp\left(\sqrt{\frac{s}{\alpha}}x\right)dx, \\
a_{21}(s) &= \int_0^1 x\exp\left(-\sqrt{\frac{s}{\alpha}}x\right)dx,\ a_{22}(s) = \int_0^1 x\exp\left(\sqrt{\frac{s}{\alpha}}x\right)dx, \\
b_1(s) &= \sqrt{\frac{\alpha}{s}}\int_0^1 \left[\left[G(\tau,s)+\Phi(\tau)\right]\int_\tau^1 \sinh\left(\sqrt{\frac{s}{\alpha}}[x-\tau]\right)dx\right]d\tau + A(s), \\
b_2(s) &= \sqrt{\frac{\alpha}{s}}\int_0^1 \left[\left[G(\tau,s)+\Phi(\tau)\right]\int_\tau^1 x\sinh\left(\sqrt{\frac{s}{\alpha}}[x-\tau]\right)dx\right]d\tau + B(s). \quad (2.22)
\end{aligned}$$

It is possible to evaluate exactly the integrals in (2.19) and (2.22). In general, one may have to resort to numerical integration in order to compute them. For exemple, the Gauss's formula (25.4.30) given in Abramowitz and Stegun [1] may be used to calculate these integrals numerically. We have the following approximations for the integrals:

$$
\begin{aligned}
&\int_0^1 \exp\left(\pm\sqrt{\frac{s}{\alpha}}x\right)dx \\
\simeq\ & \frac{1}{2}\sum_{i=1}^{N} w_i \exp\left(\pm\sqrt{\frac{s}{\alpha}}\left[x_i+1\right]\frac{1}{2}\right), \\
&\int_0^1 x\exp\left(\pm\sqrt{\frac{s}{\alpha}}x\right)dx, \\
\simeq\ & \frac{1}{2}\sum_{i=1}^{N} w_i \left(\frac{1}{2}\left[x_i+1\right]\right)\exp\left(\pm\sqrt{\frac{s}{\alpha}}\left[x_i+1\right]\frac{1}{2}\right), \\
&\int_0^x \left[G(\tau,s)+\Phi(\tau)\right]\sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\tau\right]\right)d\tau, \\
&\int_0^1 \left[\left[G(\tau,s)+\Phi(\tau)\right]\int_\tau^1 \sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\tau\right]\right)dx\right]d\tau, \\
\simeq\ & \frac{x}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{x}{2}\left[x_i+1\right];s\right)+\Phi\left(\frac{x}{2}\left[x_i+1\right]\right)\right]\sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\frac{x}{2}\left[x_i+1\right]\right]\right), \\
\simeq\ & \frac{1}{4}\sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}\left[x_i+1\right];s\right)+\Phi\left(\frac{1}{2}\left[x_i+1\right]\right)\right]\left(1-\frac{1}{2}\left[x_i+1\right]\right)\times \\
&\sum_{i=1}^{N} w_j \sinh\left(\sqrt{\frac{s}{\alpha}}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}\left[x_i+1\right]\right)x_j+\left(1+\frac{1}{2}\left[x_i+1\right]\right)\right]-\frac{1}{2}\left(x_i+1\right)\right]\right), \\
&\int_0^1 \left[\left[F(\tau,s)+\Phi(\tau)\right]\int_\tau^1 x\sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\tau\right]\right)dx\right]d\tau \\
\simeq\ & \frac{1}{4}\sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}\left[x_i+1\right];s\right)+\Phi\left(\frac{1}{2}\left[x_i+1\right]\right)\right]\left(1-\frac{1}{2}\left[x_i+1\right]\right)\times \\
&\left(\frac{1}{2}\left[\left(1-\frac{1}{2}\left[x_i+1\right]\right)x_j+\left(1+\frac{1}{2}\left[x_i+1\right]\right)\right]\right)\times \\
&\sum_{i=1}^{N} w_j \sinh\left(\sqrt{\frac{s}{\alpha}}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}\left[x_i+1\right]\right)x_j+\left(1+\frac{1}{2}\left[x_i+1\right]\right)\right]-\frac{1}{2}\left(x_i+1\right)\right]\right)
\end{aligned} \tag{2.23}
$$

where x_i and w_i are the abscissa and weights, defined as

$$
x_i : i^{th} \text{ zero of } P_n(x), \quad \omega_i = 2/\left(1-x_i^2\right)\left[P_n'(x)\right]^2.
$$

Their tabulated values can be found in[1]for different values of N.

By Stehfest's algorithm approximates $(1.17)-(1.18)$we obtain the numerical solution.

2.5 Numerical exemples

In this section, we report some results of numerical computations using Laplace transform method proposed in the previous section. These technique are applied to solve the problem defined by (2.1) – (2.3) for particular functions g, Φ, Ψ, E, M , and positive constant α.

Example 1 *We take*

$$\begin{aligned} g(x,t) &= -\exp(-(x+t)),\ 0<x<1,\ 0<t\le T \text{ and } \alpha=1,\\ \Phi(x) &= \exp(-x),\ 0<x<1,\\ E(t) &= \left(1-e^{-1}\right)\cosh(t),\ 0<t\le T,\\ M(t) &= \left(1-2e^{-1}\right)\cosh(t),\ 0<t\le T, \end{aligned}$$

In this case, the exact solution given by

$$v(x,t) = e^{-x}\cosh(t),\ 0<x<1,\ 0<t\le T.$$

The method of solution is easily implemented on the computer. Numerical results are obtained for $N = 8$ in (2.23) and $m = 5$ in (1.17), and then compared to the exact solution. For $t = 0.10$we $x \in [0.10, 0.90]$, we calculate v numericaly using the proposed method of solution and compare it with the exact solution in Table1.

x	0.10	0.30	0.50	0.70	0.90
v *exact*	0.909365376	0.744525399	0.609565841	0.499070300	0.408604202
v *numerical*	0.909826451	0.744399924	0.608625003	0.499151233	0.408250006
error	0.000507029	−0.000168530	−0.000000154	0.000162167	−0.000866843

Table 1

Example 2 *We take*

$$\begin{aligned} g(x,t) &= 0,\ 0<x<1,\ 0<t\le T \text{ and } \alpha=1,\\ \Phi(x) &= \sin(\pi x),\ 0<x<1,\\ E(t) &= \frac{2}{\pi}\exp\left(-\pi^2 t\right),\ 0<t\le T,\\ M(t) &= \frac{1}{\pi}\exp\left(-\pi^2 t\right),\ 0<t\le T, \end{aligned}$$

In this case, the excat solution given by

$$v\left(x,t\right)=\exp\left(-\pi^{2}t\right)\sin\left(\pi x\right),\ 0<x<1,\ 0<t\leq T.$$

For $t=0.10,\ x\in[0.10,0.90]$, *we calculate* v *numericaly using the proposed method of solution and compare it with the exact solution in Table2.*

x	0.10	0.30	0.50	0.70	0.90
v *exact*	0.115173056	0.301526975	0.372707838	0.301526975	0.115173056
v *numerical*	0.115157164	0.301736804	0.373047121	0.301250080	0.115109831
error	-0.000137983	0.000695887	0.000982761	-0.000918309	-0.000548956

Table2

Example 3 *We take*

$$\begin{aligned}
g\left(x,t\right) &= -\frac{2\left(x^{2}+1+t\right)}{\left(1+t\right)^{3}},\ 0<x<1,\ 0<t\leq T \text{ and } \alpha=1,\\
\Phi\left(x\right) &= x^{2},\ 0<x<1,\\
E(t) &= \frac{1}{3\left(1+t\right)^{2}},\ 0<t\leq T,\\
M(t) &= \frac{1}{4\left(1+t\right)^{2}},\ 0<t\leq T,
\end{aligned}$$

In this case, the excat solution given by

$$v\left(x,t\right)=\left(\frac{x}{1+t}\right)^{2},\ 0<x<1,\ 0<t\leq T.$$

For $t=0.1,\ x\in[0.1,0.9]$, *we calculate* v *numericaly using the proposed method of solution and compare it with the exact solution in Table 3.*

x	0.10	0.30	0.50	0.70	0.90
v *exact*	0.008264469	0.074380165	0.206611570	0.404958677	0.669421487
v *numerical*	0.008258147	0.074381271	0.206744395	0.405038382	0.668936202
error	-0.000764961	0.000014872	0.000642874	0.000196823	-0.000724931

Table3

Chapter 3

Solvability of hyperbolic problems with purely integral conditions

3.1 Statement of the problem

In the rectangular domain $Q = \Omega \times I = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$, we consider a second-order hyperbolic equation

$$\frac{\partial^2 v}{\partial t^2} - \alpha \frac{\partial^2 v}{\partial x^2} = g(x,t),\ 0 < x < 1,\ 0 < t \leq T \tag{3.1}$$

subject to the initial conditions

$$\begin{aligned} v(x,0) &= \Phi(x),\ 0 < x < 1 \\ \frac{\partial v(x,0)}{\partial t} &= \Psi(x),\ 0 < x < 1 \end{aligned} \tag{3.2}$$

and the purely integral conditions

$$\begin{aligned} \int_0^1 v(x,t)\,dx &= E(t),\ 0 < t \leq T \\ \int_0^1 xv(x,t)\,dx &= M(t),\ 0 < t \leq T \end{aligned} \tag{3.3}$$

where f, φ, ψ, E, and G are known functions, α and T are known positive constants.

