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ABSTRACT

This PhD thesis is dedicated to the study of diferential inclusions involving normal cones in Hilbert spaces. In particular, we are interested in the sweeping process. The sweeping process is a constrained diferential inclusion involving normal cones.

This work is divided conceptually in three parts:

In the first part we prove the local existence of solutions of sweeping process involving a locally prox-regular set with an upper semicontinuous set-valued perturbation contained in the Clarke subdifferential of a primal lower nice function.

In the second part we consider the free endpoint Mayer problem for a controlled Moreau process, the control acting as a perturbation of the dynamics driven by the normal cone, and derive necessary optimality conditions of Pontryagin's Maximum Principle type.

Finally, in the third part we consider the problem of minimizing a cost at the endpoint of a trajectory subject to the finite dimensional dynamics, this dynamics is a nonclassical control problem with state constraints.

Keywords: Diferential inclusions, Sweeping process, Mayer problem, Pontryagin's Maximum Principle.

RÉSUMÉ

Cette thèse est consacrée à l'étude des inclusions differentielles gouvernées par des cônes normaux dans les espaces de Hilbert. En particulier, nous nous sommes intéressés à l'étude des processus de rafle. Le processus de rafle est une inclusion differentielle avec contrainte impliquant des cônes normaux .

Ce travail est divisé conceptuellement en trois parties :

Dans la première partie nous prouvons l'existence de solutions locales pour le processus de rafle gouverné par un ensemble localement prox-régulier et avec une perturbation semicontinue supérieurement incluse dans le subdifférentiel de Clarke d'une fonction pln.

Dans la deuxième partie, nous considérons le problème de Mayer d'un point final libre pour un processus de Moreau contrôlé, le contrôle agissant comme une perturbation de la dynamique conduite par le cône normal. Nous dérivons les conditions nécessaires d'optimalité.

Finalement, dans la troisième partie, nous considérons le problème de la minimisation d'un coût au point final d'une trajectoire soumise à une dynamique en dimension finie, cette dynamique est un problème de contrôle non classique avec des contraintes sur l'état.

Mots-clés: Inclusions différentielles, Processus de rafle, Problème de Mayer, Principe de maximum de Pontryagin.

CONTENTS

GENERAL INTRODUCTION

The theory of multivalued differential equations is now well known. This Theory was introduced in the 1940s for the study of systems of equations with nonlinear partial drift and problems from mechanics. Today,this theory has become more important and more attractive. Its scope has grown considerably, and has been successful in many areas such as: the unilateral mechanics, the mathematical economy, the sciences of the engineers(non-regular electrical circuit),... more recently, it has become one of the methods important for the study of variational evolutionary inequalities, mainly those governed by the normal cone.

The aim of this thesis is to give some contributions to theory of diferential inclusions and diferential inclusions involving normal cones. In particular, we are interested at existence of solutions and necessary optimality conditions.

This thesis is composed of four chapters. The first is about concepts basics and some auxiliary results that we have used throughout this work.

In chapter two we prove the local existence of solutions of sweeping process involving a locally prox-regular set C with an upper semicontinuous set-valued perturbation

F contained in the Clarke subdifferential of a primal lower nice function g of the form

$$\dot{x}(t) \in -N_C(x(t)) + F(x(t)), \quad x(0) = x_0 \in C, a.e \quad t \in [0,T].$$

Here $N_C(\cdot)$ denotes the Clarke normal cone of C. The study requires the both quantified concepts of locally prox-regularity termed as (r,α)-prox-regularity for set C and $c - pln$ regularity for function g.

In chapter tree we consider the free endpoint Mayer problem for a controlled Moreau process, the control acting as a perturbation of the dynamics driven by the normal cone, and derive necessary optimality conditions of Pontryagin's Maximum Principle type. The results are also discussed through an example. We combine techniques from [55] and from [16], which in particular deals with a different but related control problem. Our assumptions include the smoothness of the boundary of the moving set $C(t)$, but, differently from [16], do not require strict convexity and time independence of $C(t)$. Rather, a kind of inward/outward pointing condition is assumed on the reference optimal trajectory at the times where the boundary of $C(t)$ is touched. The state space is finite dimensional.

Finally in chapter four we consider the problem of minimizing the cost $h(x(T))$ at the endpoint of a trajectory x subject to the finite dimensional dynamics

$$\dot{x} \in -N_C(x) + f(x,u), \quad x(0) = x_0,$$

where N_C denotes the normal cone to the convex set C. Such differential inclusion is termed, after Moreau, *sweeping process*. We label it as a "nonclassical" control problem with state constraints, because the right hand side is discontinuous with respect to the state, and the constraint $x(t) \in C$ for all t is implicitly contained in the dynamics.

We prove necessary optimality conditions in the form of Pontryagin Maximum Principle by requiring, essentially, that C is independent of time. If the reference trajectory is in the interior of C, necessary conditions coincide with the usual ones. In the general case, the adjoint vector is a BV function and a signed vector measure appears in the adjoint equation.

CHAPTER 1

DEFINITIONS AND PRELIMINARY RESULTS

In this chapter we describe the notation, the definitions and basic results that are going to be used throughout the thesis.
Let H be a real Hilbert space with the norm $\|.\|$ and scalar product $\langle ., . \rangle$. For $x \in H$ and $\varepsilon > 0$, let $B(x, \varepsilon) = \{y \in H : \|y - x\| < \varepsilon\}$ be the open ball centered at x with radius ε and $\overline{B}(x, \varepsilon)$ be its closure.

We denote by $d(x, C) := \inf\{\|x - y\|; y \in C\}$ the distance from $x \in H$ to a subset $C \subset H$ and $d^*(A, B) := \sup\{d(a, B) : a \in A\}$ for $A, B \subset H$.
One defines the (possibly empty) set of nearest points of y in C by

$$proj_C(x) := \{y \in C : d_C(x) = \|x - y\|\}$$

If $x \in proj_C(x)$, and $s \geq 0$, then the vector $s(y-x)$ is called (see, e.g., [26]) a proximal normal to C at x. The set of all vectors of this form is a cone which is termed the proximal normal cone of C at x. It is denoted by $N^p_C(x)$, and $N^p_C(x) = \emptyset$ whenever $x \in intC$.

Given a function $f : H \to \mathbb{R} \cup \{+\infty\}$. Let $domf := \{x \in H : f(x) < +\infty\}$ be its domain. We say that f is proper if $domf$ is nonempty.

1.1 Primal lower nice function

Let us now define primal lower nice function in a quantified way [44, 57].

Definition 1.1.1 Let f: $H \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. The function f is said to be primal lower nice (pln, for short) on an open convex set O with $O \cap domf \neq \emptyset$ if there exists some real number $c \geq 0$ such that for all $x \in O \cap dom\partial^p f(x)$ and for all $v \in \partial^p f(x)$ on has

$$f(y) \geq f(x) + \langle v, y-x \rangle - c(1 + \|v\|)\|y-x\|^2, \tag{1.1}$$

for each $y \in O$.
The real $c \geq 0$ will be called a pln constant for f over O and we will say that f is c -pln on O.

For $\overline{x} \in domf$, we say that f is pln at $\overline{x}$ whenever it is pln on some open set containing $\overline{x}$. Here $\partial^P f(x)$ denotes the proximal subdifferential of f at x (for its definition the reader is refereed for instance to [14, 33]) and $dom\partial^P f(x) := \{x \in H : \partial^P f(x) \neq \emptyset\}$.

Remark 1.1.1 *It is established in [57] that the definition above is equivalent to the pioneering definition of primal lower nice function.*

The following propositions summarizes some important properties for pln functions needed in the sequel.

Proposition 1.1.1 *[44] Assume that the function $f\colon H \to \mathbb{R} \cup \{+\infty\}$ is primal lower nice on an open set O with $O \cap dom f \neq 0$. Then for all $x \in O$, we have*

$$\partial^P f(x) = \partial^C f(x)$$

Hence, the definition of the pln property is independent of the subdifferential operator.

Here $\partial^C f(x)$ denotes the Clarke subdifferential of f at x (see for instance [14, 33] for the definition of $\partial^C f$).

1.2 Prox-regular sets

After the above definition of $c - pln$ functions, we recall the definition of local prox-regularity of sets. For a large development of this concept, the reader is referred to [52]. In this paper, we will use some results where the quantified viewpoint [43] of the local prox-regularity has been introduced.

Definition 1.2.1 *Let $C \subset \mathbb{R}^n$ be a closed smooth set and $\rho > 0$ be given. We say that C is ρ-prox-regular provided the inequality*

$$\langle \zeta, y - x \rangle \leq \frac{|y - x|^2}{2\rho} \tag{1.2}$$

holds for all $x, y \in C$, where ζ is the unit external normal to C at $x \in \partial C$.

In particular, every convex set is ρ-prox regular for every $\rho > 0$ and every set with a $C^{1,1}$-boundary is ρ-prox regular, where ρ depends only the Lipschitz constant of the gradient of the parametrization of the boundary (see [32, Example 64]). In this case, the (proximal) normal cone to C at $x \in C$ is the nonnegative half ray generated by the unit external normal, and

$$v \in N_C(x) \text{ if and only if there exists } \sigma > 0 \text{ such that } \langle v, y - x\rangle \leq \sigma |y - x|^2 \quad \forall y \in C.$$

Prox-regular sets enjoy several properties, including uniqueness of the metric projection and differentiability of the distance (in a suitable neighborhood) and normal regularity, which hold also true for convex sets, see, e.g. [32].

Definition 1.2.2 *For positive real numbers r and α, the closed set C is said to be (r, α) prox-regular at a point $\overline{x} \in C$ provided that for any $x \in C \cap B(\overline{x}, \alpha)$ and any $v \in N_C^P(x)$ such that $\|v\| \leq r$, one has*

$$x = proj_C(x + v).$$

The set C is r prox regular (resp. prox regular) at $\overline{x}$ when it (r, α) prox regular at $\overline{x}$ for some real $\alpha > 0$ (resp. for some numbers $r > 0$ and $\alpha > 0$). The set C is said to be r-uniformly prox-regular when $\alpha = +\infty$.

It is not difficult to see that the latter (r, α)-prox-regularity property of C at $x \in C$ is equivalent requiring that

$$x \in proj_C(x + rv) \text{ for all } x \in C \cap B(\overline{x}, \alpha) \text{ and } v \in N_C^P(x) \cap \mathbb{B}.$$

When the set C is (r, α)-prox-regular at $\overline{x}$, we have

$$N_C^P(x) = N_C^F(x) = N_C^L(x) = N_C^C(x) \text{ for all } x \in C \cap B(\overline{x}, \alpha).$$

The (r, α)-prox-regularity of the set C gives the following hypomonotonicity property of the truncated normal cone. Recall that a set-valued mapping $A : H \to 2^H$ is hypomonotone with constant $\sigma \geq 0$ on a subset $S \subset H$ when for all $x_1, x_2 \in S$ and $\xi_1 \in A(x_1)$ and $\xi \in A(x)$, $\langle \xi_1 - \xi_2, x_1 - x_2\rangle \geq -\sigma \|x_1 - x_2\|^2$.

Proposition 1.2.1 *[43] Let C be a closed subset of H, and $\overline{x} \in C$. Then if there exist positive real numbers r and α such that C is (r, α)-prox-regular at $\overline{x}$, then the set-valued mapping $N_C^P(\cdot) \cap \mathbb{B}$ is $\frac{1}{r}$ -hypomonotone on $C \cap B(\overline{x}, \alpha)$.*

We also recall the Gronwall Lemma for absolutely continuous solutions of differential inequalities.

Lemma 1.2.1 *(Gronwall lemma) Let $\alpha, \beta, \zeta : [T_0, T] \to R$ be three real-valued Lebesgue integrable functions. If the function $\zeta(\cdot)$ is absolutely continuous and if for almost all $t \in [T_0, T]$*

$$\dot{\zeta}(t) \leq \alpha(t) + \beta(t)\zeta(t),$$

then for all $t \in [T_0, T]$

$$\zeta(t) \leq \zeta(T_0)exp(\int_{T_0}^{t} \beta(\theta)d\theta) + \int_{T_0}^{t} \alpha(s)exp(\int_{s}^{t} \beta(\theta)d\theta)ds$$

1.3 Control Systems

Control theory provides a different paradigm. We now assume the presence of an external agent, i.e. a controller, who can actively influence the evolution of the system. This new situation is medelled by a control system, namely

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(0) = \overline{x}, \quad u(.) \in U \tag{1.3}$$

where U is the family of admissible control functions defined as

$$U = \{u : [0, T] \to \mathbb{R}^m; u \quad measurable, u(t) \in \mathbf{U} \quad for \quad a.e. \quad t\}. \tag{1.4}$$

$\mathbf{U} \subset \mathbb{R}^m$ is the set of control values.

In this case, the rate of change $\dot{x}(t)$ depends not only on the state x itself, but also on some external parameters, say $u = (u_1, ..., u_m)$, which can also vary in time. The control function $u(.)$, subject to some constraints, will be chosen by a controller in order to modify the evolution of the system and achieve certain preassigned goals-steer the system from one state to another, maximize the terminal value of one of the parameters, minimize a certain cost functional, etc... The system (1.3) can be written as a differential inclusion, namely

$$\dot{x}(t) \in F(t, x(t)) \tag{1.5}$$

where the set of possible velocities is given by

$$F(t, x) := \{y, \quad y = f(t, x(t), u(t), \quad u(t) \in \mathbf{U}\}. \tag{1.6}$$

Given an initial state$\overline{x}$, a set of admissible terminal conditions $S \subset \mathbb{R} \times \mathbb{R}^m$, and a cost function $J : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, we consider the optimization problem

$$\min_{u \in U, T > 0} J(T, x(T, u)) \tag{1.7}$$

with initial and terminal constraints

$$x(0) = \overline{x}, \quad (T, x(T)) \in S \tag{1.8}$$

This controlled problem (1.7) subject to (1.3) is called the Mayer problem.
Now we introduce the Pontryagin Maximum Principle (PMP), it can used in order to compute optimal controls and optimal trajectories. the PMP is used in optimal control theory to find the best possible control for taking a dynamical system from one state to another, especially in the presence of constraints for the state or input controls. It was formulated in 1956 by the Russian mathematician Lev Pontryagin and his students.
So the goal is derive necessary conditions for the optimality of a control $x^*(.)$. These conditions will provide a basic tool for the actual computation of optimal controls.

Theorem 1.3.1 ***(Pontryagin Maximum Principle for The Mayer problem with free terminal point).*** *Consider the optimal control problem* (1.3) - (1.7), *under the assumptions that the function $f = f(t, x, u)$ is continuous and continuously differentiable w.r.t. x. The payoff* *function J is differentiable.*

Let $u^*(.)$ be a bounded admissible control whose corresponding trajectory $x^*(.) = x(., u)$ is optimal. Call $p : [0, T] \longrightarrow \mathbb{R}^m$ the solution of the adjoint linear equation

$$\dot{p}(t) = -p(t).D_x f(t, x^*(t), u^*(t)), \quad p(T) = \nabla J(x^*(T)) \tag{1.9}$$

Then the maximality condition

$$p(t).f(t, x^*(t), u^*(t)) = \max_{w \in \mathbf{U}} \{p(t).f(t, x^*(t), w)\} \tag{1.10}$$

holds for almost every time $t \in [0, T]$.

In the above theorem,x, f, u represent column vectors, $D_x f$ is the $n \times n$ Jacobian matrix of first order partial derivatives of f w.r.t. x, while p is a row vector.

CHAPTER 2

ON EVOLUTION EQUATIONS HAVING HYPOMONOTONICITIES OF OPPOSITE SIGN GOVERNED BY SWEEPING PROCESSES

2.1 Introduction

The notion of so-called "sweeping process" was introduced by Jean Jacques Moreau in the seventies. Moreau studied in a series of seminal papers [46, 47, 49, 50] the both theoretical and numerical aspects of the sweeping process for a moving closed convex set included in a Hilbert space. There are plenty of existence and uniqueness results of the perturbed sweeping processes

$$\dot{x}(t) \in -N_C(x(t)) + F(x(t)), \quad x(0) = x_0 \in C, a.e \quad t \in [0, T], \tag{2.1}$$

in the literature (see, e.g., [1, 27, 34, 38, 61] and the references therein). The set-valued perturbations of equations of sweeping processes type have been considered for the first time by Castaing and Monteiro-Marques [23, 24]. Let us note that the convexity assumption on the set-valued perturbation is essential in the most previous works: see for example, some recent papers [34, 61, 15, 35]. In [25, 60], the authors considered the possibly nonconvex perturbation contained in the subdifferential of a convex Lipschitz function. Moreover the closed set of constraints was supposed to be compact.

It is also worth mentioning that problem (2.1) in the more general form

$$\dot{x}(t) \in -\partial f(x(t)) + F(x(t)), \quad F(x) \subset \partial g(x), \quad x(0) = x_0 \in dom f, \tag{2.2}$$

with g is φ-convex of order two and f has a φ-monotone subdifferential of order two (shortly $f \in MS(2)$) have been studied in convex and nonconvex cases , see, e.g [18, 19]. Notice that we can obtain (2.1) from (2.2), by taking the indicator function of the set of constraints. Indeed, when the set C of constraints is locally pox-regular, then the associated indicator function $f = \delta_C$ is pln (Proposition 3.31 [43]) and so $MS(2)$.

Despite the similarity of (2.1) and (2.2), the problems are quite different, since in general with $f = \delta_C$ the level set $\{x \in H; f(x) \leq r\}$ is not compact, and these were basic assumption in [18]. The question arises whether we can drop the assumption of compactness of the set of constraints. Our main result here establishes an existence result in this vein. Since the compactness assumption will be shifted from the set of constraints to the nonconvex set-valued perturbation in the case of sweeping processes problems. This compactness assumption on the set-valued perturbation is necessarily made because of the infinite dimensional character of the Hilbert space. More precisely, the ordinary differential equation generally has no solution under the sole continuity of the single-valued right hand side on the infinite dimensional Hilbert space.

In this chapter, we propose a discretization method based on the existence and uniqueness of solutions for single-valued perturbation [43] with different techniques to analyze the sweeping processes under nonconvex set-valued perturbation in Hilbert spaces. The fixed set of constraints is supposed to be possibly noncompact and locally prox-regular at the initial datum. The set-valued perturbation force is supposed to be upper semicontinuous contained in the subdifferential of *pln* function and also in a fixed compact subset.

The chapter is organized as follows. In Sect. 2.2, we recall some basic notations, definitions and useful results which are used throughout the paper. The existence of solutions are thoroughly analyzed in Sect. 2.3. An extension of the existence result to the case of shifted moving set is discussed in Sect. 2.4. Finally, Sect. 2.5 closes the paper with some concluding remarks.

2.2 Fundamental results

In this section, we summarize some preliminary results. We begin with this useful proposition.

Proposition 2.2.1 *[44] Let f: $O \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinous function, where v is an open set of H .The following are equivalent :*
(a) f is locally semiconvex, finite and locally lipschitz continuous on O;
(b) f is locally semiconvex, finite and continuous on O;
(c) f is locally bounded from above on O and pln at any point of O.

The graph of the (proximal) subdifferential of a pln function enjoys the useful closure property.

Proposition 2.2.2 *[45] Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc function which is pln at $u_0 \in \mathrm{dom}\, f$ with constants s_0, c_0, $Q_0 > 0$, and let T_0, T, v_0, η_0 be positive real numbers such that $T > T_0$ and $v_0 + \eta_0 = s_0$. Let $v(.) \in L^2([T_0, T], H)$ and $u(\cdot)$ be a mapping from $[T_0, T]$ into H. Let $(u_n(\cdot))_n$ be a sequence of mappings from $[T_0, T]$ into H and $(v_n(\cdot))_n$ be a sequence in $L^2([T_0, T], H)$. Assume that:*

1. *$\{u_n(t), n \in \mathbb{N}\} \subset \overline{B}(u_0, \eta_0) \cap \mathrm{dom}\, f$ for almost every $t \in [T_0, T]$,*

2. *$(u_n)_n$ converges almost everywhere to some mapping u with $u(t) \in \operatorname{dom} f$ for almost every $t \in [T_0, T]$,*

3. *v_n converges to v with respect to the weak topology of $L^2([T_0, T], H)$,*

4. *for each $n \geq 1$, $v_n(t) \in \partial f(u_n(t))$ for almost every $t \in [T_0, T]$.*

Then, for almost all $t \in [T_0, T]$, $v(t) \in \partial f(u(t))$.

We have the following existence and uniqueness result [43].

