PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION
AND SCIENTIFIC RESEARCH
JIJEL UNIVERSITY

FACULTY OF EXACT SCIENCES AND COMPUTER SCIENCE

DEPARTEMENT OF PHYSICS

serie :

Thesis presented to obtain diploma of Master in physics

Option

Theoritical physics

by

Kamli Marwa

Theme

Systems with constraints treated by *Faddeev and Jackiw method*

Supported on : 27 /10/2020 .

Before the jury :

President	A. Tilbi	A. Prof. At Jijel. Univ.
Reporter:	N. Farkous	A. Prof. At Jijel. Univ.
Examiner:	T. Boudjedaa	Prof. At Jijel. Univ.

Contents

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Chapter 1

General introduction

As is well known, the systems described by singular Lagrangian contain inherent constraints that appear clearly in the calculation of the conjugate momenta, where some velocities look as if it had been absorbed inside them. According to the classical view point, the conjugate momenta deduced from this Lagrangians are not all invertible with respect to velocities, which is the main indication that lead us to recognize the existance of the singular Lagrangian type besides to the standard one. Although the Legendre transformation allows us to deduce the canonical Hamiltonian, the corresponding canonical equations of motion must be modified so that they contain the constraints in question after constructing the new Hamiltonian on the same frequency. This adjustment has been made in order to have a good Hamiltonian that has no contradictory in the motion equations comparing by the one obtained by Euler-Lagrange.

Seventy years ago, a consistent analysis of constrained systems was formulated by Dirac [1], then developed by Bergmann [2] whose works were the pioneers in this treatment. This formulation considered as a standard model to theories that are characterized by constraints and offers suitable generalized brackets which is the crossing road to quantize these kind of systems [3]. Since that time, this formalism based on the classification of constraints in first and second class, weak and strong equality notions and his algorithm besides to Bergmann, has been widely used in many quantum systems [4, 5, 6], in gauge field theories, gravitational field theory, supersymmetric theory, super gravity and superstring theory...etc [7]. Although this formalism is very powerful and consistent, it necessitates considerable calculation of the basic geometric structures known as Dirac brackets. It could be that Dirac's method overloads the problem and leads to unnecessary calculations. On this direction, Faddeev and Jackiw have developed a new more economical method based on linear Lagrangians and depends on the matrix form of Euler-Lagrange equations [8]. The motion equations deduced from these Lagrangians of the first order do not contain the accelerations, where the basic geometric structure known by generalized Poisson brackets can be deduced directly from them as elements of a matrix known by their names: Fadeev-Jackiw matrix. Moreover, we can show that these brackets coincide with those obtained by Dirac's method; therefore they will be the main tools for a quantum theory.

The purpose of this thesis is to present Faddeev-Jackiw method which will serve for a good initiation to scientific research given the growing interest of physicists for methods of quantification to classical systems with the first order of lagrangians used for the presence and absence of constraints by much simpler and faster way comparing to Dirac's one for singular systems [6]. We based to show this covered aim under the shadow of comparative study proved effictevely in the illustrative applications.

This thesis contains five chapters besides to this general introduction.The next chapter will notably be a reminder to the singular Lagrangian notion with some needed tools of analytical mechanics.The third chapter will be devoted to the study of systems with constraints by Dirac's method until we will define finally its brackets [6]. In the fourth chapter, we will expose the Faddeev -Jackiw method, where we'll show the simplicity and the efficiency of this approach that lead us to the set of Dirac's brackets in one fell swoop. The fifth chapter will be two illustrative applications of particle moving on a circle and ellipse treated by the both mentioned methods in order to compare between them. Finally, we ends with the conclusion in a form of compartive study and prespective to these treatments as the last chapter.

Chapter 2

Singular Lagrangian

2.1 Introduction

To study the dynamic of a system that described by Lagrangian, we need to calculate the Euler-Lagrange equations that lead us to get finally the motion equations, where all the accelerations are expected to be expressed in functions of positions and velocities as a standard model for treatment. On the other hand, if we do not reach this expection, it is obvious that we are dealing with the opposite case where our Lagrangian seemed to be *singular*. The dilemma is in this last type of systems which is characterized by constraints presence submitted on the initial data and assumed generally to be independent of time. Besides that the Lagrangian type may be predicted from the constraints, there exists a definitive way to determine its quality from the determinant of what is known as *the Hessian matrix* . The singular Lagrangian expected to be treated in exception way that made physicists to search for methods to deal with it.

The aim of this first chapter is to give an introduction to singular Lagrangian which is the main motivation that leads us to expose two effective ways to treat its systems as we will show in the next chapters, depending on simple and illustrative examples. However, this can not be approached directly without going through important concepts in analytical mechanics seemed to be related to what is known as the Lagrangian and the Hamltonian formalism that are descibed respectively in configuration and phase spaces.

2.2 Lagrangian formalism

To describ a dynamic system, we give the Lagrangian $L\left(q_i,\dot{q}_i\right)$ with N number of freedom degrees where q_i and $\dot{q}_i$ represent coordinates and velocities respectively, while $(i=1,...,n)$. The action S between two points t_1 and t_2 is given by the expression

$$S=\int_{t_1}^{t_2} L\left(q_i,\dot{q}_i\right)dt. \tag{2.1}$$

Most of the basic equations in physics can be deduced from what we call least action principle which stipulates that the action S must be stationary, and its small variation δS tends towards zero between two close moments t_1 and t_2 verifynig conditions that $\delta q\left(t_1\right)=\delta q\left(t_2\right)=0$ .Indeed , the variation of the action is then written :

$$\begin{aligned}\delta S &= \int_{t_1}^{t_2}\delta L\left(q_i,\dot{q}_i\right)dt\\ &= \int_{t_1}^{t_2}\sum_i\left(\frac{\partial L}{\partial q_i}\delta q_i+\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right)dt,\end{aligned}$$

where we'll integrate by using

$$\delta\dot{q}_i=\delta\frac{dq_i}{dt}=\frac{d}{dt}\delta q_i \text{ and } \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i=\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\right),$$

to get

$$\delta S=\sum_i\frac{\partial L}{\partial \dot{q}_i}\delta q_i\Bigg|_{t_1}^{t_2}+\int_{t_1}^{t_2}\sum_i\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i dt,$$

taking into account the conditions at the boundary that we have already mentioned above, we arrive to

$$\delta S=\int_{t_1}^{t_2}\sum_i\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i dt,$$

this variation must be null regardless of δq_i value, this is only possible if

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}=0,\quad i=1,...,n, \tag{2.2}$$

this equations called Euler-Lagrange equation can be written by p_i as follows

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2.3}$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i}, \tag{2.4}$$

where p_i defined in (2.3) called conjugate momenta, while (2.4) is the veritable motion equation according to the sense of Newton and Lagrange.

2.3 Hamiltonian formalism

Starting from the Lagrangian and using the transformation of Legendre, we can construct the Hamiltonian wich is a new description much effective in symmetric systems than lagrangian formalism. It depends on moving from the configuration space with n dimensions to the phase one with $2n$ dimensions, by remplacing the n generalized velocities $\dot{q}_i$ according to the momenta p_i defined in (2.3), where $i = 1, ..., n$. Thus, the Hamiltonian experssion is given as follows

$$H(q_i, p_i) = p_i\dot{q}_i - L\left(q_i, \dot{q}_i\right). \tag{2.5}$$

The action principle (2.1) gives

$$\begin{aligned} S &= \int_{t_1}^{t_2} L dt \\ &= \int_{t_1}^{t_2} \left(p_i\dot{q}_i - H(q_i, p_i)\right) dt. \end{aligned} \tag{2.6}$$

The principle of least action stipulates that ($\delta S = 0$) between two times t_1 and t_2 as follows

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \delta\left(p_i\dot{q}_i - H(q_i, p_i)\right) dt \ = \int_{t_1}^{t_2} \left(\delta p_i\dot{q}_i + p_i\delta\dot{q}_i - \delta H(q_i, p_i)\right) dt \\ &= \int_{t_1}^{t_2} \left(\delta p_i\dot{q}_i + p_i\delta\dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right) dt \\ &= \int_{t_1}^{t_2} \left(\delta p_i\dot{q}_i + \frac{d}{dt}\left(p_i\delta q_i\right) - \dot{p}_i\delta q_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right) dt, \end{aligned}$$

that can be written

$$\delta S = \left(p_i\delta q_i\right)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i}\right)\delta q_i\right) dt.$$

Starting from that $\delta q\left(t_1\right) = \delta q\left(t_2\right) = 0$, the first term is null. Moreover, the variations δp_i and $\delta q_i$are independents. So to have $\delta S = 0$ we must offer that

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad i = 1, ..., n \tag{2.7}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1, ..., n, \tag{2.8}$$

which are called Hamilton's equations. These equations are principally equivalents with Euler-Lagrange equations (2.2).

2.4 General form of Poisson brackets

Defining the ordinary form of the Poisson bracket that depends on the two functions $f\left(q_i, p_i\right)$ and $g\ \left(q_i, p_i\right)$ as follows

$$\{f, g\}=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right), \tag{2.9}$$

where Poisson bracket verify the next proprieties

$$\begin{aligned}
\{f, g\} &= -\{g, f\} && \text{(Antisymmetry)}\\
\{f+h, g\} &= \{f, g\}+\{f, h\} && \text{(Linearity)}\\
\{fh, g\} &= f\{h, g\}+\{f, g\}h && \text{(Leibniz's identity)}\\
\{f,\{g, h\}\}+\{g,\{h, f\}\}+\{h,\{f, g\}\} &= 0 && \text{(Jacobi's identity)}
\end{aligned}\quad .$$

We can express Hamilton's equations as follows

$$\dot{q}_i = \{q_i, H\}, \quad i=1,...,n \tag{2.10}$$

$$\dot{p}_i = \{p_i, H\}, \quad i=1,...,n. \tag{2.11}$$

We can rewrite the formula of Poisson bracket as more general and practical form that will be used later in the next chapters

$$\{f, g\}_{GPB}=\sum_{ij} J_{ij}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}, \qquad i,\ j=1,2,.....,2n \tag{2.12}$$

where $J_{ij}=\left\{\xi_i, \xi_j\right\}$ is an antisymmetric matrix element called *structure matrix*.So, the motion equation is written as

$$\dot{f}=\{f, H\}_{GPB}$$

For our phase space, the dynamic variables are given by

$$\left(\xi_1,\ \xi_2,...,\xi_n,\ \xi_{n+1},...,\xi_{2n}\right)=\left(q_1,\ q_2,...,\ q_n,\ p_1,...,p_n\right).$$

For the dynamic variable ξ_i, we have this relation

$$\{\xi_i, f\}_{GPB}=\sum_j J_{ij}\frac{\partial f}{\partial \xi_j} \tag{2.13}$$

2.5 Singular Lagrangian

The determination of Lagrangian quality depends on the determinant of *the Hessian matrix*, that can be constructed from the differential derivative of momenta with respect to velocities, where $p_i = p_i\left(q_i, \dot{q}_i\right)$ defined by (2.3) in a system with N number of freedom degrees according to the Lagrangian $L\left(q_i, \dot{q}_i\right)$, $i = 1, ..., n$, as follows

$$dp_i = \sum_j \frac{\partial p_i}{\partial q_j} dq_j + \sum_j \frac{\partial p_i}{\partial \dot{q}_j} d\dot{q}_j, \tag{2.14}$$

and

$$\frac{dp_i}{dt} = \sum_j \frac{\partial p_i}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial p_i}{\partial \dot{q}_j} \ddot{q}_j, \tag{2.15}$$

remplacing the relation (2.3) in (2.15) we obtian

$$\frac{dp_i}{dt} = \sum_j \frac{\partial L^2}{\partial q_j \partial \dot{q}i} \dot{q}_j + \sum_j \frac{\partial L^2}{\partial \dot{q}_j \partial \dot{q}_i} \ddot{q}_j, \tag{2.16}$$

we use now the equation (2.4), we get the equality

$$\sum_j \frac{\partial L^2}{\partial q_j \partial \dot{q}_j} \dot{q}_j + \sum_j \frac{\partial L^2}{\partial \dot{q}_j \partial \dot{q}_i} \ddot{q}_j - \frac{\partial L}{\partial q_i} = 0,$$

or else

$$\sum_j W_{ij}\left(q, \dot{q}\right) \ddot{q}_j = \frac{\partial L}{\partial q_i} - \sum_j \frac{\partial L^2}{\partial q_j \partial \dot{q}_i} \dot{q}_j, \tag{2.17}$$

where W is the Hessian matrix defined by the next elements

$$W_{ij} = \frac{\partial L^2}{\partial \dot{q}_j \partial \dot{q}_i} = \frac{\partial p_i}{\partial \dot{q}_j}, \tag{2.18}$$

If $\det W \neq 0$, the marix W is invertible, it means that we can express all the $\ddot{q}_i$ as functions of $\dot{q}_i$ and q_i. This signifies that a unique solution of (E-L) equations exists, and we are dealing with non-singular Lagrangian. Contrariwise, if $\det W = 0$, the matrix W is not invertible, and the Lagrangian is seemed to be *singular.*

As we know, to pass from the Lagrangian formulation to the Hamiltonian one, it must be that all the velocities $\dot{q}_i$ expressed by functions of q_i and p_i as follows :

$$\dot{q}_i = f\left(q_i, p_i\right), \tag{2.19}$$

while the Hamiltonian (2.5) can be constructed by the Legendre transformation as

$$H = \sum_i p_i f\left(q_i, p_i\right) - L\left(q_i, f\left(q_i, p_i\right)\right). \tag{2.20}$$

It is clear that the procedure of having the Hamiltonian (2.20) is based particularly on the possibility of solving $p_i = \partial L/\partial \dot{q}_i$. This requires that the Jacobian matrix $\partial p_i/\partial \dot{q}_j$ is invertible, and it leads to

$$\frac{\partial p_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} = W_{ij}. \tag{2.21}$$

Thus, in the case of a singular Lagrangian, it is impossible to pass to the Hamiltonian formulation in a standard way. We will illustrate this point with the following example

Considering the Lagrangian with two degrees of freedom [6] as follows

$$L = \frac{1}{2}\left(\dot{x} - y\right)^2, \tag{2.22}$$

The Hessian matrix W correspondent is

$$W = \begin{pmatrix} \frac{\partial^2 L}{\partial \dot{x}\partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} \\ \frac{\partial^2 L}{\partial \dot{y}\partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{y}\partial \dot{y}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.23}$$

This Lagrangian is singular since that $\det W = 0$. The conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y \text{ and } p_y = \frac{\partial L}{\partial \dot{y}} = 0. \tag{2.24}$$

which define the momenta that are insoluble with respect to $\dot{y}$, as what it was expected for a singular Lagrangian.