3.2 Reformulation of the problem

Since integral boundary conditions are inhomogeneous, it is convenient to convert problem $(3.1)-(3.3)$ to an equivalent problem with homogeneous integral conditions. For this, we introduce a new function $u(x,t)$ as follows

$$u(x,t) = v(x,t) - r(x,t) \tag{3.4}$$

where

$$r(x,t) = E(t) + 6\left(3x^2 - 2x\right).\left(2M(t) - E(t)\right) \tag{3.5}$$

Problems $(3.1)-(3.3)$ with inhomogeneous integral conditions (3.3) can be equivalently reduced to the problem of finding a function u satisfying:

$$\frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} = f(x,t), \ 0 < x < 1, \ 0 < t \leq T \tag{3.6}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= \varphi(x), \quad 0 < x < 1 \\ \frac{\partial u(x,0)}{\partial t} &= \psi(x), \quad 0 < x < 1 \end{aligned} \tag{3.7}$$

and the purely nonlocal conditions

$$\begin{aligned} \int_0^1 u(x,t)\,dx &= 0, \ 0 < t \leq T \\ \int_0^1 xu(x,t)\,dx &= 0, \ 0 < t \leq T \end{aligned} \tag{3.8}$$

where

$$f(x,t) = g(x,t) - \left(\frac{\partial^2 r}{\partial t^2} - \alpha \frac{\partial^2 r}{\partial x^2}\right) \tag{3.9}$$

and

$$\begin{aligned} \varphi(x) &= \Phi(x) - r(x,0) \\ \psi(x) &= \Psi(x) - r(x,0) \end{aligned} \tag{3.10}$$

Hence, instead of looking for v , we simply look for u. The solution of problem $(3.1)-(3.3)$ will be obtained by the relations $(3.4)-(3.5)$

3.3 A priori estimates and its consequences

We first establish an a priori estimates. Then, the uniqueness and continuous dependence of the solution with respect to the data are immediate consequences.

Theorem 21 *If* $u(x,t)$ *is a solution of problem* $(3.6)-(3.8)$ *and* $f \in C\left(\overline{D}\right)$, *then we have*

$$\begin{aligned} & \|u\|^2_{C(0,T;L^2(0,1))} \\ \leq\ & c_1 \left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)} \right) \\ & \left\| \frac{\partial u}{\partial t} \right\|^2_{C\left(0,T;B_2^1(0,1)\right)} \\ \leq\ & c_2 \left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)} \right) \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} c_1 &= \frac{\max(1,2\alpha)\exp T}{\alpha} \\ c_2 &= \max(1,2\alpha)\exp T \end{aligned}$$

and $0 \leq \tau \leq T$.

Proof. Taking the scalar product in $B_2^1(0,1)$ of equation (3.6)and $\frac{\partial u}{\partial t}$, and integrating over $(0,\tau)$, we have

$$\begin{aligned} & \int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial t^2}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt - \alpha \int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt \\ =\ & \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt. \end{aligned} \tag{3.12}$$

The integration by parts on the left-hand side of (3.12), we obtain

$$\begin{aligned} & \frac{1}{2} \left\| \frac{\partial u(.,\tau)}{\partial t} \right\|^2_{B_2^1(0,1)} - \frac{1}{2} \|\psi\|^2_{B_2^1(0,1)} + \frac{\alpha}{2} \|u(.,\tau)\|^2_{L^2(0,1)} - \alpha \|\varphi\|^2_{L^2(0,1)} \\ =\ & \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt. \end{aligned} \tag{3.13}$$

By the Cauchy inequality, the right-hand side of (3.13) is bounded by

$$\frac{1}{2} \int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \frac{1}{2} \int_0^\tau \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{B_2^1(0,1)} dt. \tag{3.14}$$

Substitution of (3.14) into (3.13) , yields

$$\begin{aligned}
&\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(0,1)}+\alpha\left\|u(.,\tau)\right\|^2_{L^2(0,1)}\\
\leq\ &\max(1,2\alpha)\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}dt+\|\varphi\|^2_{L^2(0,1)}+\|\psi\|^2_{B_2^1(0,1)}\right)\\
&+\int_0^\tau\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)}dt\\
\leq\ &\max(1,2\alpha)\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}dt+\|\varphi\|^2_{L^2(0,1)}+\|\psi\|^2_{B_2^1(0,1)}\right)\\
&+\int_0^\tau\left(\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)}+\alpha\left\|u(.,t)\right\|^2_{L^2(0,1)}\right)dt, \qquad (3.15)
\end{aligned}$$

and by Gronwall Lemma, we have

$$\begin{aligned}
&\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(0,1)}+\alpha\left\|u(.,\tau)\right\|^2_{L^2(0,1)}\\
\leq\ &\max(1,2\alpha)\left(\int_0^T \|f(.,t)\|^2_{B_2^1(0,1)}dt+\|\varphi\|^2_{L^2(0,1)}+\|\psi\|^2_{B_2^1(0,1)}\right). \qquad (3.16)
\end{aligned}$$

Since the right-hand side of (3.16) is independent of τ, we take the supremum with respect to τ from 0 to T in the left-hand side, thus obtaining (3.11) . ■

Corollary 22 *If problem* (3.6) – (3.8) *has a solution, then this solution is unique and depends continuously on* (f,φ,ψ).

3.4 Existence of solution

Laplace transform is an efficient method for solving many differential equations and partial differential equations,The main difficulty with Laplace transform method is in inverting the Laplace domain solution into the real domain. In this section we shall apply the Laplace transform technique to find solutions of partial differential equations.

Suppose that $v(x,t)$ is defined and is of exponential order for $t\geq 0$ i.e. there exists A, $\gamma>0$ and $t_0>0$ such that $|f(t)|\leq A\exp(\gamma t)$ for $t\geq t_0$. Than the Laplace transform $V(x,s)$, exists and is given by

$$V(x,s)=\mathcal{L}\{v(x,t);t\longrightarrow s\}=\int_0^\infty v(x,t)\exp(-st)\,dt$$

where s is a positive reel parameter. Taking the Laplace transforms on both sides of (3.1), we get

$$s^2 V(x,s) - \alpha \frac{d^2}{dx^2}\left[V(x,s)\right] = G(x,s) + s\Phi(x) + \Psi(x) \tag{3.17}$$

where $G(x,s) = \mathcal{L}\{g(x,t); t \longrightarrow s\}$. Similarly, we have

$$\begin{aligned} \int_0^1 V(x,s)\,dx &= A(s) \\ \int_0^1 xV(x,s)\,dx &= B(s) \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} A(s) &= \mathcal{L}\{E(t); t \longrightarrow s\} \\ B(s) &= \mathcal{L}\{M(t); t \longrightarrow s\} \end{aligned} \tag{3.19}$$

Thus, the considered equation is reduced into a boundary-value problem governed by a second-order inhomogeneous ordinary differential equation. We obtain a general solution of (3.17) as

$$\begin{aligned} V(x,s) &= -\frac{\sqrt{\alpha}}{s}\int_0^x \left[G(\tau,s) + s\Phi(\tau) + \Psi(\tau)\ \right]\sinh\left(\frac{s}{\sqrt{\alpha}}[x-\tau]\right)d\tau \\ &\quad + C_1(s)\exp\left(-\frac{s}{\sqrt{\alpha}}x\right) + C_2(s)\exp\left(\frac{s}{\sqrt{\alpha}}x\right) \end{aligned} \tag{3.20}$$

where $C_1$and C_2 are arbitrary functions of s. Substituting (3.20) into (3.18), we have

$$\begin{aligned} & C_1(s)\int_0^1 \exp\left(-\frac{s}{\sqrt{\alpha}}x\right)dx + C_2(s)\int_0^1 \exp\left(\frac{s}{\sqrt{\alpha}}x\right)dx \\ = & \frac{\sqrt{\alpha}}{s}\int_0^1\left[\left[F(\tau,s) + s\varphi(\tau) + \psi(\tau)\right]\int_\tau^1 \sinh\left(\frac{s}{\sqrt{\alpha}}[x-\tau]\right)dx\right]d\tau \\ & +A(s), \\ & C_1(s)\int_0^1 x\exp\left(-\frac{s}{\sqrt{\alpha}}x\right)dx + C_2(s)\int_0^1 x\exp\left(\frac{s}{\sqrt{\alpha}}x\right)dx \\ = & \frac{\sqrt{\alpha}}{s}\int_0^1\left[\left[G(\tau,s) + s\Phi(\tau) + \Psi(\tau)\ \right]\int_\tau^1 x\sinh\left(\frac{s}{\sqrt{\alpha}}[x-\tau]\right)dx\right]d\tau \\ & +B(s), \end{aligned} \tag{3.21}$$

where

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{3.22}$$

and

$$\begin{aligned}
a_{11}(s) &= \int_0^1 \exp\left(-\frac{s}{\sqrt{\alpha}}x\right) dx, \\
a_{12}(s) &= \int_0^1 \exp\left(\frac{s}{\sqrt{\alpha}}x\right) dx, \\
a_{21}(s) &= \int_0^1 x\exp\left(-\frac{s}{\sqrt{\alpha}}x\right) dx, \\
a_{22}(s) &= \int_0^1 x\exp\left(\frac{s}{\sqrt{\alpha}}x\right) dx, \\
b_1(s) &= \frac{\sqrt{\alpha}}{s}\int_0^1 \left[\left[G(\tau,s) + s\Phi(\tau) + \Psi(\tau)\ \right]\int_\tau^1 \sinh\left(\frac{s}{\sqrt{\alpha}}\left[x-\tau\right]\right) dx\right] d\tau \\
&\quad +A(s), \\
b_2(s) &= \frac{\sqrt{\alpha}}{s}\int_0^1 \left[\left[G(\tau,s) + s\Phi(\tau) + \Psi(\tau)\ \right]\int_\tau^1 x\sinh\left(\frac{s}{\sqrt{\alpha}}\left[x-\tau\right]\right) dx\right] d\tau \\
&\quad +B(s).
\end{aligned} \tag{3.23}$$