Proposition 2.2.3 *Let C be an (r, α)-prox-regular set at the point $x_0 \in C$ and let any real number $\eta_0 \in]0, \alpha[$. Then for any $\overline{x} \in B(x_0, \alpha - \eta_0) \cap C$, any positive real number $\tau \leq T_0 - T$, and any mapping $h \in L^1([0, T], H)$ such that $\int_{T_0}^{T_0+\tau} \|h(s)\| ds < \eta_0/2$, the differential variational inequality*

$$\dot{x}(t) \in -N_C(x(t)) + h(t), \quad x(0) = \overline{x}, a.e \quad t \in [T_0, T_0 + \tau], \tag{2.3}$$

has an absolutely continuous solution $x : [T_0, T_0 + \tau] \to B(\overline{x}, \eta_0) \cap C$. Moreover,

$$\|\dot{x}(t)\| \leq \|\dot{x}(t) - h(t)\| + \|h(t)\| \leq 2\|h(t)\| \quad a.e \quad t \in [T_0, T_0 + \tau].$$

2.3 Main Result

In this section, we are interested in the study of the existence of solutions of evolution problem of the form

$$\dot{x}(t) \in -N_C(x(t)) + F(x(t)), \quad x(0) = x_0 \in C, a.e \quad t \in [0, T], \tag{2.4}$$

Under the following assumptions :
(HC_1) : the closed set C is (r, α)-prox-regular at the point $x_0 \in C$;
(HF_1) : $\mathcal{O} \subset H$ is an open convex set containing $\overline{B}(x_0, \eta_0)$ and $F : \mathcal{O} \to 2^H$ is an upper semicontinous set-valued mapping with nonempty weakly compact values;

(HF_2) : let $g : \mathcal{O} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinous function c-pln on $\mathcal{O}$ with $F(x) \subset \partial^C g(x)$, $\forall x \in \mathcal{O}$ such that g is locally bounded from above on $\mathcal{O}$.
By a solution of inclusion (2.1) we mean an absolutely continuous function $x(.) : [0,T] \to H, \quad x(0) = x_0 \in C$, such that the inclusion

$$\dot{x}(t) \in -N_C(x(t)) + f(t) \tag{2.5}$$

holds a.e. for some $f \in L^2([0,T], H)$ with $f(t) \in F(x(t)) \quad a.e.$
It is known that if $x(.)$ is solution of inclusion (2.1), then $x(t) \in C$ for all $t \in [0,T]$.
We note that the closed set C is locally prox-regular at x_0 (hence N_C is hypomonotone set-valued mapping), g is a pln function and the set-valued mapping F is not necessarily the whole subdifferential of g , and we take the plus sign, instead of the classical minus. Also, The system (2.1) can be considered as a nonconvex hypomonotone differential inclusion under control term $u(t) \in N_C(\cdot)$ which guarantees that the trajectory $x(t)$ always belongs to the desired (r, α)-prox-regular set C for all $t \in [0,T]$. We prove (local) existence of solutions.

Theorem 2.3.1 *Assume that H is the Hilbert space, (HC_1), (HF_1) and (HF_2) hold. Assume that for some compact set $\mathcal{K} \subset H$, one has $F(x) \subset \mathcal{K}$ for all $x \in H$.*
Then, for any $x_0 \in C$ there exists $\overline{T} > 0$ and an absolutely continuous function $x(.) : [0, \overline{T}] \rightsquigarrow B(x_0, \eta_0)$ a local solution to problem (2.1).

Proof.

First step: Construction of approximates solutions.
Let $x_0 \in C$ and and let $g : O \to \mathbb{R} \cup \{+\infty\}$ satisfy (HF_2). Then, by proposition 2.2.1 there exist $M > 0$ such that g is Lipschitzean with Lipschitz constant M on $\overline{B}(x_0, \eta_0)$ and, since $F(x) \subset \partial^p g(x)$ it follows that F is bounded by M on $\overline{B}(x_0, \eta_0)$.

Let $\overline{T} > 0$ such that

$$\overline{T} < min\{\eta_0/2M, (\alpha - \eta_0)/2M\}, \tag{2.6}$$

where α and η_0 are given by (HC_1). Our purpose is to prove that there exists $x : [0, \overline{T}] \to B(x_0, \eta_0) \cap C$ a solution to the Cauchy problem (2.1) .
Let $n \in \mathbb{N}^*$, $t_0^n = 0$ and, for $i = 1, ..., n$, let $t_i^n = i\frac{\overline{T}}{n}$. Take $y_0^n \in F(x_0)$ and define $f_0^n : [0, t_1^n[\to H$ by $f_0^n(t) = y_0^n$. Then, for a. e $t \in [0, t_1^n]$; $\|f_0^n(t)\| \le M$ and $f_0^n \in L^2([0, t_1^n], H)$.
Let us consider the problem

$$\dot{x}(t) \in -N_C(x(t)) + f_0^n(t), \quad x(0) = x_0 \in C, a.e \quad t \in [0, t_1^n]. \tag{2.7}$$

By (2.6) and proposition 2.2.3, it has an absolutely continuous solution that we denote by $x_0^n(.) : [0, t_1^n] \rightsquigarrow B(x_0, \eta_0) \cap C$ and this solution satisfies

$$\|(\dot{x_0^n})(t)\| \le 2M, a.e \quad t \in [0, t_1^n].$$

This yields for every $t \in [0, t_1^n]$ that

$$x_0^n(t) \in B(x_0, \alpha - \eta_0) \cap C,$$

because by the latter inequality and by (2.6)

$$\|x_0^n(t) - x_0\| \le \int_0^{t_1^n} \|(\dot{x_0^n})(s)ds\| \le 2M\overline{T} < \alpha - \eta_0.$$

Likewise, take $y_1^n \in F(x_0^n(t_1^n))$. Since $\|x_0^n(t_1^n) - x_0\| < \alpha - \eta_0$, we may apply proposition 2.2.3 again with $x_0^n(t_1^n)$ as initial condition at time t_1^n, $f_1^n : [t_1^n, t_2^n[\to H$ with $f_1^n(t) = y_1^n$, and this gives the existence of an absolutely continuous solution $x_1^n(.) : [t_1^n, t_2^n] \rightsquigarrow B(x_0^n(t_1^n), \eta_0) \cap C$ of the problem

$$\dot{x}(t) \in -N_C(x(t)) + f_1^n(t), \quad x(t_1^n) = x_0^n(t_1^n) \in C, a.e \quad t \in [t_1^n, t_2^n], \tag{2.8}$$

with

$$\|(\dot{x_1^n})(t)\| \le 2M, a.e \quad t \in [t_1^n, t_2^n].$$

So, for any $t \in [t_1^n, t_2^n]$ one has

$$\begin{aligned}
\|x_1^n(t) - x_0\| \leq \|x_1^n(t) - x_1^n(t_1^n)\| + \|x_1^n(t_1^n) - x_0\| &= \|x_1^n(t) - x_1^n(t_1^n)\| + \|x_0^n(t_1^n) - x_0\| \\
&= \|\int_{t_1^n}^{t} (\dot{x_1^n})(s)ds\| + \|\int_0^{t_1^n} (\dot{x_0^n})(s)ds\| \\
&\leq \int_{t_1^n}^{t} 2Mds + \int_0^{t_1^n} 2Mds \\
&= \int_0^t 2Mds \leq 2M\bar{T}.
\end{aligned} \tag{2.9}$$

The last inequality entails through(2.6)

$$x_1^n(t) \in B(x_0, \eta_0) \cap C \text{ for a.e } t \in [t_1^n, t_2^n],$$

and

$$\|x_1^n(t_2^n) - x_0\| < \alpha - \eta_0.$$

And so on. So, for $2 \leq k \leq n-1$; take $y_k^n \in F(x_{k-1}^n(t_k^n))$ with $\|x_{k-1}^n(t_k^n) - x_0\| < \alpha - \eta_0$ and define $f_k^n : [t_k^n, t_{k+1}^n[\to H$ by $f_k^n(t) = y_k^n$, one has a finite sequences of absolutely continuous mapping $x_k^n(.) : [t_k^n, t_{k+1}^n] \rightsquigarrow B(x_0, \eta_0) \cap C$ with $0 \leq k \leq n-1$, such that for each $k \in 0, ..., n-1$ (with $x_{-1}^n(t_0^n) = x_0$)

$$\begin{cases}
\dot{x}_k^n(t) \in -N_C(x_k^n(t)) + f_k^n(t), \quad a.e \quad t \in [t_k^n, t_{k+1}^n] \\
x_k^n(t) \in B(x_0, \eta_0) \cap C; \quad \forall t \in [t_k^n, t_{k+1}^n] \\
x_k^n(t_k^n) = x_{k-1}^n(t_k^n), \\
\|(\dot{x_k^n})(t)\| \leq 2M, a.e \quad t \in [t_k^n, t_{k+1}^n].
\end{cases}$$

Define $x_n(.), f_n(.) : [0, \bar{T}] \to H$ by $x_n(t) = \sum_{k=0}^{n-1} x_k^n(t)\chi_{[t_k^n, t_{k+1}^n]}(t)$ and $f_n(t) = \sum_{k=0}^{n-1} f_k^n(t)\chi_{(t_k^n, t_{k+1}^n]}(t)$, where χ_A is the characteristic function of the set A.

Second step: Differential inclusion for the approximate solution.

Obviously $x_n(.)$ is absolutely continuous with

$$\|\dot{x_n}(t)\| \leq 2M, a.e \quad t \in [0, \bar{T}], \tag{2.10}$$

and, putting

$$\begin{cases} \theta_n(\bar{T}) = \bar{T} \\ \theta_n(t) := t_k^n \quad if \quad t \in [t_k^n, t_{k+1}^n[\quad for \quad k \in 0, ..., n-1, \end{cases} \tag{2.11}$$

one has

$$\begin{cases} \dot{x}_n(t) \in -N_C(x_n(t)) + f_n(t), \quad x_n(0) = x_0, \quad a.e \quad t \in [0, \bar{T}] \\ x_n(t) \in B(x_0, \eta_0) \cap C; \quad \forall t \in [0, \bar{T}], \\ f_n(t) \in F(x_n(\theta_n(t))) \subset \partial^p g(x_n(\theta_n(t))) \subset M\bar{B}(0,1) \quad a.e \quad [0, \bar{T}], \end{cases} \tag{2.12}$$

and

$$\langle \dot{x}_n(t), \dot{x}_n(t) \rangle = \langle \dot{x}_n(t), f_n(t) \rangle \text{ a.e. } t \in [0, \bar{T}]. \tag{2.13}$$

Let us prove (2.13). Fix $t \in]0, \bar{T}[$ such that $\dot{x}_n(t)$ exists and $f_n(t) - \dot{x}_n(t) \neq 0$. (HC_1), (2.12) and the property of the normal cone on $C \cap B(x_0, \alpha)$ yield for any $u \in C$

$$\langle \frac{f_n(t) - \dot{x}_n(t)}{\|f_n(t) - \dot{x}_n(t)\|}, u - x_n(t) \rangle \leq \frac{1}{2r} \|u - x_n(t)\|^2.$$

Let δ be a real number with $0 < \delta < min\{t, \bar{T} - t\}$. By (2.12), $u =: x_n(t + \delta) \in C \cap B(x_0, \alpha)$ (note that $B(x_0, \eta_0) \subset B(x_0, \alpha)$). Then the last inequality gives

$$\langle \frac{f_n(t) - \dot{x}_n(t)}{\|f_n(t) - \dot{x}_n(t)\|}, x_n(t + \delta) - x_n(t) \rangle \leq \frac{1}{2r} \|x_n(t + \delta) - x_n(t)\|^2.$$

Taking successively $\delta > 0$ and $\delta < 0$, dividing by δ, and passing to the limit $\delta \to 0$ we successively obtain

$$\langle \frac{f_n(t) - \dot{x}_n(t)}{\|f_n(t) - \dot{x}_n(t)\|}, \dot{x}_n(t) \rangle \leq 0 \text{ and } \langle \frac{f_n(t) - \dot{x}_n(t)}{\|f_n(t) - \dot{x}_n(t)\|}, \dot{x}_n(t) \rangle \geq 0,$$

and hence (2.13).

Third step: Convergence of sequences.

We cannot use Arzela-Ascoli's theorem, as we do not know the relative compactness of the sets $\{x_n(t), n \in N^*\}$. However we use technique developed by Bounkhel and Thibault [15] to prove that $(x_n)_n$ is a Cauchy sequence in the space $C([0, \overline{T}], H)$.

Let us define $Z_n(t) := \int_0^t f_n(s)ds$. Then for all $t \in [0, \overline{T}]$ the set $\{Z_n(t), n \in N^*\}$ is contained in the strong compact set $\overline{T}\,\overline{co}(\{0\} \cup \mathcal{K})$ and so it is relatively strongly compact in H. Then by Arzela Asccoli's theorem we get the relative compactness of the set $\{Z_n, n \in N^*\}$ with respect to the uniform convergence in $C([0,\overline{T}], H)$ and so we may assume that without loss of generality that $(Z_n)_n$ converges uniformly to some mapping Z. As $\|f_n(t)\| \leq M$, we may suppose that $(f_n)_n$ converges weakly in $L^2([0,\overline{T}], H)$ to some mapping f. Then for all $t \in [0,\overline{T}]$

$$Z(t) = lim_n Z_n(t) = lim_n \int_0^t f_n(s)ds = \int_0^t f(t)dt.$$

So Z is absolutely continuous and $\dot{Z}(t) = f(t)$ for almost $t \in [0,\overline{T}]$.

Fix $m, n \in N^*$. For almost $t \in [0,\overline{T}]$, (2.12) and the property of the normal cone on $C \cap B(x_0, \alpha)$ yield for any $u \in C$

$$\langle \frac{f_n(t) - \dot{x_n}(t)}{\|f_n(t) - \dot{x_n}(t)\|}, u - x_n(t)\rangle \leq \frac{1}{2r}\|u - x_n(t)\|^2.$$

Then

$$\langle \frac{f_n(t) - \dot{x_n}(t)}{\|f_n(t) - \dot{x_n}(t)\|}, x_m(t) - x_n(t)\rangle \leq \frac{1}{2r}\|x_m(t) - x_n(t)\|^2,$$

or

$$\langle \dot{x_n}(t) - f_n(t), x_n(t) - x_m(t)\rangle \leq \frac{M}{2r}\|x_n(t) - x_m(t)\|^2. \tag{2.14}$$

Put $\varepsilon_{m,n} := \|Z_m - Z_n\|_\infty \to 0$, and $w_n(t) := x_n(t) - Z_n(t)$, we obtain

$$\begin{aligned}\langle \dot{w}_n(t), w_n(t) - w_m(t)\rangle &= \langle \dot{w}_n(t), x_n(t) - Z_n(t) - x_m(t) + Z_m(t)\rangle \\ &= \langle \dot{w}_n(t), x_n(t) - x_m(t)\rangle + \langle \dot{w}_n(t), Z_m(t) - Z_n(t)\rangle.\end{aligned}$$

From the latter equality and from (2.14) we note that

$$\langle \dot{w}_n(t), w_n(t) - w_m(t)\rangle \leq \frac{M}{2r}[\|w_n(t) - w_m(t)\| + \|Z_n(t) - Z_m(t)\|]^2 + M\|Z_n(t) - Z_m(t)\|,$$

or

$$\langle \dot{w}_n(t), w_n(t) - w_m(t)\rangle \leq \frac{M}{2r}[\|w_n(t) - w_m(t)\| + \varepsilon_{m,n}]^2 + M\varepsilon_{m,n}. \tag{2.15}$$

Repeating the same argument, we have

$$\langle \dot{w}_m(t), w_m(t) - w_n(t)\rangle \leq \frac{M}{2r}[\|w_n(t) - w_m(t)\| + \varepsilon_{m,n}]^2 + M\varepsilon_{m,n}. \tag{2.16}$$

Combining (2.15) and (2.16) we obtain

$$\langle \dot{w}_m(t) - \dot{w}_n(t), w_m(t) - w_n(t)\rangle \leq \frac{M}{r}[\|w_n(t) - w_m(t)\| + \varepsilon_{m,n}]^2 + 2M\varepsilon_{m,n}. \tag{2.17}$$

As $w_n(t) = w_n(0) + \int_0^t \dot{w}_n(s)ds = x_0 + \int_0^t [\dot{x}_n(s) - f_n(s)]ds$ then $\|w_n(t)\| \leq \|x_0\| + \overline{T}M$, and so

$$2\langle \dot{w}_m(t)-\dot{w}_n(t), w_m(t)-w_n(t)\rangle \leq \frac{2M}{r}\|w_n(t)-w_m(t)\|^2+\frac{2M}{r}\varepsilon_{m,n}^2+8\varepsilon_{m,n}\frac{M}{r}(\|x_0\|+\overline{T}M)+4M\varepsilon_{m,n}.$$

Consequently

$$\frac{d}{dt}(\|w_m(t) - w_n(t)\|^2) \leq \frac{2M}{r}\|w_n(t) - w_m(t)\|^2 + \gamma\varepsilon_{m,n},$$

where γ is some positive constant independent of m, n and t. As $\|w_m(0) - w_n(0)\| = 0$, the Gronwall inequality implies for almost t

$$\|w_m(t) - w_n(t)\|^2 \leq L\varepsilon_{m,n},$$

L is some positive constant independent of m, n and t. Hence $(w_n)_n$ converges uniformly to some mapping w and so $(x_n)_n$ converges uniformly to some mapping $x := w + Z$. Since $\|f_n(t)\| \leq M$ and $\|\dot{x}_n(t)\| \leq 2M$ a.e.on $[0, \bar{T}]$, we can assume without loss of generality that $(f_n)_{n\in\mathbb{N}^*}$ converges weakly in $L^2([0, \bar{T}], H)$ to $f \in L^2([0, \bar{T}], H)$ with $\|f(t)\| \leq M$ a.e $t \in [0, \bar{T}]$ and $(\dot{x}_n)_{n\in\mathbb{N}^*}$ converges weakly in $L^2([0, \bar{T}], H)$ to $\dot{u} \in L^2([0, \bar{T}], H)$. Also, we have

$$\theta_n(t) \to t \text{ uniformly on } [0, \bar{T}].$$

Fourth step: The limit function x is a solution of the continuous problem $\dot{x}(t) \in -N_C(x(t)) + \partial g(x(t))$.

First, we show that

$$f(t) \in \partial g(x(t)) \quad a.e \quad t \in [0, \bar{T}]. \tag{2.18}$$

To summarize, we know that

(a) $\{x_n(\theta_n(t)) : t \in [0, \bar{T}], n \in \mathbb{N}^*\} \subset \overline{B}(x_0, \eta_0)$

(b) $x([0, \bar{T}]) \subset \overline{B}(x_0, \eta_0)$ and $x_n(\theta_n(\cdot)) \to_{n\to\infty} x(\cdot)$ uniformly on $[0, \bar{T}]$,

(c) $\dot{x}_n \rightharpoonup \dot{x}$ weakly in $L^2([0, \bar{T}], H)$, and

(d) for each $n \in \mathbb{N}^*$,

$$f_n(t) \in F(x_n(\theta_n(t))) \subset \partial^p g(x_n(\theta_n(t))) \text{ for a.e. } t \in [0, \bar{T}].$$

Consequently, Theorem $VI - 4$ in [22] guarantees the inclusion $f(t) \in \overline{co}(F(x(t))) \subset \partial g(x(t)) \quad a.e \quad t \in [0, \bar{T}]$, and so (2.18).

Now we prove that $x(.)$ satisfies

$$\begin{cases} \dot{x}(t) \in -N_C(x(t)) + f(t), \quad a.e \quad t \in [0, \bar{T}] \\ x(t) \in B(x_0, \eta_0) \cap C; \quad \forall t \in [0, \bar{T}] \\ x(0) = x_0 \in C, \end{cases} \tag{2.19}$$

and

$$\langle \dot{x}(t), \dot{x}(t) \rangle = \langle \dot{x}(t), f(t) \rangle \text{ a.e. } t \in [0, \bar{T}]. \tag{2.20}$$

Sine $x_n(\cdot) \to x(\cdot)$ uniformly, passing to the limit and keeping in the mind that C is closed, we obtain $x(t) \in C$, for all $t \in [0, \bar{T}]$. Further, from the inequality

$$\| - \dot{x}_n(t) + f_n(t) \| \leq \| f_n(t) \| \leq M$$

and from (2.12) it follows for a.e. $t \in [0, \bar{T}]$

$$-\dot{x}_n(t) + f_n(t) \in N_C(x_n(t)) \cap \overline{B}(0, M) = M\partial d_C(x_n(t)). \tag{2.21}$$

Now, by Mazur's lemma, there exists a sequence $(\xi_n(.))$ which converges strongly in $L^2([0, \bar{T}], H)$ to $-\dot{x}(.) + f(.)$ with

$$\xi_n(.) \in co\{-\dot{x}_k + f_k : k \geq n\}. \tag{2.22}$$

Extracting a subsequence if necessary, we may suppose that

$$(\xi_n(t))_n \to -\dot{x}(t) + f(t) \text{ a.e.} t \in [0, \bar{T}].$$

Combining the inclusion (2.22) with the latter convergence, we obtain

$$-\dot{x}(t) + f(t) \in \cap_n \bar{co}\{-\dot{x}_k + f_k : k \geq n\} \text{ a.e.} t \in [0, \bar{T}].$$

Such an inclusion yield for almost every $t \in [0, \bar{T}]$ that

$$\langle \zeta, -\dot{x}(t) + f(t) \rangle \leq \inf_n \sup_{k \geq n} \langle \zeta, -\dot{x}_k(t) + f_k(t) \rangle \text{ for all } \zeta \in H.$$

Coming back to (2.21), it follows, for almost every $t \in [0, \bar{T}]$,

$$\begin{aligned} \langle \zeta, -\dot{x}(t) + f(t) \rangle &\leq M \limsup_n \sigma(\partial^p d_C(x_n(t)), \zeta) \\ &\leq M \limsup_n \sigma(\partial^C d_C(x_n(t)), \zeta) \quad for \quad all \quad \zeta \in H. \end{aligned}$$

Since $\sigma(\partial^C d_C(.), \zeta)$ is upper semicontinuous on H, the latter inequality entails for almost $t \in [0, \bar{T}]$,

$$\langle \zeta, -\dot{x}(t) + f(t) \rangle \leq M\sigma(\partial^C d_C(x(t)), \zeta) \quad for \quad all \quad \zeta \in H.$$

Hence, by the closedness and convexity of $\partial^C d_C(x(t))$, we obtain

$$-\dot{x}(t) + f(t) \in M\partial^C d_C(x(t)), \quad a.e \quad t \in [0, \bar{T}]. \tag{2.23}$$

The last inclusion and $x(t) \in C$ ensure (2.19). To prove (2.20), we proceed like in (2.13) by using (2.19).