Chapter 3

Dirac's method for systems with constraints

3.1 Introduction

Hamiltonian of constrained systems represents an important class of physical systems described by singular Lagrangians. In this case, our conjugate momenta will not all be invertible with respect to velocities as already mentioned in the previous chapter.The Hamiltonian can be always formulated by the Legendre transformation, but in singular systems, it must be corrected so that it contains the constraints in question multiplied by what is called *Dirac's multipliers*. As a result,the canonical Hamiltonian equations changed automatically to be equivalent with Euler-Lagrange equations.

Dirac was the first who succeeded in treating singular systems by standard and consistent manner [1]. In Dirac's formalism, the inherent constraints would be generated and called primary constraints. Due to the consistency conditions, these primary constraints may generate new constraints, called secondary constraints.This iterative way of calculating the different constraints in the Dirac formalism is called the Dirac-Bergmann algorithm that ends when we determine Dirac's multipliers. The Poisson brackets must be replaced by another brackets called Dirac brackets which are more adequate in the presence of constraints.

Thus, the aim of this chapter is to expose this algorithm step by step till we will end with Dirac brackets determination that may lead us to correct quantizations of constrained systems.

3.2 Primary constraints and the new Hamiltonian formalism

In a system that desccribed by a singular Lagrangian in wich $\det W = 0$, and the conjugate momenta are defined by (2.3), may not all be invertible to velocities.We can't work directly by standard way to get the Hamiltonian equations as we did above. Therefore, we use Dirac's method to fix the problem starting on constructing contraints as follows:

the momenta are not all independent, but there are rather some relations of the type $\phi_m(q,p) = 0$ called primary constraints, that was obtained automatically from the canonical definition of momenta $p_i = \partial L/\partial \dot{q}_i$, $i = 1, ..., n$. where M is the constraints number

$$\phi_m(q,p) = 0,\ m = 1,..,M \quad \text{where } q = (q,p) \quad \text{and} \quad M = \dim(W) - rank(W). \tag{3.1}$$

In line to the primary constraints existance, our system must be descibed by new total Hamiltonian H_T or new Lagrangian $\tilde{L}$ depend on them besides to the older canonical form of H_c or L respectively, where λ_m is the Dirac's multipliers, and the total Hamiltonian expression is given by

$$H_T(p,q) = H_c(p,q) + \lambda_m \phi_m(p,q), \tag{3.2}$$

it can be expressed also by the transformation of Legendre in the opposite direction, and allows to extract the new Lagrangian as follows

$$H_T(p,q) = p_i\dot{q}_i - \tilde{L} \ \text{ leads to } \tilde{L} = p_i\dot{q}_i - H_T(p,q) = p_i\dot{q}_i - H_c(p,q) - \lambda_m\phi_m(p,q). \tag{3.3}$$

The principle of least action stipulates that ($\delta S = 0$) between two times t_1 and t_2 giving

$$\delta S = \delta \int_{t_i}^{t_f} \tilde{L} dt = \delta \left[\int_{t_i}^{t_f} p_i\dot{q}_i - H_c(p,q) - \lambda_m\phi_m(p,q)\, dt \right] = \int_{t_i}^{t_f} \left[\delta\left(p_i\dot{q}_i - H_c\right) - \delta\left(\lambda_m\phi_m\right)\right] dt,$$

leads to

$$\delta S = \int_{t_i}^{t_f} \left[\left(\dot{q}_i - \frac{\partial H_c}{\partial q_i} - \lambda_m \frac{\partial \phi_m}{\partial p_i} \right) \delta p_i + \left(-\dot{p}_i - \frac{\partial H_c}{\partial q_i} - \lambda_m \frac{\partial \phi_m}{\partial q_i} \right) \delta q_i - \delta\lambda_m \phi_m \right] dt, \tag{3.4}$$

Since $\phi_m(q,p) = 0$ and $\delta S \to 0$, moreover, $\forall\ \delta p_i$, δq_i and $\delta \lambda_m$ that are independents, we get finally the new Hamiltonian equations

$$\dot{q}_i = \frac{\partial H_c}{\partial q_i} + \lambda_m \frac{\partial \phi_m}{\partial p_i}, \quad i = 1,..,n \tag{3.5}$$

$$\dot{p}_i = -\frac{\partial H_c}{\partial q_i} - \lambda_m \frac{\partial \phi_m}{\partial q_i}, \quad i = 1,..,n \tag{3.6}$$

$$\phi_m = 0, \qquad m = 1,..,M. \tag{3.7}$$

To have the Poisson bracktes form of these equations, we constuct the general formula of the differential equation with respect to time of the function $F = F(q,p)$ using the usual mathematical relation

$$\dot{F} = \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i, \tag{3.8}$$

using (3.5), (3.6) and (3.7) we have

$$\dot{F} = \frac{\partial F}{\partial q_i}\frac{\partial H_c}{\partial q_i} - \frac{\partial F}{\partial p_i}\frac{\partial H_c}{\partial q_i} + \left(\frac{\partial F}{\partial q_i}\frac{\partial \phi_m}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial \phi_m}{\partial q_i}\right)\lambda_m \quad ; \quad \phi_m = 0,$$

where $\dot{F}$ may take the Poisson bracket form as follows

$$\dot{F} = \{F, H_c\} + \lambda_m \{F, \phi_m\} \quad ; \quad \phi_m = 0. \tag{3.9}$$

According to Dirac, it is necessary to calculate the Poisson brackets before using the constraints $\phi_m = 0$. It is therefore convenient to rewrite the previous equation in this form

$$\dot{F} = \left(\{F, H_c\} + \lambda_m \{F, \phi_m\}\right)\big|_{\phi_m=0} \tag{3.10}$$

or

$$\dot{F} = \{F, H_T\}\big|_{\phi_m=0}. \tag{3.11}$$

Exemple

Considering the Lagrangian from [6]

$$L = \frac{1}{2}\dot{x}^2 + x\dot{y} + f(x,y).$$

Calculating the (E-L) equations

$$\dot{y} + \tfrac{\partial f}{\partial x} - \ddot{x} = 0, \qquad \tfrac{\partial f}{\partial y} - \dot{x} = 0, \tag{3.12}$$

and the conjugate momenta

$$p_x = \tfrac{\partial L}{\partial \dot{x}} = \dot{x}, \quad p_y = \tfrac{\partial L}{\partial \dot{y}} = x,$$

we have the primary constraint $\phi_1 = p_y - x = 0$. Forming the canonical Hamiltonian

$$H_c = \dot{x}p_x + \dot{y}p_y - L = \tfrac{1}{2}p_x^2 - f(x,y)$$

If we try to calculate Hamilton's equations from H_c, we will obtain equations which are not equivalent to the equations of (E-L). Indeed, we will obtain the equations

$$\begin{cases} \dot{x} = p_x \\ \dot{p}_x = \frac{\partial f}{\partial x}. \end{cases}, \quad \begin{cases} \dot{y} = 0 \\ \dot{p}_y = \frac{\partial f}{\partial y}. \end{cases} \tag{3.13}$$

Therefore, we must hamiltonize H_c i.e Finding H_T for which the corresponding hamiltonian equations will be equivalent to the E-L one.Writing H_T as follows

$$H_T = H_c + \lambda_1\phi_1 = \frac{1}{2}p_x^2 - f(x,y) + \lambda_1\left(p_y - x\right)$$

Thus, the Hamiltonian equations lead to

$$\begin{cases} \dot{x} = p_x \\ \dot{p}_x = \frac{\partial f}{\partial x} + \lambda_1 \end{cases}, \quad \begin{cases} \dot{y} = \lambda_1 \\ \dot{p}_y = \frac{\partial f}{\partial y} \end{cases}, \text{ and } \; p_y - x = 0 \tag{3.14}$$

3.3 Weak and strong equality

Dirac introduced the notion of the weak equality under that sign (" $\approx$ ") replacing the constraints condition given by $\phi_m = 0$, where the system was described by $(= ()|_{\phi_m=0})$ to express the dynamic only in the sub space of constraints, otherwise the notion of strong equality ("=") is vailable in all the space. Thus, the evolution equations may be written as follows

$$\dot{F} = \{F, H_T\}|_{\phi_m=0} \tag{3.15}$$

$$\dot{F} \approx \{F, H_T\} \approx \{F, H_c\} + \lambda_m \{F, \phi_m\}, \tag{3.16}$$

Therefore we can write the Hamiltonian equations in the form of Poisson brackets as well

$$\dot{q}_i \approx \{q_i, H_T\}, \quad \dot{p}_i \approx \{p_i, H_T\}. \tag{3.17}$$

3.4 Secondary constraints and Dirac-Bergmann algorithm

The primary constraints must be preserved over time during an evoltion, we can write

$$\tfrac{d\phi_{m'}}{dt} = \dot{\phi}_{m'} \approx 0, \quad m' = 1, ...M, \tag{3.18}$$

but according to (3.16), we'll have

$$\dot{\phi}_{m'} = \{\phi_{m'}, H_T\} \approx 0 \Leftrightarrow \{\phi_{m'}, H_c\} + \lambda_m \{\phi_{m'}, \phi_m\} \approx 0, \qquad m', m = 1, ...M. \tag{3.19}$$

That are called *consistency conditions (the CCs)*, where they are related to primary constraints here specifically.The system (3.19) is a system of non-homogeneous algebraic equations, which will help us to verify the Dirac' multipliers λ_m. In reality, the study of this system will lead us to one of the following three situations :

1) The CCs determine the Dirac's multipliers either all (all equations give values of λ_m with $m = 1, ...M$) or some (in addition to some equations which are identically true such that $0 \approx 0$). In this case, the iteration stops.

2) The CCs do not determine multipliers and gives at least one incorrect equation such as for example ($1 = 0$). In this case, there is certainly an anomaly, so it is useless to go further before modifying the Lagrangian itself, and restarting again the steps.

3) The CCs do not determine the multipliers directly, and give new different relations between p_i and the q_i described by the formula $\varphi_k(q, p) \approx 0$, $k = 1, ..., K$, that expresses a new restarting called *secondary constraints* can have also CCs according to (3.16) and need to be treated to give cases as the both that we have already mentioned besides to this one itself. The iteration stops in the end, where we may determine mutipliers.

The logical analysis above was formulated in a sequential consistent manner with restricted iteration may be stopped or continued according to the existing situation that ends by the determination of multipliers as a goal. This process is known as *The Dirac-Bergmann algorithm.*

3.5 Constraints classification

Considering $\{\phi_j \approx 0\}$ with $j = 1, ..., J = M + K$ that describs all the constraints (secondary and primary), where M is the number of primary constraints, and K the one of secondary constraints. According to Dirac we say that the function $F(q, p)$ is first class if its Poisson bracket with each of the constraints (primary or secondary) that are included under the previous relation, is null on the surface of constraints, i.e $\left\{F, \phi_j\right\} \approx 0$.Otherwise, we say that the function $F(q,p)$ is second class, if $\left\{F, \phi_j\right\} \not\approx 0$ (at least for one j).