It is possible to evaluate the integrals in (3.20) and (3.23) exactly. In general, one may have to resort to numerical integration in order to compute them, however. For example, the Gauss's formula (25.4.30) given in Abramowitz and stegun [1] may be employed to calculate these integrals numerically, we have the following approximations for the integrals:

$$\begin{aligned}
& \int_0^1 \exp\left(\pm\frac{s}{\sqrt{\alpha}}x\right)dx \\
\simeq\ & \frac{1}{2}\sum_{i=1}^{N} w_i \exp\left(\pm\frac{s}{\sqrt{\alpha}}\left[x_i+1\right]\right), \\
& \int_0^1 x\exp\left(\pm\frac{s}{\sqrt{\alpha}}x\right)dx \\
\simeq\ & \frac{1}{2}\sum_{i=1}^{N} w_i \left(\frac{1}{2}\left[x_i+1\right]\right)\exp\left(\pm\frac{s}{\sqrt{\alpha}}\left[x_i+1\right]\right), \\
& \int_0^1 \left[\left[G(\tau,s)+s\Phi(\tau)+\Psi(\tau)\right]\int_\tau^1 \sinh\left(\frac{s}{\sqrt{\alpha}}\left[x-\tau\right]\right)dx\right]d\tau \\
& \int_0^x \left[G(\tau,s)+s\Phi(\tau)+\Psi(\tau)\ \right]\sinh\left(\frac{s}{\sqrt{\alpha}}\left[x-\tau\right]\right)d\tau \\
\simeq\ & \frac{x}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{x}{2}\left[x_i+1\right];s\right)+s\Phi\left(\frac{x}{2}\left[x_i+1\right]\right)+\Psi\left(\frac{x}{2}\left[x_i+1\right]\right)\right]\times \\
& \times\sinh\left(\frac{s}{\sqrt{\alpha}}\left[x-\frac{x}{2}\left[x_i+1\right]\right]\right), \\
\simeq\ & \frac{1}{4}\sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}\left[x_i+1\right];s\right)+s\Phi\left(\frac{1}{2}\left[x_i+1\right]\right)+\Psi\left(\frac{1}{2}\left[x_i+1\right]\right)\right]\left(1-\frac{1}{2}\left[x_i+1\right]\right)\times \\
& \times\sum_{i=1}^{N} w_j \sinh\left(\frac{s}{\sqrt{\alpha}}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}\left[x_i+1\right]\right)x_j+\left(1+\frac{1}{2}\left[x_i+1\right]\right)\right]-\frac{1}{2}\left(x_i+1\right)\right]\right), \\
& \int_0^1 \left[\left[F(\tau,s)+s\varphi(\tau)+\psi(\tau)\right]\int_\tau^1 x\sinh\left(\frac{s}{\sqrt{\alpha}}\left[x-\tau\right]\right)dx\right]d\tau \\
\simeq\ & \frac{1}{4}\sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}\left[x_i+1\right];s\right)+s\Phi\left(\frac{1}{2}\left[x_i+1\right]\right)+\Psi\left(\frac{1}{2}\left[x_i+1\right]\right)\right]\left(1-\frac{1}{2}\left[x_i+1\right]\right)\times \\
& \times\left(\frac{1}{2}\left[\left(1-\frac{1}{2}\left[x_i+1\right]\right)x_j+\left(1+\frac{1}{2}\left[x_i+1\right]\right)\right]\right)\times \\
& \times\sum_{i=1}^{N} w_j \sinh\left(\frac{s}{\sqrt{\alpha}}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}\left[x_i+1\right]\right)x_j+\left(1+\frac{1}{2}\left[x_i+1\right]\right)\right]-\frac{1}{2}\left(x_i+1\right)\right]\right), \quad (3.24)
\end{aligned}$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n(x)\,,\ \omega_i\ = 2/\left(1-x_i^2\right)\left[P_n'(x)\right]^2.$$

Their tabulated values can be found in [1] for different values of N.

By Stehfest's algorithm approximates $(1.17)-(1.18)$we obtain the numerical solution.

3.5 Numerical examples

In this section, we report some results of numerical computations using Laplace transform method proposed in the previous section. These technique are applied to solve the problem defined by (3.1) – (3.3) for particular functions g, Φ, Ψ, E, M, and positive constant α. The method of solution is easily implemented on the computer, used Matlab 7.9.3 program. The numerical results in triple Table **(table 1, 2, 3)** are excellent agreement with the exact solution.

Example 4 *We take*

$$\begin{aligned} g(x,t) &= 0,\ 0<x<1,\ 0<t\le T \text{ and } \alpha = 1, \\ \Phi(x) &= \exp(-x),\ 0<x<1, \\ \Psi(x) &= 0,\ 0<x<1, \\ E(t) &= \left(1-e^{-1}\right)\cosh(t),\ 0<t\le T\ , \\ M(t) &= \left(1-e^{-1}\right)\cosh(t),\ 0<t\le T\ , \end{aligned}$$

In this case, the exact solution given by

$$v(x,t) = e^{-x}\cosh(t),\ 0<x<1,\ 0<t\le T.$$

The method of solution is easily implemented on the computer, and numerical results are obtained by $N = 8$ in (3.24) and $m = 5$ in (1.17). Then, we compared the exact solution with numerical solution. For $t = 0.10$, $x \in [0.10, 0.90]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution in **Table 1**:

x	0.10	0.30	0.50	0.70	0.90
v *exact*	0.9093654	0.7445254	0.6095658	0.490703	0.4086042
v *numerical*	0.9093851	0.7443921	0.6097452	0.500183	0.4080919
error	0.000217	−0.0001790	0.0002943	0.0022295	−0.0012538

Table 1

Example 5 *We take*

$$
\begin{aligned}
g(x,t) &= \pi^2\left(1+\pi^2\right)\exp\left(-\pi^2 t\right)\sin(\pi x),\ 0<x<1,\ 0<t\leq T \text{ and } \alpha=1,\\
\Phi(x) &= \sin(\pi x),\ 0<x<1,\\
\Psi(x) &= -\pi^2\sin(\pi x),\ 0<x<1,\\
E(t) &= \frac{2}{\pi}\exp\left(-\pi^2 t\right),\ 0<t\leq T\ ,\\
M(t) &= -\frac{1}{\pi}\exp\left(-\pi^2 t\right),\ 0<t\leq T,
\end{aligned}
$$

In this case, the exact solution given by$v(x,t)=\exp\left(-\pi^2 t\right)\sin(\pi x)$, $0<x<1$, $0<t\leq T$.

For $t=0.10$, $x\in[0.10,0.90]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution in Table 2.

x	0.10	0.30	0.50	0.70	0.90
v *exact*	0.1151730	0.3015269	0.3727078	0.3015270	0.1151730
v *numerical*	0.1150014	0.3015291	0.3728361	0.3012199	0.1152185
error	-0.0014899	0.0000073	0.0003442	-0.0010185	0.0003951

Table 2

Example 6 *We take*

$$
\begin{aligned}
g(x,t) &= \left(1+4\pi^2\right)e^{-t}\cos(2\pi x),\ 0<x<1,\ 0<t\leq T \text{ and } \alpha=1,\\
\Phi(x) &= \cos(2\pi x),\ 0<x<1,\\
\Psi(x) &= -\cos(2\pi x),\ 0<x<1,\\
E(t) &= 0,\ 0<t\leq T\ ,\\
M(t) &= 0,\ 0<t\leq T,
\end{aligned}
$$

In this case, the exact solution given by

$$v(x,t)=e^{-t}\cos(2\pi x),\ 0<x<1,\ 0<t\leq T\ .$$

For $t=0.1$, $x\in[0.1,0.9]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution in Table 3.

x	0.10	0.30	0.50	0.70	0.90
v *exact*	0.7320288	-0.2796101	-0.9048374	-0.2796101	0.7320288
v *numerical*	0.7324162	-0.2795921	-0.9047562	-0.2795421	0.7321329
error	0.0005292	-0.0000644	-0.0000897	-0.0002432	0.0001422

Table 3

Chapter 4

Solvability of the telegraph equation with purely integral conditions

4.1 Setting of the problem

The telegraph equations appear in the propagation of electrical signals along a telegraph line, digital image processing, telecommunication, signals and systems, (see Abdou and Soliman[2], Wazwaz [123]).

Consider the one-dimensional second order hyperbolic equation (telegraph equation) defined in the region $Q = \Omega \times I = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$, of the following form:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + a \frac{\partial u}{\partial t} + bu = f(x,t),\ 0 < x < 1,\ 0 < t \leq T, \tag{4.1}$$

subject to the initial conditions

$$\begin{aligned} u(x,0) &= \varphi(x),\ 0 < x < 1, \\ \frac{\partial u(x,0)}{\partial t} &= \psi(x),\ 0 < x < 1, \end{aligned} \tag{4.2}$$

and the purely integral conditions

$$\begin{aligned} \int_0^1 u(x,t)\,dx &= 0,\ 0 < t \leq T, \\ \int_0^1 xu(x,t)\,dx &= 0,\ 0 < t \leq T, \end{aligned} \tag{4.3}$$

where f, φ, ψ, E, and G are known functions, c, a, b,are constants related to resistance, inductance, capacitance and conductance of the cable and T are known positives constants.

4.2 A priori estimates

We first establish an a priori estimates. Then, the uniqueness and continuous dependence of the solution with respect to the data are immediate consequences.

Theorem 23 *If $u(x,t)$ is a solution of problem* $(4.1)-(4.3)$ *and* $f \in C\left(\overline{D}\right)$, *then we have*

$$\begin{aligned}
&\|u(.,\tau)\|^2_{L^2(0,1)} \\
\leq\ & c_1\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\,dt + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)}\right), \\
&\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)} \\
\leq\ & c_2\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\,dt + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)}\right),
\end{aligned} \tag{4.4}$$

where

$$c_1 = \frac{1}{(b+2c^2)}\max\left(1, \frac{1}{2a}, \frac{(b+2c^2)}{2}\right), c_2 = \max\left(1, \frac{1}{2a}, \frac{(b+2c^2)}{2}\right)$$

and $0 \leq \tau \leq T$.