Fifth step: The sequence $(\dot{x}_n)_n$ converges strongly in $L^2([0, \bar{T}], H)$ to $\dot{x}$.

By exploiting that g is $c - pln$, (2.13) and (2.20) we will show that the sequence $(\dot{x}_n)_n$ actually converges strongly in $L^2([0, \bar{T}], H)$ to $\dot{x}$.

Since $x : [0, \bar{T}] \to B(x_0, \eta_0)$ is absolutely continuous function and g is $M-$ Lipschitzean on $\overline{B}(x_0, \eta_0)$, the function $(g \circ x)(\cdot)$ is absolutely continuous. Fix any $t \in [0, \bar{T}]$ such that there exist both $\dot{x}(t)$ and $\frac{d}{dt}(g(x(t)))$, and consider any $\xi \in \partial^p g(x(t))$. Let h be a real positive number with $t+h \in [0, \bar{T}]$, then by (2.19) we have $x(t+h) \in B(x_0, \eta_0)$. By (HF_2), g is $c-pln$, on O ($\overline{B}(x_0, \eta_0) \subset O$), so

$$g(x(t+h)) - g(x(t)) \geq \langle \xi, x(t+h)) - x(t) \rangle - c(1 + \|\xi\|)\|x(t+h)) - x(t)\|^2.$$

Dividing by h and taking the limit $h \to 0^+$, we obtain

$$\frac{d}{dt}(g(x(t))) \geq \langle \xi, \dot{x}(t) \rangle.$$

We take h negative such that $t+h \in [0, \bar{T}]$, we get by the same argument

$$\frac{d}{dt}(g(x(t))) \leq \langle \xi, \dot{x}(t) \rangle,$$

and so

$$\frac{d}{dt}(g(x(t))) = \langle \xi, \dot{x}(t) \rangle.$$

In particular by (2.18), we have

$$\int_0^{\bar{T}} \langle f(s), \dot{x}(s) \rangle ds = g(x(\bar{T})) - g(x_0). \tag{2.24}$$

On one hand, by construction, recalling the $c-pln$ on the function g and (2.12), we have for all $k \in \{0, \ldots, n-1\}$,

$$\begin{aligned}
& g(x_n(t_{k+1}^n))) - g(x_n(t_k^n))) \\
& \geq \langle y_k^n, x_n(t_{k+1}^n) - x_n(t_k^n) \rangle - c(1 + \|y_k^n\|)\|x_n(t_{k+1}^n) - x_n(t_k^n)\|^2 \\
& = \langle y_k^n, \int_{t_k^n}^{t_{k+1}^n} \dot{x}_n(t) dt \rangle - c(1 + \|y_k^n\|)\| \int_{t_k^n}^{t_{k+1}^n} \dot{x}_n(t)\|^2 \\
& \geq \int_{t_k^n}^{t_{k+1}^n} \langle f_n(t), \dot{x}_n(t) \rangle dt - 4c(1+M)M^2(t_{k+1}^n - t_k^n)^2.
\end{aligned}$$

By adding, we obtain

$$\begin{aligned} & g(x_n(\overline{T})) - g(x_0) \\ & \geq \int_0^{\overline{T}} \langle f_n(t), \dot{x}_n(t) \rangle dt - \varepsilon_n \end{aligned} \tag{2.25}$$

with

$$\varepsilon_n = \frac{4c(1+M)M^2\overline{T}^2}{n} \to 0, \text{ as } n \to \infty.$$

Comparing (2.24) and (2.25), using the continuity of g in $x(\overline{T})$ and the uniform convergence of x_n to x, it follows that

$$\limsup_n \int_0^{\overline{T}} \langle f_n(t), \dot{x}_n(t) \rangle dt \leq \int_0^{\overline{T}} \langle f(t), \dot{x}(t) \rangle dt. \tag{2.26}$$

On the other hand, from (2.13) and (2.26) we get

$$\limsup_n \int_0^{\overline{T}} \langle \dot{x}_n(t), \dot{x}_n(t) \rangle dt = \limsup_n \int_0^{\overline{T}} \langle f_n(t), \dot{x}_n(t) \rangle dt \leq \int_0^{\overline{T}} \langle f(t), \dot{x}(t) \rangle dt. \tag{2.27}$$

Integrating (2.20) we have

$$\int_0^{\overline{T}} \langle \dot{x}(t), \dot{x}(t) \rangle dt = \int_0^{\overline{T}} \langle f(t), \dot{x}(t) \rangle dt. \tag{2.28}$$

Combining (2.27) and (2.28) we obtain

$$\limsup_n \int_0^{\overline{T}} \|\dot{x}_n(t)\|^2 dt \leq \int_0^{\overline{T}} \|\dot{x}(t)\|^2 dt.$$

By the weak lower semicontinuity of the norm, we deduce that

$$\lim_n \int_0^{\overline{T}} \|\dot{x}_n(t)\|^2 dt = \int_0^{\overline{T}} \|\dot{x}(t)\|^2 dt,$$

which implies that $(\dot{x}_n)_n$ converges to $\dot{x}$ in the strong topology of $L^2([0,\bar{T}], H)$. Therefore, there exists a subsequence, still denoted by $(\dot{x}_n)_n$ which converges point-wise a.e. to $\dot{x}$. Then, there is a Lebesgue negligible set $\mathcal{N} \subset [0, \bar{T}]$ such that for every $t \in [0, \bar{T}] \setminus \mathcal{N}$ on one hand, $(\dot{x}_n(t)) \to \dot{x}(t)$ strongly in H, on the other hand, the inclusions (2.25) hold true for each $n \in N^*$, i.e.

$$-\dot{x}_n(t) + f_n(t) \in M\partial d_C(x_n(t)). \tag{2.29}$$

Sixth step: The limit function x is a solution of the continuous problem $\dot{x}(t) \in -N_C(x(t)) + F(x(t))$.

Now, let us establish that

$$\dot{x}(t) \in -N_C(x(t)) + F(x(t)), \quad a.e \quad t \in [0, \bar{T}]. \tag{2.30}$$

Fix $t \in [0, \bar{T}] \setminus \mathcal{N}$ and let $\varepsilon > 0$. Recall that $x_n(\theta_n(t)) \to x(t)$ strongly in H. Then, from the upper semicontinuity of F, there exists $N(\varepsilon) \in N^*$ such that $F(x_n(\theta_n(t))) \subset F(x(t)) + \varepsilon\bar{B}(0,1)$ for $n \geq N_\varepsilon$. Define the set-valued mapping $G(x) := -M\partial d_C(x) + F(x(t)) + \varepsilon\bar{B}(0,1)$, $x \in H$. Note that G has a nonempty closed weakly compact values, and recalling by (HF_1) that F is upper semicontinuous with nonempty closed weakly compact values, then the graph of G is sequentially strongly-weakly closed.

We have

$$\dot{x}_n(t) \in G(x_n(t)). \tag{2.31}$$

Since $\dot{x}_n(t) \to \dot{x}(t)$ and $x_n(t) \to x(t)$, the inclusion (2.31) entails that

$$\dot{x}(t) \in G(x(t)).$$

Finally, as $\varepsilon > 0$ is arbitrary, the inclusion $x(t) \in C$ assures us that

$$\dot{x}(t) \in -M\partial^p d_C(x(t)) + F(x(t)) \subset -N_C(x(t)) + F(x(t)).$$

This completes the proof of Theorem 3.1. ■

2.4 Extension of our Result with a Moving Set

In the previous section, we have proved the existence result concerning the evolution problem (2.1). This section is devoted to extend this result to the case of a moving set $C(t)$. With the specific assumption about the displacement of the set $C(\cdot)$, we have to require a uniform $r-$prox-regularity (i.e. $\alpha = +\infty$) and not only a (r, α)-prox-regular one. Firstly, we give the analogue of Proposition 2.2.3 and (2.20) when $C(\cdot)$ is supposed to be a translation.

Proposition 2.4.1 *Let $a \in W^{1,2}([T_0, T], H)$ be a mapping such that $a(T_0) = 0$. Let C be an r-uniformly prox-regular subset of H and let $f \in L^2([T_0, T], H)$. Let $x(\cdot) : [T_0, T] \to H$ be the unique solution of the differential variational inequality (see, e.g. [34] for the existence and uniqueness of solution and (2.33))*

$$\dot{x}(t) \in -N_{C+a(t)}(x(t)) + f(t), \quad x(T_0) = x_0, a.e \quad t \in [T_0, T]. \tag{2.32}$$

Then, the following properties hold true

$$\|\dot{x}(t)\| \leq \|\dot{x}(t) - f(t)\| + \|f(t)\| \leq 2\|f(t)\| + \|\dot{a}(t)\| \quad a.e \quad t \in [T_0, T], \tag{2.33}$$

and

$$\langle \dot{x}(t), \dot{x}(t)\rangle = \langle \dot{x}(t), f(t)\rangle + \langle \dot{x}(t), \dot{a}(t)\rangle - \langle \dot{a}(t), f(t)\rangle \; a.e. \; t \in [T_0, T]. \tag{2.34}$$

Proof. Let us prove (2.34). Let $t \in [T_0, T]$ be a differentiability point of $\dot{x}(\cdot)$ and $\dot{a}(\cdot)$. Then there is some $\delta > 0$ such that

$$\langle f(t) - \dot{x}(t), x(s) - a(s) - x(t) + a(t)\rangle \leq \delta\|x(s) - a(s) - x(t) + a(t)\|^2, \text{ for all } s \in [T_0, T].$$

By dividing on $s - t$ an taking the limit as $s \downarrow t$ we derive that

$$\langle f(t) - \dot{x}(t), \dot{x}(t) - \dot{a}(t)\rangle \leq 0.$$

Repeating the preceding argument by considering $s \uparrow t$ yields similarly

$$\langle f(t) - \dot{x}(t), \dot{x}(t) - \dot{a}(t) \rangle \geq 0,$$

which gives (2.34). ■ Now we come to the extension for Theorem 2.3.1

Theorem 2.4.1 *Let H be a Hilbert space. Let $O \subset H$ is an open convex subset containing $\overline{B}(x_0, \rho)$ for some $\rho > 0$ and $F : O \to 2^H$ is an upper semicontinous set-valued mapping with nonempty weakly compact values for which there exists a proper lower semicontinous c-pln function $g : O \to \mathbb{R} \cup \{+\infty\}$ that is locally bounded from above on O such that*

$$F(x) \subset \partial^C g(x), \forall x \in O.$$

Let $a \in W^{1,2}([0,T], H)$ be a mapping such that $a(0) = 0$ and C be a nonempty r-uniformly prox-regular subset of H. We consider the set-valued mapping $C(\cdot)$ defined by $\forall t \in [0,T]$, $C(t) = C + a(t)$.
Then for all $x_0 \in C$, there exists $\overline{T} > 0$ such that the Cauchy problem

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + F(x(t)), \quad x(0) = x_0, a.e \quad t \in [0, \overline{T}], \tag{2.35}$$

has an absolutely continuous solution x and for all $t \in [0, \overline{T}]$, $x(t) \in C(t)$.

Proof. The proof is similar to the one of Theorem 2.3.1. Let $x_0 \in C$ and and let $g : O \to \mathbb{R} \cup \{+\infty\}$. Then, by proposition 2.2.1 there exist $M > 0$ such that g is Lipschitzean with Lipschitz constant M on $\overline{B}(x_0, \rho)$ and, since $F(x) \subset \partial^p g(x)$ it follows that F is bounded by M on $\overline{B}(x_0, \rho)$.
The suitable final time $\overline{T} > 0$ is taken in order that

$$\int_0^{\overline{T}} (\|\dot{a}(t)\| + 2M) dt \leq \rho. \tag{2.36}$$

By using Proposition 2.4.1, we build the same sequence $x_n(.)$, which satisfies:

$$\begin{cases} \dot{x}_n(t) \in -N_C(x_n(t)) + f_n(t), \quad x_n(0) = x_0, \quad a.e \quad t \in [0, \bar{T}] \\ \|\dot{x}_n(t)\| \leq 2M + a(t), a.e \quad t \in [0, \bar{T}] \\ f_n(t) \in F(x_n(\theta_n(t))) \subset \partial^p g(x_n(\theta_n(t))) \subset M\bar{B}(0,1) \quad a.e \quad [0, \bar{T}], \end{cases} \tag{2.37}$$

and

$$\langle \dot{x}_n(t), \dot{x}_n(t) \rangle = \langle \dot{x}_n(t), f_n(t) \rangle + \langle \dot{x}_n(t), \dot{a}(t) \rangle - \langle \dot{a}(t), f_n(t) \rangle \text{ a.e. } t \in [0, \bar{T}]. \tag{2.38}$$

Then we finish the proof as previously, in applying (2.34) and (2.38) to obtain

$$\limsup_n \int_0^{\overline{T}} \|\dot{x}_n(t)\|^2 dt \leq \int_0^{\overline{T}} \|\dot{x}(t)\|^2 dt, \tag{2.39}$$

which is the key-point of the proof. Indeed, the $c-$pln regularity on the function g gives

$$\limsup_n \int_0^{\overline{T}} \langle f_n(t), \dot{x}_n(t) \rangle dt \leq \int_0^{\overline{T}} \langle f(t), \dot{x}(t) \rangle dt. \tag{2.40}$$

On the other hand, from (2.38) and (2.40) we get

$$\limsup_n \int_0^{\overline{T}} \langle \dot{x}_n(t), \dot{x}_n(t) \rangle dt = \limsup_n \int_0^{\overline{T}} \langle \dot{x}_n(t), f_n(t) \rangle dt + \int_0^{\overline{T}} \langle \dot{x}(t), \dot{a}(t) \rangle dt - \int_0^{\overline{T}} \langle \dot{a}(t), f(t) \rangle dt \tag{2.41}$$

$$\leq \int_0^{\overline{T}} \langle f(t), \dot{x}(t) \rangle dt + \int_0^{\overline{T}} \langle \dot{x}(t), \dot{a}(t) \rangle dt - \int_0^{\overline{T}} \langle \dot{a}(t), f(t) \rangle dt. \tag{2.42}$$

From the last inequality and from (2.34) we infer that

$$\limsup_n \int_0^{\overline{T}} \|\dot{x}_n(t)\|^2 dt \leq \int_0^{\overline{T}} \|\dot{x}(t)\|^2 dt.$$

Therefore, by the weak lower semicontinuity of the norm, we deduce that

$$\lim_n \int_0^{\overline{T}} \|\dot{x}_n(t)\|^2 dt = \int_0^{\overline{T}} \|\dot{x}(t)\|^2 dt.$$

This completes the strong convergence of the sequence $(\dot{x}_n)_n$ to $\dot{x}$ in $L^2([0,\bar{T}],H)$ and we finish to show the existence of solutions in the same way as for Theorem 2.3.1. ■

2.5 Conclusions

In this chapter, the existence of local solutions for a class of evolution equations of variational type, defined by a sweeping processes and a set-valued map with nonconvex values in Hilbert space, has been studied carefully. It is remarkable that the fixed set of constraints is possibly noncompact and satisfies the weaker assumption than uniform prox-regularity, namely a quantified viewpoint of local prox-regularity. Also, an existence result for the particular case of a shifted moving set is considered.

CHAPTER 3

A MAXIMUM PRINCIPLE FOR THE CONTROLLED SWEEPING PROCESS

3.1 Introduction

Moreau's sweeping process appears as a model in several contexts and is being studied from the theoretical viewpoint since the early Seventies of last Century. The main subject of investigation continues to be the existence of solutions, under increasing degrees of generality.

Essentially, the sweeping process is an evolution differential inclusion, which models the displacement of a point subject to be dragged by a moving set in a direction normal to its boundary. Formally, the (perturbed) sweeping process is the differential inclusion

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t)), \quad t \in [0,T], \tag{3.1}$$

coupled with the initial condition

$$x(0) = x_0 \in C(0), \tag{3.2}$$

where $C(t)$ is a closed moving set, with normal cone $N_{C(t)}(x)$ at $x \in C(t)$, and the space variable belongs to a Hilbert space (to $\mathbb{R}^n$ in the present paper). If $C(t)$ is convex, or mildly non-convex (i.e., uniformly prox-regular), and is Lipschitz as a set-valued map depending on the time t, and the perturbation f is Lipschitz, then it is well known that the Cauchy problem (3.1), (3.2) admits one and only one Lipschitz solution (see, e.g., [61]). Observe that the state constraint $x(t) \in C(t)$ for all $t \in [0, T]$ is built in the dynamics, being $N_{C(t)}(x)$ empty if $x \notin C(t)$.

The present paper deals with the problem of determining necessary conditions for global minimizers of a final cost $h(x(T))$, subject to the finite dimensional controlled sweeping dynamics

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t), u(t)), \quad x(0) = x_0 \in C(0), \quad u(t) \in U, \quad t \in [0, T], \tag{3.3}$$

U being the control set and f being smooth. Given a global minimizer, we prove that for a suitable adjoint vector, which is a BV function that satisfies a natural ODE in the sense of distributions together with the usual final time transversality condition, a version of Pontryagin's Maximum Principle holds. To keep technicalities at a minimum, we do not add (further) endpoint constraints and require the final cost to be smooth. We believe that our arguments can be adapted to such more general cases, including also problems of Bolza type. Our main assumptions are the smoothness of the boundary of the moving set $C(t)$ and, more importantly, a kind of outward/inward pointing condition on $f(x_*(t), u_*(t))$ at all times t where the optimal trajectory $x_*(t)$ belongs to the boundary of $C(t)$ (see (M_1) or (M_2) below). This strong requirement is assumed in order to handle the discontinuity of the gradient of the distance $d_{C(t)}(\cdot)$ to the set $C(t)$ at boundary points. In fact, the main difficulty to be overcome in the study of necessary conditions for optimal control problems subject to (3.3) is the severe lack of Lipschitz continuity of the normal cone mapping at boundary points of $C(t)$. The outward/inward pointing condition on f indeed permits to confine this issue at a negligible time set.

Control problems driven by a dynamics which involves the sweeping process appeared rather recently. Not mentioning some works scattered in the mechanical engineering literature, some early theoretical results appeared in [56] (a Hamilton-Jacobi characterization of the value function, with C constant, later generalized in [31]) and in [53, 54] (existence and discrete approximation of optimal controls, in the related framework of rate independent processes). More recently, the papers [28, 29, 30] are devoted to the case where the control acts on the moving set, which in turn is required to have a polyhedral structure. In particular, [30] contains a set of necessary conditions for local minima which are derived by passing to the limit along suitable discrete approximations. Some partial results on necessary conditions for an optimal control problem acting on the perturbation f were obtained in [58], while the first complete achievement of this type appeared in [16]. The present paper owes to [16] several ideas. The problem studied in [16] involves a controlled ODE, coupled with a sweeping process with a constant sweeping set C. An adjoint equation together with Pontryagin's Maximum Principle are derived by passing to the limit along suitable Moreau-Yosida approximations, by penalizing also the L^2 distance to the reference control. The set C is required to be both smooth and uniformly convex and the limiting argument requires the minimizer to be a *global* one. The dynamics considered in [16] is different from (3.3), but the main difficulty – namely the discontinuity of $\nabla d_C(\cdot)$ at boundary points – is exactly the same. In [16], this issue is solved by imposing enough smoothness on ∂C and via a smooth extension of d_C up to the interior of $C(t)$. This method, however, requires the constancy and the uniform convexity of C, which in turn yields the coercivity of the Hessian of the (modified) squared distance. This last property is important to obtain a uniform L^1 bound on a sequence of approximate adjoint vectors, which provides compactness in the space of BV functions of the time variable. Our contribution is in modifying the method developed in [16] – through seemingly simpler estimates on the distance d_C based on [55] – in order to drop the requirement of uniform convexity. The price to pay is the inward/outward pointing assumption (M_1) or (M_2). A simple example permits to test our necessary conditions.

The recent results contained in [20, 21] are also worth being mentioned. In such papers the control acts both on the moving set, which is required to be polyhedral, and on the perturbation f. The problem studied is on one hand more general, and the class of minimizers to which the necessary conditions apply is larger, on the other the method requires some extra regularity assumptions on the optimal trajectory. Also, in contrast with our approach, the moving set is allowed to be nonsmooth, but its generality is weakened by the requirement to be a polyhedron. This happens because of the need of computing explicitly the coderivative of the normal cone mapping. Indeed, the method used in [20, 21] is *completely different* from the one adopted in the present paper, as it relies on passing to the limit along a suitable sequence of discrete approximations of the reference optimal trajectory. The necessary conditions obtained in [21] include a kind of adjoint equation, transversality conditions both at the initial and at the final point, and nontriviality conditions, but do not include a maximum principle. General existence and relaxation results for optimal control problems of the same nature of those investigated in [20, 21] appear in [62].

Finally, let us mention that H. Sussmann devoted a lot of work to establish the Maximum Principle in high generality, including possibly discontinuous vector fields (see, e.g., [59]). Here we rely on the special structure of the right hand side of (3.3), and develop an *ad hoc* method.

In what follows, Section 3.3 contains the statement of overall assumptions and of the main result, while Sections 3.4 to 3.7 are devoted to the proofs. In Section 3.8 an example is presented and discussed.

3.2 Preliminaries

We will consider all vectors in a finite dimensional space as column vectors.