3.6 Dirac brackets

We will assume that all the constraints of our system (primary and secondary) are secondary class. We notice that ϕ_m, $m = 1, ..., M$ the primary constraints, while ϕ_k , $k = 1, ..., K$ secondry constraints. Writing the CCs of the set of constraints, we get

$$\left\{\phi_j, H_c\right\} + \lambda_m \left\{\phi_j, \phi_m\right\} \approx 0, \quad m = 1, ..., M \;\; \text{et} \;\; j = 1, ..., J = K + M \tag{3.20}$$

where

$$H_T = H_c + \lambda_m \phi_m, \quad m = 1, ..., M.$$

Rewriting(3.20) in matrix form as follows

$$\underbrace{\begin{pmatrix} \{\phi_1, \phi_1\} & .... & \{\phi_1, \phi_M\} \\ \vdots & \ddots & \vdots \\ \{\phi_J, \phi_1\} & .... & \{\phi_J, \phi_M\} \end{pmatrix}}_{=\Omega} \underbrace{\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_M \end{pmatrix}}_{=\boldsymbol{\lambda}} \approx \underbrace{\begin{pmatrix} -\{\phi_1, H_c\} \\ \vdots \\ -\{\phi_J, H_c\} \end{pmatrix}}_{=\boldsymbol{\eta}}, \tag{3.21}$$

Or else

$$\Omega \lambda \approx \eta, \tag{3.22}$$

where Ω is a matrix of K lines and M columns. Forming now the square matrix Δ defined by

$$\Delta_{\alpha,\alpha'} = \{\phi_\alpha, \phi_{\alpha'}\} \quad , \; \alpha, \alpha' = 1, ..., J \;\; \text{where } J = M + K, \tag{3.23}$$

this matrix is antisymmetric and contains the matrix Ω as a block; explicitly

$$\Delta = \begin{pmatrix} \{\phi_1, \phi_1\} & .... & \{\phi_1, \phi_M\} & \{\phi_1, \phi_{M+1}\} & .... & \{\phi_1, \phi_J\} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \{\phi_J, \phi_1\} & .... & \{\phi_J, \phi_M\} & \{\phi_J, \phi_{M+1}\} & .... & \{\phi_J, \phi_J\} \end{pmatrix}$$

$$= \begin{pmatrix} \underbrace{\begin{matrix} 0 & .... & \{\phi_1, \phi_M\} \\ \vdots & \ddots & \vdots \\ \{\phi_J, \phi_1\} & .... & \{\phi_J, \phi_M\} \end{matrix}}_{=\Omega} & \underbrace{\begin{matrix} \{\phi_1, \phi_{M+1}\} & .... & \{\phi_1, \phi_J\} \\ \vdots & \ddots & \vdots \\ \{\phi_J, \phi_{M+1}\} & .... & 0 \end{matrix}}_{=\omega} \end{pmatrix},$$

Where ω is a matrix with J lines and $J-M$ columns. Dirac has shown that $\det(\Delta) \neq 0$ (for the demonstration, see [1]), moreover the matrix Δ must be of even dimension, because the determinant of an odd antisymmetric matrix must be null . Considering now the column vector θ at J components

$$\theta = \begin{pmatrix} \lambda_1 & .... & \lambda_M & \underbrace{0 \quad .... \quad 0}_{J-M} \end{pmatrix}^t, \tag{3.24}$$

or otherwise written

$$\boldsymbol{\theta} = \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{0} \end{pmatrix}. \tag{3.25}$$

Calculating the product $\Delta\theta$ by block as follows

$$\Delta\boldsymbol{\theta} = (\Omega\omega)\begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{0} \end{pmatrix} = \Omega\boldsymbol{\lambda}, \tag{3.26}$$

then by comparing between(3.22)and(3.26), we get

$$\Delta\boldsymbol{\theta} \approx \boldsymbol{\eta}, \tag{3.27}$$

since Δ is invertible,we can obtain

$$\boldsymbol{\theta} \approx \Delta^{-1}\boldsymbol{\eta},$$

or else

$$\boldsymbol{\theta}_\alpha \approx \Delta^{-1}_{\alpha,\alpha'}\eta_{\alpha'} \quad , \quad \alpha, \alpha' = 1, ..., J,$$

but as $\theta = (\boldsymbol{\lambda}, \mathbf{0})^t$, we deduce that

$$\theta_m = \lambda_m \approx \Delta^{-1}_{m,\alpha'}\eta_{\alpha'} \quad , \quad m = 1, ..., M \quad \text{and } \alpha' = 1, ..., J \tag{3.28}$$

$$\theta_\alpha = 0 \approx \Delta^{-1}_{\alpha,\alpha'}\eta_{\alpha'} \quad , \quad \alpha = M+1, ..., J \text{ and } \alpha' = 1, ..., J. \tag{3.29}$$

Since the matrix elements Δ are the brackets $\Delta_{\alpha,\alpha'} = \{\phi_\alpha, \phi_{\alpha'}\}$, $\alpha, \alpha' = 1, ..., J$, the elements of the inverse matrix Δ^{-1} will be noted by $\Delta^{-1}_{\alpha,\alpha'} = \{\phi_\alpha, \phi_{\alpha'}\}^{-1}$, $\alpha, \alpha' = 1, ..., J$. According to the equations (3.28),(3.29)and (3.21), we write

$$\lambda_m \approx -\{\phi_m, \phi_{\alpha'}\}^{-1}\{\phi_{\alpha'}, H_c\} \ , \ m = 1, ..., M \quad \text{and } \alpha' = 1, ..., J \tag{3.30}$$

$$0 \approx \{\phi_\alpha, \phi_{\alpha'}\}^{-1}\{\phi_{\alpha'}, H_c\} \quad , \ \alpha = M+1, ..., J \text{ and } \alpha' = 1, ..., J. \tag{3.31}$$

Recalling the evolution equation of the function $F(q,p)$ that was given by (3.16) as follows

$$\dot{F} \approx \{F, H_c\} + \lambda_m \{F, \phi_m\},$$

taking into account (3.30), we'll have

$$\begin{aligned} \dot{F} &\approx \{F, H_c\} - \{F, \phi_m\} \{\phi_m, \phi_{\alpha'}\}^{-1} \{\phi_{\alpha'}, H_c\} \\ \text{with } m &= 1, ..., M \text{ and } \alpha' = 1, ..., J, \end{aligned} \tag{3.32}$$

but according to (3.31), we have $\{\phi_\alpha, \phi_{\alpha'}\}^{-1} \{\phi_{\alpha'}, H_c\} \approx 0$, with $\alpha = M+1, ..., J$, that allows to generalize (3.32) without any problem as follows

$$\dot{F} \approx \{F, H_c\} - \{F, \phi_\alpha\} \{\phi_\alpha, \phi_{\alpha'}\}^{-1} \{\phi_{\alpha'}, H_c\}, \text{ with } \quad \alpha,\ \alpha' = 1, ..., J, \tag{3.33}$$

Dirac defined (3.33) as brackts that take his name

$$\{F, H_c\}_D = \{F, H_c\} - \{F, \phi_\alpha\} \{\phi_\alpha, \phi_{\alpha'}\}^{-1} \{\phi_{\alpha'}, H_c\}, \tag{3.34}$$

while the reduced form is given by

$$\dot{F} \approx \{F, H_c\}_D. \tag{3.35}$$

The generalization of Dirac bracket to the case of two functions f and g in phase space is

$$\boxed{\{f, g\}_D = \{f, g\} - \{f, \phi_\alpha\} \{\phi_\alpha, \phi_{\alpha'}\}^{-1} \{\phi_{\alpha'}, g\}}. \tag{3.36}$$

The consistency conditions $\{\phi_\alpha, H_T\} \approx 0$ allows to write

$$\{F, H_T\}_D = \{F, H_T\} - \{F, \phi_\alpha\} \{\phi_\alpha, \phi_{\alpha'}\}^{-1} \underbrace{\{\phi_{\alpha'}, H_T\}}_{\approx 0},$$

we obtain the equality

$$\{F, H_T\}_D \approx \{F, H_T\} \approx \dot{F}.$$

In the special case where $F = q$ or $F = p$, we obtain the Hamiltonian equations

$$\dot{q} \approx \{q, H_T\}_D \tag{3.37}$$

$$\dot{p} \approx \{p, H_T\}_D \tag{3.38}$$

Dirac brackets have properties similar to those of Poisson brackets, besides to another two properties given by

$$\{f, \phi_\alpha\}_D = 0 \qquad (\phi_\alpha \text{ second class constraint}) \text{ and } \{f, G\}_D \approx \{f, G\} \qquad (G \text{ first class function}), \tag{3.39}$$

where f depend on q and p. For the demonstration of (3.39), we can have look to [6].

The evolution equation of a quantity $F(q,p)$ is given as a function of these new brackets as

$$\dot{F} \approx \{F, H_c\}_D \,. \tag{3.40}$$

Dirac brackets have a simple interpretation, it bears the information of constrained systems inside itselfs. Otherwise, we can say that the Dirac's method takes the information on the constraint starting from the Lagrangian to give it in the end to the canonical brackets of himself.

Chapter 4

Faddeev and jackiw method for systems with constraints

4.1 Introduction

In order to search for new much simpler methods to deal with constrainted systems, Faddeev-Jackiw proposed an alternative treatment seems technically different and does not have the same Dirac's conjecture, thus it has evoked much attention [3]. Noting that the original Faddeev-Jackiw method was addressed to unconstrained systems, while Barcelos-Neto and Wotzasek had been proposed an extension called symplectic algorithm to deal with constraints systems [9, 10], that we are dealing with it in this thesis.

The Faddeev-Jackiw (F-J) formalism pursues a classical geometric treatment based on the sympletic structure of the phase space and it is only applied to first order Lagrangians, linear with respect to velocities [3]. This method is rised basically on Lagrangian formalism and the matrix form of Euler-Lagrange equations as a main source of studying, without missing an important passage in converting the Lagrangian to linear one with respect to velocities and conjugate momenta using the Legendre transformation. The matrix form of (E-L) equations lead us to introduce the *(F-J) matrix* that gives us two cases can be treated according to its determinant as we will see later.

Thus, the objective of this chapter is to treat the (F-J) matrix cases with a symplectic algorithm step by step till we will end with an invertible matrix represent the basic geometric structure called generalized Poisson brackets and coincide with Dirac's brackets, that will be the bridge to the commutators of the quantized theory, as we have already mentioned in the previous chapter, while our real aim is to make a clear comparaison later between those methods in that crossing road.

4.2 Lagrangian linearization

As we have already evoked in the preceding chapter, we will not be able to express for a singular systems all velocities (the $\dot{q}_i$) according to the coordinates (the q_i), and the conjugate momenta (the p_i) using the relations $p_i = \partial L/\partial \dot{q}_i$, $i = 1, ..., n$. As we know in this case the Hessian matrix W is not invertible. Considering $R = rank\,(W)$, this means that it is possible to reverse the equations $p_i = \partial L/\partial \dot{q}_i$ only with respect to R generalized velocities $\dot{q}_a$ with $a = 1, ..., R$, writing them as functions of the other velocities, generalized coordinates and conjugate momenta as follows : $\dot{q}_a = f_a\,(q_i, p_b, \dot{q}_s)$, $a, b = 1, ..., R$, $i = 1, .., n$, $s = R + 1, ..., n$

Since $s = n - R$, we make appear s relations noted as :

$$\phi_s = p_s - g_s\,(q_i, p_b)\,,\ b = 1, ...,\ R,\ s = R + 1, ..., n,\quad i = 1, .., n, \tag{4.1}$$

the s relations express constraints that come automatically from the system.

The associated Hamiltonian H to the Lagrangian $L\,(q_i, \dot{q}_i)$ takes the form

$$\begin{aligned} H &= p_i\dot{q}_i - L \\ &= p_a\dot{q}_a + p_s\dot{q}_s - L \\ &= p_a f_a\,(q_i, p_b, \dot{q}_s) + g_s\,(q_i, p_b)\,\dot{q}_s - L. \end{aligned} \tag{4.2}$$

The H does not depend on generalized velocities despite their apparent presence.We can prove that fact by deriving(4.2) with respect to $\dot{q}_c$, while it appears directly in illustrative example since $H = H\,(q_i, p_i)$.