Proof. Taking the scalar product in $B_2^1(0,1)$ of both sides of equation (4.1) with $\frac{\partial u}{\partial t}$, and integrating over $(0,\tau)$, we have

$$\begin{aligned}
&\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial t^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt - c^2\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt + \\
&a\int_0^\tau \left(\frac{\partial u(.,t)}{\partial t}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} + b\int_0^\tau \left(u(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} \\
=\ & \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt.
\end{aligned} \tag{4.5}$$

Integrating by parts on the left-hand side of (4.5), we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(0,1)} + \left(\frac{b}{2}+c^2\right)\|u(.,\tau)\|^2_{B_2^1(0,1)} + a\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)} dt \\
\leq\ & \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt + \frac{1}{2}\|\psi\|^2_{B_2^1(0,1)} + \left(\frac{b+2c^2}{4}\right)\|\varphi\|^2_{L^2(0,1)}
\end{aligned} \tag{4.6}$$

By the $\varepsilon-$Cauchy inequality, the first term in the right-hand side of (4.6) is bounded by

$$\frac{\varepsilon}{2}\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}dt+\frac{1}{2\varepsilon}\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)}dt.$$

We choose $\varepsilon=\frac{1}{2a}$ so that the second term will be simplified by the third term in the left-hand sid. Thus, we have

$$\begin{aligned}&\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(0,1)}+\left(b+2c^2\right)\|u(.,\tau)\|^2_{L^2(0,1)}\\ \leq\ &\frac{1}{2a}\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}dt+\|\psi\|^2_{B_2^1(0,1)}+\left(\frac{b+2c^2}{2}\right)\|\varphi\|^2_{L^2(0,1)}\end{aligned} \tag{4.7}$$

.From (4.7), we obtain estimates (4.4) . ■

Corollary 24 *If problem* (4.1) – (4.3) *has a solution, then this solution is unique and depends continuously on* (f,φ,ψ).

4.3 Existence of solution

In this section we shall apply the Laplace transform technique to find solutions of partial differential equations. Suppose that $u(x,t)$ is defined and is of exponential order for $t\geq 0$ i.e. there exists A, $\gamma>0$ and $t_0>0$ such that $|f(t)|\leq A\exp(\gamma t)$ for $t\geq t_0$. Than the Laplace transform $U(x,s)$, exists and is given by

$$U(x,s)=\mathcal{L}\{u(x,t);t\longrightarrow s\}=\int_0^\infty u(x,t)\exp(-st)\,dt,$$

where s is a positive reel parameter. Taking the Laplace transforms on both sides of (4.1), we have

$$-c^2\frac{d^2}{dx^2}[U(x,s)]+\left(s^2+as+b\right)U(x,s)=F(x,s)+(s+a)\varphi(x)+\psi(x)\ , \tag{4.8}$$

where $F(x,s)=\mathcal{L}\{f(x,t);t\longrightarrow s\}$. Similarly, we have

$$\begin{aligned}\int_0^1 U(x,s)\,dx&=0,\\ \int_0^1 xU(x,s)\,dx&=0,\end{aligned} \tag{4.9}$$

Thus, the considered equation is reduced into a boundary-value problem governed by a second-order inhomogeneous ordinary differential equation. We obtain a general solution of (4.8) as

$$\begin{aligned} U(x,s) &= -\frac{c}{\sqrt{s^2+as+b}}\int_0^x \left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\ \right]\sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right)d\tau \\ &\quad +C_1(s)\exp\left(-\frac{\sqrt{s^2+as+b}}{c}x\right)+C_2(s)\exp\left(\frac{\sqrt{s^2+as+b}}{c}x\right), \end{aligned} \tag{4.10}$$

where $C_1$and C_2 are arbitrary functions of s. Substituting (4.10) into 4.9, we get

$$\begin{aligned} &C_1(s)\int_0^1 \exp\left(-\frac{\sqrt{s^2+as+b}}{c}x\right)dx+C_2(s)\int_0^1 \exp\left(\frac{\sqrt{s^2+as+b}}{c}x\right)dx \\ &= \frac{c}{\sqrt{s^2+as+b}}\int_0^1\left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\int_\tau^1 \sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right)dx\right]d\tau, \\ &C_1(s)\int_0^1 x\exp\left(-\frac{\sqrt{s^2+as+b}}{c}x\right)dx+C_2(s)\int_0^1 x\exp\left(\frac{\sqrt{s^2+as+b}}{c}x\right)dx \\ &= \frac{c}{\sqrt{s^2+as+b}}\int_0^1\left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\int_\tau^1 x\sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right)dx\right]d\tau, \end{aligned} \tag{4.11}$$

where

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{4.12}$$

and

$$\begin{aligned} a_{11}(s) &= \int_0^1 \exp\left(-\frac{\sqrt{s^2+as+b}}{c}x\right)dx, \\ a_{12}(s) &= \int_0^1 \left(\frac{\sqrt{s^2+as+b}}{c}x\right)dx, \\ a_{21}(s) &= \int_0^1 x\left(-\frac{\sqrt{s^2+as+b}}{c}x\right)dx, \\ a_{22}(s) &= \int_0^1 x\left(\frac{\sqrt{s^2+as+b}}{c}x\right)dx, \\ b_1(s) &= \frac{c}{\sqrt{s^2+as+b}}\int_0^1\left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\int_\tau^1 \sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right)dx\right]d\tau, \end{aligned}$$

$$b_2(s) =$$

$$\frac{c}{\sqrt{s^2+as+b}}\int_0^1\left[F\left(\tau,s\right)+\left(s+a\right)\varphi\left(\tau\right)+\psi\left(\tau\right)\int_\tau^1 x\sinh\left(\frac{\sqrt{s^2+as+b}}{c}\left[x-\tau\right]\right)dx\right]d\tau, \tag{4.13}$$

It is possible to evaluate the integrals in (4.10) and (4.13) exactly. In general, one may have to resort to numerical integration in order to compute them, however. For example, the Gauss's formula (25.4.30) given in Abramowitz and stegun [1] may be employed to calculate these integrals numerically, we have

$$
\begin{aligned}
& \int_0^1 \exp\left(\pm\frac{\sqrt{s^2+as+b}}{c}x\right)dx \\
\simeq\; & \frac{1}{2}\sum_{i=1}^{N} w_i \exp\left(\pm\frac{\sqrt{s^2+as+b}}{2c}\left[x_i+1\right]\right), \\
& \int_0^1 x\exp\left(\pm\frac{\sqrt{s^2+as+b}}{c}x\right)dx \\
\simeq\; & \frac{1}{2}\sum_{i=1}^{N} w_i \left(\frac{1}{2}\left[x_i+1\right]\right)\exp\left(\pm\frac{\sqrt{s^2+as+b}}{2c}\left[x_i+1\right]\right), \\
& \int_0^x \left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\;\right]\sinh\left(\frac{\sqrt{s^2+as+b}}{c}\left[x-\tau\right]\right)d\tau \\
\simeq\; & \frac{x}{2}\sum_{i=1}^{N} w_i \left[F\left(\frac{x}{2}\left[x_i+1\right];s\right)+(s+a)\varphi\left(\frac{x}{2}\left[x_i+1\right]\right)+\psi\left(\frac{x}{2}\left[x_i+1\right]\right)\right]\times \\
& \times\sinh\left(\frac{\sqrt{s^2+as+b}}{c}\left[x-\frac{x}{2}\left[x_i+1\right]\right]\right), \\
& \int_0^1\left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\int_\tau^1 \sinh\left(\frac{\sqrt{s^2+as+b}}{c}\left[x-\tau\right]\right)dx\right]d\tau \\
\simeq\; & \frac{1}{4}\sum_{i=1}^{N} w_i\left[F\left(\frac{1}{2}\left[x_i+1\right];s\right)+(s+a)\varphi\left(\frac{1}{2}\left[x_i+1\right]\right)+\psi\left(\frac{1}{2}\left[x_i+1\right]\right)\right]\left(1-\frac{1}{2}\left[x_i+1\right]\right)\times \\
& \times\sum_{i=1}^{N} w_j\sinh\left(\frac{\sqrt{s^2+as+b}}{c}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}\left[x_i+1\right]\right)x_j+\left(1+\frac{1}{2}\left[x_i+1\right]\right)\right]-\frac{1}{2}(x_i+1)\right]\right), \\
& \int_0^1\left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\int_\tau^1 x\sinh\left(\frac{\sqrt{s^2+as+b}}{c}\left[x-\tau\right]\right)dx\right]d\tau \\
\simeq\; & \frac{1}{4}\sum_{i=1}^{N} w_i\left[F\left(\frac{1}{2}\left[x_i+1\right];s\right)+(s+a)\varphi\left(\frac{1}{2}\left[x_i+1\right]\right)+\psi\left(\frac{1}{2}\left[x_i+1\right]\right)\right]\left(1-\frac{1}{2}\left[x_i+1\right]\right)\times \\
& \times\left(\frac{1}{2}\left[\left(1-\frac{1}{2}\left[x_i+1\right]\right)x_j+\left(1+\frac{1}{2}\left[x_i+1\right]\right)\right]\right) \\
& \sum_{i=1}^{N} w_j\sinh\left(\frac{\sqrt{s^2+as+b}}{c}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}\left[x_i+1\right]\right)x_j+\left(1+\frac{1}{2}\left[x_i+1\right]\right)\right]-\frac{1}{2}(x_i+1)\right]\right)
\end{aligned}
\tag{4.14}
$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n(x)\,,\ \omega_i\ = 2/\left(1-x_i^2\right)\left[P_n'(x)\right]^2.$$

Their tabulated values can be found in [1] for different values of N.

By Stehfest's algorithm approximates $(1.17) - (1.18)$we obtain the numerical solution.