Let $C \subset \mathbb{R}^n$ be nonempty and closed. We denote by $d_C(x)$ the distance of x from C,

$d_C(x) := \inf\{|y - x| : y \in C\}$, and the metric projection of x onto C is the set of points in C which realize the infimum. Should this set be a singleton, we denote this point by $\mathrm{proj}_C(x)$. Given $\rho > 0$, we set

$$C_\rho := \{x : d_C(x) < \rho\}.$$

Prox-regular sets will play an important role in the sequel. The definition was first given by Federer, under the name of *sets with positive reach*, and later studied by several authors (see the survey paper [32]). We give only the definition for *smooth* sets, because the general case will not be relevant here. All definitions of tangent and normal cones may be found in [26], to which we refer for all concepts of nonsmooth analysis that will be touched within this paper. By a smooth set we mean a closed set C in $\mathbb{R}^n$ whose boundary is an embedded manifold of dimension $n - 1$. In this case, the tangent cone is actually an $n - 1$-dimensional vector space and the normal cone is a half ray (or a line if C has empty interior). In particular, we will consider sets whose boundary can be described as the zero set of a smooth function (at least of class $C^{1,1}$, namely of class C^1 with Lipschitz partial derivatives), with nonvanishing gradient.

Definition 3.2.1 *Let $C \subset \mathbb{R}^n$ be a closed smooth set and $\rho > 0$ be given. We say that C is ρ-prox-regular provided the inequality*

$$\langle \zeta, y - x \rangle \leq \frac{|y - x|^2}{2\rho} \tag{3.4}$$

holds for all $x, y \in C$, where ζ is the unit external normal to C at $x \in \partial C$.

In particular, every convex set is ρ-prox regular for every $\rho > 0$ and every set with a $C^{1,1}$-boundary is ρ-prox regular, where ρ depends only the Lipschitz constant of the gradient of the parametrization of the boundary (see [32, Example 64]). In this case, the (proximal) normal cone to C at $x \in C$ is the nonnegative half ray generated by the unit external normal, and

$$v \in N_C(x) \text{ if and only if there exists } \sigma > 0 \text{ such that } \langle v, y - x \rangle \leq \sigma |y - x|^2 \quad \forall y \in C.$$

Prox-regular sets enjoy several properties, including uniqueness of the metric projection and differentiability of the distance (in a suitable neighborhood) and normal regularity, which hold also true for convex sets, see, e.g. [32]. We state the main properties which we are going to use in the present paper.

Proposition 3.2.1 *Let $\rho > 0$ be given and let $C \subset \mathbb{R}^n$ be ρ-prox-regular. Then d_C is differentiable on $C_\rho \setminus C$, and*

$$\nabla d_C(x) = (x - \mathrm{proj}_C(x))/d_C(x) \ \text{ for all } x \in C_\rho \setminus C.$$

Moreover, ∇d_C is Lipschitz with Lipschitz constant 2 in $C_{\frac{\rho}{2}} \setminus C$. Finally, proj_C is well defined and is Lipschitz with Lipschitz constant 2 in $C_{\frac{\rho}{2}}$.

The proof of this Proposition can be found, e.g., in [32].

The fact that the distance from a ρ-prox-regular set C is of class $\mathcal{C}^{1,1}$ in $C_{\frac{\rho}{2}} \setminus C$ will play a fundamental role in the sequel. In particular, given a moving closed set $C(t)$, $t \in [0, T]$, we wish to discuss the differentiability of the Lipschitz map

$$x \mapsto x - \mathrm{proj}_{C(t)}(x) := P(t, x), \tag{3.5}$$

namely of the gradient ∇_x of $\frac{1}{2}d^2_{C(t)}(x)$, with respect to the state variable x, in the case where the boundary of $C(t)$ is an immersed manifold of class $\mathcal{C}^2$. Then it is well known that $d^2_{C(t)}(\cdot)$ is of class $\mathcal{C}^1$ in $C_\rho(t)$ and of class $\mathcal{C}^2$ in $C_\rho(t) \setminus C(t)$, where ρ depends only on the global lower bound of the curvature of $\partial C(t)$, see, e.g., [3, Theorem 3.1]. If $x \in \mathrm{int}\, C(t)$, then $P(t, x)$ vanishes in a neighborhood of x and so it is differentiable in the classical sense, with zero Jacobian. If $x \notin C(t)$, then $P(t, x) = d_{C(t)}(x)\nabla_x d_{C(t)}(x)$, so that

$$\nabla_x P(t, x) = d_{C(t)}(x)\nabla^2_x d_{C(t)}(x) + \nabla_x d_{C(t)}(x) \otimes \nabla_x d_{C(t)}(x) =: P_x(t, x), \tag{3.6}$$

where we recall that if v and w are column vectors, then $v \otimes w$ denotes the matrix $v\, w^\top$ and we denote by ∇^2_x the Hessian with respect to the state variable x.

Denote by $n(t,x)$ the unit external normal to $C(t)$ at x, if $x \in \partial C(t)$, and 0 if $x \in \operatorname{int} C(t)$. We will adopt the following convention:

$$\text{if } x \in \partial C(t), \text{ by writing } \nabla_x d_{C(t)}(x) \text{ we mean } n(t,x).$$

With this notation, one can extend $\nabla_x P(t,x)$ also to $x \in \partial C(t)$, by using (3.6). Of course, this does not mean that P is differentiable at $\partial C(t)$: the Clarke generalized gradient (with respect to x) of $P(t,x)$ at $x \in \partial C(t)$ is $\partial_x P(t,x) = \operatorname{co}\{0, \nabla_x P(t,x)\}$.

We will consider also the signed distance

$$d_S(t,x) := \begin{cases} d_{C(t)}(x) & \text{if } x \in C^c(t) \\ -d_{C(t)^c}(x) & \text{if } x \in C(t), \end{cases}$$

where $C(t)^c$ denotes the complement of $C(t)$. It is well known that $d_S(t,\cdot)$ is of class $\mathcal{C}^2$ around $\partial C(t)$ (see, e.g., Proposition 2.2.2 (iii) in [17]) if $\partial C(t)$ is a manifold of class $\mathcal{C}^2$ and $C(t)$ has nonempty interior. It is also easy to see that $d_S(\cdot,x)$ is Lipschitz if $C(\cdot)$ is so.

3.3 Standing assumptions and statement of the main results

The following assumptions will be valid throughout the paper.

(H_1): $C : [0,\infty) \rightsquigarrow \mathbb{R}^n$ is a set-valued map with the following properties:

- $(H_{1.1})$: for all $t \in [0,T]$, $C(t)$ is nonempty and compact and there exists $\rho > 0$ such that $C(t)$ is ρ-prox regular. Moreover, $C(t)$ has a C^3-boundary.
- $(H_{1.2})$: $C(\cdot)$ is Lipschitz, with Lipschitz constant γ.

(H_2): $U \subset \mathbb{R}^m$ is compact and convex.

(H_3): $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a single valued map with the following properties:

$(H_{3.1})$: that there exist $\beta \geq 0$ such that $|f(x,u)| \leq \beta$ for all (x,u);

$(H_{3.2})$: $f(\cdot,\cdot)$ is of class C^1;

$(H_{3.3})$: $f(\cdot,\cdot)$ is Lipschitz with constant k;

$(H_{3.4})$: $f(x,U)$ is convex for all $x \in \mathbb{R}^n$;

(H_4): $h : \mathbb{R}^n \to \mathbb{R}$ is of class C^1.

If $C(t) = \{x : g(t,x) \leq 0\}$, with $g(\cdot,x)$ Lipschitz, $g(t,\cdot)$ of class $C^{2,1}$ is possible to impose conditions directly on the map g in order to let $(H_{1.1})$ and $(H_{1.2})$ hold. This is discussed in detail in [2].

We are interested in determining necessary conditions for solutions of the following minimization problem, that we call *Problem* (P):

Minimize $h(x(T))$ subject to

$$\begin{cases} \dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t),u(t))), \\ x(0) = x_0 \in C(0) \,, \end{cases} \tag{3.7}$$

with respect to $u : [0,T] \to U$, u measurable (labeled as *admissible control*).

Let us recall that the dynamics (3.7) implicitly contains the state constraint

$$x(t) \in C(t) \qquad \forall t \in [0,T].$$

Existence of minimizers for (P) can be obtained by standard methods (even under less stringent assumptions on f), essentially thanks to the graph closedness of the normal cone to a prox-regular set (see, e.g., [32, Proposition 7]).

Let (x_*,u_*) be a minimizer. We will impose an *outward (resp. inward) pointing condition* on $f(x_*(t),u_*(t))$ with respect to the boundary of $C(t)$. To this aim, we introduce the (possibly empty) set

$$I_\partial := \{t \in [0,T] : x_*(t) \in \partial C(t)\} \tag{3.8}$$

and require that there exists $\sigma > 0$ for which either

$$\frac{\partial d_S}{\partial t}(t, x_*(t)) + \langle \nabla_x d_S(t, x_*(t)), f(x_*(t), u) \rangle \geq \sigma \quad \text{for a.e. } t \in I_\partial \text{ and for all } u \in U \qquad (M_1)$$

or

$$\frac{\partial d_S}{\partial t}(t, x_*(t)) + \langle \nabla_x d_S(t, x_*(t)), f(x_*(t), u) \rangle \leq -\sigma \quad \text{for a.e. } t \in I_\partial \text{ and for all } u \in U \qquad (M_2)$$

hold, where we recall that $d_S(t, x)$ denotes the signed distance between x and $C(t)$. Of course, if $I_\partial = \emptyset$ both conditions are automatically satisfied.

Remark 3.3.1 *More in general, we can assume that $[0, T]$ can be split into finitely many subintervals such that I_∂ does not contain their end points and in each subinterval either (M_1) or (M_2) holds. Without loss of generality, the proofs will be carried out in the case where we have only one interval and either (M_1) or (M_2) hold.*

Before stating the main result of the paper, we recall that in Section 3.2 we have given a meaning to

$$\nabla_x d_{C(t)}(x_*(t)) \qquad \text{and} \qquad \nabla_x^2 d_{C(t)}(x_*(t))$$

also for $t \in I_\partial$.

Theorem 3.3.2 *Assume that $(H_1), \ldots, (H_4)$ hold and consider the minimization problem* (3.7). *Let (x_*, u_*) be a global minimizer for which either (M_1) or (M_2) are valid. Then there exist a BV adjoint vector $p : [0, T] \to \mathbb{R}^n$, a finite signed Radon measure μ on $[0, T]$, and measurable vectors $\xi, \eta : [0, T] \to \mathbb{R}^n$, with $\xi(t) \in L^1_\mu(0, T)$, $\xi(t) \geq 0$ for μ-a.e. t and $0 \leq \eta(t) \leq \beta + \gamma$ for a.e. t, satisfying the following properties:*

• *(adjoint equation)* *for all continuous functions* $\varphi : [0,T] \to \mathbb{R}^n$

$$
\begin{aligned}
-\int_{[0,T]} \langle \varphi(t), dp(t) \rangle = & -\int_{[0,T]} \langle \varphi(t), \nabla_x d_{C(t)}(x_*(t)) \rangle \xi(t)\, d\mu(t) \\
& - \int_{[0,T]} \langle \varphi(t), \nabla_x^2 d_{C(t)}(x_*(t)) p(t) \rangle \eta(t)\, dt \\
& + \int_{[0,T]} \langle \varphi(t), \nabla_x f(x_*(t), u_*(t)) p(t) \rangle\, dt,
\end{aligned} \tag{3.9}
$$

• *(transversality condition)* $-p(T) = \nabla h(x_*(T))$,

• *(maximality condition)*

$$
\langle p(t), \nabla_u f(x_*(t), u_*(t)) u_*(t) \rangle = \max_{w \in U} \langle p(t), \nabla_u f(x_*(t), u_*(t)) w \rangle \quad \textit{for a.e. } t \in [0,T]. \tag{3.10}
$$

Further conditions, in particular on discontinuities of p or, equivalently, on Dirac masses for μ, will be discussed in Proposition 3.7.3 below. Here we observe only that on $I_0 := [0,T] \setminus I_\partial$, namely on the (possibly empty) set where $x_*(t)$ belongs to the interior of $C(t)$, p is absolutely continuous and satisfies the classical adjoint equation

$$
-\dot{p}(t) = \nabla_x f(x_*(t), u_*(t))\, p(t) \quad \text{a.e.},
$$

so that μ, ξ, and η do not play any role on that set. This is a simple consequence of (3.9).

The proof of Theorem 3.3.2 is contained in Sections 3.6 and 3.7 and is divided into several propositions, containing estimates on a sequence of adjoint vectors. Sections 3.4 and 3.5 are devoted to estimates on solutions to suitable approximations of the primal problem.

3.4 Results on the sweeping process and its regularization

Set $n(t,x)$ to be the unit external normal to $C(t)$ at $x \in \partial C(t)$ and 0 if $x \in \operatorname{int} C(t)$, and not defined if $x \notin C(t)$. Observe that $n(t,x) = \nabla_x d_S(t,x)$ for all $x \in \partial C(t)$.

For any solution x_* of (3.7), the general theory on the sweeping process (see, e.g., [61, Theorem 3.1]), yields that

$$|\dot{x}_*(t)| \leq \gamma + \beta \ \text{ for a.e } t \in [0,T]. \tag{3.11}$$

The main tool that we are going to use is an approximate control problem, where the dynamics is the Moreau-Yosida regularization of (3.7) and the cost is the original one, plus a penalization term. More precisely, the approximate problem is the following one:

For a given $\varepsilon > 0$ and a given admissible control u_*,

$$\text{Minimize} \qquad h(x(T)) + \frac{1}{2}\int_0^T |u(t) - u_*(t)|^2 dt \tag{3.12}$$

subject to

$$\dot{x}(t) \ = \ -\frac{1}{\varepsilon}\Big(x(t) - \operatorname{proj}_{C(t)}(x(t))\Big) + f(x(t), u(t)), \quad x(0) = x_0 \in C(0), \tag{3.13}$$

over all admissible controls $u : [0,T] \to U$.

We label the above problem as $(P_\varepsilon(u_*))$. By standard results, for every $\varepsilon > 0$ there exists a global minimizer u_ε. If u_ε is such a minimizer and x_ε is the solution of (3.13) with u_ε in place of u, we will refer to $(x_\varepsilon, u_\varepsilon)$ as an optimal couple for $(P_\varepsilon(u_*))$.

As a preliminary result on $(P_\varepsilon(u_*))$, we are going to prove that, thanks to the Lipschitz continuity of the metric projection onto $C(t)$ on the set C_ρ for each t, and the boundedness of the Lipschitz perturbation f, the Cauchy problem (3.13) admits one and only one solution on the interval $[0, T]$, for every fixed admissible control u. Our first result is in fact concerned with existence, uniqueness and some estimates on such solutions, uniform with respect to ε.

Proposition 3.4.1 *Let C, f, U, h be given satisfying assumptions $(H_1), \dots, (H_4)$. Let $\varepsilon_n \downarrow 0$ and let $\{u_n\}$ be a sequence of admissible controls. Then, for every n large enough, the problem* (3.13) *with ε_n in place of ε and u_n in place of u admits one and only one solution x_n on the interval* $[0, T]$*. Such solutions are Lipschitz uniformly with respect to n, with Lipschitz constant $\gamma + 2\beta$. Moreover the following estimates hold:*
for all $t_0 \in [0, T]$ and all $t \in [t_0, T]$

$$d_{C(t)}(x_n(t)) \le d_{C(t_0)}(x_n(t_0))e^{-\frac{t-t_0}{\varepsilon_n}} + \varepsilon_n(\beta + \gamma)\Big(1 - e^{-\frac{t-t_0}{\varepsilon_n}}\Big), \tag{3.14}$$

so that, in particular,

$$d_{C(t)}(x_n(t)) \le \varepsilon_n(\beta + \gamma) \quad \textit{for all } \; t \in [0, T]. \tag{3.15}$$

The proof of Proposition 3.4.1 follows the arguments developed in [55, Section 3] and will be sketched after some technical results.

First of all, let $x(t)$ be absolutely continuous and set $g(t) := d_{C(t)}(x(t))$. Recalling Lemma 3.1 in [55], we have, for a.e. $t \in [0, T]$,

$$\dot{g}(t)g(t) \le \langle \dot{x}(t), x(t) - \mathrm{proj}_{C(t)}(x(t))\rangle + \gamma g(t), \tag{3.16}$$

provided

$$d_{C(t)}(x(t)) < \rho \qquad \text{for all } \; t \in [0, T]. \tag{3.17}$$

Therefore, we obtain immediately the following Lemma.

Lemma 3.4.1 *For every $\varepsilon > 0$, let the admissible control u_ε be given and let x_ε be the corre-*

sponding solution of (3.13). *Set*

$$g_\varepsilon(t) = d_{C(t)}(x_\varepsilon(t)), \qquad t \in [0, T]$$

and assume that x_ε satisfies (3.17). *Then the estimates* (3.14) *and* (3.15) *hold, with $g_\varepsilon(t)$ in place of $d_{C(t)}(x_n(t))$ and ε in place of ε_n.*

Proof. From (3.16) we obtain

$$\dot{g}_\varepsilon(t) g_\varepsilon(t) \leq \gamma g_\varepsilon(t) - \frac{1}{\varepsilon} g_\varepsilon^2(t) + g_\varepsilon(t) |f(x_\varepsilon(t), u_\varepsilon(t))|,$$

which yields, if $g_\varepsilon(t) > 0$,

$$\dot{g}_\varepsilon(t) \leq -\frac{1}{\varepsilon} g_\varepsilon(t) + \gamma + \beta.$$

The case $g_\varepsilon(t) = 0$ can be treated exactly as in the proof of [55, Lemma 3.3]. Then the result follows from Gronwall's lemma. ∎

Proof of Proposition 3.4.1. The proof is divided into two steps. First, we assume that the final time T is small enough, namely $0 < T \leq \theta$, with

$$\theta < \frac{\rho}{3(2\beta + \gamma)}, \tag{3.18}$$

where we recall that the constants ρ, β, and γ appear in the standing assumptions $(H_{1.1})$, $(H_{1.2})$, and $(H_{3.1})$. Second, the general case will be treated.

Assume now that $T \leq \theta$ and let $\varepsilon_n \downarrow 0$ and a sequence $\{u_n\}$ of admissible controls be given. It is clear that a solution x_n of (3.13), with ε_n, resp. u_n, in place of ε, resp. u, exists and is defined on its maximal interval of existence $[0, T_n] \subseteq [0, T]$ such that $d_{C(t)}(x_n(t)) < \rho$ for all $t \in [0, T_n]$. We have from Lemma 3.4.1 that (3.15) holds on $[0, T_n]$ and so the solution x_n is unique by a standard application of Gronwall's lemma. It is also easy to see, arguing as in [55, Sect. 3], that the maximal interval of existence must be the whole of $[0,T]$. Moreover, we have for all n

$$|\dot{x}_n(t) - f(x_n(t), u_n(t))| = \left| \frac{x_n(t) - \mathrm{proj}_{C(t)}(x_n(t))}{\varepsilon_n} \right| = \frac{1}{\varepsilon_n} d_{C(t)}(x_n) \leq \gamma + \beta \quad \forall t \in [0,T],$$

from which the conclusion on the Lipschitz constant of $\dot{x}_n$ follows immediately, since f is uniformly bounded by β.

Consider now the general case. From the preceding argument, for each n there exists a solution x_n such that, for all $t \in [0,\theta]$,

$$d_{C(t)}(x_n(t)) \leq \varepsilon_n(\beta + \gamma). \tag{3.19}$$

For every n large enough, we can assume that

$$\varepsilon_n(\beta + \gamma) + \theta < \frac{\rho}{3(2\beta + \gamma)}.$$

Applying the argument used in the preceding step to the Cauchy problem

$$\begin{cases} \dot{x}(t) & = -\frac{1}{\varepsilon_n}\Big(x(t) - \mathrm{proj}_{C(t)}(x(t))\Big) + f(x(t), u_n(t)), \\ x(0) & = x_n(\theta), \end{cases}$$

we can extend x_n, keeping the property (3.19), up to the time 2θ. Since θ is independent of n, the interval $[0,T]$ can be covered after finitely many steps. ■

Our second result is concerned with compactness and passing to the limit for solutions of $(P_\varepsilon(u_*))$, as $\varepsilon \to 0$.

Proposition 3.4.2 *Let u_* be a global minimizer for the problem (P), together with the corresponding solution x_* of (3.7). Let $(x_\varepsilon, u_\varepsilon)$ be an optimal couple for the regularized minimization problem $(P_\varepsilon(u_*))$. Then there exists a sequence $\varepsilon_n \downarrow 0$ such that*

$$x_{\varepsilon_n} \to x_* \text{ weakly in } W^{1,2}([0,T];\mathbb{R}^n),$$
$$u_{\varepsilon_n} \to u_* \text{ strongly in } L^2([0,T];\mathbb{R}^m).$$

Proof. By Proposition 3.4.1 and assumptions (H_2) and $(H_{3.1})$, we find a sequence $\varepsilon_n \downarrow 0$ and an admissible control $\tilde{u}$ such that

$$\begin{aligned} &x_n := x_{\varepsilon_n} \text{ converges weakly in } W^{1,2}([0,T];\mathbb{R}^n) \text{ to some } x, \\ &u_n := u_{\varepsilon_n} \text{ converges weakly in } L^2([0,T];\mathbb{R}^n) \text{ to } \tilde{u}, \\ &\int_0^T |u_n(t) - u_*(t)|^2\,dt \quad \text{converges to some} \quad \delta \geq 0, \end{aligned} \tag{3.20}$$

and moreover (3.15) holds for every n. Observe that (3.15) implies in turn that $\mathrm{proj}_{C(t)}(x_n(t))$ is well defined and also

$$x_n(t) - \mathrm{proj}_{C(t)}(x_n(t)) \in N_{C(t)}\Big(\mathrm{proj}_{C(t)}(x_n(t))\Big) \tag{3.21}$$

for each $t \in [0,T]$ and $n \in \mathbb{N}$.