Very often, the Lagrangian is nonlinear with respect to velocities. Linearization consists in passing from this Lagrangian $L(q_i, \dot{q}_i)$ to a canonical Hamiltonian $H\,(q_i, p_i)$, to then return to have directly a linear Lagrangian $L(q_i, \dot{q}_i, p_i)$.The main controller in this process is the Legendre transformation in the both directions. In a specific way, we define the inverse of Legendre transformation as follows

$$L = p_i\dot{q}_i - H,$$

as well as the constraints (4.1), we have

$$L\,(q_i, \dot{q}_i, p_a) = p_a\dot{q}_a + g_s\,(q_i, p_a)\,\dot{q}_s - H\,(q_i, p_a)\,. \tag{4.3}$$

The Faddeev and Jackiw method consists in treating the q_i and p_a to be independents for the Lagrangian that had been constructed as we will see in the next example

Example

To explain this point well, considering the following nonlinear Lagrangian [5]

$$L = \frac{1}{2}(y\dot{x} + x\dot{y})^2 - xy. \tag{4.4}$$

The conjugate monenta are

$$\begin{aligned}
p_x &= \frac{\partial L}{\partial \dot{x}} = y(y\ \dot{x} + \dot{y}\ x) \\
p_y &= \frac{\partial L}{\partial \dot{y}} = x(x\ \dot{y} + y\ \dot{x}) \\
(y\ \dot{x} + \dot{y}\ x) &= \frac{p_x}{y} = \frac{p_y}{x} \quad (\text{constraint})\,.
\end{aligned}$$

We can deduce one constraint $p_y = \frac{x}{y}\ p_x$.Using this constraint the Hamiltonian gets the expression

$$\begin{aligned}
H &= p_x\dot{x} + p_y\dot{y} - L \\
&= p_x\dot{x} + p_y\dot{y} - \frac{1}{2}(y\dot{x} + x\dot{y})^2 + xy \\
&= p_x\left(\dot{x} + \frac{x}{y}\dot{y}\right)\frac{y^2}{y^2} - \frac{1}{2}(\frac{p_x}{y})^2 + xy \\
&= \frac{p_x^2}{y^2} - \frac{1}{2}(\frac{p_x}{y})^2 + xy \\
&= \frac{1}{2}(\frac{p_x}{y})^2 + xy,
\end{aligned}$$

H doesn't depend on velocities clearly. Now the linear Lagrangian is

$$\begin{aligned}
L &= p_i\dot{q}_i - H \\
&= p_x\dot{x} + p_y\dot{y} - \frac{1}{2}(\frac{p_x}{y})^2 - xy \\
&= p_x\dot{x} + \frac{xp_x}{y}\dot{y} - \frac{1}{2}(\frac{p_x}{y})^2 - xy
\end{aligned}$$

The independent variables are then x, y and p_x, while the momentum p_y depends on the other variables through the mentioned constraint above $p_y = \frac{x}{y}\ p_x$. We will see later that the (E-L) equations apply on the independent variables of any system according to the constraints.

4.3 Faddeev and Jackiw approach

Faddeev-Jackiw method is based on two main maneuvers

i) The linearization of the Lagrangian with respect to the generalized velocities.

ii) The inversion of the Faddeev-Jackiw matrix obtained using the (E-L) equations.

This method allows to derive the set of Dirac brackets in one fell swoop without needing to calculate any Poisson brackets separately .

The idea is to treat the independent variables (the q_i , $i = 1, .., n$ and the p_a , $a = 1, ...,$ R), on an equal footing by introducing new variables $\xi_i = q_i$, $i = 1, .., n$ and $\xi_{n+a} = p_a$ with $a = 1, ..., R$, in such a way that the Lagrangian (4.3) is written

$$L = A_J \, \dot{\xi}_J - H \quad , \quad J = 1, ..., n + R, \tag{4.5}$$

so that

$$\begin{aligned} A_a &= p_a \ , \ a = 1, ..., \ R \\ A_s &= g_s \left(q_i, p_a\right) \ , \ s = R + 1, ..., n \\ A_{n+a} &= 0. \end{aligned}$$

We write the Euler-Lagrange equations relating to the dynamic variables $(\xi_J, \dot{\xi}_J)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_J}\right) - \frac{\partial L}{\partial \xi_J} = 0. \tag{4.6}$$

We have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_J}\right) &= \frac{d}{dt} A_J = \frac{\partial A_J}{\partial \xi_I}\frac{d\xi_I}{dt} = \frac{\partial A_J}{\partial \xi_I} \, \dot{\xi}_I \\ \frac{\partial L}{\partial \xi_J} &= \frac{\partial A_I}{\partial \xi_J} \, \dot{\xi}_I - \frac{\partial H}{\partial \xi_J}, \end{aligned}$$

thus, (4.6) gives

$$\left(\frac{\partial A_I}{\partial \xi_J} - \frac{\partial A_J}{\partial \xi_I}\right) \dot{\xi}_I = \frac{\partial H}{\partial \xi_J}, \tag{4.7}$$

or else

$$f_{IJ} \, \dot{\xi}_I = \frac{\partial H}{\partial \xi_J} \quad , \quad I, J = 1, ..., n + R, \tag{4.8}$$

where

$$f_{IJ} = \frac{\partial A_I}{\partial \xi_J} - \frac{\partial A_J}{\partial \xi_I}, \tag{4.9}$$

is the element of Faddeev-Jackiw matrix f .This matrix is antisymmetric since $f_{IJ} = -f_{JI}$. Thus, two cases arise

i) if the matrix f is invertible, we can deduce from (4.8) the expression

$$\dot{\xi}_I = f_{IJ}^{-1} \frac{\partial H}{\partial \xi_J}. \tag{4.10}$$

On the other hand, Hamilton's equations must be on the form

$$\dot{\xi}_I = \{\xi_I, H\}, \tag{4.11}$$

recalling the general form of the Poisson bracket given by the equation (2.13)

$$\{\xi_I, H\} = \{\xi_I, \xi_J\} \frac{\partial H}{\partial \xi_J}, \tag{4.12}$$

it leads that

$$f_{IJ}^{-1} = \{\xi_I, \xi_J\}, \tag{4.13}$$

The bracket$\{\xi_I, \xi_J\}$ Are nothing but just the Dirac brackets obtained by Faddeev-Jackiw approach.

Exemple

Considering the nonlinear Lagrangian from [10], while we choosed $m = 1$

$$L = \frac{1}{2}\dot{q}^2 - V(q).$$

the conjugate momentum

$$p = \frac{\partial L}{\partial \dot{q}} = \dot{q}.$$

The canonical Hamiltonian

$$\begin{aligned} H &= p\dot{q} - L \\ &= p\dot{q} - \frac{1}{2}\dot{q}^2 + V(q) \\ &= \frac{1}{2}p^2 + V(q). \end{aligned}$$

Thus, the linear Lagrangian will be

$$\begin{aligned} L &= p\dot{q} - H \\ &= p\dot{q} - \frac{1}{2}p^2 - V(q). \end{aligned}$$

The independent variables q and p .The (E-L) equations are

$$\left\{ \begin{array}{l} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right) - \frac{\partial L}{\partial p} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{c} \dot{p} + \frac{\partial V}{\partial q} = 0 \\ p - \dot{q} = 0 \end{array} \right. , \tag{4.14}$$

the matrix form of (4.14) is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial V}{\partial q} \\ p \end{pmatrix}, \tag{4.15}$$

where f is

$$f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

f is invertible, thus its inverse is

$$f^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \{q,q\} & \{q,p\} \\ -\{q,p\} & \{p,p\} \end{pmatrix}$$

As we have no constraints, we get the from f^{-1} directly, the canonical Poisson brackets

$$\{q,q\} = 0, \ \ \{q,p\} = 1, \quad \{p,p\} = 0$$

ii) If f is not invertible, we may deal with two sub cases :

a- there exists supplementary conditions.

Since f is not invertible, it means that $rank(f) < n + R$, then this matrix admits $n + R$ $-rank(f)$ independents zero *mode* υ^m , $m = 1, ..., n + R\ -rank(f)$. These modes are the line vectors verifying the relation

$$\upsilon^m f = 0, \tag{4.16}$$

or explicitly

$$\upsilon_I^m f_{IJ} = 0. \tag{4.17}$$

Multipling the equation (4.8) in the left side by υ_I^m will principally give a rise to the constraints

$$\phi_m = \upsilon_I^m \frac{\partial H}{\partial \xi_J} = 0, \quad m = 1, ..., n + R - rank(f), \tag{4.18}$$

These constraints ϕ_m are relations between ξ_J that must be conserved with respect to time. We can write their derivation as follows

$$\dot{\phi}_m = \frac{d\phi_m}{dt} = \frac{\partial \phi_m}{\partial \xi_J}\dot{\xi}_J = 0.$$

Proceeding this path, we must add to the Lagrangian (4.5) terms of the form $\left(\lambda_m \frac{\partial \phi_m}{\partial \xi_J} \dot{\xi}_J\right)$, or in the form $\left(\dot{\lambda}_m \phi_m\right)$. We obtain a new linear Lagrangian according to $\dot{\xi}_J$ and $\dot{\lambda}_m$ having the expression

$$L = A_J \dot{\xi}_J + \dot{\lambda}_m \phi_m - H. \tag{4.19}$$

The $\lambda_m$are treated as new independent variables . Thus, (E-L) equations in this case will be

$$\xi_I \rightarrow \left(\frac{\partial A_I}{\partial \xi_J} - \frac{\partial A_J}{\partial \xi_I}\right) \dot{\xi}_I + \frac{\partial \phi_m}{\partial \xi_J} \dot{\lambda}_m = \frac{\partial H}{\partial \xi_J} \tag{4.20}$$

$$\lambda_m \rightarrow \frac{d\phi_m}{dt} = \frac{\partial \phi_m}{\partial \xi_J} \dot{\xi}_J = 0 \quad \text{(conservation of } \phi_m \text{ with respect to time.)} \quad , \tag{4.21}$$

in matrix form, the equations will be

$$\underbrace{\begin{pmatrix} \frac{\partial A_I}{\partial \xi_J} - \frac{\partial A_J}{\partial \xi_I} & \frac{\partial \phi_m}{\partial \xi_J} \\ \frac{\partial \phi_m}{\partial \xi_J} & 0 \end{pmatrix}}_{\text{the matrix } f} \begin{pmatrix} \dot{\xi}_I \\ \dot{\lambda}_m \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial \xi_J} \\ 0 \end{pmatrix}$$

This new matrix f is an antisymmetric square matrix of dimension $n+R+(n+R-rank(f)) = 2(n+R) - rank(f)$.

Exemple

Considering the linear Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - ax\dot{y}, \qquad a = cte \neq 0. \tag{4.22}$$

The conjugate momenta

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = \dot{x} \\ p_y &= \frac{\partial L}{\partial \dot{y}} = -ax, \end{aligned}$$

where the primary constraint is $p_y + ax = 0$. The canonical Hamiltonian is

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} - L \\ &= p_x\dot{x} + p_y\dot{y} - \left(\frac{1}{2}\dot{x}^2 - ax\dot{y}\right) \\ &= p_x^2 + p_y\dot{y} - \frac{1}{2}p_x^2 + ax\dot{y} \\ &= \frac{1}{2}p_x^2 + (p_y + ax)\,\dot{y} \end{aligned} \tag{4.23}$$

Since $p_y + ax = 0$, the Hamiltonian becomes

$$H = \frac{1}{2}p_x^2.$$

Thus, the linear Lagrangien is

$$\begin{aligned} L &= p_x\dot{x} + p_y\dot{y} - H \\ &= p_x\dot{x} + p_y\dot{y} - \frac{1}{2}p_x^2 \\ &= p_x\dot{x} - ax\dot{y} - \frac{1}{2}p_x^2. \end{aligned} \tag{4.24}$$

The independent variables here are x, y et p_x. The corresponding (E-L) equations

$$\begin{aligned} \dot{p}_x + a\dot{y} &= 0 \\ -a\dot{x} &= 0 \\ -\dot{x} + p_x &= 0 \end{aligned}$$

The matrix form of the system is given by

$$\underbrace{\begin{pmatrix} 0 & a & 1 \\ -a & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{f^{(0)}} \underbrace{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \end{pmatrix}}_{\dot{\xi}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ p_x \end{pmatrix}}_{\partial H/\partial \xi}, \tag{4.25}$$

$f^{(0)}$ is singular $rank\left(f^{(0)}\right) = 2$. Thus, this matrix admits one zero mode ; $n+R-rank(f^{(0)}) = 2+1-2 = 1$ that is given as follows (check the annex)

$$\upsilon = \begin{pmatrix} 0 & -\frac{1}{a} & 1 \end{pmatrix}. \tag{4.26}$$

Multiplying (4.25) by (4.26) on the left side, we get the next supplementary constraint

$$p_x = 0,$$

that must be preserved with respect to time, therefore we may add the term $\dot{\lambda}p_x$ to the linear Lagrangian

$$L = p_x\dot{x} - ax\dot{y} - \frac{1}{2}p_x^2 + \dot{\lambda}p_x$$

The independent variables now are x, y , p_x and λ. The corresponding (E-L) equations

$$\begin{aligned} \dot{p}_x + a\dot{y} &= 0 \\ -a\dot{x} &= 0 \\ -\dot{x} + p_x - \dot{\lambda} &= 0 \\ \dot{p}_x &= 0, \end{aligned}$$

or else

$$\underbrace{\begin{pmatrix} 0 & a & 1 & 0 \\ -a & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{f^{(1)}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_x \\ 0 \end{pmatrix}, \tag{4.27}$$

$f^{(1)}$ is invertible, where the inverse is

$$\left(f^{(1)}\right)^{-1} = \begin{pmatrix} 0 & -\frac{1}{a} & 0 & 0 \\ \frac{1}{a} & 0 & 0 & -\frac{1}{a} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{a} & -1 & 0 \end{pmatrix} \tag{4.28}$$

The generalized Poisson brackets (that are identical to Dirac's one) between the dynamic variables according to the Lagrangian of the begining are

$$\begin{aligned} \{x, x\} &= \{y, y\} = \{p_x, p_x\} = 0 \\ \{x, y\} &= -\frac{1}{a}, \quad \{x, p_x\} = 0, \ \ \{y, p_x\} = 0. \end{aligned}$$

b-There exists no supplementary constraints, but only identities of the type $(0 = 0)$ produced by multiplying the equation (4.8) in the left side by υ_I^m. This is due to the presence of gauge symmetry that we lead us to add term coincide with the Lagrangian, where we fix the gauge in certain conditions.