Chapter 5

Solvability of a Solution for Pseudohyperbolic equation with Nonlocal Boundary Condition

5.1 Introduction

Certain problems of modern physics and technology have been studied by many mathematicians for a long time *cf.* [3-13, 16-23]. Recent investigations on the nonlocal conditions include the data on the boundary which can not be measured directly. A large number of physical phenomena reduce to a work derived by initial-boundary value problem, as follows: For $(x,t) \in D = \Omega \times I$ with the bounded intervals in $\mathbb{R}_+$ as $\Omega = (0,1)$, $I = (0,T)$

$$\frac{\partial^2 v}{\partial v^2} - \alpha \frac{\partial^2 v}{\partial x^2} - \beta \frac{\partial^3 v}{\partial t \partial x^2} = g(x,t) \tag{5.1}$$

where α and β are positive constants.

Le us consider the following function $v = v(x,t)$ satisfying the Eq. (5.1) in D

$$\begin{aligned} v(x,0) &= \Phi(x), \ \frac{\partial v(x,0)}{\partial t} = \Psi(x) \qquad (x \in \Omega), \\ \int_\Omega v(x,t)\, dx &= \mu(t), \ \int_\Omega x v(x,t)\, dx = m(t) \qquad (t \in I). \end{aligned} \tag{5.2}$$

It follows from (5.2) that

$$g \in C\left(\overline{D}\right), \ \Phi, \Psi \in C^1\left(\overline{\Omega}\right), \mu \text{ and } m \in C^2\left(\overline{I}\right)$$

and the suitable conditions are as follows:

$$\int_{\Omega}\Phi\left(x\right)dx=\mu\left(0\right),\ \int_{\Omega}x\Phi\left(x\right)dx=m\left(0\right),\ \int_{\Omega}\Psi\left(x\right)dx=\mu'\left(0\right),\ \int_{\Omega}x\Psi\left(x\right)dx=m'\left(0\right).$$

5.2 Reformulation of the problem

Since nonlocal (integral) boundary conditions are inhomogeneous, it is applicable to convert the problem (5.1)-(5.2) into an equivalent problem with homogeneous nonlocal conditions. Now, we introduce an unknown function $u=u\left(x,t\right)$ subtracting the function $v=v(x,t)$ from the function $w=w(x,t)$ known in [25], as follows:

$$u\left(x,t\right)=v(x,t)-w(x,t). \tag{5.3}$$

The problem (5.1)-(5.2) can be equivalently reduced to the problem for finding the function u satisfying the following

$$\begin{aligned}\frac{\partial^2u}{\partial t^2}-\alpha\frac{\partial^2u}{\partial x^2}-\beta\frac{\partial^3u}{\partial t\partial x^2} &= f\left(x,t\right),\ \ (x,t)\in D,\\ u\left(x,0\right) &= \varphi\left(x\right),\ \frac{\partial u\left(x,0\right)}{\partial t}=\psi\left(x\right),\ x\in\Omega,\\ \int_{\Omega}u\left(x,t\right)dx &= 0,\ \int_{\Omega}xu\left(x,t\right)dx=0,\ t\in I,\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\varphi'\left(0\right) &= 0,\ \int_{\Omega}\varphi\left(x\right)dx=0,\ \psi'\left(0\right)=0,\ \int_{\Omega}\psi\left(x\right)dx=0,\\ f\left(x,t\right) &= g(x,t)-\left(\frac{\partial w}{\partial t}-\alpha\frac{\partial^2w}{\partial x^2}\right),\\ \varphi\left(x\right) &= \Phi\left(x\right)-w\left(x,0\right),\\ \psi\left(x\right) &= \Psi\left(x\right)-\frac{w\left(x,0\right)}{\partial t}.\end{aligned}$$

Hence, the solution of problem (5.1)-(5.2) will be obtained by the Eq. (5.3).

5.3 Uniqueness and continuous dependence of the solution

We first establish a priori estimates. In addition, the uniqueness and continuous based on the solution with respect to the data are immediate consequences.

Theorem 25 *If $u(x,t)$ is a solution of problem (5.4) and $f \in C\left(\overline{D}\right)$, then we have a priori estimates:*

$$\begin{aligned} \|u(.,\tau)\|^2_{L^2(\Omega)} &\leq c_1\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}\,dt + \|\varphi\|^2_{L^2(\Omega)} + \|\psi\|^2_{B_2^1(\Omega)}\right) \\ \left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)} &\leq c_2\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}\,dt + \|\varphi\|^2_{L^2(\Omega)} + \|\psi\|^2_{B_2^1(\Omega)}\right) \end{aligned} \tag{5.5}$$

where $c_1 = \frac{1}{\alpha}\max\left(1,\alpha,\frac{1}{4\beta}\right)$, $c_2 = \max\left(1,\alpha,\frac{1}{4\beta}\right)$ and $0 \leq \tau \leq T$.

Proof. It is proved by taking the scalar product in $B_2^1(\Omega)$ of the pseudohyperbolic eqaution in the Eq. (5.4), $\frac{\partial u}{\partial t}$ and integrating over $(0,\tau)$, it becomes

$$\begin{aligned} &\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial t^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt - \alpha\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt \\ &-\beta\int_0^\tau \left(\frac{\partial^3 u(.,t)}{\partial t\partial x^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt \\ = &\int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt. \end{aligned} \tag{5.6}$$

The integration by parts of the left-hand side of the Eq. (1.13) gives

$$\begin{aligned} &\alpha\|u(.,\tau)\|^2_{L^2(\Omega)} + \left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)} + 2\beta\int_0^\tau \left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{L^2(\Omega)} dt \\ = \; &2\int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt + \alpha\|\varphi\|^2_{L^2(\Omega)} + \|\psi\|^2_{B_2^1(\Omega)}. \end{aligned} \tag{5.7}$$

It follows from the Eq. (1.11) and the Eq. (1.9) that

$$\int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt \leq \frac{\varepsilon}{2}\int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}\,dt + \frac{1}{4\varepsilon}\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2(\Omega)} dt.$$

We choose $\varepsilon = \frac{1}{4\beta}$ on that it yields to

$$\begin{aligned} & \alpha \left\| u\left(.,\tau\right) \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial u\left(.,\tau\right)}{\partial t} \right\|_{B_2^1(\Omega)}^2 \\ \leq \; & \frac{1}{4\beta} \int_0^\tau \left\| f\left(.,t\right) \right\|_{B_2^1(\Omega)}^2 dt + \alpha \left\| \varphi \right\|_{L^2(\Omega)}^2 + \left\| \psi \right\|_{B_2^1(\Omega)}^2 . \end{aligned} \tag{5.8}$$

Finally, it follows from (5.8) that we obtain estimates (5.5). ■

5.4 Existence of Solution

5.4.1 Laplace transform technique

Laplace transform is widely used in the area of engineering technology and mathematical science. There are many problems whose solutions may be found in terms of the Laplace transform. In fact, it is an efficient method for solving many differential equations and partial differential equations. The main difficult of the method of the Laplace transform is in inverting the solution of the Laplace domain into the real domain. Hence we apply the technique of the Laplace transform [69, 93, 94, 96, 118] to find solutions of the problem (5.1)-(5.2).

Suppose that $v\left(x,t\right)$ is defined and is of the exponential order for $t \geq 0$, i.e. there exists A, $\gamma > 0$ and $t_0 > 0$ such that $\left|v\left(x,t\right)\right| \leq A \exp\left(\gamma t\right)$ for $t \geq t_0$. Then the Laplace transform $V\left(x,s\right)$ including the function $v\left(x,t\right)$ is introduced by

$$V\left(x,s\right) = \mathcal{L}\left\{v\left(x,t\right); t \longrightarrow s\right\} = \int_0^\infty v\left(x,t\right) \exp\left(-st\right) dt \tag{5.9}$$

where s is known as a Laplace variable. A capital letter V represents Laplace transform of function v, i.e., V is a function in the Laplace domain.

If we start at this approximation and apply Laplace transform on the both sides of the problem (5.1)-(5.2), with respect to t, then we discover

$$\begin{aligned} & \frac{d^2}{dx^2} V\left(x;s\right) - \frac{s^2}{\left(\alpha+\beta s\right)} V\left(x;s\right) = \\ & -\frac{1}{\left(\alpha+\beta s\right)} \left[G\left(x;s\right) + \frac{\partial V\left(x;0\right)}{\partial t} + sV\left(x;0\right) - \beta \frac{d^2 V\left(x;0\right)}{dx^2} \right], \end{aligned} \tag{5.10}$$

$$\begin{aligned}
V(x;0) &= \Phi(x), \frac{\partial V(x;0)}{\partial t} = \Psi(x), \\
\int_0^1 V(x;s)\,dx &= A(s), \int_0^1 xV(x;s)\,dx = B(s).
\end{aligned} \tag{5.11}$$

Using the initial conditions, it becomes

$$\begin{aligned}
&\frac{d^2}{dx^2}V(x;s) - \frac{s^2}{(\alpha+\beta s)}V(x;s) \\
&= -\frac{1}{(\alpha+\beta s)}\left[G(x;s) + \Psi(x) + s\Phi(x) - \beta\frac{d^2\Phi(x)}{dx^2}\right], \\
\int_0^1 V(x;s)\,dx &= A(s), \int_0^1 xV(x;s)\,dx = B(s)
\end{aligned} \tag{5.12}$$

Notice that

$$\begin{aligned}
V(x;t) &= \mathcal{L}\{v(x,t);t \longrightarrow s\} \\
G(x;t) &= \mathcal{L}\{g(x,t);t \longrightarrow s\} \\
A(s) &= \mathcal{L}\{\mu(t);t \longrightarrow s\} \\
B(s) &= \mathcal{L}\{m(t);t \longrightarrow s\}.
\end{aligned}$$