Let us prove first that there exists an admissible control $\bar{u}$ such that

$$\dot{x}(t) \ \in \ -N_{C(t)}(x(t)) + f(x(t), \bar{u}(t)) \ \text{ a.e. on } \ [0,T]. \tag{3.22}$$

Indeed, from

$$-\dot{x}_n(t) = \frac{x_n(t) - \mathrm{proj}_{C(t)}(x_n(t))}{\varepsilon_n} - f(x_n(t), u_n(t))$$

and (3.4), (3.21), (3.15) it follows immediately that

$$\langle -\dot{x}_n(t) + f(x_n(t), u_n(t)), y - \mathrm{proj}_{C(t)}(x_n(t))\rangle \leq \frac{\gamma+\beta}{2\rho}|y - \mathrm{proj}_{C(t)}(x_n(t))|^2 \quad \forall y \in C(t). \tag{3.23}$$

First we see that the uniform convergence of x_n to x implies by passing to the limit in (3.15) that $x(t) \in C(t)$ for all $t \in [0,T]$. Furthermore, by possibly taking a subsequence we may assume that $z_n := f(x_n, u_n)$ converges weakly in $L^2([0,T];\mathbb{R}^n)$ to some z, and by Mazur's lemma we can find a convex combination $\sum_{k=n}^{r(n)} S_{k,n}(-\dot{x}_k + z_k)$, with $\sum_{k=n}^{r(n)} S_{k,n} = 1$ and $S_{k,n} \in [0,1]$ for all k, n, which converges strongly in L^2 and pointwise a.e. to $-\dot{x} + z$.

Let now $t \in [0,T]$ and $y \in C(t)$. We have

$$
\begin{aligned}
\langle -\dot{x}(t) + z(t), y - x(t)\rangle = \Big\langle -\dot{x}(t) + z(t) - \sum_{k=n}^{r(n)} S_{k,n}(-\dot{x}_k(t) + z_k(t)), y - x(t)\Big\rangle \\
+ \sum_{k=n}^{r(n)} S_{k,n}\langle -\dot{x}_k(t) + z_k(t), y - \mathrm{proj}_{C(t)}(x_k(t))\rangle \\
+ \sum_{k=n}^{r(n)} S_{k,n}\langle -\dot{x}_k(t) + z_k(t), -x(t) + \mathrm{proj}_{C(t)}(x_k(t))\rangle.
\end{aligned}
$$

The first and the third summands in the above expression tend to zero a.e. The second one, thanks to (3.23), satisfies the estimate

$$
\sum_{k=n}^{r(n)} S_{k,n}\langle -\dot{x}_k(t) + z_k(t), y - \mathrm{proj}_{C(t)}(x_k(t))\rangle \le \frac{\gamma+\beta}{2\rho}\sum_{k=n}^{r(n)} S_{k,n}|y - \mathrm{proj}_{C(t)}(x_k(t))|^2.
$$

Thus, passing to the limit one obtains

$$
\langle -\dot{x}(t) + z(t), y - x(t)\rangle \le \frac{\gamma+\beta}{2\rho}|y - x(t)|^2 \qquad \forall y \in C(t).
$$

This proves that $\dot{x}(t) \in -N_{C(t)}(x(t)) + z(t)$ for a.e. $t \in [0,T]$. Since $f(x,U)$ is convex for all x, from the classical Convergence Theorem (see, e.g., [7, Theorem 1, p. 60]) it follows that $z(t) \in f(x(t),U)$ for a.e. t. It then follows from from Filippov's Selection Theorem (see, e.g., [63, Th. 2.3.13]) that there exists $\bar{u}(\cdot)$ such that $z(t) = f(x(t),\bar{u}(t))$. This proves (3.22).

We claim now that $(x,\bar{u}) = (x_*,u_*)$. To this aim, define x_*^n to be the unique solution of the Cauchy problem

$$
\begin{cases}
\dot{y}(t) = -\frac{1}{\varepsilon_n}\Big(y(t) - \mathrm{proj}_{C(t)}(y(t))\Big) + f(y(t),u_*(t)), \\
y(0) = x_0 \,,
\end{cases}
$$

on $[0,T]$ and observe that x_*^n converges weakly to x_* in $W^{1,2}([0,T];\mathbb{R}^n)$ (see [55, Lemma 3.6]). Since (x_n,u_n) is an optimal couple (namely, a global minimizer) for $(P_\varepsilon(u_*))$, we have

$$h(x_*^n(T)) \geq h(x_n(T)) + \frac{1}{2}\int_0^T |u_n(t) - u_*(t)|^2 dt \tag{3.24}$$

for all $n \in \mathbb{N}$. By passing to the limit in (3.24), using the weak lower semicontinuity of the integral together with our convergence properties (3.20), we obtain

$$h(x_*(T)) \geq h(x(T)) + \delta \geq h(x(T)) + \int_0^T |\tilde{u}(t) - u_*(t)|^2 dt.$$

Since x_* is a global minimizer for the problem (P), the above inequalities imply that $\delta = 0$, i.e., $u_n \longrightarrow u_*$ strongly in $L^2([0,T];\mathbb{R}^n)$.

Now, the strong convergence of u_n to u_* allows us to prove that $x_n \longrightarrow x_*$ weakly in $W^{1,2}([0,T];\mathbb{R}^n)$. Indeed, set $r_n = \frac{1}{2}|x_n(t) - x_*(t)|^2$. Then, by using the fact that $-\dot{x}_n(t) + f(x_n(t), u_n(t)) = \frac{1}{\varepsilon_n}\Big(x_n(t) - \text{proj}_{C(t)}(x_n(t))\Big) \in N_{C(t)}(x_n(t))$ for all $t \in [0,T], n \in \mathbb{N}$ together with the (hypo)monotonicity of the normal cone to $C(t)$ – which follows immediately from (3.4) – and $(H_{3.3})$, (3.11), we obtain, for all $t \in [0,T]$ and each $n \in \mathbb{N}$ large enough,

$$\begin{aligned}
\dot{r}_n(t) &= \langle -\dot{x}_n(t) + \dot{x}_*(t), -x_n(t) + x_*(t)\rangle \\
&\leq -\langle f(x_n(t), u_n(t)) - f(x_*(t), u_*(t)), x_n(t) - x_*(t)\rangle + \frac{2\gamma + 3\beta}{\rho} r_n(t) \\
&\leq K r_n(t) + k|x_n(t) - x_*(t)|\,|u_n(t) - u_*(t)|,
\end{aligned}$$

where $K := k + \frac{2\gamma+3\beta}{\rho}$. Since $r_n(0) = 0$ for each $n \in \mathbb{N}$, Gronwall's Lemma yields, for each $t \in [0,T]$,

$$r_n(t) \leq k\int_0^T |x_n(t) - x_*(t)|\,|u_n(t) - u_*(t)| e^{K(T-t)} dt,$$

which, by the strong convergence of u_n to u_*, implies that $x(t) = x_*(t)$ for all $t \in [0,T]$. ■

Remark 3.4.1 *Observe that a similar argument implies that the sequence $\{x_n\}$ is Cauchy for the uniform convergence.*

3.5 Monotonicity of the distance

Let x_* be a given optimal trajectory, and x_n be trajectories of the regularized dynamics, and recall that $I_\partial := \{t \in [0,T] : x_*(t) \in \partial C(t)\}$. The first result in this section will be crucial in order to allow some estimates involving $\nabla_x d_{C(t)}(x_n(t))$ and $\nabla_x^2 d_{C(t)}(x_n(t))$ by forbidding that $x_n(t)$ remains on $\partial C(t)$ on a set of times with positive measure. Recall that $\partial C(t)$ is the discontinuity set of $\nabla_x d_{C(t)}(x)$ and $\nabla_x^2 d_{C(t)}(x)$ as a function of x. The simplest way to satisfy this requirement is giving sufficient conditions in order to let $\frac{d}{dt} d_S(t, x_n(t))$ be nonzero for a.e. t in a suitable neighborhood of I_∂.

Proposition 3.5.1 *Assume* $(H_1), \ldots, (H_4)$*, and let the admissible control* u_**, with the corresponding solution* x_* *of* (3.7)*, be such that that there exists* $\sigma > 0$ *for which either* (M_1) *or* (M_2) *hold. Let* $\varepsilon_n \downarrow 0$ *and let* u_n *be admissible controls such that* $u_n \to u_*$ *strongly in* $L^2([0,T]; \mathbb{R}^m)$ *and the corresponding solutions of* (3.13) $x_n \to x_*$ *strongly in* $W^{1,2}([0,T]; \mathbb{R}^n)$*. Then for each* n *large enough there exists at most one time* $t_n \in [0,T]$ *such that* $x_n(t_n) \in \partial C(t_n)$*.*

Proof. We write the proof for the case (M_1), the case (M_2) being similar and easier.

Assume (M_1) and let $\delta > 0$ be such that for a.e. $t \in [0,T]$ with $d(t, I_\partial) < \delta$ and all $x, y \in \mathbb{R}^n$ with $|x - x_*(t)|, |y - x_*(t)| < \delta$, $u \in U$ we have

$$\frac{\partial d_S}{\partial t}(t,x) + \langle \nabla_x d_S(t,y), f(x,u) \rangle \geq \frac{\sigma}{2}. \tag{3.25}$$

Let $y_n(t)$ be the projection of $x_n(t)$ onto $C(t)$. Then, for a.e. $t \in [0,T]$ we have

$$\begin{aligned}
\frac{d}{dt} d_S(t, x_n(t)) &= \frac{\partial d_S}{\partial t}(t, x_n(t)) + \langle \nabla_x d_S(t, x_n(t)), \dot{x}_n(t) \rangle \\
&= \frac{\partial d_S}{\partial t}(t, x_n(t)) + \Big\langle \nabla_x d_S(t, y_n(t)) + \nabla_x^2 d_S(t, y_n(t))(x_n(t) - y_n(t)) \\
&\quad + \sum_{|\alpha|=2, \alpha \in \mathbb{N}^n} \int_0^1 (1-\tau) \partial_x^\alpha \nabla_x d_S\big(t, y_n(t) + \tau(x_n(t) - y_n(t))\big)(x_n(t) - y_n(t))^\alpha \, d\tau, \\
&\qquad - \frac{x_n(t) - y_n(t)}{\varepsilon_n} + f(x_n(t), u_n(t)) \Big\rangle,
\end{aligned} \tag{3.26}$$

where α is the multiindex $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\partial_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$, and $|\alpha|$ denotes the sum of all entries of α.

Let now $t_0 \in [0, T] \cap \left(I_\partial + (-\delta, \delta)\right)$ be such that $d_{C(t_0)}(x_n(t_0)) \leq \frac{\sigma}{4}\varepsilon_n$. Thanks to (3.14), for all n large enough we have for all $t \in [t_0, T]$

$$d_{C(t)}(x_n(t)) \leq \varepsilon_n \left(\frac{\sigma}{4} e^{-\frac{t-t_0}{\varepsilon_n}} + (\gamma + \beta)\left(1 - e^{-\frac{t-t_0}{\varepsilon_n}}\right) \right) < \delta.$$

Therefore, for all $t \in I_\partial + (-\delta, \delta)$, $t \geq t_0$, we obtain from (3.26) and the above inequality that

$$\begin{aligned} \frac{d}{dt} d_S(t, x_n(t)) \geq{}& \frac{\partial d_S}{\partial t}(t, x_n(t)) + \langle \nabla_x d_S(t, y_n(t)), f(x_n(t), u_n(t)) \rangle \\ & - (\gamma + \beta)\left(1 - e^{-\frac{t-t_0}{\varepsilon_n}}\right) - \frac{\sigma}{4} - K d_{C(t)}(x_n(t)), \end{aligned}$$

for a suitable constant K, independent of t and n. Since x_n converges uniformly to x_*, we obtain from (3.25) that $\frac{d}{dt} d_S(t, x_n(t)) > 0$ for a.e. t in a (suitably small) right neighborhood of t_0, for all n large enough. Thus there must exist at most one $t \in [0, T] \cap (I_\partial + (-\delta, \delta))$ such that $x_n(t) \in \partial C(t)$. Since in $[0, T] \setminus (I_\partial + (-\delta, \delta))$ the trajectory $x_n(t)$ belongs to $\operatorname{int} C(t)$ for all n large enough, no further crossings of $\partial C(t)$ are possible.

The second case is analogous and actually easier. In fact there will not be any crossing of $\partial C(t)$ on $(0, T]$, for all n large enough, since on a suitable neighborhood of I_∂ we will have $\frac{d}{dt} d_S(t, x_n(t)) < 0$. ■

The following simple corollaries will be useful in the discussion of necessary conditions.

Proposition 3.5.2 *Assume* (M_1)*. Then* I_∂ *is an interval and, if it is nonempty,* $\sup I_\partial = T$*.*

Proof. It is enough to show that if $t \in I_\partial$, then $[t, T] \subset I_\partial$. To this aim, assume by contradiction that there exists $t < T$ such that $t \in I_\partial$, but $\bar{t} := \sup I_\partial < T$. This means, in particular, that for all $s \in (\bar{t}, T]$ we have $d_S(s, x_*(s)) < 0$. Thus, for all such s we have

$$
\begin{aligned}
0 > d_S(s, x_*(s)) - d_S(\bar{t}, x_*(\bar{t})) &= \int_{\bar{t}}^{s} \Big(\frac{\partial d_S}{\partial t}(\tau, x_*(s)) + \langle \nabla_x d_S(\tau, x_*(\tau)), \dot{x}_*(\tau) \rangle \Big) \, d\tau \\
&= \int_{\bar{t}}^{s} \Big(\frac{\partial d_S}{\partial t}(\tau, x_*(s)) + \langle \nabla_x d_S(\tau, x_*(\tau)), f(x_*(\tau), u_*(\tau)) \rangle \Big) \, d\tau
\end{aligned}
$$

and the integrand is positive if s is close enough to $\bar{t}$, a contradiction. ∎

Proposition 3.5.3 *Assume* (M_2)*. Then* I_∂ *is at most the singleton* $\{0\}$.

Proof. Assume by contradiction that there exists $\bar{t} > 0$, with $\bar{t} \in I_\partial$. Then, for all $t < \bar{t}$ we have, on one hand,

$$
d_S(\bar{t}, x_*(\bar{t})) - d_S(t, x_*(t)) \geq 0,
$$

while on the other,

$$
\begin{aligned}
d_S(\bar{t}, x_*(\bar{t})) - d_S(t, x_*(t)) &= \int_{t}^{\bar{t}} \Big(\frac{\partial d_S}{\partial t}(\tau, x_*(s)) + \langle \nabla_x d_S(\tau, x_*(\tau)), \dot{x}_*(\tau) \rangle \Big) \, d\tau \\
&= \int_{t}^{\bar{t}} \Big(\frac{\partial d_S}{\partial t}(\tau, x_*(s)) + \langle \nabla_x d_S(\tau, x_*(\tau)), \dot{x}_*(\tau) - f(x_*(\tau), u_*(\tau)) \rangle \\
&\qquad + \langle \nabla_x d_S(\tau, x_*(\tau)), f(x_*(\tau), u_*(\tau)) \rangle \Big) \, d\tau
\end{aligned}
$$

Observe that if $x_*(\tau) \in \operatorname{int} C(\tau)$, then $\dot{x}_*(\tau) - f(x_*(\tau), u_*(\tau)) = 0$, while if $x_*(\tau) \in \partial C(\tau)$, then $\dot{x}_*(\tau) - f(x_*(\tau), u_*(\tau)) = -\delta(\tau) \nabla_x d_S(\tau, x_*(\tau))$ for a bounded nonnegative function δ. Therefore, (M_2) implies that the integrand is < 0, provided t is close enough to $\bar{t}$, yielding a contradiction. ∎

3.6 The Approximate Control Problem

Given a global minimizer u_* of the problem (P) and $\varepsilon > 0$, we recall that in Section 3.4 the approximate problem $(P_\varepsilon(u_*))$ was defined and studied.

Let u_ε be a global minimizer for $(P_\varepsilon(u_*))$. By Proposition 3.4.2, we know that, up to a subsequence, u_ε converges weakly in $L^2([0, T]; \mathbb{R}^m)$ to u_* and x_ε converges strongly in $W^{1,2}([0,T];\mathbb{R}^n)$ to the optimal trajectory x_*, namely the trajectory of (3.7) where we replace u with u_*.

In order to state necessary conditions satisfied by $(x_\varepsilon, u_\varepsilon)$, we recall that the subdifferentiability of the Lipschitz map

$$x \mapsto x - \mathrm{proj}_{C(t)}(x) := P(t,x)$$

was preliminarily discussed in Section 3.2. Here we recall only that under our standing assumptions the unit external normal to $C(t)$ at $x \in \partial C(t)$ is $\nabla_x d_S(t,x)$.

Differently from classical computations in control theory, we will not consider needle variations, but rather compute a directional derivative of the cost, following [16]. More precisely, let $(x_\varepsilon, u_\varepsilon)$ be an optimal pair for the problem (3.12) subject to (3.13) and let $\tilde{u}$ be an admissible control. For $\sigma \in [0,1]$ set $u_\sigma(t) = u_\varepsilon(t) + \sigma(\tilde{u}(t) - u_\varepsilon(t))$ and observe that, thanks to the convexity of the control set U, this is an admissible control as well. Set x_σ to be the corresponding solution of (3.13). We wish to compute the directional derivative of the cost $J(x,u;u_*)$ at $(x_\varepsilon, u_\varepsilon)$ in the direction $\tilde{u} - u_\varepsilon$. Should this derivative exist, then it would be ≥ 0, by the minimality of $(x_\varepsilon, u_\varepsilon)$, namely

$$\lim_{\sigma \to 0} \frac{J(x_\sigma, u_\sigma; u_*) - J(x_\varepsilon, u_\varepsilon; u_*)}{\sigma} \geq 0.$$

The difference quotient in the above expression consists of the summands

$$\frac{h(x_\sigma(T)) - h(x_\varepsilon(T))}{\sigma} \;+\; \frac{1}{2\sigma}\int_0^T \left(|u_\sigma(t) - u_*(t)|^2 - |u_\varepsilon(t) - u_*(t)|^2\right) dt.$$

The limit of the second summand for $\sigma \to 0$ is straightforward and equals

$$\int_0^T \langle \tilde{u}(t) - u_\varepsilon(t), u_\varepsilon - u_*(t)\rangle \, dt$$

The limit of the first summand is

$$\lim_{\sigma \to 0}\left(\left\langle \nabla h(x_\varepsilon(T)), \frac{x_\sigma(T) - x_\varepsilon(T)}{\sigma} \right\rangle + \frac{o\big(x_\sigma(T) - x_\varepsilon(T)\big)}{\sigma} \right).$$

Therefore, we are lead to compute the limit

$$\lim_{\sigma\to 0} \frac{x_\sigma(T) - x_\varepsilon(T)}{\sigma}.$$

This is classical (see, e.g., [12, Theorem 3.4]) under the assumption of continuous differentiability with respect of both x and u of the right hand side of (3.13) for a.e. t. Such assumptions are valid in our setting thanks to Proposition 3.5.1, since there exists at most one time t such that $x_\varepsilon(t)$ belongs to $\partial C(t)$. Therefore, thanks to Theorem 3.4 in [12] we have

$$\lim_{\sigma\to 0} \frac{x_\sigma(T) - x_\varepsilon(T)}{\sigma} = \Big\langle \nabla h(x_\varepsilon(T)), \int_0^T M(T,t)\nabla_u f(x_\varepsilon(t), u_\varepsilon(t))(\tilde{u}(t) - u_\varepsilon(t))\,dt \Big\rangle, \tag{3.27}$$

where $M(T,t)$ is the fundamental matrix solution of the linear O.D.E.

$$\dot{v}(t) = \Big(\frac{-1}{2\varepsilon}\nabla_x^2 d^2(x_\varepsilon(t), C(t)) + \nabla_x f(x_\varepsilon(t), u_\varepsilon(t))\Big)v(t)$$

We can write the right hand side of (3.27) as

$$\int_0^T \Big\langle M^T(T,t)\nabla h(x_\varepsilon(T)), \nabla_u f(x_\varepsilon(t), u_\varepsilon(t))(\tilde{u}(t) - u_\varepsilon(t))\Big\rangle\,dt.$$

By setting $p_\varepsilon(t)$ to be the solution of the adjoint equation

$$\begin{cases} -\dot{p}_\varepsilon(t) &= \Big(\frac{-1}{2\varepsilon}\nabla_x^2 d^2(x_\varepsilon(t), C(t)) + \nabla_x f(x_\varepsilon(t), u_\varepsilon(t))\Big)p_\varepsilon(t), \quad t \in [0,T] \\ -p_\varepsilon(T) &= \nabla h(x_\varepsilon(T)), \end{cases} \tag{3.28}$$

the sign condition on the directional derivative becomes, for all feasible $\tilde{u}$,

$$\int_0^T \Big(\langle -p_\varepsilon(t), \nabla_u f(x_\varepsilon(t), u_\varepsilon(t))(\tilde{u}(t) - u_\varepsilon(t))\rangle + \langle u_\varepsilon(t) - u_*(t), \tilde{u}(t) - u_\varepsilon(t)\rangle\Big)\,dt \geq 0.$$

Since $\tilde{u}$ is an arbitrary measurable selection from U, we obtain

$$\begin{aligned}&\langle p_\varepsilon(t), \nabla_u f(x_\varepsilon(t), u_\varepsilon(t))u_\varepsilon(t)\rangle - \langle u_\varepsilon(t) - u_*(t), u_\varepsilon(t)\rangle = \\ &\qquad \max_{u\in U}\{\langle p_\varepsilon(t), \nabla_u f(x_\varepsilon(t), u_\varepsilon(t))u\rangle - \langle u_\varepsilon(t) - u_*(t), u\rangle\} \qquad \text{for a.e. } t \in [0, T]. \end{aligned} \tag{3.29}$$

Now we recall that, thanks to Proposition (3.5.1), if $\varepsilon_n \downarrow 0$ is such that $(x_{\varepsilon_n}, u_{\varepsilon_n}) \to (x_*, u_*)$, for all n large enough the right hand side of (3.28) is single valued except for at most one point t and equals either $\nabla_x f(x_{\varepsilon_n}(t), u_{\varepsilon_n}(t))p_{\varepsilon_n}(t)$, if $x_{\varepsilon_n}(t) \in \operatorname{int} C(t)$, or

$$\Big(\frac{-1}{\varepsilon_n} d_{C(t)}(x_{\varepsilon_n})\nabla_x^2 d_{C(t)}(x_{\varepsilon_n}) + \nabla_x d_{C(t)}(x_{\varepsilon_n}) \otimes \nabla_x d_{C(t)}(x_{\varepsilon_n}) + \nabla_x f(x_{\varepsilon_n}(t), u_{\varepsilon_n}(t))\Big)p_{\varepsilon_n}(t)$$

if $x_{\varepsilon_n} \notin C(t)$. This means that we can consider (3.28) as a differential equation with a switch, which occurs (at most) at a single time t_{ε_n}. Without loss of generality, we can assume that

$$t_{\varepsilon_n} \quad \text{converges to some } \bar{t} \in I_\partial. \tag{3.30}$$

3.7 Estimates and convergence for the adjoint vectors

In this section we keep the notations of the preceding one, and we consider a sequence $\varepsilon_n \downarrow 0$ such that the conclusions of Propositions 3.4.2 and 3.5.1 are valid.