To make it clear, we recall the previous example mentioned in (4.4)

$$L = \frac{1}{2}(y\dot{x} + x\dot{y})^2 - xy, \tag{4.29}$$

where its linear form was as follows

$$L = p_x\dot{x} + \frac{xp_x}{y}\dot{y} - \frac{1}{2}(\frac{p_x}{y})^2 - xy.$$

The independent variables are then x, y and p_x, while the momentum p_y depends on the other variables through the mentioned constraint above $p_y = \frac{x}{y}\, p_x$. the (E-L) equations apply on the independent variables as follows

$$\begin{aligned} \dot{y}\frac{p_x}{y} - \dot{p}_x &= y \\ -\dot{x}\frac{p_x}{y} + \dot{p}_x\frac{x}{y} &= -\frac{p_x^2}{y^3} + x \\ \dot{x} + \dot{y}\frac{x}{y} &= \frac{p_x}{y^2}, \end{aligned}$$

their matrix form is given by

$$\underbrace{\begin{pmatrix} 0 & \frac{p_x}{y} & -1 \\ -\frac{p_x}{y} & 0 & \frac{-x}{y} \\ 1 & \frac{x}{y} & 0 \end{pmatrix}}_{f^{(0)}} \underbrace{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \end{pmatrix}}_{\dot{\xi}} = \underbrace{\begin{pmatrix} y \\ -\frac{p_x^2}{y^3} + x \\ \frac{p_x}{y^2} \end{pmatrix}}_{\partial H/\partial \xi}, \tag{4.30}$$

$f^{(0)}$ is singular of $rank\left(f^{(0)}\right) = 2$. Thus the matrix admits one zero mode; $n+R - rank(f^{(0)}) = 2+1-2 = 1$ is given as follows

$$\upsilon = \begin{pmatrix} -\frac{x}{p_x} & \frac{y}{p_x} & 1 \end{pmatrix} \tag{4.31}$$

Multiplying (4.30) by (4.31) in the left side, we get only identities of the type $(0 = 0)$, so there is no generated constraint in this case, and the matrix keeps singular to express that we are dealing exactely with the presence of gauge symmetry. We choose the gauge condition $y = 1$ by adding the term $\dot{w}(y-1)$ to the Lagrangian as follows

$$L = p_x\dot{x} + \frac{xp_x}{y}\dot{y} - \frac{1}{2}(\frac{p_x}{y})^2 - xy + \dot{w}(y-1).$$

The independant variables now x, y , p_x and w.The corresponding (E-L) equations

$$\begin{aligned} \frac{p_x}{y}\dot{y} - \dot{p}_x &= y \\ -\frac{p_x}{y}\dot{x} - \frac{x}{y}\dot{p}_x + \dot{w} &= -\frac{p_x^2}{y^3} + x \\ \dot{x} + \frac{x}{y}\dot{y} &= \frac{p_x}{y^2} \\ -\dot{y} &= 0, \end{aligned}$$

using the matrix form we get

$$\underbrace{\begin{pmatrix} 0 & \frac{p_x}{y} & -1 & 0 \\ -\frac{p_x}{y} & 0 & -\frac{x}{y} & 1 \\ 1 & \frac{x}{y} & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}}_{f^{(1)}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \\ \dot{w} \end{pmatrix} = \begin{pmatrix} y \\ -\frac{p_x^2}{y^3} + x \\ \frac{p_x}{y^2} \\ 0 \end{pmatrix}, \tag{4.32}$$

$f^{(1)}$ is invertible, and it inverse is given by

$$f^{(1)^{-1}} = \begin{pmatrix} 0 & 0 & 1 & \frac{x}{y} \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & -\frac{1}{y}p_x \\ -\frac{x}{y} & 1 & \frac{1}{y}p_x & 0 \end{pmatrix} = \begin{pmatrix} \{x,x\} & \{x,y\} & \{x,p_x\} & \{x,w\} \\ \{y,x\} & \{y,y\} & \{y,p_x\} & \{y,w\} \\ \{p_x,x\} & \{p_x,y\} & \{p_x,p_x\} & \{p_x,w\} \\ \{w,x\} & \{w,y\} & \{w,p_x\} & \{w,w\} \end{pmatrix}, \tag{4.33}$$

where then we can extract the following brackets

$$\{x,p_x\} = 1 \ , \quad \{y,p_x\} = 0, \text{ and } \{x,y\} = 0.$$

At this level, we can summarize the existence of three cases that characterize Faddeev and Jackiw method as follows

i) f is invertible and the brackets are obtained using f^{-1}as a matrix elements, and the algorithm ends here.

ii) f is not invertible and there is no generated constraints, this is a sign of gauge symmetry presence. In this case ,the supplementary conditions $\zeta_n(\xi) = 0$ are necessary in order to fix the gauge and have an invertible matrix f . We add terms to the Lagrangian (4.5) as $\dot{\omega}_n\zeta_n(\xi)$ where ω_n represent multipliers. Then we have to write the E-L equations with respect to these variables ξ_I , λ_m and ω_n. Algorithm ends when we find f^{-1}.

iii) f is not invertible and the zero modes give new constraints. We must then add them to the Lagrangian (4.19) with a different lagrangian multipliers, and restart the zero procedure.

Chapter 5

Special applications

There is no doubt that the comparaison study between Dirac's method and (F-J) approach in introducing correct brackets supposed to be the bridge to the quantize theory for constrained systems highlights effectively under the shadow of illustrative applications more than giving analysis to the general principles. In order that, we will show two applications of particle moving on circle and other one moving on ellipse. These two applications will be studied by those methods mentioned above for giving remarks later.

5.1 Applications treated by Dirac's method

5.1.1 Particale moving on a circle

Considering here a particle of mass m moving on a circle of radius $(r = a)$. We will calculate the Dirac brackets for this system. Thus, the corresponding Lagrangian is written

$$L\left(x, \dot{x}, y, \dot{y}, \mu\right) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - \mu \left(x^2 + y^2 - a^2\right) \tag{5.1}$$

where the quantity μ is treated here as an independent dynamic variable that called Lagrangian multiplier. The corresponding conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$
$$p_\mu = \frac{\partial L}{\partial \dot{\mu}} = 0$$

The Hessian matrix W corresponding is

$$W = \begin{pmatrix} \frac{\partial^2 L}{\partial \dot{x} \partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{x} \partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{x} \partial \dot{\mu}} \\ \frac{\partial^2 L}{\partial \dot{y} \partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{y} \partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{y} \partial \dot{\mu}} \\ \frac{\partial^2 L}{\partial \dot{\mu} \partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{\mu} \partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{\mu} \partial \dot{\mu}} \end{pmatrix} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.2}$$

$\det(W) = 0$, therefore Lagrangian (5.1) is singular. We pose the relation $p_\mu \approx 0$ as a primary constraint

$$\phi_1 = p_\mu \approx 0. \tag{5.3}$$

The constraint $\phi_1$is our only primary constraint, then we construct the canonical Hamiltonian

$$\begin{aligned} H_c &= p_x \dot{x} + p_y \dot{y} + p_\mu \dot{\mu} - L \\ &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + p_\mu \dot{\mu} + \mu\left(x^2 + y^2 - a^2\right), \end{aligned}$$

then the total Hamiltonian

$$H_T = H_c + \lambda_1 \phi_1, \tag{5.4}$$

where λ_1 is Dirac's multiplier. Explicitly H_T is

$$H_T = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(x^2 + y^2 - a^2\right) + \gamma p_\mu, \tag{5.5}$$

where $\lambda_1 + \dot{\mu} = \gamma$. The consistency condition for ϕ_1 is

$$\dot{\phi}_1 = \{\phi_1, H_T\} \approx 0 \Rightarrow \{p_\mu, H_T\} \approx 0 \Rightarrow -\left(x^2 + y^2 - a^2\right) \approx 0, \tag{5.6}$$

which is a new constraint, that the Lagrange multiplier had already implicitly imposed. So, we write

$$\phi_2 = a^2 - x^2 - y^2 \approx 0 \tag{5.7}$$

Likewise, the consistency condition for ϕ_2 gives

$$\dot{\phi}_2 = \{\phi_2, H_T\} \approx 0 \Rightarrow \left\{a^2 - x^2 - y^2, \frac{1}{2m}\left(p_x^2 + p_y^2\right)\right\} \approx 0 \Rightarrow -\frac{2}{m}\left(x p_x + y p_y\right) \approx 0,$$

which also a new constraint that we note by

$$\phi_3 = -\frac{2}{m}\left(x p_x + y p_y\right) \approx 0, \tag{5.8}$$

we do the same for ϕ_3, we obtain

$$\begin{aligned}\dot{\phi}_3 &= \{\phi_3, H_T\} \approx 0 \Rightarrow \left\{-\frac{2}{m}\left(xp_x + yp_y\right), \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(x^2 + y^2\right)\right\} \approx 0 \\ &\Rightarrow \frac{4\mu}{m}\left(x^2 + y^2\right) - \frac{2}{m^2}\left(p_x^2 + p_y^2\right) \approx 0,\end{aligned}$$

which is also a new constraint, can be defined as

$$\phi_4 = \frac{4\mu}{m}\left(x^2 + y^2\right) - \frac{2}{m^2}\left(p_x^2 + p_y^2\right) \approx 0. \tag{5.9}$$

Finally, if we apply again the consistency condition for ϕ_4, we get expression for γ (or else for λ_1)

$$\gamma = -\frac{4\mu}{m\left(x^2 + y^2\right)}\left(xp_x + yp_y\right). \tag{5.10}$$

The algorithm ends.

We have four constraints, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 and the Poisson brackets between these constraints can be written in the form of an antisymmetric matrix Δ whose elements are noted $\Delta_{ij} = \left\{\phi_i, \phi_j\right\}$. This matrix is known as the constraints matrix. Explicitly

$$\Delta = \begin{pmatrix} 0 & \Delta_{12} & \Delta_{13} & \Delta_{14} \\ -\Delta_{12} & 0 & \Delta_{23} & \Delta_{24} \\ -\Delta_{13} & -\Delta_{14} & 0 & \Delta_{34} \\ -\Delta_{14} & -\Delta_{24} & -\Delta_{34} & 0 \end{pmatrix}, \tag{5.11}$$

where

$$\begin{aligned}\Delta_{12} &= \{\phi_1, \phi_2\} = \left\{p_\mu, a^2 - x^2 - y^2\right\} = 0 \\ \Delta_{13} &= \{\phi_1, \phi_3\} = \left\{p_\mu, -\frac{2}{m}\left(xp_x + yp_y\right)\right\} = 0 \\ \Delta_{14} &= \{\phi_1, \phi_4\} = -\frac{4}{m}\left(x^2 + y^2\right) = -\frac{4a^2}{m} \\ \Delta_{23} &= \{\phi_2, \phi_3\} = \frac{4}{m}\left(x^2 + y^2\right) = \frac{4a^2}{m} \\ \Delta_{24} &= \{\phi_2, \phi_4\} = \frac{8}{m^2}\left(xp_x + yp_y\right) = 0 \\ \Delta_{34} &= \{\phi_3, \phi_4\} = \frac{16\mu}{m^2}\left(x^2 + y^2\right) + \frac{8}{m^3}\left(p_x^2 + p_y^2\right) = \frac{32a^2}{m^2}\mu,\end{aligned}$$

where we used the constraints as strong equalities after the computation of these brackets .