Hence, it is reduced to the boundary value problem by the inhomogeneous ordinary differential equation of second order. From this, we obtain a general solution of the Eq. (5.12), as follows:

$$\begin{aligned}
V(x;t) = &-\frac{\sqrt{\alpha+\beta s}}{s}\int_0^x \left[G(\tau;s) + \Psi(\tau) + s\Phi(\tau) - \beta\frac{d^2\Phi(\tau)}{dx^2}\right] \times \\
&\sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)d\tau + C_1(s)\exp\left(-\frac{s}{\sqrt{\alpha+\beta s}}x\right) + \\
&C_2(s)\exp\left(\frac{s}{\sqrt{\alpha+\beta s}}x\right),
\end{aligned} \tag{5.13}$$

where C_1 and C_2 are arbitrary functions of s. Substituting the Eq. (5.13) into the integral boundary conditions in the Eq. (5.12), we have

$$\begin{aligned}
&C_1(s)\int_0^1 \exp\left(-\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx + C_2(s)\int_0^1 \exp\left(\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx = \\
&\frac{\sqrt{\alpha+\beta s}}{s}\int_0^1 \left[\left[G(\tau;s) + \Psi(\tau) + s\Phi(\tau) - \beta\frac{d^2\Phi(\tau)}{dx^2}\right] \times \right. \\
&\left.\int_\tau^1 \sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)dx\right]d\tau + A(s),
\end{aligned}$$

$$C_1(s)\int_0^1 x\exp\left(-\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx + C_2(s)\int_0^1 x\exp\left(\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx =$$
$$\frac{\sqrt{\alpha+\beta s}}{s}\int_0^1\left[\left[G(\tau;s)+\Psi(\tau)+s\Phi(\tau)-\beta\frac{d^2\Phi(\tau)}{dx^2}\right]\times\right.$$
$$\left.\int_\tau^1 x\sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)dx\right]d\tau + B(s).$$

which in turn yields to

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}$$

where

$$\begin{aligned} a_{11}(s) &= \int_0^1 \exp\left(-\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx,\ a_{12}(s) = \int_0^1 \exp\left(\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx, \\ a_{21}(s) &= \int_0^1 x\exp\left(-\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx,\ a_{22}(s) = \int_0^1 x\exp\left(\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx, \end{aligned}$$

$$\begin{aligned} b_1(s) &= \frac{\sqrt{\alpha+\beta s}}{s}\int_0^1\left[\left[G(\tau;s)+\Psi(\tau)+s\Phi(\tau)-\beta\frac{d^2\Phi(\tau)}{dx^2}\right]\times\right. \\ &\left.\int_\tau^1 \sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)dx\right]d\tau + A(s), \end{aligned}$$

$$\begin{aligned} b_2(s) &= \frac{\sqrt{\alpha+\beta 1}}{s}\int_0^1\left[\left[G(\tau;s)+\Psi(\tau)+s\Phi(\tau)-\beta\frac{d^2\Phi(\tau)}{dx^2}\right]\times\right. \\ &\left.\int_\tau^1 x\sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)dx\right]d\tau + B(s), \end{aligned} \tag{5.14}$$

Thus, by evaluating all integrals appeared in the Eq. (5.13) and the Eq. (5.14), we find out the solution of the Laplace domain. This can be done for known functions G, Ψ, Φ, A, B; however, in many cases, the results of the functions are not easy to show exactly. Therefore, it is needed to numerical approximations of the integrals. As it is known, Gaussian Quadrature formula exists for computing integrals numerically (see [1]). Using this formula, we have approximate of the above integrals, as follows:

$$\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \exp\left(\pm\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx \simeq \frac{1}{2}\sum_{i=1}^{n}\omega_i\begin{pmatrix} 1 \\ \frac{1}{2}(x_i+1) \end{pmatrix}\exp\left(\pm\frac{s}{\sqrt{\alpha+\beta s}}\left(\frac{1}{2}(x_i+1)\right)\right),$$

$$\int_0^x \left[G(\tau;s) + \Psi(\tau) + s\Phi(\tau) - \beta\frac{d^2\Phi(\tau)}{dx^2}\right]\sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)d\tau \simeq$$

$$\frac{x}{2}\sum_{i=1}^{n}\omega_i\left[G\left(\frac{x}{2}(x_i+1);s\right) + \Psi\left(\frac{x}{2}(x_i+1)\right) + s\Phi\left(\frac{x}{2}(x_i+1)\right) - \beta\frac{d^2\Phi\left(\frac{x}{2}(x_i+1)\right)}{d\overline{\tau}^2}\right]\times$$
$$\sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}\left(x-\frac{x}{2}(x_i+1)\right)\right),$$

where we have used $\overline{\tau} = \frac{x}{2}(x_i+1)$.

$$\int_0^1 \left[\left[G(\tau;s) + \Psi(\tau) + s\Phi(\tau) - \beta\frac{d^2\Phi(\tau)}{dx^2}\right]\int_\tau^1 \left(\frac{1}{x}\right)\sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)dx\right]d\tau \simeq$$

$$\frac{1}{2}\sum_{i=1}^{n}\omega_i\left[G\left(\frac{1}{2}(x_i+1);s\right) + \Psi\left(\frac{1}{2}(x_i+1)\right) + s\Phi\left(\frac{1}{2}(x_i+1)\right) - \beta\frac{d^2\Phi\left(\frac{1}{2}(x_i+1)\right)}{d\overline{\tau}^2}\right]\times$$
$$\left(\frac{1-\frac{1}{2}(x_i+1)}{2}\right)\sum_{j=1}^{n}\omega_j\left(\frac{1}{\left(\frac{1-\frac{1}{2}(x_i+1)}{2}\right)x_j + \left(\frac{1+\frac{1}{2}(x_i+1)}{2}\right)}\right)\times$$

$$\sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}\left(\left(\frac{1-\frac{1}{2}(x_i+1)}{2}\right)x_j + \frac{1+\frac{1}{2}(x_i+1)}{2} - \frac{1}{2}(x_i+1)\right)\right), \tag{5.15}$$

where $\overline{\tau} = \frac{1}{2}(x_i+1)$, x_i and ω_i are defined by

$$x_i:\ i^{th} \text{ zero of } P_n(x),\ \omega_i = \frac{2}{(1-x_i^2)\left[P_n'(x_i)\right]^2}.$$

Their values can be found in [1] for different values of n.

By Stehfest's algorithm approximates (1.17) – (1.18)we obtain the numerical solution.

Remark 2 *Stehfest's method gives accurate results for many problems including diffusion problem, fractional functions in the Laplace domain. However, it fails to predict* $\exp(t)$ *type functions or those with oscillatory behavior such as sine and wave function (see [69]).*

Remark 3 *Note that more than one numerical inversion algorithm can also be performed to check the accuracy of the results.*

5.5 Numerical Examples

In this section, we perform some results of numerical computations using Laplace transform method proposed in the previous section. This technique is applied to solve the problem defined by the problem (5.1)-(5.2). The method of solution is easily applicable via the computer, is used Matlab 7.9.3 program.

Example 7 *We take*

$$\begin{aligned} g(x,t) &= -\frac{8\tanh(x+t)\left(2+\sinh^2(x+t)\right)}{\cosh^4(x+t)},\ 0<x<1,\ 0<t\le T, \\ \Phi(x) &= \frac{1}{\cosh^2(x)},\ 0<x<1, \\ \Psi(x) &= \frac{-2\tanh(x+t)}{\cosh^2(x+t)},\ 0<x<1, \\ \mu(t) &= \tanh(1+t)-\tanh t,\ 0<t\le T, \\ m(t) &= \tanh(1+t)-\ln\cosh(1+t)+\ln\cosh(t)\,,\ 0<t\le T, \end{aligned}$$

in this case exact solution is given by

$$v(x,t)=\frac{1}{\cosh^2(x+t)},\ 0<x<1,\ 0<t\le T.$$

The method of solution is easily implemented on the computer, numerical results obtained by $n=8$ in (5.15) and $m=5$ in (1.25), then we compared the exact solution with numerical solution. For $t=0.10$ and $x\in[0.10,0.90]$, we calculate u numerically using the proposed method of solution and compare it with the exact solution in Table 1.

x	0.10	0.30	0.50	0.70	0.90
u exact	0,0782785	0,0879214	0,1057214	0,1345645	0,1791275
u numerical	0,0778127	0,0878643	0,1057002	0,1340946	0,1790532
absolute error	0,0004658	0,0000517	0,0000212	0,0004699	0,0000743

Table 1

Example 8 *We take*

$$
\begin{aligned}
g(x,t) &= -\frac{4\cosh(x+t)\left(\sinh^2(x+t)-2\right)}{\sinh(x+t)},\ 0<x<1,\ 0<t\leq T,\\
\Phi(x) &= \coth^2(x),\ 0<x<1,\\
\Psi(x) &= \frac{-2\cosh(x)}{\sinh^2(x)},\ 0<x<1,\\
\mu(t) &= 1-\coth(1+t)+\coth(t),\ 0<t\leq T,\\
m(t) &= \frac{1}{2}-\coth(1+t)-\ln\sinh(1+t)-\ln\sinh(t),\ 0<t\leq T,
\end{aligned}
$$

in this case exact solution given by

$$v(x,t)=\coth^2(x+t),\ 0<x<1,\ 0<t\leq T.$$

For $t=0.10$ and $x\in[0.10,0.90]$, we calculate u numerically using the proposed method of the solution and compare it with the exact solution in Table 2:

x	0.10	0.30	0.50	0.70	0.90
u exact	25, 6693160	6, 9270684	3, 4671390	2, 2678574	1, 7240617
u numerical	25, 6692268	6, 9269887	3, 4670924	2, 2678555	1, 724061657
absolute error	0, 0000892	0, 0000797	0, 0000466	0, 0000019	0, 0000043