3.7.1 Estimates

First we prove some uniform estimates on the sequence $\{p_{\varepsilon_n}(\cdot)\}$, which will guarantee compactness in a suitable space.

Lemma 3.7.1 *The sequence $\{p_{\varepsilon_n}(\cdot)\}$ is uniformly bounded in $L^\infty([0, T]; \mathbb{R}^n)$.*

Proof. We can rewrite (3.28) as

$$-\dot{p}_{\varepsilon_n}(t) = \frac{-1}{\varepsilon_n}\nabla_x P(t, x_{\varepsilon_n}(t))p_{\varepsilon_n}(t) + \nabla_x f(x_{\varepsilon_n}(t), u_{\varepsilon_n}(t))p_{\varepsilon_n}(t),$$

where the Jacobian $\nabla_x P(t, x_{\varepsilon_n}(t))$ of $P(t, x_{\varepsilon_n}(t))$ with respect to the state variable x exists for all t different from the (possible) switching time t_{ε_n}. Recalling (3.5) and (3.6), we have, for all $t \neq t_{\varepsilon_n}$,

$$\begin{aligned}\nabla_x P(t, x_{\varepsilon_n}(t))\, p_{\varepsilon_n}(t) &= d_{C(t)}(x_{\varepsilon_n}(t))\nabla_x^2 d_{C(t)}(x_{\varepsilon_n}(t))p_{\varepsilon_n}(t)\\ &\quad + \langle \nabla_x d_{C(t)}(x_{\varepsilon_n}(t)), p_{\varepsilon_n}(t)\rangle \nabla_x d_{C(t)}(x_{\varepsilon_n}(t)).\end{aligned}$$

From this we obtain

$$\begin{aligned}-\dot{p}_{\varepsilon_n}(t) = &\frac{-d_{C(t)}(x_{\varepsilon_n}(t))}{\varepsilon_n}\nabla_x^2 d_{C(t)}(x_{\varepsilon_n}(t))p_{\varepsilon_n}(t) - \frac{1}{\varepsilon_n}\langle \nabla_x d_{C(t)}(x_{\varepsilon_n}(t)), p_{\varepsilon_n}(t)\rangle \nabla_x d_{C(t)}(x_{\varepsilon_n}(t))\\ &+ \nabla_x f(x_{\varepsilon_n}(t), u_{\varepsilon_n}(t))p_{\varepsilon_n}(t).\end{aligned}$$

Observe that either p_{ε_n} vanishes identically or it is never zero. In this case, we multiply both sides of the adjoint equation by $\frac{p_{\varepsilon_n}(t)}{|p_{\varepsilon_n}(t)|}$, and obtain

$$\begin{aligned}-\frac{d}{dt}|p_{\varepsilon_n}(t)| = &\frac{-d_{C(t)}(x_{\varepsilon_n}(t))}{\varepsilon_n}\Big\langle \frac{p_{\varepsilon_n}(t)}{|p_{\varepsilon_n}(t)|}, \nabla_x^2 d_{C(t)}(x_{\varepsilon_n}(t))p_{\varepsilon_n}(t)\Big\rangle\\ &- \frac{1}{\varepsilon_n}\frac{1}{|p_{\varepsilon_n}(t)|}\Big\langle \nabla_x d_{C(t)}(x_{\varepsilon_n}(t)), p_{\varepsilon_n}(t)\Big\rangle^2\\ &+ \Big\langle \nabla_x f(x_{\varepsilon_n}(t), u_{\varepsilon_n}(t))p_{\varepsilon_n}(t), \frac{p_{\varepsilon_n}(t)}{|p_{\varepsilon_n}(t)|}\Big\rangle.\end{aligned} \tag{3.31}$$

The second term on the right hand side of (3.31) is nonpositive, while the first one is bounded by Lemma 3.4.1, recalling that, if ε_n is small enough and $t \neq t_{\varepsilon_n}$, $x_{\varepsilon_n}(t)$ belongs to a set where $d_{C(t)}(\cdot)$ is $C^{1,1}$. Let c be a Lipschitz constant for $\nabla_x d_{C(t)}(\cdot)$ on this set. Integrating over the interval $[t, T]$ and recalling the final time condition contained in (3.28) yields

$$|p_{\varepsilon_n}(t)| - |\nabla h(x_{\varepsilon_n}(T))| \leq \int_t^T c(\gamma+\beta)|p_{\varepsilon_n}(s)|ds + \int_t^T \langle \nabla_x f(x_{\varepsilon_n}(t), u_{\varepsilon_n}(t))p_{\varepsilon_n}(t), \frac{p_{\varepsilon_n}(t)}{|p_{\varepsilon_n}(t)|}\rangle ds,$$

whence

$$|p_{\varepsilon_n}(t)| \le |\nabla h(x_{\varepsilon_n}(T))| + \Big(c(\gamma+\beta)+k\Big)\int_t^T |p_{\varepsilon_n}(s)|\,ds,$$

recalling that f is k-Lipschitz continuous. Now Gronwall's Lemma in integral form yields

$$|p_{\varepsilon_n}(t)| \le |\nabla h(x_{\varepsilon_n}(T))| e^{(c(\gamma+\beta)+k)\,(T-t)} \quad \text{for all } t \in [0,T].$$

Since $x_{\varepsilon_n}(T)$ converges uniformly to $x_*(T)$ and h is of class C^1, the proof is concluded. ∎

We deal now with a uniform L^1-bound for $\{\dot p_{\varepsilon_n}\}$. For simplicity of notation, we set $t_n := t_{\varepsilon_n}$, $x_n(t) := x_{\varepsilon_n}(t)$, $u_n(t) := u_{\varepsilon_n}(t)$, $p_n(t) := p_{\varepsilon_n}(t)$, $\delta_n(t) := d_{C(t)}(x_n(t))$, $\delta_n'(t) := \nabla_x d_{C(t)}(x_n(t))$, and finally $\delta_n''(t) := \nabla_x^2 d_{C(t)}(x_n(t))$. We observe first that, thanks to (3.15) and the fact that $x_n(t)$, for all $t \neq t_n$, belongs to a region where $d_{C(t)}(\cdot)$ is of class $C^{1,1}$,

$$\frac{\delta_n(\cdot)}{\varepsilon_n}\delta_n''(\cdot) \quad \text{is well defined and is bounded in } L^\infty([0,T];\mathbb{R}^n), \text{ uniformly with respect to } n. \tag{3.32}$$

Define now the normal component of $p_n(t)$ as

$$\xi_n(t) = \langle p_n(t), \nabla_x d_{C(t)}(x_n(t))\rangle \; (= \langle p_n(t), \delta_n'(t)\rangle), \quad t \neq t_n.$$

We have, for a.e. t (in particular $t \neq t_n$),

$$\dot\xi_n(t) = \langle \dot p_n(t), \delta_n'(t)\rangle + \langle p_n(t), \delta_n''(t)\dot x_n(t)\rangle. \tag{3.33}$$

With this notation, the primal dynamics in (3.13) and the dual one in (3.28) can be rewritten, respectively, as

$$\begin{aligned}
\dot x_n(t) &= -\frac{\delta_n(t)}{\varepsilon_n}\delta_n'(t) + f(x_n(t),u_n(t))\\
-\dot p_n(t) &= -\frac{\delta_n(t)}{\varepsilon_n}\delta_n''(t)p_n(t) - \frac{1}{\varepsilon_n}\delta_n'(t)\otimes\delta_n'(t)p_n(t) + \nabla_x f(x_n(t),u_n(t))p_n(t)\\
&= -\frac{\delta_n(t)}{\varepsilon_n}\delta_n''(t)p_n(t) - \frac{\xi_n(t)}{\varepsilon_n}\delta_n'(t) + \nabla_x f(x_n(t),u_n(t))p_n(t).
\end{aligned}$$

By inserting $\dot{p}_n(t)$ and $\dot{x}_n(t)$ from the above equations into (3.33), we obtain, for a.e. $t \in [0, T]$,

$$\begin{aligned}-\dot{\xi}_n(t) = &-\frac{\delta_n(t)}{\varepsilon_n}\langle \delta_n''(t)p_n(t), \delta_n'(t)\rangle - \frac{\xi_n(t)}{\varepsilon_n}|\delta_n'(t)|^2 \\ &+ \langle \nabla_x f(x_n(t), u_n(t))p_n(t), \delta_n'(t)\rangle \\ &- \Big\langle p_n(t), \delta_n''(t)\Big(- \frac{\delta_n(t)}{\varepsilon_n}\delta_n'(t) + f(x_n(t), u_n(t))\Big)\Big\rangle.\end{aligned} \tag{3.34}$$

In order to simplify the above relation, we observe that,

$$\delta_n''(t)\delta_n'(t) = \nabla_x^2 d_{C(t)}(x)\nabla_x d_{C(t)}(x) = 0 \quad \text{for all } t \in [0, T] \text{ and all } x \notin \partial C(t)$$

(see Lemma 3.8 in [16]), and furthermore that

$$\xi_n(t) = \xi_n(t)|\delta_n'(t)|^2 \quad \text{for all } \ t \in [0, T], t \neq t_n,$$

since if $x_n(t) \in \operatorname{int} C(t)$ then both sides are zero, while if $x_n(t) \notin C(t)$ we have $|\delta_n'(t)| = 1$. Therefore, recalling that $x_n(t) \notin \partial C(t)$ for all $t \neq t_n$, the equation (3.34) becomes, for a.e. $t \in [0, T]$ (with $t \neq t_n$),

$$\begin{aligned}-\dot{\xi}_n(t) + \frac{1}{\varepsilon_n}\xi_n(t) = &-\frac{\delta_n(t)}{\varepsilon_n}\Big[\langle \delta_n''(t)p_n(t), \delta_n'(t)\rangle + \langle p_n(t), \delta_n''(t)f(x_n(t), u_n(t))\rangle\Big] \\ &+ \langle \nabla_x f(x_n(t), u_n(t))p_n(t), \delta_n'(t)\rangle.\end{aligned} \tag{3.35}$$

Observe now that the right hand side of the above equality is bounded in $L^\infty([0, T], \mathbb{R}^n)$, uniformly with respect to n, thanks to Lemma 3.7.1 and (3.15), and to the ρ-prox-regularity of the moving set $C(\cdot)$. Observe also that Lemma 3.7.1 implies that $\xi_n(t)$ is bounded in $L^\infty(0, T)$, uniformly with respect to n. By multiplying both sides of (3.35) by $\operatorname{sign}(\xi_n(t))$ and integrating over the interval $[t, T]$, we then obtain

$$\frac{1}{\varepsilon_n}\int_t^T |\xi_n(s)|ds \leq \bar{k} \quad \text{for all } t \in [0, T] \tag{3.36}$$

for a suitable constant $\bar{k}$ independent of n.

We are now ready to obtain the L_1 uniform boundedness of the sequence $\{\dot{p}_\varepsilon\}$.

Lemma 3.7.2 *The sequence $\{\dot{p}_n(\cdot)\}$ is bounded in $L^1([0,T],\mathbb{R}^n)$, uniformly with respect to n .*

Proof. We recall that that the adjoint equation can be rewritten as

$$-\dot{p}_n(t) = -\frac{\delta_n(t)}{\varepsilon_n}\delta_n''(t)p_n(t) - \frac{\xi_n(t)}{\varepsilon_n}\delta_n'(t) + \nabla_x f(x_n(t),u_n(t))p_n(t).$$

The result follows immediately by using Lemma 3.7.1 together with (3.36), (3.32), and the assumptions on f. ■

3.7.2 Passing to the limit

We wish now to derive the equations which are satisfied by a suitable limit of the sequence $\{p_n\}$. By possibly extracting a further subsequence from $\{\varepsilon_n\}$ (without relabeling), thanks to Lemma 3.7.2 and Helly's selection theorem, we can suppose that there exists a function $p \in BV([0,T];\mathbb{R}^n)$ such that

$$p_n(t) \to p(t) \quad \text{for all } t \in [0,T]$$

(in particular $p(T) = -\nabla h(x_*(T))$) and, for all $h \in C^0([0,T];\mathbb{R}^n)$,

$$\int_0^T \langle h(t), \dot{p}_n(t)\rangle dt \to \int_0^T \langle h(t), dp\rangle.$$

We recall also that

$$\begin{array}{ll} x_n \to x_* & \text{uniformly in } [0,T] \\ \dot{x}_n^* \rightharpoonup \dot{x}^* & \text{weakly in } L^2([0,T];\mathbb{R}^n) \\ u_n \to u_* & \text{strongly in } L^2([0,T];\mathbb{R}^m) \\ u_n(t) \to u_*(t) & \text{a.e. on } [0,T], \end{array}$$

and that we have set

$$I_\partial = \{t \in [0,T] : x_*(t) \in \partial C(t)\}.$$

We define also

$$I_0 := [0,T] \setminus I_\partial = \{t \in [0,T] : x_*(t) \in \operatorname{int} C(t)\}.$$

Of course one of the two sets I_0 and I_∂ may be empty. We will proceed with our arguments, without loss of generality, by assuming that both of them are nonempty.

For every compact interval $[s,t] \subset I_0$, the adjoint equation for p_n is

$$-\dot{p}_n(\tau) = \nabla_x f(x_n(\tau), u_n(\tau))p_n(\tau), \tag{3.37}$$

since $d_{C(t)}(\cdot)$ is zero in a neighborhood of $x_*(t)$ and x_n converges to x_* uniformly. By integrating (3.37) over $[s,t]$ and using the absolute continuity of p_n, we obtain

$$p_n(s) - p_n(t) = \int_s^t \nabla_x f(x_n(\tau), u_n(\tau))p_n(\tau)\, d\tau.$$

Since p_n converges to p pointwise and is uniformly bounded, by the dominated convergence theorem we obtain

$$p(s) - p(t) = \int_s^t \nabla_x f(x_*(\tau), u_*(\tau))p(\tau)\, d\tau.$$

We have therefore proved the following

Proposition 3.7.1 *On I_0, p is absolutely continuous and satisfies the equation*

$$-\dot{p}(t) = \nabla_x f(x_*(t), u_*(t))p(t), \quad a.e.\ t \in I_0. \tag{3.38}$$

We will deal now with passing to the limit along (3.28) and obtaining necessary conditions on the whole interval $[0,T]$. The main effort will be put in passing to the limit in I_∂.

For the sake of convenience, we rewrite here the adjoint equation for p_n, recalling that we have set $\xi_n(t) = \langle \nabla_x d_{C(t)}(x_n(t)), p_n(t)\rangle$. We have

$$\begin{aligned}-\dot{p}_n(t) &= -\frac{1}{\varepsilon_n}\nabla_x d_{C(t)}(x_n(t))\xi_n(t) - \frac{d_{C(t)}(x_n(t))}{\varepsilon_n}\nabla_x^2 d_{C(t)}(x_n(t))p_n(t)\\ &\quad + \nabla_x f(x_n(t), u_n(t))p_n(t)\\ &\quad := \mathbf{I} + \mathbf{II} + \mathbf{III}\end{aligned}$$

We recall that under our assumptions this equation can be seen as and O.D.E. with (possibly) a switch, which occurs at the time t_n, and we can assume that the sequence $\{t_n\}$ has a limit point $\bar{t}$ (see (3.30)).

We discuss now passing to the limit for each summand **I**, **II**, and **III**.

I. Set $n_*(t)$ to be the unit external normal to $C(t)$ at $x_*(t)$ for all $t \in I_\partial$ and 0 for all $t \in I_0$. Observe that on every compact subset $I \subset [0, T]$ such that $\bar{t} \notin I$ we can suppose that $\nabla_x d_{C(t)}(x_n(t))$ converges to $n_*(t)$ uniformly on I. By the uniform boundedness of $\nabla_x d_{C(t)}(\cdot)$ and (3.36) we can suppose that (up to a subsequence) along $\{\varepsilon_n\}$ we have that the sequence of measures

$$\left\{\frac{\xi_n(.)}{\varepsilon_n}\nabla_x d_{C(\cdot)}(x_n(\cdot))\, dt\right\}$$

converges weakly* in the dual of $C^0([0, T]; \mathbb{R}^n)$ to a finite signed Radon measure on $[0, T]$, which can be written as

$$\xi(t)n_*(t)d\mu, \tag{3.39}$$

where μ is a finite signed Radon measure on $[0, T]$ and $\xi \in L^\infty[0, T]$, $\xi(t) \geq 0$ μ-a.e. Observe that, without loss of generality, we can suppose that $\xi(t) = 0$ on I_0.

II. Recalling (3.15),

$$\frac{d_{C(t)}(x_n(t))}{\varepsilon_n} \leq \beta + \gamma \quad \text{for all } t \in [0, T] \text{ and } n \in \mathbb{N}.$$

Recall that $\nabla_x^2 d_{C(t)}(x_*(t)) = 0$ if $t \in I_0$, and set $\nabla_x^2 d_{C(t)}(x_*(t)) = \nabla_x^2 d_S(t, x_*(t))$ if $t \in I_\partial$ (indeed, the signed distance $d_S(t, \cdot)$ is C^2 in a neighborhood of boundary points of $C(t)$, $t \in I_\partial$, see, e.g., Proposition 2.2.2 (iii) in [17], since both (M_1) and (M_2) imply that $C(t)$ has nonempty interior).

We can suppose that $\nabla_x^2 d_{C(t)}(x_n(t))$ converges uniformly to $\nabla_x^2 d_{C(t)}(x_*(t))$ on every compact $I \in [0,T]$ such that $\bar{t} \notin I$. By combining the uniform bound on $d_{C(t)}(x_n(t))/\varepsilon_n$, the uniform convergence of $\nabla_x^2 d_{C(t)}(x_n(t))$ on every compact $I \in [0,T]$ with $\bar{t} \notin I$ and the pointwise convergence of p_n, we obtain, up to a subsequence without relabeling, that

$$\frac{d_{C(t)}(x_n(t))}{\varepsilon_n} \nabla_x^2 d_{C(t)}(x_n(t)) p_n(t) \rightharpoonup \eta(t) \nabla_x^2 d_{C(t)}(x_*(t)) p(t)$$

weakly in $L^2([0,T];\mathbb{R}^n)$, where $\eta \in L^\infty[0,T]$, $0 \leq \eta(t) \leq \beta + \gamma$ a.e. Observe also that $\eta(t) \equiv 0$ on I_0.

III. Recalling Proposition 3.4.2, up to a subsequence

$$\nabla_x f(x_n(t), u_n(t)) p_n(t) \to \nabla_x f(x_*(t), u_*(t)) p(t) \text{ a.e. on } [0,T]$$

and weakly in $L^2([0,T];\mathbb{R}^n)$.

We have therefore proved that p satisfies in a weak sense a differential equation, namely (and this establishes (3.9)) we have

Proposition 3.7.2 *Let p be a weak limit of p_n in $BV([0,T];\mathbb{R}^n)$. Then $p(T) = -\nabla h(x_*(T))$ and there exist a finite Radon measure μ on $[0,T]$, and nonnegative measurable functions $\xi, \eta : [0,T] \to \mathbb{R}$ satisfying the properties $\xi \in L^1_\mu(0,T)$, $\xi(t) = 0$ on I_0 and $\xi(t) \geq 0$ on I_∂, μ-a.e., and $0 \leq \eta(t) \leq \beta + \gamma$, $\eta(t) = 0$ on I_0, a.e., such that for all continuous functions $\varphi : [0,T] \to \mathbb{R}^n$ we have*

$$\begin{aligned} -\int_{[0,T]} \langle \varphi(t), dp(t) \rangle + \int_{[0,T]} \langle \varphi(t), n_*(t) \rangle \xi(t)\, d\mu - \int_{[0,T]} \langle \varphi(t), \nabla_x f(x_*(t), u_*(t)) p(t) \rangle\, dt \\ = -\int_{[0,T]} \langle \varphi(t), \eta(t) \nabla_x^2 d_{C(t)}(x_*(t)) p(t) \rangle\, dt, \end{aligned} \tag{3.40}$$

where we recall that $n_*(t) = 0$ if $x_*(t) \in int\, C(t)$, and $n_*(t) = \nabla_x d_S(t, x_*(t))$ is the unit external normal to $C(t)$ if $x_*(t) \in \partial C(t)$.