Therefore the constraints matrix is going to be

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & -\frac{4a^2}{m} \\ 0 & 0 & \frac{4a^2}{m} & 0 \\ 0 & -\frac{4a^2}{m} & 0 & \frac{32a^2}{m^2}\mu \\ \frac{4a^2}{m} & 0 & -\frac{32a^2}{m^2}\mu & 0 \end{pmatrix} \tag{5.12}$$

the inverse is

$$\Delta^{-1} = \begin{pmatrix} 0 & \frac{2\mu}{a^2} & 0 & \frac{m}{4a^2} \\ -\frac{2\mu}{a^2} & 0 & -\frac{m}{4a^2} & 0 \\ 0 & \frac{m}{4a^2} & 0 & 0 \\ -\frac{m}{4a^2} & 0 & 0 & 0 \end{pmatrix} \tag{5.13}$$

Calculating now the Dirac's brackets of the dynamic variables using the formula (3.36) to know that

$$\{f, g\}_D = \{f, g\} - \sum_{i,j=1}^{4} \{f, \phi_i\} \Delta_{ij}^{-1} \{\phi_j, g\}. \tag{5.14}$$

For example, we calculate the bracket $\{\mu, p_\mu\}_D$

$$\begin{aligned} \{\mu, p_\mu\}_D &= 1 - \sum_{i,j=1}^{4} \{\mu, \phi_i\} \Delta_{ij}^{-1} \{\phi_j, p_\mu\} \\ &= 1 - \{\mu, \phi_1\} \Delta_{14}^{-1} \{\phi_4, p_\mu\} \\ &= 1 - \{\mu, p_\mu\} \frac{m}{4a^2} \left\{ \frac{4\mu}{m} \left(x^2 + y^2\right) \ , \ p_\mu \right\} \\ &= 1 - \frac{m}{4a^2} \{\mu, p_\mu\} \frac{4}{m} \left(x^2 + y^2\right) \\ &= 1 - \frac{m}{4a^2} \frac{4}{m} \left(x^2 + y^2\right) \quad , \quad x^2 + y^2 = a^2 \\ &= 0. \end{aligned}$$

Likewise we can obtain the bracket

$$\begin{aligned} \{x, p_x\}_D &= 1 - \sum_{i,j=1}^{4} \{x, \phi_i\} \Delta_{ij}^{-1} \{\phi_j, p_x\} \\ &= 1 - \sum_{j=1}^{4} \{x, \phi_3\} \Delta_{3j}^{-1} \{\phi_j, p_x\} - \sum_{j=1}^{4} \{x, \phi_4\} \Delta_{4j}^{-1} \{\phi_j, p_x\} \\ &= 1 - \{x, \phi_3\} \Delta_{32}^{-1} \{\phi_2, p_x\} \\ &= 1 - \left\{ x, -\frac{2}{m} \left(x p_x + y p_y\right) \right\} \frac{m}{4a^2} \left\{a^2 - x^2 - y^2, p_x\right\} \\ &= 1 - \frac{x^2}{a^2}. \end{aligned}$$

We can equaly verify that we have the Dirac's brackets as follows

$$\begin{aligned}
\{y, p_y\}_D &= 1 - \frac{y^2}{a^2}, \quad \{x, p_y\}_D = -\frac{xy}{a^2}, \quad \{y, p_x\}_D = -\frac{xy}{a^2}, \\
\{x, y\}_D &= 0, \qquad \{p_x, p_y\}_D = -\frac{1}{a^2}(xp_y - yp_x) = -\frac{1}{a^2}L_Z,
\end{aligned} \tag{5.15}$$

where L_Z is the angular momentum for the component Z.

5.1.2 Particale moving on ellipse

Considering here a particle of mass m moving on a ellipse that was centered at the origin with width $2a$ and height $2b$. We will calculate the Dirac brackets for this system.

Thus, the corresponding Lagrangian is written

$$L(x, \dot{x}, y, \dot{y}, \mu) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \mu\left(b^2x^2 + a^2y^2 - a^2b^2\right), \tag{5.16}$$

where the quantity μ is treated here as an independent dynamic variable.

The corresponding conjugate momenta are

$$\begin{aligned}
p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\
p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y} \\
p_\mu &= \frac{\partial L}{\partial \dot{\mu}} = 0
\end{aligned}$$

The Hessian matrix W is

$$W = \begin{pmatrix} \frac{\partial^2 L}{\partial \dot{x}\partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{x}\partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{x}\partial \dot{\mu}} \\ \frac{\partial^2 L}{\partial \dot{y}\partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{y}\partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{y}\partial \dot{\mu}} \\ \frac{\partial^2 L}{\partial \dot{\mu}\partial \dot{x}} & \frac{\partial^2 L}{\partial \dot{\mu}\partial \dot{y}} & \frac{\partial^2 L}{\partial \dot{\mu}\partial \dot{\mu}} \end{pmatrix} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.17}$$

$\det(W) = 0$, therefore Lagrangian (5.16) is singular. We pose the relation $p_\mu \approx 0$ as a primary constraint.i.e

$$\phi_1 = p_\mu \approx 0 \tag{5.18}$$

The constraint $\phi_1$is our only primary constraint. then we construct the canonical Hamiltonian

$$\begin{aligned}
H_c &= p_x\dot{x} + p_y\dot{y} + p_\mu\dot{\mu} - L \\
&= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + p_\mu\dot{\mu} + \mu\left(b^2x^2 + a^2y^2 - a^2b^2\right),
\end{aligned}$$

while, the total Hamiltonian

$$H_T = H_c + \lambda_1 \phi_1, \tag{5.19}$$

where λ_1 is Dirac's multiplier. Explicitly H_T is

$$H_T = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(b^2x^2 + a^2y^2 - a^2b^2\right) + \gamma p_\mu, \tag{5.20}$$

where $\lambda_1 + \dot{\mu} = \gamma$. The consistency condition for ϕ_1 is

$$\dot{\phi}_1 = \{\phi_1, H_T\} \approx 0 \Rightarrow \{p_\mu, H_T\} \approx 0 \Rightarrow -\left(b^2x^2 + a^2y^2 - a^2b^2\right) \approx 0, \tag{5.21}$$

which is a new constraint, that the Lagrangian multiplier had already implicitly imposed. So we write

$$\phi_2 = a^2b^2 - b^2x^2 - a^2y^2 \approx 0, \tag{5.22}$$

Likewise, the consistency condition for ϕ_2 gives

$$\dot{\phi}_2 = \{\phi_2, H_T\} \approx 0 \Rightarrow \left\{a^2b^2 - b^2x^2 - a^2y^2, \frac{1}{2m}\left(p_x^2 + p_y^2\right)\right\} \approx 0 \Rightarrow -\frac{2}{m}\left(b^2xp_x + a^2yp_y\right) \approx 0,$$

which also is a new constraint that we note by

$$\phi_3 = -\frac{2}{m}\left(b^2xp_x + a^2yp_y\right) \approx 0, \tag{5.23}$$

We do the same for ϕ_3, we obtain

$$\begin{aligned} \dot{\phi}_3 &= \{\phi_3, H_T\} \approx 0 \Rightarrow \left\{-\frac{2}{m}\left(b^2xp_x + a^2yp_y\right), \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(b^2x^2 + a^2y^2 - a^2b^2\right)\right\} \approx 0 \\ &\Rightarrow \frac{4\mu}{m}\left(b^4x^2 + a^4y^2\right) - \frac{2}{m^2}\left(b^2p_x^2 + a^2p_y^2\right) \approx 0, \end{aligned}$$

which is also a new constraint, can be defined as

$$\phi_4 = \frac{4\mu}{m}\left(b^4x^2 + a^4y^2\right) - \frac{2}{m^2}\left(b^2p_x^2 + a^2p_y^2\right) \approx 0, \tag{5.24}$$

Finally, if we apply again the consistency condition for ϕ_4, we get expression for γ (or else for λ_1)

$$\gamma = -\frac{4\mu}{m\left(b^4x^2 + a^4y^2\right)}\left(b^4xp_x + a^4yp_y\right). \tag{5.25}$$

The algorithm ends.

We have four constraints, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 and the Poisson brackets between these constraints can be written in the form of an antisymmetric matrix Δ whose elements are noted $\Delta_{ij} = \left\{\phi_i, \phi_j\right\}$. This matrix is known as the constraints matrix. Explicitly

$$\Delta = \begin{pmatrix} 0 & \Delta_{12} & \Delta_{13} & \Delta_{14} \\ -\Delta_{12} & 0 & \Delta_{23} & \Delta_{24} \\ -\Delta_{13} & -\Delta_{14} & 0 & \Delta_{34} \\ -\Delta_{14} & -\Delta_{24} & -\Delta_{34} & 0 \end{pmatrix}, \tag{5.26}$$

where

$$\begin{aligned}
\Delta_{12} &= \{\phi_1, \phi_2\} = \left\{p_\mu, a^2b^2 - b^2x^2 - a^2y^2\right\} = 0 \\
\Delta_{13} &= \{\phi_1, \phi_3\} = \left\{p_\mu, -\frac{2}{m}\left(b^2xp_x + a^2yp_y\right)\right\} = 0 \\
\Delta_{14} &= \{\phi_1, \phi_4\} = -\frac{4}{m}\left(b^4x^2 + a^4y^2\right) \\
\Delta_{23} &= \{\phi_2, \phi_3\} = \frac{4}{m}\left(b^4x^2 + a^4y^2\right) \\
\Delta_{24} &= \{\phi_2, \phi_4\} = \frac{8}{m^2}\left(b^4xp_x + a^4yp_y\right) \\
\Delta_{34} &= \{\phi_3, \phi_4\} = \frac{16\mu}{m^2}\left(b^6x^2 + a^6y^2\right) + \frac{8}{m^3}\left(b^4p_x^2 + a^4p_y^2\right).
\end{aligned}$$

Where we have used the constraints as strong equalities after the computation of these brackets . Therefore the constraints matrix is

$$\Delta = \begin{pmatrix} 0 & 0 & 0 & -\frac{4\left(b^4x^2+a^4y^2\right)}{m} \\ 0 & 0 & \frac{4\left(b^4x^2+a^4y^2\right)}{m} & \frac{8\left(b^4xp_x+a^4yp_y\right)}{m^2} \\ 0 & -\frac{4\left(b^4x^2+a^4y^2\right)}{m} & 0 & \frac{16\mu\left(b^6x^2+a^6y^2\right)}{m^2} + \frac{8\left(b^4p_x^2+a^4p_y^2\right)}{m^3} \\ \frac{4\left(b^4x^2+a^4y^2\right)}{m} & -\frac{8\left(b^4xp_x+a^4yp_y\right)}{m^2} & -\frac{16\mu\left(b^6x^2+a^6y^2\right)}{m^2} - \frac{8\left(b^4p_x^2+a^4p_y^2\right)}{m^3} & 0 \end{pmatrix}.$$

The inverse is

$$\Delta^{-1} = \begin{pmatrix} 0 & \frac{\mu\left(b^6x^2+a^6y^2\right)}{(b^4x^2+a^4y^2)^2} + \frac{\left(b^4p_x^2+a^4p_y^2\right)}{2m(b^4x^2+a^4y^2)^2} & -\frac{\left(b^4xp_x+a^4yp_y\right)}{2(b^4x^2+a^4y^2)^2} & \frac{m}{4(b^4x^2+a^4y^2)} \\ -\frac{\mu\left(b^6x^2+a^6y^2\right)}{(b^4x^2+a^4y^2)^2} - \frac{\left(b^4p_x^2+a^4p_y^2\right)}{2m(b^4x^2+a^4y^2)^2} & 0 & -\frac{m}{4(b^4x^2+a^4y^2)} & 0 \\ \frac{\left(b^4xp_x+a^4yp_y\right)}{2(b^4x^2+a^4y^2)^2} & \frac{m}{4(b^4x^2+a^4y^2)} & 0 & 0 \\ -\frac{m}{4(b^4x^2+a^4y^2)} & 0 & 0 & 0 \end{pmatrix}.$$

Calculating now the Dirac's brackets of the dynamic variables in the same frequency of circle application, for example, we calculate the bracket $\{\mu, p_\mu\}_D$

$$\begin{aligned}
\{\mu, p_\mu\}_D &= 1 - \sum_{i,j=1}^{4} \{\mu, \phi_i\} \Delta_{ij}^{-1} \{\phi_j, p_\mu\} \\
&= 1 - \{\mu, \phi_1\} \Delta_{14}^{-1} \{\phi_4, p_\mu\} \\
&= 1 - \{\mu, p_\mu\} \frac{m}{4\left(b^4x^2 + a^4y^2\right)} \left\{ \frac{4\mu}{m} \left(b^4x^2 + a^4y^2\right) \ , \ p_\mu \right\} \\
&= 1 - \frac{m}{4\left(b^4x^2 + a^4y^2\right)} \{\mu, p_\mu\} \frac{4}{m} \left(b^4x^2 + a^4y^2\right) \\
&= 1 - \frac{m}{4\left(b^4x^2 + a^4y^2\right)} \frac{4}{m} \left(b^4x^2 + a^4y^2\right) \\
&= 0.
\end{aligned}$$