Table 2

Example 9 *We take*

$$
\begin{aligned}
g(x,t) &= \frac{4\tanh(x+t)}{\coth^2(x+t)},\ 0<x<1,\ 0<t\leq T,\\
\Phi(x) &= \frac{1}{\coth^2(x)},\ 0<x<1,\\
\Psi(x) &= 2\tanh(x),\ 0<x<1,\\
\mu(t) &= 1-\tanh(x+t)+\tanh(t),\ 0<t\leq T,\\
m(t) &= \frac{1}{2}-\tanh(t)+\ln\cosh(t)-\ln\cosh(t),\ 0<t\leq T,
\end{aligned}
$$

in this case exact solution is given by

$$v(x,t)=\frac{1}{\coth^2(x+t)},\ 0<x<1,\ 0<t\leq T.$$

For $t = 0.10$ and $x \in [0.10, 0.90]$, we calculate u numerically using the proposed method of the solution and compare it with the exact solution in Table 3:

x	0.10	0.30	0.50	0.70	0.90
u *exact*	$0,0353069$	$0,0095278$	$0,0047689$	$0,0031193$	$0,0023714$
u *numerical*	$0,0352700$	$0,0093904$	$0,0044651$	$0,0030620$	$0,0023015$
absolute error	$0,0000369$	$-0,0001374$	$0,0003038$	$0,0000573$	$0,0000699$

Table 3

Chapter 6

Solvability of parabolic and hyperbolic integro-differential problems with purely integral conditions

6.1 Statement of the problem

In this chapter, we study only parabolic integro-differential equations. The study of hyperbolic integro-differential equations can be done by the same procedure.

We consider the equation the following parabolic integro-differential equation

$$\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \int_0^t a(t-s)\, v(x,s)\, ds,\ 0 < x < 1,\ 0 < t \leq T \tag{6.1}$$

subject to the initial condition

$$v(x,0) = \Phi(x), \quad 0 < x < 1 \tag{6.2}$$

and the integral conditions

$$\begin{aligned} \int_0^1 v(x,t)\, dx &= r(t),\ 0 < t \leq T \\ \int_0^1 x v(x,t)\, dx &= q(t),\ 0 < t \leq T \end{aligned} \tag{6.3}$$

A second problem of a hyperbolic integro-differential equation can be defined as

$$\frac{\partial^2 v}{\partial t^2}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \int_0^t a(t-s)\, v(x,s)\, ds,\ 0 < x < 1,\ 0 < t \leq T \tag{6.4}$$

subject to the initial condition

$$\begin{aligned} v(x,0) &= \Phi(x),\ 0 < x < 1 \\ \frac{\partial v(x,0)}{\partial t} &= \Psi(x),\ 0 < x < 1 \end{aligned} \tag{6.5}$$

and the integral conditions

$$\begin{aligned} \int_0^1 v(x,t)\, dx &= r(t),\ 0 < t \leq T \\ \int_0^1 x v(x,t)\, dx &= q(t),\ 0 < t \leq T \end{aligned} \tag{6.6}$$

where v is an unkown function, $r, q,\ \Phi(x)$, and $\Psi(x)$ are given functions supposed to be sufficiently regular, a is suitably defined function satisfying certain conditions that will be specified later and T is a positive constant. Some problems of modern physics and technology can be described in terms of partial differential equations with nonlocal conditions. The nonlocal term of our problem (i.e $\int_0^t a(t-s)\, v(x,s)\, ds$) appears, for instance, in the modelling of the quasistatic flexure of a thermoelastic rod, see [8, 10] firstly has been studied, by the second author with more general second-order parabolic equation or a $2m-$parabolic equation in [8, 10, 12] by using of the energy-integrals methods and the Rothe method in [101]. For other models, we refer the reader, for instance, to [6, 10, 11, 13, 14, 21, 22, 102, 94, 117],and references therein. The Problem $(6.1)-(6.3)$ is studied by the Rothe method in [24].Ang [4] has considered a one-dimensional heat equation with nonlocal (integral) conditions. The author has taken the laplace transform of the problem and then used numerical technique for the inverse laplace transform to obtain the numerical solution.

6.2 Reformulation of the problem

Since integral conditions are inhomogenous, it is convenient to convert problem (6.1) – (6.3) to an equivalent problem with homogenous integral conditions. For this, we introduce a new function $u(x,t)$ representing the deviation of the function $v(x,t)$ from the function

$$u(x,t) = v(x,t) - w(x,t), \; 0 < x < 1, \; 0 < t \leq T \tag{6.7}$$

where

$$w(x,t) = 6(2q(t) - r(t))x - 2(3q(t) - 2r(t)) \tag{6.8}$$

Problem (6.1) – (6.3) with inhomogenous integral conditions (6.3) can be equivalently reduced to the problem of finding a function u satisfying

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \int_0^t a(t-s)\, u(x,s)\, ds, \quad 0 < x < 1, 0 < t \leq T \tag{6.9}$$

the initial condition

$$u(x,0) = \varphi(x), \; 0 < x < 1 \tag{6.10}$$

and the purely integral conditions

$$\begin{aligned} \int_0^1 u(x,t)\, dx &= 0, \; 0 < t \leq T \\ \int_0^1 xu(x,t)\, dx &= 0, \; 0 < t \leq T \end{aligned} \tag{6.11}$$

and

$$\varphi(x) = \Phi(x) - w(x,0) \tag{6.12}$$

Hence, instead of solving for v, we simply look for u. The solution of problem (6.1) – (6.3) will be obtained by the relation (6.7) – (6.8).

6.3 Existence of the Solution

Laplace transform is an efficient method for solving many differential equations and partial differential equations,The main difficulty with Laplace transform method is in inverting the

Laplace domain solution into the real domain. In this section we shall apply the Laplace transform technique to find solutions of partial differential equations.

Suppose that $v(x,t)$ is defined and is of exponential order for $t \geq 0$ i.e. there exists A, $\gamma > 0$ and $t_0 > 0$ such that $|v(x,t)| \leq A\exp(\gamma t)$ for $t \geq t_0$. Than the Laplace transform $V(x,s)$, exists and is given by

$$V(x,s) = \mathcal{L}\{v(x,t);t \longrightarrow s\} = \int_0^\infty v(x,t)\exp(-st)\,dt, \tag{6.13}$$

where s is positive real parameter. Appying the Laplace transform on both sides of (6.1), we have

$$(s - A(s))V(x,s) - \frac{d^2}{dx^2}V(x,s) = s\Phi(x), \tag{6.14}$$

where $G(x,s) = \mathcal{L}\{g(x,t);t \longrightarrow s\}$. Similarly, we have

$$\begin{aligned} \int_0^1 V(x,s)\,dx &= R(s), \\ \int_0^1 xV(x,s)\,dx &= Q(s), \end{aligned} \tag{6.15}$$

where

$$\begin{aligned} R(s) &= \mathcal{L}\{r(t);t \longrightarrow s\}, \\ Q(s) &= \mathcal{L}\{q(t);t \longrightarrow s\}. \end{aligned}$$

Now, we distinguish the following cases:

Case 1: If $s - A(s) > 0$.

Case 2: If $s - A(s) < 0$.

Case 3: If $s - A(s) = 0$.

We only consider cases 2 and 3, since case 1 can be dealt with similarly as in [4]. For $(s - A(s)) = 0$, we have

$$\frac{d^2}{dx^2}V(x,s) = -s\Phi(x). \tag{6.16}$$

The general solution for case 3 is given by

$$V(x,s) = -\int_0^x \int_0^y [s\Phi(x)]\,dzdy + C_1(s)x + C_2(s). \tag{6.17}$$

Putting the integral conditions (6.15) in (6.17) we get

$$\begin{aligned}
& \frac{1}{2}C_1(s) + C_2(s) \\
= & \int_0^1 \int_0^x \int_0^y [s\Phi(x)]\, dzdy + R(s), \\
& \frac{1}{3}C_1(s) + \frac{1}{2}C_2(s) \\
= & \int_0^1 \int_0^x \int_0^y x\,[s\Phi(x)]\, dzdy + Q(s),
\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}
C_1(s) &= 12\int_0^1 \int_0^x \int_0^y x\,[s\Phi(x)]\, dzdy - \\
&\quad 6\int_0^1 \int_0^x \int_0^y [s\Phi(x)]\, dzdy + \\
&\quad 12Q(s) - 6R(s), \\
C_2(s) &= 4\int_0^1 \int_0^x \int_0^y [s\Phi(x)]\, dzdy - \\
&\quad 6\int_0^1 \int_0^x \int_0^y x\,[s\Phi(x)]\, dzdy - \\
&\quad 6Q(s) + 4R(s).
\end{aligned} \tag{6.19}$$

For case 2, that is, $(s - A(s)) < 0$, using the method of variation of parameter, we have the general solution as

$$\begin{aligned}
V(x,s) &= \frac{1}{\sqrt{A(s) - s^2}} \int_0^x (s\Phi(x)) \times \\
&\quad \sin\left(\sqrt{A(s) - s}\right)(x-\tau)\, d\tau + d_1(s)\cos\sqrt{(A(s) - s)}x + \\
&\quad d_2(s)\sin\sqrt{(A(s) - s)}x.
\end{aligned} \tag{6.20}$$

From the integral conditions (6.15) we get

$$
\begin{aligned}
& d_1(s)\int_0^1 \cos\sqrt{(A(s)-s)}x\,dx + d_2(s)\int_0^1 \sin\sqrt{(A(s)-s)}x\,dx \\
= \; & R(s) - \frac{1}{\sqrt{A(s)-s^2}}\int_0^1\int_0^x (s\Phi(x)) \times \\
& \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\,d\tau dx, \\
& d_1(s)\int_0^1 x\cos\sqrt{(A(s)-s)}x\,dx + d_2(s)\int_0^1 x\sin\sqrt{(A(s)-s)}x\,dx \\
= \; & Q(s) - \frac{1}{\sqrt{A(s)-s}}\int_0^1\int_0^x x(s\Phi(x)) \times \\
& \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\,d\tau dx.
\end{aligned}
\tag{6.21}
$$