The adjoint vector p can be proved to satisfy a bunch of further conditions in the interval I_∂.

Proposition 3.7.3 *Let p, ξ, η be given by Proposition* (3.7.2) *and set $p^N(t) = \langle p(t), n_*(t)\rangle$ for all $t \in [0, T]$. Then*

(1) *$p^N(t) = 0$ for a.e. $t \in [0, T]$, and p is absolutely continuous on I_0.*

(2) *If (M_1) holds, then (recall that $I_\partial = [\bar{t}, T]$ according to Proposition 3.5.2 and* (3.30)*) p is absolutely continuous on $(\bar{t}, T)$ and for a.e. $t \in [\bar{t}, T]$ we have*

$$-\dot{p}(t) = \langle \dot{n}_*(t), p(t)\rangle n_*(t) + \Gamma(t)p(t) - \langle \Gamma(t)p(t), n_*(t)\rangle n_*(t), \tag{3.41}$$

where $\Gamma(t) = \nabla_x f(x_*(t), u_*(t)) - \eta(t)\nabla_x^2 d_{C(t)}(x_*(t))$. Moreover, the equalities

$$p(\bar{t}-) - p(\bar{t}+) = p^N(\bar{t}-)n_*(\bar{t}) \tag{3.42}$$

$$p(T-) - p(T) = -p^N(T)n_*(T) \tag{3.43}$$

(which mean that jumps may occur only in the normal direction n_*) are valid.

(3) *If (M_2) holds, then*

$$p(0) - p(0+) = \left(p^N(0) - p^N(0+)\right)n_*(0). \tag{3.44}$$

(4) *If (M_1) holds, p is continuous at $\bar{t}$ and $|p(T-)| \le |p(T)|$.*

Remark 3.7.1 *It follows from the above Proposition that the measure μ appearing in* (3.40) *may admit a Dirac mass at most at $t = 0$ (if (M_2) holds) or at $t = T$ (if (M_1) holds).*

Proof. **(1)**. The first assertion is an immediate consequence of (3.36), which implies that the sequence $\langle p_n(t), \nabla_x d_{C(t)}(x_n(t))\rangle$ converges to 0 in $L^1(0, T)$, and the convergence of $\nabla_x d_{C(t)}(x_n(t))$ to $n_*(t)$ for all $t \in [0, T]$, $t \neq \bar{t}$ and of $p_n(t)$ to $p(t)$. The second assertion follows from Proposition 3.7.1.

(2) and **(3)**. Since $n_*(t)$ is continuous on I_∂, there exist $n - 1$ continuous unit vectors $v_1(t), \dots, v_{n-1}(t)$ such that $\mathbb{R}^n = \mathbb{R}n_*(t) \oplus \operatorname{span}\langle v_1(t), \dots, v_{n-1}(t)\rangle$. Fix $t \in (\bar{t}, T)$ and $\sigma > 0$ such that $[t-\sigma, t+\sigma] \subset (\bar{t}, T)$. Let $\varphi : [0, T] \to \mathbb{R}^n$ be continuous, with support contained in $[t-\sigma, t+\sigma]$. Set $\varphi^T(t) = \varphi(t) - \langle\varphi(t), n_*(t)\rangle n_*(t)$. By putting $\varphi^T(t)$ in place of φ in (3.40) we obtain

$$-\int_{t-\sigma}^{t+\sigma}\langle\varphi^T(s), dp(s)\rangle + \int_{t-\sigma}^{t+\sigma}\langle\varphi^T(s), n_*(s)\rangle\xi(s)\, d\mu = \int_{t-\sigma}^{t+\sigma}\langle\varphi^T(s), \nabla_x f(x_*(s), u_*(s))p(s)\rangle\, ds$$
$$-\int_{t-\sigma}^{t+\sigma}\langle\varphi^T(s), \eta(s)\nabla_x^2 d_{C(s)}(x_*(s))p(s)\rangle\, ds. \tag{3.45}$$

Observe now that $\langle\varphi^T, n_*(t)\rangle \equiv 0$, so that, by letting $\sigma \to 0$ in the above equation and using the continuity of φ^T, we obtain $\langle\varphi^T(t), p(t+) - p(t-)\rangle = 0$, namely $\langle p(t+) - p(t-), \varphi(t)\rangle = \langle p(t+) - p(t-), \langle\varphi(t), n_*(t)\rangle n_*(t)\rangle$. By taking subsequently φ such that $\varphi(t) = n_*(t)$, and $\varphi(t) = v_i(t)$, we obtain that $p(t-) - p(t+) = (p^N(t-) - p^N(t+))n_*(t)$, namely jumps of p may occur only in the direction $n_*(t)$, for all $t \in (\bar{t}, T)$. By taking $t = T$ and arguing as in (3.45), but integrating over $[T-\sigma, T]$, one immediately obtains (3.43). In order to prove (3.42), resp. (3.44), it is enough to extend $n_*(t)$ to be constantly $n_*(\bar{t})$ on $[\bar{t}-\sigma, \bar{t})$, resp. constantly $n_*(0)$ on $(0, \sigma)$, and observe that the part (1) of this Proposition together with the fact that p has bounded variation imply that $p(\bar{t}+) = p(T-) = 0$.

Fix now an interval $[s, t] \subseteq (\bar{t}, T)$. The regularity condition on $\partial C(t)$ allows us to integrate by parts on (s, t) (see (34), p. 8 in [39]), so that

$$\int_s^t\langle n_*(\tau), dp(\tau)\rangle + \int_s^t\langle\dot{n}_*(\tau), p(\tau)\rangle\, d\tau = \langle n_*(t+), p(t+)\rangle - \langle n_*(s-), p(s-)\rangle = 0, \tag{3.46}$$

where both summands in the right hand side of (3.46) vanish, as a consequence of **(1)** and of the fact that p has bounded variation, since both s and t belong to the interior of I_∂. In other words, the two measures $\langle n_*, dp\rangle$ and $\langle\dot{n}_*, p\rangle\, dt$ coincide in the open interval $(\bar{t}, T)$. Therefore, for all continuous φ, with support contained in $(\bar{t}, T)$, we obtain from

(3.45) and (3.46) that

$$
\begin{aligned}
-\int_{\bar t}^{T}\langle\varphi(t),dp(t)\rangle &= -\int_{\bar t}^{T}\langle\varphi(t),n_*(t)\rangle\,\langle n_*(t),dp(t)\rangle-\int_{\bar t}^{T}\langle\varphi^T(t),dp(t)\rangle\\
&=\int_{\bar t}^{T}\langle\varphi(t),n_*(t)\rangle\langle p(t),\dot n_*(t)\rangle\,dt\\
&\quad+\int_{\bar t}^{T}\langle\varphi(t)-\langle\varphi(t),n_*(t)\rangle n_*(t),\nabla_x f(x_*(t),u_*(t))p(t)\rangle dt\\
&\quad-\int_{\bar t}^{T}\langle\varphi(t)-\langle\varphi(t),n_*(t)\rangle n_*(t),\eta(t)\nabla_x^2 d_{C(t)}(x_*(t))p(t)\rangle dt
\end{aligned}
$$

Since φ is arbitrary, we obtain (3.41).

(4). By multiplying (3.28) by $\frac{p_n(s)}{|p_n(s)|}$ and integrating over $[\bar t-\sigma,\bar t+\sigma]$, using the fact that $d_{C(t)}(\cdot)$ is $C^{1,1}$, uniformly with respect to t, by using the same argument of the proof of Lemma 3.7.1 (see (3.31)) we obtain

$$|p_n(\bar t-\sigma)|-|p_n(\bar t+\sigma)|\le k\int_{\bar t-\sigma}^{\bar t+\sigma}|p_n(s)|ds$$

for a suitable constant k independent of n. By passing to the limit as $n\to\infty$ (along a suitable subsequence) we obtain

$$|p(\bar t-\sigma)|-|p(\bar t+\sigma)|\le 0,$$

whence

$$|p(\bar t-)|\le|p(\bar t+)|.$$

Analogously, multiplying both sides of (3.35) by sign $(\xi_n(t))$, integrating and using the fact that $\frac{|\xi_n|}{\varepsilon_n}$ is uniformly bounded in $L^1(0,T)$, we obtain by passing to the limit as $n\to\infty$ that $|p^N(\bar t-)|\le|p^N(\bar t+)|=0$. The latter vanishes, recalling **(1)**, and thus it follows that $p^N(\bar t-)=0$ as well. Recalling (3.42), p is continuous at $\bar t$. The same argument shows that $|p(T-)|\le|p(T)|$. ■

Our last task is now to the limit formulation of the maximum principle. Indeed, from (3.29) we immediately obtain, by passing to the limit for $n \to \infty$, that (3.10) holds.

Therefore, the proof of our main result is concluded.

3.8 An example

We propose a simple example, inspired by Remark 5.1 in [58], in order to test our necessary conditions.

3.8.1 Example 1

The state space is $\mathbb{R}^2 \ni (x, y)$, the constraint $C(t)$ is constant and equals $C := \{(x, y) : y \geq 0\}$, the upper half plane.

We wish to minimize $x(1) + y(1)$ subject to

$$\begin{cases} \big(\dot{x}(t), \dot{y}(t)\big) & \in -N_C\big(x(t), y(t)\big) + \big(u^x(t), u^y(t)\big) \\ \big(x(0), y(0)\big) & = \big(0, y_0\big), \qquad y_0 \geq 0, \end{cases} \tag{3.47}$$

where the controls $\big(u^x(t), u^y(t)\big)$ belong to $[-1, 1] \times [-1, -1/2] =: U$.

By inspecting the level sets of the cost $h\big((x, y)\big) = x + y$, it is natural looking for an optimal solution such that both u^x and u^y are nonpositive. If we restrict ourselves to the case where $u^y(t) < 0$ for a.e. t, then this problem satisfies all our assumptions; in particular we are in the case (M_1).

Observe first that if $y_0 \geq 1$, the constraint C does not play any role, and the optimality of the control $(-1, -1)$ is straightforward. If instead $0 \leq y_0 < 1$, then our analysis becomes relevant. Since we are in the case (M_1), there exists at most one $\bar{t}$ such that the optimal solution hits the boundary of C and after $\bar{t}$ it remains on ∂C. The external unit normal $n_*(t)$ is identically $(0, -1)$ and on ∂C, namely for $x = 0$, we have $\nabla^2_x d_C\big((0, y)\big) \equiv 0$. Thanks to Propositions (3.7.3) and (3.38) we obtain, for the optimal trajectory (x_*, y_*)

corresponding to the optimal control (u_*^x, u_*^y) and the adjoint vector (p^x, p^y), that (p^x, p^y) is absolutely continuous on $(0,1)$, $\dot{p}^x = 0$, $\dot{p}^y = 0$ a.e. on $[0,T]$, $p^x(1) = p^y(1) = -1$, p^x is continuous at $t = 1$ and $p^y(1-) + 1 = 1$, namely $p^y(1-) = 0$. Thus the adjoint vector (p^x, p^y) is :

$$\begin{aligned}
p^x(t) &= -1 \qquad \text{for all } t \in [0,1] \\
p^y(t) &= 0 \qquad \text{for all } t \in [0,1) \\
p^y(1) &= -1 \\
\mu &= -\delta_1.
\end{aligned}$$

The maximum condition reads as

$$\begin{aligned}
\langle(-1,-1),(u_*^x,u_*^y)\rangle &= \max_{|u_1|\le 1, -1\le u_2\le -1/2} \langle(-1,-1),(u_1,u_2)\rangle \quad \text{for } t = 1 \\
\langle(-1,0),(u_*^x,u_*^y)\rangle &= \max_{|u_1|\le 1, -1\le u_2\le -1/2} \langle(-1,0),(u_1,u_2)\rangle \quad \text{for } 0 \le t < 1,
\end{aligned}$$

which gives $u_*^x = -1$, while no information is available for $u_*^y(t)$. If we assume that u_*^y is continuous at $t = 1$, then the transversality condition yields and $u_*^y(1) = -1$. If we assume that u_*^y is constant, then an expected optimal control $u_*^y = -1$ is found. Of course all other optimal controls u_*^y, namely $u_*^y(t) = -1$ for $0 \le t < \bar{t}$ and $u_*^y(t) \le 0$ for $\bar{t} < t < 1$ satisfy our necessary conditions.

Remark 3.8.1 *1) The vanishing of p^y on the interval $[\bar{t}, 1]$ is somehow to be expected, since all controls $u^y \le -1/2$ (actually $u^y \le 0$) in that time interval are optimal. The vanishing of p^y on $[0, \bar{t}]$ instead makes a remarkable difference with the classical case (i.e., $C = \mathbb{R}^n$), where $p^y \equiv -1$ allows to fully determine the optimal control. It should be natural finding an adjoint vector which gives the same information as in the classical case in an interval where the optimal solution lies in the interior of C, but this feature does not follow from the method developed here.*
2) In order to have the assumption (M_1) be satisfied, we had to impose that the control u^y was negative and bounded away from zero. However, all arguments of Section 3.7 go through also in the case where u^y belongs to the interval $[-1, 1]$. In fact, the optimal control u_n^y for the approximate problem is always -1, so that the approximate solution (x_n, y_n) touches the

boundary of C only at one time.

3.9 Conclusions

Given a smooth moving set $C(\cdot)$ and smooth maps f and h, we have proved necessary optimality conditions for global minimizers of the problem (P), provided the strong inward/outward pointing conditions (M_1) or (M_2) are satisfied. Such conditions were imposed in order to deal with the discontinuity of the gradient of the distance to $C(t)$ at boundary points and actually transform the space discontinuity into a time discontinuity. A similar idea appears in [9]. An alternative approach is adopting the method developed in [16], which makes use of a smooth approximation of the distance. This approach, however, requires the uniform strict convexity and the time independence of the sweeping set.

If $C(t)$ is a moving smooth manifold without boundary, in particular has empty interior, then the squared distance $d^2_{C(t)}(\cdot)$ is of class C^2 in a whole neighborhood of $C(t)$. In this case, then, all the convergence arguments of Section 3.7 leading to Proposition 3.7.2 go through without requiring (M_1) or (M_2). Theorem 3.3.2 can be rephrased in this context, but for the sake of brevity we do not perform this task.

CHAPTER 4

NECESSARY CONDITIONS FOR A NONCLASSICAL CONTROL PROBLEM WITH STATE CONSTRAINTS

4.1 Introduction

The *sweeping process* was introduced by Moreau in the Seventies as a model for dry friction and plasticity (see [48]) and later studied by several authors. In its perturbed version, it features the differential inclusion

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t)), \quad t \in [0, T] \tag{4.1}$$

coupled with the initial condition

$$x(0) = x_0 \in C(0). \tag{4.2}$$

Here $C(t)$ is a closed moving set, with normal cone $N_{C(t)}(x)$ at $x \in C(t)$. The space variable, in this paper, belongs to $\mathbb{R}^n$. If $C(t)$ is convex, or mildly non-convex (in a sense that will not be made precise here), and is Lipschitz as a set-valued map depending on t, and the perturbation f is Lipschitz as well, then it is well known that the Cauchy problem (4.1), (4.2) admits one and only one Lipschitz solution (see, e.g., [61]). Observe that the state constraint $x(t) \in C(t)$ for all $t \in [0, T]$ is *built in the dynamics*, being $N_{C(t)}(x)$ empty if $x \notin C(t)$: should a solution $x(\cdot)$ exist, then automatically $x(t) \in C(t)$ for all t. If a control parameter u appears within f, then one is lead to study problems of the type

$$\dot{x}(t) \in -N_{C(t)}(x(t)) + f(x(t), u(t)), \quad u(t) \in U \tag{4.3}$$

subject to (4.2), aiming, for example, at

$$\text{minimizing } h(x(T)), \tag{4.4}$$

the final cost h being smooth. There is a clear difference with classical control problems with state constraints (see, e.g., [63]), where the constraint does not appear explicitly in the dynamics: in this case the right hand side of the dynamics is not Lipschitz with respect to the state variable, but indeed has only closed graph. This fact is a source of major difficulties in deriving necessary optimality conditions for (4.3), (4.4).

In recent years (see, e.g., [8], [36], [16], [30], [31], [4], and [21], and references therein) some papers dealing with control problems involving the sweeping process were published, the control appearing in the perturbation f and/or in the moving set C. Several necessary conditions were established, under different kinds of assumptions, or a Hamilton-Jacobi characterization of value function was proved. The present paper is devoted to prove a result inspired by [4] and [16]. More precisely, we prove necessary conditions of Pontryagin maximum principle type for (4.4) subject to (4.3) and (4.2), the control appearing only within f, in the case where $C(\cdot)$ is constant, smooth and convex (see Theorems 4.5.1 and 4.5.2). The case where C satisfies milder convexity assumptions and is not necessarily constant was treated in [4] with an extra assumption, while [16] contains results for a particular control problem involving a fixed smooth and *uniformly convex* set C. More preccisely, differently from [30] and [21], where discrete approximations are used, in both [16] and [4] the authors use a penalization technique. The classical Moreau-Yosida regularization allows in [4] to relax the uniform convexity assumption, at the price of requiring a strong *outward pointing condition* on f in order to treat the discontinuity of second derivatives of the squared distance function at the boundary of $C(t)$. In [16], the authors adopt a suitable smoothing of the distance, which on one hand needs $C(t)$ constant and uniformly convex and $0 \in C$, while on the other avoids imposing further compatibility assumptions between f and C. In this paper we adapt to our situation the method developed in [16] and remove the assumption of strict convexity on C. The main technical part is Section 4.

4.2 Preliminaries and assumptions

Notation. We define the distance from a set $C \subset \mathbb{R}^n$ as $d(x) = \inf\{\|y - x\| : y \in C\}$ and *signed* distance from C as $d_S(x) = d(x)$ if $x \notin C$ and $d_S(x) = -\inf\{\|y - x\| : y \notin C\}$ if $c \in C$. The normal cone to a convex set C is defined as $N_C(x) = \emptyset$ if $x \notin C$ and $N_C(x) = \{v \in \mathbb{R}^n : \langle v, y - x\rangle \leq 0 \ \ \forall y \in C\}$ if $x \in C$.

Assumptions on the set C**.** Let

$$C = \{x \in \mathbb{R}^n : g(x) \leq 0\},$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is of class C^2 with gradient $\nabla g \neq 0$ on the boundary ∂C of C, and with the Hessian matrix $\nabla^2 g(x)$ positive semidefinite for all $x \in \mathbb{R}^n$. Assume furthermore that $g(\cdot)$ is coercive, so that C is compact (and convex) and that $g(0) < 0$, so that $0 \in C$ and C has nonempty interior. Observe that under our assumption the signed distance

$d_S(x)$ from C is of class C^2 in a neighborhood of ∂C.

Assumptions on the dynamics and the cost. The control set $U \subset \mathbb{R}^n$ is compact and f is continuous and bounded, say by a constant β, and is of class C^1 with respect to x, with $\|\nabla_x f(x,u)\| \leq L$ for all x, u. The cost h is smooth.

Let now $\psi(x)$ be a C^2 smoothing of d_S in the interior of C (which is < 0 in $\operatorname{int} C$ and is such that $\nabla\psi(x)$ is the unit external normal to C at x for every $x \in \partial C$). Set also

$$\Psi(x) = \frac{1}{3}\psi^3(x)\, 1_{(0,+\infty)}(\psi(x)).$$

Observe that $\Psi(\cdot)$ is of class C^2 and convex in the whole of $\mathbb{R}^n$ and that both $\nabla\Psi(\cdot)$ and $\nabla^2\Psi(\cdot)$ vanish on C. Moreover one has, for each $x \in \mathbb{R}^n$,

$$\nabla\Psi(x) = d^2(x)\nabla d(x), \tag{4.5}$$

$$\nabla^2\Psi(x) = 2d(x)\nabla d(x) \otimes \nabla d(x) + d^2(x)\nabla^2 d(x), \tag{4.6}$$

because in C, and in particular at the points where $\nabla d(x)$ does not exist (namely, in ∂C), both sides of the above expressions vanish, and outside C they coincide.

4.3 The regularized problem

Consider the regularized dynamics

$$\dot{x}(t) = \frac{-1}{\varepsilon}\nabla\Psi(x(t)) + f(x(t),u(t)), \;\; x(0) = x_0, \tag{4.7}$$

where $\varepsilon > 0$ and $u(t) \in U$ for all t. For each given u, this Cauchy problem admits a unique solution x_ε for each $\varepsilon > 0$ on a maximal interval of existence. It is not difficult to prove that this interval is $[0,T]$ (see the proof of Proposition 4.3.1).

For every $\varepsilon > 0$ and every *global* minimizer x_*, u_* of (4.4) subject to (4.3) and (4.2), we consider the *approximate problem* $P_\varepsilon(u_*)$

$$\text{minimize } h(x(T)) + \frac{1}{2}\int_0^T \|u(t) - u_*(t)\|^2\,dt, \tag{4.8}$$

over controls u, where x is a solution of (4.7). By standard results, $P_\varepsilon(u_*)$ admits a global minimizer u_ε, with the corresponding solution x_ε. Necessary conditions of the original problem will be obtained by passing to the limit along conditions for $P_\varepsilon(u_*)$.