Likewise we can obtain the bracket

$$\begin{aligned}
\{x, p_x\}_D &= 1 - \sum_{i,j=1}^{4} \{x, \phi_i\} \Delta_{ij}^{-1} \{\phi_j, p_x\} \\
&= 1 - \sum_{j=1}^{4} \{x, \phi_3\} \Delta_{3j}^{-1} \{\phi_j, p_x\} - \sum_{j=1}^{4} \{x, \phi_4\} \Delta_{4j}^{-1} \{\phi_j, p_x\} \\
&= 1 - \{x, \phi_3\} \Delta_{32}^{-1} \{\phi_2, p_x\} \\
&= 1 - \left\{ x, -\frac{2}{m} \left(b^2 x p_x + a^2 y p_y\right) \right\} \frac{m}{4\left(b^4x^2 + a^4y^2\right)} \left\{ a^2b^2 - b^2x^2 - a^2y^2, p_x \right\} \\
&= 1 - \frac{b^4x^2}{\left(b^4x^2 + a^4y^2\right)}.
\end{aligned}$$

We can equaly verify that we have the Dirac's brackets as follows

$$\begin{aligned}
\{y, p_y\}_D &= 1 - \frac{a^4y^2}{\left(b^4x^2 + a^4y^2\right)} \ , \quad \{x, p_y\}_D = -\frac{a^2b^2xy}{\left(b^4x^2 + a^4y^2\right)}, \quad \{y, p_x\}_D = -\frac{a^2b^2yx}{\left(b^4x^2 + a^4y^2\right)}, \\
\{x, y\}_D &= 0, \qquad \{p_x, p_y\}_D = -\frac{L_Z}{\left(b^4x^2 + a^4y^2\right)}.
\end{aligned} \tag{5.27}$$

5.2 Applications treated by Fadeev and Jackiw method

5.2.1 Particle moving on a circle

The Lagrangian of the system is given by

$$L\left(x, \dot{x}, y, \dot{y}, \mu\right) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - \mu \left(x^2 + y^2 - a^2\right). \tag{5.28}$$

The correspondant conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \ , \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \ , \ p_\mu = \frac{\partial L}{\partial \dot{\mu}} = 0, \tag{5.29}$$

where $p_\mu = 0$ is the primary constraints . The canonical Hamiltonian for the system is

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} + p_\mu\dot{\mu} - L \\ &= p_x\dot{x} + p_y\dot{y} + p_\mu\dot{\mu} - \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \mu\left(x^2 + y^2 - a^2\right) \\ &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(x^2 + y^2 - a^2\right) \quad , \ p_\mu = 0. \end{aligned} \tag{5.30}$$

Thus, the linear Lagrangian will be

$$\begin{aligned} L &= p_x\dot{x} + p_y\dot{y} - H \\ &= p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \mu\left(x^2 + y^2 - a^2\right). \end{aligned} \tag{5.31}$$

We arrive to an important situation that deserves to be given some observations. if we follow directly the algorithm above, we may find as follows in the next step using our independent variables x, y, μ, p_x and p_y. The correspondent (E-L) equations lead us to

$$\begin{aligned} \dot{p}_x + 2\mu x &= 0 \\ \dot{p}_y + 2\mu y &= 0 \\ x^2 + y^2 - a^2 &= 0 \\ -\dot{x} + \frac{p_x}{m} &= 0 \\ -\dot{y} + \frac{p_y}{m} &= 0. \end{aligned}$$

Under the matrix form, we have

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}}_{=f} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\mu} \\ \dot{p}_x \\ \dot{p}_y \end{pmatrix} = \begin{pmatrix} 2x\mu \\ 2y\mu \\ a^2 - x^2 - y^2 \\ \frac{p_x}{m} \\ \frac{p_y}{m} \end{pmatrix}. \tag{5.32}$$

The calculation of the determinant of f leads that it is singular with $rank(\ f) = 4$. Therefore, this matrix admits one zero mode under this relation $n+R - rank(f) = 3+2-4 = 1$, that is given by

$$\upsilon = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{5.33}$$

Multiplying(5.32) on the left side by (5.33), we may obtain a supplementary constraint

$$\phi = a^2 - x^2 - y^2 = 0, \tag{5.34}$$

which is nothing but expresses the circle equation as it should be. However, we know that this constraint must be introduced in the Lagrangian (5.31) either like $\dot{\alpha}\phi$ or $\alpha\dot{\phi}$, where α is the Lagrangian multiplier. Thus, it is now more practical to replace easily $\mu \rightarrow \dot{\alpha}$ in the begining. By doing this, we simply introduce a total derivative to the Lagrangian

$$\mu\phi \rightarrow \mu\phi - \frac{d}{dt}(\alpha\phi) = (\mu - \dot{\alpha})\,\phi - \alpha\dot{\phi}. \tag{5.35}$$

Choosing $\mu = \dot{\alpha}$. After this digression, we then write our Lagrangian (5.31) as follows

$$L = p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \dot{\alpha}\left(x^2 + y^2 - a^2\right). \tag{5.36}$$

Our independent variables become $(x, y,\ p_x,\ p_y$ and $\alpha)$. The (E-L) equations give

$$\begin{aligned}
\dot{p}_x + 2x\dot{\alpha} &= 0\\
\dot{p}_y + 2y\dot{\alpha} &= 0\\
-\dot{x} + \frac{p_x}{m} &= 0\\
-\dot{y} + \frac{p_y}{m} &= 0\\
-2x\dot{x} - 2y\dot{y} &= 0.
\end{aligned}$$

Under the matrix form, we get

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 0 & -2x\\ 0 & 0 & 0 & -1 & -2y\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 2x & 2y & 0 & 0 & 0 \end{pmatrix}}_{f^{(0)}} \begin{pmatrix} \dot{x}\\ \dot{y}\\ \dot{p}_x\\ \dot{p}_y\\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix} 0\\ 0\\ \frac{p_x}{m}\\ \frac{p_y}{m}\\ 0 \end{pmatrix}, \tag{5.37}$$

$f^{(0)}$ is singular and $rank\left(f^{(0)}\right) = 4$, where it has one zero mode given by

$$v = \begin{pmatrix} 0 & 0 & -2x & -2y & 1 \end{pmatrix} \tag{5.38}$$

Multiplying the system (5.37) by that latter (5.38) , we obtain a new supplementary constraint

$$-\frac{2}{m}\left(xp_x + yp_y\right) = 0 \tag{5.39}$$

This constraint must be introduced in the Lagrangian of the starting (5.36). Thus, we write

$$L = p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \dot{\alpha}\left(x^2 + y^2 - a^2\right) - 2\frac{\dot{\beta}}{m}\left(xp_x + yp_y\right) \tag{5.40}$$

Our independent variables now are x, y , p_x, p_y, α and β.The corresponding (E-L) equations are

$$\begin{aligned}
\dot{p}_x + 2x\dot{\alpha} + 2\frac{p_x}{m}\dot{\beta} &= 0 \\
\dot{p}_y + 2y\dot{\alpha} + 2\frac{p_y}{m}\dot{\beta} &= 0 \\
-\dot{x} + \frac{p_x}{m} + 2\frac{x}{m}\dot{\beta} &= 0 \\
-\dot{y} + \frac{p_y}{m} + 2\frac{y}{m}\dot{\beta} &= 0 \\
-2x\dot{x} - 2y\dot{y} &= 0 \\
-\frac{2}{m}\left(\dot{x}p_x + \dot{y}p_y + x\dot{p}_x + y\dot{p}_y\right) &= 0
\end{aligned}$$

Under the matrix form, we get

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 0 & -2x & -\frac{2p_x}{m} \\ 0 & 0 & 0 & -1 & -2y & -\frac{2p_y}{m} \\ 1 & 0 & 0 & 0 & 0 & -\frac{2x}{m} \\ 0 & 1 & 0 & 0 & 0 & -\frac{2y}{m} \\ 2x & 2y & 0 & 0 & 0 & 0 \\ \frac{2p_x}{m} & \frac{2p_y}{m} & \frac{2x}{m} & \frac{2y}{m} & 0 & 0 \end{pmatrix}}_{f^{(1)}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \\ \dot{p}_y \\ \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{p_x}{m} \\ \frac{p_y}{m} \\ 0 \\ 0 \end{pmatrix} \tag{5.41}$$

Noting in the begining that the matrix $f^{(1)}$contains the matrix $f^{(0)}$ as a sub matrix. Moreover, $f^{(1)}$ is invertible and its inverse is

$$\left(f^{(1)}\right)^{-1} = \begin{pmatrix} 0 & 0 & 1 - \frac{x^2}{a^2} & -\frac{xy}{a^2} & \frac{x}{2a^2} & 0 \\ 0 & 0 & -\frac{xy}{a^2} & 1 - \frac{y^2}{a^2} & \frac{y}{2a^2} & 0 \\ \frac{x^2}{a^2} - 1 & \frac{xy}{a^2} & 0 & -\frac{L_z}{a^2} & -\frac{p_x}{2a^2} & \frac{mx}{2a^2} \\ \frac{xy}{a^2} & \frac{y^2}{a^2} - 1 & \frac{L_z}{a^2} & 0 & -\frac{p_y}{2a^2} & \frac{my}{2a^2} \\ -\frac{x}{2a^2} & -\frac{y}{2a^2} & \frac{p_x}{2a^2} & \frac{p_y}{2a^2} & 0 & -\frac{m}{4a^2} \\ 0 & 0 & -\frac{mx}{2a^2} & -\frac{my}{2a^2} & \frac{m}{4a^2} & 0 \end{pmatrix} \tag{5.42}$$

The generalized Poisson brackets of the dynamic variables contained in the symplectic matrix $\left(f^{(1)}\right)^{-1}$ are identical to the Dirac's brackets obtained by his method in the same treatment.

For example, we mention the next brackets

$$
\begin{aligned}
\{x, p_x\}_{GPB} &= 1 - \frac{x^2}{a^2} = \{x, p_x\}_D \\
\{y, p_y\}_{GPB} &= 1 - \frac{y^2}{a^2} = \{y, p_y\}_D
\end{aligned}
$$

5.2.2 Particle moving on ellipse

The Lagrangian of the system is given by

$$L\left(x, \dot{x}, y, \dot{y}, \mu\right) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - \mu\left(bx^2 + a^2y^2 - a^2b^2\right). \tag{5.43}$$

The correspondent conjugate momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}\ , \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}\ ,\ \ p_\mu = \frac{\partial L}{\partial \dot{\mu}} = 0 \tag{5.44}$$

where $p_\mu = 0$ is the primary constraints . The canonical Hamiltonian for the system is

$$
\begin{aligned}
H &= p_x\dot{x} + p_y\dot{y} + p_\mu\dot{\mu} - L \\
&= p_x\dot{x} + p_y\dot{y} + p_\mu\dot{\mu} - \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \mu\left(bx^2 + a^2y^2 - a^2b^2\right) \\
&= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \mu\left(bx^2 + a^2y^2 - a^2b^2\right) \quad ,\ \ p_\mu = 0.
\end{aligned} \tag{5.45}
$$

Thus, the linear Lagrangian will be

$$
\begin{aligned}
L &= p_x\dot{x} + p_y\dot{y} - H \\
&= p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \mu\left(bx^2 + a^2y^2 - a^2b^2\right).
\end{aligned} \tag{5.46}
$$

We arrive to an important situation that deserves to be given some observations. if we follow directly the algorithm above, we may find as follows in the next step using our independent variables x, y, μ, p_x and p_y. The correspondent (E-L) equations lead us to

$$
\begin{aligned}
\dot{p}_x + 2\mu b^2 x &= 0 \\
\dot{p}_y + 2\mu a^2 y &= 0 \\
\left(bx^2 + a^2y^2 - a^2b^2\right) &= 0 \\
-\dot{x} + \frac{p_x}{m} &= 0 \\
-\dot{y} + \frac{p_y}{m} &= 0.
\end{aligned}
$$

Under the matrix form, we have

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}}_{=f} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\mu} \\ \dot{p}_x \\ \dot{p}_y \end{pmatrix} = \begin{pmatrix} 2xb^2\mu \\ 2ya^2\mu \\ a^2b^2 - b^2x^2 - a^2y^2 \\ \frac{p_x}{m} \\ \frac{p_y}{m} \end{pmatrix}. \tag{5.47}$$

The calculation of the determinant of f leads that it is singular with $rank\,(f) = 4$. Therefore, this matrix admits one zero mode under this relation $n+R-rank(f) = 3+2-4 = 1$, that is given by

$$v = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{5.48}$$

Multiplying(5.47) on the left side by (5.48), we may obtain a supplementary constraint

$$\phi = a^2b^2 - b^2x^2 - a^2y^2 = 0 \tag{5.49}$$

which is nothing but expresses the ellipse equation as it should be. However, we know that this constraint must be introduced in the Lagrangian (5.46). As in circle application we choose $\mu = \dot{\alpha}$, and we write our Lagrangian (5.46) as follows

$$L = p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \dot{\alpha}\left(b^2x^2 + a^2y^2 - a^2b^2\right). \tag{5.50}$$

Our independent variables becomes $(x, y,\ p_x\ ,\ p_y$ and $\alpha)$.