Thus d_1, d_2 are given by

$$
\begin{pmatrix} d_1(s) \\ d_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{6.22}
$$

where

$$
\begin{aligned}
a_{11}(s) &= \int_0^1 \cos\sqrt{(A(s)-s)}x\,dx, \\
a_{12}(s) &= \int_0^1 \sin\sqrt{(A(s)-s)}x\,dx, \\
a_{21}(s) &= \int_0^1 x\cos\sqrt{(A(s)-s)}x\,dx, \\
a_{22}(s) &= \int_0^1 x\sin\sqrt{(A(s)-s)}x\,dx, \\
b_1(s) &= R(s) - \frac{1}{\sqrt{A(s)-s}}\int_0^1\int_0^x (s\Phi(x)) \times \\
& \quad \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\,d\tau dx, \\
b_2(s) &= Q(s) - \frac{1}{\sqrt{A(s)-s}}\int_0^1\int_0^x x(s\Phi(x)) \times \\
& \quad \sin\left(\sqrt{A(s)-s}\right)(x-\tau)\,d\tau dx.
\end{aligned}
\tag{6.23}
$$

If it is not possible to calculate the integrals directly, then we calculate them numerically. We approximate similarly as done in [4]. If the Laplace inversion is possibly computed directly for (6.17) and (6.20) , we get our solution explicitly. In otherwise we use the suitable approximate method and then we use the numerical inversion of the Laplace transform. Considering

$A(s) - s = k(s)$ and using Gauss's formula given in [1] we have the following appoximations of the integrals:

$$\begin{aligned}
&\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \cos \sqrt{k(s)} x dx \\
\simeq\ & \frac{1}{2} \sum_{i=1}^{N} w_i \begin{pmatrix} 1 \\ \frac{1}{2}[x_i + 1] \end{pmatrix} \cos \left(\sqrt{k(s)} \frac{1}{2} [x_i + 1] \right), \\
&\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin \sqrt{k(s)} x dx \\
\simeq\ & \frac{1}{2} \sum_{i=1}^{N} w_i \begin{pmatrix} 1 \\ \frac{1}{2}[x_i + 1] \end{pmatrix} \sin \left(\sqrt{k(s)} \frac{1}{2} [x_i + 1] \right), \\
&\int_0^x (s\Phi(x)) \sin \left(\sqrt{k(s)} \right) (x - \tau) d\tau \\
\simeq\ & \frac{x}{2} \sum_{i=1}^{N} w_i \left[s\Phi \left(\frac{x}{2} [x_i + 1] \right) \right] \\
& \sin \left(\sqrt{k(s)} \left[x - \frac{x}{2} [x_i + 1] \right] \right), \\
&\int_0^1 \left[[s\Phi(\tau)] \int_\tau^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin \left(\sqrt{k(s)} \right) (x - \tau) dx \right] d\tau \\
\simeq\ & \frac{1}{2} \sum_{i=1}^{N} w_i \left[s\Phi \left(\frac{1}{2} [x_i + 1] \right) \right] \\
& \left(\frac{1 - \frac{1}{2}[x_i + 1]}{2} \right) \sum_{i=1}^{N} w_j \begin{pmatrix} 1 \\ \frac{1 - \frac{1}{2}[x_i+1]}{2} x_j + \frac{1 - \frac{1}{2}[x_i+1]}{2} \end{pmatrix} \times \\
& \sin \left(\sqrt{k(s)} \left[\frac{1 - \frac{1}{2}[x_i + 1]}{2} x_j + \frac{1 + \frac{1}{2}[x_i + 1]}{2} - \frac{1}{2}(x_i + 1) \right] \right),
\end{aligned} \tag{6.24}$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n(x),\ \omega_i = 2/\left(1 - x_i^2\right) \left[P_n'(x) \right]^2.$$

Their tabulated values can be found in [1] for different values of N.

By Stehfest's algorithm approximates $(1.17) - (1.18)$we obtain the numerical solution.

6.4 Numerical Example

In this section, we perform some results of numerical computations using Laplace transform method proposed in the previous section. This technique is applied to solve the problem defined by the problem (6.1) – (6.3). The method of solution is easily applicable via the computer, is used Matlab 7.9.3 program.

Example 10 *We take*

$$\begin{aligned}
\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) &= \int_0^t \exp(t-s)u(x,s)\,ds, 0<x<1,\ 0<t\leq T,\\
v(x,0) &= \sin x,\ 0<x<1,\\
\int_0^1 v(x,t)\,dx &= 0,\ 0<t\leq T,\\
\int_0^1 xv(x,t)\,dx &= 0,\ 0<t\leq T,
\end{aligned}$$

in this case exact solution is given by

$$v(x,t) = \exp(-t).\cos t.\sin x,\ 0<x<1,\ 0<t\leq T.$$

The method of solution is easily implemented on the computer, numerical results obtained by $N=8$ in (6.24) and $m=5$ in (**??**), then we compared the exact solution with numerical solution. For $t=0.10$ and $x\in[0.10,0.90]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution in Table 1.

The relative error computed by $\frac{v\ numerical - v\ exact}{v\ exact}$

x	0.10	0.30	0.50	0.70	0.90
v *exact*	0.0898817	0.2660619	0.4316350	0.5800001	0.7052425
v *numerical*	0.0898818	0.2660623	0.4316355	0.5800058	0.7052395
relativ error	$-0,0000058$	$0,0000017$	$0,0000012$	$0,0000099$	$-0,0000043$

Table1

6.5 Uniqueness and Continuous dependence of the Solution

We establish an a priori estimate, the uniqueness and continuous dependence of the solution with respect to the data are immediate consequences.

Theorem 26 *If $u(x,t)$ is a solution of the Problem* $(6.9)-(6.11)$*, then we have*

$$\|u(.,\tau)\|^2_{L^2(0,1)} \leq c_1\left(\|\varphi\|^2_{L^2(0,1)}\right)$$
$$\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} \leq c_2\left(\|\varphi\|^2_{L^2(0,1)}\right), \tag{6.25}$$

where $c_1 = \exp(a_0 T)$, $c_2 = \frac{\exp(a_0T)}{1-a_0}$, $1 < a(x,t) < a_0$, *and* $0 \leq \tau \leq T$.

Proof. *Taking the scalar product in* $B_2^1(0,1)$ *of equation* (6.9) *and* u*, and integrating over* $(0,\tau)$*, we have*

$$\int_0^\tau \left(\frac{\partial u(.,t)}{\partial t}, u\right)_{B_2^1(0,1)} dt -$$
$$\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, u\right)_{B_2^1(0,1)} dt$$
$$= \int_0^\tau \left(\int_0^t a(t-s)\,u(x,s)\,ds, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt. \tag{6.26}$$

Integrating by parts on the left-hand sid of (6.26) *we obtain*

$$\frac{1}{2}\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)} +$$
$$\frac{1}{2}\|u(.,\tau)\|^2_{L^2(0,1)} - \frac{1}{2}\|\varphi\|^2_{L^2(0,1)}$$
$$= \int_0^\tau \left(\int_0^t a(t-s)\,u(x,s)\,ds, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt. \tag{6.27}$$

By the ***Cauchy inequality****, the right-hand side of* (6.27) *is bounded by*

$$\frac{a_0}{2}\int_0^t \|u(x,s)\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} ds + \frac{a_0}{2}\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} \tag{6.28}$$

Substitution of (6.28) *into* (6.27) *yields*

$$(1-a_0)\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} + \|u(.,\tau)\|^2_{L^2(0,1)} \leq$$

$$\|\varphi\|^2_{L^2(0,1)} +$$
$$\frac{a_0}{2}\int_0^t \|u(x,s)\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} ds. \tag{6.29}$$

*By the **Gronwall Lemma** we have*

$$\begin{aligned}
&(1-a_0)\left\|\frac{\partial u\left(.,t\right)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)}+\left\|u\left(.,\tau\right)\right\|^2_{L^2(0,1)}\\
\leq\ & \exp\left(a_0T\right)\left(\left\|\varphi\right\|^2_{L^2(0,1)}\right). \qquad (6.30)
\end{aligned}$$

From (6.30), *we obtain the estimates* (6.25). ■

Corollary 27 *If Problem* $(6.9)-(6.11)$ *has a solution, then this solution is unique and depends continuously on* φ.

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ملخص

في هذه الاطروحة نثبت وجود ووحدانية، والارتباط المستمرللحل بالمعطيات لبعض اصناف المسائل التطورية الخطية مع شروط غير محلية (شروط حدية تكاملية). البراهين تعتمد على تقديرات قبلية وطريقة تحويل لابلاس وفي الاخير نتحصل على الحل باستعمال تقنية عددية(لوغاريتم ستيفاست) وذلك بعكس تحويل لابلاس

RESUME

Dans cette thèse, nous démontrons l'existence, l'unicité et la dépendance continue de la solution par rapport aux données pour certaines classes des problèmes d'évolution linéaires avec des conditions non locales (conditions aux limites intégrales). Les preuves sont basée sur des estimations a priori et la méthode de transformée de Laplace. Finalement, nous obtenons la solution en utilisant la technique numérique (algorithme de Stehfest) pour inverser la transformée de Laplace.

ABSTRACT

In this thesis we prove the existence, uniqueness, and continuous dependance of solution upon the data for some classes linear evolution problems with nonlocal conditions (boundary integral conditions). The proofs are based on a priori estimates and Laplace transform method. Finally, we obtain the solution by using numerical technique (Stehfest algorithm) for inverting the Laplace transform.