4.3.1 A priori estimates for the regularized problem

Proposition 4.3.1 *Let $\varepsilon_n \to 0$ and let (u_n, x_n) be a solution of the problem P_{ε_n}. Then, up to a subsequence, u_n converges strongly in $L^2(0,T)$ to u_* and x_n converges weakly in $W^{1,2}(0,T)$ to x_*.*

Proof. Since $0 \in C$ and so $\nabla\Psi(0) = 0$, by the convexity of Ψ we obtain that $\langle \nabla\Psi(x), x\rangle \geq 0$ for all $x \in \mathbb{R}^n$. Thus

$$\begin{aligned}&\|x_n(t)\| - \|x_0\| = \\ &= \int_0^t \Big\langle \frac{x_n(s)}{\|x_n(s)\|}, \frac{-1}{\varepsilon_n}\nabla\Psi(x_n(s)) + f(x_n(s), u_n(s))\Big\rangle\,ds \quad \leq \beta,\end{aligned}$$

which, in particular, implies that x_n is defined in the whole of $[0,T]$. Moreover,

$$\begin{aligned}\|\dot{x}_n\|_{L^2}^2 &= \int_0^T \Big\langle \dot{x}_n(t), \frac{-1}{\varepsilon_n}\nabla\Psi(x_n(t)) + f(x_n(t), u_n(t))\Big\rangle\,dt \\ &= \int_0^T \Big(\frac{-1}{\varepsilon_n}\frac{d}{dt}\Psi(x_n(t)) + \langle f(x_n(t), u_n(t)), \dot{x}_n(t)\rangle\Big)\,dt \\ &= \frac{-1}{\varepsilon_n}\Psi(x_n(T)) + \frac{1}{\varepsilon_n}\Psi(x_0) + \beta\int_0^T \|\dot{x}_n(t)\|\,dt \\ &\leq \beta\sqrt{T}\|\dot{x}_n\|_{L^2},\end{aligned}$$

where we have used the fact that $x_0 \in C$ and that $\psi(x_n(T)) \geq 0$. The above estimate implies that the sequence $\dot{x}_n$ is uniformly bounded in $L^2(0,T)$. Thus, up to a subsequence, x_n converges weakly in $W^{1,2}(0,T)$ to $\bar{x}$. Observe now that the from the uniform boundedness of $\|\dot{x}_n\|_{L^2(0,T)}$ and of f, we can deduce from (4.7), thanks to (4.5), that

$$\|d(x_n(\cdot))^2\|_{L^2(0,T)} \leq K\varepsilon_n \tag{4.9}$$

for a suitable constant K. Thus $x(t) \in C$ for all t. Again up to a subsequence, u_n converges weakly in $L^2(0,T)$ to some $\bar{u}$. By using the very same argument of Proposition 4.3 in [4], one can prove that $\bar{x}$ is the solution of (4.3), (4.2) corresponding to $\bar{u}$, that $\bar{u} = u_*$, and so that $\bar{x} = x_*$, and that the convergence is indeed strong. ■

Remark. Eq. (4.9) implies that, up to a subsequence,

$$\|d(x_n(\cdot))\|_{L^2(0,T)} \leq \sqrt{TK}\,\sqrt{\varepsilon_n}. \tag{4.10}$$

From the uniform convergence of x_n (again up to a subsequence) we also get $\|d(x_n(\cdot))\|_{L^\infty} \to 0$ for $n \to \infty$. This is an important difference between this approach and the use of Moreau-Yosida approximation, which instead yields the stronger estimate $\|d(x_n(\cdot))\|_{L^\infty} \sim \varepsilon_n$ (see [55] or [4, Proposition 4.1]). Observe that the assumption that C is constant appears essential in order to obtain uniform *a priori* estimates for $\|\dot{x}_n\|_{L^2}$ within the present approach, which essentially uses the weaker penalization d^3 instead of the Moreau-Yosida one, namely d^2.

4.3.2 Necessary conditions for the regularized problem

The approximate problem $P_\varepsilon(u_*)$ satisfies the assumptions for necessary conditions of classical unconstrained optimal control problems. The same computations of Section 6 in [4] yield that for every ε and every minimizer $(u_\varepsilon, x_\varepsilon)$ there exists an absolutely continuous adjoint vector $p_n : [0,T] \to \mathbb{R}^n$ such that

$$-\dot{p}_\varepsilon(t) = \Big(\frac{-1}{\varepsilon}\nabla^2\Psi(x_\varepsilon(t)) + \nabla_x f(x_\varepsilon(t), u_\varepsilon(t))\Big)p_\varepsilon(t) \tag{4.11}$$

a.e. on $[0, T]$, together with the final condition

$$-p_\varepsilon(T) = \nabla h(x_\varepsilon(T)), \tag{4.12}$$

and the maximality condition

$$\langle p_\varepsilon(t), \nabla_u f(x_\varepsilon(t), u_\varepsilon(t))u_\varepsilon(t)\rangle - \langle u_\varepsilon(t) - u_*(t), u_\varepsilon(t)\rangle =$$
$$\max_{u\in U}\{\langle p_\varepsilon(t), \nabla_u f(x_\varepsilon(t), u_\varepsilon(t))u\rangle - \langle u_\varepsilon(t) - u_*(t), u\rangle\}$$

for a.e. $t \in [0, T]$.

4.4 Passing to the limit

From now on we consider a sequence $\varepsilon_n \to 0$ such that the minimum (u_n, x_n) of the approximate problem converges as in the statement of Proposition 4.3.1. We set $p_n := p_{\varepsilon_n}$.

4.4.1 A priori estimates for the adjoint vectors of the approximate problem

We obtain from (4.11) and (4.6) that

$$\|p_n(t)\| - \|p_n(T)\| = \frac{-1}{\varepsilon_n}\int_t^T \Bigg(\frac{\Big\langle \nabla^2\Psi(x_n(s))p_n(s), p_n(s)\Big\rangle}{\|p_n(s)\|}$$
$$+\Big\langle \nabla_x f(x_n(s), u_n(s))p_n(s), \frac{p_n(s)}{\|p_n(s)\|}\Big\rangle\Bigg) ds \leq$$
(since $\nabla^2\Psi$ is positive semidefinite and $\nabla_x f$ is bounded)
$$\leq L\int_t^T \|p_n(s)\|\, ds.$$

Recalling (4.12), we obtain from the above inequality and Gronwall's lemma that there exists a constant K_1 independent of n such that

$$\|p_n\|_\infty \leq K_1. \tag{4.13}$$

Now we address ourselves to prove an *a priori* estimate on $\|\dot{p}_n\|_{L^1}$. To this aim, we define

$$\xi_n(t) = \langle p_n(t), \nabla d(x_n(t))\rangle,$$

(whenever it makes sense, i.e., if $x_n(t) \notin \partial C(t)$), so that,

$$\dot{\xi}_n(t) = \langle \dot{p}_n(t), \nabla d(x_n(t))\rangle + \langle p_n(t), \nabla^2 d(x_n(t))\dot{x}_n(t)\rangle.$$

Set now

$$\delta_n(t) = d(x_n(t)),\ \ \delta_n'(t) = \nabla d(x_n(t)),\ \ \delta_n''(t) = \nabla^2 d(x_n(t)).$$

With this notation, thanks to (4.5) and (4.6), eq. (4.11) can be rewritten as

$$\begin{aligned} -\dot{p}_n(t) = -\frac{\delta_n^2(t)}{\varepsilon_n}\delta_n''(t)p_n(t) - 2\frac{\delta_n(t)\xi_n(t)}{\varepsilon_n}\delta_n'(t) + \\ +\nabla_x f(x_n(t), u_n(t))p_n(t). \end{aligned} \tag{4.14}$$

Inserting $\dot{p}_n$ and $\dot{x}_n$ in the expression for $\dot{\xi}_n$ we obtain (omitting the t-dependence and using the fact that $\nabla^2 d(x)\,\nabla d(x) = 0$ for all x where it makes sense, and that $\delta_n\|\nabla\psi(x_n)\|^2 = \delta_n\|\nabla d(x_n)\| = \delta_n$)

$$\begin{aligned} -\dot{\xi}_n + 2\frac{\delta_n\xi_n}{\varepsilon_n} = &-\frac{\delta_n^2}{\varepsilon_n}\langle\delta_n''p_n, \delta_n'\rangle \\ &+\langle\nabla_x f(x_n, u_n)p_n, \delta_n'\rangle - \langle p_n, \delta_n'' f(x_n, u_n)\rangle. \end{aligned}$$

Observe now that the first summand in the right hand side of the above expression is bounded in $L^1(0, T)$ uniformly with respect to n, because, recalling (4.10), $\frac{\|\delta_n^2\|_{L^1}}{\varepsilon_n}$ is uniformly bounded. In turn, the second and the third summands are seen to be bounded in $L^\infty(0, T)$, uniformly with respect to n, by invoking (4.13). By multiplying both sides by sign(ξ_n) and integrating, we thus obtain the second estimate

$$\frac{1}{\varepsilon_n}\int_t^T \delta_n(s)|\xi_n(s)|\,ds \le K_2, \tag{4.15}$$

for a suitable constant K_2, independent of n. As a consequence, all three summands in the right hand side of the adjoint equation (4.14) are bounded in $L^1(0,T)$, uniformly with respect to n, and so we reach our final estimate

$$\|\dot{p}_n\|_{L^1(0,T)} \le K_3 \tag{4.16}$$

for a suitable constant K_3 independent of n.

4.4.2 Passing to the limit along the adjoint equation

By possibly extracting a further subsequence, we can assume that the sequence of measures $\dot{p}_n\,dt$ converges weakly∗ in the sense of Radon measures to a signed vector measure μ, which is the distributional derivative of the BV function $p(t) =: \lim p_n(t)$, i.e., $\mu = dp$, where the limit of the p_n is pointwise in $[0,T]$. By arguing as in [4, Proposition 7.3], we obtain first that the sequence of measures

$$\frac{\delta_n(t)\xi_n(t)}{\varepsilon_n}\delta_n'(t)\,dt$$

converges weakly* to a finite signed vector Radon measure, which can be written as

$$\xi(t)n_*(t)\,d\nu,$$

where $\xi \in L^1_\nu(0,T)$, $\xi \ge 0$ ν-a.e., $n_*(t)$ denotes the unit outward normal vector to C at $x_*(t)$ if $x_*(t) \in \partial C$ and 0 if $x_*(t) \in \operatorname{int} C$, and ν is a finite vector measure. Moreover

$$\frac{\delta_n^2(t)}{\varepsilon_n}\delta_n''(t)p_n(t) \rightharpoonup \eta(t)\nabla^2 d(x_*(t))p(t)$$

in $L^2(0,T)$, where $\eta \in L^\infty_\nu(0,T)$ and $\eta \ge 0$ a.e., with $\eta \equiv 0$ when x_* is in $\operatorname{int} C$.

4.4.3 Passing to the limit along the maximality condition

By taking into account Proposition 4.3.1, we obtain that the limit adjoint vector p is such that

$$\langle p(t), \nabla_u f(x_*(t), u_*(t)) u_*(t)\rangle = \\ = \max_{u \in U}\{\langle p(t), \nabla_u f(x_*(t), u_*(t)) u\rangle\} \text{ for a.e. } t \in [0, T]. \tag{4.17}$$

4.5 The main result

We deduce from Sections 3 and 4 the following necessary conditions:

Theorem 4.5.1 *Under the assumptions stated in Section 2, let* (x_*, u_*) *be a global minimizer for (4.4) subject to (4.3) and to (4.2). Then there exist a BV adjoint vector* $p : [0, T] \to \mathbb{R}^n$*, together with a finite signed Radon measure* ν *on* $[0, T]$*, and measurable vectors* $\xi, \eta : [0, T] \to \mathbb{R}$ *(with* $\xi \in L^1_\nu(0, T)$*,* $\xi(t) \geq 0$ *for* ν*-a.e.* t*,* $\eta \in L^\infty(0, T)$*, and* $\eta(t) \geq 0$ *for a.e.* t*) such that* $\xi(t) = \eta(t) = 0$ *for all* t *with* $x_*(t) \in \operatorname{int} C$*, satisfying the following properties:*

- *(adjoint equation)*

  for all continuous functions $\varphi : [0, T] \to \mathbb{R}^n$

$$\begin{aligned} -\int_{[0,T]} \langle \varphi(t), dp(t)\rangle = &-\int_{[0,T]} \langle \varphi(t), n_*(t)\rangle \xi(t)\, d\nu(t) \\ &-\int_{[0,T]} \langle \varphi(t), \nabla^2_x d(x_*(t)) p(t)\rangle \eta(t)\, dt \\ &+\int_{[0,T]} \langle \varphi(t), \nabla_x f(x_*(t), u_*(t)) p(t)\rangle\, dt, \end{aligned}$$

- *(transversality condition)*

$$-p(T) = \nabla h(x_*(T)),$$

• *(maximality condition)*

$$\langle p(t), \nabla_u f(x_*(t), u_*(t)) u_*(t) \rangle =$$
$$= \max_{u \in U} \langle p(t), \nabla_u f(x_*(t), u_*(t)) u \rangle \quad \text{for a.e. } t \in [0, T].$$

Some more precise statements on the measure ν require some assumptions on the reference trajectory x_*. In fact, consider the sets

$$E_0 := \{t \in [0, T] : x_*(t) \in \operatorname{int} C\}$$
$$E_\partial := \{t \in [0, T] : x_*(t) \in \partial C\}.$$

Of course, E_0 is open and E_∂ is closed, but one has to take into account the possibility that E_∂ be irregular (e.g., totally disconnected). Such phenomenon, in stratified state constrained control theory, is sometimes referred to as Zeno phenomenon, namely the switching from a stratum (the boundary of C, in this case) to other strata (the interior of C in this case) occurs at a complicated set (see, e.g., [10]).

The following is the second part of our necessary conditions. It is only a partial result, with respect to the rich set of conditions proved in [16].

Theorem 4.5.2 *Under the assumptions stated in Section 2, let (x_*, u_*) be a global minimizer for (4.4) subject to (4.3) and to (4.2), and let p the adjoint vector given by Theorem 4.5.1. Define $p^N(t) = \langle p(t), n_*(t) \rangle$, $t \in [0, T]$. The following properties hold:*

(1) $p^N(t) = 0$ *for all $t \in E_0$, and p is absolutely continuous on E_0, where it satisfies the classical adjoint equation*

$$-\dot{p}(t) = \nabla_x f(x_*(t), u_*(t))\, p(t). \tag{4.18}$$

(2) *At every interior (or such that a left or right neighborhood is contained in E_∂) point t of*

E_∂, jumps of p may occur only in the normal direction $n_*(t)$, namely

$$p(t-) - p(t+) = \big(p^N(t-) - p^N(t+)\big) n_*(t).$$

(3) *The adjoint vector p is absolutely continuous on every open interval contained in E_∂, and for a.e. t in such interval we have*

$$\begin{aligned} -\dot{p}(t) = \langle \dot{n}_*(t), p(t)\rangle n_*(t) + \Gamma(t)p(t) - \\ -\langle \Gamma(t)p(t), n_*(t)\rangle n_*(t), \end{aligned} \tag{4.19}$$

where $\Gamma(t) = \nabla_x f(x_*(t), u_*(t)) - \eta(t)\nabla_x^2 d(x_*(t))$.

Proof. *(1).* The first assertion is obvious, since on E_0 we have $n_* \equiv 0$ and the absolute continuity together with (4.18) follow from the properties of the functions ξ and η proved in Theorem 4.5.1.

(2). Since $n_*(t)$ is continuous on E_∂, there exist $n-1$ continuous unit vectors $v_1(t), \ldots, v_{n-1}(t)$ such that $\mathbb{R}^n = \mathbb{R}n_*(t) \oplus \operatorname{span}\langle v_1(t), \ldots, v_{n-1}(t)\rangle$ for all $t \in E_\partial$. Let $t \in E_\partial$ and $\sigma > 0$ be such that $[t-\sigma, t+\sigma] \subset E_\partial$. Let $\varphi : [0,T] \to \mathbb{R}^n$ be continuous, with support contained in $[t-\sigma, t+\sigma]$. Set $\varphi^T(t) = \varphi(t) - \langle \varphi(t), n_*(t)\rangle n_*(t)$. By putting $\varphi^T(t)$ in place of φ in the adjoint equation we obtain

$$\begin{aligned} -\int_{t-\sigma}^{t+\sigma} \langle \varphi^T(s), dp(s)\rangle + \int_{t-\sigma}^{t+\sigma} \langle \varphi^T(s), n_*(s)\rangle \xi(s)\, d\nu \\ = \int_{t-\sigma}^{t+\sigma} \langle \varphi^T(s), \nabla_x f(x_*(s), u_*(s)) p(s)\rangle\, ds \\ - \int_{t-\sigma}^{t+\sigma} \langle \varphi^T(s), \eta(s)\nabla^2 d(x_*(s)) p(s)\rangle\, ds. \end{aligned} \tag{4.20}$$

Observe now that $\langle \varphi^T, n_*(t)\rangle \equiv 0$, so that, by letting $\sigma \to 0$ in the above equation and using the continuity of φ^T, we obtain $\langle \varphi^T(t), p(t+) - p(t-)\rangle = 0$, namely $\langle p(t+) - p(t-), \varphi(t)\rangle = \big\langle p(t+) - p(t-), \langle \varphi(t), n_*(t)\rangle n_*(t)\big\rangle$. By taking subsequently φ such that $\varphi(t) = n_*(t)$, and $\varphi(t) = v_i(t)$, $i = 1, \ldots, n-1$, we obtain that $p(t-) - p(t+) = \big(p^N(t-) - p^N(t+)\big) n_*(t)$, namely jumps of p may occur only in the direction $n_*(t)$, for all t in the interior of E_∂. If

t is a left or a right endpoint of E_∂, one can extend n_* as a constant to the left or to the right of t and repeat the same argument.

(3). Fix now an interval $[s,t] \subseteq E_\partial$. The regularity condition on ∂C allows us to integrate by parts on (s,t), so that

$$\begin{aligned}\int_s^t \langle n_*(\tau), dp(\tau)\rangle + \int_s^t \langle \dot{n}_*(\tau), p(\tau)\rangle \, d\tau = \\ = \langle n_*(t+), p(t+)\rangle - \langle n_*(s-), p(s-)\rangle = 0. \end{aligned} \tag{4.21}$$

Observe that both summands in the right hand side of (4.21) vanish, as a consequence of *(1)* and of the fact that p has bounded variation, since s and t belong to the interior of I_∂. In other words, the two measures $\langle n_*, dp\rangle$ and $\langle \dot{n}_*, p\rangle \, dt$ coincide in any open interval contained in E_∂. Therefore, for all continuous φ with support contained in (s,t), we obtain from (4.20) and (4.21) that

$$\begin{aligned} -\int_s^t \langle \varphi(\tau), dp(\tau)\rangle = \\ = -\int_s^t \langle \varphi(\tau), n_*(\tau)\rangle \, \langle n_*(\tau), dp(\tau)\rangle - \int_s^t \langle \varphi^T(\tau), dp(\tau)\rangle \\ = \int_s^t \langle \varphi(\tau), n_*(\tau)\rangle \langle p(\tau), \dot{n}_*(\tau)\rangle d\tau \\ + \int_s^t \Big\langle \varphi(\tau) - \langle \varphi(\tau), n_*(\tau)\rangle n_*(\tau), \nabla_x f(x_*(\tau), u_*(\tau)) p(\tau)\Big\rangle d\tau \\ - \int_s^t \Big\langle \varphi(\tau) - \langle \varphi(\tau), n_*(\tau)\rangle n_*(\tau), \eta(\tau) \nabla_x^2 d(x_*(\tau)) p(\tau)\Big\rangle d\tau \end{aligned}$$

Since φ and the interval (s,t) are arbitrary, we obtain (4.19). ■

CONCLUSIONS AND PERSPECTIVES

In this thesis, by using tools from nonsmooth and variational analysis, we have studied diferential inclusions involving normal cones in Hilbert spaces. Although the main focus of this thesis has been the sweeping process, the developed methods have allowed us to address several diferential inclusions involving normal cones.

In chapter two, we give a new approach, in which the compactness assumption is shifted from the uniformly prox-regular set to the perturbation, and prove the existence of local solutions without compactness of the set. In this chapter we ask the following open question: How can we get the validity of Theorem 2.3.1 and Theorem 2.4.1 without using specific assumptions about the displacement of the moving set and how can we get around the assumption of compactness ? The arguments (used in Sect. 2.3 and Sect. 2.4) are based on Proposition 2.4.1 and are specific to the shifted moving set case. Mainly, the property (2.34) is the bridge between the normal cone and the nonconvex set-valued perturbation and permits to get the convergence for the norm topology of space of Lebesgue square integrable functions of the sequences of derivatives. For general moving set, this property do not holds and so the existence of solutions seems to be a difficult problem. We also probably need a new approach of this question.

In chapter tree, we get a nonclassical necessary optimality conditions of Pontryagin's Maximum Principle type for a Mayer problem. Our assumptions include the smoothness of the boundary of the moving but do not require strict convexity and time independence of the set, rather, a kind of inward/outward pointing condition is assumed on the reference optimal trajectory. A related problem that can be addressed is try to remove this inward/outward pointing condition.

In chapter four, we adapt to our situation the method developed in chapter tree, remove the assumption of strict convexity on the set and inward/outward pointing condition is removed also.

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