$$\begin{aligned} \dot{p}_x + 2b^2x\dot{\alpha} &= 0 \\ \dot{p}_y + 2a^2y\dot{\alpha} &= 0 \\ -\dot{x} + \frac{p_x}{m} &= 0 \\ -\dot{y} + \frac{p_y}{m} &= 0 \\ -2b^2x\dot{x} - 2a^2y\dot{y} &= 0. \end{aligned}$$

Under the matrix form, we get

$$\underbrace{\begin{pmatrix} 0 & 0 & -1 & 0 & -2b^2x \\ 0 & 0 & 0 & -1 & -2a^2y \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 2b^2x & 2a^2y & 0 & 0 & 0 \end{pmatrix}}_{f^{(0)}} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \\ \dot{p}_y \\ \dot{\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{p_x}{m} \\ \frac{p_y}{m} \\ 0 \end{pmatrix}, \tag{5.51}$$

$f^{(0)}$ is singular and $rank\left(f^{(0)}\right) = 4$, where it has one zero mode given by

$$\upsilon = \begin{pmatrix} 0 & 0 & -2b^2x & -2a^2y & 1 \end{pmatrix}. \tag{5.52}$$

Multiplying the system (5.51) by that latter (5.52), we obtain a new supplementary constraint

$$-\frac{2}{m}\left(b^2xp_x + a^2yp_y\right) = 0. \tag{5.53}$$

This constraint must be introduced in the Lagrangian of the starting (5.50). Thus, we write

$$L = p_x\dot{x} + p_y\dot{y} - \frac{1}{2m}\left(p_x^2 + p_y^2\right) - \dot{\alpha}\left(b^2x^2 + a^2y^2 - a^2\right) - 2\frac{\dot{\beta}}{m}\left(b^2xp_x + a^2yp_y\right). \tag{5.54}$$

Our variables now are x, y , p_x, p_y, α and β.The corresponding E-L equations are

$$\begin{aligned}
\dot{p}_x + 2b^2x\dot{\alpha} + 2b^2\frac{p_x}{m}\dot{\beta} &= 0 \\
\dot{p}_y + 2a^2y\dot{\alpha} + 2a^2\frac{p_y}{m}\dot{\beta} &= 0 \\
\dot{x} - 2b^2\frac{x}{m}\dot{\beta} &= \frac{p_x}{m} \\
\dot{y} - 2a^2\frac{y}{m}\dot{\beta} &= \frac{p_y}{m} \\
-2b^2x\dot{x} - 2a^2y\dot{y} &= 0 \\
-\frac{2}{m}\left[b^2\left(\dot{x}p_x + x\dot{p}_x\right) + a^2\left(\dot{y}p_y + y\dot{p}_y\right)\right] &= 0
\end{aligned}$$

Under the matrix form, we get

$$\underbrace{\begin{pmatrix}
0 & 0 & -1 & 0 & -2b^2x & -\frac{2b^2p_x}{m} \\
0 & 0 & 0 & -1 & -2a^2y & -\frac{2a^2p_y}{m} \\
1 & 0 & 0 & 0 & 0 & -\frac{2b^2x}{m} \\
0 & 1 & 0 & 0 & 0 & -\frac{2a^2y}{m} \\
2b^2x & 2a^2y & 0 & 0 & 0 & 0 \\
\frac{2b^2p_x}{m} & \frac{2a^2p_y}{m} & \frac{2b^2x}{m} & \frac{2a^2y}{m} & 0 & 0
\end{pmatrix}}_{f^{(1)}}
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p}_x \\ \dot{p}_y \\ \dot{\alpha} \\ \dot{\beta} \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \frac{p_x}{m} \\ \frac{p_y}{m} \\ 0 \\ 0 \end{pmatrix} \tag{5.55}$$

Noting in the begining that the matrix $f^{(1)}$contains the matrix $f^{(0)}$ as a sub matrix. Moreover, $f^{(1)}$is invertible and its inverse is

$$\left(f^{(1)}\right)^{-1}=\begin{pmatrix} 0 & 0 & \frac{a^4y^2}{a^4y^2+b^4x^2} & -\frac{a^2b^2xy}{a^4y^2+b^4x^2} & \frac{b^2x}{2(a^4y^2+b^4x^2)} & 0 \\ 0 & 0 & -\frac{a^2b^2xy}{a^4y^2+b^4x^2} & \frac{b^4x^2}{a^4y^2+b^4x^2} & \frac{a^2y}{2(a^4y^2+b^4x^2)} & 0 \\ -\frac{a^4y^2}{a^4y^2+b^4x^2} & \frac{a^2b^2xy}{a^4y^2+b^4x^2} & 0 & -\frac{a^2b^2L_z}{a^4y^2+b^4x^2} & -\frac{b^2p_x}{2(a^4y^2+b^4x^2)} & \frac{b^2mx}{2(a^4y^2+b^4x^2)} \\ \frac{a^2b^2xy}{a^4y^2+b^4x^2} & -\frac{b^4x^2}{a^4y^2+b^4x^2} & \frac{a^2b^2L_z}{a^4y^2+b^4x^2} & 0 & -\frac{a^2p_y}{2(a^4y^2+b^4x^2)} & \frac{a^2my}{2(a^4y^2+b^4x^2)} \\ -\frac{b^2x}{2(a^4y^2+b^4x^2)} & -\frac{a^2y}{2(a^4y^2+b^4x^2)} & \frac{b^2p_x}{2(a^4y^2+b^4x^2)} & \frac{a^2p_y}{2(a^4y^2+b^4x^2)} & 0 & -\frac{m}{4(a^4y^2+b^4x^2)} \\ 0 & 0 & \frac{-b^2mx}{2(a^4y^2+b^4x^2)} & \frac{-a^2my}{2(a^4y^2+b^4x^2)} & \frac{m}{4(a^4y^2+b^4x^2)} & 0 \end{pmatrix}.$$

The generalized Poisson brackets of the dynamic variables contained in the symplectic matrix $\left(f^{(1)}\right)^{-1}$ are identical to the Dirac's brackets obtained by his method in the same treatment. For example, we mention the next brackets

$$\begin{aligned} \{x,p_x\}_{GPB} &= \frac{a^4y^2}{a^4y^2+b^4x^2}=1-\frac{b^4x^2}{(b^4x^2+a^4y^2)}=\{x,p_x\}_D \\ \{y,p_y\}_{GPB} &= \frac{b^4x^2}{a^4y^2+b^4x^2}=1-\frac{a^4y^2}{(b^4x^2+a^4y^2)}=\{y,p_y\}_D \end{aligned}$$

5.3 Notes and results

It must be noted that we dealt in the two above-mentioned applications with Lagrangians of the first order with two ways that are technically different of Dirac and (F-J).These both methods enabled us to reach the Dirac's brackets which considerd as important entrance to the quantize theory with fully compatible results. There is no doubt that the F-J method was much faster and more economical .We can recognize that effictively in givinig us those Dirac's brackets in one fell swoop as a matrix elements, while Dirac's conjecture gave us the same result, one by one under many Poisson brackets calculations.

It is clear also that we didn't use much steps and notions such as weak and strong equality, constraint classifications, and there is also reduction in constraints number in F-J method.

We need to mention that it is axiomatic that the brackets obtained in the ellipse application can lead us to the same one obtained for a particle moving on circle in specific condition where $a=b$.Indeed, this is what we may get clearly in our brackets.

Finally we may say that the effective role of Dirac's conjecture can't be denied, but (F-J) method is considered to be more successful and attractive in practice.

Chapter 6

General conclusion and perspectives

The importance of singular Lagrangians is fundamental to the fact that when we express the four konwn fundamental interactions in a Lagrangian or Hamiltonian framework correspond precisely to this kind of systems.

In this thesis, we studied mainly the classical dynamics of systems described by singular lagrangians by two methods namely: the Dirac-Bergmann algorithm and the Faddeev-Jackiw method. Firsteval, we started by defining the notion of singular Lagrangian through simple and illustrative examples where we recalled also the necessary concepts and methods of analytical mechanics.

For a singular Lagrangian, the conjugate momenta are not all invertible with respect to velocities. Therefore, there may exist constraints that will be generated from the system, and called primary constraints. In this case, it is necessary that the canonical equations must be corrected so that they contain the constraints in question. These primary constraints generate another constraints called secondary constraints. In fact, it was Dirac then Bergmann, who proposed an algorithm seventy years ago, which allows to determine all the constraints of the system with an iterative process by applying certain consistency conditions. Although the Dirac method is very powerful and consistent, it requires considerable computation of the basic geometric structures known as Dirac's brackets. For this reason, another method was proposed later by Faddeev and Jackiw to study standard and singular systems.This method consists firsteval to linearize the Lagrangian with respect to velocities, and then in inverting the symplectic matrix obtained using the Euler-Lagrange equations. The main advantage of Faddeev and Jackiw technique is that it manages in several situations and lead to derive directly the Dirac brackets in one fell swoop, with the fewest possible concepts, counter to Dirac's approach, where the bridge to the quantize theory need to be calculated separately moving through specific notions in the treatment.

Our comparaison study focused on the compatibility of the both methods in giving us consistent results that appear specifically in the suitability of Dirac brackets obtained by the both methods.While, some studies ([3] and [7]) indicated the possibility that this compatibility is submitted to certain conditions that we hope to investigate later.

Annex

Zero mode matrix

Zero mode matrix A are the basis vectors of the null space of this matrix $nul\,(A)$.the null space of A is defined as :

$$nul\,(A) = \{x,\ Ax = 0\}$$

We can find the basis vectors of $nul\,(A)$ easily using "scientific work place" by writing the matrix then we put the cursor just next to this matrix and follow the following path:

Compute → Matrices → Nullspace Basis

Example

Considering the matrix

$$\begin{pmatrix} 0 & a & 1 \\ -a & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \text{ null space basis: } \begin{pmatrix} 0 \\ -\frac{1}{a} \\ 1 \end{pmatrix}.$$

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Summary

There is many physical domains that concerne on the quantum study for its systems, where the Poisson brackets of the classical mechanics is considered as the main bridge to commutators in the quantize theory.This treatment is made for the standard systems, where our lagrangian is non singular.

Our study aims to treat constrained systems that is described by singular Lagrangians to get an alternative and correct entrance to the quantum study called Dirac's brackets presenting the Fadeev-Jackiw method as much simpler, faster and economical treatment to reach them comparing to Dirac's one in one fell swoop as matrix elements without going through many notions and calculations .

To show the difference between the both methods, we depended on the general analytical study of these treatments Supported by illustrative examples in addition to two special applications of a particle moving on circle and another one on ellipse.

Résumé

Il existe de nombreux domaines physiques qui concernent à l'étude quantique de leurs systèmes, où les crochets de Poisson de la mécanique classique sont considérés comme le pont principal vers les commutateurs dans la théorie de la quantification. Ce traitement est fait pour les systèmes standards, où notre Lagrangien est non singulier.

Notre étude vise à traiter des systèmes avec contraintes qui sont décrits par des Lagrangiens singuliers pour obtenir une entrée alternative et correcte à l'étude quantique appelée les crochets de Dirac présentant la méthode de Fadeev-Jackiw comme un traitement plus simple, rapide et économique pour les atteindre d'un seul coup sous forme des éléments de matrice sans passer par de nombreuses notions et calculs comparés à celui de Dirac.

Pour montrer la différence entre les deux méthodes, nous avons adopté l'étude analytique générale de ces traitements appuyée par des exemples illustratifs, en plus de deux applications spéciales d'une particule se déplaçant sur un cercle, et une autre sur un ellipse.

Keywords: Constraints, Singular Lagrangian, Dirac-Bergmann algorithm, Dirac brackets, Faddeev-Jackiw method, Symplectic matrix.

الملخص

هناك العديد من المجالات الفيزيائية التي تهتم بالدراسة الكمومية لأنظمتها، حيث تُعتبر أقواس بواسون للميكانيكا الكلاسيكية بمثابة الجسر الرئيسي للمبدلات في نظرية التكميم ، ويتم إجراء هذه المعالجة للأنظمة القياسية ، حيث يكون لاغرونجيان النظام غير شاذ.

تهدف دراستنا إلى معالجة الأنظمة المقيدة الموصوفة باللاغرانجيان الشاذ، للحصول على مدخل بديل وصحيح للدراسة الكمومية الموصوفة بأقواس ديراك مستعرضة طريقة فاديف-جاكيو كأبسط وأسرع طريقة وأكثر اقتصادا في التوصل إليها مرة واحدة كعناصر مصفوفة دون المرور بالعديد من المفاهيم والحسابات مقارنة بمنهجية ديراك.

لإظهار الاختلاف بين الطريقتين ، اعتمدنا على الدراسة التحليلية العامة لهاتين الطريقتين، مدعمة بأمثلة توضيحية، بالإضافة إلى تطبيقين خاصين لجسيم يتحرك على دائرة وآخر على قطع ناقص.